The Work of Curtis T. McMullen

Steve Smale

Curtis T. McMullen has been awarded the Fields Medal for his work in dynamics as well as for his contributions to the theory of computation, complex variables, geometry of three manifolds, and other areas of mathematics. I limit myself here to a brief discussion of some of his results.

The search for understanding of solutions of a polynomial equation has had a central and glorious place in the history of mathematics. Already the ancient Greek mathematicians had approximated the square root of two, i.e., the solution of $x^2 = 2$ by what is now called Newton's Method. Providing a solution for equations such as $x^2 + 1 = 0$ led to the introduction of complex numbers in mathematics. Group theory was introduced to understand which polynomial equations could be solved in terms of radicals. Earlier there had been such formulas for degrees 2 (the quadratic formula taught in high school), 3 and 4. For degrees greater than 4 there are no such formulae.

Instead of formulae, algorithms have been developed which produce (perhaps by complex routines) a sequence of better and better approximations to a solution of a general polynomial equation. In the most satisfactory case, iteration of a single map, Newton's Method, converges to a zero for almost all quadratic polynomials and initial points; it is a "generally convergent algorithm." But for degree 3 polynomials it converges too infrequently.

Thus I was led to raise the question as to whether there existed for each degree such a generally convergent algorithm which succeeds for all polynomial equations of that degree.

McMullen answers this question in his thesis, under Dennis Sullivan, where he shows that no such algorithm exists for polynomials of degree greater than 3, and for polynomials of degree 3 he produces a new algorithm which does converge to a solution for almost all polynomials and initial points.

Thus McMullen "finished the job" since this work answers, in degree 3, "yes," and degree greater than three, "no;" it is complete. This indicates his depth of understanding of the situation and is characteristic of his later work.

For the proof of his result McMullen establishes a rigidity theorem for full families of rational maps of $\mathbb{C}$ into $\mathbb{C}$ with no attracting cycles other than fixed points. Members of such families are conjugate by a linear fractional (Moebius) transformation. The attracting cycles condition is implied by the general convergence.

One obtains radicals by Newton's method applied to the polynomial

$$f(x) = x^d - a\,,$$

starting from any initial point. In this way solution by radicals can be seen as a special case of solution by generally convergent algorithms. This fact led Doyle and McMullen to extend Galois Theory for finding zeros of polynomials. This extension uses McMullen's thesis together with the composition of generally convergent algorithms (a "tower") and the introduction of finite Moebius groups.

They showed that the zeros of a polynomial could be found by a tower if and only if its Galois group is nearly solvable, extending the notion of solvable with the inclusion of the Moebius group A_5 (the alternating group). As a consequence, for polynomials of degree bigger than 5 no tower will succeed.

For degree 5, Doyle and McMullen construct an algorithm following some ideas dating to Felix Klein's famous lectures on the quintic and the icosahedron, and using the classical theory of invariant polynomials. Thus the power of the tower of generally convergent algorithms is found. Quite beautiful!

T. Y. Li and Jim Yorke introduced the word "chaos" into dynamics in connection with the map of population biology,

$$L_r : [0,1] \to [0,1] , \quad L_r(x) = rx(1-x) .$$

Bob May had been intrigued by this map because there was an infinite sequence of period doubling parameters r_i converging to $s = 3.57....$

Soon thereafter, Mitch Feigenbaum's work (with similar results due to Coullet-Tresser) demonstrating the universality properties of this map, helped establish the acceptance by physicists of the new discipline of dynamical systems. The sequence $(r_i - r_{i-1})/(r_{i+1} - r_i)$ has a limit, a number which is independent of the period doubling map! Key to Feigenbaum's work was the concept of renormalization and the convergence of the renormalizations of an iterate of the Feigenbaum map L_s to a fixed point F of the renormalization operator.

Let us see what renormalization means for the second iterate L^2 of $L = L_r$ for some $2 < r < 4$. So $L([0,1]) \subset [0,1]$ as above, and L has a second fixed point $q = (r-1)/r$. Define p by the conditions $0 < p < q$ and $L(p) = q$. Thus L^2 acts on $[p,q]$ (with a sign reversal) something like L on $[0,1]$. If $L^2([p,q]) \subset [p,q]$ the conditions for renormalization are present. Let A be the map $Ax = (x-q)/(p-q)$, sending $[p,q]$ onto $[0,1]$. The renormalized L^2 is given by $RL(x) = AL^2A^{-1}(x)$, where R is the renormalization operator acting on L.

For certain r one may be able to repeat this process. If one can do it indefinitely then L is called infinitely renormalizable. This is a very special situation but occurs for the Feigenbaum map L_s above.

Lanford found computer assisted proofs of the conjectures of Feigenbaum and subsequently Sullivan put them into a broader, detailed, conceptual framework, finding important relations between 1-dimensional dynamics and parts of classical function theory as Kleinian groups.

Yet the proof of fast (exponential) convergence of the renormalizations, a basic ingredient in this program, was missing until McMullen's beautiful work was published in the second of his two Annals of Math Studies in 1996. The fast convergence was necessary to yield the crucial rigidity of the theory ("$C^{1+\alpha}$ conjugacy").

With the notation as above, McMullen's result for the Feigenbaum map may be expressed by the estimate:

$$\left|R^k L_s(x) - F(x)\right| < c\beta^k\,, \quad \beta < 1\,.$$

Complex one dimensional dynamics is the study of the iterates of a polynomial map $P : \mathbb{C} \to \mathbb{C}$. This has become the most advanced and the most technical part of dynamics. Yet one simple problem may be singled out as giving some focus to this subject.

Among polynomial maps of a given degree d, are the hyperbolic ones dense? A polynomial is called hyberbolic (sometimes axiom A), if the orbits of its critical points tend under time to an attracting cycle ("including infinity").

I naively gave this as a thesis problem in the 1960's. Today it is still unsolved even for $d = 2$, but there are a number of partial results.

Quadratic dynamics may be studied for polynomials in the normalized form

$$P_c(z) = z^2 + c$$

with parameter $c \in \mathbb{C}$. The unique critical point is zero and if it tends to ∞ under iteration, the dynamics is well understood in terms of symbolic dynamics. The Mandelbrot set M is defined as the set of $c \in \mathbb{C}$ for which this is not the case. This often pictured set can be thought of as a "tree with fruit," the fruit being the components of its interior. McMullen proves in the first of his Annals of Math Studies:

If c is in a component of the interior of the Mandelbrot set which meets the real axis, then P_c is hyperbolic.

As McMullen writes, "if one runs the real axis through M, then all the fruit which is skewered is good."

Earlier Yoccoz had done an important special case, and I am ignoring here much other earlier fundamental work in complex (and real) dynamics such as Fatou, Julia, Douady and Hubbard. I am also ignoring the later work of Lyubich and Graczyk-Swiatek.

Again the ideas of renormalization play a big role in the proof but now in the context of complex maps.

To describe more precisely these ideas, the idea of a quadratic-like map is useful. A quadratic polynomial map $\mathbb{C} \to \mathbb{C}$ is a proper map of degree 2. A holomorphic proper map $f : U \to V$ of degree 2, with the closure of U a compact subset of V, and having a critical point q in U, is called quadratic-like. Here U, V are supposed simply connected open sets of the complex numbers. For example, an iterate of a quadratic polynomial restricted to an appropriate neighborhood of its critical point is often quadratic like. If moreover, the critical point of f doesn't escape (all the iterates of q are well defined), then according to Douady-Hubbard, this map is topologically conjugate to a quadratic map of the form $P_c(z) = z^2 + c$, for some c in the Mandelbrot set M.

The map $P_c(z) = z^2 + c$ with $c \in M$ is said to be renormalizable if P_c^n is quadratic-like, the critical point $0 \in U$ and 0 doesn't escape. P_c is called infinitely renormalizable if there are infinitely many values of such n. For the problem of density of hyperbolic polynomials in degree two, the case of finitely renormalizable points had been dealt with earlier by Yoccoz. McMullen's work is on the problem of infinitely renormalizable points in M. It contains an intricate analysis of the dynamics of these maps.

Moreover in these two books McMullen establishes new results in complex function theory and the geometry of 3-manifolds.

Another important result of McMullen is his proof of Kra's "Theta conjecture." Let X be a compact Riemann surface with a finite number of points removed and its associated Riemannian curvature constant at -1, in other words a hyperbolic surface. Its universal covering, $\Delta \to X$, has as its group of covering transformation, G, the fundamental group of X. Let $Q(\Delta)$ be the space of holomorphic quadratic differentials ϕ with finite norm given by $||\phi|| = \int |\phi|$ and similarly define $Q(X)$. To $\phi \in Q(\Delta)$ one may associate $\Theta\phi \in Q(X)$ by the formula $\Theta\phi = \sum g^*\phi$, the sum being over the elements g of G. This is well defined since the sum is G-invariant. The sum is the Poincaré series.

It is easily shown that the norm of this operator Θ is less than or equal to one. Kra's conjecture and McMullen's theorem asserts that in fact $||\Theta||$ is strictly less than one. But McMullen proves much more. For a general class of coverings $Y \to X$ of Riemann surfaces he characterizes those for which his conclusion is true (in terms of "amenable" covers).

Armed with this work on Kra's conjecture, he is able to make a substantial contribution to Thurston's program of introducing hyperbolic structures for a large class of 3-manifolds.

I have given a brief glimpse of what Curt McMullen has accomplished, but would like to emphasize that his work has encompassed a large realm of the kind of mathematics that lies at the cross-section of many paths of our rich culture. McMullen is not a dynamicist, not an analyst nor a geometer. He is a mathematician.

Important Publications of McMullen

P. Doyle and C. McMullen. Solving the quintic by iteration. *Acta Math.* 163(1989), 151–180.

C. McMullen. Families of rational maps and iterative root-finding algorithms. *Annals of Math.* 125(1987), 467–493.

C. McMullen. Amenability, Poincaré series and quasiconformal maps. *Invent. math.* 97(1989), 95–127.

C. McMullen. Iteration on Teichmüller space. *Invent. math.* 99(1990), 425–454.

C. McMullen. Cusps are dense. *Annals of Math.* 133(1991), 217–247.

C. McMullen. Riemann surfaces and the geometrization of 3-manifolds. *Bull. Amer. Math. Soc.* 27(1992), 207–216.

C. McMullen. *Complex Dynamics and Renormalization.* Annals of Math Studies 135, Princeton University Press, 1994.

C. McMullen. Frontiers in complex dynamics. *Bull. Amer. Math. Soc.* 31(1994), 155–172.

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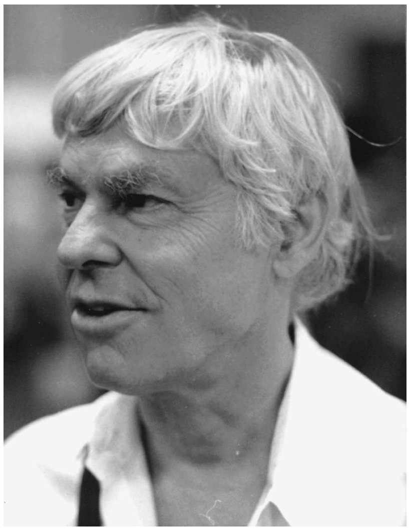

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