Geometry on the Group of Hamiltonian Diffeomorphisms

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Abstract. The group of Hamiltonian diffeomorphisms $\mathrm{Ham}(M, \Omega)$ of a symplectic manifold (M, Ω) plays a fundamental role both in geometry and classical mechanics. For a geometer, at least under some assumptions on the manifold M, this is just the connected component of the identity in the group of all isometries of the symplectic structure Ω. From the point of view of mechanics, $\mathrm{Ham}(M, \Omega)$ is the group of all admissible motions. It was discovered by H. Hofer ([H1], 1990) that this group carries a natural Finsler metric with a non-degenerate distance function. Intuitively speaking, the distance between a given Hamiltonian diffeomorphism f and the identity transformation is equal to the minimal amount of energy required in order to generate f. This new geometry has been intensively studied for the past 8 years in the framework of modern symplectic topology. It serves as a source of refreshing problems and gives rise to new methods and notions. Also, it opens up the intriguing prospect of using an alternative geometric intuition in Hamiltonian dynamics. In the present note we discuss these developments.

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1. The group of Hamiltonian diffeomorphisms. Let (M, Ω) be a connected symplectic manifold without boundary. Every smooth compactly supported function F on $M \times [0; 1]$ defines a Hamiltonian flow $f_t : M \to M$. This flow is generated by a time-dependent vector field ξ_t on M which satisfies the point-wise linear algebraic equation $\Omega(., \xi_t) = dF_t(.)$, where $F_t(x)$ stands for $F(x, t)$. Symplectomorphisms f_t arising in this way are called *Hamiltonian diffeomorphisms.* Hamiltonian diffeomorphisms form an infinite-dimensional Lie group $\mathrm{Ham}(M, \Omega)$. When $H^1_{\mathrm{comp}}(M, \mathbf{R}) = 0$ this group coincides with $\mathrm{Symp}_0(M, \Omega)$ - the identity component of the group of all symplectomorphisms in the strong Whitney topology. In general the quotient group $\mathrm{Symp}_0(M, \Omega)/\mathrm{Ham}(M, \Omega)$ is non-trivial but "quite small" [Ba]. The Lie algebra $\mathcal{A}$ of $\mathrm{Ham}(M, \Omega)$ consists of all smooth functions on M which satisfy the following normalization condition. Namely when M is open $F \in \mathcal{A}$ iff F is compactly supported, and when M is closed $F \in \mathcal{A}$ iff F has the zero mean with respect to the canonical measure on M induced by Ω. With this normalization different functions from $\mathcal{A}$ generate different Hamiltonian

vector fields. The Lie bracket on $\mathcal{A}$ is the Poisson bracket, and the adjoint action of $\mathrm{Ham}(M,\Omega)$ on $\mathcal{A}$ is the standard action of diffeomorphisms on functions.

2. HOFER'S METRIC. Consider the L_∞-norm $||F|| = \max_M F - \min_M F$ on $\mathcal{A}$. This norm is invariant under the adjoint action, and thus defines a biinvariant Finsler metric on $\mathrm{Ham}(M,\Omega)$. This Finsler metric determines in the standard way a length structure, and a pseudo-distance ρ on the group. More explicitly, let $\{f_t\}$, $t \in [0;1]$ be a path of Hamiltonian diffeomorphisms with $f_0 = \phi$ and $f_1 = \psi$. Let $F(x,t)$ be its *normalized* Hamiltonian function, that is $F(.,t) \in \mathcal{A}$ for all t. Then

$$\mathrm{length}\{f_t\} = \int_0^1 ||F(.,t)||dt,$$

and $\rho(\phi,\psi) = \inf \mathrm{length}\{f_t\}$, where the infimum is taken over all smooth paths $\{f_t\}$ which join ϕ and ψ.

A non-trivial fact is that the pseudo-distance ρ is non-degenerate, that is $\rho(\phi,\psi) \neq 0$ for $\phi \neq \psi$ (this was proved in [H1] for $\mathbf{R}^{2n}$, then extended in [P1] for some other symplectic manifolds, and finally confirmed in [LM1] in full generality). Note that the construction above goes through for any other norm on the Lie algebra which is invariant under the adjoint action, for instance for the L_p-norm. However for all $1 \le p < \infty$ the corresponding pseudo-distance is degenerate [EP].

Interestingly enough, the quantity $\rho(\mathrm{id},\phi)$ can be interpreted as the distance between a point and a subset in a linear normed space [P7]. Consider the space $\mathcal{F}$ of all smooth compactly supported functions F on $M \times S^1$ such that $F(.,t) \in \mathcal{A}$ for all $t \in S^1 = \mathbf{R}/\mathbf{Z}$. For $F \in \mathcal{F}$ denote by ϕ_F the time-one-map of the Hamiltonian flow generated by F. Every Hamiltonian diffeomorphism can be expressed in this way. Let $\mathcal{H} \subset \mathcal{F}$ be the subset of all functions H which generate loops of Hamiltonian diffeomorphisms, that is $\phi_H = \mathrm{id}$. Introduce a norm on $\mathcal{F}$ by $|||F||| = \max_t ||F(.,t)||$. It is easy to show that

$$(2.A) \qquad \rho(\mathrm{id},\phi_F) = \inf_{H\in\mathcal{H}} |||F-H|||.$$

Thus the set $\mathcal{H}$ carries a lot of information about Hofer's geometry.

We complete this section with the following open problem in the very foundation of Hofer's geometry [EP]. It is quite natural (see section 7 below) to consider the "maximum" and the "minimum" parts of Hofer's length structure separately. Namely set $\mathrm{length}_+\{f_t\} = \int_0^1 \max_x F(x,t)dt$ and $\mathrm{length}_-\{f_t\} = \int_0^1 -\min_x F(x,t)dt$, and define $\rho_+(\phi,\psi)$ and $\rho_-(\phi,\psi)$ as the infimum of positive and negative lengths respectively over all paths $\{f_t\}$ with $f_0 = \phi$ and $f_1 = \psi$. Clearly, $\rho(\phi,\psi) \ge \rho_-(\phi,\psi) + \rho_+(\phi,\psi)$. In fact, in all examples known to me *the equality* holds. It would be interesting either to prove this, or to find a counterexample.

3. DISPLACEMENT ENERGY. Consider any norm on $\mathcal{A}$ invariant under the adjoint action, and denote by ρ' the corresponding pseudo-distance. For a subset U of M denote by G_U the set of all Hamiltonian diffeomorphisms f such that $f(U) \cap U = \emptyset$. Define *the displacement energy* of U as $\rho'(\mathrm{id}, G_U)$. We use the convention that the displacement energy equals $+\infty$ when G_U is empty. Clearly this is a symplectic

invariant. It takes strictly positive values on non-empty open subsets if and only if the pseudo-metric ρ' is non-degenerate [EP]. Denote by $e(U)$ the displacement energy with respect to Hofer's metric.

Example 3.A Every symplectic manifold of dimension $2n$ admits a symplectic embedding of a standard $2n$-dimensional ball of a sufficiently small radius r. The supremum of πr^2 where r runs over such the embeddings is called *Gromov's width of the symplectic manifold.* Hofer showed [H1] that for every open subset U of the standard symplectic vector space $\mathbf{R}^{2n}$ holds $e(U) \geq \text{width}(U)$. Later on it was proved in [LM1] that $e(U) \geq \frac{1}{2}\text{width}(U)$ for every open subset U of an arbitrary symplectic manifold. Conjecturally, in the general case the factor $\frac{1}{2}$ can be removed.

Example 3.B Consider the cotangent bundle $\theta : T^*T^n \to T^n$ with a *twisted* symplectic structure $dp \wedge dq + \theta^*\sigma$, where σ is a closed 2-form on T^n. Such structures arise in the theory of magnetic fields. Denote by $Z \subset T^*T^n$ the zero section. If $\sigma = 0$ then $f(Z) \cap Z \neq \emptyset$ for every Hamiltonian diffeomorphism f (this is the famous Arnold's Lagrangian intersections conjecture proved by Chaperon, Hofer and Laudenbach-Sikorav, see [MS]). Thus $e(Z) = +\infty$. However if $\sigma \neq 0$ then Z admits a nowhere tangent Hamiltonian vector field [P2], and thus $e(Z) = 0$.

4. A paradox of Hofer's geometry. What does the metric space $\text{Ham}(M, \Omega)$ look like? Here we present two results which intuitively contradict one another, and no convincing explanation is known at present. The first one is the following *C^1-flatness phenomenon.*

Theorem 4.A [BP1]. *There exists a C^1-neighbourhood $\mathcal{E}$ of the identity in $\text{Ham}(\mathbf{R}^{2n})$ and a C^2-neighbourhood $\mathcal{C}$ of zero in $\mathcal{A}$ such that $(\mathcal{E}, \rho)$ is isometric to $(\mathcal{C}, || \, ||)$.*

The isometry takes every C^1-small Hamiltonian diffeomorphism from $\mathcal{E}$ to its classical generating function. Some generalizations can be found in [LM2].

The second result, due to J.-C. Sikorav [S] states that *every one-parameter subgroup of* $\text{Ham}(\mathbf{R}^{2n})$ *remains a bounded distance from the identity* (see discussion in §6 below). This can be interpreted as a "positive curvature type effect".

It sounds likely that in order to resolve this paradox one should understand properly the interrelation between the topology on $\text{Ham}(M, \Omega)$ which comes from Hofer's metric, and the smooth structure on the group. For instance, paths which are continuous in the metric topology can be non-continuous in the usual sense, and there is no satisfactory way to think about them. In what follows we restrict ourselves to *smooth* paths, homotopies, etc.

5. Geodesics. The C^1-flatness phenomenon above serves as the starting point for the theory of geodesics of Hofer's metric. Indeed, at least on small time intervals the geodesics should behave as the ones in the linear normed space $(\mathcal{A}, || \, ||)$. This leads to the following definition [BP1]. Consider a smooth path of Hamiltonian diffeomorphisms of (M, Ω) generated by a normalized Hamiltonian function $F(x, t)$. Assume that $||F(., t)|| \neq 0$ for all t. The path is called *quasi-autonomous* if there exist two (time-independent!) points x_+ and x_- on M such that for all t the function $F(., t)$ attains its maximal and minimal values at x_+ and x_- respectively. For instance, every one-parameter subgroup is quasi-autonomous. A path

of Hamiltonian diffeomorphisms is called *a minimal geodesic* if each of its segments minimizes length in the homotopy class of paths with fixed end points. It turns out that every minimal geodesic is quasi-autonomous [LM2]. However the converse is not true in general (see Sikorav's result above). In [H2] Hofer discovered a surprising link between minimality of paths on the group of Hamiltonian diffeomorphisms and closed orbits of corresponding Hamiltonian flows. Numerous further results in this direction (see [HZ],[BP1],[Si1],[LM2] and [Sch]) serve as a motivation for the following conjecture. A closed orbit of period c of a (time-dependent) flow $\{f_t\}$ with $f_0 = \mathrm{id}$ is a piece of the trajectory of a point $x \in M$ on a time interval $[0; c]$ such that $x = f_c x$. A closed orbit is called constant if it corresponds to a fixed point of the flow, that is $f_t x = x$ for all t.

CONJECTURE 5.A. *Let $\{f_t\}$, $t \in [0; T]$, $f_0 = \mathrm{id}$ be a quasi-autonomous path of Hamiltonian diffeomorphisms. Assume that the flow $\{f_t\}$ has no contractible non-constant closed orbits of period less than T. Then this path is a minimal geodesic.*

As an immediate consequence one gets that one-parameter subgroups should be minimal on short time intervals. In 8.A and 9.A below we describe a minimality-breaking mechanism on large time intervals which together with 5.A allows us to detect non-trivial closed orbits. In 9.B we give an example of an infinite minimal geodesic. The study of the breaking of minimality is still far from being completed. Another step in this direction was made in the framework of the theory of conjugate points (see [U],[LM2]) which deals with the local behavior of the length functional under small deformations of quasi-autonomous paths, and where an infinitesimal version of 5.A plays a crucial role.

6. DIAMETER. Here we discuss the following conjecture.

CONJECTURE 6.A. *The diameter of* $\mathrm{Ham}(M, \Omega)$ *with respect to Hofer's metric is infinite.*

The conjecture is established at present for a number of manifolds (see [LM2],[P7],[Sch]). We shall illustrate the methods in the case when (M, Ω) is a closed oriented surface endowed with an area form. In the case when the genus of M is at least 1, the conjecture was proved in [LM2] as follows. One can produce a Hamiltonian flow on M whose lift to the universal cover displaces a disc of an arbitrarily large area. For instance, take a flow which is the standard rotation in a small neighbourhood of a non-contractible curve on M. Inequality 3.A implies that such a flow goes arbitrarily far away from the identity. There is also a different proof [Sch] which is based on the analysis of closed orbits (cf. 5.A).

These methods do not work when M is the 2-sphere. This case was treated in [P7] as follows. Consider the set $\mathcal{H}$ of all 1-periodic normalized Hamiltonians which generate the identity map (see §2). Let L be an equator of S^2.

THEOREM 6.B [P7]. *For every $H \in \mathcal{H}$ there exist $x \in L$ and $t \in S^1$ such that $H(x, t) = 0$.*

Choose now an arbitrary large number c, and a time-independent normalized Hamiltonian function F such that $F(x) \geq c$ for all $x \in L$. It follows from (2.A) and 6.B above that $\rho(\mathrm{id}, \phi_F) \geq c$, and thus *the diameter of* $\mathrm{Ham}(S^2)$ *is infinite.* In particular, on S^2 (in contrast to $\mathbf{R}^{2n}$, see §4) there are unbounded one-parameter subgroups. As a by-product of this argument we get that *there exists a sequence of Hamiltonian diffeomorphisms of* S^2 *which converges to the identity in the* C^0*-topology but diverges in Hofer's metric.* Indeed, choose the function F above to be equal to a large constant outside a tiny open disc on S^2. Note that ϕ_F acts trivially outside the disc and thus is C^0-small, while $\rho(\mathrm{id}, \phi)$ can be made arbitrary large. Again, in the linear symplectic space $\mathbf{R}^{2n}$ the situation changes drastically. Hofer showed [H2] that if $\phi_i \to \mathrm{id}$ in $\mathrm{Ham}(\mathbf{R}^{2n})$ in the strong C^0-topology then $\rho(\mathrm{id}, \phi_i)$ must converge to 0. The reason is that in $\mathbf{R}^{2n}$ there is *enough room* to shorten "long" paths with "small" supports.

The proof of Theorem 6.B can be reduced to a Lagrangian intersections problem which one solves using a version of Floer Homology developed by Oh (see [O] for a survey). An important ingredient of this reduction is a detailed knowledge about the fundamental group of $\mathrm{Ham}(S^2)$. Our method works also for some four-dimensional manifolds, for instance when $M = \mathbf{CP}^2$.

7. LENGTH SPECTRUM. Let (M, Ω) be a closed symplectic manifold. For an element $\gamma \in \pi_1(\mathrm{Ham}(M, \Omega), \mathrm{id})$ set $\nu(\gamma) = \inf \mathrm{length}\{f_t\}$ where the infimum is taken over all loops $\{f_t\}$ of Hamiltonian diffeomorphisms which represent γ. In principle, Conjecture 5.A above would give a method of computing $\nu(\gamma)$ at least in some examples. The first step in this direction was made in [LM2] for the case $M = S^2$, and recently J. Slimowitz informed me about her work in progress in dimension four. Here we describe a different approach (see [P3-P6]).

The starting observation is that one can develop a sort of Yang-Mills theory for symplectic fibrations over S^2 with the structure group $\mathrm{Ham}(M, \Omega)$. The role of the Yang-Mills functional is played by the L_∞-norm of the curvature of a symplectic connection on such a fibration (see [GLS] for the definition of symplectic curvature). As expected its minimal values correspond to the length spectrum on $\mathrm{Ham}(M, \Omega)$ in the sense of Hofer's geometry. The L_∞-Yang-Mills functional was first introduced in the context of complex vector bundles by Gromov [Gr] , who called its minimal value *the K-area.*

Further, and this seems to be a specific feature of the Hamiltonian situation, the K-area of a symplectic fibration is closely related to the *coupling parameter.* The coupling is a special construction (see [GLS]) which allows one to extend the fiber-wise symplectic structure of a symplectic fibration to a symplectic form defined in the total space of the fibration. The coupling parameter is responsible for an "optimal" cohomology class of such an extension.

The final step of this approach is based on a powerful machinery of Gromov-Witten invariants [R] which provides us with obstructions to deformations of symplectic forms in cohomology. One can use it in order to compute/estimate the value of the coupling parameter in a number of interesting examples. Therefore one gets the desired information about the length spectrum in Hofer's geometry.

Let us give a precise statement relating Hofer's length spectrum to the coupling parameter. Pick up an element $\gamma \in \pi_1(\mathrm{Ham}(M, \Omega), \mathrm{id})$, and let $\{h_t\}$, $t \in S^1$ be a loop which represents γ. Define a fibration $p : P \to S^2$ as follows. Let D_+ and D_- be two copies of the disc D^2 bounded by S^1. Consider a map

$\Psi : M \times S^1 \to M \times S^1$ given by $(z, t) \to (h_t z, t)$. Define now a new manifold $P(\gamma) = (M \times D_-) \cup_\Psi (M \times D_+)$. It is clear that $P(\gamma)$ has the canonical fiber-wise symplectic form, and thus can be considered as a symplectic fibration over S^2. Moreover, homotopic loops $\{h_t\}$ give rise to isomorphic symplectic fibrations. In what follows we assume that the base S^2 is oriented, and the orientation comes from D_+.

The symplectic fibration $P(\gamma)$ carries a remarkable class $u \in H^2(P, \mathbf{R})$ called the *coupling class*. It is defined uniquely by the following two properties. Its restriction to a fiber coincides with the class of the fiber-wise symplectic structure, and its top power vanishes. Denote by a the positive generator of $H^2(S^2, \mathbf{Z})$, and by $p : P(\gamma) \to S^2$ the natural projection. Using the coupling construction one gets that for $E > 0$ large enough the class $u + Ep^*a$ is represented by a canonical (up to isotopy) symplectic form on the total space $P(\gamma)$ which extends the fiber-wise symplectic structure. Define the coupling parameter of γ as the infimum of such E. Finally, consider the positive part of Hofer's norm $\nu_+(\gamma)$ defined as the infimum of $\text{length}_+\{h_t\}$ over all loops $\{h_t\}$ representing γ (compare with the discussion at the end of section 2 above).

THEOREM 7.A [P6]. *The coupling parameter of γ coincides with $\nu_+(\gamma)$.*

Combining this theorem with the theory of Gromov-Witten invariants one gets for instance the following estimate for the length spectrum. Denote by c the first Chern class of the vertical tangent bundle to $P(\gamma)$. In other words, the fiber of this bundle at a point of $P(\gamma)$ is the (symplectic) vector space tangent to the fiber through this point. Assume that M has real dimension $2n$. Define the "characteristic number"

$$I(\gamma) = \int_{P(\gamma)} u^n \cup c.$$

It is easy to see that $I : \pi_1(\text{Ham}(M, \Omega)) \to \mathbf{R}$ is a homomorphism ([P4],[LMP]).

THEOREM 7.B [P4]. *Let (M, Ω) be a monotone symplectic manifold, that is $[\Omega]$ is a positive multiple of $c_1(TM)$. Then there exists a positive constant $C > 0$ such that $\nu(\gamma) \geq C|I(\gamma)|$ for all $\gamma \in \pi_1(\text{Ham}(M, \Omega))$.*

In other words, the homomorphism I calibrates Hofer's norm on the fundamental group. The proof of 7.B uses results from [Se]. Recently Seidel obtained a generalization of this inequality to non-monotone symplectic manifolds.

Let us mention also that there exists a surprising link between Hofer's length spectrum and spectral Riemannian geometry (see [P6]).

8. ASYMPTOTIC GEOMETRIC INVARIANTS. In applications to dynamical systems it is useful to consider asymptotic invariants arising in Hofer's geometry.

8.A. Asymptotic non-minimality [BP2]. Define a function $\mu : \mathcal{A} - \{0\} \to [0; 1]$ as follows. Take a Hamiltonian function F in $\mathcal{A}$ and consider its Hamiltonian flow $\{f_t\}$. Consider all paths on $\text{Ham}(M, \Omega)$ joining the identity with f_s which are homotopic to $\{f_t\}_{t \in [0;s]}$ with fixed end points. Denote by $\mu(F, s)$ the infimum of lengths of these paths. For instance if $\{f_t\}$ is a minimal geodesic then $\mu(F, s) =$

$s||F||$. It is easy to see that the limit

$$\mu(F) = \lim_{s \to +\infty} \frac{\mu(F,s)}{s||F||}$$

exists. This number is called the asymptotic non-minimality of F, and measures the deviation of $\{f_t\}$ from a (semi-infinite) minimal geodesic. If F generates a minimal geodesic then $\mu(F) = 1$. Consider now two subsets of M consisting of all points where the function F attains its maximal and minimal values respectively. One can show [BP2] that *if one of these subsets has finite displacement energy, then $\mu(F) < 1$, and in particular F does not generate a minimal geodesic.* Note that this method does not allow us to control the length of the time interval on which the curve $\{f_t\}$ can be shortened.

8.B. Asymptotic length spectrum [P4]. For an element $\gamma \in \pi_1(\mathrm{Ham}(M, \Omega))$ set

$$\nu_\infty(\gamma) = \lim_{k \to +\infty} \frac{1}{k} \nu(\gamma^k).$$

This is an analogue of the Gromov-Federer stable norm in Hofer's geometry. Theorem 7.B above implies that for monotone symplectic manifolds $\nu_\infty(\gamma) \geq C|I(\gamma)|$.

Example. Let M be the blow up of the complex projective plane $\mathbf{CP}^2$ at one point. Choose a Kähler symplectic structure Ω on M which integrates to 1 over a general line and to $\frac{1}{3}$ over the exceptional divisor. The periods of the symplectic form are chosen in such a way that its cohomology class is a multiple of the first Chern class of M. One can easily see that (M, Ω) admits an effective Hamiltonian action of the unitary group $U(2)$, in other words there exists a monomorphism $i : U(2) \to \mathrm{Ham}(M, \Omega)$. The fundamental group of $U(2)$ equals $\mathbf{Z}$. Let $\gamma \in \pi_1(\mathrm{Ham}(M, \Omega))$ be the image of the generator of $\pi_1(U(2))$ under i. It turns out (Abreu - McDuff) that $\pi_1(\mathrm{Ham}(M, \Omega))$ equals $\mathbf{Z}$ and is generated by γ. The direct calculation [P4] shows that $I(\gamma) \neq 0$. We conclude that the asymptotic norm ν_∞ is strictly positive for each non-trivial element of the fundamental group of $\mathrm{Ham}(M, \Omega)$.

I do not know the *precise* value of $\nu_\infty(\gamma)$ in any example where this quantity is strictly positive (for instance, in the example above). The difficulty is as follows. In all known examples where Hofer's norm $\nu(\gamma)$ can be computed precisely there exists a closed loop h which minimizes the length in its homotopy class (that is a minimal closed geodesic). It turns out however that every loop loses minimality after a suitable number of iterations. In other words the loop $\{h_{Nt}\}$ can be shortened provided the integer N is large enough [P8].

9. New intuition in Hamiltonian dynamics. A Hamiltonian flow on a symplectic manifold can be considered as a curve on the group of Hamiltonian diffeomorphisms. One may hope that geometric properties of this curve (in the sense of Hofer's metric) are related to dynamics of the flow. In this section we present three examples of such a link, and thus illustrate our thesis that the geometry on the group of Hamiltonian diffeomorphisms gives rise to a different way of thinking about Hamiltonian dynamics.

9.A. Closed orbits of magnetic fields on the torus. This example was born in discussions with V. L. Ginzburg. Consider the cotangent bundle T^*T^n endowed with a twisted symplectic structure $\Omega_\sigma = dp \wedge dq + \theta^*\sigma$ as in 3.B above. Fix a Riemannian metric g on T^n. The dynamics of a magnetic field is described by the Hamiltonian flow of the function $|p|_g^2$ with respect to Ω_σ. We claim that *if the magnetic field is non-trivial (that is $\sigma \neq 0$) then there exist non-trivial contractible closed orbits of the flow on a sequence of arbitrary small energy levels.* We refer the reader to [Gi] for a survey of related results. Here is a geometric argument. Fix $\epsilon > 0$. Choose a smooth function $r(x)$, $x \in [0; +\infty)$ which equals $x - 2\epsilon$ on $[0; \epsilon]$, vanishes on $[3\epsilon; +\infty)$ and is strictly increasing on $[0; 3\epsilon)$. Consider a normalized Hamiltonian $F(p,q) = r(|p|_g^2)$. Every non-trivial closed orbit of F corresponds to a non-trivial closed orbit of the magnetic field whose energy is less than 3ϵ. The minimum set of F coincides with the zero section and thus its displacement energy vanishes (see 3.B). From 8.A we see that the asymptotic non-minimality of F is strictly less than 1, thus F does not generate a minimal geodesic. Finally, using the assertion of 5.A (which follows in this case from a result in [LM2]) we conclude that the Hamiltonian flow of F has a non-constant contractible closed orbit.

9.B. Invariant Lagrangian tori (along the lines of [BP2], cf. [Si2]). Consider T^*T^n, this time with the standard symplectic structure $dp \wedge dq$. Let $F \in \mathcal{A}$ be a normalized Hamiltonian with $||F|| = 1$. An important problem of classical mechanics is to decide which energy levels $\{F = c\}$ carry invariant Lagrangian tori homotopic to the zero section. Define a "converse KAM" type parameter $K(F)$ as the supremum of $|c|$, where c is as above. One can show that $\mu(F,s) \geq sK(F)$ for all $s > 0$, and thus $\mu(F) \geq K(F)$. The proof is based on an analogue of theorem 6.B above. Suppose now in addition that F is non-negative and its maximum set $L = F^{-1}(1)$ is a section of T^*T^n. The estimate above shows that if L is Lagrangian then F generates a minimal geodesic. If L is not Lagrangian, then its displacement energy vanishes (cf. 3.B) and thus $\mu(F) < 1$ (see 8.A). We conclude that in this case the asymptotic non-minimality of F gives a non-trivial upper bound for the quantity $K(F)$.

9.C. Strictly ergodic Hamiltonian skew products [P8]. Let (M^{2n}, Ω) be a closed symplectic manifold. Given an irrational number α and a smooth loop $h : S^1 \to \mathrm{Ham}(M, \Omega)$, one defines a *skew product diffeomorphism* $T_{h,\alpha}$ of $M \times S^1$ by $T_{h,\alpha}(x,t) = (h_t x, t + \alpha)$. A traditional problem in ergodic theory is to construct skew products with prescribed ergodic properties associated to loops in groups (see e.g. [N] and references therein). The property we are interested in is *the strict ergodicity*. In our situation the skew product $T_{h,\alpha}$ is called strictly ergodic if it has only one invariant Borel probability measure (which is a multiple of $\Omega^n \wedge dt$). One can adjust existing ergodic methods in order to show that for a wide class of symplectic manifolds (say for simply connected ones) there exist α and h such that $T_{h,\alpha}$ is strictly ergodic. It turns out that the loops h arising in this construction are *contractible*. An attempt to understand this phenomenon gives rise to the following definition. An element $\gamma \in \pi_1(\mathrm{Ham}(M, \Omega))$ is called strictly ergodic if if there exist a number α, and a loop h representing γ such that $T_{h,\alpha}$ is strictly ergodic. It turns out that *the asymptotic norm $\nu_\infty(\gamma)$ vanishes for all strictly ergodic classes γ*. Thus the geometry on $\mathrm{Ham}(M, \Omega)$ supplies us with an obstruction to strict ergodicity. For instance, it follows from 8.B above that for the monotone blow up of $\mathbf{CP}^2$ at one point $\gamma = 0$ is the only strictly ergodic class.

10. Does the geometry on $\mathrm{Ham}(M,\Omega)$ determine (M,Ω)? Here is the simplest version of this question. Let (M,Ω) be a closed surface endowed with an area form, and let $c>1$ be a real number. Are the spaces $\mathrm{Ham}(M,\Omega)$ and $\mathrm{Ham}(M,c\Omega)$ smoothly isometric with respect to their Hofer's metrics ? Here an isometry is smooth if it sends smooth paths, homotopies etc. to the smooth ones. When $M=S^2$ the answer is negative, since these spaces have different length spectra. When the genus of M is at least 1, the length spectrum is trivial, and the answer is unknown. This open problem of 2-dimensional symplectic topology completes our journey.

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