Singularity and Regularity — Local and Global

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Abstract. There exists a smoothly bounded, pseudoconvex domain in $\mathbb{C}^2$ for which the Bergman projection fails to preserve the class of functions which are globally smooth up to the boundary. The counterexample is explained and placed in a wider context through a broader discussion of the local and global regularity of solutions to subelliptic and more degenerate partial differential equations in various function spaces.

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1 Introduction

Consider a bounded open set $\Omega \subset \mathbb{C}^n$, assumed always to have C^∞ boundary. The Bergman projection B is the orthogonal projection from $L^2(\Omega)$ (with respect to Lebesgue measure) onto the closed subspace consisting of all L^2 holomorphic functions. Our purpose is to explain and to place in a wider context the following counterexample.

Theorem 1. [8] *There exists a smoothly bounded, pseudoconvex domain $\Omega \subset \mathbb{C}^2$ for which the Bergman projection fails to preserve $C^\infty(\overline{\Omega})$.*

Barrett [1] had given a nonpseudoconvex example, but the issue is most natural for pseudoconvex domains. The first motivation was Bell and Ligocka's discovery that if $C^\infty(\overline{\Omega})$ were always preserved then any biholomorphic mapping between two (smoothly bounded) pseudoconvex domains would extend smoothly to a diffeomorphism of their closures; this in turn would have implications for the classification of domains up to biholomorphism by means of boundary invariants.[1] Secondly, it is one of many problems about the regularity of solutions of the $\bar\partial$–Neumann problem and related PDE.

This paper stresses the author's own work. Because of rigid limitations on the lengths of text and bibliography, the important contributions of many authors are slighted, including S. Baouendi, E. Bernardi, A. Bove, D. Catlin, S.-C. Chen, D. Geller, C. Goulaouic, N. Hanges, B. Helffer, A. A. Himonas, M. Derridj, V. Grušin, G. Komatsu, J. J. Kohn, G. Métivier, Pham The Lai, D. Robert, N. Sibony, D. Tartakoff, and C.-C. Yu. A more complete bibliography and discussion are in [10].

[1]The question of boundary extendibility of biholomorphic mappings remains open.

2 Some Background

Except for very symmetric domains, the best method known for analyzing the Bergman projection is by means of the $\bar{\partial}$–Neumann problem. This is a boundary value problem[2] $\Box u = f$ on Ω, with boundary conditions $u \lrcorner \bar{\partial}\rho = 0$ and $\bar{\partial} u \lrcorner \bar{\partial}\rho = 0$ on $\partial\Omega$, where u, f are $(0,1)$ forms, ρ is any defining function for Ω, $\Box = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ and $\lrcorner$ denotes the interior product of forms.

$\Box$ is simply the Laplacian times the identity matrix, but the boundary conditions are noncoercive. In $\mathbb{C}^n$ a Dirichlet condition is imposed on one of the n components of u; on each of the other components is imposed a complex Neumann condition; however the problem does not decouple into separate scalar problems, instead there is an interaction between the good (Dirichlet) component and bad (complex Neumann) components. This interaction, and consequently the regularity of solutions, depend heavily on the complex geometry of the boundary.

For any pseudoconvex, bounded, smoothly bounded domain Ω there exists for each $f \in L^2$ a unique solution $u \in L^2$ which satisfies the boundary conditions in an appropriate sense; the bounded linear operator N mapping f to u is called the Neumann operator. The Bergman projection is related to N by Kohn's formula $B = I - \bar{\partial}^* N \bar{\partial}$. In particular, if the $\bar{\partial}$–Neumann problem is *globally regular* in the sense that N preserves $C^\infty(\overline{\Omega})$, then B also preserves $C^\infty(\overline{\Omega})$. In $\mathbb{C}^2$ these properties are actually equivalent; there is a less simply formulated generalization in higher dimensions.

More commonly studied is hypoellipticity. The $\bar{\partial}$–Neumann problem is said to be hypoelliptic (in C^∞) if for every $p \in \overline{\Omega}$ and $f \in L^2(\Omega)$ which is C^∞ near p, the solution u likewise is C^∞ near p. A partial differential operator L is said to be hypoelliptic (in C^∞) in an open set U if for any distribution, $u \in C^\infty$ in any open subset of U where $Lu \in C^\infty$. These notions can be modified by replacing C^∞ by other function classes such as C^ω, the real analytic functions, or G^s, the Gevrey classes. Hypoellipticity implies global regularity.

The issue in hypoellipticity is whether N *transports* singularities in f from one place to another, while in global regularity the issue is whether N *creates* singularities out of nothing. We will argue in §5 that this point of view, though literally correct, is misleading.

Global regularity is a very weak property. A standard example is $L = \partial_{x_1} + \alpha\partial_{x_2}$ on a two-torus, where $\alpha \in \mathbb{R}$ is constant; L is globally regular, unless α has exceptional Diophantine properties, yet is never hypoelliptic. Similarly, on any compact Lie group, convolution with any distribution preserves $C^\infty(G)$.

The $\bar{\partial}$–Neumann problem is hypoelliptic if Ω is strictly pseudoconvex or more generally is of finite type. The latter condition is necessary for subellipticity, but not for hypoellipticity; see for instance [13].

For $\mathbb{C}^2$, the $\bar{\partial}$–Neumann problem is closely related to sums of squares of (two) real vector fields in a three real dimensional space.[3] Indeed, the general method of reduction to the boundary reduces matters to an equation $\Box^+ v = g$ on $\partial\Omega$, where the pseudodifferential Calderón operator $\Box^+$, near its characteristic variety

[2] In this paper only the $\bar{\partial}$–Neumann problem for forms of bidegree $(0,1)$ will be discussed.

[3] In higher dimensions matters are more subtle.

$\Sigma \subset T^*\partial\Omega$ (and modulo an elliptic factor), takes the form $\bar\partial_b \circ \bar\partial_b^*$, modulo certain lower order terms which are omitted here to simplify the exposition. Here $\bar\partial_b$ is a Cauchy-Riemann operator associated to the CR structure on $\partial\Omega$; thus the complex geometry of $\partial\Omega$ enters the problem quite directly. Locally $\bar\partial_b = X + iY$ where X, Y are everywhere linearly independent, smooth real vector fields. Thus $\bar\partial_b \circ \bar\partial_b^* = -X^2 - Y^2 + i[X,Y]$ modulo relatively harmless lower order terms. Pseudoconvexity guarantees that the principal symbol of $i[X,Y]$ is nonnegative near Σ, so it does not substantially alter the character of $-X^2 - Y^2$. Henceforth we assume always that $n = 2$.

The $\bar\partial$–Neumann problem is said to be *compact* if N is a compact mapping from L^2 to L^2. It is *exactly regular* in the Sobolev space H^s if N maps $H^s(\Omega)$ to itself, and is simply said to be exactly regular if it is exactly regular in H^s for every $s \ge 0$. A simple perturbation argument shows that for any Ω there exists $\delta > 0$ such that exact regularity holds in H^s for all $0 \le s < \delta$. Subellipticity implies compactness, which implies exact regularity, which implies global regularity in C^∞. All existing proofs of global regularity proceed by establishing exact regularity. The other two implications just stated are not reversible; nor does compactness imply hypoellipticity.

Compactness is easily shown to fail for domains in $\mathbb{C}^2$ whose boundaries contain one-dimensional complex disks. No satisfactory characterization is known; Matheos [19] has constructed Hartogs domains in $\mathbb{C}^2$ whose boundaries contain no complex disks, yet for which N is noncompact.

Global regularity can hold without compactness. It holds in the presence of sufficient symmetry, no matter how degenerate the domain. A related but deeper theorem of Boas and Straube [3] requires only an approximate symmetry: it suffices to have a smooth real vector field T on $\partial\Omega$ which is everywhere transverse to the complex tangent space, and for which $[T,X]$ and $[T,Y]$ belong everywhere to the span of X, Y (where X, Y denote the real and imaginary parts of $\bar\partial_b$ in local coordinates). Moreover a weaker approximate version of this condition still suffices [3], and is quite important.

An interesting special class of domains consists of those for which the set W of all weakly pseudoconvex boundary points is a smooth one-dimensional complex manifold with boundary. To any such domain is associated [4] a cohomology class $\alpha \in H^1(W)$, which vanishes if and only if there exists a vector field T having the required weaker version of the above commutation property. α also admits complex geometric descriptions. Consequently global regularity holds (a) whenever W is *simply connected*, and (b) (paradoxically) whenever the CR structure is *sufficiently degenerate* near W.

In the negative direction, Kiselman [18] proved that for certain nonsmooth domains with corners, both exact and global regularity fail. Barrett [2] extended the analysis to show that for the famous worm domains, exact regularity cannot hold for large s; this left open the possibility that N might map H^s to $H^{s-\varepsilon}$ for all $s \ge 0$ and $\varepsilon > 0$. Roughly speaking, he produced Kiselman's domains as limits of blowups of worm domains and used the common scaling of the two sides in the inequality $\|Nu\|_{H^s} \le C\|u\|_{H^s}$ to pass from exact regularity for worm domains to the same for Kiselman's domains.

The worm domains were originally invented by Diederich and Fornæss [14] as examples of smoothly bounded, pseudoconvex domains whose closures lack[4]. arbitrarily small pseudoconvex neighborhoods. A worm domain $\mathcal{W} \subset \mathbb{C}^2$ takes the form

$$\mathcal{W} = \{z : |z_1 + e^{i \log |z_2|^2}|^2 < 1 - \phi(\log |z_2|^2)\} \tag{1}$$

with the following properties: (i) $\mathcal{W}$ has smooth boundary and is pseudoconvex; (ii) $\phi \in C^\infty$ takes values in $[0,1]$, vanishes identically on $[-r,r]$ for some $r > 0$, and vanishes nowhere else; and (iii) $\mathcal{W}$ is strictly pseudoconvex at every boundary point where $|\log |z_2|^2| > r$. There do exist ϕ for which these properties hold [14]. The two caps, where $|\log |z_2|| > r$, serve to make $\mathcal{W}$ be bounded. Properties of worm domains include: (iv) The set of all weakly pseudoconvex points of $\partial \mathcal{W}$ is the annular complex manifold with boundary $\mathcal{A}_r = \{z : z_1 = 0 \text{ and } |\log |z_2|^2| \le r\}$. (v) The cohomology class $\alpha \in H^1(\mathcal{A}_r)$ is nonzero. (vi) There is a one-parameter global symmetry group, $\rho_\theta(z) = (z_1, e^{i\theta} z_2)$ for $\theta \in \mathbb{R}$.

3 Comments on the proof

The proof of Theorem 1 demonstrates that global regularity fails for all worm domains; moreover N and B fail to map $C^\infty(\overline{\mathcal{W}})$ to H^s, where $s(r)$ tends to zero as $r \to \infty$. Siu [24] has given an alternative proof that there exist worm domains for which B fails to map $C^\infty(\overline{\mathcal{W}})$ to a Hölder class $\Lambda_s(\overline{\mathcal{W}})$; he obtains good control on the dependence of s on r. *Grosso modo* he shows that the two caps can be chosen so that their effects on $B(f)$ cancel for a certain f, reducing matters to Kiselman's analysis. Both proofs exploit special features of worm domains and appear quite limited in scope. Only the original proof will be discussed here.

Boundary reduction leads to a global regularity problem for a pseudodifferential equation on the real three-dimensional manifold $\partial \mathcal{W}$; the pseudodifferential operator is closely analogous to $-X^2 - Y^2$ for certain real vector fields. With respect to the symmetries ρ_θ, $L^2(\partial \mathcal{W})$ decomposes by Fourier analysis into orthogonal subspaces $\mathcal{H}_j$. The equation respects this decomposition. Fixing any such j, one may identify functions in $\mathcal{H}_j$ with functions of two real variables.

A model captures the essence of the situation. Fix an open neighborhood V of $A = [-r,r] \times \{0\} \subset \mathbb{R}^2$, with coordinates (x,t). Let $L = -X^2 - Y^2 + b(x,t)$ where $X = \partial_x$, Y is a real vector field which in the region $|x| \le r$ takes the form $[a(x)t + O(t^2)]\partial_t$ with a nowhere vanishing, and $X, Y, [X,Y]$ span the tangent space everywhere on $V \backslash A$. Suppose moreover that Re $\langle Lu, u \rangle \ge c\|u\|_{L^2}^2$ for all $u \in C^2$ supported in V, and likewise for the transpose of L.

The last hypothesis mimics the $L^2(\mathcal{W})$ boundedness of N. A corresponds to the set of all weakly pseudoconvex points in $\partial \mathcal{W}$; L is hypoelliptic on $V \backslash A$. The condition $a(x) \ne 0$ corresponds to the nonvanishing of $\alpha \in H^1(\mathcal{W})$; if $a(x,t) \equiv 0$ for $|x| \le r$, then L is more degenerate but paradoxically becomes globally regular, as follows from the method of [3]. Under the hypotheses stated, there exists $u \notin C^\infty(V)$ such that $Lu \in C^\infty(V)$. The proof is quite indirect; no construction of singular solutions is known to me. Of its three steps, the principal one is:

[4]Provided that the parameter r below is $\ge \pi$.

Proposition 2. *There exists a discrete set $\Sigma \subset [0,\infty)$, with $0 \notin \Sigma$, so that for every $s \notin \Sigma$, one has $\|u\|_{H^s} \le C_s \|Lu\|_{H^s}$ for every $u \in C_0^\infty(V)$.*

The hypotheses ensure that $L^{-1} : L^2 \mapsto L^2$ is well defined and bounded. Step 2 is to show[5] that L^{-1} cannot map H_0^s to H^s for large s. Supposing the contrary, scaling $(x,t) \mapsto (x, \lambda t)$, and letting $\lambda \to \infty$ as in [2], one deduces that the limiting operator $\mathcal{L} = -\partial_x^2 - (a(x)t\partial_t)^2 + b(x,0)$ on $[-r,r] \times (0,\infty)$, with Dirichlet boundary conditions at $x = \pm r$, must be exactly regular in a (homogeneous) Sobolev space of the same order s.

Applying the Mellin transform with respect to t and conjugating by ∂_t^s, one arrives at ODEs $-\partial_x^2 - a^2(x)(s+i\tau)^2 + b(x,0)$, for $\tau \in \mathbb{R}$, with Dirichlet boundary conditions on $[-r,r]$. There must exist nonlinear eigenvalues $\sigma + i\tau$ for which there are nonzero solutions f (with $f(\pm r) = 0$); $f(x)t^{\sigma+i\tau}$ is then a solution of the two variable Dirichlet problem, and is singular at $t = 0$. Consequently L^{-1} cannot preserve H^s for $s > \sigma + \frac{1}{2}$.

Step 3 is merely to observe that if L^{-1} did map $C_0^\infty(V)$ to $C^\infty(V)$ then because L^{-1} is bounded on L^2, a density argument combined with step 1 would imply that L^{-1} maps $H_0^s(V)$ to $H^s(V)$, for all $s \notin \Sigma$, contradicting step 2.

The more intricate analysis for Step 1 divides naturally into three overlapping regions: (i) the complement of A, where L is subelliptic; (ii) the Cartesian product of $[-r,r]$ with an arbitrarily small neighborhood of 0, where the natural tool is Mellin analysis in the t coordinate and reduction to properties of the family of one dimensional Dirichlet problems described above, modulo certain error terms; and (iii) an arbitrarily narrow transitional region $r \le |x| < r + \delta$, for which little information is available; $\|u\|_{L^2} \le C\delta^{+1}\|\partial_x u\|_{L^2}$ for functions supported there. The final ingredient is an *a priori* inequality $\|\partial_x u\|_{H^s} \le C_s\|Lu\|_{H^s} + C_s\|u\|_{H^s}$. This combined with the three region analysis yields the proof.

4 Other regularity problems

Henceforth we discuss the regularity of solutions of $Lu = f$ where $L = \sum_j X_j^2$ and the X_j are real vector fields in some open set or compact manifold without boundary, denoted in either case by V. Their coefficients are assumed to belong to whichever function space we are working in. (Many of the results do however have analogues for the $\bar\partial$–Neumann problem.) Regularity in C^∞, C^ω and to a lesser extent G^s, will be discussed, in both the global and local (that is, hypoellipticity) senses. There are two types of positive results for each function space $\mathcal{F}$: (a) If L is sufficiently strong then it is hypoelliptic in $\mathcal{F}$. (b) If L is arbitrarily weak but satisfies an appropriate commutation condition then it is still hypoelliptic in $\mathcal{F}$. We assume an inequality valid for all $u \in C_0^2$:

$$\int |\hat u|^2(\xi) w^2(\xi)\, d\xi \le C \sum_j \|X_j u\|_{L^2}^2, \tag{2}$$

where $w(\xi) \to \infty$ as $|\xi| \to \infty$, suitably interpreted in the manifold case.

[5] Certain points are slurred over in the discussion for the sake of brevity.

C^∞*, global.* (a) The validity of (2) with some $w \to \infty$ is equivalent to compactness, which implies global regularity. A type (b) result is that of Boas and Straube [3]; here it is not required that $w \to \infty$.

C^∞*, local.* (a) If $w(\xi)/\log|\xi| \to \infty$ as $|\xi| \to \infty$ then L is C^∞ hypoelliptic [20]. This is sharp in general. A consequence [13] is hypoellipticity of the $\bar\partial$–Neumann problem for any domain in $\mathbb{C}^2$ for which the set of weakly pseudoconvex points is a real hypersurface $M \subset \partial\Omega$ transverse to the complex tangent space, for which the Levi form λ is $\gg \exp(-c\,\text{distance}\,(z,M)^{-1})$ for all $c > 0$. A result of type (b) is roughly as follows; for more precise statements see [17],[21] and the many references therein.

Suppose that for any ray $R \subset T^*V$ and any small conic neighborhood Γ of R there exists a scalar valued symbol $0 \le \psi \in S^0_{1,0}$ such that $\psi \equiv 0$ in some smaller conic neighborhood of R, $\psi \ge 1$ on $T^*V\backslash\Gamma$, and such that for each $\delta > 0$ there exists $C_\delta < \infty$ such that for any relatively compact open subset $U \Subset V$ and for all $u \in C^2_0(U)$ and each index i,

$$\|\,\mathrm{Op}\left[\log\langle\xi\rangle\{\psi,\sigma(X_i)\}\right]u\|^2 \le \delta \sum_j \|X_j u\|^2 + C_\delta\|u\|^2 \tag{3}$$

Then L is hypoelliptic, indeed microhypoelliptic, in V. Here $\mathrm{Op}(\cdot)$ denotes the pseudodifferential operator with the indicated symbol, and $\{\cdot\}$ the Poisson bracket.

C^ω*, local.* (a) $w(\xi) \ge c|\xi|$ is equivalent to ellipticity, which by a theorem of Petrowsky, implies analytic hypoellipticity. (b) Denote by $\Sigma \subset T^*V$ the characteristic variety of L. By assumption, Σ is conic. Assume that Σ is a manifold, and that the symbol of L vanishes to order exactly two at each point of Σ. Suppose that for each $p \in T^*V$ and each small neighborhood W of p, there exists $\psi \in C^\omega(W)$ such that $\psi(p) = 0$, $\psi > 0$ near the boundary of W, and $H_{\sigma_j}(\psi) \equiv 0$ in W, where H_{σ_j}, here and below, denotes the Hamiltonian vector field associated to the principal symbol of X_j. Then L is analytic hypoelliptic, by a theorem of Grigis and Sjöstrand[6] [16]. A closely related commutation condition appears in the work of Tartakoff.

G^s*, local.* (a) If (2) holds with $w(\xi) = |\xi|^{1/s}$ then L is hypoelliptic in the Gevrey class G^s by a theorem of Derridj and Zuily; this is the optimal condition on w. A type (b) result is in [17]. Two examples indicate the intricacy of the problem. (i) [9] In $\mathbb{R}^3$ with coordinates (x, y_1, y_2), the operator $\partial_x^2 + x^{2(m-1)1}\partial_{y_1}^2 + x^{2(n-1)}\partial_{y_2}^2$ is hypoelliptic in G^s if and only if $s \ge \max(n/m, m/n)$; however it satisfies (2) only with $w(\xi) \sim |\xi|^{1/\max(n,m)}$. (ii) [11] In $\mathbb{R}^2$ with coordinates (x,t), for $p \ge 1$, $\partial_x^2 + x^{2(m-1)}\partial_t^2 + x^{2(m-1-k)}t^{2p}\partial_t^2$ is hypoelliptic in G^s if[7] $s^{-1} \le 1 - \tilde p^{-1}(1 - m^{-1})$ where $\tilde p = p(m-1)/k$. The optimal w here is $\sim |\xi|^{1/m}$. In the positive direction these results were obtained independently and in greater generality by Matsuzawa, and were also proved by Bernardi, Bove and Tartakoff. The negative result for (ii) for $m = 2, k = 1, p = 1$ is due to Métivier. An intriguing conjecture of Treves [26] proposes to relate analytic hypoellipticity to the fine symplectic geometry of the

[6] The theorem is not formulated explicitly but does seem to be proved in [16].

[7] I am confident that this exponent can be proved to be optimal for many parameters m, p, k, by the method used in [7] to disprove analytic hypoellipticity, but have not verified the details.

characteristic variety Σ of L; these examples illustrate that at least for $s > 1$, G^s hypoellipticity is not controlled by Σ alone.

C^ω, global. The result and method in [7] show that there is no better result of type (a) than for local C^ω regularity. I know of no really satisfactory general result of type (b), although there are many particular results of that flavor.

We turn to results in the negative direction, concentrating on the C^ω case. The theory here is fragmentary, with a large gap between counterexamples and the results above. A common structure underlies the proofs. To L one associates a one-parameter family of simpler operators, $\mathcal{L}_z$; in all the results below, these are ordinary differential operators.[8] In simple cases, solutions to the ODE lead to solutions of $Lu = 0$, by separation of variables. One proves the existence of at least one nonlinear eigenvalue $\zeta \in \mathbb{C}$ for which $\mathcal{L}_\zeta$ has a nonzero solution f_ζ in the Schwartz class on $\mathbb{R}^1$.

Analytic hypoellipticity implies that all solutions of $Lu = f$ satisfy certain *uniform* Cauchy-type inequalities in terms of f. When separation of variables applies, scaling and f_ζ lead to a one-parameter family of solutions of L which violate any such Cauchy inequalities as $\lambda \to \infty$. For instance, for the Baouendi-Goulaouic example $\partial_x^2 + x^2\partial_t^2 + \partial_y^2$, one has solutions $u = \exp(i\lambda t + i\zeta\lambda^{1/2}y)f(\lambda^{1/2}x)$ where $-\zeta^2, f$ are a Hermite eigenvalue and corresponding eigenfunction. This method was pioneered by Oleĭnik and Radkevič [22], and developed much further, to situations where separation of variables does not apply directly, by G. Métivier.

Theorem 3 is a bit more complicated, and the proofs of Theorems 4 and 5 are even more intricate, because separation of variables does not apply directly. The latter two theorems rely on reasoning by contradiction. Assuming the Cauchy inequalities, the structure of the equation is used to deduce stronger *a priori* bounds on solutions. Exact solutions of $Lu_\lambda = f_\lambda$ for precisely chosen f_λ are then proved to be well controlled by solutions of a simpler related partial differential equation, which in turn can be analyzed by separation of variables. Eventually solutions which are supposed to be holomorphic in certain regions are proved to have poles, a contradiction. This reasoning has elements in common with the proof of global C^∞ irregularity for the worm domains.

THEOREM 3. [5] *Consider $L = X^2 + Y^2$ in $\mathbb{R}^3$, where X, Y are linearly independent at each point. Suppose there exists a nonconstant curve $\gamma \subset \mathbb{R}^3$ such that at each point $p \in \gamma$, the tangent vector $\dot\gamma(p)$ is in the span of X, Y, and moreover $X, Y, [X, Y]$ fail to span the tangent space to $\mathbb{R}^3$ at p. Then L is not analytic hypoelliptic.*

This is a very special case of an older conjecture of Treves [25].

Next consider $L = X^2 + Y^2$ in an open subset of $\mathbb{R}^2$, and $\tilde L = (X + iY)(X - iY)$, where X, Y do not simultaneously vanish at any point. Assume the bracket hypothesis; for $\tilde L$ we also impose a certain natural pseudoconvexity hypothesis (see [6]). The positive parts of the following theorem are special cases of an old theorem of Grušin.

[8] Barrett has studied nonlinear eigenvalue problems for elliptic PDE on smoothly bounded Riemann surfaces, which are relevant to global regularity for the $\bar\partial$–Neumann problem.

THEOREM 4. [6] *$\tilde{L}$ is microlocally analytic hypoelliptic if and only if there exist coordinates (x,t) in which* $\text{span}\{X,Y\} = \text{span}\{\partial_x, x^{m-1}\partial_t\}$*, as $C^\omega(\mathbb{R}^3)$-modules, for some $m \geq 1$. For generic*[9] *pairs X,Y, L is analytic hypoelliptic if and only if the same condition holds.*

The generalization to more than two vector fields (for L) is straightforward, but matters are much subtler in $\mathbb{R}^n$ for $n > 2$.

THEOREM 5. [7] *There exists a bounded, pseudoconvex domain $\Omega \subset \mathbb{C}^2$ with C^ω boundary, for which the Szegö projection fails to preserve $C^\omega(\partial\Omega)$.*

F. Tolli has shown that, in contrast to the C^∞ case, there exists such a domain which is strictly pseudoconvex except at a single isolated point.

5 A METRIC IN PHASE SPACE

For definiteness let $L = \sum X_j^2$ be a sum of squares of vector fields, in an open subset of $\mathbb{R}^n$. Let $\sigma_j(x,\xi)$ be the principal symbol of X_j and H_{σ_j} the associated Hamiltonian vector field in $T^*\mathbb{R}^n$. Assume the bracket hypothesis of Hörmander to hold to some order $\leq m$; define the effective symbol $\tilde{\sigma}(x,\xi)$ to be the square root of $\sum_I |\sigma_I(x,\xi)|^{2/|I|}$, where each σ_I is an iterated Poisson bracket of the functions σ_j, $I = (j_1, \dots j_{|I|})$, $1 \leq |I| \leq m$.

All the positive results above are consistent with a vague and partly conjectural principle: "energy" propagates in phase space along the integral curves of H_{σ_j}, while decaying at a rate dictated by $\tilde{\sigma}$. An analogue is the Feynman-Kac formula for $-\Delta + V$ with potential $V \geq 0$; heat propagates along Brownian paths, decaying at a relative rate proportional to V. From this point of view, global and local regularity are similar notions; the former fails when too much energy is transported from small $|\xi|$ to large $|\xi|$, whereas (micro)local regularity fails whenever too much energy is transported from any one place to another in phase space.

To make this more precise we define [12] a metric ρ_L on $T^*\mathbb{R}^n$: $\rho_L(p,q)$ is the supremum of $|\psi(p) - \psi(q)|$, over all C^1 functions $\psi : T^*\mathbb{R}^n \mapsto \mathbb{R}$ satisfying (i) $|H_{\sigma_j}\psi| \leq \tilde{\sigma}$ and (ii) $|\xi|^{-1}|\nabla_x\psi| + |\nabla_\xi\psi| \leq 1$. This definition is distinct from a phase space metric introduced by Fefferman [15] and Parmeggiani [23]; ρ_L is unchanged if L is multiplied by a constant.

Points $(x,\xi), (x',\xi')$ are said to be δ-separated if $|x-x'|+(|\xi|+|\xi'|)^{-1}|\xi-\xi'| \geq \delta$. Denote by ρ_Δ the metric associated by the above definition to the Laplacian; essentially $d\rho_\Delta^2 = |\xi|^2 dx^2 + d\xi^2$.

The results concerning C^ω/G^s hypoellipticity discussed in this paper are consistent [12] with the requirement that for each $\delta > 0$ there exists $c_\delta < \infty$ such that for all δ-separated pairs p,q, $\rho_L(p,q) \geq c_\delta \rho_\Delta^{1/s}(p,q)$ (with $s = 1$ for $C^\omega = G^1$). For example, the exponent $[1-\tilde{p}^{-1}(1-m^{-1})]^{-1}$ encountered above is exactly predicted by this comparison inequality. The same is roughly true for C^∞ hypoellipticity, with the condition $\rho_L(p,q)/\log\rho_\Delta(p,q) \to \infty$ as $\rho_\Delta(p,q) \to \infty$, for δ-separated points p,q, provided that an effective symbol $\tilde{\sigma}$ is defined in an *ad hoc* way on a

[9] See [6]. The genericity hypothesis is needed at present solely because the underlying nonlinear eigenvalue problem is not completely solved.

case by case basis. For global regularity the same remarks apply, provided merely that δ-separatedness is replaced by the assumption that $\big||\xi| - |\xi'|\big| \ge \delta|\xi| + \delta|\xi'|$.

A fundamental question, then, is to what extent ρ_L controls the hypoellipticity and global regularity of L. Skepticism is in order because only $\nabla\psi$, rather than higher-order derivatives, is taken into account. In existing proofs of hypoellipticity, ψ belongs to an appropriate symbol class; in [16], for instance, it must be analytic, with appropriate bounds as $|\xi| \to \infty$.

A delicate example is $X^2 + Y^2$ in $\mathbb{R}^3$, with coordinates (x, y, t), where $X = \partial_x + b(x, y)\partial_t$, $Y = \partial_y + a(x, y)\partial_t$, $a, b \in C^\omega$ are real, and $\partial_x a - \partial_y b \equiv x^6 + y^6 + x^2y^2$. It is shown in [12] that (i) $\rho_L(p, q) \ge c\rho_\Delta(p, q)$ for δ-separated points, but (ii) if ψ is additionally required to belong to the standard class $S^1_{1,0}$, then the modified metric which results no longer satisfies the inequality. To determine whether or not this operator is analytic hypoelliptic might well represent a substantial advance. It would also be desirable to have proofs of negative results based on the same point of view as ρ_L, rather than the nonlinear eigenvalue method.

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