Scattering Theory: Some Old and New Problems

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Abstract. Scattering theory is, roughly speaking, perturbation theory of self-adjoint operators on the (absolutely) continuous spectrum. It has its origin in mathematical problems of quantum mechanics and is intimately related to the theory of partial differential equations. Some recently solved problems, such as asymptotic completeness for the Schrödinger operator with long-range and multiparticle potentials, as well as open problems, are discussed. We construct also potentials for which asymptotic completeness is violated. This corresponds to a new class of asymptotic solutions of the time-dependent Schrödinger equation. Special attention is paid to the properties of the scattering matrix, which is the main observable of the theory.

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1. Basic notions. Let H_0 and H be self-adjoint operators on Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}$, respectively. Let $P_0^{(ac)}$ be the orthogonal projection on the absolutely continuous subspace $\mathcal{H}^{(ac)}(H_0)$ of H_0 and $J:\mathcal{H}_0\to\mathcal{H}$ be a bounded operator. The main problem of mathematical scattering theory (see e.g. [23] or [31]) is to show the existence of the strong limits

$$W^\pm = W^\pm(H,H_0;J) = s-\lim_{t\to\pm\infty}\exp(iHt)J\exp(-iH_0t)P_0^{(ac)}, \qquad (1)$$

known as the wave operators. If the limits (1) exist, then the wave operators enjoy the intertwining property $HW^\pm = W^\pm H_0$, so their ranges are contained in $\mathcal{H}^{(ac)}(H)$. In the most important case $\mathcal{H}_0=\mathcal{H}$, $J=Id$, the limit (1) is isometric and is denoted $W^\pm(H,H_0)$. The operator $W^\pm(H,H_0)$ is said to be complete if its range coincides with $\mathcal{H}^{(ac)}(H)$. This is equivalent to the existence of $W^\pm(H_0,H)$. In terms of the operators (1) the scattering operator is defined by $\mathbf{S}=(W^+)^*W^-$. It commutes with H_0 and hence reduces to multiplication by the operator-function $S(\lambda)$, known as the scattering matrix, in a representation of $\mathcal{H}_0$ which is diagonal for H_0.

In scattering theory there are two essentially different approaches. One of them, the trace-class method, makes no assumptions about the "unperturbed" operator H_0. Its basic result is the Kato-Rosenblum theorem (and its extension due to M. Birman and D. Pearson), which guarantees the existence of $W^{\pm}(H, H_0; J)$ if the perturbation $V = HJ - JH_0$ belongs to the trace class. According to the Weyl-von Neumann-Kuroda theorem this condition cannot be relaxed in the framework of operator ideals even in the case $J = Id$. The second, smooth, method relies on a certain regularity of the perturbation in the spectral representation of the operator H_0. There are different ways to understand regularity. For example, in the Friedrichs model [8] V is an integral operator with smooth kernel. Another possibility is to assume that $V = K^* K_0$ where K is H-smooth (in the sense of T. Kato which, roughly speaking, means that the function $||K \exp(-Ht) f||^2$ is integrable on $\mathbb{R}$ uniformly for $||f|| \leq 1$) and K_0 is H_0-smooth.

The assumptions of trace-class and smooth scattering theory are quite different. Thus it would be desirable to develop a theory unifying the trace-class and smooth approaches. Of course this problem admits different interpretations, but it becomes unambiguously posed in the context of applications, especially to differential operators. Suppose that $\mathcal{H} = L_2(\mathbb{R}^d)$, $H_0 = -\Delta + V_0(x)$, $H = H_0 + V(x)$ where V_0 and V are real bounded functions and $V(x) = O(|x|^{-\rho})$ as $|x| \to \infty$. Trace-class theory shows that the wave operators $W^{\pm}(H, H_0)$ exist (and are complete) if V_0 is an arbitrary bounded function and $\rho > d$. Smooth theory requires an explicit spectral analysis of the operator H_0, which is possible for special V_0 only (the simplest case $V_0 = 0$) but imposes the less stringent assumption $\rho > 1$ on the perturbation V. This raises

Problem 1 *Let $d > 1$. Do the wave operators $W^{\pm}(H, H_0)$ exist for arbitrary $V_0 \in L_\infty(\mathbb{R}^d)$ and V satisfying the bound $V(x) = O(|x|^{-\rho})$, assuming only that $\rho > 1$?*

In the event of a positive solution of Problem 1, wave operators would be automatically complete under its assumptions. We conjecture, on the contrary, that Problem 1 has a negative solution. Moreover, we expect that the absolutely continuous part of the spectrum is no longer stable in the situation under consideration.

2. The multiparticle Schrödinger operator. One of the important problems of scattering theory is the description of the asymptotic behaviour of N interacting quantum particles for large times. The complete classification of all possible asymptotics (channels of scattering) is called asymptotic completeness. Let us recall the definition of generalized N-particle Hamiltonians introduced by S. Agmon. Consider the self-adjoint Schrödinger operator $H = -\Delta + V(x)$ on the Hilbert space $\mathcal{H} = L_2(\mathbb{R}^d)$. Suppose that some finite number α_0 of subspaces X^α of $X := \mathbb{R}^d$ are given and let x^α, x_α be the orthogonal projections of $x \in X$ on X^α and $X_\alpha = X \ominus X^\alpha$, respectively. We assume that $V(x) = \sum_{\alpha=1}^{\alpha_0} V^\alpha(x^\alpha)$, where V^α is a real function of the variable x^α satisfying the short-range condition

$$|V^\alpha(x^\alpha)| \leq C(1 + |x^\alpha|)^{-\rho}, \quad \rho > 1. \tag{2}$$

Many intermediary results are valid also for long-range pair potentials satisfying

$$|V^\alpha(x^\alpha)| + (1 + |x^\alpha|)\,|\nabla V^\alpha(x^\alpha)| \leq C(1 + |x^\alpha|)^{-\rho}, \quad \rho > 0. \tag{3}$$

The two-particle Hamiltonian H is recovered if $\alpha_0 = 1$ and $X^1 = X$. The three-particle problem is distinguished from the general situation by the condition that $X_\alpha \cap X_\beta = \{0\}$ for $\alpha \neq \beta$. Clearly, $V^\alpha(x^\alpha)$ tends to zero as $|x| \to \infty$ outside of any conical neighbourhood of X_α but $V^\alpha(x^\alpha)$ is constant on planes parallel to X_α. Due to this property the structure of the spectrum of H is much more complicated than in the two-particle case.

Let us consider linear sums $X^a = X^{\alpha_1} + X^{\alpha_2} + \ldots + X^{\alpha_k}$ of the subspaces X^{α_j}. Without loss of generality, one can suppose that X coincides with one of the X^a. We denote by $\mathcal{X}$ the set of all subspaces X^a with $X^0 := \{0\} \in \mathcal{X}$ included in it but X excluded. Let x^a and x_a be the orthogonal projections of $x \in X$ on the subspaces X^a and $X_a = X \ominus X^a$, respectively. The index a (or b) labels all subspaces $X^a \in \mathcal{X}$ and, in the multiparticle terminology, a parametrizes decompositions of an N-particle system into noninteracting clusters; x^a is the set of "internal" coordinates of all clusters, while x_a describes the relative motion of clusters.

For each a define an auxiliary operator $H_a = -\Delta + V^a$ with a potential $V^a = \sum_{X^\alpha \subset X^a} V^\alpha$, which does not depend on x_a. In the representation $L_2(X) = L_2(X_a) \otimes L_2(X^a)$, $H_a = -\Delta_{x_a} \otimes I + I \otimes H^a$, where $H^a = -\Delta_{x^a} + V^a$. The operator H^a corresponds to the Hamiltonian of clusters with their centers-of-mass fixed at the origin, $-\Delta_{x_a}$ is the kinetic energy of the center-of-mass motion of these clusters, and H_a describes an N-particle system with interactions between different clusters neglected. Eigenvalues of the operators H^a are called thresholds for the Hamiltonian H. We denote by Υ the set of all thresholds and eigenvalues of the Hamiltonian H. Let P^a be the orthogonal projection in $L_2(X^a)$ on the subspace spanned by all eigenvectors of H^a. Then $P_a = I \otimes P^a$ commutes with the operator H_a. Set also $H_0 = -\Delta, P_0 = I$. The basic result of scattering theory for N-particle Schrödinger operators is the following

Theorem 2 *Let assumption* (2) *hold. Then the wave operators $W_a^\pm = W^\pm(H, H_a; P_a)$ exist and are isometric on the ranges $R(P_a)$ of projections P_a. The subspaces $R(W_a^\pm)$ are mutually orthogonal, and scattering is asymptotically complete:*

$$\bigoplus_a R(W_a^\pm) = \mathcal{H}^{(ac)}, \quad \mathcal{H}^{(ac)} = \mathcal{H}^{(ac)}(H).$$

The spectral theory of multiparticle Hamiltonians starts with the following basic result (see [19], [22]). It is formulated in terms of the auxiliary operator $A = \sum(x_j D_j + D_j x_j)$, $D_j = -i\partial_j$, $j = 1, \ldots, d$. In what follows $E(\Lambda)$ is the spectral projection of the operator H corresponding to a Borel set $\Lambda \subset \mathbb{R}$ and Q is the operator of multiplication by $(x^2+1)^{1/2}$.

Theorem 3 *Let each pair potential V^α be a sum of two functions satisfying assumptions* (2) *and* (3)*, respectively. Then eigenvalues of H may accumulate only at the thresholds of H, so the "exceptional" set Υ is closed and countable. Furthermore, for every $\lambda \in \mathbb{R} \setminus \Upsilon$ there exists a small interval $\Lambda_\lambda \ni \lambda$ such that the Mourre estimate for the commutator holds, i.e.,*

$$i([H, A]u, u) \geq c\|u\|^2, \quad c = c_\lambda > 0, \quad u \in E(\Lambda_\lambda)\mathcal{H}. \tag{4}$$

Finally, for any compact interval Λ such that $\Lambda \cap \Upsilon = \emptyset$ and any $r > 1/2$, the operator $Q^{-r}E(\Lambda)$ is H-smooth (the limiting absorption principle). In particular, the operator H does not have singularly continuous spectrum.

In the case $N = 2$ the limiting absorption principle suffices for construction of scattering theory but, for $N > 2$, one needs additional analytical information corresponding in some sense to the critical case $r = 1/2$. However, the operator $Q^{-1/2}$ is definitely not H-smooth even in the free case $H = -\Delta$. Hence we construct differential operators which improve the fall-off of functions $(\exp(-iHt)f)(x)$ for large t and x. Denote by $\langle \cdot, \cdot \rangle$ the scalar product in the space $\mathbb{C}^d$. Let $\nabla_a = \nabla_{x_a}$ be the gradient in the variable x_a and let $\nabla_a^\perp$,

$$(\nabla_a^\perp u)(x) = (\nabla_a u)(x) - |x_a|^{-2} \langle (\nabla_a u)(x), x_a \rangle x_a,$$

be its orthogonal projection in X_a on the plane orthogonal to the vector x_a. Let $\mathbf{\Gamma}_a$ be a closed cone in $\mathbb{R}^d$ such that $\mathbf{\Gamma}_a \cap X_b = \{0\}$ if $X_a \not\subset X_b$. Let $\chi(\mathbf{\Gamma}_a)$ denote its characteristic function. Our main analytical result is the following:

Theorem 4 *Suppose that the assumptions of Theorem 3 hold. Then for any a, the operator $G_a = \chi(\mathbf{\Gamma}_a) Q^{-1/2} \nabla_a^\perp E(\Lambda)$ is H-smooth.*

In particular, for the free region $\mathbf{\Gamma}_0$, where all potentials V^α are vanishing, the operator $\chi(\mathbf{\Gamma}_0) Q^{-1/2} \nabla^\perp E(\Lambda)$ is H-smooth. By analogy with the radiation conditions in the two-particle case (see e.g. [24]), we refer to the estimates of Theorem 4 as radiation estimates.

Our proof of Theorem 4 hinges on the commutator method. To that end, we construct a first-order differential operator $M = \sum (m_j D_j + D_j m_j)$, $m_j = \partial m / \partial x_j$, such that, for any a, the commutator $[H, M]$ satisfies the estimate

$$i[H, M] \geq c_1 G_a^* G_a - c_2 Q^{-2r}, \quad r > 1/2, \quad c_1, c_2 > 0, \tag{5}$$

locally (that is, sandwiched by $E(\Lambda)$). Here the "generating" function m is real, smooth and homogeneous of degree 1 for $|x| \geq 1$. It is completely determined by the geometry of the problem, that is by the collection of subspaces X^α. Roughly speaking, we set $m(x) = \mu_a |x_a|$ in a neighbourhood of every subspace X_a with neighbourhoods of all subspaces $X_b \not\supset X_a$ removed from it. In particular, $m(x) = |x|$ in a free region where all potentials are vanishing. It is important that $m(x)$ is a convex function, which implies that $i[H_0, M] \geq 0$ (up to an error $O(|x|^{-3})$). The arguments of [18] show that H-smoothness of the operator G_a is a direct consequence of estimate (5) and of the limiting absorption principle.

For the proof of asymptotic completeness we first consider auxiliary wave operators

$$W^\pm(H, H_a; M^a E_a(\Lambda)), \quad W^\pm(H_a, H; M^a E(\Lambda)), \tag{6}$$

where the "identifications" M^a are again first-order differential operators with suitably chosen "generating" functions m^a. It is important that $m^a(x)$ equals zero in some conical neighbourhood of every X_α such that $X_a \not\subset X_\alpha$. Hence coefficients of the operator $(V - V^a) M^a$ vanish as $O(|x|^{-\rho})$, $\rho > 1$, at infinity. The analysis of the commutator $[H_a, M^a]$ relies on Theorem 4. This shows that the "effective perturbation" $HM^a - M^aH_a$ can be factorized into a product K^*K_a where K is H-smooth and K_a is H_a-smooth (locally), which implies the existence of the wave operators (6). The final step of the proof is to choose functions m^a in such a way that $\sum_a m^a = m$ and to verify that the range of the operator $W^{\pm}(H, H; ME(\Lambda))$ coincides with the subspace $E(\Lambda)\mathcal{H}$. This is again a consequence of the Mourre estimate (4).

In the three-particle case Theorem 2 was first obtained, under some additional assumptions, by L. Faddeev [7] (see also [10], [27]), who used a set of equations he derived for the resolvent of H. The optimal formulation in the three particle case is due to V. Enss [5]. The approach to asymptotic completeness relying on the Mourre estimate goes back to I. Sigal and A. Soffer [25]. Our proof given in [32] is closer to that of G. M. Graf [11]. In contrast to [11] we fit N-particle scattering theory into the standard framework of the smooth perturbation theory. Its advantage is that it admits two equivalent formulations: time-dependent as discussed above, and stationary, where unitary groups are replaced by resolvents. This allows us [33, 34] to obtain stationary formulas for the basic objects of the theory: wave operators, scattering matrix, etc. The approaches of [7] and of [25, 11, 32] are quite different and at the moment there is no bridge between them.

There are several Hamiltonians similar to the N-particle Schrödinger operator H for which the methods of [11] or [32] can be tried.

Problem 5 *Develop scattering theory for the discrete version of H (the Heisenberg model) acting in the space $L_2(\mathbb{Z}^d)$. The same question is meaningful for the generalization of $H = H_0 + V$ where $H_0 = -\sum \Delta_k$, $k = 1, \dots, N$, is replaced by a more general differential operator, say, by $\sum(-\Delta_k)^2$.*

Radiation estimates similar to those of Theorem 4 are crucial also for different proofs due to S. Agmon, T. Ikebe, H. Kitada, Y. Saito (see e.g. [12] and also [14, 37]) of asymptotic completeness for the two-particle Schrödinger operator with a long-range potential and for scattering on unbounded obstacles [3, 13]. Actually, only the case of the Dirichlet boundary condition was considered in [3, 13]. Therefore the following question naturally arises:

Problem 6 *Develop scattering theory for the operator $H = -\Delta$ in the complement of an unbounded domain Ω (for example, of a paraboloid) with Neumann or more general boundary conditions on Ω.*

3. Long-range pair potentials. If H is the two-particle Schrödinger operator with a short-range potential $V(x) = O(|x|^{-\rho})$, $\rho > 1$, then, by the definition of the wave operator, for any $f^0 \in L_2(\mathbb{R}^d)$ and $f^{\pm} = W^{\pm}(H, H_0)f^0$,

$$(\exp(-iHt)f^{\pm})(x) = \exp(i\Phi(x,t))(2it)^{-d/2}\hat{f}^0(x/(2t)) + o(1), \quad t \to \pm\infty, \quad (7)$$

where $\Phi_0(x,t) = x^2(4t)^{-1}$, $\hat{f}^0$ is the Fourier transform of f^0 and $o(1)$ denotes a function whose norm tends to zero as $t \to \pm\infty$. For long-range potentials satisfying the condition

$$|D^\kappa V(x)| \le C(1 + |x|)^{-\rho-|\kappa|}, \quad \rho > 0, \quad \forall\kappa, \quad (8)$$

the relation (7) can be used for the definition of the modified wave operator $\tilde{W}^{\pm}: f^0 \mapsto f^{\pm}$. (Actually, many results in the long-range case remain valid if (8) is satisfied for $|\kappa| \leq 2$ only but we shall not dwell upon this.) In this case however the phase function $\Phi(x,t) = \Phi_V(x,t)$ depends on a potential V and is constructed as an approximate solution of the corresponding eikonal equation. It follows from asymptotic completeness that relation (7) is fulfilled for every $f \in \mathcal{H}^{(ac)}$. The asymptotics (7) shows that, for $f \in \mathcal{H}^{(ac)}$, the solution $(\exp(-iHt)f)(x)$ "lives" in the region where $|x| \sim |t|$.

Similarly, in the N-particle short-range case, for any $f_a^0 \in L_2(X_a)$ and $f_{a,k}^{\pm} = W_a^{\pm}(\psi^{a,k} \otimes f_a^0)$,

$$(\exp(-iHt)f_{a,k}^{\pm})(x) = \psi^{a,k}(x^a)\exp(i\Phi_{a,k}(x_a,t))(2it)^{-d_a/2}\hat{f}_a^0(x_a/(2t)) + o(1), \quad (9)$$

where $d_a = \dim X_a$, $\Phi_{a,k}(x_a,t) = x_a^2(4t)^{-1} - \lambda^{a,k}t$. Theorem 2 implies that every $f \in \mathcal{H}^{(ac)}$ is an orthogonal sum of vectors $f_{a,k}^{\pm}$ satisfying (9). For N-particle systems with long-range pair potentials V^α the result is almost the same if condition (8) with some $\rho > \sqrt{3}-1$ is fulfilled for all functions $V^\alpha(x^\alpha)$. In this case again every $f \in \mathcal{H}^{(ac)}$ is an orthogonal sum of vectors $f_{a,k}^{\pm}$ satisfying (9) with suitable functions $\Phi_{a,k}(x_a,t)$. This result (asymptotic completeness) was obtained by V. Enss [6] for $N=3$ and extended by J. Dereziński [4] (his method is different from [6] and uses some ideas of I. Sigal and A. Soffer) to an arbitrary number of particles (see also [26]).

4. New channels of scattering. It turns out that for some three- (and N-) particle systems with pair potentials satisfying (8) for $\rho < 1/2$, there exist channels of scattering different from (9). We rely on the following general construction [35, 36]. Suppose that $\mathbb{R}^d = X_1 \oplus X^1$, $\dim X_1 = d_1$, $\dim X^1 = d^1$, $d_1 + d^1 = d$, but we do not make any special assumptions about a potential $V(x) = V(x_1, x^1)$. Let us introduce the operator $H^1(x_1) = -\Delta_{x^1} + V(x_1, x^1)$ acting on the space $L_2(X^1)$. Suppose that $H^1(x_1)$ has a negative eigenvalue $\lambda(x_1)$, and denote by $\psi(x_1, x^1)$ a corresponding normalized eigenfunction. In interesting situations the function $\lambda(x_1)$ tends to zero slower than $|x_1|^{-1}$. Let us consider it as an "effective" potential energy and associate to the long-range potential $\lambda(x_1)$ the phase function $\Phi = \Phi_\lambda$. We prove, under some assumptions, that for every $g \in L_2(X_1)$ there exists an element $f^{\pm} \in \mathcal{H}^{(ac)}$ such that

$$(\exp(-iHt)f^{\pm})(x) = \psi(x_1, x^1)\exp(i\Phi(x_1,t))(2it)^{-d_1/2}g(x_1/(2t)) + o(1) \quad (10)$$

as $t \to \pm\infty$. The mapping $\check{g} \mapsto f^{\pm}$ ($\check{g}$ is the inverse Fourier transform of g) defines the new wave operator $\mathcal{W}^{\pm}$. It is isometric on $L_2(X_1)$, and $H\mathcal{W}^{\pm} = \mathcal{W}^{\pm}(-\Delta_{x_1})$. The ranges of $\mathcal{W}^{\pm}$ and of $\tilde{W}^{\pm}$ are orthogonal if both of these wave operators exist. The existence of solutions of the time-dependent Schrödinger equation with asymptotics (10) requires rather special assumptions which are naturally formulated in terms of eigenfunctions $\psi(x_1, x^1)$. Typically the asymptotic behaviour of $\psi(x_1, x^1)$ as $\lambda(x_1) \to 0$ has a certain self-similarity:

$$\psi(x_1, x^1) = |x_1|^{-\sigma d^1/2}\Psi(|x_1|^{-\sigma}x^1) + o(1) \quad (11)$$

for some $\Psi \in L_2(X^1)$ and $\sigma > 0$. We prove the asymptotics (10) if (11) is fulfilled for $\sigma < 1/2$. On the other hand, simple examples show that (11) for $\sigma \geq 1/2$ does not ensure existence of solutions with the asymptotics (10). It is important that $\psi(x_1, x^1)$ can be chosen as an *approximate* solution of the equation $H^1(x_1)\psi(x_1) - \lambda(x_1)\psi(x_1) = 0$.

Let us first give an example of a two-body long-range potential (see [36], for more general classes) for which the completeness of the modified wave operator is violated. Let

$$V(x_1, x^1) = -v(< x_1 >^q + < x^1 >^q)^{-\rho/q}, \quad \rho \in (0,1),\ q \in (0,2),\ v > 0, \tag{12}$$

where we use the notation $< y >= (1 + |y|^2)^{1/2}$. The function (12) is infinitely differentiable and $V(x) = O(|x|^{-\rho})$ as $|x| \to \infty$. The bound (8) is fulfilled for arbitrary κ off any conical neighbourhood of the planes X_1 and X^1. This suffices for the existence of the modified wave operator $\tilde{W}^{\pm}$. If $q = 2$, then $V(x_1, x^1)$ is a radial function, so $\tilde{W}^{\pm}$ is complete.

THEOREM 7 *Let a potential V be defined by* (12) *where $1 - \rho < q < 2(1 - \rho)$. Let Λ be any eigenvalue and Ψ be a corresponding eigenfunction of the operator $K = -\Delta_{x^1} + v\rho q^{-1}|x^1|^q$ in the space $L_2(X^1)$. Define the function $\psi(x_1, x^1)$ and the "potential" $\lambda(x_1)$ by the equations*

$$\psi(x_1, x^1) = |x_1|^{-\sigma/2}\Psi(|x_1|^{-\sigma}x^1), \quad \lambda(x_1) = -v|x_1|^{-\rho} + \Lambda|x_1|^{-2\sigma}, \tag{13}$$

where $\sigma = (\rho + q)(2 + q)^{-1}$ and set $\Phi = \Phi_\lambda$. Then the wave operator $\mathcal{W}^{\pm}$ exists and the subspaces $R(\mathcal{W}^{\pm})$, $R(\tilde{W}^{\pm})$ are orthogonal.

Let us now consider the Schrödinger operator, which describes three one-dimensional particles with one of three pair interactions equal to zero. The following result was obtained in [35].

THEOREM 8 *Let $V(x) = V^1(x^1) + V^2(x^1 - x_1)$ where $d^1 = d_1 = 1$. Suppose that $V^1 \geq 0$ is a bounded function, $V^1(x^1) = 0$ for $x^1 \geq 0$ and $V^1(x^1) = v_1|x^1|^{-r}$, $v_1 > 0$, $r \in (0,2)$, for large negative x^1. Suppose that a bounded function V^2 satisfies for some $\rho \in (0,1/2)$ and $v_2 > 0$ one of the two following conditions: 1^0 $V^2(x^2) = -v_2|x^2|^{-\rho}$ for large positive x^2; 2^0 $V^2(x^2) = v_2|x^2|^{-\rho}$ for large negative x^2. Let Λ be any eigenvalue and Ψ a corresponding eigenfunction of the equation $-\Psi'' + |v_2|\rho x^1\Psi = \Lambda\Psi$ for $x^1 \geq 0$, $\Psi(0) = 0$, extended by 0 to $x^1 \leq 0$. Define the function $\psi(x_1, x^1)$ and the "potential" $\lambda(x_1)$ by the equations* (13), *where $\sigma = (\rho + 1)/3$ and $v = -v_2$, $x_1 < 0$ in the case 1^0, $v = v_2$, $x_1 > 0$ in the case 2^0. Put $\Phi = \Phi_\lambda$. Then the wave operator $\mathcal{W}^{\pm}$ defined by equality* (10) *exists for any $g \in L_2(\mathbb{R}_{\mp})$ in the first case and for any $g \in L_2(\mathbb{R}_{\pm})$ in the second case. Moreover, the subspaces $R(\mathcal{W}^{\pm})$, $R(\tilde{W}_0^{\pm})$ and $R(\tilde{W}_\alpha^{\pm})$, $\alpha = 1, 2$, are orthogonal.*

We emphasize that for $f \in R(\mathcal{W}^{\pm})$ the solution $u(t) = \exp(-iHt)f$ of the Schrödinger equation "lives" for large $|t|$ in the region where $x_1 \sim -|t|$ in the case 1^0 or $x_1 \sim |t|$ in the case 2^0 and $x^1 \sim |t|^\sigma$ for $\sigma \in (1/3, 1/2)$. Such solutions describe a physical process where a pair of particles (say, the first and the second) interacting by the potential V^1 are relatively close to one another and the third particle is far away. This pair is bound by a potential depending on the position of the third particle, but this bound state is evanescent as $|t| \to \infty$. Thus solutions $u(t)$ for $f \in R(\mathcal{W}^{\pm})$ are intermediary between those for $f \in R(\tilde{W}_0^{\pm})$ and $f \in R(\tilde{W}_\alpha^{\pm})$.

There is however a gap between the cases when asymptotic completeness holds and when it is violated. Hence the following questions arise.

Problem 9 *Is the scattering asymptotically complete when $\rho \in [1/2, \sqrt{3}-1)$? The same question for all $\rho < \sqrt{3}-1$ if particles are, say, three-dimensional.*

Note that, under some additional assumptions, asymptotic completeness for all $\rho > 1/2$ was checked in [28, 9]. In the cases when new channels are constructed one can expect that all possible asymptotics of the time-dependent Schrödinger equation have either the form (9) or (10). In a somewhat similar situation a result of such type was established in [30]. Thus, we formulate

Problem 10 *To prove (for example, under the assumptions of Theorems 7 and 8) generalized asymptotic completeness, that is, that the ranges of all wave operators constructed exhaust $\mathcal{H}^{(ac)}(H)$.*

5. The scattering matrix. For the two-particle Schrödinger operator $H = -\Delta + V(x)$ in the space $L_2(\mathbb{R}^d)$, the scattering matrix $S(\lambda)$, $\lambda > 0$, is a unitary operator on the space $L_2(\mathbb{S}^{d-1})$. If V is a short-range potential, then the operator $S(\lambda) - Id$ is compact so the spectrum of $S = S(\lambda)$ consists of eigenvalues $\mu_n^{\pm} = \exp(\pm i\theta_n^{\pm})$, $\pm\theta_n^{\pm} > 0$, lying on the unit circle $\mathbb{T}$ and accumulating only at the point 1. Moreover, the asymptotics of the scattering phases $\theta_n^{\pm}$ is determined by the asymptotics of the potential $V(x)$ at infinity and is given by the Weyl type formula. The following assertion was established in [2].

Theorem 11 *Let $V(x) = v(x|x|^{-1})|x|^{-\rho} + o(|x|^{-\rho})$, $\rho > 1$, $v \in C^\infty(\mathbb{S}^{d-1})$, as $|x| \to \infty$. Then $n^\gamma \theta_n^{\pm}(\lambda) \to \Omega_\pm$ as $n \to \infty$, where $\gamma = (\rho-1)(d-1)^{-1}$ and $\Omega_\pm$ is some explicit functional of v and ρ, λ.*

The situation is drastically different for long-range potentials. Note that in this case modified wave operators can be defined [14, 15] by equality (1) where $J_\pm$ is a suitable pseudo-differential operator. It depends on the sign of t. The existence and completeness of the operators $W_\pm(H, H_0; J_\pm)$ follow immediately from Theorem 4, which fits the long-range scattering into the theory of smooth perturbations. In the long-range case, $S(\lambda) - Id$ is no longer compact. Moreover, its spectrum covers the whole unit circle. For simplicity we give the precise formulation only for the case $\rho > 1/2$ (see [37], for details).

Theorem 12 *Let condition (8) with $\rho > 1/2$ hold. Suppose that the function*

$$\mathbf{V}(\omega, b) = \int_{-\infty}^{\infty} \Big(V(t\omega) - V(b + t\omega) \Big) dt, \quad |\omega| = 1, \quad \langle \omega, b \rangle = 0, \tag{14}$$

satisfies the condition $|\mathbf{V}(\omega_0, t_n b_0)| \to \infty$ *for some point* ω_0, b_0 *and some sequence* $t_n \to \infty$. *Then for all* $\lambda > 0$ *the spectrum of the scattering matrix* $S(\lambda)$ *covers the unit circle.*

Our study of the scattering matrix relies on its stationary representation (in terms of the resolvent). First, using the so called microlocal or propagation estimates [20, 17, 16], we show that, up to an integral operator with C^∞-kernel, S can be considered as a pseudo-differential operator with explicit principal symbol

$$\mathbf{s}(\omega, b; \lambda) = \exp\Big(i2^{-1}\lambda^{-1/2}\mathbf{V}(\omega, \lambda^{-1/2}b)\Big), \quad |\omega| = 1, \quad \langle \omega, b\rangle = 0. \tag{15}$$

If $\rho \leq 1$, this is an oscillating function as $|b| \to \infty$, which implies Theorem 12. Note that in the short-range case the principal symbol of S equals 1 which corresponds to the Dirac-function in its kernel. In the long-range case this singularity disappears.

The kernel $s(\omega, \omega')$ of S (the scattering amplitude) is [1, 15] a C^∞-function off the diagonal. Its diagonal singularity is given by the Fourier transform of the symbol (15). It turns out [37] that for an asymptotically homogeneous function $V(x)$ of order $-\rho$, $\rho < 1$, the kernel s is a sum of a finite number of terms $s_j = w_j \exp(i\psi_j)$, where the moduli $w_j(\omega, \omega')$ and the phases $\psi_j(\omega, \omega')$ are asymptotically homogeneous functions, as $\omega - \omega' \to 0$, of orders $-(d-1)(1+\rho^{-1})/2 < -d+1$ and $1-\rho^{-1}$, respectively. Thus S is more singular than the singular integral operator. In the case $\rho = 1$, the modulus $w = |s|$ is asymptotically homogeneous of order $-d+1$, and the phase ψ of s has a logarithmic singularity on the diagonal.

In the N-particle case results on the structure of the scattering matrix S are scarce. Let us mention the one by R. Newton [21] (see also [29], for an elementary proof), which asserts that in the 3-particle case, $S = S_1 S_2 S_3 \tilde{S}$, where S_α is the scattering matrix for the Hamiltonian with only one pair interaction V^α and the operator $\tilde{S} - Id$ is compact. We conclude with

Problem 13 *Extend the above result to an arbitrary* N.

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