Spaces with Ricci Curvature Bounds

Tobias H. Colding

0. Introduction

One of the early (and very important) formulas involving Ricci curvature is the so called Bochner formula from the forties. This formula asserts that if M^n is a Riemannian manifold and $u \in C^3(M)$, then

$$\frac{1}{2}\Delta|\nabla u|^2 = |\mathrm{Hess}_u|^2 + \langle \nabla \Delta u, \nabla u\rangle + \mathrm{Ric}\,(\nabla u, \nabla u)\,. \tag{0.1}$$

Bochner used this formula to conclude that a closed n-dimensional manifold with nonnegative Ricci curvature has first Betti number, b_1, at most equal to the dimension with equality if and only if the manifold is a flat torus.

From the Bochner formula one can also obtain the Ricatti equation which in turn can be seen to yield the Laplacian comparison. This important comparison principle allows one to construct cut off functions with bounds on the Laplacian on manifolds with a lower Ricci curvature bound. It also allows one to use maximum principle methods to give a priori estimates on these manifolds. The Laplacian comparison theorem says that if $\mathrm{Ric}_{M^n} \geq (n-1)\,\Lambda$, $x \in M$ is fixed, r denotes the distance function to x, and $f : \mathbf{R} \to \mathbf{R}$ is a C^2 function then the following hold. If $f' \geq 0$, then

$$\Delta f(r) \leq \Delta_\Lambda f(r)\,, \tag{0.2}$$

and if $f' \leq 0$ then

$$\Delta_\Lambda f(r) \leq \Delta f(r)\,. \tag{0.3}$$

Here $\Delta_\Lambda f(r)$ denote the corresponding quantities on the simply connected n-dimensional space form of constant sectional curvature Λ.

It is important to note (as Calabi originally emphasized in the fifties) that the Laplacian comparison holds in a useful generalized sense, even at points where the distance function fails to be smooth i.e. on the cut locus. This point was also illustrated in the gradient estimate mentioned below and the Cheeger-Gromoll splitting theorem from the early seventies.

From the Laplacian comparison together with an integration by parts argument one can get the so called volume comparison theorem (see below for an important application of this). This comparison theorem assert that if $\mathrm{Ric}_{M^n} \geq (n-1)\,\Lambda$ then for all $x \in M$ and all $0 < s \leq t$

$$\frac{\mathrm{Vol}(B_s(x))}{\mathrm{Vol}(B_t(x))} \geq \frac{\mathrm{V}^n_\Lambda(s)}{\mathrm{V}^n_\Lambda(t)}\,. \tag{0.4}$$

Here $\mathrm{V}^n_\Lambda(s)$ is the volume of a ball of radius s in the simply connected space form with constant sectional curvature Λ. For the present survey one of the most important consequences of the volume comparison theorem is that it can be thought of as claiming the monotonicity of a density type quantity. For instance if the Ricci curvature is nonnegative then the volume comparison can clearly be restated as

$$r_0^{-n}\,\mathrm{Vol}(B_{r_0}(x))\downarrow\,, \tag{0.5}$$

for all fixed x. A particular consequence of this volume comparison theorem is the upper bound on the volume of balls due to Bishop in the early sixties. That is if $\mathrm{Ric}_{M^n}\geq (n-1)\,\Lambda$ then for all $x\in M$ and all $0<r_0$

$$\mathrm{V}^n_\Lambda(r_0)\geq \mathrm{Vol}(B_{r_0}(x))\,. \tag{0.6}$$

Another very crucial estimate on manifolds with a lower Ricci curvature bound is the gradient estimate of Cheng and Yau from the early seventies.

For instance if $\mathrm{Ric}_{M^n}\geq (n-1)\,\Lambda$, $x\in M$ and u is a harmonic function on $B_{2r_0}(x)$ then

$$\sup_{B_{r_0}(x)}|\nabla u|\leq C(n,\Lambda,r_0)\sup_{B_{2r_0}(x)}|u|\,. \tag{0.7}$$

The final inequality that we will recall is the Abresch-Gromoll inequality from the late eighties. This inequality was inspired by the Cheeger-Gromoll splitting theorem and it gives an important estimate for thin triangles. A specific case of this inequality is the following. If $\mathrm{Ric}_{M^n}\geq 0$, $n\geq 3$, x, y, $z\in M$ and $e_{x,y}(z)=\overline{x,z}+\overline{z,y}-\overline{x,y}$ then assuming $2\,\overline{x,z}\leq\overline{x,y}$,

$$\min\{\overline{x,z}^{\frac{n}{n-1}},\overline{x,z}^{\frac{1}{n-1}}\,e_{x,y}(z)\}\leq C\,h^{\frac{n}{n-1}}\,. \tag{0.8}$$

Here h is the height of the triangle, that is $h=\inf_{m\in\{\gamma_{x,y}\}}\overline{m,z}$ where $\{\gamma_{x,y}\}$ is the set of minimal geodesics connecting x, y and $C=C(n)$.

There is a natural generalization of the classical Hausdorff distance between subsets of Euclidean space to a distance function on all metric spaces. This is the Gromov-Hausdorff distance and it gives a good tool for studying metric spaces.

Suppose that (X,d_X) and (Y,d_Y) are two compact metric spaces. We say that the Gromov-Hausdorff distance between them is at most $\epsilon>0$ if there exist maps $f:X\to Y$ and $g:Y\to X$ such that

$$\forall x_1,\,x_2\in X:\;|d_X(x_1,x_2)-d_Y(f(x_1),f(x_2))|<\epsilon\,, \tag{0.9}$$

$$\forall x\in X:\;d_X(x,g\circ f(x))<\epsilon\,, \tag{0.10}$$

and the two symmetric properties in Y hold. The Gromov-Hausdorff distance between X and Y, denoted by $d_{GH}(X,Y)$, is then the infimum of all such ϵ.

Using this topology we can then say that a sequence of metric space converges to another metric space. For noncompact metric spaces there is a more useful notion of convergence which is essentially convergence on compact subsets. Namely, if (X_i,x_i,d_{X_i}) is a pointed sequence of metric spaces then we say that (X_i,x_i,d_{X_i}) converges to some pointed metric space (X,x,d_X) in the pointed Gromov-Hausdorff topology if for all $r_0>0$ fixed the compact metric spaces $B_{r_0}(x_i)$ converges to $B_{r_0}(x)$.

Gromov's compactness theorem is the statement that any pointed sequence, (M_i^n, m_i), of n-dimensional manifolds, with

$$\mathrm{Ric}_{M_i^n} \geq (n-1)\,\Lambda\,, \tag{0.11}$$

has a subsequence, (M_j^n, m_j), which converges in the pointed Gromov-Hausdorff topology to some length space (M_∞, m_∞).

The proof of this compactness theorem relies only on the volume comparison theorem. In fact it only uses a volume doubling which is implied by the volume comparison.

Finally we refer to [C5] and [Ga] for surveys related to this article and more detailed references.

1. Geometry and topology of smooth manifolds

Let us first recall some results from [C3]; see also [C1], [C2].

Theorem 1.1. [C3]. Given $\epsilon > 0$ and $n \geq 2$, there exist $\delta = \delta(\epsilon, n) > 0$ and $\rho = \rho(\epsilon, n) > 0$, such that if M^n has $\mathrm{Ric}_{M^n} \geq -(n-1)$ and $0 < r_0 \leq \rho$ with

$$\mathrm{Vol}(B_{r_0}(x)) \geq (1-\delta)\,\mathrm{V}_0^n(r_0)\,, \tag{1.2}$$

then

$$d_{GH}(B_{r_0}(x), B_{r_0}(0)) < \epsilon\, r_0\,, \tag{1.3}$$

where $B_{r_0}(0) \subset \mathbf{R}^n$.

We also have the following converse.

Theorem 1.4. [C3]. Given $\epsilon > 0$ and $n \geq 2$, there exist $\delta = \delta(\epsilon, n) > 0$ and $\rho = \rho(\epsilon, n) > 0$, such that if M^n has $\mathrm{Ric}_{M^n} \geq -(n-1)$ and $0 < r_0 \leq \rho$ with

$$d_{GH}(B_{r_0}(x), B_{r_0}(0)) < \delta\, r_0\,, \tag{1.5}$$

then

$$\mathrm{Vol}(B_{r_0}(x)) \geq (1-\epsilon)\,\mathrm{V}_0^n(r_0)\,, \tag{1.6}$$

where $B_{r_0}(0) \subset \mathbf{R}^n$.

As mentioned in the introduction Bochner used the formula (0.1) to give a bound for b_1 on closed manifolds with nonnegative Ricci curvature. In the late seventies Gromov (see also Gallot) showed that there exists an $\epsilon = \epsilon(n) > 0$ such that any closed n-manifold with $\mathrm{Ric}_M \mathrm{diam}_M^2 > -\epsilon$ has $b_1 \leq n$. Gromov also conjectured the following.

Theorem 1.7. [C3]. There exists an $\epsilon = \epsilon(n) > 0$ such that if M^n is a closed n-dimensional manifold with $\mathrm{Ric}_M\, \mathrm{diam}_M^2 > -\epsilon$ and $b_1(M) = n$, then M is homeomorphic to a torus.

For further discussion of other work related to this see [C3].

We will think of Theorems 1.1 and 1.4 as regularity results which parallel Allard's classical regularity theorem for minimal submanifolds. To explain this point further we make the following definition inspired in part by the classical work of Reifenberg.

Let (Z, d_Z) be a complete metric space. We will say that Z satisfies the (ϵ, ρ, n)-$\mathcal{G}_{\mathcal{R}}$ condition at $z \in Z$ if for all $0 < \sigma < \rho$ and all $y \in B_{\rho-\sigma}(z)$,

$$d_{GH}(B_\sigma(y), B_\sigma(0)) < \epsilon\, \sigma\,, \tag{1.8}$$

where $B_{r_0}(0) \subset \mathbf{R}^n$.

The above two theorems has the following corollary.

COROLLARY 1.9. Given $\epsilon > 0$ and $n \geq 2$, there exist $\delta = \delta(\epsilon, n) > 0$ and $\rho = \rho(\epsilon, n) > 0$, such that if M^n has $\mathrm{Ric}_{M^n} \geq -(n-1)$ and $0 < r_0 \leq \rho$ with

$$d_{GH}(B_{2r_0}(x), B_{2r_0}(0)) < \delta\, r_0\,, \tag{1.10}$$

or

$$\mathrm{Vol}(B_{2r_0}(x)) \geq (1-\delta)\, \mathrm{V}_0^n(2r_0)\,, \tag{1.11}$$

then M^n satisfies the (ϵ, r_0, n)-$\mathcal{G}_{\mathcal{R}}$ condition at x.

THEOREM 1.12. [ChC2]. Given $\epsilon > 0$ and $n \geq 2$, there exists $\delta > 0$ such that if (Z, d_Z) is a complete metric space and Z satisfies the (δ, r_0, n)-$\mathcal{G}_{\mathcal{R}}$ condition at $z \in Z$ then there exists a bi-Hölder homeomorphism $\Phi : B_{\frac{r_0}{2}}(z) \to B_{\frac{r_0}{2}}(0)$ such that for all z_1, $z_2 \in Z$,

$$r_0^{-\epsilon}\, |\Phi(z_1) - \Phi(z_2)|^{1+\epsilon} \leq d_Z(z_1, z_2) \leq r_0^{\epsilon}\, |\Phi(z_1) - \Phi(z_2)|^{1-\epsilon}\,. \tag{1.13}$$

Using the above results one can show the following stability theorem.

THEOREM 1.14. [C3], [ChC2]. If M^n is a closed n-manifold, then there exists an $\epsilon = \epsilon(M) > 0$ such that if N^n is an n-manifold with $\mathrm{Ric}_N \geq -(n-1)$ and $d_{GH}(M, N) < \epsilon$ then M and N are diffeomorphic.

A very useful result about the structure on a small but definite scale of smooth manifolds with a lower Ricci curvature bound is the following theorem whose implications will be explained in more detail in the next section.

THEOREM 1.15. [ChC1]. Given $\epsilon > 0$ and $n \geq 2$, there exist $\rho = \rho(\epsilon, n) > 0$, $\delta = \delta(\epsilon, n) > 0$ such that if M^n has $\mathrm{Ric}_{M^n} \geq -(n-1)$, $0 < r_0 \leq \rho$, and

$$(2\, r_0)^{-n}\, \mathrm{Vol}(B_{2r_0}(p)) \geq (1-\delta)\, r_0^{-n}\, \mathrm{Vol}(B_{r_0}(p))\,, \tag{1.16}$$

then

$$d_{GH}(B_{2r_0}(p) \setminus B_{r_0}(p), B_{2r_0}(v) \setminus B_{r_0}(v)) < \epsilon\, r_0\,. \tag{1.17}$$

Here v is the vertex of some metric cone $(0, \infty) \times_r X$, for some length space X.

As a particular consequence of this theorem together with the monotonicity, (0.5), we get that if M^n has nonnegative Ricci curvature and Euclidean volume growth, i.e. $V_M = \lim_{r_0 \to \infty} r_0^{-n}\, \mathrm{Vol}(B_{r_0}(x)) > 0$, then every tangent cone at infinity of M is a metric cone. Here a tangent cone at infinity is a rescaled limit of $(M, x, r_i^{-2} g)$ where $r_i \to \infty$.

Another useful result is the following almost splitting theorem.

THEOREM 1.18. [ChC1]. Given $\epsilon > 0$ and $n \geq 2$, there exist $\rho = \rho(\epsilon, n) > 0$, $\delta = \delta(\epsilon, n) > 0$ such that if M^n has $\mathrm{Ric}_{M^n} \geq -(n-1)$, $0 < r_0 \leq \rho$, x, $y \in \partial B_{r_0}(z)$, and

$$e_{x,y}(z) \leq \delta\, r_0\,, \tag{1.19}$$

then there exists some metric space X, $\overline{z} \in X \times \mathbf{R}$ such that

$$d_{GH}(B_{\delta\, r_0}(\overline{z}), B_{\delta\, r_0}(z)) < \epsilon\, r_0\,. \tag{1.20}$$

2. SINGULAR SPACES

We will now describe some results concerning spaces, (M^n_∞, m_∞) which are pointed Gromov-Hausdorff limits of sequences of manifolds satisfying (0.11). If in addition

$$\mathrm{Vol}(B_1(m_i)) \geq v > 0, \tag{2.1}$$

then we say that the sequence is noncollapsing.

To begin with, recall that M^n_∞ is a *length space* of Hausdorff dimension at most n and for all $r > 0, y_i \in M^n_i$, with $y_i \to y_\infty$, we have by [C3] (see also [ChC2], Theorem 2.13) the following which was conjectured by Anderson-Cheeger

$$\lim_{i\to\infty} \mathcal{H}^n(B_r(y_i)) = \mathcal{H}^n(B_r(y_\infty)), \tag{2.2}$$

where $\mathcal{H}^\alpha$ denotes α-dimensional Hausdorff measure. (Note that for smooth n-dimensional manifolds $\mathcal{H}^n$ is just the Riemmanian volume). As a consequence of this we have that the volume comparison also holds for noncollapsed limit spaces. That is for all $x \in M_\infty$ and all $0 < s \leq t$

$$\frac{\mathcal{H}^n(B_s(x))}{\mathcal{H}^n(B_t(x))} \geq \frac{\mathrm{V}^n_\Lambda(s)}{\mathrm{V}^n_\Lambda(t)}\,. \tag{2.3}$$

By definition, a *tangent cone*, $(T_x M^n_\infty, x_\infty)$, at $x \in M^n_\infty$, is any rescaled pointed Gromov-Hausdorff limit, $(T_x M^n, x_\infty, d_\infty)$, of some sequence, $(M^n_\infty, x, r_i^{-1} d_\infty)$, where $r_i \to 0$. Here, d_i, d_∞ denotes the metric (i.e. distance function) on M^n_i, M^n_∞. It follows from Gromov's compactness theorem that tangent cones exist at all points, $x \in M^n_\infty$, but according to [ChC2], Example 5.37, they need not be unique even when (2.1) hold. However, when (2.1) holds, every tangent cone is a metric cone, $C(X)$, on some length space, X, with $\mathrm{diam}(X) \leq \pi$; see [ChC2], Theorem 2.2.

The set of points, $\mathcal{R}$, for which the tangent cone is unique and isometric to $\mathbf{R}^k$ for some k, is called the regular set. The complement of the regular set is called singular set and denoted by $\mathcal{S}$. For $k \leq n-1$, let $\mathcal{S}_k$ denote the subset of singular points of M^n_∞ consisting of points for which no tangent cone splits off a factor, $\mathbf{R}^{k+1}$, isometrically. Then one can show using [C3] that $\bigcup_k \mathcal{S}_k = \mathcal{S}$. By Theorem 1.13 of [ChC2], have $\dim \mathcal{S}_k \leq k$, where *dim* denotes Hausdorff dimension. Moreover, when (2.1) holds, we have $\mathcal{S} \subset \mathcal{S}_{n-2}$; see [ChC2], Theorem 4.1.

The ϵ-*regular* set, $\mathcal{R}_\epsilon$, is by definition, the set of points for which some tangent cone satisfies, $d_{GH}(B_1(x_\infty), B_1(0)) < \epsilon$, where x_∞ is the vertex of $T_x M^n_\infty$ and $0 \in \mathbf{R}^n$. There exists $\epsilon(n) > 0$, such that for $\epsilon < \epsilon(n)$ the subset, $\mathcal{R}$, is homeomorphic to a smooth n-dimensional manifold and the restriction of the metric to $\mathcal{R}_\epsilon$ is bi-Hölder equivalent to a smooth Riemannian metric with exponent $\alpha(\epsilon)$ satisfying $\alpha(\epsilon) \to 1$ as $\epsilon \to 0$. Note that for all $\epsilon > 0$, we have $M^n_\infty = \mathcal{S}_{n-2} \bigcup \mathcal{R}_\epsilon$, although $\mathcal{S}_{n-2} \bigcap \mathcal{R}_\epsilon$ need not be empty.

If (2.1) hold and some tangent cone at a point is isometric to $\mathbf{R}^n$, then every tangent cone at the point has this property. Clearly, $\mathcal{R} = \cap_{\epsilon>0}\mathcal{R}_\epsilon$, but $\mathcal{R}$ need not be open. For the results of this paragraph and the preceding one, see Section 2 of [ChC2].

In case (0.11) is strengthened to

$$|\mathrm{Ric}_{M_i^n}| \leq n-1, \tag{2.4}$$

we have, using in part a theorem of Anderson, $\mathcal{R}_{\epsilon(n)} = \mathcal{R}$, for some $\epsilon(n) > 0$. In particular, $\mathcal{R}$ is open and $\mathcal{S} \subset \mathcal{S}_{n-2}$ is closed in this case. Moreover, $\mathcal{R}$ is a $C^{1,\alpha}$ Riemannian manifold — a C^∞ Riemannian manifold if in addition to (2.1) and (2.4), we assume that M_i^n is Einstein for all i. For above mentioned results, see Section 3 of [ChC2].

If the manifolds, M_i^n, are Kähler, and (0.11), (2.1) holds, then the tangent cones, $T_x M^n_\infty$, have natural complex structures on their smooth parts. If in addition to (2.1), (2.4) M_i is a convergent sequence of Kähler Einstein metrics on a fixed complex manifold then $\mathcal{H}^{n-4}(\mathcal{S}) < \infty$. Moreover, there exists $A \subset \mathcal{S}_{n-4} \setminus \mathcal{S}_{n-5}$, such that $\mathcal{H}^{n-4}(\mathcal{S}_{n-4} \setminus A) = 0$ and at points of A, the tangent cone is unique and isometric to $\mathbf{R}^{n-4} \times C(S^4/\Gamma_x)$. Where $S^3 \setminus \Gamma_x$ is a 3-dimensional space form and the order of Γ_x is bounded by a constant, $c(n, \mathcal{H}(B_1(x))$. Moreover, the isometry, between $T_x M^n_\infty$ and $\mathbf{R}^{n-4} \times C(\mathbf{S}^3/\Gamma_x) = \mathbf{C}^{k-2} \times \mathbf{C}/\Gamma$ is complex. See [ChCT1], [ChCT2] for these results.

When (2.1) does not hold it is useful to construct renormalized limit measures, ν, on limit spaces. These measures were first constructed by Fukaya. In the noncollapsed case, the limit measure exists without the necessity of passing to a subsequence, or of renormalizing the measure. The unique limit measure is just Hausdorff measure, $\mathcal{H}^n$; see Theorem 5.9 of [ChC2]. (If, for the sake of consistency, one does renormalize the measure, then one obtains a multiple of $\mathcal{H}^n$, where as usual, the normalization factor depends on the choice of base point.) However, in the collapsed case, the renormalized limit measure on the limit space can depend on the particular choice of subsequence; see Example 1.24 of [ChC2]. The renormalized limit measures play an important role in [ChC3], [ChC4], for instance in connection with the theory of the Laplace operator on limit spaces.

Let M^n satisfy $\mathrm{Ric}_{M^n} \geq (n-1)\,\Lambda$, and define the renormalized volume function, by

$$\underline{V}(x,r) := \underline{\mathrm{Vol}}(B_r(x)) := \frac{1}{\mathrm{Vol}(B_1(p))}\mathrm{Vol}(B_r(x))\,. \tag{2.5}$$

By combining the proof of Gromov's compactness theorem with the proof of the theorem of Arzela-Ascoli, we obtain:

Given any sequence of pointed manifolds, (M_i^n, m_i), for which (0.11) holds, there is a subsequence,(M_j^n, m_j), convergent to some (M_∞, m_∞) in the pointed Gromov-Hausdorff sense, and a continuous function $\underline{V}_\infty : M_\infty \times \mathbf{R}_+ \to \mathbf{R}_+$, such that if $y_j \in M_j^n, z \in M_\infty$ and $y_j \to z$, then for all $R > 0$,

$$\underline{V}_j(y_j, R) \to \underline{V}_\infty(z, R) \qquad \text{(uniformly on } B_{R_1}(p) \times [0, R_2]) . \tag{2.6}$$

One can then show that there is a unique Radon measure, ν, on M_∞ such that for all, z, R,

$$\nu(B_R(z)) = \underline{V}_\infty(z, R) . \tag{2.7}$$

In particular ν satisfies the inequality (0.4) with ν in place of Vol.

By a result in Section 2 of [ChC2] $\nu(\mathcal{S}) = 0$ also in the collapsed case. Even though the measure ν depend on the particular subsequence it is shown in [ChC3] that the collection, $\{\nu\}$, of all renormalized limit measures determines a well defined measure class i.e. ν_a is absolutely continuous with respect to ν_b, for all ν_a, ν_b.

We will conclude this section with some well known important open problems.

CONJECTURE 2.8. The interior of $M_\infty^n \setminus \mathcal{S}_{n-4}$ is a topological manifold.

CONJECTURE 2.9. Suppose that (M_i^n, x_i) are Einstein, (2.1) and (2.4) hold then $\mathcal{S} = \mathcal{S}_{n-4}$ and there exist some $C < \infty$ such that for all $x \in M_\infty$

$$\mathcal{H}^{n-4}(B_1(x) \cap \mathcal{S}) \leq C . \tag{2.10}$$

Moreover outside a subset $A \subset \mathcal{S}$ with $\mathcal{H}^{n-4}(A) = 0$ the tangent cone is unique and isometric to some orbifold.

3. ANALYSIS ON MANIFOLDS

Analysis on manifolds with nonnegative (Ricci) curvature is generally believed to resemble that on Euclidean space. For instance, in the 1970's S.T. Yau showed that the classical Liouville theorem generalized to this setting; cf. the gradient estimate of Cheng and Yau. Yau conjectured that in fact the spaces of harmonic functions of polynomial growth were finite dimensional on these spaces. It is a classical fact that $\mathcal{H}_d(\mathbf{R}^n)$ consists of harmonic polynomials of degree at most d. That is they are spanned by the spherical harmonics where the eigenvalue on $\mathbf{S}^{n-1}$ is given in terms of d.

Recall that for an open manifold M, $d > 0$, and $x \in M$ a function u is in $\mathcal{H}_d(M)$, the space of harmonic functions of polynomial growth of degree at most d, if $\Delta u = 0$ and there exists $C < \infty$ such that

$$|u| \leq C\,(1 + r^d) . \tag{3.1}$$

We note that there have been numerous interesting results in this area over the years, including work of Li and Tam, Donnelly and Fefferman, and others (see [CM1], [CM4] for detailed references).

This conjecture of Yau was settled affirmatively in [CM5]. We also obtained sharper (polynomial) estimates for the dimension and finite dimensionality in more general settings, including for certain harmonic sections of bundles and stationary varifolds.

In the case of Euclidean volume growth we obtained in [CM2] an asymptotic description of harmonic functions with polynomial growth. Namely, recall that asymptotically $u \in \mathcal{H}_d(\mathbf{R}^n)$ behaves like its highest order homogeneous part; i.e.

u is asymptotically a homogeneous separation of variables solution. We showed in [CM2] that this asymptotic description remain true in the Euclidean volume growth case. In [CM7] we showed that the Euclidean volume growth case studied in [CM2] was indeed the worst case.

In [ChCM], we showed that if a manifold with nonnegative Ricci curvature has $\dim \mathcal{H}_1(M) \geq k+1$ then any tangent cone at infinity, M_∞, splits isometrically as $\bar{M}_\infty \times \mathbf{R}^k$, where $\mathbf{R}^k$ has the standard flat metric. Combining this with the volume convergence of [C3] yields a rigidity theorem. Namely, if M^n is complete, $\mathrm{Ric}_M \geq 0$, and $\dim \mathcal{H}_1(M) = n+1$, then M^n is isometric to $\mathbf{R}^n$.

In [CM7], we gave polynomial bounds for the dimension of the space of polynomial growth L-harmonic functions, where L is a second order divergence form uniformly elliptic on a manifold with the doubling property and a scale invariant lower bound on some Neumann eigenvalue (not necessarily the first).

In the case of nonnegative Ricci curvature, we obtained the following "Weyl type" bounds.

THEOREM 3.2. [CM7]. Let $\mathrm{Ric}_M \geq 0$, and $d \geq 1$. There exist a constant C and $o(d^{n-1})$ depending on n such that

$$\dim \mathcal{H}_d(M^n) \leq C\, V_M\, d^{n-1} + o(d^{n-1})\,, \tag{3.3}$$

where $\lim_{d\to\infty} d^{1-n}\, o(d^{n-1}) = 0$.

This theorem yields immediately a Siegel type theorem for the field of rational functions on a Kähler manifold with nonnegative Ricci curvature. By looking at a cone over a compact manifold, Theorem 3.2 gives "Weyl type" eigenvalue estimates for compact manifolds; see [CM7].

In [CM6], we showed that the finite dimensionality continues to hold in other related settings; e.g., given a mean value inequality and the doubling property. As applications, in [CM6] we got finite dimensionality results for spaces of harmonic sections of bundles with nonnegative curvature and on stationary varifolds with Euclidean volume growth; see also [CM4]. The methods are flexible and give many related, but slightly different, theorems; the two following results are representative.

THEOREM 3.4. [CM6]. Let $\mathrm{Ric}_M \geq 0$, and E^k a rank k Hermitian vector bundle over M with nonnegative curvature. For all $d \geq 1$,

$$\dim \mathcal{H}_d(M^n, E) \leq C\, k\, d^{n-1}\,, \tag{3.5}$$

where $C = C(n) < \infty$.

The final result that we will mention is a weak Bernstein type theorem for stationary varifolds in Euclidean space. This is obtained from a bound on the space of harmonic functions of a given degree of growth since the restrictions of the coordinate functions are harmonic. Recall that stationary varifolds satisfies a mean value inequality.

THEOREM 3.6. [CM6]. Let $\Sigma^n \subset \mathbf{R}^N$ be a stationary n-rectifiable varifold with density at least one almost everywhere on its support and bounded from above by V_Σ, then Σ must be contained in some affine subspace of dimension at most $C = C(n)$.

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Tobias H. Colding
Courant Institute
of Mathematical Sciences
251 Mercer Street
New York, NY 10012
USA