Fourier Analysis of Null Forms and Non-linear Wave Equations

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Abstract. The non-linear terms of many equations, including Wave Maps and Yang-Mills have a special, "null", structure. In joint work with Sergiu Klainerman, I use techniques of Fourier Analysis, such as generalizations and refinements of the restriction theorem applied to null forms to study the optimal Sobolev space in which such non-linear wave equations are well posed.

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The following notation will be used: repeated indices are summed, $x^\alpha, 0 \le \alpha \le n$ are the coordinates $t, x^i, 1 \le i \le n$, ∇_α are the usual derivatives, and indeces are raised or lowered according to the Minkowski metric $-1, 1, \cdots, 1$ (i. e. raising or lowering the 0 index changes sign).

Wave maps are functions $\phi : \mathbb{R}^{n+1} \to M$ from Minkowski space $\mathbb{R}^{n+1}$ to a Riemannian manifold M with metric g which arise as critical points of the Lagrangian

$$\int_{R^{n+1}} \left(\nabla_\alpha \phi, \nabla^\alpha \phi\right)_g \tag{1}$$

The Euler-Lagrange equations of the above, written in coordiantes on M are

$$\Box \phi^i + \Gamma^i_{jk}(\phi)\big(\nabla_\alpha \phi^j, \nabla^\alpha \phi^k\big) = 0 \tag{2}$$

where $\Gamma^i_{j,k}$ are the Christoffel symbols. We see the first null form

$$Q_0(\phi, \psi) = \nabla_\alpha \phi \nabla^\alpha \psi$$

arising as part of the non-linear term. There is more going on than one can read off from (2). In the special case of $M = S^{k-1} \subset \mathbb{R}^k$ the equation (2) can also be written as

$$\Box \phi + \phi\big(\nabla_\alpha \phi \cdot \nabla^\alpha \phi\big) = 0 \tag{3}$$

with constraint $|\phi| = 1$. Take dot product with ϕ_t. Because of the constraint, the non-linear term drops out and we get conservation of energy, just as for the linear wave equation. In the special case of $n = 2$, it is still an open question whether the Cauchy problem (2) is well posed globally in time. Related to that is the question whether (2) is well posed locally in time for small data in H^1. Because of conservation of energy, such a result would prove global in time regularity for solutions of (2) with smooth data with small energy. There is a lot of evidence that using both the null condition and the geometric condition used in (3) the wave map equation should be well posed locally in time for Cauchy data in $H^{n/2}$. There has been a lot of work in recent years on this question. We currently have the result for $H^{n/2+\epsilon}$, see [K-M 4], [K-S], [G] . These results use only the null condition, and such a result fails by half a derivative for equations not satisfying the null condition, see [L]. Also, the sharp $H^{n/2}$ result cannot be true for general equations of the type (2), without using geometric information about the target manifold, as the example of geodesic solutions shows: Let $\gamma(t)$ be a geodesic on M which blows up in finite time, and let ψ be a solution of the homogeneous wave equations $\Box\psi = 0$ with $H^{n/2}$ data. Now, $\phi = \gamma(\psi)$ is a solution of (2). Since the supremum of ψ can become large, ϕ can blow up instantly.

The definitive result on equations of the type (2) which does not take the geometric condition into account is well posedness in the Besov space $B^{2,1}_{n/2}$, due to Daniel Tataru [T2]. For related applications of the geometric condition see [F-M-S], [Sh].

The Yang-Mills equations are non-linear analogues of the Maxwell equations. Let G be one of the classical compact Lie groups, and g its Lie algebra. The unknown is a connection potential $A_\alpha : \mathbb{R}^{n+1} \to g$, such that the corresponding covariant derivative $D_\alpha = \partial_\alpha + [A_\alpha,]$ satisfies

$$D^\alpha F_{\alpha,\beta} = 0$$

where the curvature $F_{\alpha,\beta} = [D_\alpha, D_\beta]$

Here we have gauge freedom: if A_α is a solution, and O is a G-valued function, then $OA_\alpha O^{-1} - \partial_\alpha O O^{-1}$ is also a solution. Thus we may impose an additional gauge condition on A_α. We choose the Coulomb gauge : $\partial^i A_i = 0$. Then we have

$$\Box A_i = -2[A_j, \partial_j A_i] + [A_j, \partial_i A_j] + \cdots$$

together with an elliptic equation for A_0. The dots turn out to be less important terms. We will now identify the null forms in the right hand side. They will involve $Q_{ij}(\phi, \psi) = \nabla_i\phi\nabla_j\psi - \nabla_j\phi\nabla_i\psi$. In fact using the divergence condition on A to express it as curl$B$, the first term is of the type $Q_{ij}(B, A)$. Similarly, the curl of the second term is of the type $Q_{ij}(A, A)$, which is all the information we need since the divergence of the whole right hand side is 0. Thus a simplified model for Yang-Mills is

$$\Box A = Q_{ij}((-\Delta)^{-1/2}A, A) + (-\Delta)^{-1/2}Q_{ij}(A, A) \tag{4}$$

The indeces of A are not important, and have been supressed.

The Yang-Mills equations in 3+1 dimensions are sub-critical. There is a conserved energy, and our local existence result implies that the time of existence of a smooth solution depends only on the energy of the initial data (and the solution stays as smooth as it started in this interval). The argument is complicated by gauge dependance, and the fact that energy differs form the H^1 norm by a lower order term, see [K-M3]. The global existence result was already known, due to Eardley and Moncrief [E-M]. However, our new techniques also give global existence in the energy space. It was shown by M. Keel [Ke], along the same lines, that there is global regularity for Yang-Mills coupled with a critical power Higgs field. This is a new global existence result, accessible only through our new local estimates.

In 4+1 dimensions, Yang-Mills are critical, and it was shown by Klainerman and Tataru that they are well posed in $H^{1+\epsilon}$ [K-T]. See also [K-M8] for a related result.

Following is a summary of the main estimates used in the above proofs.

Recall the classical Strichartz inequality gives the (optimal) estimate for a solution of $\Box\phi = 0$

$$\|(\nabla\phi)^2\|_{L^3(\mathbf{R}^3)} \leq C\big(\|\phi(0,\cdot)\|^2_{H^{3/2}(\mathbf{R}^2)} + \|\phi_t(0,\cdot)\|^2_{H^{1/2}(\mathbf{R}^2)}\big)$$

However, for a null form we have

$$\|Q(\phi,\phi)\|_{L^2(\mathbf{R}^3)} \leq C\big(\|\phi(0,\cdot)\|_{H^{5/4}(\mathbf{R}^2)} + \|\phi_t(0,\cdot)\|_{H^{1/4}(\mathbf{R}^2)}\big)$$

The proof is based on writing the L^2 norm of the quadratic form as the L^2 norm of a convolution of measures supported on the light cone, on the Fourier transform side. The symbol of the null form kills the worst singularity in the convolution. This has been generalized to the variable coefficient case by C. Sogge [So]. Some ideas in the proof were also used in [Sc-So].

Using this type of estimate one can prove that (2) is well posed in $H^{(n+1)/2}$, which is already non-trivial, is only true for equations satisfying some kind of null condition (for n=2, 3), but is not optimal. Also, the same techniques give local existence for finite energy data for Yang-Mills in 3+1 dimensions.

To get to the optimal result, that the Wave Map equation (2) is well posed in $H^{n/2+\epsilon}$ we have to make extensive use of the spaces $H_{s,\delta}$ used by Bourgain for KdV [B]; see also [Be]:

$$\|\phi\|_{s,\delta} \ = \|w_+^s w_-^\delta \tilde{\phi}\|_{L^2(d\tau d\xi)}$$

where $w_+(\tau,\xi) = 1+|\tau|+|\xi|$, $w_-(\tau,\xi) = 1+||\tau|-|\xi||$, and $\tilde{\phi}$ denotes the space-time Fourier transform. Also, let $D_\pm$ be the operator with symbol $w_\pm$. There are two advantages in working with these spaces. Functions in $H_{s,\delta}$ with $\delta > 1/2$ satisfy the same Strichartz-Pecher estimates that solutions of $\Box\phi = 0$ with H^s Cauchy data would.

In 3+1 dimensions, for instance,

$$\|\phi\|_{L^\infty(dt)L^2(dx)} \le C\|\phi\|_{0,\delta} \tag{5a}$$

is the energy estimate, and all estimates obtained by interpolating it with the (false) end-point result

$$\|\phi\|_{L^2(dt)L^\infty(dx)} \le C\|\phi\|_{1,\delta} \tag{5b}$$

are true.

Also, the argument is simplified if one also notices that, for $\delta < 1/2$ and p defined by $\frac{1}{p} = \frac{1}{2} - \delta$,

$$\|\phi\|_{L^p(dt)L^2(dx)} \le C\|\phi\|_{0,\delta} \tag{5c}$$

See [T1] for a general treatment of these spaces.

The second advantage of the spaces $H^{s,\delta}$ is that the solution to $\Box\phi = F$ with Cauchy data f_0, f_1 satisfies

$$\|\chi(t)\phi\|_{s,\delta} \le C\Big(\|F\|_{s-1,\delta-1} + \|f_0\|_{H^s} + \|f_1\|_{H^{s-1}}\Big)$$

where χ is a smooth cut-off function in time. In order to solve $\Box\phi = Q(\phi,\phi)$ for small time it suffices to solve the integral equation

$$\phi = \chi(t)\Big(W * Q + W(f_0) + \partial_t W(f_1)\Big) \tag{6}$$

W is the fundamental solution of $\Box$. This idea also goes back to Bourgain. See also [K-P-V].

In order to show that the equation (2) is well posed in H^s, for $s > 3/2$, in 3+1 dimensions, it suffices to prove an inequality of the form

$$\|Q_0(\phi,\psi)\|_{s-1,\delta-1} \le C\|\phi\|_{s,\delta}\|\psi\|_{s,\delta} \tag{7}$$

where $\delta > 1/2$

The symbol of the null form Q_0 is

$$\tau\lambda - \xi\cdot\eta = \frac{1}{2}\Big((\tau+\lambda)^2 - |\xi+\eta|^2 - \tau^2 + |\xi|^2 - \lambda^2 + |\eta|^2\Big)$$

Using this, the left hand side of (4) is dominated by the sum of terms, a typical one being

$$\|D_-^{\delta-1}((D_+^s D_-^{1/2}\phi)(D_+^{1/2}\psi))\|$$

Estimate this norm by duality, integrating against $F \in L^2$:

$$\int D_-^{\delta-1}((D_+^s D_-^{1/2}\phi)(D_+^{1/2}\psi))F$$
$$= \int (D_+^s D_-^{1/2}\phi)(D_+^{1/2}\psi))D_-^{\delta-1}F$$

The first term is in L^2, the second one in $H_{s-1/2,\delta}$ and the third one in $H_{0,1-\delta}$. Thus, it suffices to show $H_{0,1-\delta} \cdot H_{s-1/2,\delta} \subset L^2$. This is true, and follows from (5 a, b, c). In fact, for there exist p close to ∞, $q > 2$, close to 2, $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$ such that the first term is in $L^p(dt)L^2(dx)$ and the second one in $L^q(dt)L^\infty(dx)$. The original argument of [K-M4] used convolutions of measures.

An problem related to Yang-Mills, worked out in [K-M6], to show that the model

$$\Box\phi = Q_{ij}(\phi,\phi) \tag{8}$$

is well posed in $H^{3/2+\epsilon}$ in 3+1 dimensions.

The analogue of (7) is not true. There is an estimate for the symbol $|\xi \times \eta| \leq |\xi|^{1/2}|\eta|^{1/2}|\xi+\eta|^{1/2}(w_-(\tau,\xi)+w_-(\lambda,\eta)+w_-(\tau+\lambda,\xi+\eta)^{1/2}$, but after distributing derivatives as above one has to bound a troublesome term

$$\|D_-^{-1/2}((D_-^{1/2}D_+^{1/2}\phi)(D_+^s\psi))\|_{L^2}$$

By duality, this would correspond to an estimate

$$D^{-(s-1/2)}\Big(H_{0,1/2} \cdot H_{0,1/2}\Big) \subset L^2$$

($s > 3/2$). This is false, the counterexample is an adaptation of an old construction due to A. Knapp. There are other useful estimates along these lines which are true, and which are needed for (4), (8), see [K-M5], [K-T]. In 3+1 dimensions the (barely false) end-point estimates are are

$$D^{-1/2}\Big(H_{1/4,\delta} \cdot H_{1/4,\delta}\Big) \subset L^2 \tag{9a}$$

and

$$D^{-1}\Big(H_{1/2,\delta} \cdot H_{1/2,\delta}\Big) \subset L^1(dt)L^\infty(dx) \tag{9b}$$

Back to (5), we are foced to make stronger assumptions on our norms. A simplification of the original argument in [K-M6], used in [K-T], is to require, (modulo an ϵ) that, in addition to $\phi \in H_{s,1/2}$, ϕ should also satisfy

$$\|\phi\|_* = \inf\{\|F\|_{L^1(dt)L^\infty(dx)},\ |\widetilde{(D_-^{1/2}D_+^{1/2}\phi)}| \le |\tilde{F}|\} < \infty$$

These norms are constructed so that we recover (7)

$$\|Q_{ij}(\phi,\psi)\|_{s-1,\delta-1} \le C\big(\|\phi\|_{s,\delta} + \|\phi\|_*\big)\big(\|\psi\|_{s,\delta} + \|\psi\|_*\big) \tag{7'}$$

and it turns out also

$$\|D_+^{-1}D_-^{-1}Q_{ij}(\phi,\psi)\|_* \le C\big(\|\phi\|_{s,\delta} + \|\phi\|_*\big)\big(\|\psi\|_{s,\delta} + \|\psi\|_*\big)$$

These types of modified norms also work for the model Yang-Mills problem (4), to prove well posedness in $H^{1+\epsilon}$ in $4+1$ dimensions. To prove the necessary estimates one must use the analogues of 9a, 9b in $4+1$ dimensions.

References

[Be] M. Beals Self-spreading and strength of singularities for solutions to semilinear wave equations, Annals of Math 118, 1983 no1 187-214

[B] J. Bourgain Fourier transform restriction phenomena for certain lattice subsets and applications to non-linear evolution equations, I, II, Geom. Funct. Analysis 3, (1993), 107-156, 202-262.

[C-Z] D. Christodoulou, A. Tahvildar-Zadeh, On the regularity properties of spherically symmetric wave maps, Comm. Pure Appl. Math. 46, (1993) 1041-1091.

[F-M-S] A. Freire, S. Muller, M Struwe, Weak convergence of harmonic maps, Invent. Math. 130, 1997, no3, 589-617

[G] M. Grillakis, A priori estimates and regularity of non-linear waves, Proceedings of ICM, Zurich, 1994.

[H] F. Helein, Regularity of weakly harmonic maps from a surface into a manifold with symmetries, Manuscripta Math. 70, (1991), 203-218.

[Ke] M. Keel, Global existence for critical power Yang-Mills-Higgs equations in R^{3_1} Comm. in PDE 22 (1997) no 7-8 1161 – 1225

[K-P-V] K. Kenig, G. Ponce, L. Vega The Cauchy problem for the Korteveg-De Vries equation in Sobolev spaces of negative indices, Duke Math Journal 71, no 1, pp 1-21 (1994)

[K-M1] S. Klainerman and M. Machedon Space-time estimates for null forms and the local existence theorem, Comm. Pure Appl. Math, vol XLVI, 1221-1268 (1993)

[K-M2], S. Klainerman and M. Machedon On the Maxwell-Klein-Gordon equation with finite energy, Duke Math Journal, vol. 74, no. 1 (1994)

[K-M3] S. Klainerman and M. Machedon Finite energy solutions of the Yang-Mills equations in $\mathbf{R}^{3+1}$, Annals of Math. 142, 39-119 (1995)

[K-M4] S. Klainerman and M. Machedon Smoothing estimates for null forms and applications, Duke Math Journal, 81, no 1, in celebration of John Nash, 99-133 (1996) Also 1994 IMRN announcement.

[K-M5] S. Klainerman and M. Machedon with appendices by J. Bourgain and D. Tataru, Remark on the Strichartz inequality, International Math Research Notices no 5, 201-220 (1996).

[K-M6] S. Klainerman and M. Machedon Estimates for null forms and the spaces $H_{s,\delta}$ International Math Research Notices no 17, 853-865 (1996).

[K-M7] S. Klainerman and M. Machedon On the regularity properties of a model problem related to wave maps, Duke Math Journal 87, (1997) no 3, 553–589

[K-M8] S. Klainerman and M. Machedon On the optimal local regularity for gauge field theories, Differential and Integral Equations 10, (1997) no. 6, 1019–1030

[K-S] S. Klainerman, S. Selberg Remark on the optimal regularity for equations of Wave Maps type, Comm PDE. 22 (1997) no 5-6, 901-918

[K-T] S. Klainerman and D. Tataru, On the local regularity for Yang-Mills equations in $\mathbf{R}^4 + 1$, preprint

[L] H. Lindblad Counterexamples to local existence for semi-linear wave equations, Amer. J. Math, 118 (1996) no 1 1-16

[Sh] J. Shatah Weak Solutions and development of singularities for the $SU(2)\sigma$-Model, Comm. Pure Appl. Math, vol XLI 459-469 (1988)

[Sch-So] W. Schlag, C Sogge Local smoothing estimates related to the circular maximal theorem. Math Res. Letters 4 (1997) no 1-15

[So] C. Sogge On local existence for non-linear wave equation satisfying variable coefficient null condition, Comm PDE 18 (1993) n0o 11 1795-1821

[T1] D. Tataru, The X^s_θ spaces and unique continuation for solutions to the semilinear wave equation, Comm. PDE, 21 (5-6), 841-887 (1996)

[T2] D. Tataru, Local and global results for wave maps, preprint

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