Extended Dynamical Systems

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Abstract. We discuss the dynamics of dissipative systems defined in unbounded space domains, and in particular results on the existence of the semi-flow of evolution, on the development structures due to instabilities, and on large time behaviors.

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I. Introduction.

Extended dynamical systems deal with the time evolutions of systems where the spatial extension is important. One of the remarkable achievement of the theory of dynamical systems was the proof that if one considers a system in a bounded domain (for example a two dimensional incompressible fluid), then the global attracting set is a compact set with finite dimensional Hausdorff dimension although the phase space of the system is infinite dimensional (see [R.], [T.] etc.). It turns out however that the dimension of the attractor grows with the spatial extension of the system, and attractors of large dimension are not very easy to analyze at the present time. Also the theory of dynamical systems does not provide easily information about the spatial structure of the solutions.

This is not so important for small spatial extension where this spatial structure is rather simple. However systems with large spatial extension develop interesting spatial structures. One of the most common of these structures are the waves on a sea excited by a gentle wind (see [M.] for a discussion of the present status of knowledge of this major phenomenon). As a simple criteria we will say that a system is extended if the spatial size of the system is much larger than the typical size of the structures. As will be explained below, in many interesting situations the scale of the structures is well defined. For example, even during the wildest storms, the wavelength of the waves is much smaller than the size of the sea (even of some large lakes).

Note that for spatially extended systems defined in large but bounded spatial domains, all the large time information about spatial structures should be available from the study of the attractor(s), invariant measures etc. The problem is that it is not easy at all to extract spatial information out of the non linear structure of the dynamics in phase space. This is why Physicists have been looking for a long time for a direct approach which emphasizes the search for structures and their evolution.

There are other difficulties with large systems which are of more technical nature. When one tries to use bifurcation theory for example, one is lead to study the spectrum of the linearized time evolution around a stationary solution. When the spatial extension of the system becomes large, the spectrum although discrete (in a bounded domain) becomes very dense. In general this implies that bifurcation theory gives results only on a very small range of parameters. This has also unpleasant experimental consequences since a small variation of the parameter near criticality can result in a large number of eigenvalues becoming unstable.

Physicists have dealt with these difficulties since a long time. By analogy with Statistical Mechanics, one may hope that if a system has a large spatial extension, its behavior may be well approximated by the behavior of a system with infinite extension (the so called thermodynamic limit). Of course one may expect corrections from far away boundaries.

This assumption has an important technical consequence. When studying the spectrum of the linearized evolution in bounded domains one should use some adequate basis of functions (Fourier series etc.). In unbounded domains, at least for operators with constant coefficients, the spectra is easily obtained using Fourier transform which is a much more convenient tool. A lot of important results have been obtained this way by Physicists (see for example [Ch.]). In fact one could remark that whenever Fourier transform is used in a Physical problem to deal with spatial dependence, an assumption of infinite spatial extension has been made.

Note that contrary to the case of spatially finite systems, extended systems lead in general to continuous spectrum for the linearized evolution. Another difficulty, almost never mentioned in the Physics literature is the nature of phase space of extended systems. Since spectral theory will be an important tool, one would imagine working in a phase space which is for example a Sobolev space. Several interesting works have been done in that direction. However this is not the phase space one would like to use. For example, such a space does not contain waves. Therefore more natural and interesting phase spaces should be like L^∞.

The rest of this paper is organized as follows. In section 2 we will present some results on the global existence of the time evolution in extended domains. In section 3 we will discuss the instabilities of homogeneous solutions and see how they can lead to the appearance of structures at well defined scales. Finally in section 4 we will present some results dealing with questions of large time asymptotic.

II. Global Existence of the Semi-Flow.

The first mathematical question with extended systems is the problem of global existence of the semi flow of time evolution. As mentioned in the introduction, one of the difficulty is that we want to deal with a rather large phase space containing in particular wave-like solutions which do not tend to zero at infinity. Several results have been obtained in the case of Sobolev phase spaces, however the methods do not seem to apply to the phase spaces which are required for the general Physical applications. Very few results are available for only bounded initial conditions, and we will briefly describe some of these results. As in the case of dynamical systems, it is convenient to work with some simplified models which exhibit the essential phenomena without the complexity of the real equations. One such model is the so called Swift-Hohenberg equation. This equation gives the time evolution of a real field $u(t,x)$ and is given in one space dimension by

$$\partial_t u = \eta u - (1+\partial_x^2)^2 u - u^3 \ , \tag{SH}$$

where η is a real parameter. This equation was derived by Swift and Hohenberg as a model for the onset of convection [S.H.].

Another popular model whose importance will become clearer below is the so called complex Ginzburg-Landau equation. This equation describes the time evolution of a complex field $A(t,x)$ and is given by

$$\partial_t A = (1+i\alpha)\Delta A + A - (1+i\beta)A|A|^2 \ , \tag{CGL}$$

where α and β are two real parameters.

The basic problem is to prove that these equations have (nice) solutions for all time if we start with an initial condition which is only bounded (and somewhat regular). The case of the Ginzburg-Landau equation was treated first in [C.E.1] in dimension one and generalized in [C.1] and [G.V.]. Regularity of the solution was obtained in [C.3] and [T.B.]. We summarize the results in the following theorem.

Theorem II.1 ([C.1],[C.3]). *In dimension one and two, for any complex valued function A_0 of the space variable x, bounded and uniformly continuous, there is a unique solution A of the (CGL) equation with initial condition A_0. This function A is for all times bounded and uniformly continuous. Moreover, there is a positive constant $T = T(A_0,\alpha,\beta)$, and two positive constants $C = C(\alpha,\beta)$ and $h = h(\alpha,\beta)$ such that for any $t > T$, the function $A(t,\cdot)$ extends to a function analytic in the strip $|\Im z| \le h$ and satisfying*

$$\sup_{|\Im z|\le h} |A(t,z)| \le C \ .$$

In other words, the dynamics contracts the large fields to a universal invariant ball, and moreover regularity develops. In the case of dimension three and higher, estimates are presently only available for a restricted range of parameters, we refer to the original publications for more details. A similar result holds for the Swift-Hohenberg equation. The case of reaction-diffusion equations may prove more difficult to deal with, see [C.X.].

As mentioned above, in order to prove such a result one has to use techniques which are rather different from the case of bounded domains. Theorem II.1 has been proven using a local energy estimate. We only indicate the basic starting point. Note that the local (in time) existence and boundedness of the solution follows easily from the usual techniques using the contraction mapping principle. The main goal is therefore to obtain some global a-priori estimate.

Let φ be a regular function tending to zero sufficiently fast at infinity. For example

$$\varphi(x) = \frac{1}{(1+|x|^2)^d}$$

where d is the dimension. The idea is to probe the size of a bounded function by looking at the L^2 norm of this function multiplied by some translate of φ. This is reminiscent of the so called amalgam spaces.

The basic quantity to estimate is the number

$$I(t) = \sup_{x_0} I(t, x_0)$$

where

$$I(t, x_0) = \int |A(t,x)|^2 \varphi(x - x_0)\, dx \ .$$

After some simple algebra and integration by parts, one gets easily

$$\frac{d}{dt} I(t, x_0) \le K - \int |A(t,x)|^2 \varphi(x - x_0)\, dx = K - I(t, x_0) \ .$$

where K is some constant which depends only on φ and the coefficients α and β. This differential inequality tells us that after some time the quantity $I(t, x_0)$ will settle forever below $2K$. Note also that the time it takes to reach this situation can be bounded above by a quantity which depends only on $\|A_0\|_{L^\infty}$ (and of course on the coefficients α and β of the equation). The rest of the proof is based on similar but more involved estimates. We refer the reader to the original papers for more details.

III. Instabilities and Structures.

As mentioned above, instability in extended systems leads very often to the development of structures with a well defined wave length. We will illustrate this phenomenon on the one dimensional Swift-Hohenberg equation (S.H.). We first observe that for any value of the parameter η, the homogeneous function $u(t, x) = 0$ is a stationary solution. If we linearize the evolution around this solution, we get the equation

$$\partial_t v = \eta v - (1 + \partial_x^2)^2 v \ ,$$

which describes the linear time evolution of small perturbations of the homogeneous solution. This equation can be explicitly solved by taking the Fourier transform in x. One gets

$$\partial_t \hat{v}(t, k) = \omega(\eta, k) \hat{v}(t, k) \qquad \text{(III.1)}$$

where

$$\omega(\eta, k) = \eta - (1 - k^2)^2 \qquad \text{(III.2)}\ .$$

It is then easy to see that if $\eta < 0$, the solution tends to zero ($\omega < 0$), whereas if $\eta > 0$ some Fourier modes are exponentially amplified ($\omega > 0$ for k near ± 1). As mentioned above, one should be careful with the interpretation of this result in direct space since we want to work in a phase space of functions which do not decay at infinity and whose Fourier transforms are in general distributions. Nevertheless this trivial analysis will give the right intuition. It is indeed possible to prove that for the complete non-linear (S.H.) equation and $\eta < 0$, bounded initial conditions relax to zero.

The case $\eta > 0$ is of course more interesting and reminiscent of bifurcation theory. Recall that the main idea of bifurcation theory is that in phase space, the dominant part of the bifurcated branch is along the subspace of the linear problem which becomes unstable. The amplitude in that direction(s) varying slowly. For $\eta > 0$ small, we have in the (SH) equation two bands of modes of width $\mathcal{O}(\eta^{1/2})$ around ± 1 which are unstable. All other modes are linearly damped. This follows easily from (III.1) and (III.2). By analogy with standard bifurcation theory, one may expect that the coefficient in the unstable directions will vary slowly in space and time. This is indeed what can be proven.

THEOREM III.1. *There are positive numbers R, η_0, $C_1, \cdots, C_4$ such that if $\eta \in]0, \eta_0[$, if $u_0(x)$ is a real bounded uniformly continuous function such that $\|u_0\|_{L^\infty} < R$, there is a positive number $T_1 = T_1(u_0, \eta)$ such that for any $t > T_1$, the solution u of (S.H.) with initial condition u_0 satisfies*

$$\|u(t,\,\cdot\,)\|_{L^\infty} < C_1 \eta^{1/2}\ .$$

Moreover, for any $t > T_1$, there is a solution $B(s,y)$ of the real Ginzburg-Landau equation

$$\partial_s B = \partial_y^2 B + B - B|B|^2 \qquad \text{(G.L.)}$$

such that for any $t \le \tau \le t + C_2 \eta^{-1} \log \eta^{-1}$ we have

$$\|u(\tau,\,\cdot\,) - \mathcal{B}(\tau,\,\cdot\,)\|_{L^\infty} < C_3 \eta^{1/2+C_4}\ ,$$

where

$$\mathcal{B}(\tau, x) = 3^{-1/2} \eta^{1/2} e^{ix} B(\eta(\tau - t), \eta^{1/2} x/2) + c.c.$$

This result is similar to what can be obtained in bifurcation theory using normal forms. It says that the function reconstructed from the normal form (here the (G.L.) equation) reproduces well the true evolution during a large time. However, as for normal forms, we cannot expect this to be true forever since small errors due to truncation of the normal form are likely to be amplified by the unstable dynamics.

The idea of amplitude equation is rather old, and we refer to [C.H.] for references. Several versions of the above result (or similar ones) have been published in [C.E.1], [v.H.], [K.M.S.], [S.]. The original idea of shadowing of trajectories is due to Eckhaus [E.]. A new proof using a dynamical renormalization group was given in [C.2] for the case of discrete evolution equations. The renormalization group method has several advantages. First of all it provides a systematic and rigorous approach to multi-scale analysis. It also provides a proof of the above theorem in one step. In the above formulation, it was convenient to separate the initial contraction regime from the subsequent shadowing result. It turns out in the proof that they are different manifestations of the same renormalization effect.

Also one gets some information on the initial contraction phase. The fact that the size of the initial condition R can be chosen independent of η is important and in a sense is optimal since one cannot expect the result to hold for initial conditions of size much larger than unity without further hypothesis since the dynamics may well have another fixed point of order one (although for the particular case of the (S.H.) equation one can prove global attraction). Last but not least, renormalization group produces universal results. This is a nice substitute for the generic arguments of finite dimensional bifurcation theory. This explains why the Ginzburg-Landau equation appears so often in the study of instabilities of extended systems. It turns out that the associated fixed point of the renormalization group is the equation

$$\partial_s B = \partial_y^2 B - B|B|^2 ,$$

which is invariant by the rescaling $s \to L^2 s$, $y \to Ly$, $B \to LB$. The relevant unstable manifold parameterized by a real number σ is the (unnormalized) G.L. equation

$$\partial_s B = \partial_x^2 B + \sigma B - B|B|^2 .$$

We refer to [B.K.1] and [C.2] for more details and references on the renormalization group ideas and to [A.] page 212 for a general program.

IV. Large Time Behavior.

For dissipative systems in bounded domains, various notions of attractors have been introduced. One tries to describe in the phase space an invariant set which captures all the asymptotic dynamics. Various results have been proven about the compactness and finite Hausdorff dimension of such objects. For extended systems we cannot hope for such results and the definition of the global attracting set has to be modified due to the lack of compactness. Mielke and Schneider [M.S.] following an idea of Feireisl have proposed to define a global attracting set using two different topologies. One is a global topology (of the type L^∞), the other one is a local topology where one recovers compactness. We give below a variant of their result for the (CGL) equation (for other equations see [M.S.]).

Theorem IV.1. *For the (CGL) equation in dimension 1 and 2, there is a set $\mathcal{A}$ of functions analytic in a strip of width h around the real space and satisfying*

$$\sup_{|\Im z|<h} |A| < C,$$

where h and C are as in Theorem II.1 and such that

1) *$\mathcal{A}$ is closed in L^∞,*
2) *$\mathcal{A}$ is invariant by space translations,*
3) *$\mathcal{A}$ is invariant by the semi flow (S_t) of evolution of the (CGL) equation (namely $S_t(\mathcal{A}) = \mathcal{A}$ for any $t > 0$),*
4) *$\mathcal{A}$ is compact in $L^\infty(Q)$ for any cube Q,*
5) *$\mathcal{A}$ attracts any bounded set of L^∞, namely if B is a bounded set in L^∞, the L^∞ distance between $S_t(B)$ and $\mathcal{A}$ tends to zero when t tends to infinity.*

We refer to [M.S.] for the proof. We mention however that although 4) is trivial from the analyticity of the functions in $\mathcal{A}$, this compactness property is crucial in the proof of 3) and 5) together with the fact that the L^∞ norm of a function on the whole line is obtained by taking the sup of the L^∞ norms of the function on the translates of a fixed cube.

Once the global attractor of a dynamical system has been identified, one can try to give some geometrical description of this object. As mentioned before, for systems in bounded domains one tries to prove that the attractor has finite Hausdorff dimension. A natural question for extended domains is whether there is a good notion of dimension per unit volume of space. In this direction, Ghidaglia and Heron [G.H.] have given for the (CGL) equation in finite domain an upper bound on the Hausdorff dimension of the attractor which is proportional to the length of the domain in space dimension one and proportional to the surface in space dimension two. However contrary to the case of statistical mechanics, it is not clear at this moment whether a sub-additive result holds for the dimension. The main difficulty is to connect the dimension of attractors for the union of two domains.

We have recently considered this question with J.-P. Eckmann from another point of view related to signal analysis. For simplicity I will only discuss one dimensional systems although the results are true in any dimension. We start directly with the system in an unbounded domain, but we observe it in a finite window, for example the interval $[-L, L]$. This is quite natural in view of the above definition of attractor. Note however that since the functions on the attractor $\mathcal{A}$ are analytic they will be seen in any interval. This implies that the dimension of $\mathcal{A}$ in $L^\infty([-L, L])$ is infinite. Kolmogorov and Tikhomirov have studied a similar situation for some spaces of analytic functions [K.T.]. They have defined following Shannon the ϵ-entropy per unit length H_ϵ as follows. Let $\mathcal{B}$ be a subset of $L^\infty(\mathrm{R})$. For a fixed $\epsilon > 0$ one defines $N_L(\epsilon)$ as the smallest number of balls of radius at most ϵ (in $L^\infty([-L, L])$) needed to cover $\mathcal{B}$. The ϵ entropy per unit length of $\mathcal{B}$ is defined by

$$H_\epsilon(\mathcal{B}) = \lim_{L\to\infty} \frac{\log_2 N_L(\epsilon)}{L} ,$$

provided the limit exists. One is then interested at the behavior of this quantity when ϵ tends to zero. Note the exchange of limits with respect to the usual definition of dimension. For the attractor $\mathcal{A}$, if one fixes L and let ϵ tends to zero, one gets an infinite dimension. In other words, for a fixed precision ϵ, if the size of the window is too small, one gets the impression of an object of infinite dimension. As the result below indicates, there is however a cross-over length which depends on the precision ϵ beyond which one sees a finite dimension per unit length (at this fixed precision). Kolmogorov and Tikhomirov in [K.T.] proved the following estimates.

For the set $\mathcal{E}_\sigma(C)$ of entire functions satisfying

$$|f(z)| \le Ce^{\sigma|\Im z|}$$

one has

$$H_\epsilon(\mathcal{E}_\sigma(C)) \approx \frac{2\sigma}{\pi} \log_2(1/\epsilon) ,$$

where $\approx$ means that the ratio of the two quantities tend to 1 when ϵ tends to zero.

For the set $\mathcal{S}_\sigma(C)$ of functions analytic in a strip of width h around the real axis and satisfying

$$\sup_{|\Im z|\le h} |f(z)| \le C$$

one has

$$H_\epsilon(\mathcal{S}_h(C)) \approx \frac{1}{\pi h}(\log_2(1/\epsilon))^2 \ .$$

We refer to [K.T.] for the proof of these two statements.

From the previous result on the analyticity of the functions in $\mathcal{A}$ one would expect a growth of the ϵ-entropy proportional to $(\log\epsilon)^2$. It turns out that there is in a sense far less functions in $\mathcal{A}$, and in the sense of ϵ-entropy we have indeed a finite dimension per unit length.

Theorem [C.E.4]. *There is a number $c = c(\alpha,\beta) > 1$ such that for the (CGL) in dimension 1 and 2 we have*

$$c^{-1}\log_2(1/\epsilon) \le H_\epsilon(\mathcal{A}) \le c\log_2(1/\epsilon) \ .$$

Note that some functions belonging to $\mathcal{A}$ are known which are not entire. We have also obtained recently with J.-P. Eckmann a proof of existence of the topological entropy per unit volume. Moreover this quantity can also be obtained from a discrete sampling of the solutions (see [C.E.5]).

V. Conclusions.

Extended systems occur naturally in many natural questions. They appear in Physics, Chemistry, Biology, Ecology and other sciences as soon as the spatial extension of the system becomes important. We refer to [C.H.] [B.N.] and [Mu.] for some examples.

From the mathematical point of view there are many open problems. The understanding of the evolution of structures and the occurrence of spatio temporal chaos are the most challenging. There are very few results in these area where even numerical simulations are difficult to perform. As in the case of finite dimensional dynamical systems, there are two main trends of research up to now.

In the first one, one tries to understand the spatial structure of the solutions. This is quite natural near the onset of instability of the homogeneous state, where the structures play a dominant role. We refer to [B.N.] for a review of this approach. Even near onset there are important questions which are not understood. For example in dimension 2 or larger, the amplitude should be a distribution on the unit circle and up to now a global derivation of an amplitude equation has not been performed. The analysis has only been achieved under various symmetry assumptions which strongly restrict the solutions, although these solutions with symmetries are the ones which appear in experiments. In a different perspective, particular solutions with interesting physical meaning have been constructed (see for example [C.E.3] for more details).

The second trend of research is more of statistical nature and is concerned with asymptotic time evolution. The existence of interesting invariant measures is still an open problem. Beyond numerical simulations some analogies have been drawn in the spatio temporal intermittency transition with directed percolation (see [B.P.V.] for a review). Even in the case where there is no spatio temporal chaos, the asymptotic state may be non trivial as in the phase ordering kinetics problem (see [B.] where consequences of scaling hypothesis are developed). We mention however that in the problem of coupled lattice maps, interesting invariant measures have been constructed ([B.K.2] and references therein).

Finally I refer to the conclusion of Bowman and Newell in their RPM Colloquia [B.N.] for a statement on the future of this field.

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