Paths of specified length in a random k-partite graph

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Fix positive integers k and l. Consider a random k-partite graph on n vertices obtained by partitioning the vertex set into $V_i, (i = 1, \ldots, k)$ each having size $\Omega(n)$ and choosing each possible edge with probability p. Consider any vertex x in any V_i and any vertex y. We show that the expected number of simple paths of even length l between x and y differ significantly depending on whether y belongs to the same V_i (as x does) or not. A similar phenomenon occurs when l is odd. This result holds even when k, l vary slowly with n. This fact has implications to coloring random graphs. The proof is based on establishing bijections between sets of paths.

Keywords: random graphs, paths, bijections

1 Motivation

This problem arose in the analysis of algorithms for coloring random k-colorable graphs [2, 3]. Consider a random graph drawn as explained in the abstract. To separate a color class, we fix a vertex x in the largest (or smallest) V_i and and compute the number of l-paths (paths of length l), $n(x, y, l)$, between x and an arbitrary vertex y. Depending on whether y belongs to the same class as x belongs to, the expectation of this quantity differs significantly. If we can show that $n(x, y, l)$ is close to its expected value almost surely, this gives us a way of separating the class containing x. Repeating this $k - 2$ times, one gets a k-coloring. The expectation of $n(x, y, l)$ is $N(x, y, l)p^l$, where $N(x, y, l)$ is the total number of l-paths in the complete k-partite graph formed by V_is. The result stated in the abstract shows that the expectations differ significantly as required.

We do not discuss the algorithmic issues here since they have been outlined in [2]. We only prove the results stated in the abstract using only counting arguments. Even though the results are obviously true for bipartite graphs, for $k \geq 3$, it is not so straightforward. We believe the arguments used here would be of interest to know. The basic idea is to partition (for each pair of start-end vertices) the corresponding set of l-paths into groups (based on the color classes of intermediate vertices). Then, for two different

pairs, we establish an (almost) bijection between the corresponding groups. For any such mapped pair of groups, we also establish an (almost) bijection between the *l*-paths in them. This establishes the required statement.

2   Paths of specified length

Definition 2.1 *By an l-path between two vertices x and y, we mean a* simple path *of length l between x and y. A simple path is one in which no vertex appears more than once. An l-path is represented as a $(l+1)$-tuple $\langle x, v_1, \ldots, v_{l-1}, y \rangle$ of vertices such that successive vertices in this sequence belong to different partite sets V_i.*

Notations : *G* is a *complete k-partite* graph on the partite sets $V_1, \ldots, V_k$ with each $|V_i| \geq n/C$ for some constant $C \geq k$. For each *i*, n_i denotes the size $|V_i|$. For each *i*, let $W_i \doteq V_i \cup \ldots \cup V_k$. For all $i(1 \leq i \leq k-1)$, for all $x \in V_i$, for all $y \in W_i$ such that $y \neq x$, let $N(x, y, l, i)$ denote the number of *l*-paths between *x* and *y* involving only vertices from W_i. Given a tuple σ with integral component values and an integer *j*, $c(\sigma, j)$ denotes the number of times *j* appears in σ.

   We obtain the following results.

Theorem 2.1 *Assume that $n_1 \leq \ldots \leq n_k$. Let l be any fixed* even *integer ≥ 2. For all i, $1 \leq i \leq k-1$, for all $x, y \in V_i$, for all $z \in W_i - V_i$, we have*

$$N(x, z, l, i) \;=\; \Theta(n^{l-1}) \text{ if } i \leq k-2$$
$$N(x, y, l, i),\, N(x, y, l, i) - N(x, z, l, i) \;=\; \Theta(n^{l-1})$$

Proof: Consider any $i(i = 1, \ldots, k-1)$ and any $x, y \in V_i$ and $z \in V_r, r > i$ and fix these parameters. We use the factorial functions defined as follows : $(n)_0 = 1.\ (n)_l = n(n-1)\ldots(n-l+1), l \geq 1$. Let $P(x, y)$ denote the set of all *l*-paths between *x* and *y* involving only vertices from the $k - i + 1$ partite sets V_j ($i \leq j \leq k$). $P(x, z)$ is defined similarly. That is,

   $P(x, y) = \{ \langle x, v_1, \ldots, v_{l-1}, y \rangle \,|\,$ the sequence is an *l*-path between *x* and *y*$\}$.

   $P(x, z) = \{ \langle x, v_1, \ldots, v_{l-1}, z \rangle \,|\,$ the sequence is an *l*-path between *x* and *z*$\}$.

Clearly, we have $|P(x, y)| = O(n^{l-1})$ and $|P(x, z)| = O(n^{l-1})$. Also if $i = k-1$, then there are only two partite sets, namely, V_{k-1} and V_k and hence $P(x, z) = \emptyset$ and $N(x, z, l, i) = 0$. Define

   $B_l^s = \{ \langle \sigma_1, \ldots, \sigma_{l-1} \rangle \,|\, \sigma_1 \neq i,\, \sigma_{l-1} \neq i,\, i \leq \sigma_j \leq k, \sigma_j \neq \sigma_{j+1} \,\forall j \}$

   $B_l^d = \{ \langle \sigma_1, \ldots, \sigma_{l-1} \rangle \,|\, \sigma_1 \neq i,\, \sigma_{l-1} \neq r,\, i \leq \sigma_j \leq k, \sigma_j \neq \sigma_{j+1} \,\forall j \}$

In the above, the superscript *s* (or *d*) is a short notation for the word "same" (or "different"). We have $|B_l^s|, |B_l^d| \leq k^{l-1}$.
   Now $f : P(x, y) \to B_l^s$ is a mapping which identifies each *l*-path $\langle x, v_1, \ldots, v_{l-1}, y \rangle$ with the unique $(l-1)$-tuple $\langle \sigma_1, \ldots, \sigma_{l-1} \rangle$ in B_l^s where if $v_m \in V_j$ then $\sigma_m = j$. Similarly, we can define a mapping $g : P(x, z) \to B_l^d$ which identifies each *l*-path in $P(x, z)$ with a unique $(l-1)$-tuple in B_l^d. We use the elements of B_l^s (or B_l^d) to partition the set $P(x, y)$ (or $P(x, z)$) as follows.

   $P(x, y) = \bigcup_{\sigma \in B_l^s} P_\sigma$ where $P_\sigma = \{ \tau \in P(x, y) \,|\, f(\tau) = \sigma \}$.

$$P(x,z) = \bigcup_{\sigma \in B_l^d} P_\sigma \text{ where } P_\sigma = \{\tau \in P(x,z) \mid g(\tau) = \sigma\}.$$

Now, for each $\sigma \in B_l^s \cup B_l^d$, $|P_\sigma| = (\prod_{i \le j \le k} (n_j)_{c(\sigma,j)}) = (\prod_{i \le j \le k} (n_j)^{c(\sigma,j)}) \cdot [1 - o(1)]$. As a result, for each $\sigma \in B_l^s \cup B_l^d$, $|P_\sigma| = \Theta(n^{l-1})$. The $[1 - o(1)]$ factor arises not only because of factorials, but also because x, y and z have to be excluded from consideration.

Also B_l^s is non-empty and it contains at least one element, namely, the tuple $\langle r, i, r, i, \ldots, r \rangle$. Hence $N(x,y,l,i) = |P(x,y)| = \Theta(n^{l-1})$. Also, if $i \le k-2$, then there are at least 3 partite sets to be considered and hence B_l^d is non-empty. Hence $N(x,z,l,i) = |P(x,z)| = \Theta(n^{l-1})$ if $i \le k-2$.

We need to prove that $|P(x,y)| - |P(x,z)| = \Theta(n^{l-1})$. In order to prove this, it is enough to prove that the following two assertions are true.

1. $|B_l^s| \ge |B_l^d| + 1$ and

2. There exists a *one-to-one* mapping $h : B_l^d \to B_l^s$ such that for each $\tau \in B_l^d$, we have $|P_{h(\tau)}| \ge |P_\tau|[1 - o(1)]$.

We prove that the two assertions are true as follows. Now, partition B_l^s, B_l^d into

$$B_l^s = B_{l,1}^s \cup \ldots \cup B_{l,l-1}^s \cup B_{l,l}^s$$

$$B_l^d = B_{l,2}^d \cup \ldots \cup B_{l,l-1}^d \cup B_{l,l}^d$$

where

$$B_{l,l}^s = \{\sigma \in B_l^s \mid \sigma_{l-1} \ne i, \sigma_{l-1} \ne r\}.$$

$$B_{l,l}^d = \{\sigma \in B_l^d \mid \sigma_{l-1} \ne i, \sigma_{l-1} \ne r\}.$$

$$B_{l,j}^s = \{\sigma \in B_l^s \mid \sigma_{j-1} \ne i, \sigma_{j-1} \ne r, \sigma_m = i, r \text{ for } m \ge j\}, \text{ for } 2 \le j \le l-1.$$

$$B_{l,j}^d = \{\sigma \in B_l^d \mid \sigma_{j-1} \ne i, \sigma_{j-1} \ne r, \sigma_m = i, r \text{ for } m \ge j\}, \text{ for } 2 \le j \le l-1.$$

$$B_{l,1}^s = \{\langle r, i, r, i, \ldots, r \rangle\}$$

Now $B_{l,1}^d$ cannot be defined similarly since l is even. It is easy to see that the definitions form a well-defined partition of B_l^s and B_l^d. In other words, for each $\sigma \in B_l^s$, there exists a unique value of j between 1 and l such that $\sigma \in B_{l,j}^s$. Similarly, for each $\tau \in B_l^d$, there exists a unique value of j between 2 and l such that $\tau \in B_{l,j}^d$.

Now we claim that for all j such that $2 \le j \le l$, $|B_{l,j}^s| = |B_{l,j}^d|$. For $j = l$, this follows from $B_{l,l}^s = B_{l,l}^d$. For $j < l$, consider the mapping $h_j : B_{l,j}^d \to B_{l,j}^s$ defined as follows. Let $\tau \in B_{l,j}^d$ be any tuple. Then, $h_j(\tau) = \sigma$ where σ is defined as

• For all m $(1 \le m \le j-1)$, $\sigma_m = \tau_m$.

• For all m such that $j \le m \le l-1$, $\sigma_m = i$ if $\tau_m = r$ and $\sigma_m = r$ if $\tau_m = i$.

Clearly $\sigma \in B_{l,j}^s$. Also it can be verified that h_j is a one-to-one and onto mapping. Since $B_{l,j}^s$ and $B_{l,j}^d$ are finite sets, it follows that $|B_{l,j}^s| = |B_{l,j}^d|$.

Thus, we have $|B_l^s| \geq |B_l^d| + 1$ and the first assertion is true.

To prove the second assertion, define the mapping $h : B_l^d \rightarrow B_l^s$ to be as follows. For each $\tau \in B_l^d$, define $h(\tau) = h_j(\tau)$ where j is such that $\tau \in B_{l,j}^d$. Clearly, h is a one-to-one mapping since each h_j is a one-to-one mapping.

We prove that for each $\tau \in B_l^d$, we have $|P_{h(\tau)}| \geq |P_\tau|[1 - o(1)]$. Let $\tau \in B_l^d$ be any tuple and let σ denote the tuple $h(\tau)$. We know $\tau \in B_{l,j}^d$ for some j, $2 \leq j \leq l$.

If $j = l$, then we have $\sigma = \tau$ and hence $|P_\sigma| \geq |P_\tau|[1 - o(1)]$.

If $j = l-2, l-4, \ldots, 2$, then clearly, $c(\tau, m) = c(\sigma, m)$ for all values of m ($i \leq m \leq k$) and hence $|P_\sigma| \geq |P_\tau|[1 - o(1)]$.

If $j = l-1, l-3, \ldots, 3$, then clearly, $c(\tau, m) = c(\sigma, m)$ for all values of m ($i \leq m \leq k$) such that $m \neq i$, $m \neq r$. Also, $c(\sigma, r) = c(\tau, r) + 1$ and $c(\tau, i) = c(\sigma, i) + 1$. Since $n_i \leq n_r$ ($r > i$) by assumption, we have $|P_\sigma| \geq |P_\tau|[1 - o(1)]$.

Thus, we have

$$N(x,y,l,i) - N(x,z,l,i) \;=\; |P(x,y)| - |P(x,z)|$$

$$= \left| \bigcup_{\sigma \in B_l^s} P_\sigma \right| - \left| \bigcup_{\tau \in B_l^d} P_\tau \right|$$

$$= \sum_{j=l-1,\ldots,3} \left(\sum_{\sigma \in B_{l,j}^s} |P_\sigma| \right) + \sum_{j=l,l-2,\ldots,2} \left(\sum_{\sigma \in B_{l,j}^s} |P_\sigma| \right) + \sum_{\sigma = \langle r,i,\ldots,r \rangle} |P_\sigma|$$

$$- \sum_{j=l-1,\ldots,3} \left(\sum_{\tau \in B_{l,j}^d} |P_\tau| \right) - \sum_{j=l,l-2,\ldots,2} \left(\sum_{\tau \in B_{l,j}^d} |P_\tau| \right)$$

$$\geq \;\; |P_\sigma| - o(|P_\sigma|) \;\; \text{where } \sigma = \langle r,i,\ldots,r \rangle. \tag{1}$$

Thus,

$$N(x,y,l,i) - N(x,z,l,i) \;=\; \Theta((n_r)_{l/2}(n_i)_{l/2-1})$$
$$= \;\Theta(n^{l-1})$$

Hence,

$$N(x,z,l,i) \;=\; \Theta(n^{l-1}) \text{ if } i \leq k-2$$
$$N(x,y,l,i),\, N(x,y,l,i) - N(x,z,l,i) \;=\; \Theta(n^{l-1}).$$

This completes the proof of the theorem.   ∎

Using similar arguments, we can prove the following theorem also.

Theorem 2.2 *Assume that $n_1 \geq \ldots \geq n_k$. Let l be any fixed* odd *integer ≥ 3. For all i $(1 \leq i \leq k-1)$, for all $x, y \in V_i$, for all $z \in W_i - V_i$, we have*

$$N(x, y, l, i) = \Theta(n^{l-1}) \ if \ i \leq k-2$$
$$N(x, z, l, i), N(x, z, l, i) - N(x, y, l, i) = \Theta(n^{l-1})$$

3   Conclusions

1. The main result of the paper is that the number of l-paths joining a vertex x (in the largest or smallest V_i depending on the parity of l) and a vertex y differs significantly depending on where y comes from. A close look at the proof (particularly, derivation of (1)) shows that this holds even if we allow k, l and C to vary with n, provided $lk^l C^l = o(n)$.

2. It would be interesting to extend these results to structures other than simple paths. Such results can be applied to the design and analysis of efficient algorithms for random graphs (see [1] for a survey).

References

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