Reducibility and Point Spectrum for Linear Quasi-Periodic Skew-Products

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Abstract. We consider linear quasi-periodic skew-product systems on $\mathbb{T}^d \times G$ where G is some matrix group. When the quasi-periodic frequencies are Diophantine such systems can be studied by perturbation theory of KAM-type and it has been known since the mid 60's that most systems sufficiently close to constant coefficients are reducible, i.e. their dynamics is basically the same as for systems with constant coefficients. In the late 80's a perturbation theory was developed for the other extreme. Fröhlich-Spencer-Wittver and Sinai, independently, were able to prove that certain discrete Schrödinger equations sufficiently far from constant coefficients have pure point spectrum, which implies a dynamics completely different from systems with constant coefficients. In recent years these methods have been improved and in particular $SL(2,\mathbb{R})$ — related to the the Schrödinger equation — and $SO(3,\mathbb{R})$ have been well studied.

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1. Introduction.

A *linear quasi-periodic skew-product system* on $\mathbb{T}^d \times G$ is a mapping

$$(\theta, X) \longmapsto (\theta+\omega, A(\theta)X) \tag{1}$$

where θ belongs to the d-dimensional torus $\mathbb{T}^d$, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, ω is a vector in $\mathbb{R}^d$ and A is a continuous function on $\mathbb{T}^d$ with values in some matrix subgroup G of $GL(D,\mathbb{R})$, for example $SL(2,\mathbb{R})$ or $SO(3,\mathbb{R})$. This system is often given as a time-one map of a system of linear differential equation

$$\frac{d}{dt}X(t) = A(\theta+t\omega)X(t), \tag{2}$$

in which case we talk about a *time-continuous* system, and it often naturally contains parameters.

What interests us is the time-evolution of the system (1). At time n it is described by a matrix product

$$A_n(\theta) = A(\theta+(n-1)\omega)\dots A(\theta+\omega)A(\theta), \tag{3}$$

whose behavior we want to study when $n \to \infty$. We are for example interested in if the product becomes unbounded or remains bounded and in the behavior of the eigenvalues.

As example we can consider *the time-discrete Schrödinger equation*

$$-(u_{n+1}+u_{n-1})+V(\theta+n\omega)u_n \;=\; Eu_n \tag{4}$$

with spectral parameter E. This equation can be written as (1) with

$$A(\theta) \;=\; \begin{pmatrix} 0 & 1 \\ -1 & V(\theta)-E \end{pmatrix} \;\in\; SL(2,\mathbb{R}),$$

where E occurs as a free parameter. Another example is *the time-continuous Schrödinger equation*

$$-\frac{d^2}{dt^2}y(t)+V(\theta+t\omega)y(t) \;=\; Ey(t) \tag{5}$$

which can be written as a first order system of the type in (2) with

$$A(\theta) \;=\; \begin{pmatrix} 0 & 1 \\ V(\theta)-E & 0 \end{pmatrix}.$$

The fundamental solution (or monodromy matrix, or time-evolution operator, or propagator, or...) $\Phi_t(\theta,E)$ of this system is a matrix in $SL(2,\mathbb{R})$ and its time-evolution is determined by (1) if we let $A(\theta,E)=\Phi_1(\theta,E)$.

If $\omega/2\pi=(p_1/q_1,\ldots,p_d/q_d)\in\mathbb{Q}^d$ the system is *periodic* and otherwise it is *quasi-periodic*. If it is quasi-periodic one can without restriction assume that $\tilde{\omega}=(\omega,2\pi)$ is rationally independent, i.e.

$$< k,\tilde{\omega} >\neq 0 \quad \text{whenever} \quad k\in\mathbb{Z}^{d+1}\setminus 0.$$

Here we distinguish two cases. We say that the frequencies are *Diophantine* if the vector $\tilde{\omega}$ is Diophantine, i.e.

$$|< k,\tilde{\omega} >| \geq \frac{\kappa}{|\,k\,|^{\tau}}, \quad \forall k\in\mathbb{Z}^{d+1}\setminus 0 \tag{6}$$

for some $k,\tau>0$. If they are not Diophantine we say that they are *Liouville*. The set of Diophantine vectors is of full measure and the set of Liouville vectors is topologically generic, i.e. it is a dense $\mathcal{G}_\delta$.

2. Basic Concepts.

We consider first the periodic case $\omega=2\pi(p_1/q_1,\ldots,p_d/q_d)$. The time-evolution of (1) for a given θ is determined by the spectral properties, in particular the eigenvalues, of the matrix $A_q(\theta)$, where q is a common multiple of all the q_i's. The best way to describe this evolution is to transform the system to a constant coefficient system. That this is possible for a time-continuous periodic system

(2) was shown by Floquet by an easy argument. For a discrete system (1) the argument is even easier and it gives that there exists a change of variables on $(2\mathbb{T})^d \times G$, as smooth as A but only piecewise,

$$(\theta, X) \ \longmapsto \ (\theta, C(\theta)X)$$

which conjugates (1) to another skew-system on $(2\mathbb{T})^d \times G$

$$(\theta, X) \ \longmapsto \ (\theta + \omega, B(\theta)X)$$

where B is constant along the orbits $\{\theta + k\omega\}_{k\in\mathbb{Z}}$, i.e.

$$B(\theta) = B(\theta + k\omega), \quad \forall \theta \in \mathbb{T}^d, \ \forall k \in \mathbb{Z}.$$

An equivalent formulation is that there exists a matrix $C(\theta)$ such that

$$A(\theta) \ = \ C(\theta + \omega)B(\theta)C^{-1}(\theta). \tag{7}$$

(The "period-doubling" which reflects that C is defined on $(2\mathbb{T})^d$ and not on $\mathbb{T}^d$ is necessary if one doesn't want to complexify the system.)

This illustrates the concept of *reducibility*, which was first considered by Lyapunov [1]. It is not obvious what conditions one should require of the transformation C but for periodic and quasi-periodic systems we shall demand that C is defined on some finite covering of the torus and is piecewise continuous. With such a choice periodic systems are always reducible while quasi-periodic systems, as we shall discuss below, turns out not to be. If a quasi-periodic system is reducible however, then the matrix B will be independent of θ. If the transformation C is, say, analytic then we talk about *analytic reducibility*. One could also consider weaker conditions on C: a transformation that is only measurable would a priori be interesting but no results are known in this direction.

A reducible system has Floquet exponents which are nothing but the eigenvalues of the matrix B. The imaginary parts of the Floquet exponents are only defined modulo

$$\{\frac{i}{2} < k, \tilde{\omega} >: k \in \mathbb{Z}^{d+1}\}.$$

In general there is no unique and independent way to specify these imaginary parts except in the case $SL(2,\mathbb{R})$ where they are identified as $\pm$*the rotation number* [2].

The real parts of the Floquet exponents have an independent characterization as *the Lyapunov exponents* which exist for all quasi-periodic skew-products. In fact by a theorem of Oseledet's for a.e. θ there is a measurable decomposition of $\mathbb{R}^D$ into a sum of invariant subspaces

$$\mathbb{R}^D = \bigoplus_i W_i(\theta), \quad \dim W_i(\theta) = m_i \tag{8}$$

such that

$$\lim_{n\to\infty} \frac{1}{n} \log \mid A_n(\theta)\bar{u} \mid \ = \ \lambda_i, \qquad \forall \bar{u} \in W_i(\theta). \tag{9}$$

We call the λ_i's and their multiplicities m_i *the Lyapunov spectrum* of the system. If the system is reducible then the Lyapunov spectrum coincides with the real part of the spectrum of B and it is *uniform* — the decomposition (8) is continuous and the limits (9) exist for all θ. There is a somewhat weak converse of this result when all exponents have multiplicity one: if the system has uniform and simple Lyapunov spectrum and if ω is Diophantine then it is reducible [3]. The assumptions of this theorem are however hard to verify in general.

We now turn to the quasi-periodic case. In distinction to the periodic case, quasi-periodic systems are not always reducible. For example, a reducible system must be *regular* in the sense of Lyapunov, i.e.

$$\sum(\text{Lyapunov exponents}) \quad = \quad \lim_{n\to\infty}\frac{1}{n}\sum_{l=0}^{n-1}\mathrm{Re}\,(Tr(A(\theta+l\omega))$$

where the Lyapunov exponents are counted with multiplicities [1]. Examples are known at least since the 60's of irregular time-continuous quasi-periodic systems [4,5]. This notion however provides little insight into the dynamics of the system.

A reducible system cannot have a *point eigenvalue*, i.e. there cannot exist a sequence of vectors $\{v_n : n\in\mathbb{Z}\}$ in $l^2(\mathbb{Z})\otimes\mathbb{R}^D$ such that

$$v_{n+1} - A(\theta+n\omega)v_n \;=\; Ev_n, \quad \forall n\in\mathbb{Z},$$

for some constant E. Examples of time-continuous quasi-periodic systems with point eigenvalues were given in [6,7]. These examples are not smooth on the torus $\mathbb{T}^d$ however. The first smooth example came as a consequence of a famous theorem on reducibility [8]. The almost Mathieu equation

$$-(u_{n+1}+u_{n-1}) + K\cos(\theta+n\omega)u_n \;=\; Eu_n$$

with Diophantine frequencies was proven to be reducible for small enough K and for certain values of E. As a consequence of the "self-dual" character of this equation under the Fourier transform it must therefore have point eigenvalues for K large enough [9].

A reducible system must be integrable in the sense that it has an invariant foliation of the space $\mathbb{T}^d\times G$ into submanifolds whose dimension equals d plus the dimension of the center of G. In particular such a system cannot be *transitive*, much less be *ergodic*. Examples of ergodic quasi-periodic skew-products were constructed in [10]. These examples are smooth on $\mathbb{T}^d$ but the frequencies are Liouville.

Nor can reducible systems be *non-uniformly hyperbolic* because, as we mentioned above, the Lyapunov spectrum of a reducible system must be uniform. Examples of non-uniformly hyperbolic systems of Schrödinger type are given in [2,11] and of other types in [12].

Because of the existence of both reducible and non-reducible quasi-periodic systems two questions occur naturally. What is the structure of the set of reducible and non-reducible systems respectively? And what are the typical dynamical properties of the non-reducible systems. We shall provide some answers to these questions in the case when ω is Diophantine (6) and the system is analytic and either close to or far from constant coefficients.

3. CLOSE TO CONSTANT COEFFICIENTS.
Reducibility of quasi-periodic skew-products close to constant coefficients was obtained by KAM-arguments already in the 60's. The first results were proven under the assumption of sufficiently many parameters [13]. That in general only one parameter is needed became obvious in [14] — where in particular the case $Sp(n, \mathbb{R})$ is treated — and it was proven in general in [15]. These results give reducibility for all parameter values except a small but positive measure set, but the following stronger statement should be true.

CONJECTURE. *Any generic analytic one-parameter family of skew-systems* (1) *sufficiently close to constant coefficients is reducible for a.e. parameter value.*

The first verification, and the motivation, of this conjecture was done in [16], where previous results [8,17] on the quasi-periodic Schrödinger equation were extended. The main result in [16] is the following

THEOREM 1. *Assume that ω satisfies* (6) *and that V is analytic in the complex strip $\mid Im\theta \mid < r$. Then there exists a constant $\varepsilon_0 = \varepsilon_0(r, \kappa, \tau)$ such that if*

$$\sup_{|\operatorname{Im}\theta|<r} \mid V(\theta) \mid < \varepsilon_0$$

then (4) *is reducible for a.e. E and all θ, i.e. the fundamental solution can be written*

$$A_n(\theta, E) = C(\theta + n\omega, E) e^{nB(E)} C^{-1}(\theta, E),$$

with $C(\cdot, E) : (2\mathbb{T})^d \to SL(2, \mathbb{R})$ analytic and $B(E) \in sl(2, \mathbb{R})$. The set of admissible E's depends on the potential V.

The theorem was proven in the time-continuous case but the proof carries over easily to the discrete case. There is probably a corresponding result for Gevrey classes but if it holds also in $\mathcal{C}^\infty$-category or in finite differentiability is unclear.

There is also a result in [16] stating that the (possible) non-reducible systems in this one-parameter family must have Lyapunov exponents = 0, and that for generic potentials not *all* systems in the family are reducible. This non-reducibility is shown by constructing solutions that are unbounded but increases more slowly than linearly. So even near to constant coefficients there is some delicate mixture of reducible and non-reducible systems. Best known is this mixture in the compact case $SO(3, \mathbb{R})$.

4. COMPACT CASE.
Let's first observe that a matrix in $SO(3, \mathbb{R})$ has three eigenvalues $e^{\pm i\alpha}, 0$ for some real number α. Hence, if A is constant then $\mathbb{T}^d \times SO(3, \mathbb{R})$ is foliated into invariant tori of dimension $d + 1$ and the the orbits of (1) are dense on these tori if and only if α is irrational.

Assume that $A_0 \in SO(3, \mathbb{R})$ and that $A : \mathbb{T}^d \to SO(3, \mathbb{R})$ is analytic in $\mid \operatorname{Im}\theta \mid < r$. The following result is due to R. Krikorian [18].

Theorem 2. *If ω satisfies* (6) *and if $A_0 \neq 0$, then there exists an $\varepsilon_1 = \varepsilon_1(\kappa, \tau, r, A_0)$ such that if*

$$\sup_{|\mathrm{Im}\,\theta|<r} | \hat{A}(\theta) | < \varepsilon_0,$$

then the skew-product (2) *with $A(\theta) = \lambda A_0 + \hat{A}(\theta)$ is analytically reducible for a.e. λ.*

Though this is not exactly the statement of the conjecture since it only refers to a particular one-parameter family it is pretty close. It justifies therefore that we think of the reducible systems as being of "full measure" close to constant coefficients. Hence, in a measure sense the typical system has an invariant foliation into $d+1$-dimensional tori. A natural question is: what can one say about the complementary set?

On the compact manifold $\mathbb{T}^d \times SO(3, \mathbb{R})$ the dynamical system (2) preserves the product Haar measure $\mu \times \nu$. In the reducible case this measure is certainly not ergodic and there are invariant measures supported on each invariant torus. However, for the topologically generic system there is no trace of any invariant set whatsoever, not even of measurable invariant sets. This is the content of the next theorem [19]

Theorem 3. *There exists an $\varepsilon_0 = \varepsilon_0(\kappa, \tau, r, A_0)$ such that for the generic $\hat{A}(\theta)$ in*

$$\sup_{|\mathrm{Im}\,\theta|<r} | \hat{A}(\theta) | < \varepsilon_0$$

the skew-product (2) *with $A(\theta) = A_0 + \hat{A}(\theta)$ is uniquely ergodic.*

Together these two theorems give a very nice picture of the behavior of analytic systems close to constant coefficients: the reducible and uniquely ergodic systems are mixed like the Diophantine and the Liouville numbers. Is it possible that this is also the global situation? A *strong version* of this question is if it is possible to conjugate any system to a close-to-constant-coefficient system in an analytical (or possibly a weaker) topology. A *weak version* would be if any system can be approximated by a reducible system in an analytical (or possibly a weaker) topology.

The weak version is completely open, and the strong version is doubtful because of an example by M. Rychlik. The example is a skew-product on $\mathbb{T} \times SU(2, \mathbb{C})$ [20], which is even a time-one map of a $\mathcal{C}^\infty$-system, given by the matrix

$$A(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.$$

With this matrix the system (1) is not reducible, and it seems unlikely that it can be conjugated close to constant coefficient. Notice however that the system, though not reducible, has an invariant foliation into 2-dimensional tori on which the orbits are dense.

Theorem 2 also holds for general compact matrix groups [21]. A weaker result than Theorem 3 was proven for $SU(2, \mathbb{C})$ in [22].

5. FAR FROM CONSTANT COEFFICIENTS.

If little is known for systems in $\mathbb{T}^d \times SO(3,\mathbb{R})$, or in other compact groups, unless they are close to constant coefficients, we have additional information in the non-compact case $\mathbb{T} \times SL(2,\mathbb{R})$. The subharmonic argument in [2,11] gives a class of discrete Schrödinger equations (4) with large potential V which has non-zero Lyapunov exponents for all E, and therefore must be non-uniformly hyperbolic at least for some E. But more is true. The discrete Schrödinger equation on $\mathbb{T}$ with Diophantine frequencies often has a complete set of eigenvectors in $l^2(\mathbb{Z})$ if the potential is large enough. We shall give a precise formulation of this result.

Assume that V satisfies the transversality conditions

$$(10)\qquad \begin{cases} \max\limits_{0\le\nu\le s} \mid \partial_x^\nu(V(\theta+x)-V(\theta)) \mid \;\ge\; \xi \;>\; 0, & \forall\theta \\ \max\limits_{0\le\nu\le s} \mid \partial_\theta^\nu(V(\theta+x)-V(\theta)) \mid \;\ge\; \xi \inf_{m\in\mathbb{Z}} \mid x-2\pi m \mid, & \forall\theta\ \forall x. \end{cases}$$

These two conditions can be understood as requiring that the potential is always "different from a constant" and always "different from itself under translations". They are fulfilled, for appropriate values of s and ξ, for any analytic function defined in θ with no shorter period than 2π.

THEOREM 4. *Assume that ω satisfies* (6) *and that V is analytic and bounded by a constant γ in the complex strip $\mid Im\theta \mid< r$ and satisfies condition* (10). *Then there exists a constant $\varepsilon_0 = \varepsilon_0(\gamma, r, \kappa, \tau, s, \xi)$ such that if $\mid \varepsilon \mid< \varepsilon_0$*

$$(11)\qquad -\varepsilon(u_{n+1}+u_{n-1}) + V(\theta+n\omega)u_n \;=\; Eu_n$$

has a complete set of eigenvectors in $l^2(\mathbb{Z})$ for almost all θ.

The closure of the set of eigenvalues — the spectrum — is a set whose complement in $[\inf V, \sup V]$ *has measure that goes to* 0 *with ε.*

It follows then from a theorem of Kotani [23, Proposition VII.3.3] that (11) must have non-zero Lyapunov exponents for almost all E and hence must be non-uniformly hyperbolic for almost all parameter values E in the spectrum.

These systems can not be conjugated close to constant coefficients in an analytic topology because close to constants there are no non-uniformly hyperbolic systems by Theorem 1. The question if they can be approximated by reducible systems in an analytic (or possibly weaker) topology is however still open.

Theorem 3 was first proven for a "cosine"-like potential in [24,25], and in [26] for a more general difference operator. The general version given here is from [27]. It holds not only for analytic potentials but also for smooth ones belonging to some Gevrey class. It also holds if one replaces the nearest neighbor operator $(u_{n+1}+u_{n-1})$ by a symmetric operator

$$\sum_{-\infty<k<\infty} a_k(\theta+n\omega)u_{n+k}$$

where the a_k's are assumed to be analytic in $\mid \mathrm{Im}\,\theta \mid< r$ (or more generally of some Gevrey class) and decay exponentially in k.

The situation with a higher-dimensional torus is much more complex. The case $\mathbb{T}^2$ has been analyzed in [28] but complete proofs has not been published.

It is not known what other kinds of dynamics can occur for a skew-system on $\mathbb{T} \times SL(2,\mathbb{R})$ than the types described here.

The proofs of Theorem 1-3 have been obtained by ODE-methods applied to a particular skew-system (1) or (2) in order to conjugate, or try to conjugate, it to constant coefficients. The proof of Theorem 4 is different. Here one considers the Schrödinger equation (3) not as a dynamical system but as an operator on $l^2(\mathbb{Z})$ which one tries to conjugate to diagonal form. Both types of results are therefore conjugation results involving small divisor problems, but one is in finite-dimensional dynamical space and the other is in an infinite-dimensional space.

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