On Nonlinear Dispersive Equations

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Introduction: I shall describe some of the recent developments in the application of harmonic analysis to non-linear dispersive equations. In recent years this subject has generated an intense activity and many new results have been proved. My contribution to this field has been made in collaboration with Carlos E. Kenig and Luis Vega. Their scientific inspiration, which has been so rewarding for me, is surpassed only by the warmth of their friendship.

We shall be concerned with the initial value problem (IVP) for nonlinear dispersive equations of the form

$$\begin{cases} \partial_t u = iP(\nabla_x)u + F(u), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n, \\ u(x,0) = u_0(x), \end{cases} \tag{1}$$

where $P(D)$ is the constant coefficient operator defined by its real symbol $P(i\xi)$ and $F(\cdot)$ represents the nonlinearity.

We shall concentrate our attention in the following two problems:

Problem A: The problem of the minimal regularity of the data u_0 which guarantees that the IVP (1) is well-posed.

Problem B: The existence and uniqueness for the IVP (1) for some dispersive models for which classical approaches do not apply.

Let us first consider Problem A. Our notion of well-posedness includes existence, uniqueness, persistence, i.e. if $u_0 \in \mathcal{X}$ function space then the corresponding solution describes a continuous curve in $\mathcal{X}$, and lastly continuous dependence of the solution upon the data. Thus, solutions of (1) induce a dynamical system on $\mathcal{X}$ by generating a continuous flow, see [Kt].

We use classical the Sobolev spaces $\mathcal{X} = H^s(\mathbb{R}^n) = (1-\Delta)^{-s/2}L^2(\mathbb{R}^n)$, $s \in \mathbb{R}$ to measure the regularity of the data.

To illustrate our arguments we consider the IVP for the generalized Korteweg-de Vries (gKdV) equation

$$\begin{cases} \partial_t u + \partial_x^3 u + u^k \partial_x u = 0, \qquad t, x \in \mathbb{R}, \quad k \in \mathbb{Z}^+, \\ u(x,0) = u_0(x). \end{cases} \tag{1.1}$$

For $k = 1$ (KdV) the equation in (1.1) was derived by Korteweg-de Vries as a model for long waves propagating in a channel. Later, the cases $k = 1, 2$ were found to be relevant in several physical situations. Also they have been studied because of their relation to inverse scattering theory and to algebraic geometry (see [Mi] and references therein).

Local well-posedness results imply global ones via the conservation laws

$$I_2(u) = \int_{-\infty}^{\infty} u^2(x,t)dx, \qquad I_3(u) = \int_{-\infty}^{\infty} ((\partial_x u)^2 - c_k u^{k+2})(x,t)dx, \tag{1.2}$$

satisfied by solutions of (1.1), (for $k = 1, 2$ there are infinitely many I_j's, see [Mi]).

Concerning the local well-posedness of the IVP (1.1) our first result is the following.

THEOREM 1.1 ([KePoVe3]).
The IVP (1.1) is locally well-posed in $H^s(\mathbb{R})$ if

$$\begin{cases} k = 1 \quad \textit{and} \quad s > 3/4, \\ k = 2 \quad \textit{and} \quad s \geq 1/4, \\ k = 3 \quad \textit{and} \quad s > 1/12, \\ k \geq 4 \quad \textit{and} \quad s \geq (k-2)/4k.- \end{cases} \tag{1.3}$$

Observe that if $u(\cdot)$ solves the equation in (1.1) then $u_\lambda(x,t) = \lambda^{2/k} u(\lambda x, \lambda^3 t)$ is also a solution with data $u_\lambda(x,0) = \lambda^{2/k} u_0(\lambda x)$ and

$$\|D_x^s u_\lambda\|_2 = c\lambda^{s-(k-4)/2k}. \tag{1.4}$$

Thus, for $s = (k-4)/2k$ the above norm is independent of λ. The result in Theorem 1.1 for $k \geq 4$ correspond to the scaling value in (1.4) and has been shown to be optimal, see [KePoVe3] and [BKPSV]. Theorem 1.1 and the conservation laws in (1.2) imply the global well-posedness of (1.1) with $u_0 \in H^s(\mathbb{R})$, $s \geq 1$ and $k = 1, 2, 3$. For $k \geq 4$ the existence of global solution for data $u_0 \in H^1(\mathbb{R})$ of arbitrary size is unknown.

To explain our result with more details we choose the case $k = 2$, i.e. the modified Korteweg-de Vries (mKdV) equation.

THEOREM 1.2 ([KePoVe3]).
Let $k = 2$. Then for any $u_0 \in H^{1/4}(\mathbb{R})$ there exist

$$T = c\, \|D_x^{1/4} u_0\|_2^{-4}, \tag{1.5}$$

and a unique strong solution $u(t)$ of the IVP (1.1) satisfying

$$u \in C([-T,T] : H^{1/4}(\mathbb{R})), \tag{1.6}$$

and

$$\|D_x^{1/4}\,\partial_x u\|_{L^\infty_x L^2_T} \; + \; \|u\|_{L^4_x L^\infty_T} < \infty. \tag{1.7}$$

Moreover, the map data $\to$ solution, from $H^{1/4}(\mathbb{R})$ into the class defined by (1.6)–(1.7) is locally Lipschitz.

In addition, if $u_0 \in H^{s'}(\mathbb{R})$ with $s' > s$, then the above results hold with s' instead of s in the same time interval $[-T,T]$.-

The properties (1.6)–(1.7) guarantee the uniqueness of the solution and that the nonlinear term is well defined, i.e. it is at least a distribution.

In [Ka], T. Kato established the existence of a global weak solution for the IVP (1.1) with $k = 1,2,3$ and data $u_0 \in L^2(\mathbb{R}^2)$. In [GiTs], Ginibre-Tsutsumi showed that if $(1+|x|)^{3/8}u_0 \in L^2(\mathbb{R})$ then IVP (1.1) with $k = 2$ has a unique solution. Since the operator $\Gamma = x - 3t\partial_x^2$ commutes with the linear part of the equation in (1.1) one sees that Theorem 1.2 and the result in [GiTs] complement each other. Also the estimate of the life span of the local solution in (1.5) agrees with that given by the scaling argument in (1.4).

The proof of Theorem 1.2 is based on the following two sharp linear estimates, in which we introduced the notation

$$U(t)v_0(x) = \int_{-\infty}^{\infty} e^{i(t\xi^3 + x\xi)}\widehat{v}_0(\xi)d\xi. \tag{1.8}$$

In [KeRu], Kenig-Ruiz proved that

$$\left(\int_{-\infty}^{\infty} \sup_{[-1,1]} |U(t)v_0|^4 dx\right)^{1/4} \le c\|D_x^{1/4}v_0\|_2, \tag{1.9}$$

and that both indexes in (1.8), i.e. 4, 1/4 are optimal. In [KePoVe1], we showed that there exists $c > 0$ such that for any $x \in \mathbb{R}$

$$\left(\int_{-\infty}^{\infty} |\partial_x U(t)v_0|^2 dt\right)^{1/2} = c\|v_0\|_2. \tag{1.10}$$

This is a sharp version of the local smoothing effects first established by T. Kato [Ka] for solutions of the KdV equation, see also [KuFr].

In [Bo1], J. Bourgain showed that the IVP for the KdV ($k = 1$ in (1.1)) is locally (consequently globally) well-posed in L^2. His proof relies on the use of the spaces $X_{s,b}$, i.e. the completion of $S(\mathbb{R}^2)$ respect to the norm

$$\|F\|_{X_{s,b}} = \|(1+|\tau - \xi^3|)^b(1+|\xi|)^s\widehat{F}(\xi,\tau)\|_{L^2(\mathbb{R}^2)}. \tag{1.11}$$

These spaces were introduced by M. Beals [Be] in his study of propagation of singularities for solutions to semi-linear wave equations, and have been successfully used in several related works. In [KlMa] and subsequent works, Klainerman-Machedon used them to study the minimal regularity problem on the data for systems of nonlinear wave equations with nonlinearities satisfying a special structure.

In [KePoVe4], we proved that the IVP for the KdV equation ($k = 1$ in (1.1)) is locally well-posed in $H^s(\mathbb{R})$, $s > -3/4$.

THEOREM 1.3 ([KePoVe4]).

Let $s \in (-3/4, 0]$. Then there exists $b \in (1/2, 1)$ such that for any $u_0 \in H^s(\mathbb{R})$ there exist $T = T(\|u_0\|_{H^s}) > 0$ (with $T(\rho) \to \infty$ when $\rho \to 0$) and a unique solution $u(t)$ of the IVP (1.1) in the time interval $[-T, T]$ satisfying

$$u \in C([-T, T] : H^s(\mathbb{R})), \tag{1.12}$$

$$u \in X_{s,b} \subseteq L^\infty_{x,\ell oc}(\mathbb{R} : L^2_t(\mathbb{R})), \tag{1.13}$$

and

$$\partial_x(u^2) \in X_{s,b-1}, \qquad \partial_t u \in X_{s-3,b-1}. \tag{1.14}$$

Moreover, the map data $\to$ solution from $H^s(\mathbb{R})$ into the class defined by (1.12)-(1.14) is locally Lipschitz.

In addition, if $u_0 \in H^{s'}(\mathbb{R})$ with $s' > s$, then the above results hold with s' instead of s in the same time interval $[-T, T]$.-

The method of proof of Theorem 1.3 is based on bilinear estimates involving the spaces $X_{s,b}$ and elementary techniques. These techniques were motivated by the work of C. Fefferman [Fe] for the $L^4(\mathbb{R}^2)$ estimate for the Bochner-Riesz operator.

In [KePoVe4], we also established that for the case of the mKdV ($k = 2$ in (1.1)) the argument based on multilinear estimates and the use of $X_{s,b}$-spaces does not improve our result in Theorem 1.2.

The gap between the KdV result ($s > -3/4$) and that for the mKdV ($s \geq 1/4$) is somehow consistent with the Miura transformation, i.e. if v solves the mKdV equation then $u = c_1 v^2 + c_2 \partial_x v$ solves the KdV equation.

The method of proof in [KePoVe3], [Bo1],[KePoVe4], is based on the contraction principle which combined with the Implicit Function Theorem shows that the map data $\to$ solution is smooth.

In [Bo2], J. Bourgain proved that if one requires the map data $\to$ solution be smooth (C^3 suffices) then our results for the KdV ($s > -3/4$) in [KePove4] and for the mKdV ($s \geq 1/4$) in [KePoVe3] are optimal. In particular, it follows that these results cannot be improved by using only an iteration argument.

Regarding the global well-posedness of the IVP for the KdV and mKdV equations we have the following recent results. In [FoLiPo], Fonseca-Linares-Ponce showed that the IVP for the mKdV equation is globally well-posed (although not necessarily globally bounded) in $H^s(\mathbb{R})$, $s \in (3/5, 1)$. In [CoSt], Colliander-Staffilani proved the IVP for the KdV equation is globally well-posed (although not necessarily globally bounded) in $H^s(\mathbb{R})$, $s \in (-3/20, 0)$. The proofs combine ideas in [Bo3] and Theorems 1.2-1.3 described above.

Problem B.

We begin by considering the IVP for nonlinear Schrödinger equations of the form

$$(2.1) \qquad \begin{cases} \partial_t u = i\mathcal{L}u + P(u, \nabla_x u, \overline{u}, \nabla_x \overline{u}), & t \in \mathbb{R},\ x \in \mathbb{R}^n, \\ u(x,0) = u_0(x), \end{cases}$$

where $\mathcal{L}$ is a non-degenerate constant coefficient, second order operator

$$(2.2) \qquad \mathcal{L} = \sum_{j \le k} \partial_{x_j}^2 - \sum_{j>k} \partial_{x_j}^2, \qquad \text{for some} \quad k \in \{1, .., n\},$$

and $P : \mathbb{C}^{2n+2} \to \mathbb{C}$, is a polynomial of the form

$$(2.3) \qquad P(z) = P(z_1, .., z_{2n+2}) = \sum_{l_0 \le |\alpha| \le d} a_\alpha z^\alpha, \qquad l_0 \ge 2.$$

When a special form of the nonlinear term P is assumed, for example,

$$(2.4) \qquad D_{\partial_{x_j} u} P, \qquad \text{are real for } \ j = 1, .., n,$$

standard energy estimates provide the desired result. In this case, the dispersive part of the equation, the operator $\mathcal{L}$, does not play any role. Another technique used to overcome the "loss of derivatives" introduced by the nonlinear term is to present the problem in a suitable analytic function spaces, see [SiTa], [Hy].

In [KePoVe2], we proved that (2.1) is locally well-posed for "small" data, in $H^s(\mathbb{R}^n)$, for s large enough, when $l_0 \ge 3$ in (2.3), and in a weighted version of it, if $l_0 = 2$ in (2.3). This result applies to the general form of $\mathcal{L}$ in (2.2). The main idea is to use in the integral equation version of the IVP (2.1)

$$(2.5) \qquad u(t) = e^{it\mathcal{L}} u_0 + \int_0^t e^{i(t-t')\mathcal{L}} P(u, \nabla_x u, \overline{u}, \nabla_x \overline{u})(t') dt',$$

and the following estimates,

$$(2.6) \qquad \begin{cases} (i) \quad |||D^{1/2} e^{it\mathcal{L}} u_0|||_T \equiv \sup_{\mu \in \mathbb{Z}^n} \Big(\int_0^T \int_{Q_\mu} |D^{1/2} e^{it\mathcal{L}} u_0|^2 dx dt \Big)^{1/2} \le c\|u_0\|_2, \\ (ii) \ |||\nabla_x \int_0^t e^{i(t-t')\mathcal{L}} F(t') dt'|||_T \le c|||F|||_T', \end{cases}$$

where $\{Q_\mu\}_{\mu \in \mathbb{Z}^n}$ is a family cubes of side one with disjoint interiors covering $\mathbb{R}^n$, and $D = (-\Delta)^{1/2}$.

The local smoothing effect in (i) was proven by Constantin-Saut [CnSa], Sjölin [Sj], and Vega [Ve]. We proved the inhomogeneous version (ii) in [KePoVe2].

It is essential the gain of one derivative in (2.6) (ii). This allows to use the contraction principle in (2.5) and avoid the "loss of derivatives". However, the $|||\cdot|||$ norm forces the use of its dual

$$|||G|||'_T \equiv \|G\|_{l^1_\mu(L^2(Q_\mu\times[0,T]))} = \sum_{\mu\in\mathbb{Z}^n}\Big(\int_0^T\int_{Q_\mu}|G(x,t)|^2dxdt\Big)^{1/2}. \tag{2.7}$$

This factor cannot be made small by taking T small, except if $G(t)$ is small at $t=0$. It is here where the restriction on the size of the data appears.

In [HyOz], for the one dimensional case $n=1$, Hayashi-Ozawa removed the smallness assumption on the size of the data in [KePoVe2]. They used a change of variable to obtain an equivalent system with a nonlinear term independent of $\partial_x u$, which can be treated by the standard energy method.

In [Ch], for the elliptic case $\mathcal{L}=\Delta$, H. Chihara removed the size restriction on the data in any dimension. The change of variable in this case involves pseudo-differential operators ψ.d.o's. A main step in his proof is a diagonalization method in which the assumption on the ellipticity of $\mathcal{L}$ is essential.

In [KePoVe5], we removed the size restriction for the general form of the operator $\mathcal{L}$ in (2.1).

THEOREM 2.1 ([KePoVe5]). *There exist $s=s(n;P)>0$, and $m=m(n;P)>0$, such that for any $u_0\in H^s(\mathbb{R}^n)\cap L^2(\mathbb{R}^n:|x|^{2m}dx)$ the IVP (2.1) has a unique solution $u(\cdot)$ defined in the time interval $[0,T]$ satisfying that*

$$u\in C([0,T]:H^s(\mathbb{R}^n)\cap L^2(\mathbb{R}^n:|x|^{2m}dx)),\quad and\quad |||J^{s+1/2}u|||_T<\infty. \tag{2.8}$$

If $s'>s$, then the above results hold, with s' instead of s, in the same time interval $[0,T]$.

Moreover, the map data $\to$ solution from $H^s(\mathbb{R}^n)\cap L^2(\mathbb{R}^n:|x|^{2m}dx)$ into the class in (2.8) is locally continuous.-

Our argument of proof uses the Calderón-Vaillancourt class [CaVa]. This was suggested by the work of J. Takeuchi [Tk]. In order to extend the argument in [KePoVe2] to prove Theorem 2.1, we need to show that, under appropriate assumptions on the smoothness and decay of the coefficients $b_{k,j}=(b_{k,1},.,b_{k,n})$, $k=1,2$, $j=1,.,n$, the IVP for the linear Schrödinger equation with variable coefficient lower order terms

$$\begin{cases}\partial_t v=i\mathcal{L}v+b_1(x)\cdot\nabla_x v+b_2(x)\cdot\nabla_x\overline{v}+F(x,t), & t\in\mathbb{R},\ x\in\mathbb{R}^n,\\ v(x,0)=v_0\in H^s(\mathbb{R}^n),\end{cases} \tag{2.9}$$

has a unique solution $v\in C([0,T]:H^s(\mathbb{R}^n))$ such that

$$\sup_{[0,T]}\|v(t)\|_{H^s}+|||J^{s+1/2}v|||_T\le c(b_1;b_2;T)(\|v_0\|_{H^s}+|||J^{s-1/2}F|||'_T). \tag{2.10}$$

Equations of the form described in (2.1) with $\mathcal{L}$ non-elliptic arise in several situations. For example, in the study of water wave problems, Davey-Stewartson [DS], and Zakharov-Shulman [ZaSc] systems, in ferromagnetism, Ishimori system [Ic], as higher dimension completely integrable model, see [AbHa].

Consider the Davey-Stewartson (DS) system

$$\begin{cases} i\partial_t u + c_0\partial_x^2 u + \partial_y^2 u = c_1|u|^2u + c_2 u\partial_x\varphi, \\ \partial_x^2\varphi + c_3\partial_y^2\varphi = \partial_x|u|^2, \\ u(x,y,0) = u_0(x,y), \end{cases} \tag{2.11}$$

where $u = u(x,y,t)$ is a complex-valued function, $\varphi = \varphi(x,y,t)$ is a real-valued function, (when $(c_0,c_1,c_2,c_3) = (-1,1,-2,1)$ or $(1,-1,2,-1)$ the system in (1.1) is known in inverse scattering as the DSI and DSII respectively).

In the case $c_3 < 0$, $c_0 < 0$, (i.e. the equation in (1.6) is essentially not semi-linear, and the dispersive operator is non elliptic) the only available existence results are for analytic data, Hayashi-Saut [HySa], or "small" data, Linares-Ponce [LiPo]. For other results for the DS system we refer to [GhSa], [HySa], [LiPo] and references therein.

The IVP for the Ishimori system can be written as

$$\begin{cases} i\partial_t u + \partial_x^2 u \mp \partial_y^2 u = \frac{2\overline{u}((\partial_x u)^2 - (\partial_y u)^2)}{1+|u|^2} + ib(\partial_x\varphi\partial_y u - \partial_y\varphi\partial_x u), \\ \\ \partial_x^2\varphi \pm \partial_y^2\varphi = 4i\ \frac{\partial_x u\ \partial_y\overline{u} - \partial_x\overline{u}\ \partial_y u}{(1+|u|^2)^2}, \\ \\ u(x,y,0) = u_0(x,y). \end{cases} \tag{2.12}$$

The $(-,+)$ case was studied by A. Souyer [So]. The case $(+,-)$ in (2.11) was first studied by Hayashi-Saut [HySa] in a class of analytic functions which allowed them to obtain local and global existence for small analytic data. In [Hy2], N. Hayashi removed the analyticity assumptions in [HySa] by establishing the local existence and uniqueness of solutions of the IVP (2.12), for the case $(+,-)$, with small data u_0 in the weighted Sobolev space $H^4(\mathbb{R}^2) \cap L^2((x^2+y^2)^4 dxdy)$.

In a forthcoming article [KePoVe6] we remove the smallness assumption in [Hy2]. In particular, we prove the local existence and uniqueness of solutions of the IVP (2.12) with $(+,-)$ sign for data of arbitrary size in a weighted Sobolev space. Several problems have to be overcome to extend our approach in [KePoVe5] to this case. First, we have to deal with operators which are ψ.d.o. only in one variable. In particular, to establish the local smoothing effects described in (2.6) we shall need the operator valued version of the sharp Gårding inequality. Another difficulty of our approach is that for the linearized system associated to (2.12) the coefficients of the first order terms do not decay in both variables. One has terms of the form $a(x,y)\partial_x u$ where the coefficient $a(\cdot)$ is a smooth function with decay only in the x-variable. However, a careful analysis, consistent with Mizohata's condition in [Mz], shows that this one variable decay suffices.

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