Random and Deterministic Perturbations of Nonlinear Oscillators

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Abstract. Perturbations of Hamiltonian systems are considered. The long-time behavior of such a perturbed system, even in the case of deterministic perturbations, is governed, in general, by a stochastic process on a graph related to the Hamiltonian. We calculate the characteristics of the process for systems with one degree of freedom and consider some applications and generalizations.

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Consider an oscillator with one degree of freedom:

$$\ddot{q}_t + f(q_t) = 0, \quad q_0 = q, \quad \dot{q}_0 = p. \tag{1}$$

Let $F(q) = \int_0^q f(u)\,du$ be the potential and $H(p,q) = \frac{p^2}{2} + F(q)$ be the Hamilton function of the oscillator. One can rewrite (1) as the system:

$$\dot{p}_t = -f(q_t) = -\frac{\partial H}{\partial q}, \quad \dot{q}_t = p_t = \frac{\partial H}{\partial p}. \tag{2}$$

We assume that the potential $F(q)$ is a smooth generic function: $f(q) = F'(q)$ is assumed to be continuously differentiable, $f(q)$ has a finite number of zeros, $|f(q)| + f'(q)| \neq 0$, and the values of $F'(q)$ at different critical points are different. Let also $\lim_{|q|\to\infty} F(q) = \infty$. A typical example of $H(p,q)$ and of the phase picture is shown in Fig. 1.

Let $C(z) = \big\{x = (p,q) \in \mathbf{R}^2 : H(x) = z\big\}$ be the z-level set of $H(x)$. Since $H(x)$ is generic, $C(z)$ consists of a finite number $n = n(z)$ of connected components. Let Γ be the graph homeomorphic to the set of all connected components of the level sets of $H(x)$ provided with the natural topology (see Fig. 1b). The vertices $O_1, \ldots, O_m$ of Γ correspond to the critical points of $H(x)$. Let $I_1, \ldots, I_n$ be the edges of the graph. A vertex $O_k \in \Gamma$ is called exterior if O_k belongs just to one edge. The other vertices are called interior (vertices O_2 and O_4 in Fig. 1b). Each interior vertex belongs to 3 edges. We write $I_i \sim O_k$ if O_k is one of the ends of I_i. The value of the Hamiltonian H and the number of an edge k define a point of Γ, so that the pairs (H,k) form a global coordinate system on Γ. Define

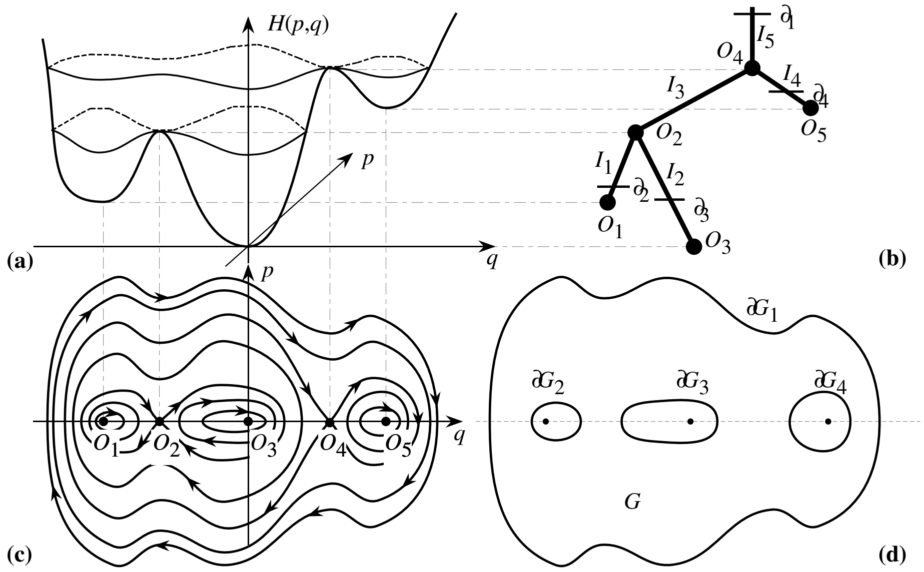

Figure 1.

a metric $\rho(\,\cdot\,,\,\cdot\,)$ on Γ: If $y_1 = (H_1, k)$ and $y_2 = (H_2, k)$ are points of the same edge $I_k \subset \Gamma$, we put $\rho(y_1, y_2) = |H_2 - H_1|$. The distance between any y_1, $y_2 \in \Gamma$ is defined as the length of the path connecting y_1 and y_2. Such a path is unique since Γ is a tree.

Consider the map $Y : \mathbf{R}^2 \to \Gamma$, $Y(x) = \big(H(x), k(x)\big) \in \Gamma$, where $k(x)$ is the number of the edge $I_{k(x)} \subset \Gamma$ containing the point of Γ corresponding to the component $C\big(H(x)\big)$ containing $x \in \mathbf{R}^2$. Let $C_k(z) = Y^{-1}(z, k)$, $(z, k) \in \Gamma$. Note that $H(x)$, as well as $k(x)$, are first integrals of system (2): $H(p_t, q_t) \overset{t}{\equiv} H(p_0, q_0)$, $k(p_t, q_t) \overset{t}{\equiv} k(p_0, q_0)$. If $H(p, q)$ has more than one minimum, then these first integrals are independent.

The Lebesgue measure in $\mathbf{R}^2$ is invariant with respect to the flow $X_t \equiv (p_t, q_t)$. If z is not a critical value of $H(x)$, then $C_k(z)$ consists of one periodic trajectory. The normalized invariant density of the flow X_t on $C_k(z)$ with respect to the length element $d\ell$ on $C_k(z)$ is

$$\Big(T_k(z)\big|\nabla H(x)\big|\Big)^{-1}, \quad x \in C_k(z),$$

where

$$T_k(z) = \oint\limits_{C_k(z)} \frac{d\ell}{\big|\nabla H(x)\big|}$$

is the period of the revolution along $C_k(z)$.

Consider now the perturbed system:

$$\ddot{q}_t^\varepsilon + f(q_t^\varepsilon) = \varepsilon\beta(\dot{q}_t^\varepsilon, q_t^\varepsilon) + \sqrt{\varepsilon}\sigma(\dot{q}_t^\varepsilon, q_t^\varepsilon) \circ \dot{W}_t. \tag{3}$$

Here W_t is the Wiener process in $\mathbf{R}^1$, functions $\beta(p,q)$ and $\sigma(p,q)$ are supposed to be bounded and continuously differentiable, $0 < \sigma(p,q)$, $0 < \varepsilon \ll 1$. The stochasitc term $\sigma(\dot{q}_t^\varepsilon, q_t^\varepsilon) \circ \dot{W}_t$ in (3) is understood in the Stratanovich sense. The deterministic part of the perturbation $\varepsilon\beta(\dot{q}, q)$ is a kind of friction. A typical and interesting example is $\beta = -\dot{q}$.

Equations (3) can be written as the system

$$\begin{aligned} \dot{p}_t^\varepsilon &= -f(q_t^\varepsilon) + \varepsilon\beta(p_t^\varepsilon, q_t^\varepsilon) + \sqrt{\varepsilon}\sigma(p_t^\varepsilon, q_t^\varepsilon) \circ \dot{W}_t; \\ \dot{q}_t^\varepsilon &= p_t^\varepsilon. \end{aligned} \tag{4}$$

The pair $(p_t^\varepsilon, q_t^\varepsilon) = X_t^\varepsilon$ forms a Markov diffusion process in $\mathbf{R}^2$. The generator A of X_t^ε for a smooth function $g(p,q)$, $(p,q) \in \mathbf{R}^2$, coincides with the differential operator

$$L^\varepsilon g(p,q) = p\frac{\partial g}{\partial q} - f(q)\frac{\partial g}{\partial p} + \varepsilon\beta(p,q)\frac{\partial g}{\partial p} + \frac{\varepsilon}{2}\frac{\partial}{\partial p}\left(\sigma^2(p,q)\frac{\partial g}{\partial p}\right).$$

We are interested in the behavior of the process X_t^ε for $0 < \varepsilon \ll 1$. On any finite time interval $[0,T]$, one can write down an expansion of X_t^ε in the powers of $\sqrt{\varepsilon}$, if $f(q)$, $\beta(p,q)$ and $\sigma(p,q)$ are smooth enough. But, actually, the long time behavior of X_t^ε is, as a rule, of interest. The finite time interval expansion does not help on time intervals of order ε^{-1}, $\varepsilon \downarrow 0$, when the perturbations become essential.

A typical example of a problem of interest is the exit problem. Let G be a bounded domain in $\mathbf{R}^2$. The most interesting case is when G is bounded by trajectories of the non-perturbed system. In Fig. 1, the boundary of the domain G consists of four components ∂G_1, ∂G_2, ∂G_3, ∂G_4. Each of them is a periodic trajectory of system (2). Let $\gamma = Y(G) \subset \Gamma$ and $\partial_i = Y(\partial G_i)$, $i = 1,2,3,4$. Let $\tau^\varepsilon = \min\{t : X_t^\varepsilon \notin G\}$ be the exit time from G. It is not difficult to check that $\tau^\varepsilon \sim \varepsilon^{-1}$ as $\varepsilon \downarrow 0$. Let $\psi(x)$, $x \in \partial G$, be continuous. Calculation of $E_x\tau^\varepsilon = u^\varepsilon(x)$, $P_x\{\tau^\varepsilon < t\} = u^\varepsilon(t,x)$, $E_x\psi(X_\tau^\varepsilon) = v^\varepsilon(x)$, where E_x and P_x mean the expectation and the probability for solutions of (4) starting at $x = (p,q) \in \mathbf{R}^2$, are of interest. Of course, since $X_t^\varepsilon = (p_t^\varepsilon, q_t^\varepsilon)$ is a diffusion process governed by the operator L^ε, one can write down a boundary problem for each of those functions $u^\varepsilon(x)$, $u^\varepsilon(t,x)$, $v^\varepsilon(x)$. Say, $u^\varepsilon(x)$ is the solution of the problem:

$$\begin{aligned} L^\varepsilon u^\varepsilon(p,q) &= p\frac{\partial u^\varepsilon}{\partial q} - f(q)\frac{\partial u^\varepsilon}{\partial p} + \varepsilon\beta(p,q)\frac{\partial u^\varepsilon}{\partial p} + \frac{\varepsilon}{2}\frac{\partial}{\partial p}\left(\sigma^2(p,q)\frac{\partial u^\varepsilon}{\partial p}\right) \\ &= -1, \quad (p,q) \in G, u^\varepsilon(p,q)\big|_{\partial G} = 0. \end{aligned} \tag{5}$$

But even numerical solution of problem (5), because of degeneration of the equation and smallness of $\varepsilon > 0$, is not simple, and the asymptotic approach is the most appropriate.

Since $\tau^\varepsilon \sim \varepsilon^{-1}$, to deal with finite time intervals as $\varepsilon \downarrow 0$, we rescale the time. Put $\tilde{X}_t^\varepsilon = X_{t/\varepsilon}^\varepsilon$, $\tilde{\tau}^\varepsilon = \varepsilon\tau^\varepsilon$. Then $\tilde{X}_t^\varepsilon = (\tilde{p}_t^\varepsilon, \tilde{q}_t^\varepsilon)$ is the solution of the system

$$\begin{aligned} \dot{\tilde{p}}_t^\varepsilon &= -\frac{1}{\varepsilon}f(\tilde{q}_t^\varepsilon) + \beta(\tilde{p}_t^\varepsilon, \tilde{q}_t^\varepsilon) + \sigma(\tilde{p}_t^\varepsilon, \tilde{q}_t^\varepsilon) \circ \dot{\tilde{W}}_t; \\ \dot{\tilde{q}}_t^\varepsilon &= \frac{1}{\varepsilon}\tilde{p}_t^\varepsilon. \end{aligned} \tag{6}$$

Here $\tilde{W}_t^\varepsilon$ is a new Wiener process. We will omit the tilde in the Wiener process.

One can single out the fast and the slow components in the process $\tilde{X}_t^\varepsilon$. The fast component is, basically, the motion along the non-perturbed trajectory. In a vicinity of a periodic trajectory $C_k(z)$, the fast motion, asymptotically as $\varepsilon \downarrow 0$, can be characterized by the invariant density $\Big(T_k(z)\big|\nabla H(x)\big|\Big)^{-1}$, $x \in C_k(z)$.

Taking into account that $H(x)$ and $k(x)$ are first integrals of the non-perturbed system, the slow motion can be described by the projection $Y(\tilde{X}_t^\varepsilon) = \big(H(\tilde{X}_t^\varepsilon), k(\tilde{X}_t^\varepsilon)\big)$ of $\tilde{X}_t^\varepsilon$ on Γ. If we are interested in the asymptotics of $u^\varepsilon(x) = \varepsilon^{-1}E_x\tilde{\tau}^\varepsilon$ as $\varepsilon \downarrow 0$, then it is sufficient to study just the slow component $Y_t^\varepsilon = Y(X_t^\varepsilon)$ as $\varepsilon \downarrow 0$ since $\tilde{\tau}^\varepsilon = \min\{t : Y_t^\varepsilon \notin \gamma\}$, $\gamma = Y(G)$. Therefore, the slow component is, in a sense, the most important for long-time behavior of the process X_t^ε, $0 < \varepsilon \ll 1$. Note, however, that if we are interested in $v^\varepsilon(x) = E_x\psi(X_{\tau^\varepsilon}^\varepsilon)$ and $\psi(x)$ is not a constant on one of the components of ∂G, then the fast component is involved in the behavior of $v^\varepsilon(x)$ as $\varepsilon \downarrow 0$ (compare with [F-W 2] Theorem 2.3 and the remark afterward).

Thus, the problem of long-time behavior of X_t^ε as $\varepsilon \downarrow 0$, to some extent, can be reduced to the asymptotic behavior of the process $Y_t^\varepsilon = Y(\tilde{X}_t^\varepsilon)$ on the graph Γ as $\varepsilon \downarrow 0$.

We prove (see [F-Web 1]) that the process Y_t^ε, $0 \le t \le T$, for any $T < \infty$ converge weakly as $\varepsilon \downarrow 0$ in the space of continuous functions $[0,T] \to \Gamma$ to a continuous Markov process Y_t on Γ. A complete description of all continuous Markov processes on a graph is given in [F-W 1,2]. A continuous Markov process Y_t on $\Gamma = \{I_1, \dots, I_n; O_1, \dots, O_m\}$ is determined by a family of second order elliptic (maybe, generalized) operators $L_1, \dots, L_n$, governing the process inside the edges, and by gluing conditions at the vertices.

To calculate the operator L_k governing the limiting process Y_t inside $I_k \subset \Gamma$, apply the Ito formula to $H(\tilde{X}_t^\varepsilon) \equiv H(\tilde{p}_t^\varepsilon, \tilde{q}_t^\varepsilon)$:

$$
\begin{aligned}
H(\tilde{X}_t^\varepsilon) - H(x) = \int_0^t \frac{\partial H}{\partial p}(\tilde{X}_s^\varepsilon)\sigma(\tilde{X}_s^\varepsilon)\,dW_s + \frac{1}{2}\int_0^t \sigma^2(\tilde{X}_s^\varepsilon)\frac{\partial^2 H}{\partial p^2}(\tilde{X}_s^\varepsilon)\,ds \\
+ \frac{1}{2}\int_0^t \frac{\partial H}{\partial p}\frac{\partial \sigma^2}{\partial p}(\tilde{X}_s^\varepsilon)\,ds + \int_0^t \frac{\partial H}{\partial p}\beta(\tilde{X}_s^\varepsilon)\,ds.
\end{aligned}
\tag{7}
$$

The stochastic integral in (7) is taken in Ito sense. Before $H(\tilde{X}_s^\varepsilon)$ changes a little, the trajectory $\tilde{X}_s^\varepsilon$ makes (for $0 < \varepsilon \ll 1$) many rotations along the periodic trajectory of the non-perturbed system. Therefore, the second, the third, and the fourth terms in the right-hand side of (7) are equivalent respectively to

$$
\frac{t}{2T(H(x))}\oint\limits_{C_k(H(x))} \frac{\sigma^2(x)H''_{pp}(x)\,d\ell}{|\nabla H(x)|}, \quad \frac{t}{2T(H(x))}\oint\limits_{C_k(H(x))} \frac{\sigma^2(x)'_pH'_p(x)\,d\ell}{|\nabla H(x)|},
$$
$$
\frac{t}{2T(H(x))}\oint\limits_{C_k(H(x))} \frac{\beta(x)H'_p(x)\,d\ell}{|\nabla H(x)|}, \quad 0 < \varepsilon \ll t \ll 1.
$$

To average the stochastic integral in (7), note that because of the selfsimilarity properties of the Wiener process, this integral is equal to

$$\overline{W}\left(\int_0^t \sigma^2(\tilde{X}_s^\varepsilon)\big(H_p'(\tilde{X}_s^\varepsilon)\big)^2\, ds\right),$$

where $\overline{W}_t$ is an appropriate Wiener process. Using this representation, one can check that the stochastic integral is equivalent to

$$\overline{W}\left(\frac{t}{T_k(H(x))}\oint_{C_k(H(x))}\frac{\sigma^2(x)(H_p'(x))^2\, d\ell}{|\nabla H(x)|}\right), \quad 0<\varepsilon\ll t\ll 1.$$

Using the divergence theorem, we have:

$$\oint_{C_k(z)}\frac{\sigma^2(x)(H_p'(x))^2\, d\ell}{|\nabla H(x)|} = \int_{G_k(z)}\left(\sigma^2(x)H_p'(x)\right)_p' dx := A_k(z),$$

where $G_k(z)$ is the domain in $\mathbf{R}^2$ bounded by $C_k(z)$, $z\in\mathbf{R}^1$. It is easy to check that

$$\frac{dA_k(z)}{dz} = \oint_{C_k(z)}\left[\frac{(\sigma^2(x))_p' H_p'(x) + \sigma^2(x)H_{pp}''(x)}{|\nabla H(x)|}\right] d\ell.$$

Combining all these facts, we conclude from (7) that, starting at a point of $I_k\subset\Gamma$, until the first exit from I_k, the limiting process Y_t is governed by the operator

$$L_k = \frac{1}{2T_k(z)}\frac{d}{dz}\left(A_k(z)\frac{d}{dz}\right) + \frac{1}{T_k(z)}B_k(z)\frac{d}{dz},$$

where

$$B_k(z) = \oint_{C_k(z)}\frac{\beta(x)H_p'(x)}{|\nabla H(x)|}\, d\ell = \int_{G_k(z)}\beta_p'(x)\, dx.$$

In particular, if the perturbation is just the white noise ($\sigma(x)\equiv 1$, $\beta(x)\equiv 0$), then the limiting process in I_k is governed by the operator

$$L_k = \frac{1}{2S_k'(z)}\frac{d}{dz}\left(S_k(z)\frac{d}{dz}\right),$$

where $S_k(z)$ is the area of the domain $G_k(z)\subset\mathbf{R}^2$ bounded by $C_k(z)$; $S_k'(z) = T_k(z)$ is the period of rotation along $C_k(z)$.

To calculate the gluing conditions at the vertices, assume for a moment that $\beta(x)\equiv 0$. Then the Lebesgue measure Λ in the plane is invariant for $\tilde{X}_t^\varepsilon$ for any $\varepsilon>0$. Therefore, the projection $\mu(s)=\Lambda(Y^{-1}(s))$, $s\subset\Gamma$, of the Lebesgue measure on Γ, defined by the mapping $Y:\mathbf{R}^2\to\Gamma$, is invariant for the processes $Y_t^\varepsilon = Y(\tilde{X}_t^\varepsilon)$ on Γ for any $\varepsilon>0$. Thus, the measure $\mu(s)$, $s\subset\Gamma$, is invariant for the limiting process Y_t on Γ. It turns out that among the diffusion processes on Γ governed by operators L_k inside the edges $I_k\subset\Gamma$, there exists just one process for which the invariant measure coincides with $\mu(s)$. This allows one to calculate the gluing conditions in the case $\beta(x) \equiv 0$. One can check that the exterior vertices are inaccessible for the limit process Y_t, and therefore, no additional gluing conditions should be imposed there. The interior vertices are accessible in a finite time inspite of the degeneration of the diffusion coefficients at the vertices.

To describe the gluing conditions at an interior vertex O_k, note that $Y^{-1}(O_k)$ is a ∞-shaped curve γ shown in Fig. 2. The curve γ consists of the trajectories γ_1, γ_2, and of the equilibrium point O_k of the non-perturbed system. Let G_1 and G_2 be the domains bounded by γ_1 and γ_2, respectively. Let $I_{k_0} \subset \Gamma$ be the edge corresponding to the trajectories surrounding γ (like the trajectory ϕ_0 in Fig. 2);

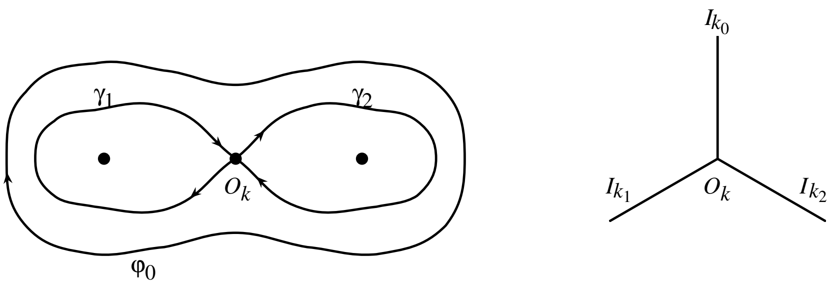

Figure 2.

$I_{k_1} \subset \Gamma$ corresponds to periodic trajectories inside γ_1 which are close to γ_1, and $I_{k_2} \subset \Gamma$ corresponds to trajectories inside γ_2 close to γ_2; I_{k_0}, I_{k_1}, $I_{k_2} \sim O_k$. Put

$$\beta_{ki} = \int\limits_{G_i} \frac{\partial}{\partial p}\left(\sigma^2(p,q)\frac{\partial H(p,q)}{\partial p}\right) dp\, dq, \quad i = 1,2, \quad \beta_{k0} = \beta_{k1} + \beta_{k2}.$$

Then a bounded and continuous on Γ function $u(y)$, $y \in \Gamma$, which is smooth inside the edges, belongs to the domain of definition of the generator A of the limiting process Y_t on Γ if the function $L_k u(z,k)$, $(z,k) \in \Gamma$, is continuous on Γ, and at any interior vertex $O_k \in \Gamma$

$$\beta_{k1} D_1 u(O_k) + \beta_{k2} D_2 u(O_k) = \beta_{k0} D_0 u(O_k),$$

where D_i is the operator of differentiation in z along I_{k_i}, $i = 0,1,2$. The operators L_k together with the gluing conditions at the vertices define the limiting process Y_t on Γ in a unique way.

Now, if $\beta(p,q) \not\equiv 0$ in the perturbation term, one can check, using the Cameron-Martin-Girsanov formula, that the gluing conditions are the same as for $\beta(p,q) \equiv 0$.

To complete the proof, one should also check that the family of processes $Y_t^\varepsilon = Y(\tilde{X}_t^\varepsilon)$, $0 \le t \le T$, is tight in the weak topology and that the limiting process is a Markov one. The tightness follows, roughly speaking, from the at most linear growth of the coefficients in (7). The Markov property can be proved using some *a priori* bounds for the operator L^ε (see [F-Web1]).

This result allows one to calculate in an explicit form the main terms as $\varepsilon \downarrow 0$ of many interesting characteristics of the process X_t^ε ([F-Web1]). A slight generalization of these results allows one to consider also perturbations of the nonlinear pendulum defined by the equation $\ddot{q}_t + \sin q_t = 0$, ([F-Web2]).

Suppose now that we have just deterministic perturbations: $\sigma(x) \equiv 0$ in equation (6). Let, for brevity, the Hamiltonian have just one saddle point, so that the phase picture for the non-perturbed system is as in Fig. 3a, and let $b_p'(p,q) < 0$, $(p,q) \in \mathbf{R}^2$. The perturbations lead to the picture in Fig. 3b: the perturbed system

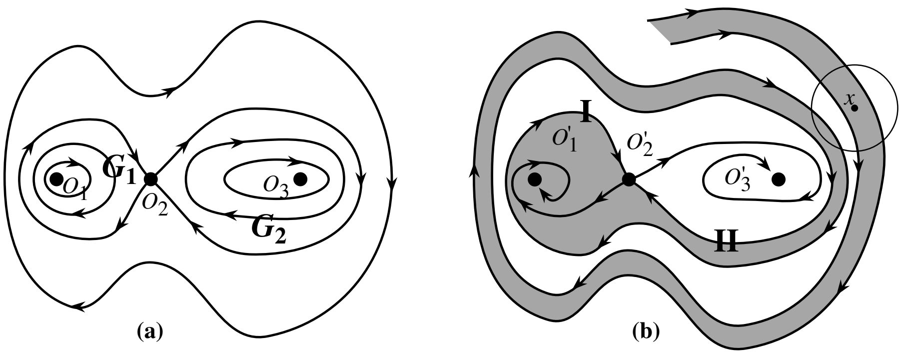

Figure 3.

has a saddle point in a point O_2' which is close to O_2; the equilibrium points O_1, O_3 will be replaced by asymptotically stable points O_1', O_3', which are close to O_1, and O_3, respectively, when $0 < \varepsilon \ll 1$. Two separatrices I and II enter O_2'. They divide the exterior $\mathcal{E}$ of the ∞-shaped curve connected with O_2 in two ribbons. One of these ribbons consists of points attracted to O_1'; another ribbon is attracted to O_3' (see Fig. 3b). The width of each of these ribbons is of order ε as $\varepsilon \downarrow 0$. When ε becomes smaller, they are moving closer and closer to the ∞-shaped curve. Therefore, any point $x \in \mathcal{E}$ alternatively belongs to a ribbon attracted to either O_1' or to O_3' as $\varepsilon \downarrow 0$. This means that the perturbed trajectory X_t^ε starting at $x \in \mathcal{E}$, is attracted alternatively to O_1' or O_3' when $\varepsilon \downarrow 0$.

The slow motion of the perturbed system in this case is again the projection on the graph Γ related to $H(x)$: $Y_t^\varepsilon = Y(X_{t/\varepsilon}^\varepsilon)$. The averaging procedure shows that the limiting slow motion $\overline{Y}_t$ is a deterministic motion inside each of the edges of the graph Γ:

$$\dot{z}_t = \frac{1}{T_k(z_t)} B_k(z_t), \quad \overline{Y}_t = (z_t, k) \in I_k, \quad k = 1,2,3. \tag{8}$$

If we start from a point x with a large enough $H(x)$, and $B_k(z) < 0$ if (z,k) is not a vertex, then the deterministic trajectory hits the vertex O_2 corresponding to the saddle point of $H(x)$ in a finite time. After that, the trajectory of the limiting slow motion goes to one of the two edges attached to O_2 along which H is decreasing.

To which of these two edges the trajectory goes depends on the initial point in a very sensitive way. One can show that the measure of the set of initial points from a neighborhood U of a point x, $U \subset \mathcal{E}$, attracted to O_1 (to O_3) is proportional to $\int\limits_{G_1} \beta_p'(x)\,dx$ ($\int\limits_{G_2} \beta_p'(x)\,dx$) as $\varepsilon \downarrow 0$, where G_1 and G_2 are the left and the right part of the set in $\mathbf{R}^2$ bounded by the ∞-shaped curve. This was briefly mentioned in [A]. The proof is available in [B-F]. If the graph corresponding to $H(x)$ has a more complicated structure and the "friction" $\beta(p,q)$ is allowed to change the sign, the situation can be more complicated: the limiting slow motion can "remember more of its past" (see [B-F]).

There is another way to regularize the problem: Instead of random perturbation of the initial point, one can add a random perturbation to the equation. Let $\sigma(p,q)$ in (6) be replaced by $\sqrt{\kappa}\tilde{\sigma}(p,q)$, where $\kappa > 0$ is a small parameter. Let $\tilde{X}_t^{\varepsilon,\kappa}$ be the solution of (6) with such a replacement. Consider the double limit of the slow component $Y_t^{\varepsilon,\kappa} = Y(\tilde{X}_t^{\varepsilon,\kappa})$, $0 \le t \le T$: first as $\varepsilon \downarrow 0$ for a fixed $\kappa > 0$, and then as $\kappa \downarrow 0$. The first limit gives us the diffusion process Y_t^κ on Γ, which was described above. Now we consider the limiting behavior of Y_t^k, $0 \le t \le T$, as $\kappa \downarrow 0$. As it is proved in [B-F], this limit (in the sense of weak convergence) exists, independent of the perturbations (of the choice of functions $\sigma(p,q)$) and coincides with the process $\overline{Y}_t$ desribed above: Inside the edges it is a deterministic motion governed by (8), and it branches at each interior vertex O_k to one of the edges attached to O_k, along which H is decreasing, with certain probabilities which are expressed through $H(x)$ and $\beta(x)$ in a way similar to that descried above. The behavior of the limiting slow component after touching an interior vertex O_k is independent now of the past (see [B-F]). The independence of the process $\overline{Y}_t$ of the characteristics of the random perturbations, as well as the fact that the limiting process is the same as occurs if the initial conditions are perturbed, shows that the "randomness" of the limiting slow component is an intrinsic property of the Hamiltonian system and its deterministic perturbations. The random perturbation here is just a way of regularization.

The perturbations in equations (6) are included just in one component. Therefore, the corresponding differential operator $\varepsilon^{-1}L^\varepsilon$ is degenerate. This leads to certain additional difficulties in the proof of Markov property for the limiting process. One can consider non-degenerate perturbations and replace the oscillator by an arbitrary Hamiltonian system with one degree of freedom:

$$\dot{\tilde{X}}_t^\varepsilon = \frac{1}{\varepsilon}\overline{\nabla}H(\tilde{X}_t^\varepsilon) + \beta(\tilde{X}_t^\varepsilon) + \sigma(\tilde{X}_t^\varepsilon) \circ \dot{W}_t, \quad \tilde{X}_t^\varepsilon = x \in \mathbf{R}^2. \tag{9}$$

Here W_t is the Wiener process in $\mathbf{R}^2$, $\beta(x)$ is a smooth bounded vector field in $\mathbf{R}^2$, and $\sigma(x)$ is a 2×2 matrix with smooth bounded entries, $\det\sigma(x) \neq 0$. The Hamiltonian function $H(x)$ is assumed to be smooth, generic, and $\lim_{|x|\to\infty} H(x) = \infty$.

Let $\Gamma = \{I_1, \ldots, I_n; O_1, \ldots, O_m\}$ be the graph corresponding to $H(x)$ and $Y(x) = \big(H(x), k(x)\big)$ be the corresponding mapping $\mathbf{R}^2 \to \Gamma$. Then one can prove [F-W2,3] that the slow component of the process $\tilde{X}_t^\varepsilon$, which is $Y(\tilde{X}_t^\varepsilon)$, $0 \le t \le T$, converges weakly as $\varepsilon \downarrow 0$ to a diffusion process Y_t on Γ. The process Y_t is governed inside I_k, $k \in \{1,\dots,n\}$, by the operator

$$L_k = \frac{1}{2T_k(z)}\frac{d}{dz}\left(A_k(z)\frac{d}{dz}\right) + \frac{1}{T_k(z)}B_k(z)\frac{d}{dz}, \quad T_k(z) = \oint\limits_{C_k(z)} \frac{d\ell}{|\nabla H(x)|},$$

$$A_k(z) = \int\limits_{G_k(z)} \operatorname{div}\big(a(x)\nabla H(x)\big)\,dx, \quad a(x) = \sigma(x)\sigma^*(x), \quad B_k(z) = \int\limits_{G_k(z)} \operatorname{div}\beta(z)\,dx. \tag{10}$$

Here $C_k(z) = Y^{-1}(z,k)$, $G_k(z)$ is the domain in $\mathbf{R}^2$ bounded by $C_k(z)$, $(z,k) \in \Gamma \setminus \{O_1,\dots,O_m\}$.

To define the process Y_t for all $t \geq 0$, we should add the gluing conditions at the vertices. The gluing conditions are defined by he domain of definition $D_{\mathfrak{a}}$ of the generator $\mathfrak{A}$ of the process Y_t: a continuous and smooth inside the edges function $f(g)$, $y \in \Gamma$, belongs to $D_{\mathfrak{a}}$, if $L_k f(z,k)$, $y = (z,k) \in \Gamma$, is continuous on Γ and at any interior vertex $O_k \in \Gamma$

$$\sum_{i=1}^{3} \pm \beta_{ki} D_i f(O_k) = 0, \tag{11}$$

where $\beta_{ki} = \lim_{(z,k_i)\to O_k} A_{k_i}(z)$; I_{k_1}, I_{k_2}, $I_{k_3} \sim O_k$; the "+" ("−") sign in front of β_{ki} is taken if H grows (decreases) as the point approaches O_k along I_{k_i}, $i \in \{1,2,3\}$, (see [F-W2,3]).

This result allows one to calculate in a rather explicit form the main term as $\varepsilon \downarrow 0$ of the solution of the following Dirichlet problem:

$$\frac{\varepsilon}{2}\operatorname{div}\big(a(x)\nabla u^\varepsilon(x)\big) + \varepsilon\beta(x)\cdot\nabla u^\varepsilon(x) + \overline{\nabla} H(x)\cdot\nabla u^\varepsilon(x) = 0, x \in G, u^\varepsilon(x)\big|_{\nabla G} = \psi(x).$$

Here $G \subset \mathbf{R}^2$ is as in Fig. 1, $\psi(x)$ is a continuous function on ∂G.

It follows from [F-W2,3] that $\lim u^\varepsilon(x) = v\big(H(x), k(x)\big)$, where $v(z,k)$ is the solution of the Dirichlet problem in $\gamma = Y(G) \subset \Gamma$

$$L_k v(z,k) = 0, (z,k) \in \gamma \setminus \{O_1,\dots,O_m\}, v(\partial_k) = \overline{\psi}_k, \quad k \in \{1,2,3,4\},$$

satisfying the gluing conditions described above. Here $\partial_k = Y(\partial G_k)$, $k = 1,2,3,4$, $\partial\gamma = (\partial_1,\partial_2,\partial_3,\partial_4)$,

$$\psi_k = \left(\oint\limits_{\partial G_k} \frac{a(x)\nabla H(x)\cdot\nabla H(x)}{|\nabla H(x)|}\,d\ell\right)^{-1} \oint\limits_{\partial G_k} \frac{\psi(x)\big(a(x)\nabla H(x)\cdot\nabla H(x)\big)}{|\nabla H(x)|}\,d\ell,$$

$k \in \{1,2,3,4\}$.

The Dirichlet problem in γ can be solved explicitly.

Consider now the case of pure deterministic perturbations: $\sigma(x) \equiv 0$ in (9). Let for brevity $B_k(z)$, defined in (10), be negative if (z,k) is not an exterior vertex. This, in particular, implies that the perturbed system is not Hamiltonian. We can again "regularize" the problem adding small random perturbations to the initial conditions or to the equation and then consider the double limit [B-F].

To consider perturbations of the equation, replace the matrix $\sigma(x)$ in (8) by $\sqrt{\kappa}\sigma(x)$, $\kappa > 0$. Let $\tilde{X}_t^{\varepsilon,\kappa}$ be the solution of equation (8). Consider the projection $Y_t^{\varepsilon,\kappa} = Y(\tilde{X}_t^{\varepsilon,\kappa})$ of $\tilde{X}_t^{\varepsilon,\kappa}$ on Γ. Then, for each $\kappa > 0$, the processes $Y_t^{\varepsilon,\kappa}$, $0 \le t \le T$, converge weakly as $\varepsilon \downarrow 0$ to the process Y_t^{κ} on Γ, which was described above. Let now $\kappa \downarrow 0$. One can check that processes Y_t^{κ}, $0 \le t \le T$, converge weakly to a process $Y_t = (z_t, k_t)$ on Γ as $x \downarrow 0$. Inside any edge $I_k \subset \Gamma$, the process Y_t is deterministic motion governed by equation (8) with $B_k(z)$ defined in (10). If Y_t touches an interior vertex $O_k \in \Gamma$, it leaves O_k without any delay along one of the edges I_{k_1}, $I_{k_2} \sim O_k$, along which H is decreasing, with probabilities P_{k1}, P_{k2};

$$P_{ki} = \frac{|B_{k_i}(O_k)|}{|B_{k_1}(O_k)| + |B_{k_2}(O_k)|}, \quad \big|B_{k_i}(O_k)\big| = \lim_{(z,k_i)\to O_k} \big|B_{k_i}(z)\big|, \quad i = 1, 2,$$

independently of the past [B-F].

A special case of this problem when $a(x)$ is the unit matrix was studied in [W].

If we consider the perturbations of the form

$$\dot{X}_t^{\varepsilon,x} = \overline{\nabla} H(X_t^{\varepsilon,x}) + \varepsilon\beta(X_t^{\varepsilon,x}) + \sqrt{\varepsilon\kappa}\zeta_t,$$

where ζ_t is a stationary process with strong enough mixing properties, and the process ζ_t is not degenerate in a certain sense, then, because of a central-limit-theorem type result, we can expect the same process Y_t as the limit of $Y(X_{t/\varepsilon}^{\varepsilon,\kappa})$ as first $\varepsilon \downarrow 0$ and then $\kappa \downarrow 0$.

These results can be applied to some non-linear problems for second order elliptic and parabolic equations. Consider, for example, reaction-diffusion in a stationary incompressible fluid in $\mathbf{R}^2$:

$$\frac{\partial u^\varepsilon(t,x)}{\partial t} = \frac{\varepsilon}{2}\Delta u^\varepsilon + \overline{\nabla} H(x) \cdot \nabla u + f(u^\varepsilon), \quad t > 0, \quad x \in \mathbf{R}^2, \quad u^\varepsilon(0,x) = g(x) \ge 0. \tag{12}$$

Here $H(x)$ is the stream function of a stationary flow. We assume that $H(x)$ is generic and $\lim_{|x|\to\infty} H(x) = \infty$. The initial function is assumed to be continuous. Let for brevity $g(x)$ has a compact support. Let Γ be the graph related to $H(x)$ and $Y(x) : \mathbf{R}^2 \to \Gamma$ be the corresponding mapping. If $f(u) \equiv 0$, it follows from the results formulated in this paper [FW2], that $u^\varepsilon(t/\varepsilon, x) \to v(t, Y(x))$, where $v(t,y)$ is the solution of a Cauchy problem on $[0,\infty) \times \Gamma$ with appropriate gluing conditions at the vertices.

But if the reaction term $f(u)$ is included in the equation, one should use a different time scale. Let, for instance, $f(u) = c(u)u$ is of Kolmogorov-Petrovskii-Piskunov type: $c(u) > 0$ for $u < 1$, $c(u) < 0$ for $u > 1$, and $c(0) = \max_{u\ge 0} c(u)$. Then $\lim_{\varepsilon\downarrow 0} u^\varepsilon(t/\sqrt{\varepsilon}, x) = w(t, Y(x))$, where $w(t,y)$, $t > 0$, $y = (z,k) \in \Gamma$, is a step function with the values 0 and 1. To describe the set, where $w(t,y)$ is equal to 1, introduce a Riemannian metric ρ on Γ corresponding to the form

$$ds^2 = \frac{T_k(z)}{A_k(z)} dz^2, \quad T_k(z) = \oint\limits_{C_k(z)} \frac{d\ell}{|\nabla H(x)|}, \quad A_k(z) = \int\limits_{G_k(z)} \Delta H \, dx.$$

Note that this form has singularities at the vertices, but those singularities are integrable. Let $\gamma = Y(\operatorname{supp} g) \subset \Gamma$. Then, $w(t,y) = 1$ on the set

$$\left\{ y \in \Gamma : \rho(y,\gamma) < t\sqrt{2c(0)} \right\}, \quad t > 0,$$

and $w(t,y) = 0$ outside of the closure of this set. This is a result of an interplay between the averaging and the large deviations for process $X^\varepsilon_{t/\varepsilon}$, where X^ε_t is the process in $\mathbf{R}^2$ governed by the linear part of the operator in the right-hand side of (12).

Applications of the ideas discussed in this paper to small viscosity asymptotics for the stationary Navier-Stokes equations one can find in [F2].

Applications to an optimal stabilization problem are available in [D-F].

I will briefly consider now some generalizations. First, consider a Hamiltonian system on a two-dimensional torus. A generic Hamiltonian system on a 2-torus has the following structure: it has a finite number of loops such that inside those loops, trajectories behave like in a part of $\mathbf{R}^2$. The exterior $\mathcal{E}$ of the union of the loops is one ergodic class so that the trajectories of the system are dense in $\mathcal{E}$ (see references in [F1]). Therefore, the graph Γ related to this system has a special vertex O_0 which corresponds to the whole set $\mathcal{E}$. Consider now small white noise perturbations of the system. The Lebesgue measure on the torus is invariant for the perturbed process and the projection of this measure on Γ is invariant for the slow component. This implies that the limiting slow component spends at $O_0 \in \Gamma$ a positive time proportional to the relative area of $\mathcal{E}$. Therefore the gluing conditions at O_0 are a little different from the conditions at other vertices or form conditions considered above (see [F-W1], [F1]).

Perturbations of certain Hamiltonian systems on 2-torus may lead also to processes on graphs with loops, but not just trees as in the case of systems in $\mathbf{R}^2$.

Finally, we consider briefly perturbations of Hamiltonian systems with many degrees of freedom:

$$\begin{aligned} \dot{X}^\varepsilon_t &= \overline{\nabla} H(X^\varepsilon_t) + \sqrt{\varepsilon}\dot{W}_t + \varepsilon\beta(X^\varepsilon_t), \\ X^\varepsilon_0 &= x \in \mathbf{R}^{2n}, x = (p_1, \dots, p_n; q_1, \dots, q_n). \end{aligned} \tag{13}$$

Here W_t is the $2n$-dimensional Wiener process. $\beta(x)$ is a smooth vector field in $\mathbf{R}^{2n}$, $0 < \varepsilon \ll 1$. If $n > 1$, the non-perturbed system may have additional smooth first integrals: $H_1(x) = H(x)$, $H_2(x)$, $\dots$, $H_\ell(x)$. Let $C(z) = \{x \in \mathbf{R}^{2n} : H_1(x) = z_1, \dots, H_\ell(x) = z_\ell\}$, $z = (z_1, \dots, z_\ell) \in \mathbf{R}^\ell$. If the non-perturbed system X^0_t has a unique "smooth" invariant measure on each $C(z)$, $z \in \mathbf{R}^\ell$, then the slow component can be described by the evolution of the first integrals. In an appropriate time scale, the slow component converges to a diffusion process Y_t, $0 \le t \le T$. The diffusion and drift coefficients of Y_t can be calculated using the standard averaging procedure. We have such an example when considering a system of independent oscillators with one degree of freedom

$$\begin{aligned} \dot{X}^\varepsilon_k(t) &= \overline{\nabla} H_k(X^\varepsilon_k(t)) + \varepsilon\beta_k(X^\varepsilon_1(t), \dots, X^\varepsilon_n(t)) + \sqrt{\varepsilon}\sigma_k \dot{W}_k(t), \\ x_k &= (p_k, q_k) \in \mathbf{R}^2, \quad k = 1, \dots, n, \end{aligned} \tag{14}$$

with $H_k(p,q) = a_k p^2 + b_k q^2$ and $a_k,\ b_k > 0$, $k \in \{1,\dots,n\}$, such that the frequencies of the oscillators are incommensurable. Here $W_k(t)$ are independent two-dimensional Wiener processes, σ_k are non-degenerate 2×2 matrices, $\beta_k \in \mathbf{R}^2$. But, in general, if H_k are not quadratic forms, the frequences are changing with the energy and resonances appear. This problem, in the case of deterministic perturbations, was studied by many authors (see [AKN]). The approaches used in the deterministic case allow one to obtain some results on stochastic perturbations as well.

Let $H_k(x)$, $x \in \mathbf{R}^2$, $k = 1,\dots,n$, be generic and $\lim_{|x|\to\infty} H_k(x) = \infty$. Let Γ_k be the graph related to $H_k(x)$ and $Y_k : \mathbf{R}^2 \to \Gamma_k$ be the corresponding mapping. The slow component Y_t^ε of the process $\big(X_1^\varepsilon(t),\dots,X_n^\varepsilon(t)\big) = X_t^\varepsilon$ is defined as the process $Y_t^\varepsilon = \big(Y_1(X_1^\varepsilon(t/\varepsilon)),\dots,Y_n(X_n^\varepsilon(t/\varepsilon))\big)$, on $\Xi = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n$. Under some mild additional conditions, the processes Y_t^ε, $0 \le t \le T$, converge as $\varepsilon \downarrow 0$ weakly to a process Y_t on Ξ. Inside the n-dimensional pieces of Ξ, where $\sum_{k=1}^n \big|\nabla H_k(x_k)\big| \neq 0$, the process Y_t is described by the averaging procedure. To define the gluing conditions, assume, first, that $\beta_k(x) \equiv 0$, $k = 1,\dots,n$. Then the process X_t^ε is just a collection of n independent processes $X_k^\varepsilon(t)$, each with one degree of freedom. The slow component $Y_k(X_k^\varepsilon(t/\varepsilon))$ of X_k^ε converges, as we already know, to a process $Y_k(t)$ on Γ_k with the gluing conditions described above. Thus, we know what is the limiting slow component for X_t^ε in the case $\beta_k(x) \equiv 0$, $k \in \{1,\dots,n\}$. Using the Cameron-Martin-Girsanov formula, one can check that, if $\beta_k(x) \not\equiv 0$ are bounded and matrices σ_k are non-degenerate, than the gluing conditions will be, in a sense, the same. This allows to give a complete description of the limiting slow component for X_t^ε as a diffusion process on Ξ [F-W4].

Similar to the case of one degree of freedom, this result enable us to show that, under some additional conditions, the long-time behavior of deterministic systems close to Hamiltonian has a stochastic nature. Consider weakly coupled oscillators with one degree of freedom:

$$\begin{aligned} \dot X_k^\varepsilon(t) &= \overline{\nabla} H_k(X_k^\varepsilon(t)) + \varepsilon\beta_k\big(X_1^\varepsilon(t),\dots,X_n^\varepsilon(t)\big), \\ X_k(0) &= x_k \in \mathbf{R}^2, \quad k \in \{1,\dots,n\}, \quad 0 < \varepsilon \ll 1. \end{aligned} \tag{15}$$

The slow motion for this system is the projection of $X^\varepsilon(t) = \big(X_1^\varepsilon(t),\dots,X_n^\varepsilon(t)\big)$ on $\Xi : Y_t^\varepsilon = Y(X_{t/\varepsilon}^\varepsilon)$. As in the one-degree-of-freedom case, the processes Y_t^ε does not converge as $\varepsilon \downarrow 0$. But one can regularize the problem, adding small noise to the equation: Replace σ_k in (14) by $\sqrt{\kappa}\sigma_k$, and let $X_t^{\varepsilon,\kappa}$ be the solution of (14) after this change. The processes $Y_t^{\varepsilon,\kappa} = Y(X_{t/\varepsilon}^{\varepsilon,\kappa})$, $0 \le t \le T$, converge as $\varepsilon \downarrow 0$, for a fixed $\kappa > 0$, to a diffusion process Y_t^κ on Ξ, under some additional conditions. Then one can check that the processes Y_t^κ, $0 \le t \le T$, converge as $\kappa \downarrow 0$ to a process Y_t on Ξ. The process Y_t is deterministic inside the n-dimensional pieces of Ξ and has some stochastic behavior on the edges. The process Y_t is independent of the choice of matrices σ_k, so that it is determined by the intrinsic properties of system (15), but not by the random perturbations.

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