Evolution of Hypersurfaces by Their Curvature in Riemannian Manifolds

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Abstract. We study hypersurfaces in Riemannian manifolds moving in normal direction with a speed depending on their curvature. The deformation laws considered are motivated by concrete geometrical and physical phenomena and lead to second order nonlinear parabolic systems for the evolving surfaces. For selected examples of such flows the article investigates local and global geometric properties of solutions. In particular, it discusses recent results on the singularity formation in mean curvature flow of meanconvex surfaces (joint with C. Sinestrari) and applications of inverse mean curvature flow to asymptotically flat manifolds used for the modelling of isolated systems in General Relativity (joint with T. Ilmanen).

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1 The evolution equations

Let $F_0 : \mathcal{M}^n \to \mathrm{I\!R}^{n+1}$ be a smooth immersion of an n–dimensional hypersurface $\mathcal{M}_0^n = F_0(\mathcal{M}^n)$ in a smooth Riemannian manifold $(N^{n+1}, \bar{g}), \quad n \geq 2$. We study one–parameter families of immersions $F : \mathcal{M}^n \times [0, T[\to (N^{n+1}, \bar{g})$ of hypersurfaces $\mathcal{M}_t^n = F(\cdot, t)(\mathcal{M}^n)$ satisfying an initial value problem

$$\frac{\partial F}{\partial t}(p,t) = -f\nu(p,t), \qquad p \in \mathcal{M}^n,\ t \in [0, T[, \tag{1.1}$$

$$F(\cdot, 0) = F_0, \tag{1.2}$$

where $\nu(p,t)$ is a choice of unit normal at $F(p,t)$ and $f(p,t)$ is some smooth homogeneous symmetric function of the principal curvatures of the hypersurface at $F(p,t)$.

We are interested in the case where $f = f(\lambda_1, \cdots, \lambda_n)$ is monotone with respect to the principal curvatures $\lambda_1, \cdots, \lambda_n$ such that (1.1) is a nonlinear parabolic system of second order. The interaction between geometric properties of the data

$\mathcal{M}_0^n$, $(N^{n+1},\bar{g})$ and the local and global behaviour of solutions leads to many interesting phenomena and has applications to a number of models in mathematical physics.

Typical examples considered here are the mean curvature flow $f=-H=-(\lambda_1+\cdots+\lambda_n)$, the inverse mean curvature flow $f=H^{-1}$ and fully nonlinear flows such as the the Gauss curvature flow $f=-K=-(\lambda_1\cdots\lambda_n)$ or the harmonic mean curvature flow, $f=-(\lambda_1^{-1}+\cdots+\lambda_n^{-1})^{-1}$. We investigate some new developments in the mathematical understanding of these evolution equations and include some applications such as the use of the inverse mean curvature flow for the study of asymptotically flat manifolds in General Relativity.

To fix notation, let $\bar{g}=\{\bar{g}_{\alpha\beta}\}_{0\leq\alpha,\beta\leq n}$, $\bar{\nabla}$ and $\bar{\mathrm{R}}\mathrm{iem}=\{\bar{R}_{\alpha\beta\gamma\delta}\}$ be the metric, the connection and the Riemann curvature tensor of the target manifold respectively, where the indices sometimes refer to local coordinates $\{y^\alpha\}$ and sometimes to a suitable local orthonormal frame $\{e_\alpha\}$. We write $\bar{g}^{-1}=\{\bar{g}^{\alpha\beta}\}$ for the inverse of the metric and use the Einstein summation convention to sum over repeated indices. The Ricci curvature $\bar{\mathrm{R}}\mathrm{ic}$ and scalar curvature $\bar{\mathrm{R}}$ of $(N^{n+1},\bar{g})$ are then given by

$$\bar{R}_{\alpha\beta}=\bar{g}^{\gamma\delta}\bar{R}_{\alpha\gamma\beta\delta},\qquad \bar{R}=\bar{g}^{\alpha\beta}\bar{R}_{\alpha\beta},$$

and the sectional curvatures (in an orthonormal frame) are computed as $\bar{\sigma}_{\alpha\beta}=\bar{R}_{\alpha\beta\alpha\beta}$.

If $F:\mathcal{M}^n\to(N^{n+1},\bar{g})$ is a smooth hypersurface immersion, we denote by $g=\{g_{ij}\}_{1\leq i,j\leq n}$, ∇, Riem the induced metric, connection and intrinsic curvature. In an adapted local orthonormal frame $e_1,\cdots,e_n,\nu$ with unit normal ν the second fundamental form $A=\{h_{ij}\}$ is then at each point given by the symmetric matrix

$$h_{ij}\ =<\bar{\nabla}_{e_i}\nu,e_j>=-<\nu,\bar{\nabla}_{e_i}e_j>,$$

such that the eigenvalues $\lambda_1,\cdots,\lambda_n$ are the principal curvatures of the hypersurface at this point, leading to the classical scalar invariants mentioned earlier.

If the initial hypersurface is smooth and closed, ie compact without boundary, it is wellknown that a smooth solution of the flow exists for a short time, provided on the initial surface the speed function f is elliptic in the sense that $\partial f/\partial\lambda_i<0$, $1\leq i\leq n$. In particular, mean curvature flow always admits shorttime solutions and inverse mean curvature flow admits shorttime solutions for meanconvex initial data, whereas the Gauss curvature flow and harmonic mean curvature flow require the initial data to be convex $\big(\lambda_i>0\big)$.

Working in the class of surfaces where shorttime existence is guaranteed, the interesting task is to understand the longterm change in the shape of solutions, and to characterise their asymptotic behaviour both for large times and near singularities. For this purpose evolution equations have to be established for all relevant geometric quantities, in particular for the second fundamental form.

Theorem 1.1 *On any solution $\mathcal{M}_t^n=F(\cdot,t)(\mathcal{M}^n)$ of (1.1) the following equations hold:*

(i) $\frac{\partial}{\partial t}g_{ij}=2fh_{ij}$,

(ii) $\frac{\partial}{\partial t}(d\mu) = fH(d\mu)$,

(iii) $\frac{\partial}{\partial t}\nu = -\nabla f$,

(iv) $\frac{\partial}{\partial t}h_{ij} = -\nabla_i\nabla_j f + f\big(h_{ik}h_j^k - \bar{R}_{0i0j}\big)$,

(v) $\frac{\partial}{\partial t}H = -\Delta f - f\big(|A|^2 + \bar{\mathrm{R}}\mathrm{ic}(\nu,\nu)\big)$.

Here $d\mu$ is the induced measure on the hypersurface, the index 0 stands for the normal direction and Δ is the Laplace–Beltrami operator with respect to the time-dependent induced metric on the hypersurface.

Notice that $-\Delta f - f\big(|A|^2 + \bar{\mathrm{R}}\mathrm{ic}(\nu,\nu)\big) = Jf$ is the Jacobi operator acting on f, as is wellknown from the second variation formula for the area. The relations above are consequences of the definitions and the Gauss–Weingarten relations. To convert the evolution equations for the curvature into parabolic systems on the hypersurface, we introduce for each speed function f the nonlinear operator L_f by setting

$$L_f u = L_f^{ij}\nabla_i\nabla_j u := -\frac{\partial \hat{f}}{\partial h_{ij}}\nabla_i\nabla_j u,$$

where $\hat{f}$ is the symmetric function f considered as a function of the h_{ij}. Note that for mean curvature flow $L_H = \Delta$ is the Laplace–Beltrami operator, for inverse mean curvature flow $f = H^{-1}$ we have $L_f = (1/H^2)\Delta$ and in general L_f is an elliptic operator exactly when f is elliptic. Using then the crucial commutator relations for the second derivatives of the second fundamental form one derives after long but straightforward calculations

COROLLARY 1.2 *On any solution $\mathcal{M}_t^n = F(\cdot,t)(\mathcal{M}^n)$ of (1.1) the second fundamental form h_{ij} and the speed f satisfy*

$$\begin{aligned}
\frac{\partial}{\partial t}h_{ij} &= L_f^{kl}\nabla_k\nabla_l h_{ij} - \frac{\partial^2 f}{\partial h_{kl}\partial h_{pq}}\nabla_i h_{kl}\nabla_j h_{pq} \\
&+ \frac{\partial f}{\partial h_{kl}}\big\{h_{kl}h_{im}h_{mj} - h_{km}h_{il}h_{mj} + h_{kj}h_{im}h_{ml} - h_{km}h_{ij}h_{ml} \\
&+ \bar{R}_{kilm}h_{mj} + \bar{R}_{kijm}h_{ml} + \bar{R}_{mijl}h_{km} + \bar{R}_{0i0j}h_{kl} - \bar{R}_{0k0l}h_{ij} + \bar{R}_{mljk}h_{im} \\
&+ \bar{\nabla}_k\bar{R}_{0jil} + \bar{\nabla}_i\bar{R}_{0ljk}\big\} + f\big(h_{ik}h_j^k - \bar{R}_{0i0j}\big), \\
\frac{\partial}{\partial t}f &= L_f^{ij}\nabla_i\nabla_j f - f\frac{\partial f}{\partial h_{ij}}\big(h_{ik}h_j^k + \bar{R}_{0i0j}\big).
\end{aligned}$$

The curvature terms in this nonlinear reaction-diffusion system provide the key for understanding the interaction between geometric properties of the hypersurface and the ambient manifold. They are the tool to study these geometric phenomena with analytical means. We will now describe some recent developments for selected choices of f: Section 2 discusses the formation of singularities in the mean curvature flow, especially in the case of mean convex surfaces. In section 3 some fully nonlinear equations like Gauss curvature flow are considered.

Finally in section 4 it is explained how the inverse mean curvature flow provides an approach to the Penrose inequality for the total mass of an asymptotically flat manifold.

2 SINGULARITIES OF THE MEAN CURVATURE FLOW

In the case of mean curvature flow $f = -H$ it is well known [19] that for closed initial surfaces the solution of (1.1)–(1.2) exists on a maximal time interval $[0, T[$, $0 < T \leq \infty$. If $T < \infty$, as is always the case in Euclidean space, the curvature of the surfaces becomes unbounded for $t \to T$. One would like to understand the singular behaviour for $t \to T$ in detail, having in mind a possible controlled extension of the flow beyond such a singularity. See [22] for a review of earlier results concerning local and global properties of mean curvature flow. We will not discuss singularities in weak formulations of the flow, a good reference in this direction is [32].

Since the shape of possible singularities is a purely local question, we may restrict attention to the case where the target manifold is Euclidean space. Nevertheless, in the light of an abundance even of homothetically shrinking examples with symmetries, the possible limiting behaviour near singularities seems in general beyond classification at this stage.

In recent joint work of C. Sinestrari and the author [26], [27] the additional assumption of nonnegative mean curvature is used to restrict the range of possible phenomena, while still retaining an interestingly large class of surfaces. We derive new a priori estimates from below for all elementary symmetric functions of the principal curvatures, exploiting the one-sided bound on the mean curvature. The estimates turn out to be strong enough to conclude that any rescaled limit of a singularity is (weakly) convex.

Define by

$$S_k(\lambda) = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}$$

the elementary symmetric functions of the principal curvatures with $S_1 = H$. Then [27] establishes the estimates

THEOREM 2.1 *(H.-Sinestrari) Suppose $F_0 : \mathcal{M} \to \mathbb{R}^{n+1}$ is a smooth closed hypersurface immersion with nonnegative mean curvature. For each k, $2 \leq k \leq n$, and any $\eta > 0$ there is a constant $C_{\eta,k}$ depending only on n, k, η and the initial data, such that everywhere on $\mathcal{M} \times [0, T[$ the estimate*

$$S_k \geq -\eta H^k - C_{\eta,k} \tag{2.1}$$

holds uniformly in space and time.

The proof proceeds by induction on the degree k of S_k and relies heavily on the algebraic properties of the elementary symmetric functions, the structure of the curvature evolution in this particular flow and the Sobolev inequality for hypersurfaces. In each step of the iteration an a priori estimate is proved for a quotient

$$Q_k = \frac{S_k}{S_{k-1}}$$

of consecutive elementary symmetric polynmials, making use of the concavity properties of this function. Using techniques in [20] and [29] the result can be extended to starshaped surfaces in $\mathbb{R}^{n+1}$ and to hypersurfaces in Riemannian manifolds.

Similarly as in the theory of minimal surfaces the structure of singularities is then studied by blowup methods, in this case by parabolic rescaling in space and time, compare [15], [21], [26]. Since η is arbitrary in the above estimate and the mean curvature $S_1 = H$ tends to infinity near a singularity, the scaling invariance is broken in inequality (2.1) and implies that near a singularity each S_k becomes nonnegative after appropriate rescaling:

COROLLARY 2.2 *Let $\mathcal{M}_t$ be a mean convex solution of mean curvature flow on the maximal time interval $[0, T[$ as in Theorem 1.1. Then any smooth rescaling of the singularity for $t \to T$ is convex.*

The structure of the rescaled limit depends on the blowup rate of the singularity: If the quantity $\sup(T-t)|A|^2$ is uniformly bounded (type I singularity), the rescaling will yield a selfsimilar, homothetically shrinking solution of the flow which is completely classified in the case of positive mean curvature, see [21] and [22]. If the quantity $\sup(T-t)|A|^2$ is unbounded (type II singularity), the rescaling of the singularity can be done in such a way that an "eternal solution" (ie defined for all time) of mean curvature flow results where the maximum of the curvature is attained on the surface. In the convex case such solutions were shown by Hamilton to move isometrically by translations, [16]. Hence, combining the classification of type I singularities in [22], the result of Hamilton and the convexity result in Corollary 2.2, one derives a description of all possible singularities (type I and type II) in the mean convex case, compare [27].

Open problems which have to be adressed for the future goal of continuing the flow by surgery concern the classification of convex translating solutions, Harnack estimates for the mean curvature, more precise estimates on the rate of convergence as well as higher order asymptotics near singularities. Some guidance on the possible higher order behaviour near singularities can be taken from the degenerate examples constructed in [6]. A Harnack estimate for the mean curvature has so far only been obtained in the convex case [16], which is too restrictive for many applications. The work of Hamilton on the Ricci flow [15] has a close relation to the mean curvature flow and indicates a strategy for the extension of the flow past singularities once stronger estimates are available [17].

We conclude this section with the one-dimensional case, where an embedded curve is evolving in the plane or on some smooth surface by the curve shortening flow. The remarkable articles of Grayson on this flow [13],[14] show by a number of global arguments that for embedded curves no finite time singularity can occur unless the whole curve contracts to a single point.

The structure of all possible singularities in this case is now well understood: There are no embedded type I singularities except the shrinking circle, which is the desired outcome, and the only possible rescaling of a type II singularity is a so called grim reaper curve given by $y = \log\cos x$. To prove Graysons result it is therefore sufficient to give an argument excluding this last curve as a possible limiting shape. Such an argument is provided both by Hamilton in [18], where an isoperimetric estimate for the area in subdivisions of the enclosed region is shown, and by the author in [23], where a lower bound for the ratio between extrinsic and intrinsic distances on the evolving curve is proved.

To describe the last result let $F: S^1 \times [0,T] \to \mathbb{R}^2$ be a closed embedded curve moving by the curve shortening flow. If $L = L(t)$ is the total length of the curve, the intrinsic distance function l along the curve is smoothly defined only for $0 \leq l < L/2$, with conjugate points where $l = L/2$. We therefore define a smooth function $\psi: S^1 \times S^1 \times [0,T] \to \mathbb{R}$ by setting

$$\psi := \frac{L}{\pi}\sin(\frac{l\pi}{L}).$$

With this choice of ψ, and with d being the extrinsic distance between two points on the curve, the isoperimetric ratio d/ψ approaches 1 on the diagonal of $S^1 \times S^1$ for any smooth embedding of S^1 in $\mathbb{R}^2$ and the ratio d/ψ is identically one on any round circle.

THEOREM 2.3 *Let $F: S^1 \times [0,T] \to \mathbb{R}^2$ be a smooth embedded solution of the curve shortening flow (1.1). Then the minimum of d/ψ on S^1 is nondecreasing; it is strictly increasing unless $d/\psi \equiv 1$ and $F(S^1)$ is a round circle.*

Clearly the estimate prevents a grim reaper type singularity. The proof uses the maximum principle on the cross product of the curve with itself. It is an open problem whether similar lower order estimates can be used for the study or exclusion of certain singularities in higherdimensional flows.

3 FULLY NONLINEAR FLOWS

The Gauss curvature flow, where the speed $f = -K = -(\lambda_1 \cdots \lambda_n)$ is the product of the principle curvatures, was first introduced by Firey [10] as a model for the changing shape of a tumbling stone being worn from all directions with uniform intensity. The flow is parabolic only in the class of convex surfaces and much more nonlinear in its analytic behaviour than the mean curvature flow. Tso [30] proved existence, uniqueness and convergence of closed convex hypersurfaces to a point for this flow without however determining the limiting shape of the contracting surface. The conjecture of Firey (1974) that the limiting shape is that of a sphere regardless of the initial data, was only recently confirmed by Andrews [2]:

THEOREM 3.1 *(Andrews) Let $\mathcal{M}_0^2$ be a smooth closed strictly convex initial surface in $\mathbb{R}^3$. Then there is a unique smooth solution of (1.1) with $f = -K$ on the time interval [0,T[, where $T = V(\mathcal{M}_0^2)/4\pi$ is determined by the enclosed volume of the initial surface, and the surfaces converge to a round sphere after appropriate rescaling.*

The corresponding result for mean curvature flow was obtained earlier by the author in [19] and for a large class of speed functions f including the harmonic mean curvature flow by Andrews in [1]. If the Gauss curvature K is replaced by some power K^α, a whole new range of interesting phenomena appears. If the homogeneity is 1, ie $\alpha = 1/n$, Chow proved contraction to a point and roundness of the limiting shape, [8]. In [5] Andrews shows that in the interval $1/(n+2) < \alpha \leq 1/n$ there is at least some smooth limiting shape at the end of the contraction, while for small values of α a degeneration of the surface near the end of the contraction is expected.

In the special case $\alpha = 1/(n+2)$, the evolution equation (1.1) becomes affine invariant. In line with the results just mentioned Andrews [3] proves by an extension of Calabi's estimate on the cubic ground form that convex initial data contract smoothly to a point in finite time, with ellipsoids as the natural unique limiting shape. As a consequence he derives an elegant proof of the affine isoperimetric inequality. Compare also the work of Sapiro and Tannenbaum [28] on the affine evolution of curves, which has applications in image processing.

For convex hypersurfaces in general Riemannian manifolds speedfunctions f such as the harmonic mean curvature or other quotients of elementary symmetric functions seem to have the best algebraic behaviour. In mean curvature flow the derivatives of the ambient curvature in the evolution equations of Corollary 1.2 are analytically hard to control, compare the dependance of the main result in [20] on these terms. For harmonic mean curvature flow and flows of similar structure Andrews derives an optimal convergence result for hypersurfaces having sufficiently positive principal curvatures in relation to the ambient curvature, [4]. In particular, he shows that such flows contract convex hypersurfaces in manifolds of positive sectional curvature to a point and gives a new argument for the classical 1/4-pinching theorem.

All speedfunctions considered so far were pointing in the same direction as the mean curvature vector, corresponding to contractions in the case of convex surfaces. In the last section we consider an expanding version of the flow.

4 THE INVERSE MEAN CURVATURE FLOW

The inverse mean curvature flow $f = H^{-1}$ is well posed for surfaces of positive mean curvature and characterised by its property that the area element is growing exponentially at each point: From Theorem 1.1(i) we have $\partial/\partial t(d\mu) = d\mu$. In particular, the total area of a smooth closed evolving surface is completely determined by its initial area:

$$|M_t^n| = |M_0^n| \exp(t).$$

The standard example is the exponentially expanding sphere of radius $R(t) = R(0)\exp(t/n)$. Further interesting properties of the flow follow from the evolution equation for the mean curvature H, which we derive from the evolution equation for the speed f.

$$\frac{\partial H}{\partial t} = \frac{\Delta H}{H^2} - \frac{2|\nabla H|^2}{H^3} - \frac{|A|^2}{H} - \frac{\bar{R}ic(\nu,\nu)}{H}.$$

Due to the negative sign of the $|A|^2$-term we get from this equation by a simple application of the parabolic maximum principle the remarkable property that the mean curvature H is uniformly bounded in terms of its initial data and the Ricci curvature of the ambient manifold. This is in strong contrast to the mean curvature flow, where the blowup of the mean curvature causes the singularities studied in section 2. For the inverse mean curvature flow the critical behaviour occurs where $H \to 0$ and the speed becomes infinite. In Euclidean space it is clear that the maximum of the mean curvature is decreasing and the same is true for any L^p-norm.

In case $n = 2$ this property of the flow can be extended to closed surfaces in arbitrary three-manifolds of nonnegative scalar curvature: For any two-surface $\Sigma^2 \subset (N^3, \bar{g})$ the so called Hawking quasi-local mass of Σ^2 is defined as the geometric quantity

$$m_H(\Sigma^2) := \frac{|\Sigma^2|^{1/2}}{(16\pi)^{3/2}} \left(16\pi - \int_{\Sigma^2} H^2 \, d\mu\right),$$

and a computation based on the evolution equation for the mean curvature, the area element of the surface and the Gauss-Bonnet formula shows that for a solution M_t^2 of the inverse mean curvature flow

$$\frac{d}{dt}\int_{M_t^2} H^2 \, d\mu = 4\pi\chi(M_t^2) + \int_{M_t^2} -2\frac{|\nabla H|^2}{H^2} - \frac{1}{2}H^2 - \frac{1}{2}(\lambda_1 - \lambda_2)^2 - \bar{R} \, d\mu.$$

Hence, if the surface M_t^2 is connected and the scalar curvature $\bar{R}$ of the three-manifold is nonnegative, we have

$$\frac{d}{dt}\int_{M_t^2} H^2 \, d\mu \leq \frac{1}{2}\left(16\pi - \int_{M_t^2} H^2 \, d\mu\right)$$

and hence the Hawking quasi-local mass is nondecreasing along the inverse mean curvature flow:

$$\frac{d}{dt} m_H(M_t^2) \geq 0.$$

A major reason for the interest in the inverse mean curvature flow comes from the interpretation of this purely geometric fact in General Relativity: The spatial part of the exterior of an isolated gravitating system (like a star, black hole or galaxy) is modelled by the end of an asymptotically flat Riemannian 3–manifold with nonnegative scalar curvature as above. Here an end of a Riemannian 3-manifold $(N^3, \bar{g})$ is called *asymptotically flat* if it is realized by an open set that is diffeomorphic to the complement of a compact set K in $\mathbb{R}^3$, and the metric tensor $\bar{g}$ of M satisfies

$$|\bar{g}_{ij} - \delta_{ij}| \leq \frac{C}{|x|}, \qquad |\bar{g}_{ij,k}| \leq \frac{C}{|x|^2}, \qquad \bar{R}ic \geq -\frac{C\bar{g}}{|x|^2},$$

as $|x| \to \infty$. The derivatives are taken with respect to the Euclidean metric $\delta = \{\delta_{ij}\}$ on $\mathbb{R}^3 \setminus K$. On such asymptotically flat ends a concept of total mass or energy is defined by a flux integral through the sphere at infinity,

$$m := \lim_{r\to\infty} \frac{1}{16\pi} \int_{\partial B_r^\delta(0)} (\bar g_{ii,j} - \bar g_{ij,i}) n^j \, d\mu_\delta,$$

which is a geometric invariant, despite being expressed in coordinates. It is finite precisely when the scalar curvature $\bar R$ of $\bar g$ satisfies

$$\int_{N^3} |\bar R| < \infty,$$

and from a physical point of view it is meant to measure both matter content and gravitational energy of the isolated system. Compare the joint papers [24][25] of the author and T. Ilmanen for references to these facts. The Hawking quasi-local mass defined above is used as a geometric concept for the energy of a three-dimensional region contained inside a two-dimensional surface, motivated by the fact that for large approximately round spheres S_R^2 it is true that $m_H(S_R^2) \to m$. Furthermore, since in the physically simplest case the outer boundary of a black hole can be represented by a minimal two-surface inside the given three-manifold, the inverse mean curvature flow can provide a relation between the size of the black hole and the total energy m: If there is a smooth connected solution of the inverse mean curvature flow starting from a minimal surface $M_0^2 \subset N^3$, (the apparent horizon of the black hole) and expanding smoothly to large round spheres where $m_H(M_t^2) \to m$, then by the monotonicity result above we have the inequality

$$\frac{1}{4\sqrt{\pi}} |M_0^2|^{1/2} = m_H(M_0^2) \le m.$$

This relation between the size of the outermost black hole and the total energy of an isolated gravitating system is the Riemannian Penrose inequality, which sharpens the positive mass theorem. The argument just described was first put forward by Geroch, [12].

The crucial question concerns of course the existence of such a solution to the flow by inverse mean curvature. For starshaped surfaces of positive mean curvature in $\mathbb{R}^{n+1}$ Gerhardt [11] and Urbas [31] show that the necessary estimates for complete regularity of the flow can be established and they prove longterm existence as well as asymptotic roundness in this class.

Without an assumption like starshapedness it is quite clear that singularities have to occur in certain situations. For example, the solution evolving from a thin symmetric torus can not exist forever, due to the upper bound on H some blowup in the speed H^{-1} must occur for such initial data. Similar examples can be constructed in the class of two-spheres making it clear that there cannot be a smooth solution for the flow in the general situations that are of natural interest in physics.

To overcome these difficulties [24] introduces a weak concept of solution for the flow which still retains the crucial monotonicity of the Hawking mass. The weak concept is a level-set formulation of (1.1), where the evolving surfaces are given as level-sets of a scalar function u via

$$M_t^2 = \partial\{x | u(x) < t\},$$

and (1.1) is replaced by the degenerate elliptic equation

$$\mathrm{div}_N \left(\frac{\nabla u}{|\nabla u|} \right) = |\nabla u|,$$

where the left hand side describes the mean curvature of the level-sets and the right hand side yields the inverse speed. This formulation in divergence form admits locally Lipschitz continuous solutions and is inspired by the work of Evans-Spruck [9] and Chen-Giga-Goto [7] on the mean curvature flow. Using elliptic regularisation and a minimization principle we show existence of a locally Lipschitz-continuous solution with level-sets of nonnegative mean curvature of class $C^{1,\alpha}$, still satisfying monotonicity of the Hawking quasi-local mass, compare [24]. The solution allows the phenomenon of fattening, which corresponds to jumps of the surfaces and is desirable for our main application. We thus succeed in adapting Geroch's original argument and derive the following sharp lower bound for the mass:

THEOREM 4.1 *(H.-Ilmanen) Let N^3 be a complete, connected 3-manifold. Suppose that*

(i) *N^3 has nonnegative scalar curvature,*

(ii) *N^3 is asymptotically flat in the sense above with ADM mass m,*

(iii) *The boundary of N^3 is compact and consists of minimal surfaces, and N^3 contains no other compact minimal surfaces.*

Then $m \geq 0$, and

$$16\pi m^2 \geq |\Sigma^2|,$$

where $|\Sigma^2|$ is the area of any connected component of ∂N^3. Equality holds if and only if N^3 is one-half of the spatial Schwarzschild manifold.

The *spatial Schwarzschild manifold* is the manifold $\mathbb{R}^3 \setminus \{0\}$ equipped with the metric $\bar{g} := (1 + m/2|x|)^4 \delta$, representing the spatial exterior region of a single static black hole of mass m.

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