O-Minimality

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In the paper [11] Tarski makes the following *observation*: every subset of the set of real numbers $\mathbb{R}$ definable in the ordered ring $\langle \mathbb{R}; +, \cdot, 0, 1, < \rangle$ (which I shall henceforth denote by $\bar{\mathbb{R}}$) is a finite union of open intervals and points. This is certainly an easy consequence of his famous quantifier elimination theorem ([12]) - that every subset of $\mathbb{R}^n$ definable in $\bar{\mathbb{R}}$ is semi-algebraic, i.e. definable by a *quantifier free* formula - and must have seemed a relatively unimportant one at the time. However, it turned out to be a remarkable insight. For in the 1980's van den Dries showed that most of the qualitative geometric and topological finiteness properties enjoyed by the class of semi-algebraic sets actually follow from this observation alone. Indeed, many such properties, e.g. finite cell-decomposition theorems in the continuous category, do not even require the ring structure, although some do, e.g. finite cell-decomposition theorems in the differentiable category and finite triangulation theorems.

The property described in Tarski's observation is now known as *o-minimality* and, as was shown by Knight, Pillay and Steinhorn in [7] and [10], it can be fruitfully considered in quite general situations: a structure $\mathcal{M} = \langle M, <, ... \rangle$, where $<$ is a dense, linear order (without endpoints) of the domain M, is called *o-minimal* if every definable (without parameters) subset of M is a finite union of points and open intervals (with endpoints in $M \cup \{\pm\infty\}$).

It is a suprising and non-obvious fact that o-minimality is preserved under elementary equivalence. This is one of the main results of [7] and is typical of the "uniformity-in-parameters" that crops up frequently in this subject: it is equivalent to the statement that for any formula $\phi(x_1, ..., x_n, y)$ of the language of $\mathcal{M}$, there is a natural number N depending only on ϕ such that the set $\{b \epsilon M : \mathcal{M} \models \phi[a_1, ..., a_n, b]\}$ is the union of at most N open intervals and points *for any choice* of parameters $a_1, .., a_n \epsilon M$.

More generally, one can also deduce from the assumption of o-minimality that there are only finitely many homeomorphism types amongst sets of the form $\{< b_1, ..., b_r >\} \epsilon M^r : \mathcal{M} \models \psi[a_1, ..., a_n, b_1, ..., b_r]\}$ as $< a_1, .., a_n >$ varies over M^n, where $\psi(x_1, ..., x_n, y_1, .., y_r)$ is a formula of the language of $\mathcal{M}$. (Here, M^r is equipped with the product topology and M with the order topology.) Similar results and, indeed, a definitive account of the foundations of the general theory of o-minimality can be found in van den Dries' recent book [3].

Of course, this general theory is only worthwhile if there are interesting examples (other than $\bar{\mathbb{R}}$ and its reducts) and it is my main aim in this short note to state a result that provides a rich source of o-minimal expansions of $\bar{\mathbb{R}}$.

Let $\tilde{\mathbb{R}}$ be any expansion of the real ordered field $\bar{\mathbb{R}}$ with language $\tilde{L}$ say. Call a formula ψ of $\tilde{L}$ *tame* if there exists a natural number N (depending only on ψ) such that whenever the free variables of ψ are partitioned into two classes, say $\psi = \psi(x_1, ..., x_m, y_1, ., y_r)$, then the set $\{< b_1, ..., b_r > \epsilon \mathbb{R}^r : \mathcal{M} \models \psi[a_1, ..., a_n, b_1, .., b_r]\}$ has at most N connected components for any choice of $< a_1, ..., a_n > \epsilon \mathbb{R}^n$. Then I know of no counterexample to the following

Conjecture

With $\tilde{\mathbb{R}}$ as above, if every quantifier free formula of $\tilde{L}$ is tame then $\tilde{\mathbb{R}}$ is o-minimal (which, in fact, implies that every formula of $\tilde{L}$ is tame - see [7] again).

I am, however, rather sceptical.

In order to state my result in this direction it is convenient to introduce a unary connective, denoted C, to our language, with truth condition:-

$\tilde{\mathbb{R}} \models (C\phi)[a_1, .., a_n]$ if and only if $< a_1, .., a_n >$ lies in the closure (in $\mathbb{R}^n$) of the set $\{< b_1, ..., b_n > \epsilon \mathbb{R}^n : \tilde{\mathbb{R}} \models \phi[b_1, ..., b_n]\}$.

Clearly C is already definable in $\tilde{L}$ (for interpretations expanding $\bar{\mathbb{R}}$) but the point is that we have the following

Theorem (Wilkie, [14]).

Let $\tilde{\mathbb{R}}$ be as above and suppose that every quantifier free formula is tame. Then so is any formula that can be obtained from quantifier free formulas by finitely many applications of conjunction, disjunction, existential quantification and the connective C. Further, if we also assume that $\tilde{\mathbb{R}}$ has the form $< \bar{\mathbb{R}}, \mathcal{F} >$ where $\mathcal{F}$ is a collection of infinitely differentiable functions from $\mathbb{R}^n$ to $\mathbb{R}$ (for various n's), then *any* formula of $\tilde{L}$ is equivalent (in $\tilde{\mathbb{R}}$) to one of this type, and hence (since a connected subset of $\mathbb{R}$ is an interval) $\tilde{\mathbb{R}}$ is o-minimal.

The reason for proving a theorem of this type was that the tameness condition on the quantifier free definable sets was established for a wide class of examples through the work of Khovanskii ([6], but see also [5] in conjunction with [8]). He showed that it holds for structures of the form $< \bar{\mathbb{R}}, f_1, ..., f_p >$ where $f_1, ..., f_p : \mathbb{R}^n \to \mathbb{R}$ are infinitely differentiable functions (actually, the following implies they are analytic) satisfying a system of partial differential equations of the form

$$\frac{\partial f_i}{\partial x_j} = P_{i,j}\,(x_1, ..., x_n, f_1, ..., f_i),\ 1 \leq i \leq p, 1 \leq j \leq p,$$

where each $P_{i,j}\ (x_1, .., x_n, y_1, ..., y_i)$ is a polynomial with real coefficients. (The sequence $f_1, ..., f_p$ is then called a *Pfaffian chain* on $\mathbb{R}^n$).

Thus, by the theorem, these structures are o-minimal.

In particular, $\langle \bar{\mathbb{R}}, exp \rangle$ is o-minimal, where $\exp(x) = e^x$ is the exponential funtion (take $p = n = 1, P_{1,1}(x_1, y_1) = y_1$). In fact, this result appears in [1] although some of the arguments in that paper are, to my mind, incomplete. However, the main idea there is fundamentally sound and was studied extensively by my student S. Maxwell (see[9]) before I finally adapted it to establish the theorem above. Perhaps I should also mention that in the case of $\langle \bar{\mathbb{R}}, exp \rangle$ we now have better information (see[13]): every definable set is *existentially* definable (from which o-minimality follows very easily from Khovanskii's result). However, nothing like this is known for expansion of $\bar{\mathbb{R}}$ by general Pfaffian chains.
I conclude with an application of the general uniformity result mentioned earlier. Clearly, if we take $\mathcal{M} = \bar{\mathbb{R}}$ then we can deduce that for any n, k there is $N = N(n, k)$ such that there are at most N homeomorphism types of sets of the form $P^{-1}(0)$ where $P : \mathbb{R}^n \to \mathbb{R}$ is a real polynomial of total degree at most k. This was actually proved by Hardt (see[4]) before o-minimality came on the scene. Now van den Dries noticed that if we take $\mathcal{M} = \langle \bar{\mathbb{R}}, exp \rangle$ then Hardt's result may be improved by using a trick of Khovanskii's. Namely, we take the *exponents* of the variables in P, as well as the coefficients, as *parameters* - which we can do as long as we bound the number of monomial terms in P. We then obtain the result that for any n, k there is $N = N(n, k)$ such that there are at most N homeomorphism types of sets of the forms $P^{-1}(0)$ where $P : \mathbb{R}^n \to \mathbb{R}$ is a real polynomial (of arbitray degree but) which is the sum of at most k monomials (i.e. terms of the form $ax_1^{q_1} \cdots x_n^{q_n}$ for $a \epsilon \mathbb{R}$ and $q_1, ..., q_n \epsilon \mathbb{N}$, although, in fact, we could also allow $q_1, ..., q_n \epsilon \mathbb{R}$). (Remark: for $n = 1$ this is usually attributed to Descartes).

More recently Coste ([2]) has proved a uniformity result for the homeomorphism types of definable *functions* in o-minimal structures and this gives a corresponding result for the homeomorphism types of polynomials with a restricted number of monomial terms.

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