Induced Hyperbolicity for One-Dimensional Maps

Grzegorz Świątek[1]

Abstract. We present a review of Yoccoz partitions and their relation with induced dynamics. A new way of interpreting the construction is shown, based on external Yoccoz partitions. These are governed by linear dynamics and hence much easier to handle. As an example of the usefulness of this method we prove the following result: For almost every parameter c on the boundary of the Mandelbrot set, in the sense of the harmonic measure, the map z^2+c satisfies the Collet-Eckmann condition.

1991 Mathematics Subject Classification: 30C10, 30D05
Keywords and Phrases: Yoccoz partition, Collet-Eckmann condition

1 Yoccoz Partitions

1.1 Historical outline

In the early 1980s M. Jakobson proved a theorem which asserted that in the logistic family $x \to ax(1-x)$, the mapping has a probabilistic absolutely continuous invariant measure for a set of parameters a with positive measure, see [10]. The crucial step of the proof was the construction of an expanding map Φ, defined on the union of countably many intervals, so that on each connected component of its domain Φ was an iterate of the original quadratic map, and the range of this restriction was a fixed interval. Hence, from the original clearly non-expanding transformation, a uniformly expanding one was constructed by taking iterates in a piecewise fashion.

In the early 1990s J.-Ch. Yoccoz proved a theorem about local connectivity of Julia sets of some quadratic polynomials, including many Julia sets which contained the critical point. He also showed that for this class of polynomials certain combinatorial data, which for real maps reduces to the kneading sequence, determines that polynomial uniquely. The proof is based on the construction of partitions of the phase space which are not far from being Markov: most pieces are mapped in a univalent way onto other pieces, except for the pieces which contain the critical point. The partitions were subjected to a process of infinite inductive refinement under which the critical pieces shrank to the point.

It was then observed that a powerful tool for studying both unimodal maps of the interval and complex polynomials in the plane is obtained when these ideas

[1]Partially supported by NSF Grant DMS-9704368

are applied jointly. When appropriate iterations of the original polynomial are applied on pieces of Yoccoz partitions, the result is a map which is expanding on all pieces except for the one which contains the critical point. Such transformations are called *induced mappings*. When a sequence of increasingly refined partitions is considered, the critical branches disappear in the limit and an expanding map of Jakobson's type is obtained. The original construction of Jakobson required exclusion of unsuitable parameters without regard for their topological dynamics, although a set of positive measure was left at the end. The new approach based on Yoccoz' discovery works for all polynomials without neutral or attracting periodic orbits, with the only exception of a well defined class of infinitely tunable ones.

This, or a closely related approach, played the key role in the solution of several important problems. For real quadratic polynomials, and by extension for unimodal maps with non-degenerate critical point, the infinitely tunable case turned out to be quite manageable, due to the phenomenon of the so-called a priori bounds, discovered by D. Sullivan. The combination of inducing and a priori bounds was the basis of the result about local connectivity of Julia sets for all real unimodal polynomials, see [14]. The same ingredients, and another phenomenon related to inducing and known as the *decay of geometry*, were used in [7] to prove that periodic windows are dense in the logistic family. Alternative proofs of both results can be found in [15].

Finally, there is a new result we wish to present which use inducing applied to complex polynomials. Recall that the *Collet-Eckmann* condition for a quadratic polynomial $f_c(z) := z^2 + c$ is that

$$\lim\inf_{n\to\infty} \frac{\log|Df_c^n(c)|}{n} > 0 \; .$$

Let χ denote the harmonic measure on the boundary of the Mandelbrot set.

THEOREM 1.1 *For χ-almost every point c, $z^2 + c$ satisfies the Collet-Eckmann condition.*

Theorem 1.1 is joint with J. Graczyk and implies the result earlier announced by Graczyk, Smirnov and the author, that for χ-almost every c and every $\alpha > 0$

$$\sum_{i=0}^{\infty} |Df_c^i(c)|^{-\alpha} < \infty \; .$$

The Collet-Eckmann condition for rational maps has been studied, see [5] and [16]. One of the results of these papers is that a map which satisfies the Collet-Eckmann condition has the Julia set with Hausdorff dimension less than 2. Thus, we get a corollary complementary to a theorem of which asserts that a complex polynomial in the residual subset of the boundary of the Mandelbrot set has the Julia set with Hausdorff dimension equal to 2, see [17].

STRUCTURE OF THE PAPER. Section 1 is a review of results about Yoccoz partitions in the context of their connections with inducing. Section 2 contains some new material, including an outline of the proof of Theorem 1.1.

1.2 Refining of partitions

Suppose that f is a rational mapping of the Riemann sphere. Consider a set B_0 which is a union of disjoint Jordan domains $B_0^1, \cdots, B_0^k$ chosen is such a way that each B_0^i contains exactly one critical point of f. Furthermore, assume that $f^n(\partial B_0) \cap B_0 = \emptyset$ for all $n > 0$. The question of how B_0 can be found will be adressed in Section 2. For now, let us describe how this initial partition can be refined into an infinite sequence of partitions.

We will regard B_0^i, $i = 0, \cdots, k$ as *Yoccoz pieces* of order 0. Next, we proceed recursively. If B_m^i is a Yoccoz piece of order m, then any connected component of $f^{-1}(B_m^i)$ is a Yoccoz piece of order $m+1$.

Trivially, all Yoccoz pieces of order m are disjoint and each is mapped by f^m onto some B_0^i as a proper holomorphic map. A more interesting property is that if B_m^i and $B_{m'}^{i'}$ have a non-empty intersection, then one of these pieces is contained in the other. To see this, assume first that $m = 0$. Then, if the claim were violated, we would have $m' > 0$ and $B_{m'}^{i'} \cap \partial B_0^i \neq \emptyset$. But then after m' iterations that part of the boundary of B_0^i would come back to B_0, contrary to our hypothesis. In general, the claim follows by induction with respect to m.

The disjoint union of all Yoccoz pieces is often called a *Yoccoz puzzle*. The discovery of Yoccoz was proving that puzzles derived from a suitably chosen B_0 for quadratic polynomials are often generating. By "generating" we mean that for every point z of the Julia set of f, whenever the ω-limit set of z contains some critical points, one can find a nesting sequence of Yoccoz pieces which intersect down to $\{z\}$. For quadratic polynomials, this will happen whenever the polynomial is non-tunable with all periodic orbits repelling. In some cases, a single Yoccoz puzzle is not enough to be generating, but one can construct a sequence of Yoccoz puzzles and show that they are jointly generating. This happens in the proof of local connectivity of Julia sets for real unimodal polynomials, see [14].

Induced dynamics. The Yoccoz puzzle does not have canonical dynamics. We might try the natural map which sends $z \in B_m^i$ to $f(z) \in f(B_m^i)$ which is usually well defined since $f(B_m^i)$ is a Yoccoz piece of order $m-1$. This breaks down, however, when $m = 0$. The only way to continue is to "drop" z to some Yoccoz piece of higher order, and since a point typically belongs to infinitely many pieces, one faces a difficult choice. A general scheme for endowing Yoccoz pieces with dynamics is as follows.

Let us call a *Yoccoz partition* any collection $\mathcal{B}$ of pairwise disjoint Yoccoz pieces. On any element B_m^i choose an integer t_m^i subject to the condition $0 \leq t_m^i \leq m$. Then the map which is defined on the union of elements of the partition by

$$\phi(z) = f^{t_m^i}(z)$$

for $z \in B_m^i$ will be described as *induced* by f on the Yoccoz partition $\mathcal{B}$. The restriction of ϕ to any connected component of its domain will be called a *branch* of ϕ. This scheme is quite general. In particular, it incorporates various inducing constructions for unimodal maps, see [10] or [9]. If one applies these constructions to a unimodal real quadratic polynomial, one gets induced maps on Yoccoz partitions restricted to the real line.

Suppose now that ζ is a branch induced by f on some Yoccoz piece, and ϕ is an unrelated map induced on some Yoccoz partition. Then $\phi \circ \zeta$, considered on the set of all points where it is well defined, is automatically an induced map defined on another Yoccoz partition. The consequence of this remark is that once a single induced map on some Yoccoz partition was obtained, we can use it as a starting point for an "inducing construction" in which we "refine" branches by composing them with other induced maps. Everything which we can obtain in this way will still be induced on Yoccoz partitions. Authors often don't talk about a Yoccoz puzzle directly, but instead proceed to define an inducing process. If such a process begins with a map induced on a Yoccoz partition, it implicitly belongs to our framework.

GEOMETRY OF CRITICAL PIECES. Yoccoz pieces which contain a critical point of f will be designated as *critical.* The geometry of critical Yoccoz pieces has been particularly carefully investigated. A curious discovery in this area was the so-called *decay of geometry* for quadratic polynomials.

To explain the setting, recall the mapping f with critical points $Z_1, \cdots, Z_k$ and the corresponding system of order 0 Yoccoz pieces B_0^i, with $Z_i \in B_0^i$. We will regard B_0^i as the 0-th generation of critical pieces. Recursively, suppose that the p-th generation of critical Yoccoz pieces D_p^i with $D_p^i \ni Z_i$ has been defined. Then D_{p+1}^i is the largest piece inside D_p^i, not equal to D_p^i, which contains Z_i and is mapped onto some D_p^j by an iterate of f. This iterate will be denoted with $f^{r_{p+1}^i}$. For each $1 \leq i \leq k$ we choose a subsequence $p_q(i)$ with $p_1(i) = 1$ and

$$p_q(i) = \min\{p : \; r_p^i > r_{p_{q-1}(i)}^i\}$$

for $q > 1$. This set could be empty, in which case $p_q(i)$ are all undefined from some point on. A trivial reason why it might happen is when D_p^i are defined for only finitely many p. But D_p^i may also be defined for all p, and $p_q(i)$ may still be undefined from some q on if the map is tunable.

DECAY OF GEOMETRY. *Suppose that f is a quadratic polynomial and consider the sequence, perhaps finite, of critical pieces D_{p_q}. For every $\alpha > 0$ there is $C > 0$ for which the following estimate holds. If $mod(D_0 \setminus \overline{D}_1) \geq \alpha$, then for every $q \geq 1$ for which p_q is well defined*

$$mod(D_{p_q - 1} \setminus \overline{D}_{p_q}) \geq Cq \; .$$

This fact depends on f having only one simple critical point. It is may not be true when the critical point is degenerate, see [12], and is generally not true in the presence of more than one critical point, see [18]. In both counterexamples the sequence of moduli not only does not increase at a linear rate, but remains bounded for all q, which are infinitely many. The decay of geometry has been established by a sequence of works. In the form in which we state it, or equivalent, it was shown in [6] for Yoccoz pieces of real quadratic polynomials, in [15] for complex polynomials, and in [8] in a more general context of holomorphic box mappings. The decay of geometry plays a role in the proof the periodic windows are dense in the logistic family, see [7] and [15].

In his original approach Yoccoz relied on a weaker version of the decay of geometry, which simply states that whenever the sequence of critical pieces D_{p_q} is infinite, intersect down to $\{0\}$. Even this relies on exponent 2. While Yoccoz' work remains unpublished, a similar reasoning which shows the importance of exponent 2 is presented in [1] for cubic polynomials.

2 External Induced Maps

2.1 Construction

Invariant curves in the phase space. Let us go back to the question which was pushed aside in the previous section, and namely how to obtain the Yoccoz pieces of level 0. The construction will be described for polynomials of the form $f(z) = z^d + c$ with $d > 1$. Since there is only one piece of order 0, we will denote it with B_0 rather than B_0^1 which would in keeping with the notation of Section 1. A trivial solution would be to choose as B_0 the interior of a geometric disk centered at 0 with sufficiently large radius. Yoccoz partitions obtained in this way would not be interesting in the case when the Julia set is connected, since they would simply form a sequence of topological disks nesting down to the filled-in Julia set. A better construction, due to Yoccoz, uses external rays of the Julia set.

Historically, the idea of constructing dynamical partitions using some invariant curves had certainly appeared before Yoccoz. Back in [3], one finds a construction in which a chain of preimages of a point is considered. When these points are joined by arcs, and invariant curve can be obtained, which will often land at a periodic repelling point. External rays were considered more recently, [2]. In addition to providing an invariant family of curves, they show a connection with the Riemann map of the complement of the connectedness locus in the parameter space.

Construction of the initial partition. A fixed point of the map $z^d + c$ is called *admissible* if it is repelling and is the landing point of finitely many external rays, all of which are smooth and have non-zero external arguments. The parameter c is *admissible* provided that $z^d + c$ has an admissible fixed point. It should be emphasized that the orbit of c may escape to ∞; all that is needed for the construction to work is that c be admissible. Note that an admissible fixed point persists under a small perturbation of parameters, and that the external arguments of rays which converge at such a point don't change. Any repelling fixed point which is not the landing point of the ray with external argument 0 is going to be admissible unless a ray which lands there is not smooth. For $d = 2$, this restricts non-admissible parameters to those which lie on external rays of the Mandelbrot set landing on the boundary of the main hyperbolic component. The set of such external arguments is of zero measure, see [4], Remark C.5. For the definition of external rays and basic facts about them, see [13].

The rays which converge at an admissible fixed point divide the plane into several sectors, one of which, say S_0 contains 0. Choose a level curve of the Green function of the Julia set, of a level L large enough so that it separates 0 from ∞. The intersection of the interior connected component of the complement of this curve with S_0 is the initial Yoccoz piece B_0. It is trivial to verify that B_0 has all properties assumed in Section 1.

BOUNDARIES OF YOCCOZ PIECES. The boundary of every Yoccoz piece B^i_m is a finite union of analytic arcs, some of which are preimages of the rays while others are arcs of the level curve of level L/d^m. Assuming that the level of the critical point is less than L/d^m, each arc of the level curve intercepts rays with external arguments from an open interval. In this way, with every Yoccoz piece B^i_m we can associate its "external" piece b^i_m defined as the interior of the set of external arguments for which the corresponding rays hit the piece. Every external Yoccoz piece is a finite union of intervals. The set of external Yoccoz pieces still has the property that two of them are either disjoint, or one is contained in the other. Even more interesting is the connection between the dynamics induced by f on the Yoccoz pieces and the dynamics induced by the map $T(x) = dx \mod 1$ on external Yoccoz pieces. If $f^{m'}(B^i_m) = B^{i'}_{m-m'}$, then $T^{m'}(b^i_m) = b^{i'}_{m-m'}$.

INTRINSIC CONSTRUCTION OF EXTERNAL YOCCOZ PIECES. What is most striking is that external Yoccoz pieces can be constructed without reference do the dynamics of f, provided that B_0 has been chosen, which determines b_0, and that an external argument of 0 is known, say γ. When c is not in the connectedness locus, then 0 in fact has d external rays that meet it. The point γ can be chosen to be any of them and the choice will not affect the construction, since it is only $T(\gamma)$ that matters.

Assume first that c is not in the connectedness locus, at level $\lambda > 0$ with respect to the Green function of its Julia set, and choose $L >> \lambda$ to construct the initial Yoccoz piece B_0. Then b_0 establishes the set of external Yoccoz pieces of level 0. Now suppose that pieces b^i_m of level m have been defined. The critical piece of level $m+1$, which contains γ, is simply $T^{-1}(b^{i_0}_m)$ where $b^{i_0}_m \ni T(\gamma)$, provided that such $b^{i_0}_m$ exists. Any other piece is $T^{-1}_j(b^i_m)$ where T^{-1}_j is an inverse branch of T mapping onto $S^1 \setminus \{\gamma\}$. By induction, we check that this is the "right" construction in the following sense. For $m < \frac{\log(L/\lambda)}{\log d}$ each b^i_m is the set of external rays intercepted by some B^i_m, and the piece b^i_m which contains γ corresponds in this way to the critical piece of level m.

Now let c tend to the connectedness locus along a fixed external ray in the parameter plane. This means that the external argument of c in the *phase* plane of $z^d + c$ remains fixed, see [2], and hence γ can be chosen fixed. At the same time, λ tends to 0. Since γ and b_0 are fixed, external Yoccoz pieces and the dynamics induced on them by T don't change at all. The correspondence with regular Yoccoz pieces holds for larger and larger levels m. In the limit, for c in the closure of the ray and on the boundary of the connectedness locus, the correspondence holds for all m, unless Yoccoz pieces B^i_m undergo a discontinuous change (in Hausdorff topology). Such a change is only possible if the orbit of 0 hits the boundary of some Yoccoz piece, and hence of B_0. But now the limit value c_0 is in the connectedness locus, so the only point on the boundary of B_0 which can be met by the critical orbit is the fixed point. As we see, the correspondence holds except for a set of external rays of the connectedness locus with rational external arguments (recall that by custom external arguments are parametrized by the circle of *circumference* 1.)

SUMMARY OF THE CORRESPONDENCE. The foregoing discussion can be summarized as follows.

PROPOSITION 1 *Let γ be irrational and c be an admissible parameter in the intersection of the boundary of the connectedness locus with the closure of the external ray with argument γ. Consider an admissible fixed point q of $z^d + c$ and let b_0 denote the arc of external arguments of all rays contained in the sector which is cut off by two adjacent rays landing at q and contains* 0. *Then there is a one-to-one correspondence between Yoccoz pieces constructed from the rays converging at q and some equipotential curve and external Yoccoz pieces of the map $T(x) = dx$ mod 1 with distinguished point γ and with the order* 0 *piece b_0. Key properties of this correspondence are:*

- *it conjugates the action on Yoccoz pieces by f to the action on external Yoccoz pieces by T,*
- *it respects inclusion and inclusion with closure,*
- *critical Yoccoz pieces correspond exactly to the external pieces which contain γ.*

The usefulness of this correspondence lies in the fact that the dynamics of T is much simpler and hence external Yoccoz pieces can be kept track of more easily than Yoccoz pieces. Properties of external Yoccoz puzzles which hold typically with respect to the Lebesgue measure on γ translate to properties which are typically true with respect to the harmonic measure on the boundary of the Mandelbrot set. This is the main line of the proof of Theorem 1.1.

2.2 COLLET-ECKMANN CONDITION

We will now sketch main steps of the proof of Theorem 1.1. Consider a connected component A of the set of admissible parameters. In the quadratic case, the initial external Yoccoz piece b_0 remains fixed throughout A. Since the whole boundary of the Mandelbrot set, except for the boundary of main cardioid which is of zero harmonic measure, is in the admissible set, it will be enough to restrict our considerations once and for all to A. In b_0, pick a point γ. This corresponds to choosing an external ray of the Mandelbrot set. There is a map $\Gamma : b_0 \to A \cap \partial M$ well defined almost everywhere, which associates to γ the landing point of the external ray with argument $T(\gamma)$. The harmonic measure on $A \cap \partial M$ is simply $\Gamma_*(\lambda)$, where λ is the Lebesgue measure on the circle.

The main construction. For any $\gamma \in b_0$ and irrational consider the sequence d_q of all external pieces which contain γ, in the order of inclusion. Hence, $d_0 := b_0$. For c in the Mandelbrot set and in the closure of the external ray with argument γ, note the corresponding sequence of all critical pieces D_q. Let $k(q)$ be the smallest positive k for which $T^k(d_q) \ni \gamma$. Clearly, $k(q)$ form a non-decreasing sequence. An external Yoccoz piece b will be called *nested* if whenever $b \subset d_q$ for some q, then $\overline{b} \subset d_q$. An analogous definition applies to Yoccoz pieces induced by f and by Proposition 1 the correspondence between external and regular pieces observes the property of being nested.

Proposition 2 *There is a set $A' \subset A$, with $A \setminus A'$ of zero measure, with the following properties. For every $\gamma \in A'$, infinitely many critical pieces are defined. Furthermore, for every such γ there are a positive integer m_0, a finite Yoccoz partition $\mathcal{B}$ with all pieces nested, and a sequence n_j with the following properties:*

1. *for every j, $T^{k(n_j)}$ maps d_{n_j} onto some d_q, with $q \leq m_0$; moreover, $T^{k(n_j)}(\gamma)$ belongs to some element of $\mathcal{B}$,*
2. $$\lim_{j\to\infty} \frac{k(n_{j+1})}{k(n_j)} = 1 \ ,$$
3. $$\limsup_{j\to\infty} \frac{k(n_j)}{j} < \infty \ .$$

Derivation of the Collet-Eckmann condition. These conditions are seen to imply the Collet-Eckmann condition at all points c in the boundary of the Mandelbrot set intersected with the closure of the external ray with argument γ. The first condition is easily understood in terms of the corresponding maps induced by f. It says that for each j the critical piece D_{n_j} is mapped by $f_c^{k(n_j)}$ onto a large D_q, while 0 is mapped into a piece which corresponds to an element of $\mathcal{B}$ in the sense of Proposition 1. Since $\mathcal{B}$ consists of nested pieces and is finite, it follows that $f^{k(n_j)}(0)$ is separated from the boundary of D_q by some positive distance independent of j. This plays a double role. First, the distortion at the critical value c is bounded, so that

$$\log |Df^{k(n_j)-1}(c)| \geq K_1 \operatorname{mod}(D_{m_0} \setminus \overline{D}_{n_j}) \tag{1}$$

with $K_1 > 0$. Secondly, since $f^{k(n_j)}(0)$ gets into a nested piece of some fixed Yoccoz partition, the modulus surrounding this piece in D_q will be pulled back, resulting in

$$\operatorname{mod}(D_{n_j} \setminus \overline{D}_{n_j+q_0}) \geq K_2$$

for all j, K_2 positive and q_0 constant. Hence,

$$\liminf \frac{\operatorname{mod}(D_{m_0} \setminus \overline{D}_{n_j})}{j} > 0 \, .$$

Together with the last assertion of Proposition 2 this implies

$$\liminf \frac{\operatorname{mod}(D_{m_0} \setminus \overline{D}_{n_j})}{k(n_j)} > 0 \, .$$

From this and estimate (1) we see that

$$\liminf \frac{\log |Df^{k(n_j)-1}(c)|}{k(n_j)} > 0 \, .$$

The second claim of Proposition 2 easily shows that the same condition in fact holds for all k, not just the subsequence $k(n_j)$, and hence the Collet-Eckmann condition is established.

A PROBABILISTIC REASONING. As to the proof of Proposition 2, it is useful to think of the "excursion times" from the large scale, $k(n_{j+1})-k(n_j)$, as independent random variables. Let us also forget at first that γ has to be mapped into an element of $\mathcal{B}$, instead just try to make sure that $T^{k(n_j)}$ maps to the large scale. The probabilistic interpretation is intuitively natural, because the next excursion depends mostly on $T^{k(n_j)}(\gamma)$, and that it is independent of the length of previous excursions, since they all end with γ mapped somewhere in the large box with uniform distribution. With that interpretation, it is enough to see that these random variables have finite expectation and variance and use the strong law of large numbers. These estimates follows rather easily once $\mathcal{B}$ is chosen carefully. The interpretation of the excursion times as independent random variables requires a careful construction. Details will be provided in a forthcoming paper.

THE ROLE OF EXPONENT 2. Theorem 1.1 is stated only for the quadratic family. However, the proof does not rely on exponent 2 is an essential way. In particular, the decay of geometry of critical Yoccoz pieces is never used a priori, it simply follows from the conditions of Proposition 2. The reason why the theorem is not stated for the family $z^d + c$ is that certain basic facts remain to be checked in this more general case. For example, it should be proved that almost all parameters in the complement of the connectedness locus are admissible.

REFERENCES

[1] Branner, B. & Hubbard, J.H.: *The iteration of cubic polynomials. Part II: patterns and parapatterns*, Acta Math. 169 (1992), 229-325

[2] Douady, A.: *Systemès Dynamiques Holomorphes*, Astérisque 105 (1982), 39-63

[3] Fatou, P.: *Sur les équations fonctionelles*, Bull. Soc. Math. Fr. 47 (1919), 161-271; 48 (1920), 33-94; 208-314

[4] Goldberg, L. & Milnor, J.: *Fixed points of polynomial maps. Part II. Fixed point portraits*, Ann. Sc. Éc. Norm. Sup. 26 (1993), 51-98

[5] Graczyk, J. & Smirnov, S.: *Collet, Eckmann, & Hölder*, Invent. Math. 133 (1998), 69-96

[6] Graczyk, J. & Świątek, G.: *Induced expansion for quadratic polynomials*, Ann. Scient. Éc. Norm. Sup. 29 (1996), 399-482

[7] Graczyk, J. & Świątek, G.: *Generic hyperbolicity in the logistic family*, Ann. of Math 146 (1997), 1-56

[8] Graczyk, J. & Świątek, G.: *Holomorphic box mappings*, preprint IHES (1996), to appear in Astérisque

[9] Guckenheimer, J. & Johnson, S.: *Distortion of S-unimodal maps*, Ann. of Math. 132 (1990), 71-130

[10] Jakobson, M.: *Absolutely continuous invariant measures for one-parameter families of one-dimensional maps*, Commun. Math. Phys. 81 (1981), 39-88

[11] Jakobson, M. & Świątek, G.: *Metric properties of non-renormalizable S-unimodal maps*, Ergod. Th. Dyn. Sys. 14 (1994), 721-755

[12] Keller, G. & Nowicki, T.: *Fibonacci maps re(al) visited*, Erg. Th. & Dyn. Sys. 15 (1995), 97-130

[13] Levin, G. & Przytycki, F.: *External rays to periodic points*, Israel Jour. Math. 94 (1995), 29-57

[14] Levin, G. & Van Strien, S.: *Local connectivity of the Julia set of real polynomials*, manuscript (1994), to appear in Ann. of Math.

[15] Lyubich, M.: *Dynamics of complex polynomials, part I-II*, Acta Math., 178 (1997), 185-297

[16] Przytycki, F. & Rohde, S.: *Porosity of Collet-Eckmann Julia sets*, Fund. Math. 155 (1998), 189-199

[17] Shishikura, M.: *The Hausdorff dimension of the boundary of the Mandelbrot set and Julia sets*, Ann. of Math. 147 (1998), 225-267

[18] Świątek, G. & Vargas, E.: *Decay of geometry in the cubic family*, Penn State preprint (1996), to appear in Erg. Th. Dyn. Sys.

Grzegorz Świątek
The Pennsylvania State University
Mathematics Department
209 Mc Allister
University Park, PA 16802, USA