Lefschetz Fibrations in Symplectic Geometry

S. K. Donaldson

Keywords and Phrases: Lefschetz pencil, symplectic manifolds

Section 1. Linear systems.

One of the cornerstones of complex geometry is the link between *positivity* of curvature and *ampleness*. Let X be a compact complex manifold and $L \to X$ be a holomorphic line bundle over X. Suppose that L has a unitary connection whose curvature form is $-2\pi i\omega$ where ω is a positive $(1,1)$-form on X. Then for large k the line bundle L^k has many holomorphic sections. More precisely, the holomorphic sections define a projective embedding of X (Kodaira). This provides a passage from the discussion of abstract complex manifolds to concrete algebro-geometric models. If one chooses some linear subspace of the holomorphic sections of L^k one gets birational maps into smaller projective spaces: for example, if X has complex dimension 2 then it may be immersed in $\mathbf{CP}^4$ with a finite number of double points and can be mapped to a hypersurface in $\mathbf{CP}^3$ with "ordinary singularities". Perhaps the simplest case of all is that covered by Bertini's Theorem: if the intersection of the zero sets of all the holomorphic sections is empty (i.e if the linear system has no base points), then the zero-set of a generic section is a smooth hypersurface in X.

A familiar instance of these ideas occurs when X is a compact Riemann surface and we consider two sections of L^k. The ratio of these sections is a meromorphic function on X, i.e. a branched covering map $X \to \mathbf{CP}^1$. If the sections are sufficiently general then this map has a very simple local structure. There are a finite number of critical values $b_\alpha \in S^2 \cong \mathbf{CP}^1$; for each α there is a corresponding critical point $x_\alpha \in X$; the restriction of the map to $X \setminus \{x_\alpha\}$ is a covering map and around each point x_α the map is modelled, in suitable local co-ordinates, on the standard example $z \mapsto z^2$. The Riemann surface X, and the branched covering map, can be recovered from the data consisting of the configuration of points b_α in the Riemann sphere and the *monodromy*, a homomorphism from the fundamental group of the punctured sphere $S^2 \setminus \{b_\alpha\}$ to the permutation group of d objects, the sheets of the covering.

More generally one has the notion of a "Lefschetz pencil" on a higher dimensional complex variety. The ratio of two, sufficiently general, sections of our line bundle is a meromorphic function, which defines a holomorphic map from the complement of a codimension-2 submanifold $A \subset X$. Alternatively, we get a map from the blow-up $\tilde{X}$ of X along A to $\mathbf{CP}^1$. Again there are a finite number of critical points, around which the map is modelled on the quadratic function $(z_1, \dots, z_n) \mapsto z_1^2 + \dots + z_n^2$. The monodromy in this situtation is more complicated:

parallel transport around loops in the punctured sphere defines a homomorphism into the mapping class group of the fibre, that is, the group of diffeomorphisms of the fibre modulo isotopy.

The main purpose of this contribution is to report on extensions of these classical ideas in complex geometry to the more general setting of *symplectic* manifolds. In the next section we will describe some of the main results, and the ideas of the proofs, and in the final section we will make some more general comments.

Section 2.1 The symplectic case: techniques.

Now let (V,ω) be a compact symplectic manifold, of dimension $2n$. We suppose that the de Rham cohomology class $[\omega]$ is an integral class, so we can choose a C^∞ line bundle $L \to V$ with $c_1(L) = [\omega]$. To mimic the classical case we can begin by choosing an almost-complex structure on V, algebraically compatible with the symplectic form. There is also a unitary connection on L with curvature $-i\omega$. This gives a notion of a "holomorphic" section of L: we can define a $\overline{\partial}$-operator on L, using the connection and the almost-complex structure and a (local) section s is holomorphic if $\overline{\partial} s = 0$. The problem is that, for $n > 1$ and for generic almost-complex structures, one expects this definition to be vacuous in that there will be no non-trivial holomorphic sections. This is because the generalised Cauchy-Riemann equation $\overline{\partial} s = 0$ is over-determined and the compatibility condition which is needed to have local solutions is precisely the integrability of the almost-complex structure. This contrasts with the much-studied theory of holomorphic maps from a Riemann surface into an almost-complex manifold, where the integrability of the almost-complex structure does not make a great difference to the local theory of solutions. The way around this problem is to study certain approximately holomorphic sections of the line bundle, or more precisely of the tensor power L^k, for large k. The integer k is the crucial parameter throughout the discussion, and it is convenient to work with the family of Riemannian metrics g_k on V, where g_k is associated to the symplectic form $k\omega$ in the usual fashion. Thus the diameter of (V, g_k) is $O(\sqrt{k})$ but on a ball of g_k-radius 1 the almost-complex structure is close to the standard flat model, as $k \to \infty$. For any $C > 0$ we set

$$H_{k,C} = \{s \in \Gamma(L^k) : \|\overline{\partial} s\|, \|\nabla \overline{\partial} s\|, \|\nabla^2 \overline{\partial} s\| \leq C \sqrt{k}^{-1} \|s\|\},$$

where all norms are L^∞, computed using the metric g_k. Elements of $H_{k,C}$ are a substitute for the holomorphic sections in the classical case. One shows that, for a suitable C depending on the geometry of X and for $k >> 0$, there is a large supply of sections in $H_{k,C}$. This is quite elementary: the sections can be constructed as linear combinations of sections concentrated in balls, of a fixed g_k-radius, in X. The fundamental model, which serves as a prototype for the influence of curvature on holomorphic geometry, is the case of $\mathbf{C}^n$, with the standard flat metric. Then there is a holomorphic section σ of the corresponding Hermitian line bundle over $\mathbf{C}^n$ which decays rapidly at infinity:

$$|\sigma(z)| = e^{-|z|^2}.$$

(In the several complex variables literature this phenomena is often described in the equivalent language of weighted L^2 norms.)

The classical theory sketched in Section 1 involves, beyond the existence of a plentiful supply of holomorphic sections, holomorphic versions of various familiar transversality statements. For example Bertini's theorem is a holomorphic version of Sard's Theorem, and the proof of the existence of Lefschetz pencils is a variant of the proof of the existence of Morse functions. The price that must be paid for the freedom to work with only approximately holomorphic sections is that one needs refinements of such transversality theorems, involving explicit estimates. These have interest in their own right. Results in this direction were obtained by Yomdin [5], although the precise statements needed are somewhat different. For simplicity consider the case of a holomorphic function f on the unit ball B^{2n} in $\mathbf{C}^n$. The familiar Sard theorem asserts that the regular values of f are dense in $\mathbf{C}$. For $\epsilon > 0$ we say that a point $w \in \mathbf{C}$ is an ϵ-regular value of f over a subset $K \subset B^{2n}$ if there are no points $z \in K$ with both $|f(z) - w| < \epsilon$ and $|\partial f(z)| < \epsilon$. The question we wish to answer is this: given any $w' \in \mathbf{C}$, how close is w' to an ϵ-regular value of f? An answer is provided by the following statement:

PROPOSITION. *There is a constant p such that for all holomorphic functions f on B^{2n} with $\|f\|_{L^\infty} \leq 1$, any $w' \in \mathbf{C}$ and $\epsilon \in (0, 1/2)$ there is an ϵ-regular value w of f over the interior ball $\frac{1}{2}B^{2n}$ with*

$$|w - w'| \leq (\log(\epsilon^{-1}))^p \epsilon.$$

One way of thinking of this result is that one would really like to have the stronger and simpler statement

$$|w - w'| \leq C.\epsilon,$$

but the factor $\log(\epsilon)^{-1}$ grows slowly as $\epsilon \to 0$, so the result stated in the Proposition serves almost as well. (The writer does not know whether the stronger statement is true or not.) The point to make is that the standard proofs of Sard's Theorem are not well-adapted to proving quantitative refinements of this kind, and the proof goes, following the idea of Yomdin, by approximating the function by polynomials and using facts about the complexity of real-algebraic sets.

SECTION 2.2: THE SYMPLECTIC CASE: MAIN RESULTS.

The first result proved using these ideas [2] is, roughly speaking, a symplectic version of Bertini's Theorem. For large k it is shown that one can choose an approximately holomorphic section $s \in H_{k,C}$ such that $|\partial s| > \delta \|s\|$ on the zero-set Z_s, for a fixed $\delta > 0$, independent of k. It follows that Z_s is a symplectic submanifold of V, i.e. the restriction of the form ω is nondegenerate on Z_s. Thus we have

THEOREM. *If (V, ω) is a compact symplectic manifold and $[\omega]$ is an integral class then for large k the Poincaré dual of $k[\omega]$ is represented by a symplectic codimension-2 submanifold.*

(This result can be compared with a much sharper but more specialised theorem of Taubes [4], proved shortly afterward using the Seiberg-Witten equations, which asserts that if V is a symplectic 4-manifold with $b_2^+(V) > 1$ then $-c_1(V)$ is represented by a symplectic surface in V.)

This result was extended, in a number of directions, by D. Aroux [1]. One striking extension was a result about the asymptotic uniqueness of the symplectic submanifold which is constructed. Let us return to the classical case of complex geometry. Then it is clear that the discriminant set $\Delta \subset H^0(L^k)$, consisting of sections whose zero-set is not transverse, is a complex analytic variety. In particular the complement of Δ is *connected*. Thus if s_0, s_1 are two sections whose zero-sets Z_0, Z_1 are transverse, there is a isotopy of the ambient manifold taking Z_0 to Z_1. This is an important principle in complex geometry. It means, for exammple, that at the level of differential topology one can unambigously talk about "a smooth hypersurface of degree d in $\mathbf{CP}^n$", without specifying precisely which polynomial is used in the definition. Of course this contrasts with the case of real algebraic geometry, where the topological type does vary with the polynomial. Aroux's extension of this principle to the symplectic case made use of the notion of an asymptotic sequence (s_k), $s_k \in \Gamma(L^k)$, of sections of the kind whose existence is established in the result above. He proves that

Theorem.

If J, J' are two almost-complex structures on V compatible with ω and $(s_k), (s'_k)$ are two asymptotic sequences of approximately holomorphic sections, with respect to J, J', then for large k there is a symplectic isotopy of V mapping the zero set of s_k to that of s'_k.

We now go on to consider the symplectic analogue of the classical theory of "pencils", generated by a pair of sections.

Definition. *A topological Lefschetz pencil on a symplectic manifold (V,ω) is given by the following data.*

(i) a codimension-2 symplectic submanifold $A \subset V$, (ii) a finite set of points $x_\alpha \in V \setminus A$, (iii) a differentiable map $f : V \setminus A \to S^2$ such that f is a submersion on $V \setminus A \setminus \{x_\alpha\}$.

The map f is required to conform to the following standard models. At a point $a \in A$ we can choose local complex co-ordinates z_i such that A is locally defined by the equations $z_1 = z_2 = 0$ and f is given locally by the map $(z_1, \dots, z_n) \mapsto z_1/z_2 \in \mathbf{CP}^1 \cong S^2$. At a point x_α we can choose local complex co-ordinates on V, and a complex co-ordinate centred on $f(x_\alpha) \in S^2$ such that the map is given locally by $(z_1, \dots, z_n) \mapsto z_1^2 + \dots + z_n^2$.

Then we have

Theorem. *If (V,ω) is a symplectic manifold with $[\omega]$ integral then for large k V admits a topological Lefschetz pencil, in which the fibres are symplectic subvarieties, representing the Poincaré dual of $k[\omega]$.*

There is also an asymptotic uniqueness statement, in the same vein as Aroux's result.

Let us spell out more explicitly what this theorem says, concentrating on the case of a 4-dimensional symplectic manifold V. In this case A is just a finite set of points. If $\tilde{V}$ is the blow-up of V at these points then f defines a smooth map from $\tilde{V}$ to S^2 whose generic fibre is a compact Riemann surface, of genus g say. There are a finite number of singular fibres, passing through the critical points x_α. Writing $b_\alpha = f(x_\alpha)$, we have a differentiable monodromy

$$\rho : \pi_1(S^2 \setminus \{b_\alpha\}) \to \Gamma_{g,h}$$

where h is the number of points in A and $\Gamma_{g,h}$ is the mapping class group of a surface of genus g with h marked points. The standard theory from complex geometry adapts with little change to give a detailed "local" picture of this monodromy. Fix a base point P in $S^2 \setminus \{b_\alpha\}$ and let γ_α be a standard generator of the fundamental group of the punctured sphere, winding once around b_α. Then the monodromy $\rho(\gamma_\alpha)$ is the *Dehn twist* $T[\delta_\alpha]$ of the fibre $\Sigma = f^{-1}(P)$ in a *vanishing cycle* $\delta_\alpha \subset \Sigma$, an embedded loop in Σ. The conclusion is, in brief, that any smooth 4-manifold which admits a symplectic structure with integral periods may be constructed from combinatorial data consisting of a marked Riemann surface Σ and a suitable set of loops δ_α in Σ, the essential requirement being that the product

$$T[\delta_1] \circ T[\delta_2] \circ \cdots \circ T[\delta_N]$$

be the identity in the mapping class group.

SECTION 3: DISCUSSION.

There are a number of directions in which one could hope to extend and fill-out and these results. First one could look at linear systems of other dimensions, and hope to prove analogues of the classical theorems in complex geometry for these cases. There is recent work of Aroux in this direction. Second, we should mention work of Gompf which provides a converse to the discussion above of symplectic Lefschetz pencils on 4-manifolds. Gompf shows that the total space of a 4-dimensional topological Lefschetz fibration, satisfying some mild numerical conditions, admits a symplectic structure. Putting everything together, one might expect to get a completely combinatorial-topological description of symplectic 4-manifolds. One natural set of questions involves the dependence on the parameter k. For example if $Z_k \subset V$ is a symplectic hypersurface representing $k[\omega]$ one can hope to describe a hypersurface $Z_{2k} \subset V$ representing $2k[\omega]$ by considering deformations of a singular space $Z_k \cup Z_k'$, where Z_k' is obtained by applying a generic small perturbation to Z_k. There is a similar discussion for Lefschetz pencils. If this project was carried through one could refine the asymptotic uniqueness statement into an explicit "stabilisation" mechanism.

The implications of this line of work for symplectic topology are unclear at present. Although it seems quite practical to "reduce" many fundamental questions about symplectic manifolds to combinatorial-topological problems, the latter seem very difficult to attack directly. Many of the difficulties have to do with the complexity of the braid groups which act as automorphisms of the fundamental groups of punctured Riemann spheres. In the case of 4-manifolds we may consider the set $\mathcal{R}$ of the representations of $\pi_1 = \pi_1(S^2 \setminus \{b_\alpha\})$ into the mapping class group $\Gamma_{g,h}$ which correspond to topological Lefschetz fibrations (i.e. which map each standard generator to a Dehn twist). Then the (spherical) braid group B_N acts on $\mathcal{R}$ and the natural invariants of the fibrations are the orbits under this action. Thus one would like are computable invariants which detect these orbits. One can view this problem as a higher dimensional analogue of the classical theory for branched covers of the Riemann sphere. In that case the issue is to classify transitive representations of π_1 into the permutation group on d-elements which map each generator to a transposition, modulo the action of the braid group, and a theorem of Hurwicz states that there is just one orbit.

On the positive side, it is worth pointing out that there are many similarities between the ideas that occur in this theory and those developed in the past few years by P. Seidel [3], in the framework of symplectic Floer theory. In both cases the Dehn twists, and their higher-dmensional generalisations, play a prominent role. These generalised Dehn twists are defined as follows. If L is an embedded Lagrangian m-sphere in a symplectic manifold W^{2m} a neighbourhood of L in W can be identified with a neighbourhood of the zero section in TS^m. The one can define a compactly-supported diffeomorphism of TS^m, using the geodesic flow composed with the antipodal map. This can then be transported to a symplectomorphism $\tau_L : W \to W$. Seidel shows that, when $m = 2$ in many cases the squares τ_L^2 are not symplectically isotopic to the identity, although they are so differentiably, thus revealing some of the rich structure of symplectic mapping class groups. On the other hand, these same symplectomorphisms occur as the monodromy of Lefschetz pencils of a symplectic manifold of dimension $2(m+1)$. They may also be analysed from the point of view of the braid group action on a Lefschetz pencil for W: the diffeomorphism arises from the action of a standard braid on a pair of identical monodromies. For these, and other, reasons, it seems possible that there may be some fruitful interaction between the symplectic Floer theory and the general Lefschetz pencil description of symplectic manifolds.

REFERENCES

1. D. Aroux, *Asymptotically holomorphic families of symplectic submanifolds*, Geometric and Functional Analysis (to appear).
2. S. K. Donaldson, *Symplectic submaifolds and almost-complex geometry*, Jour. Differential Geometry **44** (1996), 666-705.
3. P. Seidel, *The symplectic isotopy problem*, Oxford D.Phil. Thesis (1997).
4. C. H. Taubes, *More constraints on symplectic manifolds from the Seiberg-Witten equations*, Math. Research Letters **2** (1995), 9-14.
5. B. Yomdin, *The geometry of critical and near-critical values of differentiable mappings*, Math. Annalen **104** (1983), 495-515.

S. K. Donaldson
Department of Mathematics,
Stanford University,
CA 94305, USA