Hyperbolicity, Stability, and the Creation of Homoclinic Points

Shuhei Hayashi

1 The connecting lemma

The importance of connecting invariant manifolds by small perturbations of dynamical systems has been realized through the solution of the C^1 Stability and Ω-Stability Conjectures for diffeomorphisms, respectively by Mañé ([M3]) and Palis ([P2]). Moreover, the extension of their results was done through the creation of homoclinic points by C^1 small perturbation ([H1]).

Let M be a compact manifold without boundary and let $\mathrm{Diff}^1(M)$, resp. $\mathcal{X}^1(M)$, be the set of C^1 diffeomorphisms of M, resp. vector fields on M, with the C^1 topology. Denote by X_t, $t \in \mathrm{R}$, the C^1 flow on M generated by $X \in \mathcal{X}^1(M)$. A set $\Lambda \subset M$ is *hyperbolic* for $f \in \mathrm{Diff}^1(M)$, resp. $X \in \mathcal{X}^1(M)$, if it is compact, invariant, i.e., $f(\Lambda) = \Lambda$, resp. $X_t(\Lambda) = \Lambda$ for all $t \in \mathrm{R}$, there exists a continuous splitting $TM|\Lambda = E^s \oplus E^u$, resp. $TM|\Lambda = E^0 \oplus E^s \oplus E^u$ that is invariant under Df, resp. DX_t for all $t \in \mathrm{R}$, and there exist constants $K > 0$, $0 < \lambda < 1$ such that

$$||(Df^n)|E^s(x)|| \leq K\lambda^n, \quad n \geq 0$$

and

$$||(Df^{-n})|E^u(x)|| \leq K\lambda^n, \quad n \geq 0,$$

resp.

$$E^0(x) = \mathrm{R} \cdot X(x)$$
$$||(DX_t)|E^s(x)|| \leq K\lambda^t, \quad t \geq 0$$

and

$$||(DX_{-t})|E^u(x)|| \leq K\lambda^t, \quad t \geq 0$$

for all $x \in \Lambda$. In particular, Λ is called *isolated* if there exists a compact neighborhood U of Λ such that

$$\bigcap_{n \in \mathbf{Z}} f^n(U) = \Lambda,$$

resp.

$$\bigcap_{t \in \mathbf{R}} X_t(U) = \Lambda.$$

We say that p is a *homoclinic point* associated to Λ if

$$p \in W^s(\Lambda) \cap W^u(\Lambda) - \Lambda.$$

Theorem 1.1 [H2]. If a C^1 dynamical system has an almost homoclinic sequence associated to an isolated hyperbolic set Λ, then there exists a dynamical system C^1 arbitrarily close to the original one, coinciding with it in a neighborhood of Λ, and having a homoclinic point associated to Λ.

For the definition of almost homoclinic sequences, see [H2]. For instance, a sequence of periodic orbits outside Λ accumulating on Λ gives an almost homoclinic sequence associated to Λ. In the proof of Theorem 1.1, we only use Pugh's perturbation technique in his Closing Lemma and don't need the hyperbolicity of Λ. So, we can apply the perturbation in Theorem 1.1 to more general situation. In particular, we get Theorem 1.2 below.
For $X \in \mathcal{X}^1(M)$ and a point $x \in M$, the *ω-limit set* of x, $\omega(x)$ is defined by $\omega(x) = \{y \in M | \exists t_i \to +\infty \text{ such that } X_{t_i}(x) \to y\}$; the *$\alpha$-limit set* of x, $\alpha(x)$ is defined similarly with $t_i \to -\infty$ instead of $t_i \to +\infty$.

Theorem 1.2 [H3]. Let $\mathcal{U}$ be a neighborhood of $X \in \mathcal{X}^1(M)$ and $p, q \in M$ with $q \in \omega(p) - \omega(q)$ be given. Then, there exists $Y \in \mathcal{U}$ coinciding with X outside an arbitrarily small neighborhood of $\{X_t(q) | -T \leq t \leq 0\}$ for some $T(\mathcal{U}, q, X) > 0$ and having an orbit including p and $\{X_t(q) | t \geq 0\}$.

There still remains the other type of connecting problem even for the C^1 case.

Problem. For p and q respectively belonging to the unstable and stable manifolds of a hyperbolic singularity (or periodic orbit), if $\omega(p)$ meets $\alpha(q)$, then is it possible to have a homoclinic point associated to it by a C^1 small perturbation?

This problem is mentioned in [PM] and [Pu]. Pugh gave an example in [Pu] showing that it is not always possible even for a C^1 vector field when the ambient manifold is not compact. Theorem 1.3 below is not the complete solution of the problem when the manifold is compact, but, using it together with Theorem 1.2, we get a C^1 Make or Break Lemma (Theorem 1.4), which gives an affirmative answer to a question suggested by Mañé. Denote by $\mathrm{Sing}(X)$ and $\mathrm{Per}(X)$ respectively the set of singularities of X and that of periodic points of X.

Theorem 1.3 [H3]. Let $\mathcal{U}$ be a neighborhood of $X \in \mathcal{X}^1(M)$ and $p, q \in M$ with $\omega(p) \cap \alpha(q) - (\mathrm{Sing}(X) \cup \mathrm{Per}(X)) \neq \emptyset$ be given. Then, for each $\tilde{p} \in \omega(p) \cap \alpha(q) - (\mathrm{Sing}(X) \cup \mathrm{Per}(X))$, there exists $Y \in \mathcal{U}$ coinciding with X outside an arbitrarily small neighborhood of $\{X_t(\tilde{p}) | 0 \leq t \leq T\}$ for some $T(\mathcal{U}, \tilde{p}, X) > 0$ and having an orbit including p and q.

Theorem 1.4 [H3]. Given $p, q \in M$ with $\omega(p) \cap \alpha(q) \neq \emptyset$, there exists a vector field Y C^1 close to $X \in \mathcal{X}^1(M)$ such that either (a) Y has an orbit including p and q, or (b) $\omega(p) \cap \alpha(q) = \emptyset$.

2 THE STABILITY CONJECTURE The concept of (structural) stability goes back to the work of Andronov and Pontryagin [AP] in 1937. They considered the necessary and sufficient conditions for vector fields on a two-dimensional disk to be structurally stable. Here the *structural stability* deals with the topological persistence under small perturbations of the orbit structure of a dynamical system, which is expressed by a homeomorphism of the ambient manifold sending orbits of the initial one onto orbits of the perturbed system preserving their orientations.

In the late fifties, Peixoto extended this characterization to closed orientable surfaces and subsequently proved the density of stable two-dimensional flows. At this point, Smale thought that perhaps such kind of result could be true in any dimension for both diffeomorphisms and flows. To that end, he formulated what is now called a Morse-Smale system: the limit set consists of finitely many fixed or periodic hyperbolic orbits with their stable and unstable manifolds being transverse. And he conjectured that (a) they are (structurally) stable and (b) they are dense among all C^r dynamical systems, $r \geq 1$. Part (b) of the conjecture was soon shown not to be true, due to the existence (and persistence) of transversal homoclinic orbits. Remarkably, Smale responded by discovering that a transversal homoclinic orbit implies the existence of a new prototype of dynamics: the horseshoe transformation, whose limit set consists of a Cantor set in which the (infinitely many) periodic orbits are dense. In the mid sixties, motivated by Smale's questions, Anosov proved that globally hyperbolic systems are stable. Soon afterwards, Palis and Smale proved that the Morse-Smale systems are stable, so (at least) part (a) of Smale's initial conjecture is correct. Their methods were quite distinct from those of Anosov, since for the Morse-Smale systems there are several hyperbolic "pieces" (fixed or periodic orbits), with stable and unstable manifolds of different dimensions. At this point, putting together their result and that of Anosov, Palis and Smale formulated in 1967 the celebrated Stability Conjecture: a system is stable if its limit set is hyperbolic and all the stable and unstable manifolds are transversal. Instead of hyperbolicity of the limit set, we can require the nonwandering set to be hyperbolic and the periodic orbits to be dense on it (*Axiom A* or *hyperbolic* systems).

In the beginning of 1970's, Robbin, de Melo, and Robinson proved that the properties of Axiom A together with the transversality condition between stable and unstable manifolds (the strong transversality condition) is sufficient for the structural stability.

THEOREM 2.1 (Robbin [R], de Melo [dM], Robinson [Ro]). *C^1 dynamical systems satisfying Axiom A and the strong transversality condition are C^1 structurally stable.*

In fact, a celebrated conjecture, the so-called Stability Conjecture had been raised by Palis and Smale [PSm] in the late 1960's, saying that the condition is the necessary and sufficient conditions for structural stability. The sufficient condition was proved relatively early, but it took much longer time to solve the converse. After contributions by many mathematicians, it was finally solved for the C^1 case by Mañé for diffeomorphisms and by Hayashi for flows. (See also Wen [W].)

THEOREM 2.2 (Mañé [M3], Hayashi [H2]). *C^1 structurally stable dynamical systems satisfy Axiom A and the strong transversality condition.*

The Ω-stability is a stability restricted to the nonwandering set (so it is weaker than the structural stability), and there is a similar conjecture, the so-called Ω-Stability Conjecture. See Palis [P2] and Hayashi [H2] for the proof. Thus, Palis-Smale's conjecture characterizing stable dynamical systems has completed (for the C^1 case). As a consequence, it turns out that the two concepts, hyperbolicity and stability are essentially equivalent to each other for C^1 dynamical systems of a compact manifold.

As to Theorem 2.2 for flows, the biggest difficulty is the existence of singularities (which are all hyperbolic by stability). In fact, if a sequence of periodic points is accumulating on a singularity, similar arguments to the diffeomorphism case cannot be applied, and if singularities are separated from periodic points, taking the time-one map, parallel arguments to those of diffeomorphism case are available in principle. The separation, the crucial step in the proof of Theorem 2.2 for flows, is obtained by Theorem 1.1. In fact, a sequence of periodic points accumulating on a (hyperbolic) singularity gives an almost homoclinic sequence, so Theorem 1.1 implies that a homoclinic point associated to the singularity can be created by a C^1 small perturbation, which belongs to an unstable saddle connection and contradicts the stability of the vector field. Thus we get the following: for C^1 stable vector fields, singularities are not in the closure of the set of periodic points. In other words, singularities are isolated in the nonwandering set.
However, there is still an essential difference between diffeomorphisms and flows; that is, even though the set of periodic points with the same index (the dimension of the stable subspace) is hyperbolic (then it can be decomposed into disjoint finite union of isolated hyperbolic sets each of which has a dense orbit), a periodic point with the same index might appear far from the original hyperbolic set by arbitrarily small perturbation, which never occurs in stable diffeomorphisms. This phenomenon cause a difficulty in proving the hyperbolicity, but we can take a dense subset in the set of stable vector fields in which the phenomenon never occurs. So, by a similar method to the diffeomorphism case, hyperbolicity of each vector field in the dense subset is obtained. After that, every stable vector field is finally proved to be hyperbolic.

3 BEYOND HYPERBOLICITY

As mentioned in Section 2, Smale expected that the stable systems would be dense in the set of all dynamical systems. This "dream" has collapsed through many examples: there are open sets of unstable or even Ω-unstable systems (see [PT]). Still, one can ask:

CONJECTURE 3.1 (Palis). The set of Morse-Smale dynamical systems together with the systems having a transversal homoclinic point forms a dense subset in the space of dynamical systems.

Since Axiom A systems which are not Morse-Smale ones have transversal homoclinic points, this conjecture can be also considered as a step toward another conjecture by Palis [PT]:

CONJECTURE 3.2 (Palis). Every diffeomorphism on a compact manifold can be approximated by a diffeomorphism satisfying Axiom A or else by one exhibiting a bifurcation involving the creation of homoclinic points (homoclinic bifurcation).

This is in the direction of Palis' program aiming at the global understanding of dynamical systems in the complement of the closure of the set of hyperbolic (or stable) ones. One of his main conjectures is to ask if densely one has finitely many attractors with Sinai-Ruelle-Bowen invariant measures and whose basins cover Lebesgue almost all points in the ambient manifold. Moreover, the attractors should be stochastically stable (see [P3]). In a probabilistic way that would rescue the lost dream of the sixties mentioned in the beginning of this section. Another way is to find a dense subset in the complement having a dynamical feature and find out some mechanism to investigate the bifurcations around each element of the dense subset. It is known that homoclinic tangencies (nontransversal intersection of a stable and an unstable manifold of the same hyperbolic fixed point or periodic point) yields rich phenomena of nonhyperbolic dynamics, such as infinitely many sinks and strange attractors. See [P3] and [PT] for more on this program.

For the two-dimensional case, Conjecture 3.2 was solved recently by Pujals and Sambarino [PS]; that is, every C^1 surface diffeomorphism can be approximated by Axiom A diffeomorphism or else by one exhibiting a homoclinic tangency . In higher dimensions, there are examples of open sets of nonhyperbolic diffeomorphisms where elements exhibit no homoclinic tangencies. In fact, Diaz [D] constructed examples which is obtained after unfolding a three-dimensional cycles with two hyperbolic fixed points p and q of saddle type having different indices containing a transversal intersection of $W^u(p)$ and $W^s(q)$ and a nontransversal orbit of intersection between $W^u(q)$ and $W^s(p)$.

As a first step to have the hyperbolicity in the complement of the closure of the set of diffeomorphisms exhibiting a homoclinic tangency, it is natural to consider showing the existence of dominated splitting on the supports of ergodic probability measures. Here a *dominated splitting* on a compact invariant set Λ is a continuous, f-invariant (i.e., invariant under the derivative of f) splitting

$$TM|\Lambda = E \oplus F$$

such that there exist constants $m \in \mathrm{Z}^+$, $0 < \lambda < 1$ satisfying

$$||(Df^m)|E(x)|| \cdot ||(Df^{-m})|F(f^m(x))|| < \lambda.$$

We know that there exist the Lyapunov splittings in a dense subsets of the supports of ergodic measures by Oseledec's theorem. Let us recall the Oseledec's theorem. Denote by $\Lambda(f)$ for $f \in \mathrm{Diff}^1(M)$ the set of points satisfying the following properties: there exists a splitting $T_xM = \bigoplus_{i=1}^m E_i(x)$ (the Lyapunov splitting at x) and numbers $\lambda_1(x) > \cdots > \lambda_m(x)$ (the Lyapunov exponents at x) such that $\lim_{n\to\pm\infty} \frac{1}{n} \log ||(D_xf^n)v|| = \lambda_i(x)$ for every $1 \le i \le m$ and $0 \ne v \in E_i(x)$. Oseledec proved that $\mu(\Lambda(f)) = 1$ for every f-invariant probability measure μ on the Borel σ-algebra of M. Here $E_i(x)$ $(1 \le i \le m)$ are just measurable functions of x. In this direction, there is a theorem by Mañé [M2], saying that for C^1 generic f (elements in a residual subset of $\mathrm{Diff}^1(M)$), there is a residual subset $\mathcal{R}$ in the space of ergodic measures $\mathcal{M}_e(f)$ of f such that each $\mu \in \mathcal{R}$ has a dominated splitting on its support $s(\mu)$ coinciding with the Lyapunov splitting at μ-a.e. point of $s(\mu)$. As Mañé mentioned in [M2], generic elements of $\mathcal{M}_e(f)$ fail to reflect the dynamical complexity of f in the sense that C^1 generically, the entropy $h_\mu(f)$ is zero for generic μ. For instance, $h_\mu(f) = 0$ when μ is supported on a single periodic orbit. So he suggested to work in the space $\mathcal{M}_e^c(f) = \{\mu \in \mathcal{M}_e(f) | h_\mu(f) > c\}$ and prove that generic measures in $\mathcal{M}_e^c(f)$ satisfy a strong form of Oseledec's theorem.
The following result is in the direction of the combination of proposals of Mañé and Palis. Let $\mathcal{H}^1(M)$ be the set of C^1 diffeomorphisms exhibiting a homoclinic tangency.

Theorem 3.3 [H5]. There is a dense subset $\mathcal{D}$ in the complement of $\overline{\mathcal{H}^1(M)}$ such that if $f \in \mathcal{D}$, for every $\mu \in \mathcal{M}_e(f)$ which is not supported on a single periodic orbit, either $\lim_{n\to\pm\infty} \frac{1}{n} \log ||(D_xf^n)v|| = 0$ for μ-a.e. x and every $0 \ne v \in T_xM$ or there exist dominated splittings

$$TM|s(\mu) = E^- \oplus F,$$

$$TM|s(\mu) = E \oplus F^+$$

such that

$$E^-(x) = \bigoplus_{\lambda_j(x)<0} E_j(x), \qquad F = \bigoplus_{\lambda_j(x)\ge 0} F_j(x),$$

$$E(x) = \bigoplus_{\lambda_j(x)\le 0} E_j(x), \qquad F^+ = \bigoplus_{\lambda_j(x)>0} F_j(x)$$

at μ-a.e. x of $s(\mu)$.

For the proof, we need an improved version of Mañé's ergodic closing lemma ([M1]) and a theorem in Pesin theory. Using this theorem and Theorem 1.1, we get the following result toward the solution of Conjecture 3.1.

THEOREM 3.4 [H5]. In the complement of the closure of the set of diffeomorphisms exhibiting a transversal homoclinic point together with the Morse-Smale ones, every diffeomorphism can be C^1 approximated by one having an ergodic measure whose support has dominated splittings as in Theorem 3.3.

ACKNOWLEDGMENT: The author would like to thank Professor J. Palis for his many suggestions on this paper.

REFERENCES

[A] D. Anosov, *Geodesic flows on closed Riemannian manifolds of negative curvature*, Proc. Steklov Inst. 90 (1967) (Amer. Math. Soc. Transl. 1969).

[AP] A.A. Andronov and L. Pontryagin, *Systems Grossiers*, Dokl. Akad. Nauk. USSR 14 (1937), 247–251.

[dM] W. de Melo, *Structural stability of diffeomorphisms on two-manifolds*, Inventiones Math. 21 (1973), 233–246.

[D] L. Diaz, *Robust nonhyperbolic dynamics and heterodimensional cycles*, Ergod. Th. and Dynam. Sys. 15 (1995), 291–315.

[H1] S. Hayashi, *Diffeomorphisms in $\mathcal{F}^1(M)$ satisfy Axiom A*, Ergod. Th. and Dynam. Sys. 12 (1992), 233–253.

[H2] ——, *Connecting invariant manifolds and the solution of the C^1 stability and Ω-stability conjectures for flows*, Ann. of Math. 145 (1997), 81-137.

[H3] ——, *A C^1 Make or Break Lemma*, preprint.

[H4] ——, *On non-singular vector fields in $\mathcal{G}^1(M)$*, in preparation.

[H5] ——, in preparation.

[M1] R. Mañé, *An ergodic closing lemma*, Ann. of Math. 116 (1982), 503–540.

[M2] ——, *Oseledec's Theorem from the Generic Viewpoint*, in the Proceeding of the International Congress of Mathematicians, Warszawa 1983, 1269–1276.

[M3] ——, *A proof of the C^1 Stability Conjecture*, Publ. Math. IHES 66 (1988), 161–210.

[P1] J. Palis, *On the Contribution of Smale to Dynamical Systems*, From Topology to Computation: Proceedings of the Smalefest (M. Hirsch, J. Marsden and M. Shub, eds.), Springer-Verlag, 1993, pp. 165–178.

[P2] ——, *On the C^1 Ω-Stability Conjecture*, Publ. Math. IHES 66 (1988), 211–215.

[P3] ——, *A global view of dynamics and a conjecture on the denseness of finitude of attractors*, to appear in Astérisque.

[PM] J. Palis and W. de Melo, *Geometric theory of dynamical systems*, Springer-Verlag, 1982.

[PSm] J. Palis and S. Smale, *Structural stability theorems.* Global Analysis, Proc. Sympos. Pure Math. vol. 14, Amer. Math. Soc., 1970, 223–231.

[PT] J. Palis and F. Takens, *Hyperbolicity and Sensitive Chaotic Dynamics at Homoclinic Bifurcations*, Cambridge University Press, Cambridge, New York. Springer-Verlag, 1987.

[Pe] M. Peixoto, *Structural stability on two dimensional manifolds*, Topology 1 (1962), 101–120.

[Pu] C. Pugh, *The C^1 Connecting Lemma*, J. Dyn. and Diff. Eq. 4 (1992), 545–553.

[PR] C. Pugh and C. Robinson, *The C^1 Closing Lemma, including Hamiltonians*, Ergod. Th. and Dynam. Sys. 3 (1983), 261–313.

[PS] E. Pujals and M. Sambarino, *Homoclinic tangencies and hyperbolicity for surfaces diffeomorphisms: a conjecture of Palis*, preprint.

[R] J. Robbin, *A structural stability theorem*, Ann. of Math. 94 (1971), 447–493.

[Ro] C. Robinson, *Structural stability of vector fields*, Ann. of Math. 99 (1974), 154–175.

[W] L. Wen, *On the C^1 stability conjecture for flows*, J. Diff. Eq. 129 (1996), 334–357.

Shuhei Hayashi
School of Commerce,
Waseda University
Shinjuku, Tokyo 169-50,
Japan