Analytic Aspects of Quasiconformality

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Abstract. We discuss recent advances in quasiconformal mappings.

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1. Quasiconformal mappings

Quasiconformal mappings are homeomorphisms which on the infinitesimal scale preserve, up to uniform bounds, relative sizes and shapes of nearby objects. These local bounds then lead to strong global constraints. The need to understand such quantities arises in a variety of different areas of geometric analysis such as hyperbolic geometry, complex dynamics, differential equations, analysis on manifolds, Gromov hyperbolic groups and so on. Hence in many respects these mappings live naturally in geometric settings.

On the other hand, the development of basic properties of quasiconformal mappings themselves usually requires considerations that are analytic in nature. In this talk we shall discuss recent advances in understanding of the fundamentals of quasiconformal mappings. In particular, we shall see how these reflect and yield new information on other topics in analysis.

There are several possible ways to give a precise meaning to the intuitive notion of quasiconformality, i.e. that infinitesimal distortion is uniform in all directions. The most "elementary" is the *metric definition*: We say that a homeomorphism $f : D \mapsto D'$, where D, D' are domains in $\mathbf{R}^n$, is quasiconformal if there exists a constant $H < \infty$ such that

$$H_f(x) \equiv \limsup_{r\to 0} \frac{\max\{|f(x)-f(y)| : |x-y| = r\}}{\min\{|f(x)-f(z)| : |x-z| = r\}} \leq H, \quad x \in D. \tag{1}$$

According to the *analytic definition*, the homeomorphism f is quasiconformal if $f \in W^{1,n}_{loc}(D)$ and the directional derivatives satisfy

$$\max_\alpha |\partial_\alpha f(x)| \leq K \min_\alpha |\partial_\alpha f(x)| \qquad a.e. \;\; x \in D \tag{2}$$

for a constant $K < \infty$. Quantifying this we speak of K*-quasiconformal* mappings if (2) holds. The equivalence of the analytic and metric definitions follows essentially from the Rademacher-Stepanoff theorem; for details in n dimensions see the work of Gehring [G1].

The essential feature of quasiconformality is that the infinitesimally bounded distortions (1), (2) give strong global constraints. This leads to the notion of quasisymmetry: a mapping $f: A \to B$, $A, B \subset \mathbf{R}^n$, is called *quasisymmetric* if

$$\frac{|f(x)-f(y)|}{|f(x)-f(z)|} \leq \eta(\frac{|x-y|}{|x-z|}) \tag{3}$$

for all points $x, y, z \in A$ and for some continuous strictly increasing function $\eta : \mathbf{R}_+ \to \mathbf{R}_+$ with $\eta(0) = 0$. It is clear that (3) implies (1); for mappings of the whole $\mathbf{R}^n$ (and in general locally) the converse is also true [G1], [TV].

A recent surprising result of Heinonen and Koskela shows that in fact the assumption (1) can be considerably weakened

THEOREM 1.1. [HK1] *Suppose $H < \infty$ and $f : \mathbf{R}^n \mapsto \mathbf{R}^n$ is a homeomorphism for which*

$$\liminf_{r\to 0} \frac{\max\{|f(x)-f(y)| : |x-y| = r\}}{\min\{|f(x)-f(z)| : |x-z| = r\}} \leq H, \quad x \in \mathbf{R}^n.$$

Then f is quasisymmetric. In particular f is quasiconformal.

The fact that one can replace *lim sup* by *lim inf* is very useful; the result has immediate applications e.g. in rigidity questions in holomorphic dynamics [PR]. Furthermore, the notions (1), (3) are well defined in all metric spaces and the argument of Theorem 1.1 is based on the notion of discrete modulus combined with considerations of a general nature. Consequently, a version of the result extends to a large family of spaces, such as the length metric spaces that for some $q > 1$ satisfy a general $(1, q)$ Poincare-inequality and posses a q-regular measure, see [HK1], [HK2], [BK].

In the Euclidean two dimensional situation, a special flavour is added by Beltrami differential equation

$$\overline{\partial} f(x) = \mu(x)\, \partial f(x), \quad a.e.\ x \in D \tag{4}$$

which in $\mathbf{R}^2$ is equivalent to the inequality (2). Here μ is the complex dilatation with $|\mu(x)| \leq \frac{K-1}{K+1} < 1$ a.e. $x \in D$. In particular, in two dimensions quasiconformal considerations interact strongly with the theory of linear elliptic PDE's.

Naturally one can consider also non-homeomorphic functions satisfying (2): We say that a function f is *K-quasiregular* if firstly $f \in W^{1,n}_{loc}(D)$ and secondly the condition (2) holds at a.e. $x \in D$. In particular, the n-integrability of the derivatives guarantees that the Jacobian determinant J_f is locally integrable.

According to the fundamental theorems of Reshetnyak [Re], all quasiregular mappings are open and discrete. In many respects quasiregular mappings form in $\mathbf{R}^n$ the natural geometric counterpart of the theory of analytic functions, c.f. [Ri1].

Note also that in dimension two each quasiregular mapping factors as a composition of an analytic function and a quasiconformal homeomorphism.

2. Regularity

One of the cornerstones in the quasiconformal theory is that from the weak assumptions (1), (2) one gains improved regularity, i.e. improved integrability properties of the derivatives. In plane this fact was shown by Bojarski [Bj] and in higher dimension by Gehring [G2].

It is natural to search here for the best possible degrees of regularity. This is particularly rewarding since such bounds will lead for instance to optimal results on metric distortion properties. It turns out that they will also have consequences on different topics outside the field.

Conversely, in a dual manner one is led to ask how much can the regularity assumption $f \in W^{1,n}_{loc}$ be weakened. For quasiconformal mappings it is in fact enough to assume $f \in W^{1,1}_{loc}$, see [LV], [IKM]. However, for the noninjective quasiregular mappings one needs certain degrees of higher integrability. To state the problem more precisely, let us call a mapping $f \in W^{1;q}_{loc}(D)$ *weakly K-quasiregular* if (2) is satisfied at *a.e.* $x \in D$. The question is then to decide how small can we take q in order to still deduce that f is (strongly) quasiregular, in particular open and discrete. Optimal bounds for the q's yield then e.g. sharp quasiregular removability results.

In the case of two dimensions one has now an essentially complete understanding of these topics. To a large degree such properties are reduced to the following recent work of Astala on the distortion of area.

Theorem 2.1. [As2] *For each $K-$quasiconformal mapping f of $\mathbf{R}^2$ fixing $0, 1$ and ∞, we have*

$$|f(E)| \le M_K |E|^{1/K}, \quad E \subset \mathbf{R}^2, \tag{5}$$

where M_K depends only on K.

The implications to the regularity of quasiconformal mappings are then as follows.

Corollary 2.2. *If f is a $K-$quasiconformal mapping in a domain $D \subset \mathbf{R}^2$ then $f \in W^{1,p}_{loc}(D)$ for all $p < \frac{2K}{K-1}$.*

In fact, since $|\partial_\alpha f|^2 \le K J_f$ a.e, the bound of Theorem 2.1 is equivalent to $J_f \in L^{K/(K-1)}_{weak}$. Locally we have also the reverse Hölder estimates

$$\Big(\frac{1}{|B|}\int_B J_f^p dx\Big)^{1/p} \le C\,\Big(\frac{1}{|B|}\int_B J_f dx\Big), \quad p < \frac{K}{K-1}, \tag{6}$$

for the Jacobian of a quasiconformal mapping f in a domain $D \subset \mathbf{R}^2$. The constant C depends only on p, K and $\mathrm{dist}(B, \partial D)/\mathrm{diam}(B)$.

As an example, the radial mapping

$$f_0(x) = x|x|^{\frac{1}{K}-1} \tag{7}$$

is $K-$quasiconformal in $\mathbf{R}^2$ but $f_0 \notin W^{1,p_o}_{loc}$ for $p_o = \frac{2K}{K-1}$. Therefore the regularity given by Corollary 2.2 is the best possible.

As mentioned above, the optimal regularity results yield also quantitative bounds on metric distortion properties. According to Ahlfors [Ah2] and Mori [Mo] $K-$quasiconformal mappings are $1/K$-Hölder continuous. This follows from (5) and (3) when one chooses $E = B(x, |x-y|)$. More importantly, we can control the distortion of Hausdorff-dimension under quasiconformal deformations.

COROLLARY 2.3. [As2] *If f is K-quasiconformal in $\mathbf{R}^2$, then for any set $E \subset \mathbf{R}^2$*

$$\frac{1}{K}\Big(\frac{1}{dim(E)} - \frac{1}{2}\Big) \leq \frac{1}{dim(fE)} - \frac{1}{2} \leq K\Big(\frac{1}{dim(E)} - \frac{1}{2}\Big). \tag{8}$$

Moreover, for any $0 < t < 2$ and any $K \geq 1$, there are sets E with $dim(E) = t$ and K-quasiconformal mappings f such that the equality holds in the above left (or respectively, right) estimate.

Let us then consider the regularity properties of weakly quasiregular mappings. In the plane the case of weakly 1-quasiregular mappings is simple; for higher dimensions the problem is more subtle and we return to it later. In two dimensions each such mapping is a weak solution of $\overline{\partial} f = 0$ and if $f \in W^{1,1}_{loc}$ then by Weyl's lemma f is holomorphic. However, for $K > 1$ the $W^{1,1}_{loc}$-regularity is not enough [IM]. Such examples combined with Corollary 2.2 and the measurable Riemann mapping theorem give

COROLLARY 2.4. *Let $1 < K < \infty$ and $D \subset \mathbf{R}^2$. Then every weakly K-quasiregular mapping, contained in a Sobolev space $W^{1,q}_{loc}(D)$ with $\frac{2K}{K+1} < q \leq 2$, is quasiregular in D.*

For each $q < \frac{2K}{K+1}$ there are weakly K-quasiregular mappings $f \in W^{1,q}_{loc}(\mathbf{R}^2)$ which are not quasiregular.

Thus only the borderline case $q = \frac{2K}{K+1}$ remains open; it is conjectured that we obtain the strong quasiregularity also in this situation. See [AIS] where the conjecture is reduced to open questions on the Beurling transform.

By the factorization properties in $\mathbf{R}^2$, the higher integrability estimates of quasiconformal mappings are also the basis for the removability results of bounded quasiregular functions. A refinement of Corollary 2.3 gives the following counterpart of the classical Painlevé-theorem.

THEOREM 2.5. [As3] *If $E \subset \mathbf{R}^2$ has Hausdorff $\frac{2}{K+1}$-measure zero, then the set E is removable for all bounded $K-$quasiregular functions.*

Moreover, for each $t > \frac{2}{K+1}$ there are sets E of dimension $dim_H(E) = t$ not removable for some bounded $K-$quasiregular functions.

3. ELLIPTIC EQUATIONS

Quasiconformal mappings are well-known to be closely connected, in many different ways, to elliptic differential equations. In two dimensions this connection is especially effective since the governing equations (4) are linear. Indeed, the measurable Riemann mapping theorem, providing homeomorphic solutions to all Beltrami equations (4) with $\|\mu\|_\infty < 1$, is the basis of the theory of two-dimensional quasiconformal mappings.

Similarly the results of the previous sections have consequences on elliptic equations. For instance, by results of Bers, Lavrentiev and others, the solutions to $\nabla \cdot \sigma \nabla u = 0$ can be interpreted as components of quasiregular mappings, yielding sharp smoothness and removability estimates. Furthermore, let us consider in more details another example, the nonlinear systems in $\mathbf{R}^2$. Identifying $\mathbf{R}^2$ with $\mathbf{C}$, take a measurable function $H : \mathbf{C}\, x\, \mathbf{C} \to \mathbf{C}$ such that for all z, a, b

$$H(z,0) \equiv 0 \quad and \quad |H(z,a) - H(z,b)| \leq k|a-b| \tag{9}$$

with a constant $0 \leq k < 1$. Then the equation

$$\overline{\partial} w(z) = H(z, \partial w(z)) + h(z), \quad z \in D, \tag{10}$$

covers all uniformly elliptic linear first order systems for $w = u + iv$ as well as general nonlinear systems $\Phi(z, \overline{\partial} w(z), \partial w(z)) = 0$ that are elliptic in the sense of Lavrentiev; c.f. [BI1].

Assuming in (10) that $h \in L^p(\mathbf{C})$, let us study the existence and uniqueness of solutions w such that $\nabla w \in L^p(\mathbf{C})$. Here we need the Beurling transform $S : L^p(\mathbf{C}) \to L^p(\mathbf{C})$, $1 < p < \infty$,

$$(Sf)(z) = -\frac{1}{2\pi i} \int_{\mathbf{C}} \frac{f(w)dw \wedge d\overline{w}}{(z-w)^2} \tag{11}$$

which intertwines the $\overline{\partial}$ and ∂ derivatives, $S(\overline{\partial} w) = \partial w$ for $\nabla w \in L^p(\mathbf{C})$.

The recent work of Astala, Iwaniec and Saksman, which applies quasiconformal coordinate changes and the reverse Hölder inequalities (6) shows

Theorem 3.1. [AIS] *Under the assumption (9) the nonlinear singular integral operator $\mathcal{B} : L^p(\mathbf{C}) \to L^p(\mathbf{C})$, $\mathcal{B}g = g - H(\,\cdot\,, Sg)$, is invertible, and in fact a bi-Lipschitz homeomorphism on $L^p(\mathbf{C})$, whenever $1 + k < p < 1 + \frac{1}{k}$.*

These bound on p are optimal, even for smooth linear equations. For instance, for each $p \geq 1 + 1/k$ there are $h \in L^p(\mathbf{C})$ and $\mu \in C^\infty(\mathbf{C})$ with $\|\mu\|_\infty = k$ which oscillate at ∞ so that the non-homogeneous Beltrami equation $\overline{\partial} w - \mu \partial w = h$ admits no solutions with $\nabla w \in L^p(\mathbf{C})$.

Another application to elliptic equations of a completely different nature was established by Nesi [N] who proved that the quasiconformal area distortion can be used to determine the optimal bounds in certain G-closure problems.

4. holomorphic motions

A picture of planar quasiconformal mappings would not be complete without mentioning the holomorphic motions. In studying the stability phenomena in complex dynamics Mañé, Sad and Sullivan coined the following effective and elegant notion.

DEFINITION 4.1. *Let* $\Delta = \{z \in \mathbf{C} : |z| < 1\}$. *A function* $\Phi : \Delta \times A \to \overline{\mathbf{C}}$ *is called a holomorphic motion of a set* $A \subset \overline{\mathbf{C}}$ *if*

(i) for any fixed $z \in A$*, the map* $\lambda \mapsto \Phi(\lambda, z)$ *is holomorphic in* Δ*,*
(ii) for any fixed $\lambda \in \Delta$*, the map* $z \mapsto \Phi_\lambda(z) = \Phi(\lambda, z)$ *is injective and*
(iii) the mapping Φ_0 *is the identity on* A*.*

Typical examples of holomorphic motions arise in deformations of Kleinian groups and dynamical systems of rational functions. The "λ−lemmas" of Mañé, Sad and Sullivan [MSS] and Slodkowski [Sl] give them strong and unexpected rigidity properties. In fact, any holomorphic motion is of the form

$$\Phi(\lambda, z) = f^{\mu_\lambda}(z), \quad z \in A. \tag{12}$$

where f^{μ_λ} is a homeomorphic solution of the equation $\overline{\partial} f = \mu_\lambda \, \partial f$ in $\mathbf{C}$ and the coefficient $\mu = \mu_\lambda \in L^\infty$ depends holomorphically on the parameter λ.

The converse is also true, as solutions to (4) depend holomorphically on μ, c.f. (14). We see that general holomorphic motions are precisely the same as (holomorphic families of) quasiconformal mappings, one is just a different representation of the other. As an immediate application this relation note that [MSS], [Sl] and (8) give

COROLLARY 4.2. . *Given a holomorphic motion* $\Phi : \Delta \times E \to \overline{\mathbf{C}}$ *of a subset* $E \subset \overline{\mathbf{C}}$ *write* $E_\lambda = \Phi_\lambda(E)$*. Then*

$$\frac{1-|\lambda|}{1+|\lambda|}\Big(\frac{1}{dim_H(E)} - \frac{1}{2}\Big) \le \frac{1}{dim_H(E_\lambda)} - \frac{1}{2} \le \frac{1+|\lambda|}{1-|\lambda|}\Big(\frac{1}{dim_H(E)} - \frac{1}{2}\Big). \tag{13}$$

For some sets E *and motions* Φ *either one of the bounds holds as an equality.*

5. SINGULAR INTEGRALS AND HIGHER DIMENSIONAL REGULARITY

Beltrami equation $\overline{\partial} f = \mu \partial f$ connects quasiconformal mappings to the singular integrals and, in particular, to the Beurling transform (11). If μ has compact support and the quasiconformal mapping f is properly normalized, then we have

$$\overline{\partial} f = (I - \mu S)^{-1}\mu, \qquad \partial f(z) = 1 + (I - S\mu)^{-1} S(\mu). \tag{14}$$

The expressions are well defined since S is an isometry on $L^2(\mathbf{C})$ and $\|\mu\|_\infty < 1$.

Consequently, quasiconformal distortion properties are equivalent to bounds on the Beurling transform. For instance, an approach to Theorem 2.1 by Eremenko and Hamilton [EH] yields the following optimal estimate: Let B be a disk in $\mathbf{R}^2$ and suppose $E \subset B$. Then

$$\int_{B\setminus E} |S(\chi_E)| dx \le |E| \log(\frac{|B|}{|E|}) \tag{15}$$

The equality holds here when E is a subdisk with the same center as B. Duality gives also sharp exponential integrability for the Beurling transform of bounded functions. If $|\omega(z)| \le \chi_{_B}(z)$ *a.e.* then $|\{z \in B : |\Re S\omega(z)| > t\}| \le Ce^{-t}$.

However, the important question of the precise value of the L^p-norm of the Beurling transform remains still open; the best estimate so far is due to Banuelos and Wang [BW], based on probabilistic methods. It has been conjectured that $\|S\|_p = \max\{p-1, 1/(p-1)\}$. Combined with (14) this would give a new proof the regularity results 2.2-2.5.

Recently Iwaniec and Martin [IM] achieved a breakthrough in applying the theory of singular integrals in higher dimensional quasiconformality. The approach applies and developes the work of Donaldson and Sullivan [DS] on quasiconformal structures on 4-manifolds.

The starting point here is to use the differential forms. Let $\Lambda^l = \Lambda^l(\mathbf{R}^n)$ be the l'th exterior power of $\mathbf{R}^n$. Then the Hodge star operator $* : \Lambda^l \to \Lambda^{n-l}$ with respect to the standard innerproduct of $\mathbf{R}^n$ is given by $\alpha \wedge *\beta = (\alpha, \beta)$. Let d be the exterior derivative $d : C^\infty(\Lambda^l) \to C^\infty(\Lambda^{l+1})$ on (smooth) l-forms of $\mathbf{R}^n$. Its formal adjoint d^* is given by $d^* = (-1)^{nl+n+1} * d* : C^\infty(\Lambda^l) \to C^\infty(\Lambda^{l-1})$.

Next, each linear operator A on $\mathbf{R}^n$ extends naturally to an operator $A_\# : \Lambda^l \to \Lambda^l$. In particular, this is true for the (formal) derivative $Df(x)$ at *a.e.* $x \in \mathbf{R}^n$ of a weakly quasiregular mapping f. If $G_f(x) = Df(x)^t Df(x) J_f(x)^{-n/2}$ is the dilatation matrix of f at x, linear algebraic considerations show that

$$\big(G_f(x)\big)_\# * Df(x)^t_\# = J_f(x)^{(2l-n)/n} Df(x)^t_\# * . \tag{16}$$

Furthermore, [IM] proves that if $\alpha \in C^\infty(\Lambda^{l-1})$ has linear coefficients and $f \in W^{1,lp}_{loc}$, $p \geq 1$, then as distributions

$$d(f^*\alpha) = f^*(d\alpha). \tag{17}$$

As a first consequence let us see how this machinery can be applied to the regularity theory in even dimensions. For weakly 1-quasiregular mappings Iwaniec and Martin prove the following precise form of the Liouville theorem.

Theorem 5.1. [IM] *Suppose $n > 2$ is even. Let $f \in W^{1,n/2}_{loc}(D)$, $D \subset \mathbf{R}^n$, be weakly 1-quasiregular. Then f is the restriction of a Möbius transformation.*

Moreover, for all $p < n/2$ there are non-continuous weakly 1-quasiregular mappings in $f \in W^{1,p}_{loc}(\mathbf{R}^n)$.

Indeed, for f in Theorem 5.1 the matrix dilatation $G \equiv Id$. If $l = n/2$ and $\alpha \in C^\infty(\Lambda^{l-1})$ has linear coefficients, then $f^*d\alpha = Df(x)^t_\# d\alpha$ and from (16), (17) we deduce that $f^*d\alpha$ has vanishing d and d^* derivatives. The assumption $f \in W^{1,n/2}_{loc}(D)$ justifies the use of Weyl's lemma and hence as a harmonic function $f^*d\alpha$ is C^∞-smooth. It follows that the same is true for the Jacobian derivative J_f. Earlier proofs of the Liouville theorem [BI2] complete then the argument.

The connection to singular integrals comes from the Hodge theory. Denote by $L^p(\mathbf{R}^n, \Lambda^l)$ the space of l forms with p-integrable coefficients. Each such form w admits the decomposition $w = d\alpha + d^*\beta$ where $d^*\alpha = d\beta = 0$. Therefore we can define

$$S : L^p(\mathbf{R}^n, \Lambda^l) \to L^p(\mathbf{R}^n, \Lambda^l), \quad S(w) = d\alpha - d^*\beta. \tag{18}$$

It turns out that (18) defines a singular integral operator resembling in many ways the two dimensional Beurling transform, for details see [IM]. In particular, S is an isometry on L^2 and bounded on L^p, $1 < p < \infty$. In fact, if f is weakly quasiregular and $G_f(x)$ is its dilatation matrix as above, one may define also the counterpart of the complex dilatation $\mu : L^p(\mathbf{R}^n, \Lambda^l) \to L^p(\mathbf{R}^n, \Lambda^l)$ by

$$\mu_f = \frac{(G_f)_\# - Id}{(G_f)_\# + Id}$$

If α is an l-form with linear coefficients, $l = n/2$, multiply $f^*\alpha$ by a test function $\phi \in C_0^\infty(\mathbf{R}^n)$. Then for forms α such that the "conformal part" $d^+\alpha \equiv \frac{1}{2}(Id + (-i)^l *)da = 0$, one obtains [IM] a representation similar to (14),

$$d(\phi f^*\alpha) = (Id + S)(Id - \mu S)^{-1}\omega, \tag{19}$$

where one can control the L^p-properties of ω. In consequence, a following estimate of Caccioppoli type is obtained; crucial here is that the integrability exponent r can be also be below n.

THEOREM 5.1. [IM] *Suppose n is even and $D \subset \mathbf{R}^n$. Then there are exponents $p_0 < n < p_1$, both depending only on n and K, such that if $f \in W^{1,p}_{loc}(D)$ is weakly K-quasiregular with $p_0 < p < p_1$, then*

$$\int_D |\phi Df|^p \le C(n,K) \int_D |f|^p |\nabla \phi|^p \tag{20}$$

for all test functions $\phi \in C_0^\infty(D)$

In fact, (20) follows for those p's for which $\|\mu\|_\infty \|S\|_{L^{2p/n}(\Lambda^l)} < 1$.

Essential in the above argument is that for $l = n/2$ the matrix dilatation operates linearly on $Df_\#$, c.f. (16). Hence for odd dimensions one necessarily needs nonlinear arguments. Iwaniec [I1] resolved the problem with the help of a nonlinear Hodge theory. In a subsequent work [I2] he obtained the following beautiful refinement.

THEOREM 5.3. [I2] *For each $n \ge 2$ there is an exponent $p_0(n) < n$ such that for all $F \in W^{1,p}(\mathbf{R}^n, \mathbf{R}^n)$ with $p > p_0(n)$ we have*

$$\left| \int_{\mathbf{R}^n} |DF|^{p-n} J_f(x) dx \right| \le \lambda_p(n) \int_{\mathbf{R}^n} |DF|^p dx \tag{21}$$

where $\lambda_p(n) < 1$. Moreover, for n even this holds for $p_0(n) = \frac{n}{2}$.

In general dimensions $n \ge 2$ we obtain then the Cacciopoli type estimates (20) for all weakly K-quasiregular mappings, for exponents p with $\lambda_p(n)K < 1$, by choosing $F = \phi f$ in (21).

As a consequence we obtain removability and regularity results for quasiregular mappings, complementing the higher integrability theorems of Gehring [G2].

COROLLARY 5.4. *Let $1 < K < \infty$ and $D \subset \mathbf{R}^n$. Then there is a number $q_1 < n$ such that every weakly K-quasiregular mapping, contained in a Sobolev space $W^{1,q}_{loc}(D)$ with $q_1 < q$, is quasiregular in D.*

Corollary 5.5. *For all $K \geq 1$ there is a $\delta = \delta(n, K) > 0$ such that all sets $E \subset \mathbf{R}^n$ of dimension $dim(E) < \delta$ are removable for bounded $K-$quasiregular mappings.*

In the converse direction Rickman [Ri2] shows that there are Cantor sets $E \subset \mathbf{R}^3$ of arbitrarily small Hausdorff dimension that are not removable for some bounded quasiregular mappings. Very recently Bishop [Bi] extended the result to quasiconformal mappings.

In conclusion, for $n > 2$ the optimal bounds for q_1, δ in Corollaries 5.4 and 5.5 are still open. However, Iwaniec [I2] connects this with problems in nonlinear elasticity and, in particular, with convexity questions. Recall that a function of matrices $\mathcal{F} : M^{nxm} \to \mathbf{R}$ is *quasiconvex* if $\mathcal{F}(A)|D| \leq \int_D \mathcal{F}(A + D\psi)$ for all $A \in M^{nxm}$ and $\psi \in C_0^\infty(\mathbf{R}^n, \mathbf{R}^m)$. Quasiconvexity governs the lower semicontinuity of the functionals $I(u) = \int_D \mathcal{F}\big(Du(x)\big)dx$ in the appropriate Sobolev spaces and hence understanding the notion is a fundamental problem in higher dimensional calculus of variations. An explicit necessary condition is that of rank-one convexity, i.e. that $t \mapsto \mathcal{F}(A + tB)$ is convex for all rank-one matrixes B. However, Sverak [Sv] found examples showing that in general rank-one convexity is not sufficient for quasiconvexity when $n \geq 2$ and $m \geq 3$.

Developing methods towards finding the precise bounds [I2] proves that the functions

$$\mathcal{F}_p(A) = |1 - \frac{n}{p}||A|^p - |A|^{p-n} detA, \quad p > \frac{n}{2},$$

are rank-one convex in all dimensions $n \geq 2$. This gives support to the conjecture that the optimal bound in (21) is $\lambda_r(n) = |1 - \frac{n}{p}|$, in other words that $\mathcal{F}_p$ is quasiconvex at $A = 0$. If that is indeed the case, then the optimal regularity bounds of Corollaries 2.2 - 2.5 generalize to all dimensions n, i.e. 5.4, 5.5 hold with $q = \frac{nK}{K+1}$, $\delta = \frac{n}{K+1}$ and K-quasiconformal mappings have locally $p-$integrable derivatives for $p < \frac{nK}{K-1}$. Combined with arguments originally due to Burkholder [Bu] this would also prove the above mentioned conjecture for the L^p-norms of the Beurling transform.

It seems evident that further advances in quasiconformal regularity require a deeper understanding of the notion of quasiconvexity in the plane and as well as under special symmetries in higher dimensions.

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