Scaling Limit of Particle Systems, Incompressible Navier-Stokes Equation and Boltzmann Equation

Horng-Tzer Yau*

Abstract. We review recent work on derivations of the Euler, incompressible Navier-Stokes and Boltzmann equations from scaling limits of microscopic dynamics.

1991 Mathematics Subject Classification: Primary 82C22; Secondary 60K35,82C05, 82C10.
Keywords and Phrases: Euler equations, incompressible Navier-Stokes equations, Boltzmann equation, Hamiltonian dynamics, stochastic dynamics, Schrödinger equation.

I. Introduction

Macroscopic equations such as the Euler equations, Navier-Stokes equations or Boltzmann equation are usually derived through a continuum formulation of conservation of mass and momentum or in the last case, by idealizing the collision process. But, they also have a more fundamental origin in the microscopic equations of Newton or Schrodinger. The main question is whether this assertion can be put on a firm mathematical foundation and whether macroscopic concepts such as the viscosity, the nonlinearity, and the dissipation of the entropy can be understood microscopically. There are other important questions about many-body dynamics such as fluctuations, time-dependent correlations and behavior of tagged particles which are naturally formulated only on the microscopic level, but due to the restriction of the length of this review, we shall address only the first question here.

In statistical physics, continuum quantities such as density, velocity, and energy have microscopic versions which assume their macroscopic, deterministic values through the law of large numbers. Therefore, in order the equations describing the evolution of the macroscopic quantities to be exact, certain limits have to be taken, with suitably chosen scalings of space, time, and other macroscopic parameters of the systems. So the first step in the derivation of such equations is a choice of scaling. Denote coordinates by lower case letters (x, t) in the microscopic scale; by capital letters (X, T) in the macroscopic scale. We put the system in a cube of

* Partially supported by U. S. National Science Foundation grants 9703752

size L in d-dimensional space with periodic boundary condition and we will usually assume $d = 3$. Denote the particles by $(x_1, \cdots, x_N, v_1, \cdots, v_N)$ with the density (in the microscopic unit, i.e., number of particles per microscopic unit volume) $\rho = N/L^d$. Let ε be the ratio between the microscopic unit and the macroscopic unit (say, $\varepsilon \sim 10^{-8}$). There are typically three choices of scalings:

$$\left\{ \begin{array}{ll} \text{Grad} & \rho = \varepsilon, (X,T) := (x\varepsilon, t\varepsilon) \\ \text{Euler} & \rho = 1, (X,T) := (x\varepsilon, t\varepsilon) \\ \text{Diffusive} & \rho = 1, (X,T) := (x\varepsilon, t\varepsilon^2) \end{array} \right\} \Longrightarrow \left\{ \begin{array}{lcl} & & \text{finite} \\ \text{collisions} & : & \varepsilon^{-1} \\ \text{per particle} & & \varepsilon^{-2} \end{array} \right\} \tag{1.1}$$

The Euler and diffusive limits will be referred to as hydrodynamic limits. The typical number of collisions is finite for the Grad limit; infinite in the hydrodynamic limits. Hence the Grad limit is the closest to free motion without collisions. Essentially due to this feature, O. Lanford [12] proved the convergence of the hard core billiards to the Boltzmann equation in the Grad limit in short time based on the BBGKY hierarchy. Lanford's work, though restrictive in many ways, remains the only rigorous result on the scaling limits of many-body Hamiltonian systems with no unproven assumptions.

II. Euler Equations

At present there is no rigorous derivation of Euler equations from Hamiltonian mechanics. Unlike the Grad limit, the Euler limit involves an infinite number of collisions and the typical behavior is governed by the stationary (equilibrium, invariant) states, which are assumed to be Gibbs in the famous Boltzmann Hypothesis. More precisely:

BOLTZMANN HYPOTHESIS : The invariant (stationary) measures of many body classical dynamics are *Gibbs* $\sim e^{-\beta H}$. In particular, the typical velocity distributions of different particles are uncorrelated (Weak Boltzmann Hypothesis).

The Boltzmann Hypothesis is strictly speaking *incorrect* because there are singular invariant measures. We believe that these singular invariant measures can be removed by regularity assumption such as *finite specific entropy*, i.e., entropy per microscopic unit volume is finite. The following theorem is a joint work with S. Olla and S. Varadhan [15].

THEOREM. *Assume the weak Boltzmann Hypothesis holds for invariant measures with finite specific entropy. Suppose the Euler equation has a smooth solution in* $[0, T]$. *Then the empirical density, velocity. and energy converge to the solution of the Euler equations in* $[0, T]$ *with probability one.*

Recall that classical dynamics are characterized by a Hamiltonian

$$H(x,v) = \frac{1}{2}\sum_{\alpha=1}^{N} \|v_\alpha\|^2 + \sum_{\alpha<\beta\le N} V(x_\alpha - x_\beta) \tag{2.1}$$

with V a two-body potential and the Liouville equation

$$\partial_t f_{N,t}(x_1, \cdots, x_N, v_1, \cdots, v_N) = \mathcal{L}^* f_{N,t} \tag{2.2}$$

where $f_{N,t}$ is the density (w.r.t. the standard Lebesgue measure) of the system at time t and the Liouville operator is given by

$$-\mathcal{L}^* = \mathcal{L} = \sum_{\alpha=1}^{N} \left[\frac{\partial H}{\partial v_\alpha} \frac{\partial}{\partial x_\alpha} - \frac{\partial H}{\partial x_\alpha} \frac{\partial}{\partial v_\alpha} \right]$$

with the adjoint taken w.r.t. the standard Lebesgue measure.

For a given configuration $\omega = (x_1, \cdots, x_N, v_1, \cdots, v_N)$ the empirical density and velocity (which rigorously speaking are measures) are defined by

$$\hat{\rho}_{\epsilon,\omega}(X) = N^{-1} \sum_{\alpha=1}^{N} \delta(X - \varepsilon x_\alpha)\,,$$

$$\hat{v}_{\varepsilon,\omega}(X) = N^{-1} \sum_{\alpha=1}^{N} \delta(X - \varepsilon x_\alpha) v_\alpha\,,$$

where δ is the standard delta function on Euclidean space. We shall say $\hat{\rho}_{\varepsilon,\omega(T/\varepsilon)}(X)$ has a density $\rho(X,T)$ if for any test function J on the unit torus,

$$\int J(\varepsilon x) \hat{\rho}_{\epsilon,\omega(T/\varepsilon)}(X) dX = N^{-1} \sum_{\alpha=1}^{N} J(\,\varepsilon\, x_\alpha(T/\varepsilon)\,) \to \int J(X) \rho(X,T) dX$$

as $\varepsilon \to 0$. Similarly for the velocity,

$$N^{-1} \sum_{\alpha=1}^{N} J(\,\varepsilon\, x_\alpha(T/\varepsilon)\,)\, v_\alpha(T/\varepsilon) \to \int J(X)\,(\rho v)(X,T) dX.$$

To obtain the Euler equation, we differentiate the velocity

$$\frac{d}{dT} \int J(X) \rho(X,T) v(X,T) dX \sim \varepsilon^{-1} \frac{d}{dt} N^{-1} \sum_{\alpha=1}^{N} J(\varepsilon x_\alpha) v_\alpha$$

$$= -(2N)^{-1} \sum_{\alpha=1}^{N} \varepsilon^{-1} J(\varepsilon x_\alpha) \frac{\partial H}{\partial x_\alpha} + \cdots$$

$$= -(2N)^{-1} \sum_{\alpha=1}^{N} \nabla J(\varepsilon x_\alpha) \underbrace{\sum_{\beta \neq \alpha} \frac{x_\alpha - x_\beta}{\varepsilon} \cdot (\nabla V) \left(\frac{x_\alpha - x_\beta}{\varepsilon} \right)}_{\text{MICRO CURRENT}} + \cdots \tag{2.3}$$

(the micro current appearing here is only a main term for illustration of the idea). Recall the Euler equations:

$$\frac{d\rho}{dt} + \nabla\,(\rho v) = 0$$
$$\frac{d\,(\rho v)}{dt} + \nabla\,[\rho v \otimes v + P] = 0$$
$$\frac{d\,(\rho e)}{dt} + \nabla\,[\rho e\, v - v\, P] = 0$$

Here the pressure P is a function of density, velocity and energy and is determined by the equation of state from the equilibrium Gibbs measure. So in order to obtain the Euler equations we need to show that

$$\text{MICRO CURRENT} \to \text{MACRO CURRENT} \, (= P(\hat{\rho}_{\varepsilon,\omega}, \hat{v}_{\varepsilon,\omega}, \hat{e}_{\varepsilon,\omega})) \tag{2.4}$$

in the limit $\varepsilon \to 0$. This equality is understood in the sense of law of large numbers w.r.t. the density of the systems $f_{N,t}$ at time t, i.e.,

$$\begin{aligned} N^{-1} \int f_{N,t}(\omega) \Big| \sum_{\alpha=1}^{N} \nabla J(\varepsilon x_\alpha) \\ \times \Big[\sum_{\beta \neq \alpha} \frac{x_\alpha - x_\beta}{\varepsilon} \cdot (\nabla V) \left(\frac{x_\alpha - x_\beta}{\varepsilon} \right) - P(\hat{\rho}_{\varepsilon,\omega}, \hat{v}_{\varepsilon,\omega}, \hat{e}_{\varepsilon,\omega}) \Big] \Big| d\omega \to 0 \end{aligned} \tag{2.5}$$

where $d\omega = dx_1 dv_1 \cdots dx_N dv_N$.

The density $f_{N,t}$ satisfies the Liouville equation (2.2). At the present time we have essentially no estimate on this equation and the required identity has not been proved. To appreciate the difficulties, we list a few comments on the Liouville equation:

- It conserves L^p norm and positivity (thus f_N can be considered as a probability density) but L^p norm is not useful since $\| f_N \|_p \sim e^{CN}$ which is a huge number.
- There is no elliptic theory for classical dynamics.
- The BBGKY method works only for perturbation of free dynamics and thus is only useful for the Grad limit for which $\rho \sim \varepsilon$.

Instead of approaching it via elliptic estimates or L^p theory, a useful way to establish (2.5) is to consider the ergodic property of the Hamiltonian systems. The key observation, due to Morrey [14], is that (2.5) holds if we replace $f_{N,t}$ by any Gibbs measure (with Hamiltonian H (2.1)), or more generally, if "locally" $f_{N,t}$ is a Gibbs measure of the Hamiltonian H. If we can prove that "locally" $f_{N,t}$ is a equilibrium measure with finite specific entropy, we have proved (2.5) provided that we assume the Boltzmann Hypothesis. So the proof of Theorem 2.1 consists of two main ingredients: 1. Prove that the weak Boltzmann hypothesis implies the Boltzmann hypothesis. 2. Clarify the precise meaning of the word "locally" and eliminate the possibility of meso-scale fluctuation. The method we used for 2 is the relative entropy method.

Recall that for any two probability densities the relative entropy is defined by

$$S(f|g) = \int f \log(f/g) d\omega$$

Suppose f_t is a solution of the Liouville equation and ψ_t is any density. Then

$$\partial_t S(f_t|\psi_t) = \int f_t \, \{ \, \psi_t^{-1} \, [\mathcal{L}^* - \partial_t] \, \psi_t \, \} \, d\omega \tag{2.6}$$

This identity can be checked easily from the Liouville equation. It also has a version for Markov processes:

$$\partial_t S(f_t|\psi_t) = -D(f_t|\psi_t) + \int f_t \, \{ \, \psi_t^{-1} \, [\mathcal{L}^* - \partial_t] \, \psi_t \, \} \, d\omega \tag{2.7}$$

where $D(f|\psi)$ is the Dirichlet form of f w.r.t. ψ and is nonnegative [21, 15]. Now recall the entropy inequality (or the Jensen inequality) which states that for any function W,

$$\int fW d\omega \leq S(f|\psi) + \log \int \psi \exp(W) \ d\omega$$

Thus from (2.6),

$$\partial_t S(f_t|\psi_t) \leq S(f_t|\psi_t) + \log \int \psi_t \exp\left\{ \ \psi_t^{-1} \left[\mathcal{L}^* - \partial_t\right] \psi_t \ \right\} d\omega$$

If we have

$$N^{-1} \log \int \psi_t \exp\left\{ \ \psi_t^{-1} \left[\mathcal{L}^* - \partial_t\right] \psi_t \ \right\} d\omega \to 0 \tag{2.8}$$

then the relative entropy can be controlled on the relevant time scale and this will imply the estimate (2.5) and thus the Euler equations. Note that the left hand side of (2.8) is independent of f_t so the remaining argument in [15] can be summarized as showing that (2.8) holds iff ψ_t is a local Gibbs state with density, velocity and energy chosen according to the Euler equations (Note: As it is, (2.8) is incorrect; some arguments using ergodicity of the Hamiltonian dynamics are also needed). This is essentially a dynamical variational approach because we solve the problem by guessing a good trial function which in this case is the local Gibbs state.

III THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

The Navier-Stokes equations are the next order corrections to the Euler equations. In order to derive them one needs to show that

$$\text{MICRO CURRENT} \to \text{MACRO CURRENT} + \varepsilon\nu\nabla\hat{v}_{\varepsilon,\omega} + o(\varepsilon) \tag{3.1}$$

where the currents are given by (2.3) and (2.4) and ν is the viscosity. Since there is an ε appearing in the viscosity term, (3.1) is in a sense *the next order correction to the Boltzmann hypothesis*! From the expression for the micro current in (2.3), it is hard to even imagine how the viscosity correction arises. This difficulty was recognized decades ago by Dobrushin, Lebowitz, and Spohn, among others. Recent work [20, 7, 8, 13] has given us a good understanding of the nature of (3.1), though a rigorous proof from the Hamiltonian dynamics is still very far off.

The equation for the leading order terms of (3.1) is (2.4) and it holds w.r.t. Gibbs measures in the sense of law of large numbers. The difficulty to justify (2.4) rigorously for Hamiltonian dynamics (i.e. (2.5)) is to prove that the solutions to the Liouville equation are locally stationary and all stationary measures are Gibbs. On the other hand, one can check easily that (3.1) is *incorrect* w.r.t any Gibbs measures with Hamiltonian H. Indeed, (3.1) is a "dynamical identity". It can be interpreted physically via the linear response theory or the Green-Kubo formula (see [17] for an account). A more mathematical interpretation is through the fluctuation-dissipation equation which we now explain.

Roughly speaking, the fluctuation-dissipation equation states that

$$\text{MICRO CURRENT} \to \text{MACRO CURRENT} + \varepsilon\nu\nabla\hat{v}_{\varepsilon,\omega} + \varepsilon\mathcal{L}g + o(\varepsilon) \tag{3.2}$$

for some function $g(\omega)$, where $\mathcal{L}$ is the Liouville operator. In other words, (3.1) is correct only up to a quotient of the image of the Liouville operator. The image of the Liouville operator is understood as fluctuation, negligible in the relevant scale after *time average*: for any bounded function g

$$\varepsilon \int_0^t ds \int f_{s,N}(\omega)\,(\varepsilon \mathcal{L} g)(\omega)\, d\omega = \varepsilon^2 \int [f_{t,N} - f_{0,N}](\omega)\, g(\omega)\, d\omega \sim \varepsilon^2$$

and is thus negligible to the first order in ε, the relevant scale.

It is difficult to work on "next order correction" and thus we turn to the incompressible Navier-Stokes (INS) equations

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p + \nabla \nu \nabla u, \qquad \nabla \cdot u = 0. \tag{3.3}$$

The INS equations are invariant under the *incompressible scaling*,

$$x \to \varepsilon x, \quad t \to \varepsilon^2 t, \quad u \to \varepsilon^{-1} u, \quad p \to \varepsilon^2 p, \tag{3.4}$$

under which (3.2) becomes

$$\text{MICRO CURRENT} \to \text{MACRO CURRENT} + \nu \nabla \hat{v}_{\varepsilon,\omega} + \mathcal{L} g \tag{3.5}$$

Notice that both the viscosity and the function g are unknown and (3.5) determines both. We interpret (3.5) as a decomposition of the space of microscopic currents into a direct sum of the space of macroscopic currents, the gradient of the velocity representing the dissipation and the image of the Liouville operator representing the fluctuation.

Equation (3.5) is extremely difficult to solve as it requires inversion of the Liouville operator. A class of more manageable *stochastic lattice gas* models were introduced in a joint work with R. Esposito and R. Marra [8]. Even for these, (3.5) requires the inversion of a nonsymmetric operator in infinite dimensions with a complex interaction. If the generator $\mathcal{L}$ is symmetric, i.e., the dynamics is reversible, (3.5) can be solved by formulating the problem in an appropriate space so that it reduces to inverting a self-adjoint operator. This formulation, due to S. Varadhan [20], is already quite sophisticated since the terms appearing in (3.5) do not live in a natural space. On the other hand, in order to obtain the INS equations, the dynamics has to retain essential features of the Hamiltonian dynamics; this forces us into nonzero drifts and therefore nonreversibility. The invertibility in the nonsymmetric case is very subtle [13]. Dimension comes into play, and we believe that (3.5) has no solution at all for dimension $d \leq 2$.

In the models of [8] particles have velocities in a chosen finite set and at each site of the lattice at most one particle of each velocity is allowed. The dynamics consists of two parts: Random walks and binary collisions between particles. The random walk part of the dynamics requires only that particles with velocity v should have the mean drift v. The binary collisions conserve momentum. Note that conservation of energy is not important here because the INS equations are equations of velocity alone. The combined dynamics should have good ergodic properties and also restore rotational symmetry in the limit. The restoration of the rotational symmetry is not trivial because the lattice structure breaks the symmetry. Sets of velocities and dynamics satisfying all the requirements can be found in [8].

The main result in [8] states that (3.5) has a solution (in a suitable sense) for $d \geq 3$ and if the INS equations have a strong solution up to a fixed time T then the rescaled empirical velocity densities (measures)

$$v_{\varepsilon,\omega}(X) := \varepsilon^{d-1} \sum_x \delta(X - \varepsilon x) \sum_v v\eta(x,v) \tag{3.6}$$

converges to that solution. Here $\eta(x,v) \in \{0,1\}$ is the number of particles of velocity v at site x. Notice the blowup of the velocity by ε^{-1} in accordance with the scaling (3.4).

The assumption that the INS equations have a strong solutions has a long history in their derivation from more basic models. Derivations of the INS equations from the Boltzmann equation go back already a century to Chapman, Enskog and Hilbert, and were made rigorous in the seventies [4,5]. However the removal of the smoothness assumption has not been so easy. A program [3] of deriving the weak (Leray) solutions from the DiPerna-Lions solutions of the Boltzmann equation remains far from complete, due to a lack of good estimates. Though it was believed that the analysis of particle systems would be even more difficult because they are essentially infinite dimensional, in a joint work with J. Quastel [16] we have been able to remove this obstacle.

THEOREM 3.1. *Let P_ε be the distributions of the empirical momentum densities (3.6). Then P_ε are precompact (as a set of probability measures with respect to a suitable topology) and any weak limit is supported entirely on weak solutions of the INS equations satisfying the energy inequality.*

Theorem 3.1 is proven only for $d = 3$. The restriction $d \leq 3$ is for technical reasons; *the restriction $d \geq 3$, however, is intrinsic.* Since the macroscopic velocity is defined through the law of large numbers in statistical physics, it inherits a small fluctuation from the central limit theorem, which is of order $\varepsilon^{d/2}$. When we blow up the velocity by ε^{-1} in the incompressible limit (3.6), this term becomes of order one or larger for dimensions $d \leq 2$ and thus the macroscopic velocity is not well defined in this limit. Note that this argument applies to any dynamics including the Hamiltonian dynamics.

Though (3.5) determines the viscosity, it is important to have an independent characterization, traditionally expressed as a time integral of current-current correlation functions, which up to constants is given by:

$$\nu = \int_0^\infty \big\langle \text{ MICRO CURRENT } (t=0); \text{ MICRO CURRENT } (t=s) \big\rangle \, ds \tag{3.7}$$

where $\langle f;g\rangle = \langle fg\rangle - \langle f\rangle\langle g\rangle$ is the correlation function and the expectation is w.r.t. lattice gas dynamics starting from *equilibrium.* This is called the Green-Kubo formula and is proved rigorously in [13, 8] for $d \geq 3$. For dimension $d \leq 2$, the Green-Kubo formula (3.7) diverges, (3.5) has no solution, and the time scaling is faster than diffusive. We are thus forced to conclude that the two dimensional INS equations cannot be obtained as the incompressible limit of *any* microscopic dynamics.

A large deviation principle was also given in [16]. One main step in [16] is a proof of the energy estimate for the INS equations directly from the lattice gas dynamics by implimenting a renormalizatin group analysis. The techincally most demanding points, the large field problems in the standard field theory and the large fluctuation here, are controlled by the entropy method [10] and and the logarithmic Sobolev inequality [22]. The entropy method is an infinite dimensional version of the energy method in PDE; the logarithmic Sobolev inequality plays the role of the usual Sobolev inequalities.

IV QUANTUM DYNAMICS

Most problems concerning classical or stochastic dynamics have corresponding quantum versions. They are however often too difficult to study. The classical or stochastic dynamics are governed by the evolution of a probability density; the quantum dynamics by a complex wave function. Although both dynamics are linear, the physics in the quantum case is given by the square of the wave function, breaking the superposition law. Furthermore, the evolution of a wave function is determined by its phase which is very hard to control. We mention here a result on the quantum Lorentz gases [6] to give some flavor of quantum dynamics.

Classical Lorentz gases model a classical particle in an environment of fixed scatterers distributed randomly (or periodically). The question is the time evolution of this particle for a typical configuration of the scatterers. Denote by $\omega = (x_\alpha), \alpha = 1, \cdots, N$, the configuration of scatterers in a cube of width L. We are interested in the Grad limit (1.1) with $\varrho = N/L^d$ denoting the density of the scatterers. The typical number of collisions is now of the order $t\rho \sim 1$. It was proved in [9,19, 1] that its time evolution converges to a linear Boltzmann equation

$$\partial_T F_T(X,V) + V \cdot \nabla_X F_T(X,V) = \int \sigma(U,V) \left[F_T(X,U) - F_T(X,V)\right] dU, \quad (4.1)$$

where F is the phase space density and $\sigma(U,V)$ is the scattering cross section.

The quantum Lorentz gases can be obtained by simply replacing the classical dynamics by the quantum dynamics. More precisely, let $V_0(x)$ be a fixed "nice" function. The Schrödinger equation governing the quantum particle is given by

$$i\partial_t \psi_t = H_{N,L}\psi_t, \quad \psi_{t=0} = \psi_0, \quad (4.2)$$

where the Hamiltonian is given by

$$H_{N,L} = H := -\Delta/2 + V_\omega, \qquad V_\omega(x) = \sum_{\alpha=1}^{N} V_0(x - x_\alpha). \quad (4.3)$$

The classical phase space density of a wave function ψ is defined through the Wigner transform:

$$W_\psi(x,v) := \int \overline{\psi(x+z/2)}\psi(x-z/2)e^{ivz}dz.$$

The scaling is the same as in the classical case,

$$W_\psi^\varepsilon(X,V) := W_\psi(X/\varepsilon, V). \tag{4.4}$$

Notice that the velocity is not rescaled. The Wigner transform typically has no definite sign, and the associated Husimi function or coherent states are needed to define a positive density, but we will not go into these details here.

Let $\psi_{\omega,t}^\varepsilon$ be the solution to the Schrödinger equation (4.2), (4.3) with initial data ψ_0^ε. Suppose that the initial data is of the following form

$$\psi_0^\varepsilon(x) = \varepsilon^{3/2} h(\varepsilon x) e^{iu_0 x},$$

for some smooth functions h so that as $\varepsilon \to 0$ the rescaled Wigner transform $W_{\psi_0^\varepsilon}^\varepsilon(X,V)dXdV$ converges weakly to $|h(X)|^2\delta(V-u_0)dXdV =: F_0(X,V)dXdV$ as probability measures on R^{2d}. Then in dimension $d=3$ and for V_0 small enough (so that there is no bound state) our main result with L. Erdős [6] is that *for any $T>0$,*

$$EW_{\psi_{\omega,T/\varepsilon}^\varepsilon}^\varepsilon W(X,V)dXdV \to F_T(X,V)dXdV$$

weakly as $\varepsilon \to 0$ and $F_T(X,V)$ satisfies the linear Boltzmann equation (3.6) with initial data $F_0(X,V)$ and effective collision kernel σ given by the quantum scattering operator of the potential V_0.

A simple example illustrates the difference between the classical and the quantum dynamics. Suppose that the particle in a Lorentz gas has one collision. Classically we simply choose a scatterer and the particle collides with it. In quantum mechanics, we have from the Duhamel formula

$$\psi_t = e^{-itH}\psi_0 = e^{-itH_0}\psi_0 - i\int_0^t e^{-i(t-s)H_0} V e^{-isH_0}\psi_0\, ds + \cdots$$

where V_ω is the potential given in (4.3) and $H_0 = -\Delta/2$. The one collision term is the second term on the right hand side which, for simplicity, we write as $\sum_{\alpha=1}^N \psi_{t,\alpha}$. Notice that instead of collision with a scatterer in classical dynamics, it is now a sum of collisions with all scatterers! Since we have to square the wave function to get physical quantities, we need to show that the overlaps (or interference) of off-diagonal terms

$$\langle\, \psi_{t,\alpha},\, \psi_{t,\beta}\, \rangle$$

are very small. Stationary phase methods show they are small, but the number of the off-diagonal terms is much larger than that of diagonal terms. The analysis of this competition is very simple in this first term but very complicated in higher order terms. It nevertheless can be carried out rigorously to all orders [18, 11]. However such results are restricted to the weak coupling limit (a semiclassical limit) and short time. Instead we renormalize the perturbation theory so that we can consider the Grad limit to obtain the quantum scattering kernel. Furthermore, we truncate the Duhammel formula and estimate the error terms to remove the short-time restriction and thus we obtain results global in time [6].

Acknowledge: It is a great pleasure to thank J. Quastel for critical readings and commments of the manuscript.

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Horng-Tzer Yau
Courant Institute of
mathematical sciences
New York University
New York, NY, 10012, USA.
yau@math.nyu.edu