# OPEN DYNAMICAL SYSTEMS AND THEIR CONTROL

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ABSTRACT. A mathematical framework for studying open dynamical systems is sketched. Special attention is given in the exposition to linear time-invariant differential systems. The main concepts that are introduced are the behavior, manifest and latent variables, controllability, and observability. The paper ends with a discussion of control, which is viewed as system interconnection.

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## 1 INTRODUCTION

The purpose of this presentation is to explain some of the main features of the theory of open dynamical systems. The adjective *'open'* refers to systems that interact with their environment. This interaction may take the form of exchange of a physical quantity as mass or energy, or it may simply consist of exchange of information. Closed dynamical systems have been studied very extensively in mathematics. Typically these lead to models of the general form $\frac{d}{dt}x = f(x)$. The evolution of such systems is completely determined by the dynamical laws (expressed by the vector-field of $f$) and the initial state $x(0)$. In open dynamical systems, however, the evolution of the system variables is determined by the dynamical laws, the initial conditions, and, in addition, by the influence of the environment. This may for instance take the form of an external input function that drives the system. Examples of application areas where this interaction with the environment is essential are signal processing and control. Whereas in signal processing it is reasonable to view the input function as a given (or stochastically described) time-function, this is not the case in application areas as control, since in this case the input function is usually generated by a mechanism which selects the input on the basis of the evolution of output variables in the system itself. This feature leads to *'feedback'* which forms the central concept of control, ever since the subject came into existence.

## 2 Dynamical systems

A first goal is to put forward a notion that serves to describe open dynamical systems mathematically. A framework that has shown to be quite effective, both in terms of generality and applicability, is called the *'behavioral approach'*. One of its main features is that it does not start from an input/output structure or map, nor from a state space model. Instead, any family of trajectories parameterized by time is viewed as a dynamical system. The theory underlying this approach has been treated in [16, 17, 12]. Here we can only describe a few of the bare essentials.

A *dynamical system* $\Sigma$ is triple $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ with $\mathbb{T} \subset \mathbb{R}$ the time-set, $\mathbb{W}$ the signal space, and $\mathfrak{B} \subset \mathbb{W}^{\mathbb{T}}$ the *behavior*. The intuition behind this definition is that $\mathbb{T}$ is the set of relevant time-instances; $\mathbb{W}$ is the set in which the signals, whose dynamic relation $\Sigma$ models, take on their values; the behavior $\mathfrak{B}$ specifies which signals $w : \mathbb{T} \to \mathbb{W}$ obey the laws of the system. The time-set $\mathbb{T}$ equals for example $\mathbb{R}$ or $\mathbb{R}_+$ in continuous-time, and $\mathbb{Z}$ or $\mathbb{Z}_+$ in discrete-time systems. Important properties of dynamical systems are linearity and time-invariance; $\Sigma$ is said to be *linear* if $\mathbb{W}$ is a vector space and $\mathfrak{B}$ is a linear subspace of $\mathbb{W}^{\mathbb{T}}$, and *time-invariant* (assuming $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$) if $\sigma^t \mathfrak{B} = \mathfrak{B}$ for all $t \in \mathbb{T}$, where $\sigma^t$ denotes the $t$-shift (defined by $(\sigma^t f)(t') := f(t'+t)$). There is much interest in generalization from a time-set that is a subset of $\mathbb{R}$ to domains with more independent variables (e.g., time and space). These 'dynamical' systems have $\mathbb{T} \subset \mathbb{R}^n$, and are referred to as *n-D* systems.

## 3 Differential systems

The 'ideology' of the behavioral approach is based on the belief that in a model of a dynamical (physical) phenomenon, it is the behavior $\mathfrak{B}$, i.e., a set of trajectories $w : \mathbb{T} \to \mathbb{W}$, that is the central object of study. But, this set of trajectories must be specified somehow, and it is here that differential (and difference) equations enter the scene. Of course, there are important examples where the behavior is specified in other ways (for example, in Kepler's laws for planetary motion), but differential equations are certainly the most prevalent specification of behaviors encountered in applications. For $\mathbb{T} = \mathbb{R}$, $\mathfrak{B}$ then consists of the solutions of a system of differential equations as $f(w, \frac{d}{dt}w, \ldots, \frac{d^N}{dt^N}w) = 0$. We call these *differential* systems. Of particular interest (at least in control, signal processing, and circuit theory, etc.) are systems with a signal space that is a finite-dimensional vector space and behavior described by linear constant-coefficient differential equations. The fact that non-trivial new things can be said about such systems, which from a mathematical point of view may appear very simple, is due to the many meaningful new concepts originating from the interaction of systems with their environment.

A *linear time-invariant differential system* is a dynamical system $\Sigma = (\mathbb{R}, \mathbb{W}, \mathfrak{B})$, with $\mathbb{W}$ a finite-dimensional (real) vector space, whose behavior consists of the solutions of

$$R(\frac{d}{dt})w = 0, \tag{1}$$

with $R \in \mathbb{R}^{\bullet \times \bullet}[\xi]$ a real polynomial matrix. Of course, the number of columns of $R$ equals the dimension of $\mathbb{W}$. The number of rows of $R$, which represents the number of equations, is arbitrary. In fact, when the row dimension of $R$ is less than its column dimension, $R(\frac{d}{dt})w = 0$ is an under-determined system of differential equations which is typical for models in which the influence of the environment is taken into account. The definition of a solution of $R(\frac{d}{dt})w = 0$ is an issue. There is much to be said for considering solutions in $\mathcal{L}^{\mathrm{loc}}(\mathbb{R}, \mathbb{W})$ and interpreting $R(\frac{d}{dt})w$ as a distribution. This allows steps, ramps, etc., which are often used in engineering applications. Nevertheless, for ease of exposition, we define the behavior to be

$$\{w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{W}) \mid R(\frac{d}{dt})w = 0\}. \tag{2}$$

We denote this behavior as $\ker(R(\frac{d}{dt}))$, the set of linear time-invariant differential systems by $\mathfrak{L}^{\bullet}$, and those with $\dim(\mathbb{W}) = \mathtt{w}$ by $\mathfrak{L}^{\mathtt{w}}$. Whence $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B}) \in \mathfrak{L}^{\mathtt{w}}$ means that there exists a $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ such that $\mathfrak{B} = \ker(R(\frac{d}{dt}))$. We call $R(\frac{d}{dt})w = 0$ a *kernel representation* of $\Sigma$. Note that we may as well write $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, instead of $\Sigma \in \mathfrak{L}^{\mathtt{w}}$, since the time-axis ($\mathbb{R}$) and the signal space ($\mathbb{R}^{\mathtt{w}}$) are evident from this notation.

Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$. Define the *consequences* of $\mathfrak{B}$ to be the set $\mathcal{N}_{\mathfrak{B}} := \{n \in \mathbb{R}^{\mathtt{w}}[\xi] \mid n^T(\frac{d}{dt})\mathfrak{B} = 0\}$. It is easy to see that $\mathcal{N}_{\mathfrak{B}}$ is an $\mathbb{R}[\xi]$-submodule of $\mathbb{R}^{\mathtt{w}}[\xi]$, that for $\mathfrak{B} = \ker(R(\frac{d}{dt}))$, $\mathcal{N}_{\mathfrak{B}}$ equals the submodule spanned by the transposes of the rows of $R$, and that there is a one-to-one relation between $\mathfrak{L}^{\mathtt{w}}$ and the $\mathbb{R}[\xi]$-submodules of $\mathbb{R}^{\mathtt{w}}[\xi]$. This property, however, depends on the fact that we used $\mathfrak{C}^{\infty}$-solutions. The same one-to-one correspondence holds with distributional solutions, but not with $\mathfrak{C}^{\infty}$- (or distributional) solutions with compact support. A problem that remains unsolved is to give a crisp characterization for subspaces of $\mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ to be elements of $\mathfrak{L}^{\mathtt{w}}$. In the discrete-time case, the analogous systems can be nicely specified: $\mathfrak{B}$ must be a linear, shift-invariant subspace of $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}}$, and closed in the topology of point-wise convergence [16, 17].

The one-to-one relationship between certain classes of dynamical systems and certain submodules has been studied in other situations as well [9, 11, 10]. For example, it holds for constant-coefficient *PDE*'s. Let $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi_1, \xi_2, \ldots, \xi_n]$ be a polynomial matrix in $n$ variables. It induces the *PDE*

$$R(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n})w = 0 \tag{3}$$

in the functions $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mapsto (w_1(x), w_2(x), \ldots, w_{\mathtt{w}}(x)) \in \mathbb{R}^{\mathtt{w}}$. Define the behavior of this *PDE* as

$$\{w \in \mathfrak{C}^{\infty}(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}}) \mid R(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n})w = 0\}. \tag{4}$$

It turns out that, as in the case with one independent variable, there is again a one-to-one relation between these behaviors and the $\mathbb{R}[\xi_1, \xi_2, \ldots, \xi_n]$-submodules spanned by the rows of $R$ [11]. Analogous, but technically more involved, results have been obtained for time-varying linear systems with hyper-functions as solutions and the ring of time-varying differential operators having coefficients in $\mathbb{R}(t)$ without poles on the real axis [10].

## 4 Latent variables and elimination

Mathematical models of complex systems are usually obtained by viewing the system (often in a hierarchical fashion) as an interconnection of subsystems, modules (standard components), for which a model can be found in a database. This principle of *tearing* and *zooming*, combined with *modularity*, lies at the basis of what is called *object-oriented* modelling, a very effective computer assisted way of model building used in many engineering domains. An important aspect of these object-oriented modelling procedures is that they lead to a model that relates the variables whose dynamic relation one wants to model (we call these *manifest* variables) to auxiliary variables (we call these *latent* variables) that have been introduced in the modelling process, for example as variables that specify the interconnection constraints. For differential systems this leads to equations as

$$f_1(w, \frac{d}{dt}w, \dots, \frac{d^N}{dt^N}w, \ell, \frac{d}{dt}\ell, \dots, \frac{d^N}{dt^N}\ell) = f_2(w, \frac{d}{dt}w, \dots, \frac{d^N}{dt^N}w, \ell, \frac{d}{dt}\ell, \dots, \frac{d^N}{dt^N}\ell),$$

relating the (vector of) manifest variables $w$ to the (vector of) latent variables $\ell$. In the linear time-invariant case this becomes

$$R(\frac{d}{dt})w = M(\frac{d}{dt})\ell, \tag{5}$$

with $R$ and $M$ polynomial. Define the *manifest* behavior of (5) as

$$\{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \exists \ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet) \text{ such that } R(\frac{d}{dt})w = M(\frac{d}{dt})\ell\}. \tag{6}$$

We call (5) *latent variable* representation of (6). The question occurs whether (6) is in $\mathfrak{L}^{\mathtt{w}}$. This is the case indeed.

Theorem 1 : *For any real polynomial matrices* $(R, M)$ *with* $\text{rowdim}(R) = \text{rowdim}(M)$*, there exists a real polynomial matrix* $R'$ *such that the manifest behavior of* $R(\frac{d}{dt})w = M(\frac{d}{dt})\ell$ *has the kernel representation* $R'(\frac{d}{dt})w = 0$.

The above theorem is called the *elimination theorem*. Its relevance in object-oriented modelling is as follows. A model obtained this way usually involves very many variables and equations, among them many algebraic ones. The elimination theorem tells that the latent variables may be eliminated and that the number of equations can be reduced to no more than the number of manifest variables. Of course, the order of the differential equation goes up in the elimination process.

The theoretical basis that underlies the elimination theorem is the *fundamental principle*. It gives necessary and sufficient conditions for solvability for $x \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet)$ in the equation $F(\frac{d}{dt})x = y$ with $F \in \mathbb{R}^{\bullet\times\bullet}[\xi]$ and $y \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet)$ given. Define the *annihilators* of $F$ as $\mathcal{K}_F := \{n \in \mathbb{R}^{\text{rowdim}(F)} \mid n^T F = 0\}$. The fundamental principle states that $F(\frac{d}{dt})x = y$ is solvable if and only if $n^T(\frac{d}{dt})y = 0$ for all $n \in \mathcal{K}_F$. This immediately yields the elimination theorem. For the case at hand, it is rather easy to prove the fundamental principle, but there are interesting generalizations where it is a deep mathematical result. For example, for the constant-coefficient *PDE*'s, and for the time-varying linear systems discussed in section 3. Thus the elimination theorem also holds for these classes of systems. The elimination problem has also been studied For nonlinear systems [4].

## 5 Controllability

An important property in the analysis and synthesis of open dynamical systems is controllability. Controllability refers to be ability of transferring a system from one mode of operation to another. By viewing the first mode of operation as undesired and the second one as desirable, the relevance to control and other areas of applications becomes clear. The concept of controllability has been introduced around 1960 in the context of state space systems. It is one of the notions that is endogenous to control theory. The classical definition runs as follows. The system described by the controlled vector-field $\frac{d}{dt}x = f(x,u)$ is said to be controllable if $\forall a, b, , \exists u$ and $T \geq 0$ such that the solution to $\frac{d}{dt}x = f(x,u)$ and $x(0) = a$ yields $x(T) = b$. One of the elementary results of system theory [1] states that the finite-dimensional linear system $\frac{d}{dt}x = Ax + Bu$ is controllable if and only if the matrix $[B \; AB \; A^2B \cdots A^{\dim(x)-1}B]$ has full row rank. Various generalizations of this result to time-varying, to nonlinear (involving Lie brackets) [7, 8, 2, 15], and to infinite-dimensional systems exist [3].

A disadvantage of the notion of controllability as formulated above is that it refers to a particular representation of a system, notably a state space representation. Thus a system may be uncontrollable either for the intrinsic reason that the control has insufficient influence on the system variables, or because the state has been chosen in an inefficient way. It is clearly not desirable to confuse these reasons. In the context of behavioral systems, a definition of controllability has been put forward that involves the system variables directly.

Let $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ be a dynamical system with $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$, and assume that is time-invariant. $\Sigma$ is said to be *controllable* if for all $w_1, w_2 \in \mathfrak{B}$ there exists $T \in \mathbb{T}, T \geq 0$ and $w \in \mathfrak{B}$ such that $w(t) = w_1(t)$ for $t < 0$ and $w(t) = w_2(t-T)$ for $t \geq T$. Thus controllability refers to the ability to switch from any one trajectory in the behavior to any other one, allowing some time-delay.

Two questions that occur are the following: What conditions on the parameters of a system representation imply controllability? Do controllable systems admit a particular representation in which controllability becomes apparent? For linear time-invariant differential systems, these questions are answered in the following theorem.

Theorem 2 : *Let $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B}) \in \mathfrak{L}^{\mathtt{w}}$. The following are equivalent:*

1. *The system $\Sigma$ is controllable;*
2. *The polynomial matrix $R$ in a kernel representation $R(\frac{d}{dt})w = 0$ of $\mathfrak{B}$ satisfies* $\operatorname{rank}(R(\lambda)) = \operatorname{rank}(R)$ *for all $\lambda \in \mathbb{C}$;*
3. *The behavior $\mathfrak{B}$ is the image of a linear constant-coefficient differential operator, that is, there exists a polynomial matrix $M \in \mathbb{R}^{\mathtt{w}\times\bullet}[\xi]$ such that $\mathfrak{B} = M(\frac{d}{dt})\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\operatorname{coldim}(M)})$;*
4. *The compact support trajectories of $\mathfrak{B}$ are dense (in the $\mathfrak{C}^\infty$-topology) in $\mathfrak{B}$;*
5. *The $\mathbb{R}[\xi]$-module $\mathbb{R}^{\mathtt{w}}[\xi]/\mathcal{N}\mathfrak{B}$ is torsion-free.*

There exist various algorithms for verifying controllability of a system $\Sigma \in \mathfrak{L}^\bullet$ starting from the coefficients of the polynomial matrix $R$ in a kernel (or a latent variable) representation of $\Sigma$, but we will not enter into these algorithmic aspects.

A point of the above theorem that is worth emphasizing is that controllable systems admit a representation as the manifest behavior of the latent variable system of the special form

$$w = M(\frac{d}{dt})\ell. \tag{7}$$

We call this an *image* representation. It follows from the elimination theorem that every system in image representation can be brought in kernel representation. But not every system in kernel representation can be brought in image representation: it is precisely the controllable ones for which this is possible.

The controllability issue has been pursued for many other classes of systems. In particular (more difficult to prove) generalizations have been derived for differential-delay [14, 6], for nonlinear, for *n-D* systems [13, 9], and, as we will discuss soon, for *PDE*'s. Systems in an image representation have received much attention recently for nonlinear differential-algebraic systems, where they are referred to as *flat* systems [5]. Flatness implies controllability, but the exact relation remains to be discovered.

We now explain the generalization to constant-coefficient *PDE*'s. Consider the system defined by (3,4). This system is said to be *controllable* if for all $w_1, w_2$ in the behavior (4) and for all open subsets $O_1, O_2$ of $\mathbb{R}^n$ with disjoint closure, there exists $w$ in (4) such that $w|_{O_1} = w_1|_{O_1}$ and $w|_{O_2} = w_2|_{O_2}$. The following result has been obtained in [11].

THEOREM 3 : *The following statements are equivalent:*

1. *(3) defines a controllable system;*

2. *(4) admits an image representation, i.e., there exists a polynomial matrix* $M \in \mathbb{R}^{\mathtt{w}\times\bullet}[\xi_1, \xi_2, \ldots, \xi_n]$ *such that (4) equals*

$$M(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n})\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathrm{coldim}(M)});$$

3. *The trajectories of compact support are dense in (4).*

It is a simple consequence of this theorem that a scalar *PDE* in one function (i.e., with $\mathrm{rowdim}(R) = \mathrm{coldim}(R) = 1$) with $R \neq 0$ cannot be controllable. It can be shown, on the other hand, that Maxwell's equations (in which case $\mathrm{rowdim}(R) = 8$ and $\mathrm{coldim}(R) = 10$) are controllable. Note that an image representation corresponds to what in mathematical physics is the existence of a *potential function*. An interesting aspect of the above theorem therefore is the fact that it identifies the existence of a potential function with the system theoretic property of controllability and concatenability of behaviors.

## 6 Observability

The notion of observability was introduced hand in hand with controllability. In the context of the input/state/output system $\frac{d}{dt}x = f(x,u), y = h(x,u)$, it refers to the possibility of deducing, using the laws of the system, the state from observation of the input and the output. The definition that is used in the behavioral context is more general in that the variables that are observed and the variables that need to be deduced are kept general.

Let $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ be a dynamical system, and assume that $\mathbb{W}$ is a product space: $\mathbb{W} = \mathbb{W}_1 \times \mathbb{W}_2$. Then $w_1$ is said to be *observable* from $w_2$ in $\Sigma$ if $(w_1, w_2') \in \mathfrak{B}$ and $(w_1, w_2'') \in \mathfrak{B}$ imply $w_2' = w_2''$. Observability thus refers to the possibility of deducing the trajectory $w_1$ from observation of $w_2$ and from the laws of the system ($\mathfrak{B}$ is assumed to be known).

The theory of observability runs parallel to that of controllability. We mention only the result that for linear time-invariant systems, $w_1$ is observable from $w_2$ if and only if there exists a set of consequences of the system behavior of the following form that puts observability into evidence: $w_1 = R_2'(\frac{d}{dt})w_2$.

## 7 Control

In order to illustrate the idea of the nature of control that we would like to transmit in this presentation, consider the system configuration depicted in figure 1. In the top part of the figure, there are two systems, shown as proverbial black-boxes with terminals. It is through their terminals that systems interact with their environment. The black-box imposes relations on the variables that 'live' on its terminals. These relations are formalized by the behavior of the system in the black-box. The system to the left in figure 1 is called the *plant*, the one to the right the *controller*. The terminals of the plant consist of *to-be-controlled variables* $w$, and *control variables* $c$. The controller has only terminals with the control variables $c$. In the bottom part of the figure, the control terminals of the plant and of the controller are connected. Before interconnection, the variables $w$ and $c$ of the plant have to satisfy the laws imposed by the plant behavior. But, after interconnection, the variables $c$ also have to satisfy the laws imposed by the controller. Thus, after interconnection, the restrictions imposed on the variables $c$ by the controller will be transmitted to the variables $w$. Choosing the black-box to the right so that the variables $w$ have a desirable behavior in the interconnected black-box is, in our view, the basic problem of control. This point of view is discussed with examples in [18].

In the remainder of this paper we describe one simple controller design problem in this setting. Let the variables $w$ be partitioned into two sets: $w = (d, z)$ with the $d$'s exogenous disturbances, and the $z$'s endogenous to-be-controlled variables. Assume that the plant is a linear time-invariant differential system with behavior $\mathcal{P} \in \mathfrak{L}^{\mathtt{d}+\mathtt{z}+\mathtt{c}}$, called the *plant behavior*. Assume further that the exogenous disturbances $d$ are *free* in $\mathcal{P}$, that is, that for all $d \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{d}})$ there exist $(z, c)$ such that $(d, z, c) \in \mathcal{P}$. Now consider the controller, also assumed to be a linear time-invariant differential system, with behavior $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$, called the *controller* *behavior*. With the controller put into place, the behavior of the to-be-controlled variables becomes

$$\mathcal{K} = \{(d,z) \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{d}+\mathtt{z}}) \mid \exists c \in \mathcal{C} \text{ such that } (d,z,c) \in \mathcal{P}\}. \tag{8}$$

By the elimination theorem, $\mathcal{K} \in \mathfrak{L}^{\mathtt{d}+\mathtt{z}}$. We call $\mathcal{K}$ the *controlled behavior*.

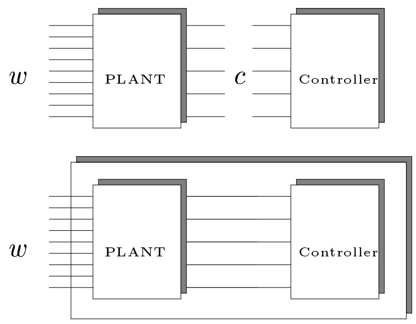


Figure 1: Controller interconnection

The controller $\mathcal{C}$ usually has to satisfy certain practical implementability constraints, perhaps as a signal processor that transforms sensor outputs into actuator inputs, or using physical energy-based constraints, etc. Here, we assume that the controller can be any linear time-invariant differential system that leave the exogenous disturbances free. This is the case if and only if its behavior $\mathcal{C}$ has the property that for all $d \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{d}})$, there exists $(z,c)$ such that both $(d,z,c) \in \mathcal{P}$ and $c \in \mathcal{C}$. We call this set of controllers *admissible controllers*, and denote it by $\mathfrak{C}$.

The control problem that now emerges is that of choosing, for a given plant $\mathcal{P}$, an admissible controller $\mathcal{C} \in \mathfrak{C}$ such that the controlled behavior $\mathcal{K}$ meets certain specifications. We consider *pole placement*, which, as we shall see, implies *stabilization*. We now explain what this means. Consider the controlled system $\mathcal{K} \in \mathfrak{L}^{\mathtt{d}+\mathtt{z}}$. When the controller which generates $\mathcal{K}$ is admissible, $d$ must be free in $\mathcal{K}$. This implies that $\mathcal{K}$ has a kernel representation $P(\frac{d}{dt})z = Q(\frac{d}{dt})d$ with $P$ a polynomial matrix of full row rank. Define the *characteristic polynomial* $\pi_{\mathcal{K}}$ of $\mathcal{K}$ as follows. If $P$ is not square (and hence wide) $\pi_{\mathcal{K}} := 0$. Otherwise, $\pi_{\mathcal{K}} = \det(P)$ where it is assumed that $P$ is chosen such that $\det(P)$ is monic. We call the roots of $\pi_{\mathcal{K}}$ the *poles* of $\mathcal{K}$. If $\pi_{\mathcal{K}} \neq 0$, then the behavior $\mathcal{K}_0 = \{(d,z) \in \mathcal{K} \mid d = 0\}$ is finite-dimensional, and the exponents of its exponential responses are the roots of $\pi_{\mathcal{K}}$.

Note that controllability can be defined when there are more variables in the model than just those that need to be concatenated. Similarly, observability can also be defined when there are more variables in the model that just the observed and the to-be-deduced ones. The definitions are evident. We now state necessary and sufficient conditions for pole assignability.

Theorem 4 : *Let the plant behavior $\mathcal{P} \in \mathfrak{L}^{\mathtt{d}+\mathtt{z}+\mathtt{c}}$ be given. Then there exists, for any monic polynomial $r \in \mathbb{R}[\xi]$, an admissible controller $\mathcal{C} \in \mathfrak{C}$ such that the resulting controlled system $\mathcal{K} \in \mathfrak{L}^{\mathtt{d}+\mathtt{z}}$ has $\pi_{\mathcal{K}} = r$ if the exogenous to-be-controlled variables $z$ are (i) controllable in $\mathcal{P}_0 := \{(d,z,c) \in \mathcal{P} \mid d = 0\}$, and (ii) observable from $c$ in $\mathcal{P}$.*

The controlled behavior $\mathcal{K}$ is said to be *stable* if $(d,z) \in \mathcal{K}$ and $d = 0$ implies that $w(t) \to 0$ as $t \to \infty$. Obviously $\mathcal{K}$ is stable if and only if $\pi_{\mathcal{K}}$ is a Hurwitz polynomial. The above theorem gives controllability and observability conditions that are sufficient for stabilizability. Pole placement and stabilization are very coarse controller design specifications. But also other, more refined, design specifications, for example $H_\infty$-control and robust stability, can be treated in this setting.

These results generalize the classical state space pole placement results in a number of ways. However, we regard the main contribution of the above theorem to be the underlying idea of control. We view *interconnection* as the principle of control. It supersedes the special case of trajectory selection and optimization (often called *open-loop (optimal) control*, and the (very important) special case of feedback control (often called *intelligent control*), in which a signal processor uses the plant sensor outputs in order to select the plant actuator inputs. The latter area is the classical view of control and will undoubtedly gain in importance for technological applications as logical devices and on-line computation becomes cheaper, more reliable, and more powerful. However, by considering interconnection as the basic principle of control, the scope of the subject and its relevance to the design of physical systems can be enhanced in meaningful directions, by making the (optimal) design of subsystems, i.e., integrated system design, as the aim and the domain of the subject.

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