Difficulties in the Passage from Secondary to Tertiary Education

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Abstract. For an important part of those students who take mathematics courses at the tertiary level, the transition from secondary to tertiary education presents major difficulties. This is true whether the students are specializing in mathematics or are registered in a program for which mathematics is a service subject. The purpose of this paper is to identify some relevant difficulties related to this passage and to examine possible causes. Such a study can be done from a broad spectrum of perspectives which will be commented briefly: epistemological and cognitive, sociological and cultural, didactical. We also consider actions which could help to improve the situation.

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The passage from secondary to tertiary mathematics education is determined by procedures varying considerably from one country to the other, and even within one country, from one institution to another. But whatever the context, this transition often presents major difficulties for an important part of those students who take mathematics courses at the tertiary level. This is true whether the students being considered are specializing in mathematics or are registered in a program for which mathematics is a service subject.

The problem of the transition to the post-secondary level in by no means a new issue in mathematics education. For instance, the very first volume in the Unesco series *New Trends in Mathematics Teaching* includes a report from a conference devoted to this problem (see [18]). This same topic was also discussed in various settings at ICME congresses — see for instance the paper by Cross [5] presented at ICME-4, as well as the report [13] of Action Group 5 at ICME-6. But still today the secondary–tertiary transition can be seen as a major stumbling block in the teaching of mathematics.

This paper, prepared in connection with a round-table discussion at ICM'98, is concerned with various groups of students taking mathematics courses at the university level: students of science (vg, mathematics, physics, chemistry), engineering, economics, preservice secondary school teacher education, etc. After presenting an overview of the respective points of view of the student and of the teacher on the passage from secondary to tertiary education in mathematics, we shall consider three different types of difficulties then encountered by students: epistemological/cognitive; sociological/cultural; didactical. We conclude the paper with some possible actions, both from an institutional and from a pedagogical perspective, which could help to improve the conditions under which the transition takes place.

1 THE POINT OF VIEW OF THE STUDENT

In order to better assess the perception that students may have of the transition from secondary to tertiary mathematics, a questionnaire was recently given to various first-year groups in our respective universities, asking students for their opinion about three possible types of sources for the difficulties they might have encountered with university mathematics: (i) difficulties linked to the way teachers present mathematics at the university level and to the organization of the classroom; (ii) difficulties coming from changes in the mathematical ways of thinking at the higher level; and (iii) difficulties arising from the lack of appropriate tools to learn mathematics. Students were asked to express their degree of agreement with various statements on a five-item Likert scale (from 1/total disagreement/ to 5/total agreement/, 3 being a neutral point). Here are some of the outcomes of this informal survey.

First, it should be stressed that the perception of students can vary considerably according to the type of mathematics they are taking and the program of study to which they belong. This is the case for instance for the results obtained at Université Laval when students were asked for their overall perception of how they went themselves through the secondary-tertiary transition. From a cohort of 250 students, 91 (36%) were in partial or total agreement (items 4 and 5 on the Likert scale) with the statement *"Transition to university mathematics was difficult for me"*, while 127 (51%) expressed disagreement (items 1 and 2 on the Likert scale). However if the results are considered according to the program of study of the students, the picture gets quite diversified. In the following table, we compare three different groups of students from Université Laval, namely[1]

- Group I: *students specializing in mathematics* (first-year and final-year);
- Group II: *preservice secondary school mathematics teachers* (first-year);
- Group III: *engineering students* (first-year).

[1] It should be noted that transition to university education in Québec typically happens as students are age 19, after a two-year intermediate level following secondary school (the so-called "cégep" level). Students entering university are divided into groups, already in their first year, according to their specific domain (mathematics, physics, engineering, etc.).

"Transition to university mathematics was difficult for me."								
Likert scale	Group I		Group II		Group III		Totals	
1	7	(12%)	3	(4%)	35	(30%)	45	(18%)
2	17	(28%)	19	(26%)	46	(39%)	82	(33%)
3	14	(23%)	5	(7%)	11	(9%)	30	(12%)
4	17	(28%)	37	(51%)	15	(13%)	69	(28%)
5	5	(8%)	8	(11%)	9	(8%)	22	(9%)
no reply	0	(0%)	0	(0%)	2	(2%)	2	(1%)
Totals	60	(100%)	72	(100%)	118	(100%)	250	(100%)

One notes that 22 out of 60 mathematics students (37%) agree that the transition was difficult for them (items 4 and 5 on the Likert scale), as opposed to 45 out of 72 (63%) for the preservice teachers and only 24 out of 118 (20%) in the case of engineering students. Analogous differences can be seen when considering other items of the scale.

The three groups often reacted also quite differently to more targeted questions. For instance,

- more than 85% of the students of mathematics and 75% of the preservice secondary school teachers from Université Laval see assessment at the university level as bearing upon more abstract mathematics than previously (the replies from these two cohorts give a mean of 4,2 on a scale of 5), as opposed to only 38% of the engineering students (mean of 3,0);
- similarly, more than 55% of the non-engineers see the mathematics problems they have to solve at the university level as substantially more difficult than at the secondary level (mean of 3,5), which is the case for only 28% of the engineering students (mean of 2,7).

In fact, a clear outcome of the data from Université Laval is that the transition to university mathematics appears much smoother for engineering students than for preservice secondary school teachers or for students of the mathematics program. (The same questionnaire was used with first-year and final-year students of the undergraduate mathematics program; although the answers were not identical, the variations observed appear far less significant than when comparing with future engineers or secondary school teachers.)

The questionnaire was used in France (Université de Versailles and Université de Montpellier) but gave rise to a somewhat different response from the students: much less diversity was observed in the patterns of answers than at Université Laval. Possibly this results from the fact that the French education system attracts a majority of the best students in special classes ("classes préparatoires") leading to the "grandes écoles", so that university-bound students form a rather homogeneous group. It is interesting to note that, from a cohort of 190 university students in the first year of a scientific program, more than 70% are in partial or total agreement with the following statements:

- *I am not used to proofs and abstract developments*;
- *I would prefer to have a textbook, as in secondary school*;

and more than 66% agree with the statements:

- *it is not always clear what is expected of me regarding what is seen in the classroom*;
- *we are not indicated what is essential and what is accessory*;
- *teachers are too abstract, they don't care to present concrete examples*;

and, more surprisingly,

- *there is not enough time spent in the classroom.*

Two groups of students from the Universidad Complutense de Madrid were given the questionnaire, namely some 70 students from the first three years at the Facultad de Matemáticas and 100 first-year and fifth-year students from the Facultad de Ciencias Económicas. The answers collected are quite similar to those from France, the statements receiving a degree of agreement greater than 3,5 on the Likert scale being those which deal with such aspects as: the high level of abstraction, the use of proofs in the mathematical development, the lecturing style (the fast pace, the ignorance of where it is heading), the abstract nature of part of what is being assessed on exams, the need for textbooks.

The questionnaire invited students to write comments on their own. Needless to say, the spectrum of opinions expressed is extremely wide, but it is interesting to consider a few comments made spontaneously by students of Université Laval. Some are quite severe on the university teachers:

- *Many university teachers do not care whether we understand or not what they are teaching us.*
- *A majority of teachers do not understand that we do not understand.*
- *It is hard for them to make us understand what is evident for them.*
- *Passing from secondary school to university mathematics was not as hard as I was told. But what makes it somewhat hard are the changes in the teachers: many of them are not at all suited for teaching. Here, we have teachers who are topnotch mathematicians. But their pedagogical skills will never outmatch those of my high school teachers.*

Other comments have to do with the background of the students or the autonomy expected of them:

- *It seems that I am lacking a lot of prerequisites. It is as if I should know 100% of my high school maths.*
- *In high school, I never learned to do proofs, and now it seems to be taken for granted that we know how to do proofs.*

- *My answers to these questions would vary considerably according to the courses and the instructors. But a general trend is that courses include many many topics which are covered very quickly, so that we need to work a lot on our own outside the classroom.*

But quite a few students did express a positive opinion about their encounter with university mathematics, reflecting the fact that the transition gives no or little problems to a number of students:

- *I appreciate much more university math, because we try to understand where the results we are using, and were using in high school, come from.*

- *Going from high school to university did not raise special problems for me, as the level of difficulty of high school math prepared us well for that.*

2 PERCEPTIONS OF THE UNIVERSITY TEACHER

When seen from the point of view of the university teacher, the transition from secondary to tertiary mathematics is considered to be problematic for a majority of students. Such is the observation we have made from an informal survey of a small number of teachers regularly involved in the teaching of first-year university mathematics. We think that this survey, however limited, still provides us with a good idea of the perceptions of a lot of our university colleagues.

Those involved in the teaching of first-year university mathematics are often rather dissatisfied with the weaknesses they perceive in their students. Many have the feeling that students are not interested in the mathematics itself covered in a course, but only in succeeding at the exams — this might be especially the case in contexts where mathematics is used as a sieve for accessing to other professional fields, for instance for admission to the medicine or law school. University teachers deplore the lack of prerequisite knowledge which makes the beginning at the tertiary level painful and difficult for many of their students; even the contents indicated in the secondary syllabi (where there is such a thing common to secondary school students) cannot be taken as understood and mastered. They also deplore the learning style of students, many of whom have concentrated in the past on the acquisition of computational skills (often, it must be said, so to meet the requirements of university entrance examinations). They lament over the thinking and working habits of their students in mathematics, their lack of organization and of mathematical rigour, as well as their difficulty in acquiring and consolidating knowledge through personal work.

Acquisition of a certain level of autonomy in learning is often seen by university teachers as the main stumbling block in the secondary-tertiary passage. Zucker [27] has expressed as follows the idea that significant individual activity outside the mathematics class becomes an absolute necessity when moving to the higher level: "The fundamental problem is that most of our current high school graduates don't know how to *learn* or even what it means to learn (a fortiori to understand) something. In effect, they graduate high school feeling that learning must come down to them from their teachers. [...] *That the students must also* *learn on their own, outside the classroom, is the main feature that distinguishes college from high school.*"

3 Types of difficulties in the secondary-tertiary transition

The nature of the difficulties related to the passage from secondary to post-secondary mathematics and the reasons for their occurrences can be seen from a broad spectrum of perspectives.

3.1 Epistemological and cognitive difficulties

As shown by many research works, an important conceptual leap takes place, with respect to the mathematical contents taught and asked practices, when passing from the secondary to the tertiary level. This transition corresponds to a significant shift in the kind of mathematics to be mastered by students: the mathematics is different not only because the topics are different, but more to the point because of an increased depth, both with respect to the technical abilities needed to manipulate the new objects and to the conceptual understanding underlying them. This shift has sometimes been described as corresponding to a move from *elementary* to *advanced mathematical thinking* (see Tall [23]) : secondary school students often succeed in mathematics by relying on their ability to perform algorithms and in spite of a lack of a real understanding of the mathematical concepts with which they are working; they may then experience substantial difficulties, when moving to the tertiary level, in being able to participate by themselves in the process of mathematical thinking, and not merely learn to reproduce mathematical information. In a word, they may have problems in becoming autonomous, mathematically speaking. Moreover, it is no more possible to limit themselves to put isolated theorems in practice, they need to enter into deeper and richer thought processes.

A word of caution is in order here: in a given course, the needs of the students, as perceived by them, are dictated mainly by the exams. In a context where assessment is not congruent with the intended level of the course, being in fact lower, then it would be totally possible for the students, should this be known to them, to succeed in the course without entering into more advanced mathematical thinking. Success in such a context would by no means testify to adequate learning. What we have in mind here is a system where the gap between the level of the course and that of assessment is not too important.

In many countries, the passage to tertiary mathematics coincide with the introduction of new abstract notions such as vector spaces or formalized limits. This is a difficult step because these notions are not in the strict continuity of what students already know (even though vector spaces take their origin in the "spaces of (physical) vectors" $\mathbf{R}^2$ or $\mathbf{R}^3$). We can speak of these notions as "unifying and generalizing concepts", in the following sense (see [7]): such concepts unify and generalize different methods, tools and objects existing previously in a variety of settings; they are formal concepts which unify the various objects from which they have been abstracted. They have not necessarily been created to solve new problems, but to make the solution of many problems easier or more similar to each other. Moreover, these concepts represent a change of perspective which induces a sophisticated change of level in mental operations.

Other concepts are acquiring a different status, when passing from one level of education to another. For instance, the equality of elementary arithmetic becomes in high school algebra a notion of identity. And in analysis equality incorporates the complex idea of "local infinite proximity", i.e., of an arbitrarily good approximation, such as in the expression $\lim_{x\to a} f(x) = L$. The manipulation of equalities in such a context rests in an essential way on inequalities, which fact contributes to the difficulties linked to the passage from algebraic to analytic thinking (see Artigue [1]).

Students entering tertiary education are facing, in the words of Tall [24, p. 495], "a difficult transition, from a position where concepts have an intuitive basis founded on experience, to one where they are specified by formal definitions and their properties reconstructed through logical deductions". Consequently proofs acquire a new and important status. They have to be complete and established through logical deductions from the formal definitions and properties. Only elementary logic is necessary for overcoming this difficulty, but this is far above what is being asked in secondary mathematics. Moreover, the basic logic one uses in mathematics is different from the ordinary logic of everyday life, just as the mathematical language differs from the natural language.

Even for those students already familiar with proofs, new difficulties may arise. For instance existence proofs are notoriously difficult for most students: on the one hand, it is not easy for them to recognize their need, as this type of situation is rarely raised in secondary mathematics — when given a problem, high school students can (almost) always take it for granted that it has a solution; and also existence questions are difficult to solve because one often has to imagine a certain mathematical object — an analysis-synthesis approach can be useful here, but is not always easy to implement. Sufficiency arguments are generally difficult because there is often a choice to be made. Sometimes, a proof requires not only to apply directly a theorem in a particular case, but also to adapt or even to transform a theorem before recognizing and/or using it. In other occasions, a proof involves a multi-stage process. For instance one often encounters situations in analysis where in order to find a limit, a given expression (vg, a sum or an integral) must be "broken" into two parts to be treated separately by different methods; a strong qualitative intuition is essential for one to succeed in such an approach.

Research shows that when facing a new complex mathematical task or notion for which intuition may not be sufficient to represent the situation, some students react by introducing simplistic procedures, like trying to reduce everything to algorithms, or by developing for themselves simplistic models — such is the case for instance with the notion of limit, as observed by Robert [19]. Other students' errors are more linked to the mathematical domain involved. For example, many difficulties encountered by students in analysis have to do with the structure of $\mathbf{R}$, especially the order relation. In algebra, students do not realize all the consequence of structures in terms of the constraints thus being introduced, because structures refer to a new idea. For instance, they are surprised (and even bewildered) by the proof — and even more by the need of a proof — of the fact that in a group there exists only one identity element, which is at the same time the only "left" and the only "right" identity element; or by the fact that there is only one group with three elements. This latter example indicates that students are often unaware of the impact of a given definition on the "degree of freedom" of the elements: in a group with three elements, there is simply no room for freedom!

The above comments deal with epistemological and cognitive difficulties which, in a certain way, are "intrinsic" to mathematics, since they concern the change in the type of mathematics to be mastered by students as they move to tertiary education. We would now like to consider some "extrinsic" difficulties.

One such difficulty has to do with the students themselves: there is a substantial heterogeneity in the mathematical background of students entering university education. Some students are fully ready for the transition to the tertiary level, but others are not. And university teachers often do not care to make sure, at the entrance to university, that each one of them masters the basic notions and skills required for an understanding of their course.

Other difficulties concern more directly the university teachers; for instance, the expectations they might have regarding their students: many university teachers develop a distorted image of students and tend to identify their "average" student with an ideal student who has successfully attended a highly scientific track in one of the best secondary schools. But in an actual class this kind of student may be only a very small minority, or even not exist at all. Teachers must also be sensitive to the importance of making explicit to the students what exactly they are doing and learning, and where they are heading. In other words, they must provide students with identifiable goals, not expecting such insights to emerge naturally by themselves.

Another difficulty concerning university teachers is that they expect students to develop from the beginning an active attitude toward "doing mathematics". But students are often not prepared for this kind of work. The situation is vividly documented in an inquiry which involved several classes of Italian first-year university students enrolled in scientific faculties [2]. More than one third of these students (who had learned many proofs in elementary geometry during their secondary school years) share the belief that *if a proof of some theorem in elementary geometry has been produced by a secondary school pupil (say in grade 9 or 10), then even a clever university student is not entitled to check the correctness or the incorrectness of the proposed proof, without the help of books or experts.* They believe that the authority of a professional mathematician is needed, since *only he knows whether the proposed argument is true.*

A last type of cognitive difficulties we would like to consider is linked to an indispensable organization (or reorganization) of knowledge by students. In order to reach the "advanced mathematical thinking" capacities which are expected of them, students must acquire "the ability to distinguish between mathematical knowledge and meta-mathematical knowledge (e.g. of the correctness, relevance, or elegance of a piece of mathematics)" [21, p. 131], they must come to stand back from the computations and to contemplate the relations between concepts.

It is not possible for students, even through extensive personal work, to have met all possible types of problems pertaining to a specific topic. They thus need to develop a global view bringing forth the connections they need to make. But even when students expect that they will have to modify their view of certain mathematical objects and establish links between them, they often encounter great difficulty in doing so because of a lack of organization of their knowledge. A typical instance of such a situation is found in linear algebra. Students may have learned to solve $f(u) = ku$ using determinants, and also to find eigenvalues. But they may well be unable to recognize these concepts in other situations. For instance they may directly meet the equation $f(w) = qw$ arising from a certain context not immediately linked to eigenvalues; but in order to solve this equation with the techniques they now know, they need to recognize that the crux of the problem has to do with eigenvalues. Still more difficult, after having studied straight lines invariant under an affine application, they must learn to link this problem to the eigenvalues of the associated linear application.

3.2 Sociological and cultural difficulties

A second type of difficulties concerns sociological and cultural factors, especially as seen from an institutional point of view. There is a great diversity of such difficulties and local differences can be quite important, which makes it very difficult to present regularities. We limit ourselves here to a few aspects.

More often than not the size of groups at the tertiary level can be very large, especially in the first year, so that a student is often only one in a crowd. For many students, this represents a major change with respect to secondary school, as was clearly shown in the answers to our questionnaire mentioned in Section 1. While some students deal quite easily with the new environment, others find that moving from a "human-size" high school, where most people know each other, to the anonymity of a large university campus is quite a frightful experience. It is only in the rare case that the student will be known as an individual to the teacher. Moreover, groups may be re-formed every semester, so that there is often little or no "sense of community" developed in the classroom. As a consequence, it is very difficult for students to receive help either from the teacher, who frequently has very little time available, or from peer students. And in contexts where students have access to teaching assistants, this systems often prove to be rather unreliable for a variety of reasons (lack of familiarity of the assistant with the content of the course, lack of perspective, problems of communication because of a language barrier, etc.).

Moreover, notwithstanding the size of groups, some students are not comfortable with the climate which may prevail in the classroom. Here is what Tobias writes about science students who do not pursue science study — but this may well apply to mathematics students: "Some students don't decide to reject science per se. They reject the culture of competition that they see as an unavoidable aspect of undergraduate science study. These students don't drop science because they fail in the competition. Often they do very well. Rather for them issues of 'culture' [...] are as important as the actual subject matter of their studies. They value such qualities as love for one's subject and intrinsic motivation in one's work, and want these qualities to be part of their academic efforts. They see the culture of college science study, in contrast, as emphasizing extrinsic rewards like getting good grades, and objective goals like getting into graduate or medical school." [25, p. 74] In such a competitive atmosphere, the attention of students concentrates on success at the exams, and not on learning.

In many countries, the democratization of teaching has had as a consequence that many weak students are getting access to university. For such students, their relation to knowledge is often not up to what is being expected of them: they meet difficulties in reaching the required level of abstraction and they confine themselves in mere actions and applications of recipes, unaware of the conceptual shift they must accomplish. As a result of this fragility, these students are in a great need of a highly personalized relationship with their instructor, which would allow for the numerous explanations they require. But the current structure of university makes almost impossible such a contact.

A frequent difficulty with students taking mathematics as a service subject is their underestimation of the role of mathematics with respect to their future career (we do not have in mind here the screening role sometimes forced upon mathematics in certain fields). Many students will have chosen a field of specialization in university in which they were not expecting to have to study mathematics. They will often be at a loss to relate their calculus or linear algebra course to their foreseen profession. The instructor needs to address this issue and convince students of the importance of mathematics for their career.

A final cultural difficulty we would like to mention concerns the general conception of the task of teachers at the university level. The lack of pedagogical awareness of some teachers may stem from the fact that they "are expected to conduct research, and thus their motivation and commitment to teaching may not be as strong as that of secondary school teachers, whose sole responsibility is teaching." [9, p. 676] Moreover university teachers, in most cases, have received their professional training as if their only occupation in mathematics is research. Consequently, they have to develop by themselves the pedagogical and communication capabilities they need with their students — and this is an arduous task! The professional reward system in university mathematics is almost universally focused on success in research, and not in pedagogy. The situation is surely much less dramatic in "teaching-oriented" than in "research-oriented" universities, but still this can be seen as a major impediment to the pedagogical dedication of university teachers on a large scale.

3.3 DIDACTICAL DIFFICULTIES

In this Section we ask ourselves to what point the style of teaching and the performance of teachers, at the university level, might be the cause for difficulties experienced by students. It is quite clear that some of these difficulties arise from the way students have been practicing and learning mathematics at the secondary level; for instance, many students arriving at university do not know how to take notes during a lecture, how to read a textbook, how to plan for the study of a topic, which questions to ask themselves before they get asked by the teacher. The solution is not for university teachers to get closer to the secondary style of teaching in this respect, as students would not get prepared to become autonomous learners. It would take us too far to discuss here how the teaching and learning of mathematics in the secondary school could be changed to improve the situation; moreover, most of the people taking part in this round-table in one way or another are much nearer to university teaching than to teaching at the secondary level.

Among the various circumstances related to university teachers that might cause some problems to their students, we consider the following.

• *Lack of pedagogical and didactical abilities.* "I know the subject and this is sufficient" is here the customary underlying philosophy. In many places it is a rather common belief among university teachers that all one needs in order to teach mathematics at the university level is to deeply know and understand the subject. However, the one who teaches well a subject at any level is not necessarily the one who knows it the most deeply, but the one who achieves that students learn those ideas and methods they should learn. This requires from the teacher many different skills (mathematical as well as didactical ones) that are rarely present in a spontaneous form. It is very important for teachers to be aware of their own possible deficiencies and to try to remedy them.

• *Lack of adequate models.* It often happens that the university teacher, especially the young one — who is usually also the one charged with the responsibility of teaching entering students —, looks at the university professor as an example to imitate. But much too often this is rather a counterexample showing how NOT to teach mathematics. Our university culture has stimulated in many places mathematicians to disregard, if not to despise, any preoccupation about teaching.

• *Disregard for the importance of the methodology of the subject.* Study and work in mathematics require a different kind of approach than study of, say, history or chemistry. Perhaps it belongs to the secondary school teacher to introduce the student to the style of work needed in each one of the subjects. But since this is usually not done, this specificity should be contemplated during the initial years at the tertiary level.

• *Lack of innovative teaching methods.* Many teachers tend to confine themselves to "unimaginative teaching methods" [21, p. 129], the style of teaching most frequently practiced at the university level being that of a lecture presentation of polished mathematics ("the teacher talks and the student takes notes"). It is sad that many university teachers have never heard, for instance, of the so-called "Moore method" or possible modifications of it (see [4]), or of many other different ways to actively engage students, individually or in groups, in the "discovery" and the development of the subject. Such approaches can give a much more exact measure of what each student is able to do and moreover motivate them more intensely towards the study of mathematics. Finally, it has to be acknowledged that while the recent developments in computer hardware and software (vg, symbolic and/or graphical software) have led quite a few teachers to rethink, totally or partially, their approach to various topics in mathematics, a majority of teachers have never considered seriously how these new tools could be used so to foster students participation, inside and outside the classroom.

• *Carelessness in the design of the course.* University teachers often pay little attention to the actual knowledge and preparation of their students and do not care about the pace which would be at every concrete moment the most adequate for the majority of the students. They may offer no clear guidelines about the course content or the exact objectives their students have to meet. Students sometimes miss well-defined material support (vg, a textbook or duplicated lecture notes) on which to rely — it seems that any decent support may be better, for the profit of the student, than the best set of notes the student would have to take hastily in class, thereby loosing opportunity for active learning. Teachers often offer little help to students, through frequent examples, exercises or problems, for digesting the subject and acquiring a concrete idea about the really important concepts and problems of the subject.

• *Lack of feedback procedures.* In the typical university classroom, there is not much interaction which might help the teacher to know, while the course is still in progress, to what extent what has been "taught" has really been "learned", and why it is so. Due to a lack of know-how (or perhaps of interest), it is often only at the very end of the course that teachers get a picture of their groups, sometimes to find out, perhaps by means of some rather unrealistic examination, that almost all of what they thought students had learnt is completely absent from their minds.

• *Lack of assessment skills.* A crucial component of the process of teaching and learning mathematics is to resort to an adequate way of assessing students' work, designed for their benefit and stimulus. But such an assessment scheme is far from trivial to implement. It is sad to observe that many university teachers make no effort to familiarize themselves with different ways of evaluating students, while many others perhaps learn such methods just after many years of ad-hoc experimentation. Very seldom are alternate assessment processes used, such as portfolios, oral interviews, discussions, proposals of open-ended questions given in advance to students so they have the opportunity to think about the problems in an autonomous study, etc. The easiest solution to evaluation, and possibly the poorest one, is of course the written examination — eventually, for large groups, one which the machine can take care of.

4 Possible actions in order to help alleviating these difficulties

We would like to conclude this paper by listing possible actions which might help alleviating the difficulties in the passage from secondary to tertiary mathematics education. We do not claim that all these actions can actually be implemented, nor that they should have the desired effect. Still it is our hope that such a list, however limited and succinct, can foster discussion around the issues raised in the paper. Some of these actions concern institutional aspects surrounding the transition, while others deal with pedagogical ones.

• Establish a better dialogue between secondary educators and tertiary educators. Such a dialogue can (and must) take place both inside and outside "formal channels" of communication. An interesting example of such a dialogue on a national level is provided by the interest recently aroused in the USA mathematical community by the undergoing revision of the so-called "NCTM Standards" (see

[10]). A more local example is to be found in [6].

• Provide students with orientation activities. This can begin already in secondary school, for example by setting up activities in order to help students individually to choose the track that seems the most appropriate for them, in the perspective of their future university career. In-coming university students should be welcomed with information helping them to better understand the place of mathematics in their university education. In a given course, an orientation document can allow to make explicit to the students the expectations of the teacher, for example that students should work right from the very first day of classes. An example of such a document distributed to students in a first-year calculus course is given in [27, p. 865].

• Provide students with individualized help. One possible solution to the fact that teachers of large groups are often totally unable to provide individual support to students is to create a "Students Help Center" in mathematics. The fact that this becomes a highly visible institutionalized activity could make it a little easier to find the necessary financial support.

• Disseminate information about "success stories". A number of institutions are renowned for their exceptional pedagogical performance. Such situations should be better documented, so to help others to develop the necessary local "culture" (commitment to graduating the students admitted, support provided to students, accessibility of professors, etc.) Successful programs in undergraduate mathematics in the USA are presented in [17] and [26].

• Change the context of the transition step. For instance, have the secondary school courses in the upper grades be delivered at a "higher" and more abstract level, so to get closer to the university teaching style. Or, in the opposite direction, make the first-year university courses closer to secondary school teaching style, i.e., delivered at a "lower" and more intuitive level. Or even have both secondary and university courses change drastically in style and content, taking into account, for instance, the possibilities offered by new technologies and the emerging needs of society (see [12] and [15]). Another approach is to create "bridge courses" for specific groups of students, between secondary and tertiary education, in order to help them to fill their gaps with regard to content, methodology and skills. Or still to introduce selective entrance examination to university, in order to ensure a more homogeneous audience for mathematics classes. A report on the use of a diagnostic placement testing in helping entering university students to choose an appropriate sequence of calculus courses is given in [14].

• Create a context propitious to faculty development. Universities have the responsibility of providing faculty members with a context fostering their general pedagogical development, and especially their awareness of the difficulties experienced by students. Those interested in changing their teaching practices need support, training, team-work, access to forums where pedagogical issues are discussed, etc. It is important that teachers be encouraged to take lots of small initiatives that work — eventually the process will lead to a larger result.

• Help students use resources. All the information cannot come from the teacher in the classroom. Students must get used to choosing, reading and understanding on their own appropriate mathematical information in various forms:

textbooks, library materials, internet, etc.

• Change the "culture" of the students. Whether they are specializing in mathematics or taking mathematics as a service subject, students must come to appreciate the mode of thought specific to mathematics. They need to learn why pure mathematics is at least as important as applied mathematics, and that many of the connections between science and mathematics involve theoretical concepts better understood from the point of view of mathematics. They must realize that mathematics is above all a question of ideas and insight, and not mere techniques — although technical skills do play an important role. Teachers must be aware of the need for students to develop insight; they should not expect that this will come naturally by itself from the experience of solving quantitative problems.

• Change the "culture" of the teachers. Traditional lecturing is just one style of teaching — and often not the most appropriate one. Using different ways of teaching can help students develop different ways of learning. One can propose students alternative types of work, like small group discussions; knowledge understood by the best students can be shared with the others, not in order to give the solutions to problems, but to illustrate what it means to do mathematics. In aiming at helping students change their perception of mathematics and make the transition to "advanced mathematical thinking", teachers must realize that "the formalizing and systematizing of the mathematics is the final stage of mathematical thinking, not the total activity". [24, p. 508–509]

• Establish a better dialogue between mathematicians and users of mathematics. The case of mathematics taught as a service subject (see [3] and [11]) needs special attention. Mathematics should be taught to, say, engineering students by someone who has an adequate understanding of the role played by mathematics in engineering and who can relate mathematics to the interests of the students. Contacts with specialists of the specific domain is essential. It was remarked by Murakami [16, p. 1680] that "it is difficult to see how people of such profile might easily come out of the present educational establishment in any significant numbers".

• Meta-cognitive actions. Students' success is linked to a great extent to their capacity of developing "meta-level" skills allowing them, for instance, to self-diagnose their difficulties and to overcome them, to ask proper questions to their tutors, to optimize their personal resources, to organize their knowledge, to learn to use it in a better way in various modes and not only at a technical level (see [7], [8], [20], [22]). For this to happen, teachers have to make explicit to the students the emergence of new "rules" in mathematics (vg, new concepts) and in learning (vg, need for organization instead of pure memory). Teachers also have to build problems adapted to the various modes of thinking they want the students to acquire, and not only present them problems dealing with technical aspects or preparing for the examination day.

• "Less is more." Decrease the quantity of content covered (provided the teacher does have some control over the content of a course), and engage students in a deeper and more adequate understanding.

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