Uniform Asymptotics for Orthogonal Polynomials

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Abstract. We consider asymptotics of orthogonal polynomials with respect to a weight $e^{-Q(x)}dx$ on $\mathbb{R}$, where either $Q(x)$ is a polynomial of even order with positive leading coefficient, or $Q(x) = NV(x)$, where $V(x)$ is real analytic on $\mathbb{R}$ and grows sufficiently rapidly as $|x| \to \infty$. We formulate the orthogonal polynomial problem as a Riemann-Hilbert problem following the work of Fokas, Its and Kitaev. We employ the steepest descent-type method for Riemann-Hilbert problems introduced by Deift and Zhou, and further developed by Deift, Venakides and Zhou, in order to obtain uniform Plancherel-Rotach-type asymptotics in the entire complex plane, as well as asymptotic formulae for the zeros, the leading coefficients and the recurrence coefficients of the orthogonal polynomials. These asymptotics are also used to prove various universality conjectures in the theory of random matrices.

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Let $w(x)dx = e^{-Q(x)}dx$ be a measure on the real line. Denote by $\pi_n(x, Q) = \pi_n(x) = x^n + \dots$ the n-th monic orthogonal polynomial with respect to the measure, and by $p_n(x, Q) = p_n(x) = \gamma_n \pi_n(x)$, $\gamma_n > 0$, the normalized n-th orthogonal polynomial, or simply the n-th orthogonal polynomial, i.e.

$$\int_{\mathbb{R}} p_n(x) p_m(x) e^{-Q(x)} dx = \delta_{n,m} \qquad , n, m \in \mathbb{N}. \tag{1}$$

Furthermore, denote by $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ the coefficients of the associated three term recurrence relation, namely, $xp_n(x) = b_n p_{n+1}(x) + a_n p_n(x) + b_{n-1} p_{n-1}(x)$, $n \in \mathbb{N}$, and denote by $x_{1,n} > x_{2,n} > \dots > x_{n,n}$ the roots of p_n.

In [8], the authors considered the case where

$$Q(x) = \sum_{k=0}^{2m} q_k x^k, \qquad q_{2m} > 0, \quad m > 0, \tag{2}$$

is a polynomial of even degree with a positive leading coefficient, and in [7] the case where

$$\begin{aligned} &Q(x) = NV(x), \qquad V(x) \quad \text{is real analytic on } \mathbb{R}, \\ &\text{and} \qquad \frac{\mathrm{V(x)}}{\log(\mathrm{x}^2+1)} \to \infty \qquad \text{as} \quad \mathrm{x} \to \infty. \end{aligned} \tag{3}$$

In [8], the authors are concerned with the asymptotics as $n \to \infty$ of the leading coefficient γ_n, the recurrence coefficient a_n, b_n and the zeros x_{jn}, as well as Plancherel-Rotach-type asymptotics for the orthogonal polynomials p_n, i.e asymptotics for $p_n(zc_n+d_n)$ uniformly for all $z \in \mathbb{C}$, where c_n, d_n are certain quantities related to the so-called Mhaskar-Rahmanov-Saff numbers (see (5) below). The name "Plancherel-Rotach" refers to [17] in which the authors prove asymptotics of this type for the classical case of Hermite polynomials. In [7], the authors are concerned with the asymptotics of γ_n, a_n, b_n and $p_n(z; NV)$ in the case $c^{-1}N \leq n \leq cN$ for some $c > 1$, as $N \to \infty$. These asymptotics are crucial ingredients in proving a variety of universality conjectures in random matrix theory (see [7]).

Due to the page restrictions in these Proceedings, we limit our considerations to a description of the results in [8]. Plancherel-Rotach-type asymptotics for polynomial orthogonal with respect to exponential weights of the above type, play a central role in various questions of weighted approximation on the line (see e.g. [15]). In order to prove our results we use a reformulation of the orthogonal polynomial problem as a Riemann-Hilbert problem, due to Fokas, Its and Kitaev [13, 14] (see below). This Riemann-Hilbert problem is then analyzed in turn asymptotically using the non-commutative steepest-descent method introduced by Deift and Zhou in [11], and further developed in [12] and [9], and placed eventually in a general form by Deift, Venakides and Zhou in [10]. In [8], and particularly in [7], a basic role is played by the results on the equilibrium measure (see below) obtained by Deift, Kriecherbauer and Ken McLaughlin in [5]. In this paper we will only have the opportunity to give a very rough sketch of the steepest descent method: full details can be found in [8]. For the case of varying weights $e^{-NV(x)}dx$, we must, alas, refer the reader to [7], for both a detailed description of the results as well as their proofs, and the connection to random matrix theory. The methods in [7] are similar to those in [8], but require additional technical considerations. In the special case where V is an even quartic polynomial, the results in [7] should be compared with the results of Bleher and Its [2], who were the first to use the steepest-descent method in [11] to study the asymptotics of orthogonal polynomials via a Riemann-Hilbert problem. Some of the results in [7] and in [8] were announced in [6].

There is a vast literature on asymptotic questions for orthogonal polynomials. The list of researchers who have made important contributions close to the results of [7] and [8], includes, in addition to Plancherel and Rotach, and Bleher and Its, Bauldry, Chen, Criscuolo, Della Vechia, Geronimo, Ismail, Lubinsky, Magnus, Maskar, Mastroiani, Mate, Nevai, Rahmanov, Saff, Sheen, Totik and Van Assche, but there are many others. Again, we do not have the opportunity to describe their work in any detail. Fortunately there is an excellent review [15]: also, a detailed description of the work of the above authors related to the present paper is given in [8].

Henceforth we will assume that the potential $Q(x)$ is of the form (2). The statement of our results involves the *n-th Mhaskar-Rahmanov-Saff numbers* (in short: MRS-numbers [16], [18]) α_n, β_n which can be determined from the equations

$$\frac{1}{2\pi}\int_{\alpha_n}^{\beta_n}\frac{Q'(t)(t-\alpha_n)}{\sqrt{(\beta_n-t)(t-\alpha_n)}}dt = n, \qquad \frac{1}{2\pi}\int_{\alpha_n}^{\beta_n}\frac{Q'(t)(\beta_n-t)}{\sqrt{(\beta_n-t)(t-\alpha_n)}}dt = -n, \quad (4)$$

and in particular the interval $[\alpha_n, \beta_n]$ whose width and midpoint are given by

$$c_n := \frac{\beta_n-\alpha_n}{2}, \qquad d_n := \frac{\beta_n+\alpha_n}{2}. \quad (5)$$

For the weights under consideration it is straightforward to prove the existence of the MRS-numbers for sufficiently large n. Moreover, they can be expressed in a power series in $n^{-\frac{1}{2m}}$. We obtain

$$c_n = n^{\frac{1}{2m}}\sum_{l=0}^{\infty} c^{(l)} n^{-\frac{l}{2m}}, \qquad d_n = \sum_{l=0}^{\infty} d^{(l)} n^{-\frac{l}{2m}}, \quad (6)$$

where the coefficients $c^{(l)}, d^{(l)}$ can be computed explicitly. From now on we will assume that n is sufficiently large for (6) to hold.

Statement of Results

To simplify the analysis, we normalize the interval $[\alpha_n, \beta_n]$ to be $[-1, 1]$ by making the linear change of variable

$$\lambda_n : \mathbb{C} \to \mathbb{C} : z \mapsto c_n z + d_n, \quad (7)$$

which takes the interval $[-1, 1]$ onto $[\alpha_n, \beta_n]$, and we work with the function

$$V_n(z) := \frac{1}{n} Q(\lambda_n(z)). \quad (8)$$

The function V_n is again a polynomial of degree $2m$ with leading coefficient $(mA_m)^{-1} > 0$, whereas all other coefficients tend to zero as n tends to ∞.

We present our results in terms of the well-known equilibrium measure μ_n (see e.g. [19]) with respect to V_n which is defined as the unique minimizer in $\mathcal{M}_1(\mathbb{R}) = \{\text{probability measures on } \mathbb{R}\}$ of the functional

$$I^{V_n} : \mathcal{M}_1(\mathbb{R}) \to (-\infty, \infty] : \mu \mapsto \int_{\mathbb{R}^2} \log|x-y|^{-1} d\mu(x)d\mu(y) + \int_{\mathbb{R}} V_n(x) d\mu(x). \quad (9)$$

The equilibrium measure and the corresponding variational problem emerge naturally in our asymptotic analysis of the Riemann-Hilbert problem. The minimizing measure is given by

$$d\mu_n(x) = \frac{1}{2\pi}\sqrt{1-x^2}\, h_n(x) \mathbf{1}_{[-1,1]}(x) dx, \quad (10)$$

where $\mathbf{1}_{[-1,1]}$ denotes the indicator function of the set $[-1, 1]$ and h_n is a polynomial of degree $2m-2$,

$$h_n(x) = \sum_{k=0}^{2m-2} h_{n,k} x^k, \quad (11)$$

and the coefficients $h_{n,k}$ can be expanded in an explicitly computable power series in $n^{-\frac{1}{2m}}$.

Finally, to state our first theorem, we define

$$l_n := \frac{1}{\pi}\int_{-1}^{1}\sqrt{1-t^2}\,h_n(t)\log|t|dt - V_n(0), \tag{12}$$

which also has an explicitly computable power series in $n^{-\frac{1}{2m}}$.

Asymptotics of the leading and recurrence coefficients of the orthogonal polynomials p_n

Theorem 13. *In the above notation we have*

$$\gamma_n\sqrt{\pi c_n^{2n+1}e^{nl_n}} = 1 - \frac{1}{n}\left(\frac{4h_n(1)-3h_n'(1)}{48h_n(1)^2} + \frac{4h_n(-1)+3h_n'(-1)}{48h_n(-1)^2}\right) + \mathcal{O}\left(\frac{1}{n^2}\right),$$
$$\frac{b_{n-1}}{c_n} = \frac{1}{2} + \mathcal{O}\left(\frac{1}{n^2}\right), \quad a_n = d_n + \frac{c_n}{2n}\left(\frac{1}{h_n(1)} - \frac{1}{h_n(-1)} + \mathcal{O}\left(\frac{1}{n}\right)\right). \tag{14}$$

In all three cases there are explicit integral formulae for the error terms which all have an asymptotic expansion in $n^{-\frac{1}{2m}}$, e.g. $\mathcal{O}\left(\frac{1}{n}\right) = \frac{1}{n}\left(\kappa_0 + \kappa_1 n^{-\frac{1}{2m}} + \dots\right)$. The coefficients of these expansions can be computed via the calculus of residues by purely algebraic means.

Next we will state the Plancherel-Rotach type asymptotics of the orthogonal polynomials p_n, i.e. the limiting behavior of the rescaled n-th orthogonal polynomial $p_n(\lambda_n(z))$, as n tends to infinity and $z \in \mathbb{C}$ remains fixed. We will give the leading order behavior and produce error bounds which are uniform in the entire complex plane $\mathbb{C}$.

Notation: In the following, $(\cdot)^\alpha$, $\alpha \in \mathbb{R}$, denotes the principal branch of the α^{th} root. On the other hand, we will reserve the notation $\sqrt{a}$ for nonnegative numbers a, and we always take $\sqrt{a}$ nonnegative: thus $\sqrt{1-x^2}$, $-1 \le x \le 1$ in (10) is positive.

Plancherel-Rotach Asymptotics

We state our second theorem in terms of the function

$$\psi_n : \mathbb{C}\setminus((-\infty,-1]\cup[1,\infty)) \to \mathbb{C} : z \mapsto \frac{1}{2\pi}(1-z)^{1/2}(1+z)^{1/2}h_n(z). \tag{15}$$

The function ψ_n is an analytic extension of the density of μ_n on $(-1,1)$ to $\mathbb{C}\setminus((-\infty,-1]\cup[1,\infty))$ and is thus closely linked to the equilibrium measure (cf. (10)). We show that there exist analytic functions f_n, $\tilde{f}_n$ in a neighborhood of 1, respectively -1, satisfying

$$
\begin{aligned}
(-f_n(z))^{3/2} &= -n\frac{3\pi}{2}\int_1^z \psi_n(y)dy, \quad \text{for } |z-1| \text{ small}, \quad z \notin [1,\infty). \\
(\tilde{f}_n(z))^{3/2} &= n\frac{3\pi}{2}\int_{-1}^z \psi_n(y)dy, \quad \text{for } |z+1| \text{ small}, \quad z \notin (-\infty,-1].
\end{aligned}
\tag{16}
$$

As $p_n(z) = \overline{p_n(\bar{z})}$, it is sufficient to describe the asymptotics of $p_n(c_n z + d_n)$ in the closed upper half plane $\overline{\mathbb{C}}_+$. Depending on a small parameter δ, we divide $\overline{\mathbb{C}}_+$ into six *closed* regions, as shown in Figure 17 below. We only describe the asymptotics in $A_\delta, C_{1,\delta}, C_{2,\delta}$ and B_δ. The asymptotics in $D_{j,\delta}$, $j = 1, 2$, is of a similar form to that in $C_{j,\delta}$, $j = 1, 2$ respectively, with $\tilde{f}_n$ replacing f_n. Let $Ai(z)$ denote the Airy function [1, 10.4].

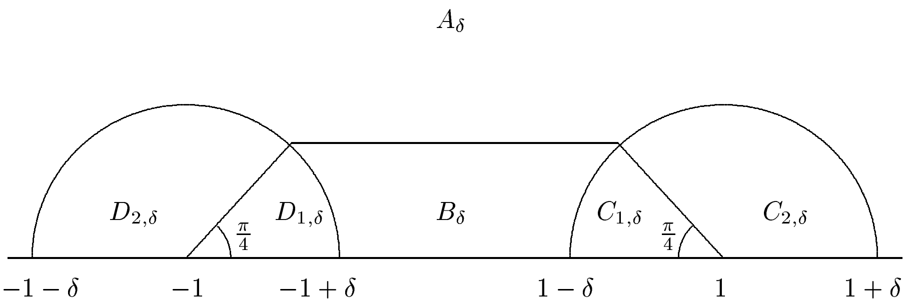

Figure 17. Different asymptotic regions for $p_n(c_n z + d_n)$ in $\overline{\mathbb{C}_\pm}$.

Theorem 18. *There exists a δ_0 such that for all $0 < \delta \leq \delta_0$ the following holds (see Figure 17):*

(i) For $z \in A_\delta$:

$$
\begin{aligned}
p_n(c_n z + d_n)e^{-\frac{1}{2}Q(c_n z + d_n)} &= \sqrt{\frac{1}{4\pi c_n}} \exp\left(-n\pi i \int_1^z \psi_n(y)dy\right) \\
&\quad \times \left(\frac{(z-1)^{1/4}}{(z+1)^{1/4}} + \frac{(z+1)^{1/4}}{(z-1)^{1/4}}\right)\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right).
\end{aligned}
\tag{19}
$$

(ii) For $z \in B_\delta$:

$$
\begin{aligned}
p_n(c_n z + d_n)e^{-\frac{1}{2}Q(c_n z + d_n)} &= \sqrt{\frac{2}{\pi c_n}}(1-z)^{-1/4}(1+z)^{-1/4} \\
\times \Bigg\{ \cos\left(n\pi \int_1^z \psi_n(y)dy + \frac{1}{2}\arcsin z\right)&\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right) \\
+ \sin\left(n\pi \int_1^z \psi_n(y)dy + \frac{1}{2}\arcsin z\right)&\mathcal{O}\left(\frac{1}{n}\right)\Bigg\}.
\end{aligned}
\tag{20}
$$

(iii) For $z \in C_{1,\delta}$:

$$\begin{aligned} & p_n(c_n z + d_n)e^{-\frac{1}{2}Q(c_n z + d_n)} && (21)\\ =&\sqrt{\frac{1}{c_n}}\left\{\left(\frac{(z+1)^{1/4}}{(z-1)^{1/4}}(f_n(z))^{1/4}Ai(f_n(z))\right)\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right)\right.\\ &\left.-\left(\frac{(z-1)^{1/4}}{(z+1)^{1/4}}(f_n(z))^{-1/4}Ai'(f_n(z))\right)\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right)\right\}. && (22)\end{aligned}$$

(iv) For $z \in C_{2,\delta}$:

$$\begin{aligned} p_n(c_n z + d_n)e^{-\frac{1}{2}Q(c_n z + d_n)} &= \sqrt{\frac{1}{c_n}}\left\{\frac{(z+1)^{1/4}}{(z-1)^{1/4}}(f_n(z))^{1/4}Ai(f_n(z))\right.\\ -\quad &\left.\frac{(z-1)^{1/4}}{(z+1)^{1/4}}(f_n(z))^{-1/4}Ai'(f_n(z))\right\}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right). \qquad (23)\end{aligned}$$

All the error terms are uniform for $\delta \in$ compact subsets of $(0, \delta_0]$ and for $z \in X_\delta$, where $X \in \{A, B, C_1, C_2\}$. There are integral formulae for the error terms from which one can extract an explicit asymptotic expansion in $n^{-\frac{1}{2m}}$.

REMARKS:

(a) Some of the expressions in Theorem 18 are not well defined for all $z \in \mathbb{R}$ (see e.g. $(z-1)^{1/4}$, $\int_1^z \psi_n(y)dy$). In these cases we always take the limiting expressions as z is approached from the upper half-plane.

(b) The function arcsin is defined as the inverse function of

$$\sin : \{z \in \mathbb{C} : |Re(z)| < \frac{\pi}{2}\} \to \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty)).$$

ASYMPTOTIC LOCATION OF THE ZEROS

In order to state our result on the location of the zeros, we denote the zeros of the Airy function Ai by $0 > -\iota_1 > -\iota_2 > \dots$. Recall that the all the zeros of Ai lie in $(-\infty, 0)$, so that there exists a largest zero $-\iota_1 < 0$. Furthermore, note that $[-1,1] \ni x \mapsto \int_x^1 \psi_n(t)dt \in [0,1]$ is bijective and we define its inverse function to be $\zeta_n : [0,1] \mapsto [-1,1]$.

THEOREM 24. *The zeros $x_{1,n} > x_{2,n} > \ldots > x_{n,n}$ of the n-th orthogonal polynomials p_n satisfy the following asymptotic formulae:*

(i) Fix $k \in \mathbb{N}$. Then

$$\frac{x_{k,n} - d_n}{c_n} = 1 - \left(\frac{2}{h_n(1)^2}\right)^{1/3}\frac{\iota_k}{n^{2/3}} + \mathcal{O}\left(\frac{1}{n}\right), \qquad \text{as } n \to \infty, \tag{25}$$

and

$$\frac{x_{n-k,n} - d_n}{c_n} = -1 + \left(\frac{2}{h_n(-1)^2}\right)^{1/3}\frac{\iota_k}{n^{2/3}} + \mathcal{O}\left(\frac{1}{n}\right), \qquad \text{as } n \to \infty. \tag{26}$$

(ii) There exist constants $k_0, C > 0$, such that for all $k_0 \leq k \leq n - k_0$ the following holds:

$$\frac{x_{k,n} - d_n}{c_n} \in \left(\zeta_n \left(\frac{6k-1}{6n} \right), \zeta_n \left(\frac{6k-5}{6n} \right) \right). \tag{27}$$

$$\left| \frac{x_{k,n} - d_n}{c_n} - \zeta_n \left(\frac{6k-3}{6n} + \frac{1}{2\pi n} \arcsin \left(\zeta_n \left(k/n \right) \right) \right) \right| \leq \frac{C}{n^2[\alpha(1-\alpha)]^{4/3}}, \tag{28}$$

where $\alpha := k/n$.

(iii) There exists a constant $C_1 > 0$ such that

$$\frac{1}{C_1} < \frac{x_{k,n} - x_{k+1,n}}{c_n[nk(n-k)]^{1/3}} < C_1 \qquad \text{for all } 1 \leq k \leq n-1. \tag{29}$$

REMARKS:

(a) Using the asymptotic expansion for the error terms in Theorem 18 one can of course approximate the k-th zero $x_{k,n}$ of the orthogonal polynomial p_n to arbitrary accuracy.

(b) Note that the error term in (28) is at most of order $\mathcal{O}(n^{-2/3})$. Furthermore it is obvious that for any compact subset K of $(0,1)$, there exists a constant C_K, such that the error term in (28) is bounded by C_K/n^2, as long as $\alpha = k/n \in K$.

As noted earlier, our approach to the asymptotic problem for orthogonal polynomials, is based on the reformulation of the orthogonal polynomial problem as a Riemann-Hilbert problem due to Fokas, Its and Kitaev (see [13], [14]: a specialized version appeared also in [4]).

A general reference for Riemann-Hilbert problems is, for example, [3]. Let Σ be an oriented contour in $\mathbb{C}$.

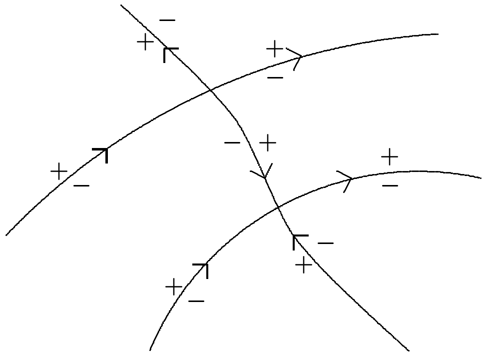

Figure 30.

As indicated in the Figure, the $(+)$-side (resp, $(-)$-side) of the contour lies to left (resp, right) as one moves along the contour in the direction of the orientation. Let v be a given map from Σ to $Gl(k, \mathbb{C})$. We say that $m = m(z)$ is a solution of the Riemann-Hilbert problem (Σ, v) if

- $m(z)$ is analytic in $\mathbb{C} - \Sigma$,

- $m_+(z) = m_-(z)v(z)$, $z \in \Sigma$,

where $m_\pm(z) = \lim_{z' \to z z' \in (\pm)-side} m(z')$. The matrix v is called the *jump matrix* for the Riemann-Hilbert problem. If in addition

- $m(z) \to I$ as $z \to \infty$,

we say that the the Riemann-Hilbert problem is normalized at infinity.

THEOREM 31. ([13, 14]) *Let $w : \mathbb{R} \to \mathbb{R}_+$ denote a function with the property that $w(s)s^k$ belongs to the Sobolev space $H^1(\mathbb{R})$ for all $k \in \mathbb{N}$. Suppose furthermore that n is a positive integer. Then the Riemann-Hilbert problem on $\Sigma = \mathbb{R}$, oriented from $-\infty$ to $+\infty$,*

$$
\begin{gathered}
Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2\times 2} \quad \text{is analytic}, \quad Y_+(s) = Y_-(s) \begin{pmatrix} 1 & w(s) \\ 0 & 1 \end{pmatrix} \text{ for } s \in \mathbb{R}, \\
Y(z) \begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} = I + \mathcal{O}\left(\frac{1}{|z|}\right), \qquad \text{as } |z| \to \infty,
\end{gathered} \tag{32}
$$

has a unique solution, given by

$$
Y(z) = \begin{pmatrix} \pi_n(z) & \int_{\mathbb{R}} \frac{\pi_n(s)w(s)}{s-z} \frac{ds}{2\pi i} \\ -2\pi i \gamma_{n-1}^2 \pi_{n-1}(z) & \int_{\mathbb{R}} \frac{-\gamma_{n-1}^2 \pi_{n-1}(s) w(s)}{s-z} ds \end{pmatrix}, \tag{33}
$$

where π_n denotes the n-th monic orthogonal polynomial with respect to the measure $w(x)dx$ on $\mathbb{R}$ and $\gamma_n > 0$ denotes the leading coefficient of the n-th orthogonal polynomial $p_n = \gamma_n \pi_n$. Furthermore, there exist Y_1, $Y_2 \in \mathbb{C}^{2\times 2}$ such that

$$
\begin{aligned}
&Y(z) \begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} = I + \frac{Y_1}{z} + \frac{Y_2}{z^2} + \mathcal{O}\left(\frac{1}{|z|^3}\right), \qquad \text{as } |z| \to \infty, \\
&\text{and} \quad \gamma_{n-1} = \sqrt{(Y_1)_{21}/-2\pi i}, \qquad \gamma_n = 1/\sqrt{-2\pi i (Y_1)_{12}}, \\
&\qquad a_n = (Y_1)_{11} + (Y_2)_{12}/(Y_1)_{12}, \qquad b_{n-1} = \sqrt{(Y_1)_{21}(Y_1)_{12}},
\end{aligned} \tag{34}
$$

where a_n, b_n are the recurrence coefficients associated to the orthogonal polynomials p_n.

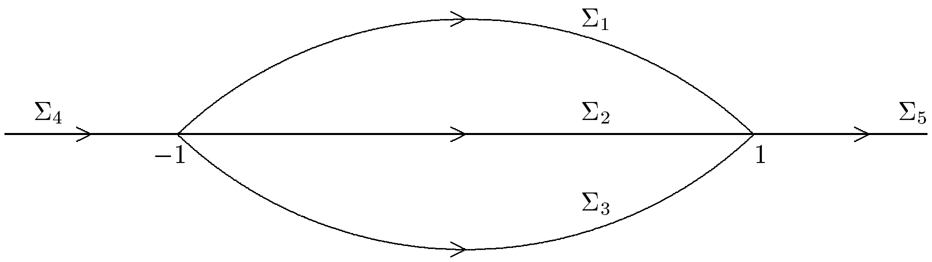

Figure 35. The contour Σ_S.

We are interested in the case where $w(x) = e^{-Q(x)}$, and $Q(x)$ satisfies (2). Theorem 31 converts the problem of computing the asymptotics of $\gamma_n, a_n, b_n, \ldots$ into a problem of computing the asymptotics of the Riemann-Hilbert problem (32) as $n \to \infty$. As indicated, this is achieved by using the steepest descent method for Riemann-Hilbert problems introduced in [11], and further developed in [12]. We conclude with a brief sketch of the method, which involves a sequence of transformations of the Riemann-Hilbert problem:

(i) RESCALING: $Y \to U_n(z) \equiv \begin{pmatrix} c_n^{-n} & 0 \\ 0 & c_n^n \end{pmatrix} Y(c_n z + d_n)$, where c_n, d_n are related to the MRS-numbers as in (5).

(ii) introduction of the "g-FUNCTION" which is the analog for the Riemann-Hilbert problem of the phase function of linear WKB theory: $U \to T(z) \equiv e^{-nl\sigma_3/2} U(z) e^{-n(g(z)-l/2)\sigma_3}$ where σ_3 is the Pauli matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $l = l_n$ is given in (12). The function $g(z)$ is analytic in $\mathbb{C} \setminus \mathbb{R}$, has asymptotics $g(z) \sim \log z$ as $z \to \infty$ and is uniquely determined as in [10] by requiring that $g_\pm(z) \equiv \lim_{\epsilon \to 0^+} g(z \pm i\epsilon)$ satisfy certain equalities and inequalities ("Phase Conditions") on $\mathbb{R}$. A simple computation shows that $T(z)$ is the solution of the following Riemann-Hilbert problem, normalized at infinity:

- $T(z)$ is analytic in $\mathbb{C} \setminus \mathbb{R}$,
- $T_+(z) = T_-(z) \begin{pmatrix} e^{-n(g_+(z)-g_-(z))} & e^{n(g_+(z)+g_-(z)-V_n(z)-l)} \\ 0 & e^{n(g_+(z)-g_-(z))} \end{pmatrix}$ for $z \in \mathbb{R}$,
- $T(z) = I + O(\frac{1}{|z|})$ as $z \to \infty$.

(iii) involves a FACTORIZATION of the jump matrix and a DEFORMATION of the contour: $T \to S$. The 2×2 matrix function $S = S(z)$ solves a Riemann-Hilbert problem on a contour of type Σ_S as in Figure 35. Now the Phase Conditions in (ii) are chosen PRECISELY to ensure that the jump matrix v_S for S on $\Sigma_1, \Sigma_3, \Sigma_4$ and Σ_5, converges exponentially to the identity matrix as $n \to \infty$, whereas $v_S = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ on $\Sigma_2 = [-1, 1]$. Thus as $n \to \infty$, we expect that S converges to the solution of the simple Riemann-Hilbert problem ($\Sigma_2 = [-1,1], v = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$), which may be solved in turn in terms of elementary radicals.

The final step (iv) involves the construction, following [12], of a PARAMETRIX for S at the points of self-intersection $\{-1, 1\}$ of Σ_S: $S \to R$. Although $v_S \to I$ on $\Sigma_1 \cup \Sigma_3 \cup \Sigma_4 \cup \Sigma_5$, the convergence is not uniform and is slower and slower near 1 and -1. This is the central analytical difficulty in the method, and requires delicate consideration. The parametrix for S is chosen so that R solves a Riemann-Hilbert problem on an extended contour $\Sigma_R \supset \Sigma_S$ with a jump matrix v_R satisfying $\|v_R - I\|_{L^\infty(\Sigma_R)} \to 0$ as $n \to \infty$. By standard Riemann-Hilbert methods, R can then be solved in terms of a Neumann series, and retracing the steps $R \to S \to T \to U \to Y$, we obtain the asymptotics for $\gamma_n, a_n, b_n, x_{kn}$ and $p_n(c_n z + d_n)$ as advertised in Theorem 13, 18 and 24.

Finally we note that it is a remarkable piece of luck that the phase condition in (ii) above can be expressed simply in terms of the equilibrium measure $d\mu_n$ corresponding to $V_n(z)$ as in (9), (10) above. Indeed, if we set $g(z) = \int \log(z - x) d\mu_n$, then it turns out that the Euler-Lagrange variational equations for μ_n, the minimizing measure in (9), are EQUIVALENT to the desired phase condition on g. In this way we construct the g-function in terms of the equilibrium measure.

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