Counterexamples to the Seifert Conjecture

To my Son Greg

Krystyna Kuperberg[1]

Abstract. Since H. Seifert proved in 1950 the existence of a periodic orbit for a vector field on the 3-dimensional sphere S^3 which forms small angles with the fibers of the Hopf fibration, several examples of aperiodic vector fields on S^3 have been produced as well as results showing that in some situations a compact orbit must exists. This paper surveys presently known types of vector fields without periodic orbits on S^3 and on other manifolds.

1991 Mathematics Subject Classification: Primary 58F25; Secondary 57R25, 35B10, 58F18

Keywords and Phrases: dynamical system, plug, periodic orbit, minimal set, PL foliation

1 Introduction: The Seifert Conjecture.

A *dynamical system* or a *flow* on a metric space X is a *topological group action* of the additive group of reals $\mathbb{R}$ on X, or equivalently a continuous map $\Phi : \mathbb{R}\times X \to X$ such that $\Phi(0,p) = p$ and $\Phi(t+s,p) = \Phi(s,\Phi(t,p))$. If M is a smooth or real analytic manifold and Φ is differentiable, then $\frac{d\Phi}{dt}|_{t=0}$ is the *vector field of* Φ and is in the same smoothness category as Φ. By a standard integration theorem, a C^1 vector field on a closed manifold can be integrated to produce a corresponding dynamical system. A *trajectory* or an *orbit* of a point p is the image of $\Phi(\mathbb{R}\times\{p\})$ in X. A compact trajectory is *periodic*: either consisting of a *fixed point* or homeomorphic to S^1. A dynamical system, or equivalently a vector field, is *aperiodic* if it contains no compact trajectories. A non-compact trajectory is a one-to-one image of $\mathbb{R}$. A compact non-empty set A is *minimal*, if A is the union of trajectories, and no proper subset of A has these properties. A compact orbit is an example of a minimal set. A minimal set is always the closure of a trajectory, but

[1]The author was supported in part by the NSF grants DMS-9401408 and DMS-9704558.

not every set containing a trajectory as a dense subset is minimal. Throughout this paper, it is assumed that all considered vector fields are non-singular, or equivalently, that the dynamical systems possess no fixed points.

Any closed 3-manifold has Euler characteristic zero and hence admits a non-singular vector field. The Hopf fibration of the 3-dimensional sphere S^3 yields a dynamical system on S^3 whose every trajectory is circular. A small perturbation can easily eliminate all but one periodic orbit. In 1950, H. Seifert [34] proved the following:

Theorem 1 *Suppose that $\mathcal{V}$ is a continuous vector field on S^3 satisfying the uniqueness of solution condition. Then there is an $\epsilon > 0$ such that if the vectors of $\mathcal{V}$ form angles smaller than ϵ with the fibers of the Hopf fibration, then $\mathcal{V}$ has a least one periodic solution.*

Subsequently, Seifert asked whether every dynamical system on S^3 has a periodic trajectory. The conjecture that the answer is "yes," under the natural C^1 differentiability assumption, became known as the Seifert conjecture. Further developments resulted in a stronger statement of the problem, see [39] and [31]:

A modified Seifert conjecture: Every C^1 dynamical system on S^3 possesses a minimal set of covering dimension 1.

The table below illustrates the existing counterexamples to the Seifert conjecture and the modified Seifert conjecture:

flows on S^3	not volume preserving	volume preserving
discrete circular trajectories	C^ω (F. W. Wilson)	C^∞ (G. Kuperberg)
aperiodic, 1-dimensional minimal sets	C^1 (P. A. Schweitzer) $C^{3-\epsilon}$ (J. Harrison)	C^1 (G. Kuperberg)
2-dimensional minimal sets	C^ω (G. Kuperberg, K. Kuperberg)	—

2 Discrete closed orbits and the application of plugs.

The presently known examples of aperiodic dynamical systems on S^3 are based on constructions of aperiodic plugs which are used to locally modify a given dynamical system with discrete periodic orbits in order to break these orbits without forming new ones. An example of an n-dimensional C^r plug, $1 \leq r \leq \infty$, can be described as follows. Let $\mathcal{V}$ be a constant vector field on $\mathbb{R}^n$ parallel to a given line L. Suppose that F is an $(n-1)$-dimensional compact connected manifold with boundary allowing an embedding of the Cartesian product of F and the interval I, $F \times I$, in $\mathbb{R}^n$ in such a way that for $p \in F$, $\{p\} \times I$ is a straight line segment parallel to L. A *plug* is a C^r vector field $\mathcal{F}$ on $F \times I$ which coincides with $\mathcal{V}$ in a neighborhood of the boundary $\partial(F \times I)$ and satisfies two additional conditions: 1. there is a trajectory whose positive limit set is inside the set $F \times I$ (*trapped trajectory*); 2. if a trajectory of $\mathcal{F}$ passes through $F \times I$, then it contains a pair of points $(p, 0)$ and $(p, 1)$ (*matched ends*). The definition of a *twisted plug* is similar, but the requirement on $\mathcal{F}$ on the side boundary is relaxed so that $\mathcal{F}$ is tangent to $(\partial F) \times I$ and there are no minimal sets in the side boundary. A chart in a manifold containing a set homeomorphic to $F \times I$ on which the vector field is conjugate to a constant vector field parallel to the fiber I is replaced by an aperiodic plug matching the end points of its trajectories to the end points of trajectories in the chart. If a segment of a circular orbit is replaced by a trapped orbit, then the periodicity is removed.

Plugs are also defined for real analytic vector fields, piecewise linear foliations, and higher dimensional foliations, see [31], [20] and [21]. As remarked by W. Thurston [35], by the Morrey-Grauert theorem asserting that two analytic manifolds which are diffeomorphic are analytically diffeomorphic, real analytic plugs can be used to alter vector fields and foliations on real analytic manifolds.

One of the two basic properties of a plug, the "trapped trajectory," dates to a classical example of a fixed point free homeomorphism of an acyclic compact subset of $\mathbb{R}^3$ given by K. Borsuk [1] in 1935. In 1966, in a fundamental paper [39] F. W. Wilson introduced a special kind of symmetry of vector fields which implies the other important property of plugs, the "matched ends." He proved the following:

Theorem 2 (Wilson) *Every C^∞ n-manifold without boundary, of Euler characteristic zero or non-compact, admits a C^∞ dynamical system with a discrete collection of minimal sets. Each of the minimal sets is an $(n-2)$-torus $S^1 \times \cdots \times S^1$, and every trajectory originates (resp. limits) on one of these tori.*

Wilson's theorem is actually valid in the C^ω category and it implies that a C^ω analogue to the Seifert conjecture for higher dimensional spheres of odd dimensions does not hold. The minimal sets are of codimension 2 and hence the resulting flows in higher dimensions are aperiodic. In a subsequent paper [28], he and P. B. Percell consider another use of a plug in a flow on a closed manifold: a single plug can capture all trajectories.

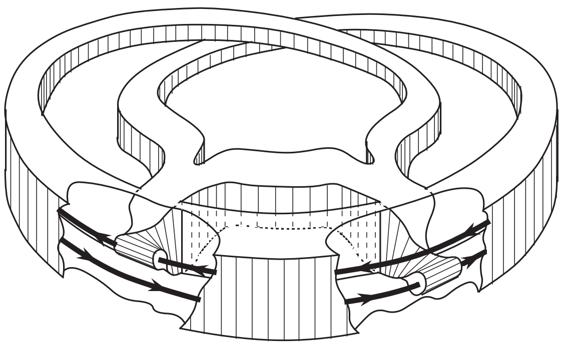
The method of "chopping up" trajectories was also used in [22] (see also [23]) to demonstrate the existence of flows with uniformly bounded orbits, specifically:

Theorem 3 *There exists an aperiodic dynamical system on $\mathbb{R}^3$ with each orbit of diameter smaller than 1.*

In dimension 3, Wilson's plug has circular orbits. His theorem asserts the existence of a real analytic vector field with finitely many circular orbits on any closed 3-manifold. A different method is used by G. Kuperberg in [21] to establish a similar fact for volume preserving dynamical systems. He constructs a twisted plug with two circular trajectories on a set homeomorphic to the solid torus $S^1 \times D^2$, copies of which he inserts into the torus $S^1 \times S^1 \times S^1$ furnished with the irrational flow whose every orbit is dense. By the Wallace-Lickorish theorem, any closed orientable 3-manifold can be obtained from any other closed orientable 3-manifold by an integral surgery on a finite link of tori $S^1 \times D^2$. Surgery on non-compact manifolds is handled on a locally finite link. The insertion of each of the Dehn twisted plugs introduces two circular orbits. The constructions of [21] are volume preserving and yield the following:

THEOREM 4 *Every orientable boundaryless 3-manifold possesses a C^∞ volume preserving dynamical system with a discrete collection of circular trajectories.*

3 COUNTEREXAMPLES TO THE SEIFERT CONJECTURE.



This section lists the known examples of aperiodic flows on S^3 with respect to the degree of differentiability and other properties. In each case a plug is constructed and inserted into a dynamical system on S^3 with one circular orbit. The plug breaks the orbit.

3.1 SCHWEITZER'S VECTOR FIELD.

The first counterexample to the Seifert conjecture came from P. A. Schweitzer in 1972 (published in 1974, see [31]). Schweitzer's construction of an aperiodic C^1 plug is very geometric and astonishing in its simplicity. Unlike Wilson's plug, this vector field is defined on $F \times I$, where F is a non-planar punctured torus. The symmetry guaranteeing the matched ends condition is modeled on two parallel Denjoy minimal sets on which the flow moves in the opposite directions.

THEOREM 5 (Schweitzer) *S^3 admits an aperiodic C^1 vector field.*

3.2 HARRISON'S DIAMOND CIRCLES.

Since the Denjoy vector field on a smooth surface $S^1 \times S^1$ cannot be of class C^2, it seemed impossible to improve the degree of differentiability of Schweitzer's example. J. Harrison [11] embeds the torus $S^1 \times S^1$ in dimension 3 sacrificing the smoothness of the embedding in order to improve the differentiability of the flow on the minimal set. The Denjoy homeomorphism on one of the S^1 factors follows the "diamond circle" pattern.

THEOREM 6 (Harrison) *S^3 admits an aperiodic $C^{3-\epsilon}$ vector field.*

Harrison's construction is limited by the dimension of S^3; thus her method cannot produce a C^3 counterexample to the Seifert conjecture.

3.3 A real analytic counterexample.

The idea behind a smooth aperiodic plug [19] (see also [7]) is to reinsert a Wilson-type plug in itself to cause a recursive breaking of the periodic trajectories. A simple condition prevents the formation of new circular trajectories, even if subjected to the repetitious process of recursion. In [20], G. Kuperberg and K. Kuperberg give specific polynomial formulas for self-insertion performed on a real analytic plug. One of the more interesting features of this construction is that the only minimal set is 2-dimensional, thus the vector field is aperiodic. Hence:

Theorem 7 *There is a C^ω counterexample to the modified Seifert conjecture.*

3.4 Volume preserving aperiodic flows.

H. Hofer [13] proved that a C^1 Reeb vector field on S^3 possesses a closed orbit. This result put a new light on questions related to Hamiltonian flows and volume preserving flows on S^3. In [21], G. Kuperberg adjusts the flow around the Denjoy minimal set in Schweitzer's C^1 plug to make a volume preserving aperiodic C^1 plug, even though the Denjoy dynamical system on $S^1 \times S^1$ is not area preserving. This gives a volume preserving flow without periodic trajectories on S^3, and by Theorem 4, on other 3-manifolds:

Theorem 8 *Every orientable 3-manifold without boundary admits an aperiodic C^1 volume preserving dynamical system.*

At this moment, it is not known whether the differentiability of the volume preserving counterexample to the Seifert conjecture can be improved in a similar fashion as in Harrison's work. However, the intricate formulas of [21] and elaborate computations of [11] emphasize the difficulty in obtaining a C^2 volume preserving aperiodic 3-dimensional plug.

Although the method of self-insertion described [19] and [20] allows quite a lot of flexibility and yields various flows with different degrees of C^r differentiability, the resulting plugs are not volume preserving if $r \geq 1$.

3.5 The structure of minimal sets.

The minimal sets in the counterexamples to the Seifert conjecture, [31], [11] and [21], based on the Denjoy flow, are all homeomorphic to the Denjoy minimal set. The mirror-image symmetry introduced by Wilson is very essential to these flows and always creates two minimal sets. In effect, no example of an aperiodic volume preserving plug with only one minimal set exists.

The plugs described in [19] and [20] contain only one minimal set and every closed 3-manifold admits an analytic flow with only one minimal set. If the construction is at least C^1, then there is a large set of trajectories limiting on the minimal set, preventing the flow from being volume preserving. In contrast to Schweitzer's example, the minimal set is not isolated in the sense of Matsumoto, i.e., every neighborhood contains trajectories that do not belong to the minimal set. It is not known if the minimal set in these constructions (C^1 or better) can be of dimension 1. A C^0 dynamical system of this type with a 1-dimensional minimal set is given in [20]. In general, if the minimal set is 1-dimensional, then, like the Denjoy sets and solenoids, it is locally homeomorphic to the Cartesian product of the Cantor set and the interval.

In all of the above examples, each of the minimal sets is the inverse limit of polyhedra. Thus a useful tool for classifying these minimal sets is the first cohomology group.

3.6 Flows in higher dimensions.

By Theorem 2, differentiable n-manifolds of Euler characteristic zero or non-compact without boundary, $n \geq 4$, admit smooth aperiodic dynamical systems whose minimal sets are of codimension 2. In particular, this is true for odd dimensional spheres S^n, $n \geq 5$. [20] strengthens Wilson's result:

Theorem 9 *If M is a closed differentiable or C^ω n-manifold, $n \geq 3$, admitting a dynamical system in the same smoothness category, then there exists an aperiodic dynamical system on M, in the same smoothness category, with only one minimal set whose dimension is $n-1$.*

Theorem 10 *If M is a differentiable or C^ω manifold without boundary, of dimension at least 3, admitting a dynamical system in the same smoothness category, and $\mathcal{U}$ is an open cover of M, then there exists an aperiodic dynamical system on M, in the same smoothness category, whose orbits are contained in the elements of $\mathcal{U}$, and whose minimal sets have codimension 1.*

The Hamiltonian version of the Seifert conjecture in dimension 3 has not been solved yet, but there are interesting examples in higher dimensions. In 1994, V. Ginzburg [8] and M. Herman [12] independently constructed examples of smooth compact hypersurfaces without closed characteristics in $\mathbb{R}^{2n}$, $n \geq 4$, resolving the case of Hamiltonian flows on spheres of dimension 7 or higher. At the same time, M. Herman [12] found a $C^{3-\epsilon}$ counterexample to the Hamiltonian Seifert conjecture in dimension 5 (i.e., for a compact hypersurface in $\mathbb{R}^6$). In 1997, V. Ginzburg [10] improved the previous results and obtained a smooth proper function $H : \mathbb{R}^{2n} \to \mathbb{R}$, for $2n \geq 6$, with a regular level set on which the Hamiltonian flow has no closed orbits.

3.7 Piecewise linear flows.

PL dynamical systems are thoroughly examined by G. Kuperberg in [21]. A measure on a PL manifold is *simplicial* relative to a triangulation T if on each simplex the measure is given by a linear embedding of the simplex in Euclidean space. The following analogue of Moser's theorem [25], given in [21], demonstrates that simplicial measures are the PL analogue of volume forms:

Theorem 11 *Two simplicial measures on a connected* PL *manifold M with the same total volume are equivalent by a* PL *homeomorphism. Moreover, any simplicial measure is locally* PL*-Lebesgue.*

The results of [21] related to volume preserving PL flows are:

Theorem 12 *Every orientable 3-manifold without boundary possesses a transversely measured* PL *flow with discrete periodic trajectories.*

Theorem 13 *There is a* PL*, measured, integrally Dehn-twisted plug $\mathcal{D}$ with two closed circular orbits.*

Theorem 14 *Every orientable 3-manifold without boundary possesses a transversely measured* PL *dynamical system with a discrete collection of circular trajectories.*

The main result for PL dynamical systems in [20] is:

Theorem 15 *Let M be a* PL *manifold of dimension $n \geq 3$, $1 \leq k \leq n-1$, and let $\mathcal{U}$ be an open cover of M. A* PL *flow on M can be modified in a* PL *fashion so that the orbits are contained in the elements of $\mathcal{U}$, there are no circular orbits, and all minimal sets are k-dimensional.*

As a corollary, in dimension 3 we have:

Theorem 16 *For $k = 1, 2$, every orientable 3-manifold without boundary admits an aperiodic* PL *flow such that all minimal sets are k-dimensional.*

4 Higher dimensional foliations.

A *k-foliation* on an n-manifold M is an atlas of charts in $\mathbb{R}^n$ that preserve the parallel k-plane foliation of $\mathbb{R}^n$, which is a partition of $\mathbb{R}^n$ into translates of flat $\mathbb{R}^k \subset \mathbb{R}^n$. M is then a *k-foliated manifold*. The foliation structure is in a given category, such as smooth, if the gluing maps are simultaneously in the same category and preserve k-planes.

In [20], G. Kuperberg and K. Kuperberg generalize Theorems 9, 10 and 15 to higher dimensional foliations as follows:

Theorem 17 *If M is a continuous, C^∞, C^ω, or* PL *closed manifold of dimension ≥ 3 admitting a dynamical system in the same smoothness category, then there exists an aperiodic dynamical system on M in the same smoothness category, with exactly one minimal set which is of codimension 1.*

Theorem 18 *If M is a continuous, C^∞, C^ω, or* PL *manifold without boundary of dimension ≥ 3 admitting a dynamical system in the same smoothness category, and $\mathcal{U}$ is an open cover of M, then there exists an aperiodic dynamical system on M in the same smoothness category, whose orbits are contained in the elements of $\mathcal{U}$, and whose minimal sets have codimension 1.*

The above theorems do not carry much information for codimension 1 foliations. There are many results relating to opening closed leaves of such foliations. In particular, S. P. Novikov [27] proved that every C^2 codimension 1 foliation of S^3 has a closed leaf (later extended to continuous foliations), while P. A. Schweitzer [32] showed that it is always possible to modify any codimension 1 foliation in dimension 4 or higher in a C^1 fashion so that it has no compact leaf.

5 Existence of closed orbits

A sequence of fundamental results related to contact forms, Hamiltonian dynamics, and periodic orbits was preceded by a paper of H. Seifert [33] who established the existence of periodic solutions on a fixed energy surface for some Hamiltonians. J. Martinet proved in [24] that every smooth compact 3-manifold possesses a contact form. A tremendous amount of work in this field was done by J. Moser [26], I. Ekeland and J.-M. Lasry [3], A. Weinstein [37], [38], P. Rabinowitz [29], [30], C. Viterbo, Y. Eliashberg, W. Thurston, H. Hofer, E. Zehnder, and others (see [4], [5], [14], [15], [17], [18] for multiple papers and authors). Of particular importance is the 1987 paper by C. Viterbo [36] with a proof of the Weinstein conjecture in $\mathbb{R}^{2n}$: a hypersurface of contact type carries a closed characteristic; and, in relation to both the Seifert and the Weinstein conjectures, the 1993 result of H. Hofer [13] who proved the existence of a closed orbit for a C^1 Reeb vector field on S^3. Subsequently, H. Hofer, K. Wysocki, and E. Zehnder [16] proved that every Reeb vector field on S^3 has an unknotted periodic orbit. K. Cieliebak [2] and V. Ginzburg [9] studied both the existence of periodic orbits and opening closed orbits. J. Etnyre and R. Ghrist [6] proved the Seifert conjecture in hydrodynamics: the C^ω plug [20] cannot be parallel to its curl under any metric.

References

[1] K. Borsuk, *Sur un continu acyclique qui se laisse transformer topologiquement en lui même sans points invariants*, Fund. Math. 24 (1935), 51-58.

[2] K. Cieliebak, *Symplectic boundaries: creating and destroying closed characteristics*, Geom. Funct. Anal. 7 (1997), 269-321.

[3] I. Ekeland and J.-M. Lasry, *On the number of periodic trajectories for a Hamiltonian flow on a convex energy surface*, Ann. of Math. 112 (1980), 283-319.

[4] Y. Eliashberg and H. Hofer, *A Hamiltonian characterization of the three-ball*, Diff. Int. Eqs. 7 (1994), 1303-1324.

[5] Y. Eliashberg and W. P. Thurston, *Confoliations*, University Lecture Series, Amer. Math. Soc., 1997.

[6] J. Etnyre and R. Ghrist, *Contact topology and hydrodynamics*, xxx.lanl.gov e-Print archive, http://front.math.ucdavis.edu/math.DG/9708111.

[7] É. Ghys, *Construction de champs de vecteurs sans orbite périodique (d'après Krystyna Kuperberg)*, Séminaire Bourbaki 78, Juin 1994.

[8] V. L. Ginzburg, *An embedding $S^{2n-1} \to \mathbb{R}^{2n}$, $2n-1 \geq 7$, whose Hamiltonian flow has no periodic trajectories*, Internat. Math. Res. Notices (1995), 83-98.

[9] V. L. Ginzburg, *On the existence and non-existence of closed trajectories for some Hamiltonian flows*, Math. Zeit. 223 (1996), 397-409.

[10] V. L. Ginzburg, *A smooth counterexample to the Hamiltonian Seifert conjecture in* $\mathbb{R}^6$, Internat. Math. Res. Notices (1997), 641-650.

[11] J. Harrison, C^2 *counterexamples to the Seifert conjecture*, Topology 27 (1988), 249-278.

[12] M. Herman, Communication at the conference in honor of A. Douady *Geometrie Complexe et Systemes Dynamiques*, Orsay, France, 1995.

[13] H. Hofer, *Pseudoholomorphic curves in symplectizations with applications to the Weinstein conjecture in dimension three*, Invent. Math. 114 (1993), 515-563.

[14] H. Hofer and C. Viterbo, *The Weinstein conjecture in cotangent bundles and related results*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 15 (1988), 411-445.

[15] H. Hofer and C. Viterbo, *The Weinstein conjecture in the presence of holomorphic spheres*, Comm. Pure Appl. Math. 45 (1992), 583-622.

[16] H. Hofer, K. Wysocki and E. Zehnder, *Unknotted periodic orbits for Reeb flows on the three-sphere*, Topol. Methods Nonlinear Anal. 7 (1996), 219-244.

[17] H. Hofer and E. Zehnder, *Periodic solutions on hypersurfaces and a result by C. Viterbo*, Invent. Math. 90 (1987), 1-9.

[18] H. Hofer and E. Zehnder, *Symplectic invariants and Hamiltonian dynamics*, Birkhäuser Verlag, Basel, 1994.

[19] K. Kuperberg, *A smooth counterexample to the Seifert conjecture*, Ann. of Math. 140 (1994), 723-732.

[20] G. Kuperberg and K. Kuperberg, *Generalized counterexamples to the Seifert conjecture*, Ann. of Math. 144 (1996), 239-268.

[21] G. Kuperberg, *A volume-preserving counterexample to the Seifert conjecture*, Comment. Math. Helv. 71 (1996), 70-97.

[22] K. M. Kuperberg and C. S. Reed, *A dynamical system on* $\mathbb{R}^3$ *with uniformly bounded trajectories and no compact trajectories*, Proc. Amer. Math. Soc. 106 (1989), 1095-1097.

[23] K. Kuperberg, W. Kuperberg, P. Minc and C. S. Reed, *Examples related to Ulam's fixed point problem*, Topol. Methods Nonlinear Anal. 1 (1993), 173-181.

[24] J. Martinet, *Formes de contact sur les variétés de dimension 3*, Lecture Notes in Math. 209 (1971), 142-163.

[25] J. Moser, *On the volume elements on a manifold*, Trans. Amer. Math. Soc. 120 (1965), 286-294.

[26] J. Moser, *Periodic orbits near an equilibrium and a theorem of A. Weinstein*, Comm. Pure Appl. Math. 29 (1976), 727-747.

[27] S. P. Novikov, *Topology of foliations*, Trans. Math. Moscow Soc. 14 (1967), 268-296.

[28] P. B. Percell, F. W. Wilson, *Plugging flows*, Trans. Amer. Math. Soc. 233 (1977), 93-103.

[29] P. H. Rabinowitz, *Periodic solutions of Hamiltonian systems*, Comm. Pure Appl. Math. 31 (1978), 157-184.

[30] P. H. Rabinowitz, *Periodic solutions of a Hamiltonian system on a prescribed energy surface*, J. Diff. Eqs 33. (1979), 336-352.

[31] P. A. Schweitzer, *Counterexamples to the Seifert conjecture and opening closed leaves of foliations*, Ann. of Math. 100 (1974), 386-400.

[32] P. A. Schweitzer, *Codimension one foliations without compact leaves*, Comment. Math. Helv. 70 (1995), 171-209.

[33] H. Seifert, *Periodische Bewegungen mechanischer Systeme*, Math. Zeit. 51 (1948), 197-216.

[34] H. Seifert, *Closed integral curves in 3-space and isotopic two-dimensional deformations*, Proc. Amer. Math. Soc. 1 (1950), 287-302.

[35] W. P. Thurston, Electronic communication (e-mail).

[36] C. Viterbo, *A proof of Weinstein's conjecture in* $\mathbb{R}^{2n}$, Ann. Inst. H. Poincaré Anal. Non Linéaire 4 (1987), 337-356.

[37] A. Weinstein, *Periodic orbits for convex Hamiltonian systems*, Ann. of Math. 108 (1978), 507-518.

[38] A. Weinstein, *On the hypotheses of Rabinowitz' periodic orbit theorems*, J. Diff. Eqs. 33 (1979), 353-358.

[39] F. W. Wilson, *On the minimal sets of non-singular vector fields*, Ann. of Math. 84 (1966), 529-536.

Krystyna Kuperberg
Department of Mathematics
Auburn University
Auburn, AL 36849-5310
USA