Discrete Analogues of Singular and Maximal Radon Transforms

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Abstract. We describe recent results concerning ℓ^p estimates for certain discrete operators and the application of methods of analytic number theory in the treatment of these operators.

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We would like to discuss recent joint work with E. M. Stein concerning estimates for certain "discrete" operators of harmonic analysis, the difference between these operators and analogous older "continuous" operators, and the role ideas of analytic number theory play in resolving the extra difficulties arising in studying these discrete operators.

We begin by recalling the continuous operators we have in mind. For each x in R^ℓ, we let $\gamma(x,t)$ be a smooth k-dimensional surface passing through x. That is $\gamma(x,t)$ is a smooth mapping of $R^\ell \times R^k \to R^\ell$ with $\gamma(x,0) = x$. We also let $K(t)$ be a smooth Calderon-Zygmund kernel on R^k. That is $K(t)$ is smooth away from the origin, for $0 < a < b$, $\int_{a \le |t| \le b} K(t)dt = 0$, and for positive λ, $K(\lambda t) = \lambda^{-k} K(t)$. We set

$$Sf(x) = \int f(\gamma(x,t))K(t)dt,$$

and

$$Mf(x) = \sup_R \frac{1}{|B(R)|} \int_{B(R)} f(\gamma(x,t))dt.$$

The following is a rough version of the type of result we have in mind.

Theorem 1: [Christ, Nagel, Stein, Wainger]. See [CNSW].

If $\gamma(x,t)$ satisfies an appropriate curvature condition, S is locally bounded in $L^p(R^\ell)$, $1 < p < \infty$ and Mf is locally bounded in $L^p, 1 < p$.

Here $B(R)$ is the ball in R^k of radius R centered at the origin, and $|B(R)|$ denotes its measure. S and M are called the singular and maximal Radon transforms respectively.

To make the rough statement correct one has to modify the definitions of S and M by introducing appropriate cut off functions. For our purposes it will not be necessary to know the precise formulation of the curvature condition,but for the sake of completeness we give two of several equivalent formulations. One way of expressing the curvature condition is in terms of vector fields. It can be shown

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that for any smooth $\gamma(x,t)$ there is a unique family of vector fields X_α on R^ℓ so that we have an asymptotic formula

$$\gamma(x,t) \sim \exp[\Sigma t^\alpha X_\alpha](x)$$

$\alpha = (\alpha_1, \cdots, \alpha_k)$ with $\alpha_1, \ldots, \alpha_k$ integers. Here exp is the ordinary exponential map and the meaning of $\sim$ is that if we only include terms with $\alpha_1 + \cdots + \alpha_k \leq N$, the error is $\mathcal{O}((t)^{N+1})$. Then the curvature condition is satisfied if the X_α and their commutators span R^ℓ at every x. If $\gamma(x,t)$ is real analytic, the curvature condition can be expressed in terms of invariant manifolds of the flow $t \to \gamma(x,t)$. If $\gamma(x,t)$ is real analytic the curvature condition is satisfied if for no x there is a small piece of submanifold passing through x of positive codimension invariant under the flow $t \to \gamma(x,t)$. If $\gamma(x,t)$ is smooth the curvature condition may be expressed by saying there is no submanifold of positive codimension invariant to infinite order in an appropriate sense.

We have the following corollary of Theorem 1.

COROLLARY: *If $\gamma(x,t)$ is real analytic and f is in $L^p, 1 < p$*

$$\lim_{\epsilon \to 0} \frac{1}{|B(\epsilon)|} \int_{B(\epsilon)} f(\gamma(x,t))dt = f(x) \quad a.e.$$

Theorem 1 has a history of over 30 years, and others have contributed steps leading to the proof. Among these people are Fabes, Geller, Greanleaf and Uhlman, D. Mueller, Phong, Ricci, and Riviere. See references cited in [CNSW].

The conclusion of Theorem 1 does not hold for an arbitrary smooth $\gamma(x,t)$. See [NW] and [SW1]. The conclusion may however hold in some cases where the curvature condition fails. For example, the conclusions hold if $\gamma(x,t) = x + \Gamma(t)$ and $\Gamma(t)$ is a straight line through the origin. Some of the people who considered the problem of obtaining L^p estimates for S and M when the curvature condition fails are Carbery, H. Carlsson, Christ, Cordoba, Duoandikoetxea, Nagel, Rubio de Francia, Seeger, Vance, Wainger, Weinberg, and Ziesler. See references cited in [CWW] and [WWZ].

The effect of curvature is more dramatic in a related question – that of spherical averages, and we digress to discuss this problem. Denote by

$$Af(x) = \sup_{r>0} \int_\Sigma |f(x - ry'))d\sigma(y')$$

where $\sum$ is the unit sphere in R^ℓ, $\ell \geq 2$, and $d\sigma(y')$ is normalized rotationally invariant measure on Σ.

THEOREM 2: [STEIN $\ell \geq 3$, BOURGAIN $\ell = 2$]

$$\|Af\|_{L^p} \leq C(\ell,p)\|f\|_{L^p}$$

if $p > \frac{\ell}{\ell-1}$ and $\ell \geq 2$.

See [S], [B1] and also [MSS]. As a corollary one finds that as $r \to 0$

$$\int_\Sigma f(x-ry')d\sigma(y') \to f(x) \qquad \text{a.e.}$$

if f is in $L^p(R^\ell), p > \frac{\ell}{\ell-1}$.

To see the effect of curvature consider

$$Bf(x) = \sup_{r>0} \int_{Q_r} f(x-y')dq_r(y')$$

where Q_r is the boundary of a cube of diameter r and faces parallel to the coordinate hyperplanes, and dq_r is $\ell-1$ dimensional Lebesgue measure on Q_r normalized so that Q_r has measure 1. Let U be the set of all points which are on those hyperplanes which are parallel to a fixed coordinate hyperplane and at a rational distance from it. Take f to be the characteristic function of U. Then $f=0$ a.e. and $Bf = 1/2^\ell$ a.e. so there can be no analogue of Theorem 2 in this setting.

We now describe the discrete analogues of S and M. Let $P(x,t)$ be a polynomial mapping from $R^\ell \times R^k$ with integer coefficients. Denote by Z^ℓ the lattice points in R^ℓ, that is points with integral coordinates. Let f be a function defined on Z^ℓ. For m in Z^ℓ, set

$$\mathcal{S}f(m) = \sum_{\substack{n\in Z^k\\ n\neq 0}} K(n)f(P(m,n)),$$

and

$$\mathcal{M}f(m) = \sup_R \sum_{\substack{n\in Z^k\\ |n|\leq R}} |f(P(M,n))|.$$

We are then interested in obtaining estimates for $\mathcal{S}$ and $\mathcal{M}$ in $\ell^p(Z^\ell)$. The first results were in the translation invariant case, namely the case that $P(m,n) = m - Q(n)$ where Q is a polynomial mapping from R^k to R^ℓ with integer coefficients. (In the continuous situation results in the translation invariant case were also obtained many years before Theorem 1 was proved in full generality). The known results in this translation invariant case are the following:

Theorem 3: Arkhipov and Oskolkov [1987] for $k=1$, Stein and Wainger [1990] for general k. See [A0] and [SW2].

$$\|\mathcal{S}f\|_{\ell^2} \leq A\|f\|_{\ell^2}.$$

Theorem 4: Bourgain [1988-1989]. See [B2].

$$\|\mathcal{M}f\|_{\ell^p} \leq A_p\|f\|_{\ell^p}, 1<p.$$

Theorem 5: Stein and Wainger [1990]. See [SW2].

$$\|\mathcal{S}f\|_{\ell^p} \leq A_p\|f\|_{\ell^r}, \qquad \frac{3}{2} < p < \frac{5}{2}.$$

There is a recent result in which the operator is not translation invariant. For example if $\ell = 2$ and $k = 1$ we might take

$$P(m_1, m_2, n) = (m_1 - n, m_2 - nm_1^2).$$

More generally we assume $m = (m_1, m_2)$ with m_1 in Z^k and m_2 in $Z^{\ell-k}$ and we consider operators commuting with translations in the m_2 directions. That is $P(m,n) = (m_1 - n, m_2 - Q(m_1, n))$ where Q is a polynomial mapping from $R^k \times R^k \to R^{\ell-k}$ with integer coefficients. In this situation we have the following result.

THEOREM 6: STEIN AND WAINGER [1997]. See [SW3].

$$\|\mathcal{S}f\|_{\ell^2(Z^\ell)} \leq A\|f\|_{\ell^2(Z^\ell)}.$$

We would also like to mention two related results. Suppose p_n denotes the nth prime. For m an integer, set

$$\mathcal{M}f(m) = \sup_R \frac{1}{R} \sum_{n=1}^{R} |f(m - p_n)|.$$

Then we have the following result.

THEOREM 7: WIERDL [1988]. See [W].

$$\|\mathcal{M}f\|_{\ell^p(Z)} \leq A_p\|f\|_{\ell^p(Z)}, p > 1.$$

Finally there is a partial analogue of Theorem 2. We let $N(\rho)$ denote the number of lattice points on the sphere of radius ρ in R^ℓ. ($N(\rho) = 0$ unless ρ is the square root of an integer). For m in Z^ℓ, let

$$\mathcal{A}_s f(m) = \sup_{\substack{\rho \\ s \leq \rho \leq 2s}} \frac{1}{N(\rho)} \sum_{|n|=\rho} |f(m-n)|.$$

We then have the following result.

THEOREM 8: MAGYAR [1996]. See [M].
For $\ell \geq 5$ and $p > n/(n-2)$

$$\|\mathcal{A}_s f\|_{\ell^p(Z^\ell)} \leq C_p\|f\|_{\ell^p(Z^\ell)}.$$

Theorem 3) and Theorem 7) have applications to ergodic theory. Let T be a measure preserving invertible transformation on a probability space Ω, and set $\tau f(x) = f(Tx)$. Then Theorem 7 is an important ingredient in Bourguin's ergodic theorem. An important special case of Bourgain's theorem is the following:

THEOREM 9: BOURGUIN 1988-1989. See [B2].

For any integer r and f in $L^p(\Omega), p > 1$,

$$\frac{1}{N}\sum_{n=1}^{N} \tau^{n^r} f(x)$$

converges almost everywhere in Ω.

A similar result holds if the sequence n^r is replaced by the sequence of primes.

We want to deal with the following question:

QUESTION: What is the difference between the continuous and discrete problems?

SHORT ANSWER: The difference between sums and integrals.

One of the most dramatic differences between sums and integrals can be seen by considering two functions $A(s) = \int_1^\infty \frac{dt}{t^s}$ and $B(s) = \sum_{n=1}^\infty \frac{1}{n^s}$. Both $A(s)$ and $B(s)$ are defined for Re $s > 1$ and have meromorphic continuations to the entire complex plane. But there the similarity stops. $A(s) = \frac{1}{s-1}$, and $B(s)$ is not bounded for s away from 1. And in fact the correct growth of $B(s)$ is one of the hardest problems in mathematics.

More to the point, certain changes of variables in integrals have no analogues for sums, and in fact estimates for integrals provide a wrong guess for analogous sums. Let λ be large and set $A(\lambda) = \int_a^b e^{2\pi i\lambda x^2}dx$ and $B(\lambda) = \sum_{a\le n\le b} e^{2\pi i\lambda n^2}$. To study $A(\lambda)$ we make a change of variables $u = \sqrt{\lambda}x$, and see $A(\lambda) = \frac{1}{\sqrt{\lambda}}\int_{a'}^{b'} e^{2\pi i u^2}du$. In effect we have normalized the situation to the case that the coefficient of u^2 is 1. Normalization procedures amounting to changes of variables in integrals, though of a more complicated nature, play an important part in the proof of Theorem 1, and these changes of variables are not available in the discrete problems. It of course follows that $|A(\lambda)| \le \frac{C}{\sqrt{\lambda}}$. On the other hand if we take λ to be an integer, $B(\lambda) \sim b - a$.

We now wish to compare continuous and discrete operators. Perhaps the easiest operators to consider are those of fractional integration of imaginary order. For j a positive integer let

$$C_j f(x) = \int_1^\infty f(x-y^j)\frac{dy}{y^{1+i\gamma}} \quad \text{and} D_j f(m) = \sum_{n=1}^\infty f(m-n^j)\frac{1}{n^{1+i\gamma}},$$

with γ real. The proof of the $L^p(R)$ boundedness of C_1 and the $\ell^p(Z)$ boundedness of D_1 are similar. The change of variables $u = y^j$ reduces the study of C_j to C_1. No such change of variables is possible for D_j and in fact D_j for $j \ge 2$ is much different from C_j or D_1 (which as we have said are similar).

We would like to consider the difference in proving the ℓ^2 boundedness D_1 and D_2. Let $\mu_j(\theta) = \sum_{n=1}^\infty \frac{1}{n^{1+i\gamma}}\exp(2\pi i n^j\theta)$ for $j = 1$ and $j = 2$. To show D_j is bounded in ℓ^2 it is sufficient (and necessary) to show $\mu_j(\theta)$ is a bounded function. Let

$$S_N^j(\theta) = \sum_{1\le n\le N} \exp 2\pi i n^j\theta,$$

QUESTION: For what θ is

$$|S_N^j(\theta)| \le AN^{1-\delta}\ ? \tag{1)}$$

If 1) holds for an interval of θ, we may sum by parts in the expression for $\mu_j(\theta)$ and conclude that $\mu_j(\theta)$ is bounded in that range of θ. Let us compare $S_n^1(\theta)$ and $S_N^2(\theta)$ at a rational point $\theta = \frac{p}{q}$ with $(p,q) = 1$. (For simplicity take N to be a multiple of q). We are then considering $S_N^j = \sum_{n=1}^N \exp 2\pi i n^j \frac{p}{q}$. We want to write $n = mq + \ell$ where m runs from 1 to $\frac{N}{q}$ and ℓ goes from 1 to q. Then

$$S_N^j\left(\frac{p}{q}\right) = \sum_{m=1}^{\frac{N}{q}} \sum_{\ell=1}^{q} \exp 2\pi i (mq+\ell)^j \frac{p}{q}.$$

$$\exp 2\pi i (mq+\ell)^j \frac{p}{q} = \exp 2\pi i (u + \ell^j)\frac{p}{q},$$

where u is an integer divisible by q. So

$$S_N^j\left(\frac{p}{q}\right) = \sum_{m=1}^{\frac{N}{q}} \sum_{\ell=1}^{q} \exp 2\pi i \ell^j \frac{p}{q},$$

and the sum on ℓ is independent of m. Thus

$$\begin{aligned} S_N^j\left(\tfrac{p}{q}\right) &= \tfrac{N}{q} \cdot \textstyle\sum_{\ell=1}^{q} \exp 2\pi i \ell^j \tfrac{p}{q} \\ &= \tfrac{N}{q} G_j(p,q) \end{aligned}$$

where $G_j(p,q) = \sum_{\ell=1}^q \exp 2\pi i \ell^j \frac{p}{q}$. If $j = 1$ $G_j(p,q) = 0$. If $j = 2$, $G_j(p,q)$ is not necessarily 0. In fact

$$|G_2(p,q)| = \begin{cases} \sqrt{q} & \text{if } q \text{ is odd} \\ \sqrt{2q} & \text{if } q \equiv 0 \pmod 4 \\ \sqrt{q} & \text{if } q \equiv 2 \pmod 4). \end{cases}$$

So

$$S_N^2\left(\frac{p}{q}\right) = \frac{N}{q} G_2(p,q) \ne \mathcal{O}(N^{1-\delta}) \tag{2)}$$

in general. The upshot is that to prove the boundedness of $\mu_1(\theta)$ we need to use the cancellation in $\Sigma \frac{1}{n^{1+i\gamma}}$ only when θ is near an integer, while to prove the boundedness of $\mu_2(\theta)$ we need to use the cancellation of $\Sigma \frac{1}{n^{1+i\gamma}}$ "near" each rational $\frac{p}{q}$.

The motivation for the proof of the boundedness of $\mu_j(\theta)$ for $j \ge 2$ comes from ideas of Hardy, Littlewood, Ramanujan and Vinogradov in analytic number theory. A typical problem concerns the number of solutions in positive integers of the equation $k = n_1^r + \cdots + n_\ell^r$ for fixed integers r and ℓ. Let us denote by $H(k)$ the number of solutions. Let $S_N(\theta) = \sum_{n=1}^N \exp 2\pi i n^r \theta$. Then

$$H(k) = \int_0^1 e^{-2\pi i k\theta}[S_N(\theta)]^\ell d\theta, \tag{3)}$$

for

$$[S_N(\theta)]^\ell = \sum_{n_1,n_2,\ldots,n_\ell} \exp 2\pi i(n_1^r + \cdots + n_\ell^r)\theta.$$

We then get a contribution to the integral in 3) exactly when $k = n_1^r + \cdots + n_\ell^r$ for some choice of integers $n_1, n_2, \ldots, n_\ell$. The idea of Hardy and Littlewood is that the main contribution to the integral in 3) comes from small intervals around rationals with denominators that are small compared to N and that if θ is in such an interval a convenient approximation to $S_N(\theta)$ can be found. See for example [HL]. Notice that 2) suggests that for θ "near" a rational with large denominator q, that is $q > N^\epsilon$. there is a non-trivial estimate for $S_N^j(\theta)$. However in the derivation of 2) we also assumed that $q << N$. One can observe that if $q >> N^j$, all the exponentials, $\exp 2\pi i n^j \frac{p}{q}$, would point in the same direction so that no cancellation could occur in the sum for $S_N^j(\theta)$. So to obtain cancellation in the sum for $S_N^j(\theta)$, we require θ to be near $\frac{a}{q}$ with $N^\epsilon \leq q \leq N^{j-\theta}$. In fact one can prove the following lemma which will be important in the sequel.

Lemma 10: *For every $\epsilon > 0$, there are constants A and δ (depending on j) such that if*

$$|\theta - \frac{p}{q}| \leq \frac{1}{q^2},\ (p,q) = 1, \quad \textit{and} \quad N^\epsilon \leq q \leq N^{j-\epsilon},$$

then

$$|S_N^j(\theta)| \leq AN^{1-\delta}.$$

See [V], where the estimate is stated in a more precise form. In our discussion of $\mu_2(\theta)$ below we shall see why good convenient approximations can be made to sums like $S_N(\theta)$ if θ is near a rational with small denominator.

The idea of using number theoretic methods to study these discrete problems was due independently to Arkhipov and Oskolkov [AO] and Bourgain [B2]. Let us show how to prove $\mu_2(\theta)$ is bounded. To simplify the notation, we shall take $\gamma = \frac{2\pi}{\ln 2}$ so that $\int_1^2 \frac{dt}{t^{1+i\gamma}} = 0$. We want to write

$$\mu_2(\theta) = \sum_{\substack{p,q \\ (p,q)=1}} M^{p,q}(\theta) + \quad \text{Error} \tag{4)}$$

where $M^{p,q}$ is the contribution from rationals $\frac{p}{q}$ near θ, where in some sense q should have small denominator. To this end we fix θ and write

$$\mu_2(\theta) = \sum_{j\geq 1} H_j(\theta)$$

where

$$H_j(\theta) = \sum_{2^j \leq n < 2^{j+1}} \frac{1}{n^{1+i\gamma}} \exp 2\pi i n^2 \theta.$$

We then set

$$M^{p,q}(\theta) = \sum_{\substack{j \\ |\theta - \frac{p}{q}| < 2^{-j(2-\epsilon)j}, q < 2^{\epsilon j}}} H_j(\theta).$$

Lemma 10 (together with Dirichilet's principle) implies that

$$\mu_2(\theta) = \sum_{p,q} M^{(p,q)}(\theta) + \quad \text{bounded error.}$$

Next we want to show that

$$M^{(p,q)}(\theta) = (\text{small in } q) \cdot \text{Integral} + \text{Error},$$

We will then be able to make appropriate changes of variables in the integral. In fact we will see that

5) $$M^{(p,q)}(\theta) = \frac{1}{q} G(p,q) I(2^{2j}(\theta - \frac{a}{q})) \quad + \quad \text{bounded error},$$

where $I(\phi) = \int_1^2 e^{2\pi i t^2 \phi} \frac{dt}{t^{1+i\gamma}}$ and $G(p,q) = \sum_{j=1}^q e^{2\pi i j^2 \frac{p}{q}}$.

Let us assume 5) is true for the moment. Then

6) $$|G(p,q)| \leq A q^{1-\delta}$$

by Lemma 10. Also since $\int_1^2 \frac{dt}{t^{1+i\gamma}} = 0$

7) $$|I(\phi)| \leq A|\phi|.$$

Now a change of variables shows that for $1 \leq s \leq 2$

$$|\int_1^s e^{2\pi i t^2 \phi} dt| \leq \frac{A}{\sqrt{|\phi|}}.$$

So integrating by parts we see

8) $$|I(\phi)| \leq \frac{A}{|\phi|^{1/2}}.$$

Finally it is possible to show that for a fixed θ

9) the number of (p,q) that occur with $2^s \leq q < 2^{s+1}$ is uniformly bounded.

If the estimates 6,7,8, and 9 are substituted into 5), it is easy to see that $\mu_2(\theta)$ is bounded. So we are faced with trying to write $M^{(p,q)}$ as a product $\frac{1}{q} G(p,q) \cdot$ Integral.

$$M^{p,q}(\theta) = \sum_j{}'' H_j(\theta).$$

Recall that if we write $\theta = \frac{9}{q} + \beta$, then for j to occur in Σ'', $|\beta| < 2^{-(2-\epsilon)j}$ and $q < 2^{\epsilon j}$. Again

$$H_j = \sum_{2^j < n < 2^{j+1}} \frac{1}{n^{1+i\gamma}} exp(2\pi i n^2 \theta).$$

We write $n = mq + \ell$ with $0 \leq \ell \leq q-1$. Then since q is small

$$\text{10)} \qquad \frac{1}{n^{1+i\gamma}} = \frac{1}{(mq+\ell)^{1+i\gamma}} \sim \frac{1}{(mq)^{1+i\gamma}}$$

since $\ell < q$ is small. A more subtle point is that

$$\text{11)} \qquad \exp 2\pi i n^2 \theta = \exp 2\pi i m^2 q^2 \beta \cdot \exp 2\pi i \ell^2 \frac{p}{q} + \mathcal{O}(2^{-j/2})$$

(if ϵ is sufficiently small). To see 11) note that $n^2\theta = (mq+\ell)^2(\frac{a}{q}+\beta) = m^2q^2\beta + \ell^2\frac{a}{q} + 2m\ell q\beta +$ integer, and now since $m < 2^j$, $q < 2^{\epsilon j}$ and $|\beta| < 2^{-(2-\epsilon)j}$ the term $2m\ell q\beta$ may be dropped by making an error $\mathcal{O}(2^{-j/2})$, which gives 11. (When we come to the non-translation invariant problems we will arrive at an analogous point, however we will not have control on the size of m i.e. $m < 2^j$ which will cause a major difficulty.)

Thus 11) is established, and in $\sum''$ we may replace H_j by

$$\text{12)} \qquad \overline{H}_j = \frac{1}{q^{1+i\gamma}} G(p,q) \sum_{\frac{2^j}{q} \leq m < \frac{2^{j+1}}{q}} \frac{1}{m^{1+i\gamma}} \exp 2\pi i m^2 q^2 \beta,$$

Using once again the facts that $m < 2^j, q < 2^{\epsilon j}$ and $|\beta| < 2^{-(2-\epsilon)j}$, we see that we may replace the sum in (2) by an integral making an error which is $\mathcal{O}(2^{-j/2})$. Then a change of variables in the integral gives us 5). The proof that $\mu_2(\theta)$ is bounded is now complete.

Estimates for the maximal function as well as ℓ^p estimates are much more difficult because it does not suffice to deal with one fixed θ.

Let us try to see what is involved in proving the ℓ^2 boundedness in a non-translation invariant case. For (m,ℓ) in Z^2, and f defined on Z^2, we set

$$Sf(m,\ell) = \sum_{\substack{n \\ m-n\neq 0}} f(n, \ell - m^2 n) \cdot \frac{1}{m-n}.$$

We wish to show

$$\text{13)} \qquad \|Sf\|_{\ell^2(Z^2)} \leq A \|f\|_{\ell^2(Z^2)}.$$

For f defined on Z^1, we set

$$\text{14)} \qquad S_\theta f(m) = \sum_{\substack{n \\ m-n\neq 0}} \exp(2\pi i m^2 n \theta) \frac{f(n)}{m-n}.$$

By using a well known technique of taking the Fourier transform in the ℓ variable and using Plancheral's theorem, we see that to prove 13 it suffices to show

$$15) \qquad \|S_\theta f\|_{\ell^2(Z)} \le A\|f\|_{\ell^2(Z)},$$

uniformly in θ. We shall try to follow the lines of the proof of the boundedness of $\mu_2(\theta)$. The main idea is to replace the formula 5) by writing S_θ as a tensor product of an operator variant of the expression $\frac{1}{q}G(a,q)$ and an integral operator. We proceed with an operator valued version of the treatment of $\mu_2(\theta)$ above. We define operator valued analoges of the H_j, namely

$$16) \qquad H_j(\theta)f(m) = \sum_{\substack{n \\ 2^j \le |m-n| < 2^{j+1}}} \exp(2\pi i m^2 n\theta) \quad \frac{f(m)}{m-n}$$

and set

$$M^{(p,q)}(\theta)f(m) = \sum_{\substack{j \\ |\theta - \frac{a}{q}| \le 2^{-(3-\epsilon)j}, q < 2^{\epsilon j}}} H_j(\theta)f(m).$$

Then it is possible to prove an operator valued version of Lemma 10 so that

$$S_\theta = \sum_{p,q} M^{(p,q)}(\theta) \ + \ \text{bounded operator.}$$

We now want to write in analogy with 5)

$$M^{p,q}(\theta) \sim \frac{1}{q} G(p,q) \otimes I_\theta$$

where $G(p,q)$ is an operator valued analogue of G and I_θ is an integral operator. In analogy with the argument proving 5) in the expression 16) for H_j, we set

$$m = m_1 q + \mu \quad \text{and} \quad n = n_1 q + \nu.$$

Following the lines of the argument in the translation invariant case we would like to write $\theta = \frac{a}{q} + \beta$, and would like to say that

$$17) \qquad \begin{aligned} &\exp 2\pi i \ \ (m_1 q + \mu)^2 (n_1 q + \nu)(\tfrac{a}{q} + \beta) \\ &\qquad = \exp 2\pi i m_1^2 n_1 q^3 \cdot \exp 2\pi i \mu^2 \nu \tfrac{a}{q} + \ \text{small error.} \end{aligned}$$

Unfortunately we can not do this because while we have control on the size of $m_1 - n_1$, we have no control of the size of m_1 or n_1. And even if we could prove 17) we could not replace a sum on n_1 by an integral because we have no estimate on the size of m_1 and n_1. The main idea in getting around this difficulty is to note that in dealing with $M^{p,q}(\theta)$, $2^j < \left(\frac{1}{\beta}\right)^{\frac{1}{3-\epsilon}}$. Thus to obtain an estimate for $M^{p,q}(\theta)$, it suffices to obtain estimates of translates of the operators $M^{p,q}$ where however m and n can be assumed to be at most $\left(\frac{1}{\beta}\right)^{\frac{1}{3-\epsilon}}$. We refer to [SW3] where the complicated details are carried out.

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