Rigidity and Inflexibility in Conformal Dynamics

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1 Introduction

This paper presents a connection between the rigidity of hyperbolic 3-manifolds and universal scaling phenomena in dynamics.

We begin by stating an inflexibility theorem for 3-manifolds of infinite volume, generalizing Mostow rigidity (§2). We then connect this inflexibility to dynamics and discuss:

- The geometrization of 3-manifolds which fiber over the circle (§2);
- The renormalization of unimodal maps $f : [0,1] \to [0,1]$ (§4),
- Real-analytic circle homeomorphisms with critical points (§5), and
- The self-similarity of Siegel disks (§6).

Chaotic sets for these four examples are shown in Figure 1. The snowflake in the first frame is the limit set Λ of a Kleinian group Γ acting on the Riemann sphere $S^2_\infty = \partial\mathbb{H}^3$. Its center c is a *deep point* of Λ, meaning the limit set is very dense at microscopic scales near c. Because of the inflexibility and combinatorial periodicity of $M = \mathbb{H}^3/\Gamma$, the limit set is also self-similar at c with a universal scaling factor.

The remaining three frames show deep points of the (filled) Julia set for other conformal dynamical systems: the Feigenbaum polynomial, a critical circle map and the golden ratio Siegel disk. Our goal is to explain an inflexibility theory that leads to universal scaling factors and convergence of renormalization for these examples as well.

The qualitative theory of dynamical systems, initiated by Poincaré in his study of celestial mechanics, seeks to model and classify stable regimes, where the topological form of the dynamics is locally constant. In the late 1970s physicists discovered a rich, universal structure in the onset of instability. One-dimensional dynamical systems emerged as elementary models for critical phenomena, phase transitions and renormalization.

In pure mathematics, Mostow and others have developed a rigidity theory for compact manifolds M^n of constant negative curvature, $n \geq 3$, and other quotients of symmetric spaces. This theory shows M is determined up to isometry by $\pi_1(M)$

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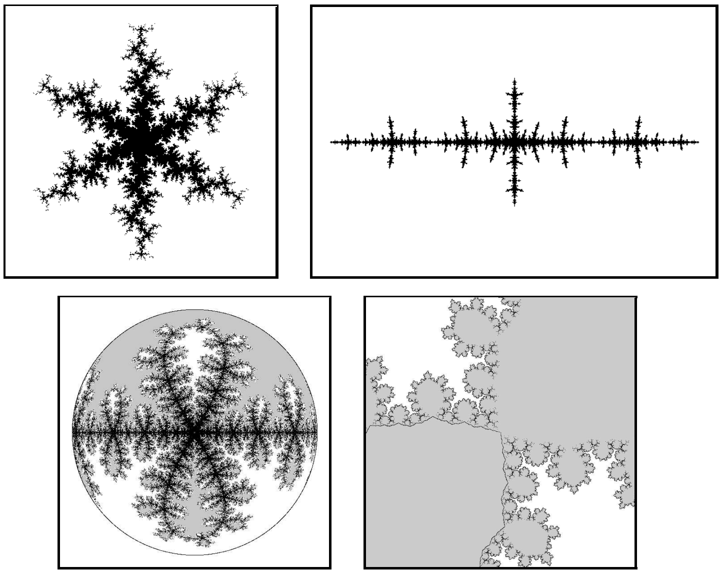

Figure 1. Dynamical systems with deep points: a totally degenerate Kleinian group, the Feigenbaum polynomial, a critical circle map and the golden mean Siegel disk.

as an abstract, finitely-presented group. Remarkably, rigidity of M is established via the ergodic theory of $\pi_1(M)$ acting on the boundary of the universal cover of M.

In our case, $M = \mathbb{H}^3/\Gamma$ is a hyperbolic 3-manifold, the boundary of its universal cover $\mathbb{H}^3$ is isomorphic to S^2, and the action of $\pi_1(M) \subset \mathrm{Isom}^+(\mathbb{H}^3) = PSL_2(\mathbb{C})$ on S^2 is conformal. Similarly, upon complexification, 1-dimensional dynamical systems give rise to holomorphic maps on the Riemann sphere $\widehat{\mathbb{C}} \cong S^2$. Hyperbolic space $\mathbb{H}^3$ enters the dynamical picture as a means to organize *geometric limits* under rescaling (§3). The universality observed by physicists can then be understood, as in the case of 3-manifolds, in terms of rigidity of these geometric limits.

We conclude with progress towards the classification of hyperbolic manifolds (§7), where geometric limits also play a central role.

2 Hyperbolic 3-manifolds and fibrations

A *hyperbolic manifold* is a complete Riemannian manifold with a metric of constant curvature -1. Mostow rigidity states that any two closed, homotopy equivalent hyperbolic 3-manifolds are actually isometric.

In this section we discuss a remnant of rigidity for *open* manifolds. Let $\mathrm{core}(M) \subset M$ denote the *convex core* of M, defined as the closure of the set of geodesic loops in M. The manifold M satisfies $[r, R]$-*injectivity bounds*, $r > 0$, if for any $p \in \mathrm{core}(M)$, the largest embedded ball $B(p, s) \subset M$ has radius $s \in [r, R]$.

Let $f : M \to N$ be a homotopy equivalence between a pair of hyperbolic 3-manifolds. Then f is a K-*quasi-isometry* if, when lifted to the universal covers,

$$\begin{aligned} \operatorname{diam}(\widetilde{f}(B)) &\leq K(\operatorname{diam} B + 1) \quad \forall B \subset \widetilde{M}, \quad \text{and} \\ \operatorname{diam}(\widetilde{f}^{-1}(B)) &\leq K(\operatorname{diam} B + 1) \quad \forall B \subset \widetilde{N}. \end{aligned}$$

A diffeomorphism $f : M \to N$ is an *asymptotic isometry* if f is exponentially close to an isometry deep in the convex core. That is, there is an $A > 1$ such that for any nonzero vector $v \in T_p M$, $p \in \operatorname{core}(M)$, we have

$$\left| \log \frac{|Df(v)|}{|v|} \right| \leq C A^{-d(p, \partial \operatorname{core}(M))}.$$

In [Mc2] we show:

Theorem 2.1 (Geometric Inflexibility) *Let M and N be quasi-isometric hyperbolic 3-manifolds with injectivity bounds. Then M and N are asymptotically isometric.*

Mostow rigidity is a special case: if M and N are closed, then any homotopy equivalence $M \sim N$ is a quasi-isometry, injectivity bounds are automatic, and $\partial \operatorname{core}(M) = \emptyset$, so an asymptotic isometry is an isometry.

To sketch the proof of Theorem 2.1, recall any hyperbolic 3-manifold M determines a conformal dynamical system, namely the action of its fundamental group $\pi_1(M)$ on the sphere at infinity $S^2_\infty = \partial \mathbb{H}^3$ for the universal cover $\widetilde{M} \cong \mathbb{H}^3$. The *limit set* $\Lambda \subset S^2_\infty$ is the chaotic locus for this action; its convex hull covers the core of M. The action is properly discontinuously on the rest of the sphere, and the quotient $\partial M = (S^2_\infty - \Lambda)/\pi_1(M)$ gives a natural Riemann surface at infinity for M.

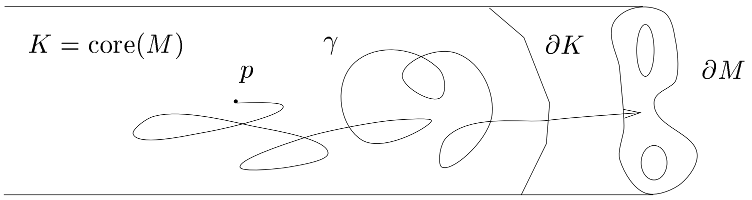

Figure 2. An observer deep in the convex core sees a kaleidoscopic view of ∂M.

A quasi-isometric deformation of M determines a quasiconformal deformation v of ∂M, which in turn admits a (harmonic) visual extension V to an equivalent deformation of M. The strain $SV(p)$ is the average of the ellipse field $Sv = \overline{\partial} v$ over all visual rays γ from p to ∂M. By our injectivity bounds, γ corkscrews chaotically before exiting the convex core. Thus the ellipses of Sv on ∂M appear in random orientations as seen from p (Figure 2). This randomness provides abundant cancellation in the visual average, and we find the metric distortion

$\|SV(p)\|$ is exponentially small compared to $\|Sv\|_\infty$. Thus V is an infinitesimal asymptotic isometry.

In dimension 3, any two quasi-isometric hyperbolic manifolds are connected by a smooth path in the deformation space, so the global theorem follows from the infinitesimal version.

Inflexibility is also manifest on the sphere at infinity. Let us say a local homeomorphism ϕ on $S^2_\infty \cong \widehat{\mathbb{C}}$ is $C^{1+\alpha}$*-conformal* at z if the complex derivative $\phi'(z)$ exists and

$$\phi(z+t) = \phi(z) + \phi'(z) \cdot t + O(|t|^{1+\alpha}).$$

We say $x \in \Lambda \subset S^2_\infty$ is a *deep point* if Λ is so dense at x that for some $\beta > 0$,

$$B(y,s) \subset B(x,r) - \Lambda \implies s = O(r^{1+\beta}).$$

It is easy to see that a geodesic ray $\gamma \subset \mathbb{H}^3$ terminating at a deep point in the limit set penetrates the convex hull of Λ at a linear rate. ¿From the inflexibility theorem we find:

Corollary 2.2 *Let M and N satisfy injectivity bounds, and let $\phi : S^2_\infty \to S^2_\infty$ be a quasiconformal conjugacy between $\pi_1(M)$ and $\pi_1(N)$. Then ϕ is $C^{1+\alpha}$-conformal at every deep point of the limit set of $\pi_1(M)$.*

The inflexibility theorem is motivated by the following application to 3-manifolds that fiber over the circle. Let S be a closed surface of genus $g \geq 2$ and let $\psi \in \mathrm{Mod}(S)$ be a pseudo-Anosov mapping class. Let

$$T_\psi \;=\; S \times [0,1]/\{(x,0) \sim (\psi(x),1)\}$$

be the 3-manifold fibering over the circle with fiber S and monodromy ψ. By a deep theorem of Thurston, T_ψ is hyperbolic. To find its hyperbolic structure, let $V(S)$ denote the variety of representations $\rho : \pi_1(S) \to \mathrm{Isom}(\mathbb{H}^3)$, and define

$$\mathcal{R} : V(S) \to V(S)$$

by $\mathcal{R}(\rho) = \rho \circ \psi_*^{-1}$. We refer to $\mathcal{R}$ as a *renormalization operator*, because it does not change the group action on $\mathbb{H}^3$, only its marking by $\pi_1(S)$.

Let $QF(S) \overset{Q}{\cong} \mathrm{Teich}(S) \times \mathrm{Teich}(\overline{S}) \subset V(S)$ denote the space of quasifuchsian groups, and define

$$M(X,\psi) = \lim_{n\to\infty} Q(X, \psi^{-n} Y), \quad \text{for any } (X,Y) \in \mathrm{Teich}(S) \times \mathrm{Teich}(\overline{S}).$$

Then $M = M(X,\psi)$ has injectivity bounds, its convex core is homeomorphic to $S \times [0,\infty)$, and the manifolds M and $\mathcal{R}(M)$ are quasi-isometric. By the inflexibility theorem there is an asymptotic isometry $\Psi : M \to M$ in the homotopy class of ψ, so the convex core of M is asymptotically periodic. As n tends to ∞, the marking of $\mathcal{R}^n(M)$ moves into the convex core at a linear rate, and we find:

Theorem 2.3 *The renormalizations $\mathcal{R}^n(M(X,\psi))$ converge exponentially fast to a fixed-point M_ψ of $\mathcal{R}$.*

Since $\mathcal{R}(M_\psi) = M_\psi$, the map ψ is realized by an isometry α on M_ψ, and the quotient $T_\psi = M_\psi/\langle\alpha\rangle$ gives the desired hyperbolic structure on the mapping cylinder of ψ.

This iterative construction of T_ψ hints at a dynamical theory of the action of $\mathrm{Mod}(S)$ on the variety $V(S)$, as does the following result [Kap]:

Theorem 2.4 (Kapovich) *The derivative $D\mathcal{R}_\psi$ is hyperbolic on the tangent space to $V(S)$ at M_ψ for all pseudo-Anosov mapping classes on closed surfaces.*

The snowflake in the first frame of Figure 1 is a concrete example of the limit set Λ for a Kleinian group $\Gamma = \pi_1(M(X,\psi))$ as above. In this example S is a torus, made hyperbolic by introducing a single orbifold point $p \in S$ of order 3; and $\psi = \left(\begin{smallmatrix}1&1\\1&2\end{smallmatrix}\right) \in SL_2(\mathbb{Z}) \cong \mathrm{Mod}(S)$ is the simplest pseudo-Anosov map. The suspension of $p \in S$ gives a singular geodesic $\gamma \subset T_\psi$ forming the orbifold locus of the mapping torus of ψ.

The picture is centered at a deep point $c \in \Lambda$ fixed by an elliptic element of order 3 in Γ. The limit set Λ is a nowhere dense but very furry tree, with six limbs meeting at c. By general results, Λ is a locally connected dendrite, with Hausdorff dimension two but measure zero [CaTh], [Th1, Ch. 8], [Sul1], [BJ1]; in fact by [BJ2] we have $0 < \mu_h(\Lambda) < \infty$ for the gauge function $h(r) = r^2|\log r \log\log\log r|^{1/2}$.

One can easily construct a quasiconformal automorphism ϕ of Γ, with $\phi(c) = c$ and $\phi\circ\gamma = \psi_*(\gamma)\circ\phi$ for all $\gamma \in \Gamma$. By Corollary 2.2, ϕ is $C^{1+\alpha}$-conformal at c, and we find:

Theorem 2.5 *The limit set Λ is self-similar at each elliptic fixed-point in Λ, with scaling factor $\phi'(c) = e^L$. Here L is the complex length of the singular geodesic γ on T_ψ.*

In particular the self-similarity factor e^L is inherited from the geometry of the rigid manifold T_ψ, and it is universal across all manifolds $M(X,\psi)$ attracted to M_ψ under renormalization.

3 Geometric limits in dynamics

In this section we extend the inflexibility of Kleinian groups and their limit sets to certain other conformal dynamical systems $\mathcal{F}$ and their Julia sets J, where we will find:

> *The conformal structure at the deep points of J is determined by the topological dynamics of $\mathcal{F}$.*

Consider the space $\mathcal{H}$ of all holomorphic maps $f : U(f) \to V(f)$ between domains in $\widehat{\mathbb{C}}$. Introduce a (non-Hausdorff) topology on $\mathcal{H}$ such that $f_n \to f$ if for any compact $K \subset U(f)$, we have $K \subset U(f_n)$ for all $n \gg 0$ and $f_n|K \to f|K$ uniformly.

A *holomorphic dynamical system* is a subset $\mathcal{F} \subset \mathcal{H}$. Given a sequence of dynamical systems $\mathcal{F}_n \subset \mathcal{H}$, the *geometric limit* $\mathcal{F} = \limsup \mathcal{F}_n$ consists of all maps $f = \lim f_{n_i}$ obtained as limits of subsequences $f_{n_i} \in \mathcal{F}_{n_i}$.

To bring hyperbolic space into the picture, identify $\widehat{\mathbb{C}}$ with the boundary of the Poincaré ball model for $\mathbb{H}^3$, let $\mathrm{F}\mathbb{H}^3$ be its frame bundle, and let $\omega_0 \in \mathrm{F}\mathbb{H}^3$ be a standard frame at the center of the ball. Given any other $\omega \in \mathrm{F}\mathbb{H}^3$, there is a unique Möbius transformation g sending ω_0 to ω, and we define

$$(\mathcal{F}, \omega) = g^*(\mathcal{F}) = \{g^{-1} \circ f \circ g \;:\; f \in \mathcal{F}\}.$$

In other words, $(\mathcal{F}, \omega)$ is $\mathcal{F}$ as 'seen from' ω.

We say $\mathcal{F}$ is *twisting* if it is essentially nonlinear — for example, if there exists an $f \in \mathcal{F}$ with a critical point, or if $\mathcal{F}$ contains a free group of Möbius transformations.

Given a closed set $J \subset \widehat{\mathbb{C}}$, we say $(\mathcal{F}, J)$ is *uniformly twisting* if $\limsup(\mathcal{F}, \omega_n)$ is twisting for any sequence $\omega_n \in \mathrm{F}(\mathrm{hull}(J))$, the frame bundle over the convex hull of J in $\mathbb{H}^3$. Informally, uniform twisting means $\mathcal{F}$ is quite nonlinear at every scale around every point of J.

For a Kleinian group, the pair $(\Gamma, \Lambda(\Gamma))$ is uniformly twisting iff $M = \mathbb{H}^3/\Gamma$ has injectivity bounds. Thus geometric inflexibility, Corollary 2.2, is a special case of [Mc2]:

Theorem 3.1 (Dynamic Inflexibility) *Let $(\mathcal{F}, J)$ be uniformly twisting, and let ϕ be a quasiconformal conjugacy from $\mathcal{F}$ to another holomorphic dynamical system $\mathcal{F}'$. Then ϕ is $C^{1+\alpha}$-conformal at all deep points of J.*

The next three sections illustrate how such inflexibility helps explain universal scaling in dynamics.

4 Renormalization of interval maps

Let $f : I \to I$ be a real-analytic map on an interval. The map f is *quadratic-like* if $f(\partial I) \subset \partial I$ and f has a single quadratic critical point $c_0(f) \in \mathrm{int}(I)$. The basic example is $f(x) = x^2 + c$ on $[-a, a]$ with $f(a) = a$. We implicitly identify maps that are linearly conjugate.

If an iterate $f^p|L$ is also quadratic-like for some interval L, with $c_0(f) \in L \subset I$, then we can take the least such $p > 1$ and define the *renormalization* of f by

$$\mathcal{R}(f) = f^p|L.$$

The order of the intervals $L, f(L), \ldots, f^p(L) = L \subset I$ determines a permutation $\sigma(f)$ on p symbols.

The map f is *infinitely renormalizable* if the sequence $\mathcal{R}^n(f)$ is defined for all $n > 0$. The *combinatorics* of f is then recorded by the sequence of permutations $\tau(f) = \langle \sigma(\mathcal{R}^n(f)) \rangle$. We say f has *bounded combinatorics* if only finitely many permutations occur, and *periodic combinatorics* if $\tau(\mathcal{R}^q f) = \tau(f)$ for some $q \geq 1$.

THEOREM 4.1 *Let $f : I \to I$ be infinitely renormalizable, with combinatorics of period q. Then $\mathcal{R}^{qn}(f) \to F$ exponentially fast as $n \to \infty$, where F is the unique fixed-point of the renormalization operator $\mathcal{R}^q$ with the same combinatorics as f.*

For example, the Feigenbaum polynomial $f(x) = x^2 - 1.4101155\cdots$, arising at the end of the cascade of period doublings in the quadratic family, has $\tau(f) = \langle (12), (12), (12), \ldots \rangle$. Under renormalization, $\mathcal{R}^n(f)$ converges exponentially fast to a solution of the functional equation

$$F \circ F(x) = \alpha^{-1} F(\alpha x).$$

To formulate the speed of convergence more completely, extend $f : I \to I$ to a complex analytic map on a neighborhood of $I \subset \mathbb{C}$, and let $F : W \to \mathbb{C}$ denote the maximal analytic continuation of the renormalization fixed-point. Then we find there is an $A > 1$ such that for any compact $K \subset W$, we have

$$\sup_{z \in K} |\mathcal{R}^n(f)(z) - F(z)| = O(A^{-n}),$$

where $\mathcal{R}^n(f)$ is suitably rescaled.

Now suppose only that f has bounded combinatorics. Under iteration of f, all but countably many points in I are attracted to the postcritical Cantor set

$$P(f) = \overline{\bigcup_{n>0} f^n(c_0(f))} \subset I.$$

THEOREM 4.2 *Let f and g be infinitely renormalizable maps with the same bounded combinatorics. Then $f|P(f)$ and $g|P(g)$ are $C^{1+\alpha}$-conjugate.*

Thus quantitative features of the attractor $P(f)$ (such as its Hausdorff dimension) are determined by the combinatorics $\tau(f)$.

These universal properties of quadratic-like maps were observed experimentally and linked to renormalization by Feigenbaum and Coullet-Tresser in the late 1970s. A program for applying complex quadratic-like maps to renormalization was formulated by Douady and Hubbard in the early 1980s. Sullivan introduced a wealth of new ideas and established the convergence $\mathcal{R}^{nq}(f) \to F$ [Sul3], [Sul4]. The inflexibility theory gives a new proof yielding, in addition, exponential speed of convergence and $C^{1+\alpha}$-smoothness of conjugacies.

Our approach to renormalization is via *towers* [Mc2]. For simplicity we treat the case of the Feigenbaum polynomial f. By Sullivan's *a priori* bounds, the sequence of renormalizations $\langle \mathcal{R}^n(f) \rangle$ is compact, and all limits are complex quadratic-like maps with definite moduli. Passing to a subsequence we can arrange that $\mathcal{R}^{n+i}(f) \to f_i$ and obtain a tower

$$\mathcal{T} = \langle f_i \ : \ i \in \mathbb{Z} \rangle \quad \text{such that } f_{i+1} = f_i \circ f_i \ \forall i.$$

The Julia set $J(\mathcal{T}) = \bigcup J(f_i)$ is dense in $\mathbb{C}$, and we deduce that $\mathcal{T}$ is rigid — it admits no quasiconformal deformations. Convergence of renormalization, $\mathcal{R}^n(f) \to F$, then easily follows.

The rapid speed of convergence of renormalization comes from inflexibility of the one-sided tower $\mathcal{T} = \langle f, f^2, f^4, \ldots\rangle$. To establish this inflexibility, we first show the full dynamical system $\mathcal{F}(f) = \{f^{-i} \circ f^j\}$ contains copies of f^{2^n} near every $z \in J(f)$ and at every scale. Thus $(\mathcal{F}(f), J(f))$ is *uniformly twisting.* Next we use expansion in the hyperbolic metric on $\mathbb{C} - P(f)$ to show $c_0(f)$ is a *deep point* of $J(f)$. Finally by Theorem 3.1, a quasiconformal conjugacy ϕ from f to $\mathcal{R}(f) = f \circ f$ is actually $C^{1+\alpha}$-conformal at the critical point. At small scales ϕ provides a nearly linear conjugacy from $\mathcal{R}^n(f)$ to $\mathcal{R}^{n+1}(f)$, and exponential convergence follows.

The second frame of Figure 1 depicts the Julia set of the infinitely renormalizable Feigenbaum polynomial f, centered at its critical point. The Julia set $J(f)$ is locally connected [JH], [LS]; it is still unknown if $\operatorname{area}(J(f)) = 0$.

Milnor has observed that the Mandelbrot set M is quite dense at the Feigenbaum point $c = -1.4101155\ldots \in \partial M$ and at other fixed-points of tuning [Mil], and it is reasonable to expect that c is a deep point of M. Lyubich has recently given an elegant proof of the hyperbolicity of renormalization at its fixed-points, including a new proof of exponential convergence of $\mathcal{R}^n(f)$ via the Banach space Schwarz lemma, and a proof of Milnor's conjecture that blowups of M around the Feigenbaum point converge to the whole plane in the Hausdorff topology [Lyu].

5 Critical circle maps

A *critical circle map* $f : S^1 \to S^1$ is a real-analytic homeomorphism with a single cubic critical point $c_0(f) \in S^1$. A typical example is the *standard map*

$$f(x) = x + \Omega + K\sin(x), \qquad x \in \mathbb{R}/2\pi\mathbb{Z}, \quad \Omega \in \mathbb{R}$$

with $K = -1$ and $c_0 = 0$. These maps arise in KAM theory and model the disappearance of invariant circles [FKS], [Lan], [Rand], [Mak], [DGK]. Another class of examples are the rational maps

$$f(z) = \lambda z^2 \frac{z-3}{1-3z}, \quad |\lambda| = 1, \tag{5.1}$$

acting on $S^1 = \{z \;:\; |z| = 1\}$ with $c_0(f) = 1$.

If $f : S^1 \to S^1$ has no periodic points, then it is topologically conjugate to a rigid rotation by angle $2\pi\rho(f)$, where the *rotation number* $\rho(f)$ is irrational [Y]. The behavior of f is strongly influenced by the continued fraction of its rotation number,

$$\rho(f) = 1/(a_1 + 1/(a_2 + 1/(a_3 + \cdots))), \quad a_i \in \mathbb{N}.$$

By truncating the continued fraction we obtain rational numbers $p_n/q_n \to \rho(f)$. We say $\rho(f)$ is of *bounded type* if $\sup a_i < \infty$.

Theorem 5.1 (de Faria-de Melo) *Let f_1, f_2 be two critical circle maps with equal irrational rotation numbers of bounded type. Then f_1 and f_2 are $C^{1+\alpha}$-conjugate.*

We sketch the proof from [dFdM]. Consider a complex analytic extension of $f(z)$ to a neighborhood of S^1. Let the *Julia set* $J(f)$ be the closure of the set of periodic points of f. As for maps of the interval, one finds the critical point $c_0(f)$ is a *deep point* of $J(f)$, and the full dynamical system $(\mathcal{F}(f), J(f))$ is *uniformly twisting*. Because of the good arithmetic of $\rho(f)$, the forward orbit of the critical point is spread evenly along S^1, so in fact the Julia set is deep at *every* point on the circle. To complete the proof, one constructs a quasiconformal conjugacy between f_1 and f_2, and then applies the inflexibility Theorem 3.1 to deduce that $\phi|S^1$ is $C^{1+\alpha}$.

To bring renormalization into the picture, it is useful to work on the universal cover $\mathbb{R}$ of $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. One can then treat the lifted map $f : \mathbb{R} \to \mathbb{R}$ and the deck transformation $g(x) = x + 2\pi$ on an equal footing. The maps (f, g) form a basis for a subgroup $\mathbb{Z}^2 \subset \operatorname{Diff}(\mathbb{R})$, and any matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathbb{Z})$ determines a *renormalization operator* by

$$\mathcal{R}(f, g) = (f^a g^b, f^c g^d).$$

When the continued fraction of $\rho(f)$ is periodic, one can choose $\mathcal{R}$ such that $\mathcal{R}^n(f, g)$ converges exponentially fast to a fixed-point of renormalization (F, G). For the more general case where $\rho(f)$ is of bounded type, a finite number of renormalization operators suffice to relate any two adjacent levels of the tower $\mathcal{T} = \langle f^{q_n} \rangle$.

The third frame in Figure 1 depicts the Julia set of the rational map $f(z)$ given by equation (5.1), with $\lambda \approx -0.7557 - 0.6549i$ chosen so $\rho(f)$ is the golden ratio. The picture is centered at the deep point $c_0(f) \in J(f)$. Petersen has shown $J(f)$ is locally connected [Pet]; it is an open problem to determine if $\operatorname{area}(J(f)) = 0$.

Levin has proposed a similar theory for critical circle *endomorphisms* such as $f(z) = \lambda z^3(z-2)/(1-2z)$ [Lev].

6 THE GOLDEN-RATIO SIEGEL DISK

Let $f(z) = \lambda z + z^2$, where $\lambda = e^{2\pi i\theta}$.

Siegel showed that f is analytically conjugate to the rotation $z \mapsto \lambda z$ on a neighborhood of the origin when θ is Diophantine ($|\theta - p/q| > C/q^n$). The *Siegel disk* D for f is the maximal domain on which f can be linearized. For θ of bounded type, Herman and Świątek proved that ∂D is a quasicircle passing through the critical point $c_0(f) = -\lambda/2$ [Dou1], [Sw]. In particular, the critical point provides the only obstruction to linearization.

Now suppose θ is a quadratic rational such as the golden ratio:

$$\theta = \frac{\sqrt{5}-1}{2} = 1/(1 + 1/(1 + 1/(1 + \cdots))).$$

Then the continued fraction of θ is preperiodic; there is an $s > 0$ such that $a_{n+s} = a_n$ for all $n \gg 0$. Experimentally, a universal structure emerges at the transition from linear to nonlinear behavior at ∂D [MN] [Wid]. In [Mc4] we prove:

Theorem 6.1 *If θ is a quadratic irrational, then the boundary of the Siegel disk D for f is self-similar about the critical point $c_0(f) \in \partial D$.*

More precisely, there is a map $\phi : (\overline{D}, c_0) \to (\overline{D}, c_0)$ which is a $C^{1+\alpha}$-conformal contraction at the critical point, and locally conjugates f^{q_n} to $f^{q_{n+s}}$.

Theorem 6.2 *Let f and g be quadratic-like maps with Siegel disks having the same rotation number of bounded type. Then $f|\overline{D}_f$ and $g|\overline{D}_g$ are $C^{1+\alpha}$ conjugate.*

For instance, let D_a be the Siegel disk for $f_a(z) = \lambda z + z^2 + az^3$. Then the Hausdorff dimension of ∂D_a is *constant* for small values of a. As for the Julia set we have:

Theorem 6.3 *If θ has bounded type, then the Hausdorff dimension of the Julia set of $f(z) = e^{2\pi i\theta}z + z^2$ is strictly less than two.*

A blowup of the golden ratio Siegel disk, centered at the critical point $c_0(f) \in \partial D$, is shown in the final frame of Figure 1. The picture is self-similar with a universal scaling factor $1.8166\ldots$ depending only on the rotation number. The Julia set of f is locally connected [Pet]. Recently Buff and Henriksen have shown that the golden Siegel disk contains a Euclidean triangle with vertex resting on the critical point [BH]; empirically, an angle of approximately 120° will fit.

The mechanism of rigidity for Siegel disks is visible in the geometry of the *filled* Julia set $K(f) = \{z \ : \ f^n(z)$ remains bounded for all $n > 0\}$. Under iteration, every point in the interior of $K(f)$ eventually lands in the Siegel disk, and $\partial K(f) = J(f)$. The gray cauliflower forming the interior of $K(f)$ in Figure 1 is visibly dense at the critical point. In fact $c_0(f)$ is a *measurable* deep point of $K(f)$, meaning

$$\frac{\text{area}(K(f) \cap B(c_0, r))}{\text{area}(B(c_0, r))} = 1 - O(r^\beta), \quad \beta > 0. \tag{6.1}$$

For the proof of Theorem 6.2, one starts with a quasiconformal conjugacy ϕ from f to g furnished by the theory of polynomial-like maps [DH]. Since f and g have the same linearization on their Siegel disks, we can assume ϕ is conformal on D_f. But then ϕ is conformal throughout $\operatorname{int} K(f)$. By (6.1) the conformal behavior dominates near $c_0(f)$, and we conclude ϕ is $C^{1+\alpha}$-conformal at the critical point. This smoothness is spread to all points of ∂D_f using the good arithmetic of θ.

The self-similarity of ∂D is established similarly, using a conjugacy from f^{q_n} to $f^{q_{n+s}}$.

The dictionary. Table 3 summarizes the parallels which emerge between hyperbolic manifolds, quadratic-like maps on the interval, critical circle maps and Siegel disks. This table can be seen as a contribution to Sullivan's dictionary between conformal dynamical systems [Sul2], [Mc1].

7 Surface groups and their geometric limits

For a complete classification of conformal dynamical systems, one must go beyond the bounded geometry of the preceding examples, and confront short geodesics,

<table>
<tr><th>Hyperbolic manifolds</th><th>Interval maps</th><th>Siegel disks/
Circle maps</th></tr>
<tr><td>Discrete surface group
$\Gamma \subset PSL_2(\mathbb{C})$
$M = \mathbb{H}^3/\Gamma$</td><td>$\mathbb{R}$-quadratic polynomial
$f(z) = z^2 + c$</td><td>Nonlinear rotation
$f(z) = \lambda z + z^2$ or
$\lambda z^2(z-3)/(1-3z)$</td></tr>
<tr><td>Representation
$\rho : \pi_1(S) \to \Gamma$</td><td>Quadratic-like map
$f : U \to V$</td><td>Holomorphic commuting
pair (f, g)</td></tr>
<tr><td>Ending lamination
$\epsilon(M) \in \mathcal{GL}(S)$</td><td>Tuning invariant
$\tau(f) = \langle \sigma(\mathcal{R}^n(f)) \rangle$</td><td>Continued fraction
$\theta = [a_1, a_2, \cdots]$, $\lambda = e^{2\pi i \theta}$</td></tr>
<tr><td>Inj. radius$(M) > r > 0$</td><td>Bounded combinatorics</td><td>Bounded type</td></tr>
<tr><td>Cut points in Λ
$= \bigcup_1^\infty$ (Cantor sets)</td><td colspan="2">Postcritical set $P(f) = \overline{\bigcup f^n(c)}$, $f'(c) = 0$
= (Cantor set)   = (circle or quasi-circle)</td></tr>
<tr><td>($\mathbb{R}$-tree of $\epsilon(M), \pi_1(S)$)</td><td>(proj lim $\mathbb{Z}/p_i, x \mapsto x + 1$)</td><td>($\mathbb{R}/\mathbb{Z}, x \mapsto x + \theta$)</td></tr>
<tr><td>$\Lambda(\Gamma)$ is locally connected</td><td>$J(f)$ is locally connected</td><td>$J(f)$ is locally connected</td></tr>
<tr><td>area $\Lambda(\Gamma) = 0$</td><td colspan="2">area$(J(f)) = 0$?</td></tr>
<tr><td>Inj. radius $\in [r, R]$ in core(M)</td><td colspan="2">$(\mathcal{F}(f), J(f))$ is uniformly twisting</td></tr>
<tr><td>Mapping class $\psi \in \mathrm{Mod}(S)$</td><td>Kneading permutation</td><td>Automorphism $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of $\mathbb{Z}^2$</td></tr>
<tr><td colspan="3">Renormalization Operators</td></tr>
<tr><td>$\mathcal{R}(\rho) = \rho \circ \psi^{-1}$</td><td>$\mathcal{R}(f) = f^p(z)$</td><td>$\mathcal{R}(f, g) = (f^a g^b, f^c g^d)$</td></tr>
<tr><td colspan="3">Stable Manifold of Renormalization</td></tr>
<tr><td>M = asymptotic fiber</td><td>f = limit of doublings</td><td>θ = golden ratio</td></tr>
<tr><td>Elliptic points deep in $\Lambda(\Gamma)$</td><td colspan="2">Critical point $c_0(f)$ deep in $J(f)$ or $K(f)$</td></tr>
<tr><td>$\rho \circ \psi^{-n}$, $n = 1, 2, 3 \ldots$</td><td>f^n, $n = 1, 2, 4, 8, 16, \ldots$</td><td>$f^n$, $n = 1, 2, 3, 5, 8, \ldots$</td></tr>
<tr><td>Geometric limit of $\mathcal{R}^n(\rho)$</td><td>Quadratic-like tower
$\langle f_i \;:\; i \in \mathbb{Z} \rangle$; $f_{i+1} = f_i \circ f_i$</td><td>Tower of commuting pairs</td></tr>
<tr><td>Hyperbolic 3-manifold $S \times [0,1]/\psi$
fibering over the circle</td><td colspan="2">Fixed-points of
Renormalization</td></tr>
<tr><td colspan="3">Conformal structure is $C^{1+\alpha}$-rigid at deep points $\Longrightarrow$
Renormalization converges exponentially fast</td></tr>
<tr><td>M is asymptotically rigid</td><td colspan="2">$J(f)$ is self-similar at the critical point $c_0(f)$</td></tr>
</table>

Table 3.

unbounded renormalization periods and Liouville rotation numbers. We conclude with an example of such a complete classification in the setting of hyperbolic geometry.

Let S be the compact surface obtained by removing a disk from a torus. Let $AH(S) \subset V(S)$ be the set of discrete faithful representations such that $\rho(\pi_1(\partial S))$ is parabolic. A representation $\rho : \pi_1(S) \to \Gamma$ in $AH(S)$ gives a hyperbolic manifold $M = \mathbb{H}^3/\Gamma$ homeomorphic to $\operatorname{int}(S) \times \mathbb{R}$. To each end of M one can associate an *end invariant*

$$E^{\pm}(M) = \begin{cases} \partial^{\pm}(M) \in \operatorname{Teich}(S) & \text{or} \\ \epsilon^{\pm}(M) \in \mathbb{P}\mathcal{ML}(S). \end{cases}$$

In the first case the end is naturally completed by a hyperbolic punctured torus $\partial^{\pm}(M)$; in the second case the end is pinched along a simple curve or lamination $\epsilon^{\pm}(M)$.

Identifying $\operatorname{Teich}(S) \cup \mathbb{P}\mathcal{ML}(S)$ with $\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{R} \cup \infty$, we may now state:

Theorem 7.1 (Minsky) *The pair of end invariants establishes a bijection*

$$E : AH(S) \to \overline{\mathbb{H}} \times \overline{\mathbb{H}} - \overline{\mathbb{R}} \times \overline{\mathbb{R}}$$

with E^{-1} continuous.

Corollary 7.2 *Each Bers' slice of $AH(S)$ is bounded by a Jordan curve naturally parameterized by $\mathbb{R} \cup \infty$, with rational points corresponding to cusps.*

Corollary 7.3 *Geometrically finite manifolds are dense in $AH(S)$.*

Theorem 7.1 establishes a special case of Thurston's *ending lamination conjecture* [Mc1, §4]. We remark that E is *not* a homeomorphism, and indeed $AH(S)$ is not even a topological manifold with boundary [Mc3, Appendix].

The proof of Theorem 7.1 from [Min] can be illustrated in the case $E(M) = (\tau, \lambda)$, with $\tau \in \mathbb{H}$ and $\lambda \in \mathbb{R}$ an irrational number with continued fraction $[a_1, a_2, \ldots]$. By rigidity of manifolds in $\partial AH(S)$, it suffices to construct a quasi-isometry

$$\phi : M \to M(a_1, a_2, \ldots)$$

from M to a model Riemannian manifold explicitly constructed from the ending invariant. The quasi-isometry is constructed piece by piece, over blocks M_i of M corresponding to terms a_i in the continued fraction.

The construction yields a description not only of manifolds in $AH(S)$, but also of their geometric limits, which we formulate as follows.

Theorem 7.4 *Every geometric limit $M = \lim M_n$, $M_n \in AH(S)$, is determined up to isometry by a sequence $\langle a_i, i \in I \rangle$, where*

- *$I \subset \mathbb{Z}$ is a possibly infinite interval,*
- *$a_i \in \operatorname{Teich}(S) \cup \{*\}$ if i is an endpoint of I; and*
- *$a_i \in \{1, 2, 3, \ldots, \infty\}$ otherwise.*

Here $\langle a_i \rangle$ should be thought of as a generalized continued fraction, augmented by Riemann surface data for the geometrically finite ends of M. (The special point $\{*\}$ is used for the triply-punctured sphere.)

For example, the sequence $\langle a_i \rangle = \langle \ldots, \infty, \infty, \infty, \ldots \rangle$ determines the periodic manifold

$$M_\infty \cong \operatorname{int}(S) \times \mathbb{R} - \left(\bigcup_{\mathbb{Z}} \gamma_i \times \{i\} \right),$$

where $\gamma_i \subset S$ are simple closed curves and $i(\gamma_i, \gamma_{i+1}) = 1$. These curves enumerate the rank two cusps of M_∞. Geometrically, M_∞ is obtained from the Borromean rings complement $S^3 - B$ (itself a hyperbolic manifold) by taking the $\mathbb{Z}$-covering induced by the linking number with one component of B.

In general the coefficients $\langle a_i \rangle$ in Theorem 7.4 specify how to obtain M by Dehn filling the cusps of M_∞. Compare [Th2, §7].

Corollaries 7.2 and 7.3 are reminiscent of two open conjectures in dynamics: the local connectivity of the Mandelbrot set, and the density of hyperbolicity for complex quadratic polynomials.

Quadratic polynomials, however, present an infinite variety of parabolic bifurcations, in contrast to the single basic type occurring for punctured tori. This extra diversity is reflected in the topological complexity of the boundary of the Mandelbrot set, versus the simple Jordan curve bounding a Bers slice.

Parabolic bifurcations can be analyzed by Ecalle cylinders [Dou2] and parabolic towers [Hin], both instances of geometric limits as in §3. A complete understanding of complex quadratic polynomials will likely entail a classification of all their geometric limits as well.

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