Maximal Averages and Packing of One Dimensional Sets

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Abstract. We discuss recent work of several authors on the Kakeya needle problem and other related problems involving nonexistence of small sets containing large families of one dimensional objects.

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My purpose here is to summarize some recent work in real analysis related to Kakeya type maximal functions. I will take a fairly narrow point of view; specifically, will only consider the classical situations of lines and circles, and will not discuss the recent work on related problems involving oscillatory integrals, to be found for example in [2] and [18]. For a more detailed survey see [22].

The basic open problem in this area, known as the Kakeya problem, has several (morally but not formally equivalent) formulations. We state them below in increasing order of "strength." One defines a Kakeya set to be a compact set $E \subset \mathbb{R}^n$ which contains a unit line segment in each direction,

$$\forall e \in \mathbb{P}^{n-1} \exists x \in \mathbb{R}^n : x + te \in E \; \forall t \in [-\frac{1}{2}, \frac{1}{2}]$$

where we regard $\mathbb{P}^{n-1}$ as being the unit sphere with antipodal points identified. If δ is a small positive number and $f : \mathbb{R}^n \to \mathbb{R}$ then one defines the Kakeya maximal function of f, $f_\delta^* : \mathbb{P}^{n-1} \to \mathbb{R}$ via

$$f_\delta^*(e) = \sup_a \frac{1}{|T_e^\delta(a)|} \int_{T_e^\delta(a)} |f(x)| dx$$

where $T_e^\delta(a)$ is the cylinder centered at a with length 1, cross section radius δ and axis in the e direction. Also define the δ-entropy $\mathcal{N}_\delta(E)$ to be the maximum possible cardinality for a δ-separated subset. Then the following are all open questions if $n \geq 3$.

1. Is it true that if E is a Kakeya set in $\mathbb{R}^n$ then $\limsup_{\delta \to 0} \frac{\log \mathcal{N}_\delta(E)}{\log \frac{1}{\delta}} = n$?
2. Is it true that a Kakeya set in $\mathbb{R}^n$ must have Hausdorff dimension n?

3. Is the following estimate true?

$$\forall \epsilon > 0 \exists C_\epsilon : \|f_\delta^*\|_{L^n(\mathbb{P}^{n-1})} \leq C_\epsilon \delta^{-\epsilon} \|f\|_{L^n(\mathbb{P}^{n-1})} \tag{1}$$

We discuss below the partial results that have been proved on this problem and some work on a class of related problems involving circles in the plane.

Background

Let me mention some results that were proved by 1990.

1. Two dimensional Kakeya maximal theorem, cf. Davies [7], Cordoba [6], Bourgain [2]. In $n = 2$ dimensions the above three statements are known to be true. The first two were proved in [7] while the last was proved in [6] in a slightly different formulation and at the beginning of [2] as stated; the latter paper also introduced the particular definition of Kakeya maximal function adapted above.

Statement 3. in $\mathbb{R}^2$ can be proved by an elementary geometric-combinatorial argument exploiting the fact that two lines intersect in at most one point, and the size of the intersection of the corresponding tubes is determined by the angle of intersection: if e_1 and e_2 determine an angle θ, then $T_{e_1}^\delta(a_1) \cap T_{e_2}^\delta(a_2)$ is contained in a tube of length $\frac{\delta}{\delta+\theta}$. This was the approach in [6]. Alternately, it can be proved using the Plancherel theorem (e.g. [2]).

2. L^p estimates for the X-ray transform. In higher dimensions the strongest result connected with 1,2,3 which was proved before 1990 was the "space-time" estimate of Drury [8] and Christ [4], which was motivated by a similar result of Oberlin-Stein for the Radon transform. We explain this briefly. "Space-time" is ad hoc terminology but it is convenient and is intended to convey the analogy with estimates for the wave equation in space-time. (Indeed, it is possible to view the X-ray transform as a Fourier integral operator, although we do not take this point of view here)

There is a hierarchy of possible partial results on (1), namely the conjectural bounds ($1 \leq p \leq n$)

$$\forall \epsilon \exists C_\epsilon : \|f_\delta^*\|_{L^q(\mathbb{P}^{n-1})} \leq C_\epsilon \delta^{-(\frac{n}{p}-1+\epsilon)} \|f\|_p, \; q = (n-1)\frac{p}{p-1} \tag{2}$$

which would follow from (1) by interpolating with the trivial $\|f_\delta^*\|_\infty \lesssim \delta^{-(n-1)}\|f\|_1$. Notice that the partial result becomes stronger as p increases. Now let G be the space of lines in $\mathbb{R}^n$. Then G can be identified with the tangent bundle to $\mathbb{P}^{n-1}$ by mapping a line ℓ to its direction e and its closest point to the origin x, which is orthogonal to e, and one gives G the resulting volume form etc. The X-ray transform of a function f is the function $Xf : G \to \mathbb{R}$ defined by $Xf(\ell) = \int_\ell f$. There is a natural splitting of directions, so it is natural to consider estimates for the operator X from L^p (or L^p Sobolev spaces $W^{p,\alpha}$) to mixed norm spaces $L_e^q(L_x^r)$, where $e \in \mathbb{P}^{n-1}$ and $x \perp e$. The Kakeya conjecture in form 3. is equivalent to the assertion that X maps $W_{loc}^{n,\epsilon}$ to $L_e^n(L_x^\infty)$ for each $\epsilon > 0$. On the other hand, it is shown in [4] that the pure L^p estimate $\|Xf\|_{L^{n+1}(G)} \leq C\|f\|_{\frac{n+1}{2}}$ is valid. From this, one can easily conclude (2) with $p = \frac{n+1}{2}$.

In [2], Bourgain gave a different, combinatorial approach not going through the space-time estimate, and used it to obtain (2) for $p = \frac{n+1}{2} + \epsilon_n$ (actually, he assumed $q = p$ in (2)) where $\epsilon_3 = \frac{1}{3}$ and ϵ_n is given by an inductive formula. This bound has since been improved in [19] and [3] as we will explain below.

3. Spherical maximal theorem of Stein-Bourgain. Let σ be surface measure on the unit sphere $S^{n-1} \subset \mathbb{R}^n$ and let

$$\mathcal{M}f(x) = \sup_r \int |f(x + r\omega)| d\sigma(\omega)$$

Then

$$\|\mathcal{M}f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)},\ p > \frac{n}{n-1} \tag{3}$$

Stein [17] proved this in three or more dimensions, Bourgain [1] in two dimensions, and independently of Bourgain, Marstrand [10] proved the following geometric consequence or special case analogous to formulations 1. and 2. of the Kakeya problem: a set in $\mathbb{R}^2$ containing a circle with each center has positive measure.

The techniques involved in proving (3) are two fold:

Fourier analysis: the Plancherel theorem and stationary phase asymptotics for $\hat{\sigma}$, e.g. the fact that $|\hat{\sigma}(x)| \lesssim |x|^{-\frac{n-1}{2}}$.

Geometry: we restrict the discussion here to the two dimensional case. The issue, which is not as trivial as it may sound, is to understand how thin annuli intersect. In contrast to the situation for the two dimensional Kakeya problem, the shape of the intersection of two annuli is not determined by the arguments of the maximal function, i.e. centers of the circles, but depends also on how the circles are drawn, and the area will be largest when they are tangent. Let $C(x,r)$ be the circle with center x and radius r and $C_\delta(x,r)$ its δ-neighborhood. We will always assume for simplicity that $\frac{1}{2} \leq r \leq 2$ and $|x| < \frac{1}{4}$. This assumption precludes "external" tangencies so two circles $C(x_1,r_1)$ and $C(x_2,r_2)$ are tangent precisely when the quantity $\Delta((x_1,r_1),(x_2,r_2)) = ||x_1 - x_2| - |r_1 - r_2||$ is equal to zero. we will say they are δ-tangent at a if the parameter Δ is $\leq \delta$ and $a \in C_\delta(x_1,r_1) \cap C_\delta(x_2,r_2)$. If we assume that $|x_1 - x_2| + |r_1 - r_2|$ is bounded from below then the intersection will have area $\approx \frac{\delta^2}{\sqrt{\Delta+\delta}}$. Compensating for this is a significant fact discovered by Marstrand (the "three circle lemma"; [10], Lemma 5.2) which is a quantitative version of the circles of Appolonius. We state only a special case. Fix three circles $C_i = C(x_i,r_i)$. Consider a set of (x,r) with $|x - x_i| + |r - r_i|$ bounded from below and such that $C(x,r)$ is δ-tangent to each C_i at three points whose mutual distances are bounded from below. This set of (x,r) is then contained in the union of two balls of radius approximately δ. This is proved in the following way: (i) a version of the circles of Appolonius covers the limiting case $\delta = 0$ - there are at most two circles tangent (in the above sense) to three given circles at distinct points, and (ii) the $\delta-$tangent circles must be close to one of these circles, as may be seen by applying the inverse function theorem in an appropriate manner.

Roughly, although various different arguments are possible, the Fourier analysis arguments work best in higher dimensions, while in two dimensions either a purely geometric approach or a combination of the two is used. The first was Marstrand's approach based on the three circle lemma, and was recently extended to a proof of (3) by Schlag [14]. The second was Bourgain's approach.

RECENT WORK RELATED TO BOURGAIN-MARSTRAND

It turns out that quite a bit of more detailed information can be obtained by combining geometric facts like the 3-circle lemma with some combinatorial techniques. This work was largely motivated by the following question which arises naturally in connection with the three dimensional Kakeya problem. Indeed the special case of the inequality (1) for functions in $\mathbb{R}^3$ invariant by rotations around the x_3 axis is a two dimensional problem which turns out to be a variant on (4) below.

Suppose a set in $\mathbb{R}^2$ contains a circle of every radius. Then must it have Hausdoff dimension two?

It is known that such a set can have measure zero so one expects to be working with an "almost maximal inequality," i.e. a bound for averages over δ-neighborhoods of circles with less than power dependence on δ, analogous to (1). The relevant maximal function is the following one: $\mathrm{M}_\delta f(r) = \sup_x \frac{1}{|C_\delta(x,r)|}\int_{C_\delta(x,r)}|f|$ which we regard as having domain $[\frac{1}{2},2]$. The following result was proved in [20]:

$$\forall\epsilon\exists C_\epsilon : \|\mathrm{M}_\delta f\|_3 \leq C_\epsilon \delta^{-\epsilon}\|f\|_3 \tag{4}$$

It follows from this that the answer to the above question is affirmative.

Prior to [20] several other related results were proved. The basic technique used is from a paper of Kolasa and the author [9] which was written in 1994, and can be described in the following way. The difficulty is to control intersections between δ-annuli, and the main difficulty in doing this occurs when the annuli are δ-tangent. Accordingly one needs to control the number of δ-tangencies among annuli. Marstrand's lemma makes it possible to view this as a continuum analogue of the following discrete problem: given N circles, bound the number of pairs of tangent circles, assuming a nondegeneracy condition such as that no three circles are tangent at a point. The circles of Appolonius and the "Zarankiewicz problem" in elementary graph theory give a bound $\mathcal{O}(N^{5/3})$ which was used in [9] to prove the partial result on (4) obtained by interpolating with an L^1 to L^∞ estimate as in (2) and then setting $p = \frac{8}{3}$. Later on the author found the paper [5] whose techniques imply a bound $\mathcal{O}(N^{\frac{3}{2}+\epsilon})$ in the discrete problem, and with some effort [20] one can obtain from this a proof of (4). In the intervening time, Schlag [13] was able to prove a sharp L^p to L^q almost maximal estimate in the setting of Bourgain's theorem using a combination of this technique and the Plancherel theorem. His result is

$$\forall\epsilon\exists C_\epsilon : \|M_\delta f\|_5 \leq C_\epsilon \delta^{-\epsilon}\|f\|_{\frac{5}{2}}$$

where $M_\delta f(x) = \sup_{\frac{1}{2}\leq r\leq 2} \frac{1}{|C_\delta(x,r)|}\int_{C_\delta(x,r)}|f|$. This was then extended using different techniques to a space-time estimate by Schlag-Sogge [16]. With hindsight these results are also corollaries of (4), see [20], p. 987. A further result (see [20]) is that a set in $\mathbb{R}^2$ containing circles whose centers contain a set of dimension $\alpha \leq 1$ will have dimension at least $\alpha + 1$. This has recently been improved (in a certain sense) by T. Mitsis [11]: a set containing circles whose centers have dimension $> \frac{3}{2}$ will have positive measure.

Approaches to Kakeya

In the rest of the article we will explain what is known about the Kakeya problem. At present the following results are known:

1. Estimate (2) holds when $p = \frac{n+2}{2}$, in particular Kakeya sets have dimension at least $\frac{n+2}{2}$. This result is from [19].

2. In the three dimensional case, an improvement of the latter result to a mixed norm space-time type estimate [21]. This can be described as follows: interpolate in an appropriate manner between the Drury-Christ estimate $\|Xf\|_{n+1} \lesssim \|f\|_{\frac{n+1}{2}}$ and the conjecture (1). This results in a collection of conjectural bounds for the X ray transform from $W^{p,\epsilon}(\mathbb{R}^n)$ for any $\epsilon > 0$ to the mixed $L^q_e(L^r_x)$ spaces on the space of lines G. For given p, the estimate on L^p improves over the corresponding estimate (2), in the same sense as the result of [8] improves over the $p = \frac{n+1}{2}$ case of (2). It turns out [21] that one can prove the mixed norm estimate when $n = 3$, $p = \frac{5}{2}$ (hence $q = \frac{10}{3}$, $r = 10$).

3. The Hausdorff dimension of a Kakeya set in $\mathbb{R}^n$ is at least $\alpha(n-1)+1$ for suitable explicit $\alpha > \frac{1}{2}$. This result and a related result for the Kakeya maximal function are from very recent work of Bourgain [3]. It is clearly a substantial improvement in high dimensions although, as of this writing, the argument does not give anything new in three dimensions.

We briefly describe the idea of [19] (which can be considered a variant on an idea in [2]), as it applies to the entropy formulation 1. of the Kakeya problem. Namely, if E is a Kakeya set then $\mathcal{N}_\delta(E) \geq C_\epsilon \delta^{-\frac{n+2}{2}+\epsilon}$ for any $\epsilon > 0$. To prove this consider a maximal δ-separated subset $\{e_j\}$ of $\mathbb{P}^{n-1}$. For each j there is a segment in the e_j direction contained in E and we let T_j be the cylinder obtained by "thickening" it by δ. For an appropriately chosen N, if half the points in each T_j belong to $< N$ other T_i's, then one immediately gets a lower bound on the volume of the union (hence on $\mathcal{N}_\delta(E)$) since $\sum_j |T_j| \approx 1$. On the other hand, if half the points of some T_j belong to $\geq N$ other T_i's, then one obtains a large family of tubes intersecting a line segment. Each of these belongs to a δ-neighborhood of an essentially unique 2-plane through the line segment and then one can obtain a lower bound for the volume of the union by applying the two dimensional results. The proof in [21] is also based on a (quite complicated) elaboration of this idea.

The above argument is rather unsophisticated. It is tempting to think that one should be able to incorporate techniques related to [5], but this appears difficult to do. We refer though to [15] which contains an analogue of the three circle lemma and to [22] for some further discussion and references.

Bourgain [3] uses a different type of combinatorics. We finish by stating one of his lemmas and explaining how it implies an improved partial result in formulation 1. of the Kakeya problem; corresponding improvements in the other formulations are also in [3] but require some further ideas. It is not really used that the set contains an entire line segment in each direction, just that it contains three well separated points in arithmetic progression on such a line segment. The lemma in question is

Lemma Let A and B be subsets of $\mathbb{Z}^n$ for some n, Γ a subset of $A \times B$ and define $S = \{a + b : (a,b) \in \Gamma\}$, $D = \{a - b : (a,b) \in \Gamma\}$. Assume that A, B and S have cardinality less than N. Then D has cardinality less than $CN^{2-\epsilon}$. Here $\epsilon > 0$ is an explicit numerical constant, and in particular is independent of n.

The value of ϵ is given in [3]. We note that the question of the relative size of sumsets and difference sets is a deep question in combinatorial number theory and refer the reader to Ruzsa's work, for example the survey article [12].

Given the lemma, one can see that a Kakeya set E satisfies $\mathcal{N}_\delta(E) \geq \delta^{-\alpha(n-1)}$ with $\alpha > \frac{1}{2}$ in the following way [3]. Let G be the lattice $\delta\mathbb{Z}^n \subset \mathbb{R}^n$, and for each of the segments $\{x + te : |t| \leq \frac{1}{2}\}$ in the definition of Kakeya set, let x^+ and x^- be the elements of G closest to $x + \frac{1}{2}e$ and $x - \frac{1}{2}e$ respectively. Let A be the set whose elements are the various x^+ and x^- and define $\Gamma \subset A \times A$ to be the set of pairs (x^+, x^-); then let S be the set of sums $x^+ + x^-$. Evidently, $|A| \lesssim \mathcal{N}_\delta(E)$, and in addition, $|S| \lesssim \mathcal{N}_\delta(E)$, since the midpoint $\frac{1}{2}(x^+ + x^-)$ is within $C\delta$ of $x \in E$. But it is equally clear that each point of $\mathbb{P}^{n-1}$ is within $C\delta$ of some difference $x^+ - x^-$. Thus $\delta^{-(n-1)} \lesssim \mathcal{N}_\delta(E)^{2-\epsilon}$, as claimed.

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