Phenomena of Compensation and Estimates for Partial Differential Equations

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Abstract. Quantities like the Jacobian determinant of a mapping play an important role in several partial differential equations in Physics and Geometry. The algebraic structure of such nonlinearities allow to improve slightly the integrability or the regularity of these quantities, sometimes in a crucial way. Focused on the instance of $\frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}$, where a and $b \in H^1(\mathbb{R}^2)$, we review some results obtained on that quantity for 30 years and applications to partial differential equations arising in Geometry, in particular concerning the conformal parametrisations of constant mean curvature surfaces and the harmonic mappingss between Riemannian manifolds.

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For 30 years, many remarkable properties concerning some nonlinear quantities like Jacobian determinants of mappings or the scalar product of a divergencefree vector field by the gradient of a function has been observed and used. One instance is the continuity with respect to the weak convergence in L^2. The basic example is the following : if $a_k \rightharpoonup a$ weakly and $b_k \rightharpoonup b$ weakly in $H^1(\mathbb{R}^m)$, then $\{a_k, b_k\}_{\alpha\beta} := \frac{\partial a_k}{\partial x^\alpha}\frac{\partial b_k}{\partial x^\beta} - \frac{\partial a_k}{\partial x^\beta}\frac{\partial b_k}{\partial x^\alpha}$ converges to $\{a, b\}_{\alpha\beta}$ in the distribution sense. The discovery and the study of such properties is the subject of the theory of compensated compactness of F. Murat and L. Tartar [Mu], which became a powerful tool in the theory of homogeneisation and the study of quasiconvex functionals. These technics has been recently enlarged, after R. Di Perna, by P. Gérard [Gé] and L. Tartar [Ta2] independentely in a microlocal context.

We want to tell here a story parallel to compensated compactness' one.

1 H-surfaces

It began with the study of surfaces of constant mean curvature H in the Euclidean space $\mathbb{R}^3$. Let D^2 be the unit disk in the plane $\mathbb{R}^2$. A local conformal parametrisation $X \in H^1(D^2, \mathbb{R}^3)$ satisfies

$$\Delta X = 2H\frac{\partial X}{\partial x} \times \frac{\partial X}{\partial y} \text{ weakly in } H^1(D^2, \mathbb{R}^3), \tag{1}$$

where $V \times W$ is the standard vectorial product in $\mathbb{R}^3$. H. Wente proved that each weak solution of (1) is smooth ($\mathcal{C}^\infty$) [W1]. The crucial step of his proof this to prove that a solution of (1) is continuous. It relies on the particular structure of the right-hand side of (1). For instance, the first component :

$$\Delta X^1 = 2H\left(\frac{\partial X^2}{\partial x}\frac{\partial X^3}{\partial y} - \frac{\partial X^2}{\partial y}\frac{\partial X^3}{\partial x}\right)$$

is a Jacobian determinant. Later in the the beginning of the eighties, in papers from H. Wente [W2] and H. Brezis, J.-M. Coron [BrC], it became clear that the main point in Wente's proof relies on the following. Let $a, b \in H^1(D^2, \mathbb{R})$ and $\phi \in L^1(D^2, \mathbb{R})$ be a weak solution of

$$\left\{\begin{array}{rcll} -\Delta\phi & = & \{a,b\} := \dfrac{\partial a}{\partial x}\dfrac{\partial b}{\partial y} - \dfrac{\partial a}{\partial y}\dfrac{\partial b}{\partial x} & \text{on } D^2 \\ \phi & = & 0 & \text{on } \partial D^2. \end{array}\right. \tag{2}$$

Then ϕ is actually in $H^1(D^2) \cap \mathcal{C}^0(D^2)$ and we have the following : there exist some positive constants C_∞ and C_2 such that

$$||\phi||_{L^\infty} \leq C_\infty ||da||_{L^2} ||db||_{L^2}, \tag{3}$$

$$||d\phi||_{L^2} \leq C_2 ||da||_{L^2} ||db||_{L^2}. \tag{4}$$

Both estimations are not true in general if we replace the right hand side of (2) by an arbitrary bilinear function of a and b : we would then only obtain that $\phi \in W^{1,p} \cap L^q$ with $1 \leq p < 2$ and $1 \leq q < \infty$. Here the algebraic structure of $\{a, b\}$ is very important and allows us to do many manipulations such as

$$\{a,b\} = \frac{\partial}{\partial x}\left(a\frac{\partial b}{\partial y}\right) - \frac{\partial}{\partial y}\left(a\frac{\partial b}{\partial x}\right)$$

- the basic trick in the proof.

Remark *Estimates (3) and (4) lead to other inequalities, similar to the isoperimetric inequality in $\mathbb{R}^3$, see [BrC].*

2 Estimates in refined spaces

In the beginning of the eighties, L. Tartar observed other nice properties on $\{a, b\}$ in the framework of fluid dynamics [Ta1]. And in 1989, S. Müller showed that if u is any function in $W^{1,m}(\mathbb{R}^m, \mathbb{R}^m)$ such that $\det(du)$ is nonnegative a.e. , then, $\det(du)\log(1 + \det(du)) \in L^1(\mathbb{R}^m)$, which improves slightly the naive observation that $\det(du) \in L^1(\mathbb{R}^m)$ [Mü]. We say that $\det(du)$ is in $L^1\log L^1(\mathbb{R}^m)$. The proof of that fact relies also on the use of the isoperimetric inequality in $\mathbb{R}^m$. Notice that if $m = 2$ and $u = (a, b)$, then $\det(du)$ is just $\{a, b\}$.

A few time later, R. Coifman, P.-L. Lions, Y. Meyer and S. Semmes proved actually that if u is any function in $W^{1,m}(\mathbb{R}^m, \mathbb{R}^m)$, then $\det(du)$ belongs to the generalized *Hardy space* $\mathcal{H}^1(\mathbb{R}^m)$ [CLMS]. It includes S. Müller's result, for it was known that any nonnegative function in $\mathcal{H}^1(\mathbb{R}^m)$ is in $L^1\log L^1(\mathbb{R}^m)$. These authors obtained similar results: for instance, if $B \in L^2(\mathbb{R}^m, \mathbb{R}^m)$ is a divergence free vector field and $V \in H^1(\mathbb{R}^m, \mathbb{R})$, then

$$\nabla V.B \in \mathcal{H}^1(\mathbb{R}^m), \tag{5}$$

the exact analog of the "div-curl lemma" of F. Murat and L. Tartar [Mu].

To make sense it is worth to say what is the generalized Hardy space (see [St]). Several definition coexists. One is the following. Let $f \in L^1(\mathbb{R}^m)$, define

$$f^\star(x) := \sup_{t>0} \left| \int_{\mathbb{R}^m} f(x-y)\phi(\frac{y}{t})\frac{dy}{t^m} \right|,$$

where $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^m)$ is a function such that $\int_{\mathbb{R}^m} f\phi = 1$. Then

$$\mathcal{H}^1(\mathbb{R}^m) := \{f \in L^1(\mathbb{R}^m) / f^\star \in L^1(\mathbb{R}^m)\}.$$

We endow this space with the norm

$$||f||_{\mathcal{H}^1} = ||f||_{L^1} + ||f^\star||_{L^1}.$$

Notice that, through as theorem of C. Fefferman and E. Stein, $BMO(\mathbb{R}^m)$ is the dual space of $\mathcal{H}^1(\mathbb{R}^m)$ ([F], [FSt]). The main property of $\mathcal{H}^1(\mathbb{R}^m)$ is that there exists many linear operators (like the Riesz transform) which are continuous on L^p spaces for $1 < p < \infty$, but not on L^1. But these operators are continuous on $\mathcal{H}^1(\mathbb{R}^m)$.

3 Applications to partial differential equations in geometry

Many applications of these properties have been obtained in the theory of harmonic maps.

Harmonic maps into a sphere

A first example is my result on the regularity of weakly harmonic maps between a two dimensional domain Ω and the two-sphere $S^2 \subset \mathbb{R}^3$ [H1]. These are maps $u \in H^1(\Omega, S^2) := \{v \in H^1(\Omega, \mathbb{R}^3)/|v| = 1 \text{ a.e. }\}$ which are weak solutions of

$$\Delta u + u|du|^2 = 0, \text{ weakly in } H^1(\Omega, \mathbb{R}^3). \tag{6}$$

Here, no Jacobian determinant appears at first glance and the knowledge that $u|du|^2 \in L^1$ is unuseful. The point is to use another equivalent form of the equation which is the conservation law

$$\frac{\partial}{\partial x}\left(u \times \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(u \times \frac{\partial u}{\partial y}\right) = 0, \text{ weakly in } H^1(\Omega, \mathbb{R}^3). \tag{7}$$

This relation was already observed and used independentely by several authors ([Che], [Sh], [KRS]). Assume without loss of generality that Ω is simply connected. We can "integrate" this equation and we deduce that $\exists B \in H^1(\Omega, \mathbb{R}^3)$ such that

$$\left\{ \begin{array}{rcl} \dfrac{\partial B}{\partial x} & = & u \times \dfrac{\partial u}{\partial y} \\ \dfrac{\partial B}{\partial y} & = & -u \times \dfrac{\partial u}{\partial x}. \end{array} \right. \tag{8}$$

Now, using the fact that $|u|^2 = 1$ a.e., which implies that $\langle u, \frac{\partial u}{\partial x}\rangle = \langle u, \frac{\partial u}{\partial y}\rangle = 0$, we can rewrite (6) as

$$\begin{array}{rcl} -\Delta u^i & = & \langle u^i \dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial x}\rangle + \langle u^i \dfrac{\partial u}{\partial y}, \dfrac{\partial u}{\partial y}\rangle \\ & = & \langle u^i \dfrac{\partial u}{\partial x} - u\dfrac{\partial u^i}{\partial x}, \dfrac{\partial u}{\partial x}\rangle + \langle u^i \dfrac{\partial u}{\partial y} - u\dfrac{\partial u^i}{\partial y}, \dfrac{\partial u}{\partial y}\rangle. \end{array}$$

We recognize in the last expression components of $u \times \frac{\partial u}{\partial x}$ and $u \times \frac{\partial u}{\partial y}$. Thus, using (8),

$$-\Delta u^i = -\{u^j, B^k\} - \{B^j, u^k\}, \tag{9}$$

for any (i, j, k) which is a circular permutation of $(1, 2, 3)$. Now equation (9) is similar to (1) and allows us to prove continuity of u using Wente's estimate. The smoothness of u follows from the classical elliptic theory.

This result generalizes in a straightforward way if we replace the target manifold S^2 by a sphere of arbitrary dimension or a homogeneous manifold, once one realized that the conservation law (7) is a consequence of the symmetries of S^2, using *Noether's theorem* (see [H2]).

This result has also been extended to to the case where the domain Ω is also of higher dimension by L. C. Evans [E]. He proved that, if Ω is an open subset of $\mathbb{R}^m$ is a weakly stationary map into a sphere, then u is smooth in $\Omega \setminus \mathcal{S}$, where $\mathcal{S}$ is a closed subset whose Hausdorff measure of dimension $m - 2$ vanishes - a weakly stationary map is a weakly harmonic map satisfying the extra condition that $\int_\Omega |d(u \circ \phi_t)|^2 = \int_\Omega |du|^2 + o(t)$, for all smooth family of diffeomorphisms ϕ_t acting on Ω, such that ϕ_0 is the identity mapping.

Evans' proof relies on the same arguments, plus the following: the extra condition leads to a monotonicity formula which provides an estimate in BMO. On the other hand, equations like (9) gives estimates in Hardy spaces, through the results of [CLMS]. These estimates complete exactly because of the duality between $\mathcal{H}^1$ and BMO.

Remark *It is possible to avoid to use the difficult duality result about $\mathcal{H}^1$ and BMO by direct estimates obtained by S. Chanillo [Cha]. Even more recently, more direct proofs without using that duality has been constructed by P. Hajlasz, P. Strzelecki [HS] and A. Chang, L. Wang, P. Yang [CWY].*

Harmonic maps into arbitrary manifolds

It has been possible to extend the previous results for weakly harmonic maps into arbitrary manifolds $\mathcal{N}$. The difficulty is that in general $\mathcal{N}$ is not symmetric and we cannot apply Noether's theorem to construct conservation laws. In dimension 2, I did prove that weakly harmonic maps on a surface, into an arbitrary smooth compact manifold without boundary is smooth, generalizing the preceeding results for spheres [H3]. After, F. Bethuel generalized Evans' result to weakly stationary maps into arbitrary manifolds [Be].

Let $\mathcal{N}$ be a smooth compact Riemannian manifold without boundary. Thanks to the Nash-Moser theorem, we can assume that $\mathcal{N}$ is isometrically embedded in $\mathbb{R}^N$. We define $H^1(\Omega, \mathcal{N})$ to be the set of functions u in $H^1(\Omega, \mathbb{R}^N)$ such that $u \in \mathcal{N}$ a.e. Then weakly harmonic maps $u \in H^1(\Omega, \mathcal{N})$ are the solutions in the distribution sense of the system

$$\Delta u + A(u)(du, du) = 0, \tag{10}$$

where $A(u)(.,.)$ is the second fundamental form of the embedding of $\mathcal{N}$ in $\mathbb{R}^N$. It is a bilinear form on the tangent space to $\mathcal{N}$ at u, with values in the normal subspace to $\mathcal{N}$ at u. Such maps are critical points of the restriction of the functional

$$E(u) = \int_\Omega |du|^2 dx$$

on $H^1(\Omega, \mathcal{N})$. In proving regularity results, the point is to exploit the Euler-Lagrange equation with suitable test-functions, which in some sense are able to measure, to calibrate the possible wild behaviour of a given weak solution. One instance of wild behaviour we have in mind is like the map $(x, y) \longmapsto (\cos(\log(r)),$ $\sin(\log(r)),0)$, from $\mathbb{R}^2$ to S^2, where $r = \sqrt{x^2 + y^2}$: it is harmonic on $\mathbb{R}^2 \setminus \{0\}$ and its image turns along a great circle faster and faster as (x, y) goes to 0. One would like to prove that such a singularity (or something which looks asymptotically like that) does not exists (it actually has an infinite energy). So how to measure such a wild winding ? If $\mathcal{N}$ is S^2, we just take the test function $u \times \phi$, where $\phi \in H^1 \cap L^\infty(\Omega, \mathbb{R}^3)$ and we recover the trick given by Noether's theorem in writing the equation as the conservation law (7). In other cases, we need to construct test functions doing the same job, namely calibrating the possible winding of u. This obtained by using an orthonormal frame on $\mathcal{N}$, moving along u in the "more parallel way". This last requirement means that, although it is not possible in general to construct a covariantly parallel moving frame, it is possible to minimize the covariant derivative of that moving frame along u. The good news are that the obstruction for constructing a covariantly parallel moving frame along u is the curvature of $\mathcal{N}$ or more precisely the pull-back of the curvature two-form by u. But this pull-back is just a combination of two-order minors of the kind $\{a,b\}$, in the Hardy space! This is done by the following construction.

We start with a given smooth orthonormal moving frame $\tilde{e}(m) = (\tilde{e}_1, ..., \tilde{e}_n)(m)$ defined globally on $\mathcal{N}$ (m being here a point on $\mathcal{N}$), a smooth section of the bundle $\mathcal{F}$ of orthonormal tangent frames over $\mathcal{N}$. In many cases, such a section does not exists globally, because of topological obstructions. Nevertheless, it is possible to reduce ourself to such a situation, through some geometrical argument. Then, for any map $u \in H^1(\Omega, \mathcal{N})$, we consider the composed moving frame $\tilde{e} \circ u$, a section of the pull-back bundle $u^\star\mathcal{F}$, together with all the gauge transformations of $\tilde{e} \circ u$, i.e. for all $R \in H^1(\Omega, SO(n))$, we consider the new frame $e^R(z) = \tilde{e} \circ u(z).R(z)$ for a.e. $z \in \Omega$, or

$$e_a^R(z) = \sum_{b=1}^n \tilde{e}_b[u(z)].R_a^b(z).$$

We choose among all e^R's those who minimize the functional

$$F(e^R) := \int_\Omega \sum_{a,b=1}^n [\langle \frac{\partial e_a^R}{\partial x}, e_b^R \rangle^2 + \langle \frac{\partial e_a^R}{\partial y}, e_b^R \rangle^2] dxdy.$$

We call a *Coulomb moving frame* such a frame. It satisfies the Euler-Lagrange equation

$$\frac{\partial}{\partial x} \langle \frac{\partial e_a^R}{\partial x}, e_b^R \rangle + \frac{\partial}{\partial y} \langle \frac{\partial e_a^R}{\partial y}, e_b^R \rangle = 0, \tag{11}$$

another conservation law. This equation can be used as (7): some manipulations shows that $\exists A_b^a \in H^1(\Omega)$ such that

$$\begin{cases} \dfrac{\partial A_a^b}{\partial x} & = & \langle \dfrac{\partial e_a^R}{\partial y}, e_b^R \rangle \\ \dfrac{\partial A_a^b}{\partial y} & = & -\langle \dfrac{\partial e_a^R}{\partial x}, e_b^R \rangle, \end{cases}$$

and that ΔA_a^b is a sum of Jacobian determinants of the type $\{a,b\}$. Namely ΔA_a^b times the volume form on Ω is the pull-back by u of a closed two-form on $\mathcal{N}$ related to the curvature form. This improves slightly the regularity of e^R. In particular, we deduce that the L^2 connection coefficients $\langle \frac{\partial e_a^R}{\partial x}, e_b^R \rangle$ and $\langle \frac{\partial e_a^R}{\partial y}, e_b^R \rangle$ are in fact in the Lorentz space $L^{(2,1)}$, a slight refinement of the usual L^2 space (actually it is the dual space to $L^{(2,\infty)}$, known as weak L^2) (see [StW], [Hu], [BL]). Notice that the above construction did not use at all the hypothesis that u is weakly harmonic.

Now, if we assume that u is weakly harmonic, we will work with the projection of equation (10) on the Coulomb moving frame. We hence get a first order, Cauchy-Riemann system $\frac{\partial \alpha^a}{\partial \bar{z}} = \sum_{b=1}^{n} \omega_b^a \alpha^b$, where the α^a's are complex numbers representing the derivatives of u and the ω_b^a's are also complex numbers representing connection coefficients. The preliminary work on the Coulomb moving frame ensures us that the ω_b^a's are in $L^{(2,1)}$, instead of L^2. This is enough to prove that u is locally Lipschitz and then that u is smooth.

The regularity theorem of F. Bethuel combines in a delicate way these arguments and Evans' ones. For more details on all of that, see [Be] and [H4].

Conformal parametrisations of surfaces

In her thesis, T. Toro, proved the surprising (and difficult) result that the graph of a map ϕ in $H^2(\Omega, \mathbb{R})$, where Ω is an open subset of $\mathbb{R}^2$, is a Lipschitz submanifold, i.e. that there exists local bilipschitz parametrisations of the graph of ϕ. Actually she proved the more general result that this is true for any surface Σ whose mean curvature is a L^2 function on Σ [Tor]. Then, a simpler approach has been found by S. Müller and V. Švérak [MüŠ]. They proved that if Σ is a surface whose mean curvature function belongs to $L^2(\Sigma)$, then a conformal parametrisation of Σ is a bilipschitz function. Their result follows from the observation that, for a local conformal parametrisation $X : D^2 \longrightarrow \Sigma$, if we denote (e_1, e_2) an orthonormal frame such that $dX = e^f(e_1 dx + e_2 dy)$, then

$$\Delta f = u^\star \Omega, \tag{12}$$

where Ω is the curvature two-form on Σ. Thus Δf looks like a Jacobian determinant $\{a, b\}$ and the Wente estimate, or the Coifman, Lions, Meyer, Semmes results implies boundedness of f in L^∞, meaning that X is Lipschitz.

4 The best constants

Going back to Wente's result on the disk D^2, it is natural to generalize this inequality to arbitrary two-dimensional domain Ω in the plane, or on a Riemannian surface $(\mathcal{M}, g)$ and to look for the best constants in (3) and (4). If

$$-\Delta_g \phi = \{a, b\} \text{ on } \mathcal{M}, \tag{13}$$

we call

$$\begin{aligned}\mathcal{C}_\infty(\mathcal{M}, g) = \inf\{ & osc(\phi)/\phi \text{ is a solution of (13)},\\ & \text{where } (a,b) \in H^1(\mathcal{M}, \mathbb{R}^2), ||da||^2_{L^2} + ||db||^2_{L^2} = 2\},\end{aligned}$$

$$\begin{aligned}\mathcal{C}_2(\mathcal{M}, g) = \inf\{ & ||d\phi||^2_{L^2}/\phi \text{ is a solution of (13)},\\ & \text{where } (a,b) \in H^1(\mathcal{M}, \mathbb{R}^2), ||da||^2_{L^2} + ||db||^2_{L^2} = 2\}.\end{aligned}$$

A priori, $\mathcal{C}_\infty(\mathcal{M},g)$ and $\mathcal{C}_2(\mathcal{M},g)$ should depend on $\mathcal{M}$ and on the metric g. A first observation is that (13) is invariant under conformal transformations of $(\mathcal{M},g)$. Thus $\mathcal{C}_\infty(\mathcal{M},g)$ and $\mathcal{C}_2(\mathcal{M},g)$ depend only on the conformal structure of $(\mathcal{M},g)$. Moreover, F. Bethuel and J.-M. Ghidaglia proved that these constants were bounded by a universal one [BeG]

Recently the precise evaluation of these constants were completed by S. Baraket and P. Topping for $\mathcal{C}_\infty(\mathcal{M},g)$ [Ba], [Top] and by Y. Ge for $\mathcal{C}_2(\mathcal{M},g)$ [Ge]. We have that

- $\mathcal{C}_\infty(\mathcal{M},g)=\frac{1}{2\pi}$, for all $(\mathcal{M},g)$.
- $\mathcal{C}_2(\mathcal{M},g)=\sqrt{\frac{3}{16\pi}}$ if $\partial\mathcal{M}$ is non empty and $\mathcal{C}_2(\mathcal{M},g)=\sqrt{\frac{3}{32\pi}}$ if $\partial\mathcal{M}$ is empty.

Both result relies on the optimal isoperimetric inequality (on the plane for $\mathcal{C}_\infty(\Omega)$ and in $\mathbb{R}^3$ for $\mathcal{C}_2(\Omega)$).

Back to the beginning

The search for the optimal constant $\mathcal{C}_2(\mathcal{M},g)$ leads to a variational problem very similar to the search for the optimal constant in Sobolev embedding of $H^1(\mathbb{R}^m)$ in $L^{\frac{2m}{m-2}}(\mathbb{R}^m)$. First this problem is invariant under conformal transformations. Moreover critical points of the functional $||d\phi||^2_{L^2}$ under the constraint that $||da||^2_{L^2}+||db||^2_{L^2}=2$, satisfies the following Euler-Lagrange equation: there exists a Lagrange multiplier $\lambda\in(0,\infty)$ such that

$$u=\begin{pmatrix}\sqrt{\lambda}a\\ \sqrt{\lambda}b\\ \lambda\phi\end{pmatrix}$$

is a weak solution of

$$\Delta u=2\frac{\partial u}{\partial x}\times\frac{\partial u}{\partial y},$$

the equation of conformal parametrisations of constant mean curvature surfaces (see [H4], [Ge]). Hence we are led to another variational formulation of that geometrical problem. Y. Ge obtained several existence results on this problem, by constructing minimizing and non-minimizing solutions [Ge].

5 References

[Ba] S. Baraket, *Estimations of the best constant involving the L^∞ norm in Wente's inequality*, Ann. Fac, Sciences Toulouse (1997).

[BL] J. Bergh, J. Löfström, *Interpolation spaces, an introduction*, Springer Verlag Berlin, 1976.

[Be] F. Bethuel, *On the singular set of stationary harmonic maps*, Manuscripta Mathematica 78 (1993), 417-443.

[BeG] F. Bethuel, J.-M. Ghidaglia, *Improved regularity for solutions of elliptic equations involving Jacobian and applications*, J. Maths pures et Appliquées 72 (1993), 441-475.

[BrC] H. Brezis, J.-M. Coron, *Multiple solutions of H-systems and Rellich's conjecture*, Comm. Pure Appl. Math. 37 (1984), 149-187.

[CWY] S.-Y. A. Chang, L. Wang, P. C. Yang, *Regularity of harmonic maps*, preprint 1998.

[Cha] S. Chanillo, *Sobolev inequalities involving divergence free maps*, Comm. PDE's 16 (1991), 1969-1994.

[Che] Y. Chen, *The weak solutions to the evolution problems of harmonic maps*, Math. Zeitschrift 201 (1989), 69-74.

[CLMS] R. Coifman, P.-L. Lions, Y. Meyer, S. Semmes, *Compensated compactness and Hardy spaces*, J. Math. Pure Appl. 72 (1993), 247-286.

[E] L. C. Evans, *Partial regularity for stationary harmonic maps into spheres*, Arch. Rat. Mech. Anal. 116 (1991), 101-163.

[F] C. Fefferman, *Characterisation of bounded mean oscillations*, Bull. Am. Math. Soc. 77 (1971), 585-587.

[FSt] C. Fefferman, E. Stein, *H^p spaces of several variables*, Acta Mathematica 129 (1972), 137-193.

[Ge] Y. Ge, *Estimation of the best constant involving the L^2 norm in Wente's inequality and compact H-surfaces into Euclidean space*, COCV (1998).

[Gé] P. Gérard, *Microlocal defect measures*, preprint Unversité Paris-Sud (1990).

[HS] P. Hajlasz, P. Strzelecki, *Subelliptic p-harmonic maps into spheres and the ghost of Hardy spaces*, preprint 1997.

[H1] F. Hélein, *Régularité des applications faiblement harmoniques entre une surface et une sphère*, C. R. Acad. Sci. Paris 311 (1990), 519-524.

[H2] F. Hélein, *Regularity of weakly harmonic maps from a surface into a manifold with symmetries*, Manuscripta Mathematica 70 (1991), 293-318.

[H3] F. Hélein, *Régularité des applications faiblement harmoniques entre une surface et une variété riemannienne*, C. R. Acad. Sci. Paris 312 (1990), 591-596.

[H4] F. Hélein, *Applications harmoniques, lois de conservation et repères mobiles*, Diderot éditeur, 1996; *Harmonic maps, conservations laws and moving frames*, Diderot éditeur, 1997.

[Hu] R.A. Hunt, *On L(p,q) spaces*, L'enseignement mathématique XII (1966), 249-276.

[KRS] J. Keller, J. Rubinstein, P. Sternberg, *Reaction-diffusion processes and evolution to harmonic maps*, SIAM J. Appl. Math. 49 n. 6 (1989), 1722-1733.

[Mü] S. Müller, *Higher integrability of determinants and weak convergence in L^1*, J. Reine Angewandte Math. 412 (1990), 20-34.

[MüŠ] S. Müller, V. Švérak, *On surfaces of total finite curvature*, J. Diff. Geometry 42 (1995), 229-258.

[Mu] F. Murat, *Compacité par compensation*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 53 (1978), 1092-1099.

[Sh] J. Shatah, *Weak solutions and developments of singularities of the SU(2) σ-model*, Comm. Pure and Appl. Math. 41 (1988), 459-469.

[St] E. Stein, *Harmonic analysis*, Princeton University Press, Princeton, 1993.

[StW] E. Stein, G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton University Press, Princeton, 1970.

[Ta1] L. Tartar, *Remarks on oscillations and Stokes' equation*, in Lec. Notes in Physics 230, Springer 1984.

[Ta2] L. Tartar, *H-measures and applications*, in Proc. Int. Cong. Math. Kyoto, Japan 1990, MSJ (1991).

[Top] P. Topping, *The optimal constant in Wente's L^∞ estimates*, Comm. Math. Helv. (1997).

[Tor] T. Toro, *Surfaces with generalized second fundamental form in L^2 are Lipschitz manifolds*, J. Diff. Geometry 39 (1994), 65-101.

[W1] H. Wente, *An existence theorem for surfaces of constant mean curvature*, J. Math. Anal. Appl. 26 (1969), 318-344.

[W2] H. Wente, *Large solutions to the volume constraint Plateau problem*, Arch. Rat. Mech. Anal. 75 (1980), 59-77.

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