Finite-Size Scaling in Percolation

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Abstract. This work is a detailed study of the phase transition in percolation, in particular of the question of finite-size scaling: Namely, how does the critical transition behavior emerge from the behavior of large, finite systems? Our results rigorously locate the proper window in which to do critical computation and establish features of the phase transition. This work is a finite-dimensional analogue of classic work on the critical regime of the random graph model of Erdös and Rényi.

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1. Introduction

This paper gives an overview and discussion of some recent results of Borgs, Chayes, Kesten and Spencer [BCKS2] on finite-size scaling and incipient infinite clusters in percolation.

We consider bond percolation in a finite subset Λ of the hypercubic lattice $\mathbb{Z}^d$. Nearest-neighbor bonds in Λ are occupied with probability p and vacant with probability $1-p$, independently of each other. Let p_c denote the bond percolation threshold in $\mathbb{Z}^d$, namely the value of p above which there exists an infinite connected cluster of occupied bonds. As a function of the size of the box Λ, we determine the scaling window about p_c in which the system behaves critically. For our purposes, criticality is characterized by the behavior of the distribution of sizes of the largest clusters in the box. We show how these clusters can be identified with the so-called incipient infinite cluster—the cluster of infinite expected size which appears at p_c. It turns out that these results can be established axiomatically from hypotheses which are mathematical expressions of the purported scaling behavior in critical percolation. Moreover, these hypotheses can be explicitly verified in two dimensions. In this brief overview, I will omit all details of the proofs of the [BCKS2] results, focusing instead on the motivation, the hypotheses and a few of the implications of these results. The reader is referred to [BCKS1] and [BCKS2] for more details and for related results which are not included here. Some of the discussion here closely parallels that of [CPS].

2. The Motivation

The motivation for the [BCKS2] work was threefold.

The Random Graph Model

The original motivation for this work was to obtain an analogue of known results on the so-called random graph model of Erdös and Rényi ([ER1], [ER2]; see also [B2]). The random graph model is simply the percolation model on the complete graph, i.e., it is a model on a graph of N sites in which each site is connected to each other site independently with uniform probability $p(N)$. Physicists would call this a mean-field percolation model. It turns out that the model has particularly interesting behavior if $p(N)$ scales like $p(N) \approx c/N$ with $c = \Theta(1)$. Here, as usual, $f = \Theta(N^\alpha)$ means that there are nonzero, finite constants c_1 and c_2, of equal sign, such that $c_1 N^\alpha \leq f \leq c_2 N^\alpha$.

Let $W^{(i)}$ denote the random variable representing the size of the i^{th} largest cluster in the system. Erdös and Renyi showed that the model has a *phase transition* at $c = 1$ characterized by the behavior of $W^{(1)}$. It turns out that, with probability one,

$$W^{(1)} = \begin{cases} \Theta(\log N) & \text{if} \quad c < 1 \\ \Theta(N^{2/3}) & \text{if} \quad c = 1 \\ \Theta(N) & \text{if} \quad c > 1. \end{cases}$$

Moreover, for $c > 1$, $W^{(1)}/N \to \theta(c) > 0$, while for $c = 1$, $W^{(1)}$ has a nontrivial distribution (i.e., $W^{(1)}/N^{2/3} \not\to$ constant), again with probability one. The smaller clusters have the same behavior as the largest for $c \leq 1$, but different behavior for $c > 1$: For $i > 1$, $W^{(i)} = \Theta(\log N)$ for all $c \neq 1$, while at $c = 1$, $W^{(i)} = \Theta(N^{2/3})$. The $\Theta(N)$ cluster for $c > 1$ is clearly the analogue of the infinite cluster in percolation on finite-dimensional graphs; here it is called the *giant component*. As we will see, the $\Theta(\log N)$ clusters are analogues of finite clusters in ordinary percolation. The $\Theta(N^{2/3})$ clusters will turn out to be the analogues of the so-called *incipient infinite cluster* in percolation. The work on the regime $c \neq 1$ appeared already in the original papers of Erdös and Rényi ([ER1], [ER2]); the correct behavior for $c = 1$ was derived many years later by Bollobás [B1].

In the past decade, there has been a great deal of work and remarkable progress on the random graph model. Much of this work culminated in the combinatoric tour de force of Janson, Knuth, Luczak and Pittel [JKLP]. Using remarkably detailed calculations, it was shown that shown that the correct parameterization of the critical regime is

$$p(N) = \frac{1}{N} + \frac{\lambda_N}{N^{4/3}},$$

in the sense that if $\lim_{N\to\infty}|\lambda_N| < \infty$, then $W^{(i)} = \Theta(N^{2/3})$ for all i, and furthermore each $W^{(i)}$ has a nontrivial distribution (which was actually calculated in [JKLP]). On the other hand, if $\lim_{N\to\infty}\lambda_N = -\infty$, then $W^{(2)}/W^{(1)} \to 1$ with probability one, whereas if $\lim_{N\to\infty}\lambda_N = +\infty$, then $W^{(2)}/W^{(1)} \to 0$ and $W^{(1)}/N^{2/3} \to +\infty$ with probability one. The largest component in the regime with $\lambda_N \to +\infty$ is called the *dominant component*. As we will see, it has an analogue in ordinary percolation.

The initial motivation for the [BCKS2] work was to find a finite-dimensional analogue of the above results. To this end, we considered d-dimensional percolation in a box of linear size n, and hence volume $N = n^d$. We asked how the size of the largest cluster in the box behaves as a function of n for $p < p_c$, $p = p_c$ and $p > p_c$. Also, we asked whether there is a window $p(n)$ about p_c such that the system has a nontrivial cluster size distribution within the window.

Finite-Size Scaling

The considerations of the previous paragraph lead us immediately to the question of *finite-size scaling* (FSS). Phase transitions cannot occur in finite volumes, since all relevant functions are polynomials and thus analytic; nonanalyticities only emerge in the infinite-volume limit. What quantities should we study to see the phase transition emerge as we go to larger and larger volumes?

Before the [BCKS2] work, this question had been addressed rigorously only in systems with first-order transitions—transitions at which the correlation length and order parameter are discontinuous ([BK], [BI]). Finite-size scaling at second-order transitions is more subtle due to the fact that the order parameter vanishes at the critical point. For example, in percolation it is believed that the infinite cluster density vanishes at p_c. However, physicists routinely talk about an incipient infinite cluster at p_c. This brings us to our third motivation.

The Incipient Infinite Cluster

At p_c, there is no infinite cluster with probability one, but the expected size of the cluster of the origin is infinite. Physicists call this finite object of infinite expected size, the *incipient infinite cluster* (IIC).

In the mid-1980's there were two attempts to construct rigorously an object that could be identified as an incipient infinite cluster. Kesten [K] proposed to look at the conditional measure in which the origin is connected to the boundary of a box centered at the origin, by a path of occupied bonds: $P_p^n(\cdot) = P_p(\cdot \mid 0 \leftrightarrow \partial[-n,n]^d)$. Here, as usual, $P_p(\cdot)$ is product measure at bond density p. Observe that, at $p = p_c$, as $n \to \infty$, $P_p^n(\cdot)$ becomes mutually singular with respect to the unconditioned measure $P_p(\cdot)$. Nevertheless, Kesten found that

$$\lim_{n\to\infty} P_{p_c}^n(\cdot) = \lim_{p \searrow p_c} P_p(\cdot \mid 0 \leftrightarrow \infty).$$

Moreover, Kesten studied properties of the infinite object so constructed and found that it has a nontrivial fractal dimension which agrees with the fractal dimension of the physicists' incipient infinite cluster.

Another proposal was made by Chayes, Chayes and Durrett [CCD]. They modified the standard measure in a different manner than Kesten, replacing the uniform p by an inhomogeneous $p(b)$ which varies with the distance of the bond b from the origin:

$$p(b) = p_c + \frac{c}{1 + \mathrm{dist}(0,b)^\zeta}.$$

The idea was to enhance the density just enough to obtain a nontrivial infinite object. [CCD] found that when $\zeta = 1/\nu$, where ν is the so-called correlation length exponent, the measure $P_{p(x)}$ has some properties reminiscent of the physicists' incipient infinite cluster.

In the work to be discussed here, [BCKS2] propose yet a third rigorous incipient cluster—namely the largest cluster in a box. This is, in fact, exactly the definition that numerical physicists use in simulations. Moreover, it will turn out to be closely related to the IICs constructed by Kesten and Chayes, Chayes and Durrett. Like the IIC of [K], the largest cluster in a box will have a fractal dimension which agrees with that of the physicists' IIC. Also, the [BCKS2] proofs rely heavily on technical estimates from the IIC construction of [K]. More interestingly, the form of the scaling window $p(n)$ for the [BCKS2] problem will turn out to be precisely the form of the enhanced density used to construct the IIC of [CCD].

3. Definitions and Preliminaries

We briefly review some standard definitions and notation for percolation on $\mathbb{Z}^d$ (see e.g., [CPS]). Let $C(x)$ denote the occupied cluster of the site $x \in \mathbb{Z}^d$, and let $|C(x)|$ denote its size. The order parameter is the infinite cluster density

$$P_\infty(p) = P_p(|C(0)| = \infty),$$

and the standard susceptibility is the expected finite cluster size

$$\chi^{\text{fin}}(p) = E_p(|C(0)|, |C(0)| \neq \infty).$$

Here, as usual, E_p denotes expectation with respect to P_p. The finite cluster point-to-point connectivity function is

$$\tau^{\text{fin}}(x, y; p) = P_p(C(x) = C(y), |C(x)| < \infty),$$

The exponential rate of decay of this connectivity defines the correlation length $\xi(p)$:

$$1/\xi(p) = -\lim_{|x|\to\infty} \frac{1}{|x|} \log \tau^{\text{fin}}(0, x; p)$$

where the limit is taken with x along a coordinate axis. Another point-to-point connectivity, which for $p > p_c$ behaves much like τ^{fin}, is

$$\tau^{\text{cov}}(x, y; p) = P_p(|C(x)| = \infty, |C(y)| = \infty) - P_\infty^2(p).$$

Notice that

$$\chi^{\text{fin}}(p) = \sum_x \tau^{\text{fin}}(0, x; p).$$

Similarly, we can define another susceptibility,

$$\chi^{\text{cov}}(p) = \sum_x \tau^{\text{cov}}(0, x; p).$$

Another connectivity function is the point-to-box connectivity function

$$\pi_n(p) = P_p(\exists x \in \partial[-n, n]^d \text{ s.t. } C(0) = C(x)).$$

We also introduce the quantity

$$s(n) = (2n)^d \, \pi_n(p_c)\,.$$

It will turn out that $s(n)$ represents the size of the largest critical clusters on scale n. Finally, the cluster size distribution is described by

$$P_{\geq s}(p) = P_p(|C(0)| \geq s).$$

We next recall the definitions of some of the standard power laws expected to characterize the scaling behavior of relevant quantities in percolation, noting that the existence of these power laws has not yet been rigorously established in low dimensions. We define $F(p) \approx |p - p_c|^\alpha$ to mean $\lim_{p \to p_c} \log F(p)/\log|p - p_c| = \alpha$, and implicitly assume that the approach is identical from above and below threshold, unless noted otherwise. Similarly, we use the notation $F(n) \approx n^\alpha$ to mean $\lim_{n\to\infty} \log G(n)/\log n = \alpha$. The power laws of relevance to us are

$$P_\infty(p) \approx |p - p_c|^\beta, \quad p > p_c\,,$$

$$\chi^{\mathrm{fin}}(p) \approx |p - p_c|^{-\gamma}\,,$$

$$\xi(p) \approx |p - p_c|^{-\nu}\,,$$

$$P_{\geq s}(p_c) \approx s^{-1/\delta}$$

and

$$\pi_n(p_c) \approx n^{-1/\rho}\,.$$

Note that the last relation implies

$$s(n) \approx n^{d_f} \qquad \text{with} \qquad d_f = d - 1/\rho\,.$$

Here we use the notation d_f to indicate that the power law of $s(n)$ characterizes the fractal dimension of the incipient infinite cluster.

For rigorous work, it is often convenient to replace the correlation length by the finite-size scaling correlation length, $L_0(p)$, introduced in [CCF]. Define the rectangle crossing probability: $R_{L,M}(p) = P_p\{\,\exists$ occupied bond crossing of $[0,L]\times[0,M]\cdots\times[0,M]$ in the 1-direction$\}$. Observing that, for $p < p_c$, $R_{L,3L}(p) \to 0$ as $L \to \infty$, we define

$$L_0(p) = L_0(p,\epsilon) = \min\{L \geq 1 \mid R_{L,3L}(p) \leq \epsilon\} \quad \text{if } p < p_c\,.$$

It can be shown [CCF] that the scaling behavior of $L_0(p,\epsilon)$ is essentially the same as that of the standard correlation length $\xi(p)$: for $0 < \epsilon < a(d)$, there exist constants $c_1 = c_1(d)$, $c_2 = c_2(d,\epsilon) < \infty$ such that

$$\frac{1}{L_0(p,\epsilon)} \leq \frac{1}{\xi(p)} \leq \frac{c_1 \log L_0(p,\epsilon) + c_2}{L_0(p,\epsilon) - 1}, \quad p < p_c\,.$$

Hereafter we will assume that $\epsilon < a(d)$; we usually suppress the ϵ-dependence in our notation. For $p > p_c$, [BCKS2] define $L_0(p, \epsilon)$ in terms of finite-cluster crossings in an annulus; the reader is referred to [BCKS2] for precise definitions and properties of the resulting length. Another important quantity in the high-density phase of percolation is the surface tension $\sigma(p)$; see [ACCFR] for the precise definition. By analogy with the definition of a finite-size scaling correlation length below threshold, [BCKS2] define a finite-size scaling inverse surface tension as

$$A_0(p) = A_0(p, \epsilon) = \min\{L^{d-1} \geq 1 \mid R_{L,3L}(p) \geq 1 - \epsilon\} \quad \text{if } p > p_c\,.$$

Again, see [BCKS2] for properties of $A_0(p)$.

4. The Scaling Axioms and the Results

The [BCKS2] results are established under a set of axioms which we can explicitly verify in two dimensions and which we expect to be true whenever the dimension does not exceed the upper critical dimension d_c (presumably $d_c = 6$). We call these axioms the *Scaling Axioms* since they are characterizations of the scaling behaviors implicitly assumed in the physics literature. In this section, we will review the axioms and a few of the results from [BCKS2]. Much of this treatment is taken almost verbatim from a preliminary version of [BCKS2] and [CPS].

The Scaling Axioms

Several of the axioms involve the length scales $L_0(p)$ and $A_0(p)$, and therefore implicitly involve the constant ϵ. [BCKS2] assume that the axioms are true for all $\epsilon < \epsilon_0$, where $\epsilon_0 = \epsilon_0(d)$ depends on a so-called rescaling lemma.

The axioms are written in terms of the equivalence symbol $\asymp$. Here $F(p) \asymp G(p)$ means that $C_1 F(p) \leq G(p) \leq C_2 F(p)$ where $C_1 > 0$ and $C_2 < \infty$ are constants which do not depend on n or p, as long as p is uniformly bounded away from zero or one, but which may depend on the constants ϵ, $\tilde{\epsilon}$ or x appearing explicitly or implicitly in the axioms. The [BCKS2] scaling axioms are

(I) $\quad L_0(p) \to \infty \quad$ as $p \downarrow p_c$;

(II) For $0 < \widetilde{\epsilon} < \epsilon_0$, $x \geq 1$ and $p > p_c$,
$A_0(p) \asymp L_0^{d-1}(p) \asymp L_0^{d-1}(p, \widetilde{\epsilon}; x)$;

(III) There are constants $D_1 > 0, D_2 < \infty$ such that
$D_1 \leq \pi_n(p)/\pi_n(p_c) \leq D_2$ if $n \leq L_0(p)$;

(IV) There are constants $D_3 > 0, \rho_1 > 2/d$, such that
$\pi_{kn}(p_c)/\pi_n(p_c) \geq D_3 k^{-1/\rho_1}, \quad n, k \geq 1$;

(V) There exists a constant D_4 such that for $p > p_c$,
$\chi^{\text{cov}}(p) \leq D_4 L_0^d(p) \pi^2_{L_0(p)}(p_c)$ and $\chi^{\text{fin}}(p) \leq D_4 L_0^d(p) \pi^2_{L_0(p)}(p_c)$;

(VI) For $p > p_c$,
$\pi_{L_0(p)}(p_c) \asymp P_\infty(p)$;

(VII) There exist constants $D_5, D_6 < \infty$ such that for $p < p_c$ and $k \geq 1$,
$P_{\geq ks(L_0(p))}(p) \geq D_5 e^{-D_6 k} P_{\geq s(L_0(p))}(p)$.

Let us briefly discuss the interpretation of the axioms. The first tells us that the approach to p_c is critical—i.e., continuous or second-order—from above p_c. Axiom (II) is the assumption of equivalence of length scales above p_c: The second part of it asserts the equivalence of the finite-size scaling lengths at various values of $x \geq 1$ and $\epsilon \in (0, \epsilon_0)$. The first part of it, i.e. $A_0(p) \asymp L_0^{d-1}(p)$, is called Widom scaling. It is equivalent to a hyperscaling relation the surface tension and correlation length exponents.

The third axiom formalizes a central element of the conventional scaling wisdom. Scaling theory asserts that whenever the system is viewed on length scales smaller than the correlation length, it behaves as it does at threshold. Axiom (III) asserts that this is the case for the connectivity function $\pi(p)$. Axiom (IV) implies that the connectivity function $\pi_n(p)$ has a bound of power law behavior at threshold. Of course, scaling theory assumes a pure power law with exponent $-1/\rho$. Axioms (V) and (VI) imply hyperscaling and scaling relations among the critical exponents. In terms of exponents, (V) is equivalent to the hyperscaling relation $d\nu = 2\beta + \gamma$, while (VI) is equivalent to the scaling relation $\nu/\rho = \beta$. Finally, Axiom (VII) gives a bound on the exponential decay rate of the cluster size distribution below p_c.

THEOREM 0 ([BCKS2]). *The Scaling Axioms (I)–(VII) hold in dimension $d = 2$.*

The proof of this theorem is technically quite complicated. It involves essentially the most complicated constructions which have been done for two-dimensional percolation.

A Few Results

In order to state the [BCKS2] results, we need to define a scaling window in which the system behaves critically, i.e. an analogue of the function $p(N)$ in the random graph problem. For us, this is described by the function

$$g(p,n) := \begin{cases} -\frac{n}{L_0(p)} & \text{if } p < p_c \\ 0 & \text{if } p = p_c \\ \frac{n}{L_0(p)} & \text{if } p > p_c. \end{cases}$$

It will turn out that a sequence of systems with density p_n behaves critically, subcritically, or supercritically— as far as size of large clusters is concerned— in finite boxes if, as $n \to \infty$, $g(p_n, n)$ remains bounded, tends to $-\infty$, or tends to ∞, respectively. If this is the case, we say that the sequence of systems is inside, below or above the scaling window, respectively.

We again use the symbol $\asymp$, this time for two sequences a_n and b_n of real numbers . We write $a_n \asymp b_n$ if $0 < \liminf_{n\to\infty} a_n/b_n \leq \limsup_{n\to\infty} a_n/b_n < \infty$.

Our first theorem characterizes the scaling window in terms of the *expectation* of the largest cluster sizes.

THEOREM 1 ([BCKS2]). *Suppose that Axioms (I)–(VII) hold.*

i) If $\{p_n\}$ is inside the scaling window, i.e., if $\limsup_{n\to\infty} |g(p_n, n)| < \infty$, and $i \in \mathbb{N}$, then

$$E_{p_n}\{W_{\Lambda_n}^{(i)}\} \asymp s(n) \, .$$

ii) If $\{p_n\}$ is below the scaling window, i.e., $g(p_n, n) \to -\infty$, then

$$E_{p_n}\{W^{(1)}_{\Lambda_n}\} \asymp s(L_0(p_n)) \log \frac{n}{L_0(p_n)}\,.$$

iii) If $\{p_n\}$ is above the scaling window, i.e., $g(p_n, n) \to \infty$, then

$$\frac{E_{p_n}\{W^{(1)}_{\Lambda_n}\}}{|\Lambda_n| P_\infty(p_n)} \to 1 \qquad as \qquad n \to \infty\,,$$

and

$$\frac{E_{p_n}\{W^{(2)}_{\Lambda_n}\}}{|\Lambda_n| P_\infty(p_n)} \to 0 \qquad as \qquad n \to \infty\,.$$

Assuming the existence of critical exponents and monotonicity of various quantities, Theorem 1 says that the scaling window is of the form

$$p_n = p_c \pm \frac{c}{n^{1/\nu}},$$

that inside the window

$$W^{(1)} \approx n^{d_f}, \quad W^{(2)} \approx n^{d_f}, \quad \cdots$$

while above the window

$$\begin{aligned} W^{(1)} &\approx n^d P_\infty\,, \\ W^{(1)}/n^{d_f} &\to \infty\,, \\ W^{(2)}/W^{(1)} &\to 0, \end{aligned}$$

and below the window

$$W^{(1)}/n^{d_f} \to 0$$

where, in fact,

$$W^{(1)} \approx \xi^{d_f} \log n/\xi\,.$$

The above results hold in expectation.

[BCKS2] also prove analogues of statements (i)–(iii) of the theorem for convergence in probability, rather than in expectation. Furthermore, within the scaling window, we get results on the distribution of cluster sizes which show that the distribution does not go to a delta function. This is to be contrasted with the behavior above the window, where the cluster size distribution approaches its expectation, with probability one. All of these additional results require some delicate second moment estimates. The reader is referred to [BCKS2] for precise statements of these results and for their proofs.

One final result is worth mentioning, since it is used in the proofs of the other results and is of interest in its own right. It concerns the number of clusters on scales $m < n$. Before stating the result, it should be noted that, due to statement

(i) of Theorem 1, the "incipient infinite cluster" inside the scaling window is not unique, in the sense that $W^{(2)}_{\Lambda_n}$ is of the same scale as $W^{(1)}_{\Lambda_n}$. This should be contrasted with the behavior of $W^{(2)}_{\Lambda_n}/W^{(1)}_{\Lambda_n}$ above the scaling window (see statement (iii)), a remnant of the uniqueness of the infinite cluster above p_c. The next theorem relates the non-uniqueness of the "incipient infinite cluster" inside the scaling window to the property of scale invariance at p_c. Basically, it says that the number of clusters of scale m in a system of scale n is a function only of the ratio n/m. How can this hold on *all* scales m? The only way it can be true is if the system has a fractal-like structure with smaller clusters inside holes in larger clusters. The theorem concerns the number $N_\Lambda(s_1, s_2)$ of clusters with size between s_1 and s_2.

Theorem 2 ([BCKS2]). *Assume that Axioms (I)–(IV) are valid. Let $\{p_n\}$ lie inside the scaling window. Then there exist strictly positive, finite constants σ_1, σ_2, C_1 and C_2 (all depending on the sequence $\{p_n\}$) such that*

$$C_1\left(\frac{n}{m}\right)^d \le E_{p_n}\{N_{\Lambda_n}(s(m), s(km))\} \le C_2\left(\frac{n}{m}\right)^d,$$

provided m and k are strictly positive integers with $k \ge \sigma_1$ and $\sigma_2 m \le n$.

5. Interpretation of the Results

How can we understand the form of the window? As explained earlier, the system is expected to behave critically whenever the length scale is less than the correlation length. Indeed, this is the content of Axiom (III). But the boundary of this region is given by

$$n \approx \tilde\lambda\xi \approx \tilde\lambda|p-p_c|^{-\nu}, \qquad \text{i.e.} \quad p \approx p_c \pm \frac{\lambda}{n^{1/\nu}},$$

where $\tilde\lambda$, λ are constants. This is of course precisely the content of Theorem 1.

What would these results say if we attempted to apply them in the case of random graph model (to which they of course do not rigorously apply)? Let us use the hyperscaling relation $d\nu = \gamma + 2\beta$ and the observation that the volume N of our system is just n^d, to rewrite the window in the form

$$p_n = p_c \pm \frac{\lambda}{n^{1/\nu}} = p_c\left(1 \pm \frac{c}{n^{1/\nu}}\right) = p_c\left(1 \pm \frac{c}{N^{1/d\nu}}\right) = p_c\left(1 \pm \frac{c}{N^{1/(\gamma+2\beta)}}\right).$$

Similarly, let us use the hyperscaling relation $d_f/d = \delta/(1+\delta)$ to rewrite the size of the largest cluster as

$$W^{(1)} \approx n^{d_f} \approx N^{d_f/d} \approx N^{\delta/(1+\delta)}.$$

Noting that the random graph model is a mean-field model, we expect (and in fact it can be verified) that $\gamma = 1$, $\beta = 1$ and $\delta = 2$. Using also $p_c = 1/N$, we have a window of the form

$$p(N) = \frac{1}{N} \pm \frac{c}{N^{4/3}},$$

and within that window

$$W^{(1)} \approx N^{2/3},$$

just the values obtained in the combinatoric calculations on the random graph model.

The results also have implications for finite-size scaling. Indeed, the form of the window tells us precisely how to locate the critical point, i.e. it tells us the correct region about p_c in which to do critical calculations. Similarly, $W^{(1)} \approx N^{2/3}$ tells us how to extrapolate the scaling of clusters in the critical regime.

Finally, the results tell us that we may use the largest cluster in the box as a candidate for the incipient infinite cluster. Within the window, it is not unique, in the sense that there are many clusters of this scale. However, outside the window (even including a region where p is not uniformly greater than p_c as $n \to \infty$), there is a unique cluster of largest scale. This is the analogue of what is called the *dominant component* in the random graph problem.

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