Quantum Cohomology and its Application

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1 A brief historical reminiscence

In a few years, quantum cohomology has grown to an impressive field in mathematics with relations to different fields such as symplectic topology, algebraic geometry, quantum and string theory, integrable systems and gauge theory. The development in last few years has been explosive. Now, the foundations have been systematically studied and the ground is secure. Many people contributed to the development of quantum cohomology. I am fortunate to be involved in it from the beginning. Quantum cohomology is such a diverse field that it is impossible to make a complete survey in 45 minutes. I will make no attempt to do so. Instead, I will review some of topics where I made some contributions in last several years.

The development of quantum cohomology has roughly two distint periods: an early pioneer period (91-93) and more recent period of technical sophistication (94-present).

First of all, there are two terminologies: Quantum cohomology, Gromov-Witten invariants. Strictly speaking, quantum cohomology is a special case of the theory of Gromov-Witten invariants. However, the terms are commonly used to mean the same thing and we shall use them interchangebly. Roughly, quantum cohomology studies the following Cauchy-Riemann equation. Let V be a $2n$-dimensional smooth manifold and ω be a symplectic form,i.e., ω^n defines a volume form. We can choose a family of ω-tamed almost complex structure J. J is ω-tamed iff $\omega(X, JX) > 0$ for any nonzero tagent vector X. We want to study the solution space (moduli space) of nonlinear elliptic PDE $\bar{\partial}_J f = 0$ and construct topological invariants of the symplectic manifold (V, ω). The motivation of this problem goes back to two great theories in the 80's, Donaldson's gauge theory and Gromov's theory of pseudo-holomorphic curves. In the summer of 91, I visited Bochum with intention to work with A. Floer on gauge theory. After his tragic death, my gauge theory project went nowhere. It was in this summer that my career took a dramatic turn. After Floer's death, H. Hofer was my main contact person. We had some stimulating conversations where he explained to me Gromov-theory of pseudo-holomorphic curves. I was struck by the obvious resemblance between gauge theory and the theory of pseudo-holomorphic curves. I decided to learn more about it and Hofer recommended to me McDuff's survey paper [Mc]. After reading her paper, I immediately saw how to define a

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Donaldson-type invariant using pseudo-holomorphic curves. Since I was primarily motivated by Donaldson theory, I named them Donaldson-type invariants and now these invariants become commonly known as Gromov-Witten invariants or GW-invariants.

Technically speaking, GW-invariants are harder to study than Donaldson invariants since the compactification of moduli space of pseudo-holomorphic curves is more complicated. For any one with a background in Donaldson theory, it is probably not difficult to define such an invariant. But it is much harder to find interesting examples to show that they are nontrivial. I spent many fruitless hours searching algebraic surfaces to find such examples. Back then, a misperception was that the theory of pseudo-holomorphic curves is a theory about lower dimensional manifolds. Luckly, I came across a group of algebraic geometers working on Mori theory in Max Planck institute in the same summer. I was impressed by beautiful relation between these two subjects. It prompted me to abandon 4-manifolds and study symplectic 6-manifolds instead. After this change of strategy, I quickly found the examples of algebraic 3-folds having the same classical invariant with different new invariants [R2]. The same idea leaded to another paper to generalize some of Mori's results to symplectic manifolds [R3]. Later was extended to Calabi-Yau 3-folds by P. Wilson [Wi2].

A short time ago, some of remarkable progress has been made in physics by Witten for topological quantum field theory. One example of his topological quantum field theory is topological sigma model. However, in 91-92, symplectic geometers were unaware of it. The main motivation of studying these invariants was to distinguish symplectic manifolds. The first version of new invariant I defined was very limited due to the technical difficulty of counting multiple-cover maps. In the early 92, I spent several monthes on thinking about how to overcome this difficulty. Finally I realized that the perturbed Cauchy-Riemann equation $\bar\partial_J f = \nu$ introduced by Gromov can be used to give an appropriate account of mutilple covered maps. However, the invariants defined by perturbed equation have a different form from previous invariants.

Let Riemann surface be S^2. S^2 has a nontrivial automorphism $SL_2(C)$, which acts on the moduli space. To obtain compactness of moduli space, we need to divide it by SL_2C action. However, if we consider the perturbed equation $\bar\partial_J f = \nu$. The group SL_2C no longer acts on the moduli space. One way to deal with this problem is to impose the condition that f maps $0, 1, \infty$ to some codimension 2 submanifolds. In the fall of 92, I met D. Morrison in a conference in southern California, he explained to me Witten's topological sigma model [W1]. I realized that this new version of the invariants is precisely the correlation function of topological sigma model. These results appeared in [R1] in early 93, which contains a construction of genus zero topological sigma model invariants.

The new link to the topological sigma model brought tremendous insight to Gromov-Witten invariants. The general properties of topological quantum field theory predicted that these invariants must satisfy a set of axioms (Quantum cohomology axioms). The next logical step was to establish a mathematical theory of these invariants, namely proving these axioms. It was clear that this is a nontrivial task which needs some new analysis about pseudo-holomorphic curves.

Furthermore, topological sigma model also contains an analgous theory for higher genus pseudo-holomorphic curves. These higher genus invariants had not been studied before. Moreover, there is an evidence that they are different from the enumerative invariants in algebraic geometry. This was very mysterious to me. In the summer of 93, I met Gang Tian in Germany. We soon started the massive task of a systematic study of Gromov-Witten invariants. By December of 93, our work on a mathematical theory of Gromov-Witten invariants on semi-positive symplectic manifolds was virtually completed. Our results was first appeared in an announcement [RT] in January of 1994 and then in two papers [RT1], [RT2].

Up to the end of 93, quantum cohomology was very much a subject of symplectic topology. It was desirable to have an algebro-geometric treatment since most of examples are Kahler manifolds. Algebraic geometry is very sensitive to compactification. On the other hand, symplectic topology is not so sensitive to compactification due to its topological nature. In the early 90, Parker-Wolfson-Ye [PW], [Ye] obtained a delicate compactification of moduli space of pseudo-holomorphic curves as the product of their effort to prove Gromov compactness theorem using bubbling off analysis. Their compactification now is commonly known as the moduli space of stable maps, a name given by Kontsevich, who was the first one to really understand the importance of stable maps. He made an important observation that the moduli space of genus zero stable maps of homogeneous spaces is a smooth orbifold, where classical techniques apply. In early 94, Kontsevich and Mannin [KM] introduced stable maps and quantum cohomology axioms to the algebraic geometry community. [FP] further popularized quantum cohomology among algebraic geometers. Since then, quantum cohomology has attracted an increasing number of young algebraic geometers. Strictly speaking, the algebro-geometric treatment of Gromov-Witten invariants so far was still short to what we had already accomplished using symplectic methods. It was clear that one needed new ideas and techniques to go beyond homogeneous spaces. The next key step was taken by Li and Tian [LT2], where they used a sophisticated excessive intersection technique (normal cone) (See [B] for a different treatment). As a result, they can dispense the semi-positivity condition in the case of algebraic manifolds. Soon after, a new range of techniques were developed by [FO], [LT3], [R4], [S1] to extend GW-invariants to general symplectic manifolds. Recently, Li-Tian [LT4] and Seibert [S2] showed that the algebraic and symplectic definitions of GW-invariants agree. This completed the first stage of the development of quantum cohomology.

2 Theory of Gromov-Witten invariants

To define GW-invariants, we start from a ω-tamed almost complex structure J. Consider the moduli space of pairs (Σ, f), where $\Sigma \in \mathcal{M}_{g,k}$ is a marked Riemann surface of genus g, with k marked points and $f : \Sigma \to V$ satisfies equation $\bar{\partial}_J f = 0$. We call f a J-holomorphic map or a J-map. f carries a fundamental class $[f] \in H_2(V, \mathbf{Z})$. We use $\mathcal{M}_A(g, k, J)$ to denote the moduli space of (Σ, f) with $[f] = A$. The first step is to compactify $\mathcal{M}_A(g, k, J)$. By Parker-Wolfsen-Ye, we can compactify it by the moduli space of stable maps. Recall that we can compactify $\mathcal{M}_{g,k}$ by adding the stable Riemann surfaces. A stable Riemann surface is a connected (singular) Riemann surface with arithmetic genus g and k-marked points such that each component is stable, i.e., $2g+k \geq 3$. We use $\overline{\mathcal{M}}_{g,k}$ to denote the moduli space of stable Riemann surfaces of genus g and k-marked points.

DEFINITION 2.1: *A J-holomorphic stable map is a pair* (Σ, f), *where (i)* Σ *is a connected (possibly singular) Riemann surface with arithmetic genus* g *and* k*-marked points; (ii)* $f: \Sigma \to V$ *is* J*-holomorphic; (iii)* (Σ, f) *satisfies the stability condition that any constant component is stable. (A constant component is one where the restriction of* f *is a constant map.)*

Let $\overline{\mathcal{M}}_A(V, g, k, J)$ be the space of stable maps. By Parker-Wolfson-Ye [PW], [Ye], $\overline{\mathcal{M}}_A(g, k, J)$ is compact. There are two obvious maps

$$\Xi_{g,k} : \overline{\mathcal{M}}_A(V, g, k, J) \to V^k, \tag{2.1}$$

$$\chi_{g,k} : \overline{\mathcal{M}}_A(V, g, k, J) \to \overline{\mathcal{M}}_{g,k}. \tag{2.2}$$

Here $\Xi_{g,k}$ is defined by evaluating f at the marked point and $\chi_{g,k}$ is defined by successively contracting the unstable component of the domain of stable maps. Let $\alpha_i \in H^*(V, \mathbf{R})$ and $K \in H^*(\overline{\mathcal{M}}_{g,k}, \mathbf{R})$ be a differential form. The GW-invariants are intuitively defined as

$$\Psi^V_{(A,g,k,)}(K; \alpha_1, \cdots, \alpha_k) = \int_{\overline{\mathcal{M}}_A(V,g,k,J)} \chi^*_{g,k}(K) \wedge \Xi^*_{g,k} \prod_i \alpha_i. \tag{2.3}$$

Of course, the above formula only makes sense if $\overline{\mathcal{M}}_A(V, g, k, J)$ is a smooth, oriented orbifold, which is almost never the case. The whole development of GW-invariants is to overcome this difficulty.

The initial approach was a homological approach taken in [R1], [RT1], [RT2]. Here, we consider the dual picture, namely the Poincare dual K^*, α^* of K, α. It is a classical fact that intergration corresponds to intersection of homological cycle K^*, α^*. This approach was accomplished for semi-positive symplectic manifolds which includes most of interesting examples like Fano and Calabi-Yau 3-folds. One consequence of this approach is that the genus zero GW-invariants are integral. This property is still difficult to obtain from recent more powerful techniques.

The second approach was using a cohomological approach where we directly make sense of the integration. There are several methods. A conceptually simple method is as follows [R4], [S1]. By omiting the J-holomorphic condition, we obtain an infinite dimensional space $\overline{\mathcal{B}}_A(V, J, g, k)$ (configuration space). One first constructs a finite dimensional vector bundle $\mathcal{E}$ over $\overline{\mathcal{B}}_A(V, J, g, k)$ [S1]. Then we can construct a triple (U, E, S) such that (i) $U \subset \mathcal{E}$ is a finite dimensional smooth open orbifold ; (ii) E is a finite dimensional bundle over U; (iii) S is a proper section of E such that $S^{-1}(0) = \overline{\mathcal{M}}_A(V, g, k, J)$. Let Θ be a Thom form of E, we can replace (2.3) by

$$\Psi^V_{(A,g,k)}(K; \alpha_1, \cdots, \alpha_k) = \int_U S^*\Theta \wedge \chi^*_{g,k}(K) \wedge \Xi^*_{g,k} \prod_i \alpha_i. \tag{2.4}$$

The triple (U, E, S) is called a virtual neighborhood of $\overline{\mathcal{M}}_A(V, g, k, J)$. Ψ is independent of J, virual neighborhood. It depends only on the cohomology classes of K, α_i. A deep fact is that Ψ satisfies a set of quantum cohomology axioms as follows.

Assume $g = g_1 + g_2$ and $k = k_1 + k_2$ with $2g_i + k_i \geq 3$. Fix a decomposition $S = S_1 \cup S_2$ of $\{1, \cdots, k\}$ with $|S_i| = k_i$. Then there is a canonical embedding $\theta_S : \overline{\mathcal{M}}_{g_1,k_1+1} \times \overline{\mathcal{M}}_{g_2,k_2+1} \mapsto \overline{\mathcal{M}}_{g,k}$, which assigns to marked curves $(\Sigma_i; x_1^i, \cdots, x_{k_1+1}^i)$ $(i = 1, 2)$, their union $\Sigma_1 \cup \Sigma_2$ with $x_{k_1+1}^1$ identified to $x_{k_2+1}^2$ and remaining points renumbered by $\{1, \cdots, k\}$ according to S. There is another natural map $\mu : \overline{\mathcal{M}}_{g-1,k+2} \mapsto \overline{\mathcal{M}}_{g,k}$ by gluing together the last two marked points.

Choose a homogeneous basis $\{\beta_b\}_{1 \leq b \leq L}$ of $H_*(Y, \mathbf{Z})$ modulo torsion. Let (η_{ab}) be its intersection matrix. Note that $\eta_{ab} = \beta_a \cdot \beta_b = 0$ if the dimensions of β_a and β_b are not complementary to each other. Put (η^{ab}) to be the inverse of (η_{ab}).

There is a natural map $\pi : \overline{\mathcal{M}}_{g,k} \to \overline{\mathcal{M}}_{g,k-1}$ as follows. For $(\Sigma, x_1, \cdots, x_k) \in \overline{\mathcal{M}}_{g,k}$, if x_k is not in any rational component of Σ which contains only three special points, then we define

$$\pi(\Sigma, x_1, \cdots, x_k) = (\Sigma, x_1, \cdots, x_{k-1}), \tag{2.5}$$

where a distinguished point of Σ is either a singular point or a marked point. If x_k is in one of such rational components, we contract this component and obtain a stable curve $(\Sigma', x_1, \cdots, x_{k-1})$ in $\overline{\mathcal{M}}_{g,k-1}$, and define $\pi(\Sigma, x_1, \cdots, x_k) = (\Sigma', x_1, \cdots, x_{k-1})$.

QUANTUM COHOMOLOGY AXIOMS:

I: Let $[K_i] \in H_*(\overline{\mathcal{M}}_{g_i,k_i+1}, \mathbf{Q})$ $(i = 1, 2)$ *and* $[K_0] \in H_*(\overline{\mathcal{M}}_{g-1,k+2}, \mathbf{Q})$*. For any* $\alpha_1, \cdots, \alpha_k$ *in* $H_*(V, \mathbf{Z})$*, then we have*

(2.6)

$$\Psi^Y_{(A,g,k)}(\theta_{S*}[K_1 \times K_2]; \{\alpha_i\}) =$$
$$\epsilon \sum_{A=A_1+A_2} \sum_{a,b} \Psi^Y_{(A_1,g_1,k_1+1)}([K_1]; \{\alpha_i\}_{i \leq k_1}, \beta_a) \eta^{ab} \Psi^Y_{(A_2,g_2,k_2+1)}([K_2]; \beta_b, \{\alpha_j\}_{j > k_1})$$

with $\epsilon := (-1)^{deg(K_2) \sum_{i=1}^{k_1} deg(\alpha_i)}$,

$$\Psi^Y_{(A,g,k)}(\mu_*[K_0]; \alpha_1, \cdots, \alpha_k) = \sum_{a,b} \Psi^Y_{(A,g-1,k+2)}([K_0]; \alpha_1, \cdots, \alpha_k, \beta_a, \beta_b) \eta^{ab} \tag{2.7}$$

II: Suppose that $(g, k) \neq (0, 3), (1, 1)$.

(1) For any $\alpha_1, \cdots, \alpha_{k-1}$ *in* $H_*(Y, \mathbf{Z})$*, we have*

$$\Psi^Y_{(A,g,k)}(K; \alpha_1, \cdots, \alpha_{k-1}, [V]) = \Psi^Y_{(A,g,k-1)}([\pi_*(K)]; \alpha_1, \cdots, \alpha_{k-1}) \tag{2.8}$$

(2) Let α_k be in $H_{2n-2}(Y, \mathbf{Z})$, then

$$\Psi^Y_{(A,g,k)}(\pi^*(K); \alpha_1, \cdots, \alpha_{k-1}, \alpha_k) = \alpha_k^*(A)\Psi^Y_{(A,g,k-1)}(K; \alpha_1, \cdots, \alpha_{k-1}) \tag{2.9}$$

where α_k^* is the Poincare dual of α_k.

III: Ψ^V is a symplectic deformation invariant.

Axioms I, II are due to Witten [W1], [W2] and Axiom III is due to Ruan [R2].

The genus zero GW-invariants can be used to define a quantum multiplication as follows. First we define a total 3-point function

$$\Psi^V(\alpha_1, \alpha_2, \alpha_3) = \sum_A \Psi^V_{(A,0,3)}(pt; \alpha_1, \alpha_2, \alpha_3)q^A, \tag{2.10}$$

where $q^A \in \wedge_V$ is an element of ring of formal power series. Then, we define a quantum multiplication $\alpha \times_Q \beta$ over $H^*(V, \wedge_V)$ by the relation

$$(\alpha \times_Q \beta) \cup \gamma[V] = \Psi^V(\alpha_1, \alpha_2, \alpha_3), \tag{?}$$

where $\cup$ represents the ordinary cup product. An important observation is that

$$\alpha \times_Q \beta = \alpha \cup \beta + \text{lower order quantum corrections.}$$

Hence, this quantum product is often called a deformed product. The 3-point function did not use all the genus zero GW-invariant. An extension of previous construction is to define

$$\Psi^V_w(\alpha_1, \alpha_2, \alpha_3) = \sum_A \sum_{k \geq 3} \frac{1}{(k-3)!} \Psi^V_{(A,0,k)}(\overline{\mathcal{M}}_{0,k}; \alpha_1, \alpha_2, \alpha_3, w, \cdots, w). \tag{2.11}$$

Then we can define a family of quantum product

$$(\alpha \times^w_Q \beta) \cup \gamma[V] = \Psi^V_w(\alpha, \beta, \gamma). \tag{2.12}$$

When $w = 0$, we obtain classical quantum product. The Axiom I for g=0 implies that quantum product $\times^w_Q$ is associative. The associativity has far reaching consequences in enumerative geometry, integrable system and mirror symmetry [Ti]. The previous theory can be generalized in a number of directions, for example, for a family of symplectic manifold and symplectic manifold with a group action [R4]. In the later case, the equivariant theory plays an important role in the recent work about mirror symmetry.

3 SURGERY AND GLUING THEORY

Many examples of quantum cohomology have been computed. I refer to [QR] for a list of examples. I believe that the most important future research direction is to develop general technique to compute GW-invariant instead of computing specific examples. Surgery plays a prominent role in geometry and topology. In fact, it is conjectured that one can connect any two Calabi-Yau 3-folds by a sequence of surgeries called flops and extremal transitions. The famous Mori program of birational geometry is basically a surgery theory. On the other hand, surgery has been used in symplectic topology to construct many new examples [Go2], [MW]. Therefore, it is very important to use surgery to study quantum cohomology. This requires a gluing theory of pseudo-holomorphic curves. While we have several choices of surgeries, a particularly useful one in the application to symplectic topology and algebraic geometry is symplectic cutting-symplectic norm sum [L]. Such a gluing technique has been recently established by Li-Ruan [LR] and Ionel-Parker [IP].

Suppose that X admits a local Hamiltonian circle action. Then, we can cut X along a level set and collapse the circle action on the boundary. Then, we obtain a pair of symplectic manifolds X^+, X^- called symplectic cuttings of X. X^+, X^- contains a common codimension 2 symplectic submanifold Z with opposite first Chern class of their normal bundle. Many algebro-geometric surgeries can be interpreted as symplectic cutting, where the Hamiltonian circle action is usually given by complex multiplication. The gluing theory describes the behavior of pseudo-holomorphic curves under stretching the "neck" (the region carring circle action). In the limit, pseudo-holomorphic curves break as pseudo-holomorphic holomorphic curves in X^+, X^- with possibly several components. Moreover, these curves could intersect Z with high tangency condition. Moreover. some component could lie in Z.

To capture these new phenomena from gluing theory, we can introduce a relative Gromov-Witten invariant [LR](see [IP] for a related invariant). Choose a tamed almost complex structure J such that Z is almost complex. Then, one can define *relative stable maps* with prescribed tangency condition on Z. Then one can use the above virtual neighborhood method to define *relative GW-invariants*. There is a natural map from the moduli space of relative stable maps into the moduli space of stable maps. However, this map is not surjective in general. The difference counts the discrepency between relative and absolute invariants, which is caused precisely by the stable maps whose components lie in Z. In favorable circumstances, relative invariants are easy to compute or can be related to regular GW-invariants.

Then, general gluing theory shows that Gromov-Witten invariants of symplectic manifolds can be related to relative invariants of its symplectic cutting. The general formula is complicated and probably not very useful. In applications, we often encounter the situation that most of the relative invariants vanish and it is much easier to count them. Then, we get formula for the GW-invariants. Here are some applications. Recall that a minimal model is an algebraic variety with terminal singularities and nef canonical bundle. In the dimension 3, two different minimal models are connected to each other by flops. By applying gluing theory to the flop, Li-Ruan showed

THEOREM 3.1: *Any two smooth three dimensional minimal models have isomorphic quantum cohomology.*

However, it is well-known that they can have different ordinary cohomology. This establishes the first quantum birational invariant. Furthermore, Li-Ruan derived various formulas of quantum cohomology under extremal transition, which are important in mirror symmetry. Moreover, Ionel-Parker use this technique to give an elegant proof of Caparosa-Harris formula of number of curves in $\mathbf{P}^2$ and Bryant-Liang's formula of number of curves in K3-surfaces. I have no doubt that the gluing theory will yield more important applications towards quantum cohomology.

4 PROBLEMS AND CONJECTURES

I believe that the future success of quantum cohomology theory depends on its applications. Clearly, the ability to apply quantum cohomology also depends on our understanding of GW-invariants. For quantum cohomology itself, I believe that the biggest problem is our poor understanding of its functorial properties. The reason cohomology is very useful is its naturality. Namely, a continuous map induces a homomorphism on cohomology. Although we have calculated many examples, it help us little on this problem.

QUANTUM NATURALITY PROBLEM: *What are the "morphisms" of symplectic manifolds where quantum cohomology is natural?*

Li-Ruan [LR] suggests that this problem is tied to so called *small transition*, which is the composition of a small contraction and smoothing. Incidentaly, small contractions are the most difficult operations in birational geometry. However, [LR] suggests that they are easiest in quantum cohomology.

I believe that there is a deep relation between quantum cohomology and birational geometry. Theorem 3.1 suggests

QUANTUM MINIMAL MODEL CONJECTURE: *Theorem 3.1 holds in any dimension.*

This leads to many more questions. For example, one can attempt to find quantum cohomology of a minimal model without knowing minimal model. This problem requires a thorough understanding of blow-up type formula of quantum cohomology. Since quantum cohomology is a deformation invariant, one can try to relax the birational classification by allowing deformation, which we call deformation birational classification. Then, one replace contraction by extremal transition. One can try to construct minimal models using extremal transition. Quantum cohomology should play an important role in this new category. It is even more exciting that such a deformation-birational minimal model program has a natural analogy in symplectic manifolds.

There are many outstanding problems in the quantum cohomology. Let me list several examples, Virasoro conjecture [EHX], quantum hyperplane conjecture [Kim], mirror surgery conjecture [LR], conjectures of characterizations of uniruled varieties and rational connected varieties [KO]. It seems that possible applications are numerous and future is bright for quantum cohomology.

Over the years, I have been benefited from generous help of many people.

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