Operads and Algebraic Geometry

M. Kapranov

Abstract. The study (motivated by mathematical physics) of algebraic varieties related to the moduli spaces of curves, helped to uncover important connections with the abstract algebraic theory of operads. This interaction led to new developments in both theories, and the purpose of the talk is to discuss some of them.

1991 Mathematics Subject Classification: 14H10, 18C15, 08C9
Keywords and Phrases: operads, superpositions, moduli spaces.

1. Operads.

The modern concept of an operad originated in topology [25] but the underlying ideas can be traced back at least to Hilbert's 13th problem which (in the way it came to be understood later) can be stated as follows.

(1.1) Let P be some class of functions considered in analysis (e.g., continuous, smooth, algebraic etc.). Is it possible to express any function $f : \mathbf{R}^n \to \mathbf{R}$ of class P, depending of $n \geq 3$ variables, as a superposition of functions of class P, depending on 1 or 2 variables only?

This question involves the *superposition operation* on functions of several variables: we can substitute some n functions $g_1(x_{1,1}, ..., x_{1,a_1})$, ..., $g_n(x_{n,1}, ..., x_{n,a_n})$ for arguments of another function $f(x_1, ..., x_n)$, thereby getting a new function $f(g_1, ..., g_n)$ depending on $a_1 + ... + a_n$ variables. (Here the x_{ij}, as well as the values of the functions, belong to some fixed set X.) This generalizes the observation that functions of one variable can be composed: $(f(x), g(x)) \mapsto f(g(x))$.

For a set X, maps $X^n \to X$ are also called n-ary operations on X. A collection $\mathcal{P}$ of such maps (with possibly varying n) is called an *operad*, if it is closed under arbitrary superpositions as well as under permutations of variables. Thus $\mathcal{P}(n)$, the n-ary part of $\mathcal{P}$, is acted upon by the symmetric group S_n. One can view $\mathcal{P}$ as a multivariable analog of a (semi)group of transformations of X.

As with groups, the actual working definition [25] splits the above naive one into two. First, one defines an abstract operad $\mathcal{P}$ as a collection of sets $\mathcal{P}(n), n \geq 0$ with S_n acting on $\mathcal{P}(n)$, equipped with an element $1 \in \mathcal{P}(1)$ (the unit) and maps

$$\mathcal{P}(n) \times \mathcal{P}(a_1) \times ... \times \mathcal{P}(a_1) \to \mathcal{P}(a_1, ..., a_n) \tag{1.2}$$

satisfying the natural associativity and equivariance conditions, as well as the conditions for the unit, see [25]. Then, one defines a $\mathcal{P}$-algebra as a set X together with S_n-equivariant maps $\mathcal{P}(n) \to \mathrm{Hom}(X^n, X)$ which take (1.2) into the actual superpositions of operations.

It was soon noticed that one can replace sets in the above definitions with objects of any symmetric monoidal category $(\mathcal{C}, \otimes)$, by using $\otimes$ instead of products of sets. The categories $\mathcal{C}$ used in practice include two groups of examples:

(1.3) $\mathcal{T}op$ (topological spaces), $\mathcal{V}ar$ (algebraic varieties, say over $\mathbf{C}$), $\mathcal{S}t$ (algebraic stacks), with $\otimes$ being the Cartesian product.

(1.4) $\mathcal{V}ect_k$ (vector spaces over $\mathbf{C}$), $dg\mathcal{V}ect_k$ (dg-vector spaces, i.e., bounded cochain complexes of finite-dimensional spaces), $g\mathcal{V}ect_k$ (graded spaces, i.e., complexes with zero diferential), with $\otimes$ being the tensor product. Operads in these categories are called, respectively, linear, dg- or graded operads.

If $\mathcal{P}$ is an operad in $\mathcal{T}op$ or $\mathcal{V}ar$, then the topological homology spaces $H_\bullet(\mathcal{P}(n), \mathbf{C})$ form a graded operad denoted $H_\bullet(\mathcal{P})$. Similarly, the chain complexes of the $\mathcal{P}(n)$ form a dg-operad.

2. Role of operads in algebraic geometry.

The source for the recent interest in operads in algebraic geometry is the synthesis of several different approaches, which we recall.

A. Abstract-algebraic approach. Many familiar algebraic structures can be described as algebras over appropriate operads. This use of operads is the traditional approach of "universal algebra".

Example 2.1. Let $\mathcal{G}r(n)$ be the Weyl group of the root system B_n, i.e., the semidirect product of S_n and $\{\pm 1\}^n$; for $n = 0$ set $\mathcal{G}r(0) := \{\mathrm{pt}\}$. The collection of the $\mathcal{G}r(n)$ can be made into an operad $\mathcal{G}r$ (in the category of sets) so that any group G is naturally a $\mathcal{G}r$-algebra. The maps $G^n \to G$ corresponding to elements of $\mathcal{G}r(n)$, are of the form $(x_1, ..., x_n) \mapsto x_{\sigma(1)}^{\epsilon_1} ... x_{\sigma(n)}^{\epsilon_n}$, $\sigma \in S_n$, $\epsilon_i \in \{\pm 1\}$. More generally, an arbitrary $\mathcal{G}r$-algebra is the same as a semigroup with involution $*$ satisfying $(ab)^* = b^* a^*$.

Example 2.2. We have linear operads $\mathcal{A}s, \mathcal{C}om, \mathcal{L}ie$ whose algebras (in the category $\mathcal{V}ect$) are respectively, associative, commutative or Lie algebras. Explicitly, $\mathcal{L}ie(n)$ is the subspace in the free Lie algebra on $x_1, ..., x_n$ formed by elements multihomogeneous of degrees (1, ..., 1), and similarly for the other classes of algebras, see [11]. For example, each $\mathcal{C}om(n) \simeq \mathbf{C}$ while $\dim(\mathcal{A}s(n)) = n!$.

B. Algebro-geometric examples. These examples are elaborations on the idea of gluing Riemann surfaces, present in the string theory for a long time [30]. But the operadic approach to this idea is surprisingly useful.

Example 2.3. Let $\overline{M}_{g,n}$ be the moduli stack of stable n-pointed curves of genus g, see [2]. Set $\overline{\mathcal{M}}_0(n) = \overline{M}_{0,n+1}$ and $\overline{\mathcal{M}}(n) = \coprod_g \overline{M}_{g,n+1}$. The $\overline{\mathcal{M}}_0(n)$, $n \geq 2$ are in fact algebraic varieties. The $n+1$ marked points on $C \in \overline{\mathcal{M}}(n)$ are denoted by $(x_0, ..., x_n)$ and the group S_n acts by permutations of $x_1, ..., x_n$. The collection of $\overline{\mathcal{M}}(n), n \geq 2$, is naturally made into an operad $\overline{\mathcal{M}}$ in $\mathcal{S}t$ while $\overline{\mathcal{M}}_0 = \{\overline{\mathcal{M}}_0(n)\}$ forms an operad in $\mathcal{V}ar$. The maps (1.2) take $(C, D_1, ..., D_n)$ into the reducible curve obtained by identifying the 0th point of D_i with the ith point of C.

EXAMPLE 2.4. Denote by $\widetilde{M}_{g,n}$ be the set of isomorphism classes of Riemann surfaces of genus g with boundary consisting of n circles, together with a smooth identification of each boundary component with S^1. It is naturally an infinite-dimensional topological space. As before we set $\widetilde{\mathcal{M}}_0(n) = \widetilde{M}_{0,n+1}$, $\widetilde{\mathcal{M}}(n) = \coprod_g \widetilde{M}_{g,n+1}$. Gluing Riemann surfaces together along the boundary makes $\widetilde{M}$ and $\widetilde{M}_0$ into operads in the category of topological spaces.

Note that the above two examples are in some sense, dual to each other, as $\widetilde{M}_{g,n+1}$ projects naturally to the open stratum (the locus of smooth curves) $M_{g,n+1} \subset \overline{M}_{g,n+1}$. When $g=0$, the complement to $M_{0,n+1}$ is precisely formed by the images of the maps (1.2).

The above does not of course exhaust all types of operads of algebro-geometric nature. A quite different class of examples was developed in [20].

C. RELATIONS OF A AND B. By taking homology of operads from B, we get graded operads which, remarkably, are related to the operads from A.

Each $\overline{\mathcal{M}}_0(n)$ is a smooth irreducible projective variety of dimension $n-2$. Let $q_n \in H_{2(n-2)}(\overline{\mathcal{M}}_0(n), \mathbf{C})$ be the fundamental class. It is clear that q_2 generates the suboperad $H_0(\overline{\mathcal{M}}_0) \subset H_\bullet(\overline{\mathcal{M}}_0)$ isomorphic to $\mathcal{C}om$. Thus an $H_\bullet(\overline{\mathcal{M}}_0)$-algebra is a commutative algebra with extra structure, and this extra structure was described explicitly by M. Kontsevich and Y.I. Manin [19].

THEOREM 2.5. *An $H_\bullet(\overline{\mathcal{M}}_0)$-algebra is the same as a graded vector space A with multilinear totally symmetric (in the graded sense) operations $(x_1, ..., x_n)$, $n \geq 2$ of degree $2(n-2)$ satisfying the generalized associativity conditions:*

$$(2.6) \qquad \sum_{[n]=S_1 \coprod S_2} \pm((a,b,x_{S_1}),c,x_{S_2}) \quad = \quad \sum_{[n]=S_1 \coprod S_2} \pm(a,(b,c,x_{S_1}),x_{S_2}),$$

where $x_S, S \subset [n]$ means the unordered set of $x_i, i \in S$ and $\pm$ is given by the Koszul sign rules. In particular, (x_1, x_2) is a commutative associative multiplication.

The condition (2.6) is the well known WDVV relation. The theory of Gromov-Witten invariants such as it was developed in [1,2, 19, 24] gives the following fact.

THEOREM 2.7. *For any smooth projective variety V the homology space $H_\bullet(V, \mathbf{C})$ has a natural structure of an algebra over the operad $H_\bullet(\overline{\mathcal{M}})$, in particular, it has the structure specified in Theorem 2.5.*

By contrast, the homology of the open moduli spaces is related to the operad $\mathcal{L}ie$ and its generalizations. First, an old result of F. Cohen [3] implies that $H_{\max} M_{0,n+1} \simeq \mathcal{L}ie(n) \otimes \mathrm{sgn}_n$, as an S_n-module. More generally, E. Getzler [8] defined an operad structure on the collection of $\mathcal{G}(n) = H_\bullet(M_{0,n+1}, \mathbf{C})[2-n] \otimes \mathrm{sgn}_n$ by using the Poincaré residue maps and proved the following.

THOREM 2.8. *A $\mathcal{G}$-algebra is the same as a graded vector space A together with totally antisymmetric products $[x_1, ..., x_n]$ of degree $2-n$, $n \geq 2$, satisfying the generalized Jacobi identities:*

$$\sum_{i \leq i < j \leq k} \pm\big[[a_i, a_j], a_1, ..., \widehat{a}_i, ..., \widehat{a}_j, ..., a_k, b_1, ..., b_l\big]$$

$$= \begin{cases} \big[[a_1, ..., a_k], b_1, ...b_l\big], & \textit{if } l > 0; \\ 0, & l = 0. \end{cases}$$

In particular, $(A, [x_1, x_2])$ *is a graded Lie algebra.*

D. Operads and trees. Maps (1.2) represent a single instance of superposition of elements of an operad (e.g., functions in several variables). By applying them several times, we get *iterated superpositions*, which are best described by their "flow charts", similar to those used by computer programmers.

More precisely, call an n-tree a tree T with $n+1$ external edges which are divided into n "inputs" and one output and such that the inputs are numbered by $1, ..., n$. Such T has orientation according to the flow from the inputs to the output; in particular, the edges adjacent to any vertex v, are separated into two subsets $\mathrm{In}(v)$ and $\mathrm{Out}(v)$, the latter consisting of one element. Given an operad $\mathcal{P}$, the S_m-action on $\mathcal{P}(m)$ allows us to speak about sets $\mathcal{P}(I)$, where I is any m-element set (any identification $I \to [m]$ identifies $\mathcal{P}(I)$ with $\mathcal{P}(m)$). Now, a flow chart with n inputs for $\mathcal{P}$ is an n-tree T together with assignment, for any vertex v of T, of an element $q_v \in \mathcal{P}(\mathrm{In}(v))$. This data produces an iterated superposition of the q_v, belonging to $\mathcal{P}(n)$.

So the combinatorics of trees is closely connected to all questions related to operads and superpositions. It is worth pointing out that the first paper of Kolmogoroff [16] on the Hilbert superposition problem for continuous functions used trees in an essential way.

On the other hand, J. Harer and R. Penner [12, 26] constructed a cell decomposition of the moduli space of curves in which cells are parametrized by graphs (with some extra structure). This led M. Kontsevich [17] to introduce certain purely combinatorial chain complexes formed of summands labelled by graphs, which turned out to be very important in such diverse questions as quasiclassical approximation to the Chern-Simons invariant of 3-manifolds and cohomology of infinite-dimensional Lie algebras.

An observation of V. Ginzburg and the author [11] was that the tree parts of Kontsevich's graph complexes can be very easily interpreted and generalized in the language of operads.

More precisely, if $\mathcal{A} = \mathcal{A}(n), n \geq 2$ is any collection of dg-vector spaces with S_n-actions, the *free operad* $F_{\mathcal{A}}$ generated by $\mathcal{A}$ consists of all possible formal iterated superpositions of elements of $\mathcal{A}$, i.e.,

$$F_{\mathcal{A}}(n) = \bigoplus_{n-\mathrm{trees}\ T} \ \bigotimes_{v \in \mathrm{Vert}(T)} \mathcal{A}(\mathrm{In}(v)).$$

(A similar definition can also be given for operads in (1.3) if we replace $\bigoplus$ with $\coprod$ and $\bigotimes$ with the Cartesian product.)

For a collection $\mathcal{A}$ as above its suspension $\Sigma\mathcal{A}$ constists of shifted complexes with twisted S_n-action: $(\Sigma\mathcal{A})(n) = \mathcal{A}(n)[n-1] \otimes \mathrm{sgn}_n$. (Meaning: if $\mathcal{A}$ is in fact an operad and A is an $\mathcal{A}$-algebra, then the $A[1]$ is a $\Sigma\mathcal{A}$-algebra.)

Theorem 2.9. *(a) Let* $\mathcal{P}$ *be a dg-operad with* $\mathcal{P}(0) = 0$, $\mathcal{P}(1) = \mathbf{C}$ *and* $\mathcal{P}^*$ *be the collection of the dual dg-spaces* $\mathcal{P}(n)^*, n \geq 2$. *Then the components of the free*

operad $F_{\Sigma\mathcal{P}^*}$ admit natural differentials with respect to which they form a new dg-operad $\mathbf{D}(\mathcal{P})$ called the cobar-dual to $\mathcal{P}$. We have a canonical quasiisomorphism $\mathbf{D}(\mathbf{D}(\mathcal{P})) \to \mathcal{P}$.
(b) We have quasi-isomorphisms $\mathbf{D}(\mathcal{A}s) \simeq \mathcal{A}s$, $\mathbf{D}(\mathcal{C}om) \simeq \mathcal{L}ie$, $\mathbf{D}(\mathcal{L}ie) \simeq \mathcal{C}om$.

3. Koszul duality

The functor $\mathbf{D}$ from Theorem 2.9 can be viewed as a kind of cohomology theory on the category of operads. In particular, when $\mathcal{P}$ is just a linear operad (has trivial dg-structure), $H^\bullet(\mathbf{D}(\mathcal{P}(n))$ provides information about generators, relations and higher syzygies of $\mathcal{P}$. There is a class of operads for which $\mathbf{D}(\mathcal{P})$ is especially simple.

In [11], a linear operad $\mathcal{P}$ with $\mathcal{P}(0) = 0, \mathcal{P}(1) = \mathbf{C}$, was called *quadratic*, if the following conditions hold:

(3.1) $\mathcal{P}$ is generated by the binary part, i.e., every element of every $\mathcal{P}(n)$ is a sum of iterated superpositions of elements of $\mathcal{P}(2)$, so that the morphism $\phi_\mathcal{P} : F_{\mathcal{P}(2)} \to \mathcal{P}$ is surjective.

(3.2) All the relations among the binary generators follow from those holding in the ternary part, i.e., the "ideal" $\mathrm{Ker}(\phi_\mathcal{P})$ is generated by $\mathrm{Ker}(\phi_\mathcal{P}(3) : F_{\mathcal{P}(2)}(3) \to \mathcal{P}(3))$.

Thus a quadratic operad $\mathcal{P}$ can be described by giving a vector space $V = \mathcal{P}(2)$ of generators, equipped with S_2-action and an S_3-invariant subspace of relations $R \subset \mathcal{F}_V(3)$. We will write $\mathcal{P} = Q(V, R)$. The *Koszul dual* operad $\mathcal{P}^!$ is defined as $\mathcal{P}^! = Q(V^* \otimes \mathrm{sgn}, R^\perp)$. This is a natural analog of Koszul duality for algebras as defined by Priddy [27].

Theorem 3.3. *The operads $\mathcal{A}s$, $\mathcal{C}om$, $\mathcal{L}ie$ are quadratic, and their Koszul duals are: $\mathcal{A}s^! = \mathcal{A}s$, $\mathcal{C}om^! = \mathcal{L}ie$, $\mathcal{L}ie^! = \mathcal{C}om$.*

The duality between commutative and Lie algebras, as a meta-mathematical principle, goes back at least to the work of D. Quillen on rational homotopy theory. Later, V. Drinfeld suggested to look for some tangible reasons behind this principle. The explanation provided by Theorem 3.3 is so far the most elementary: it exhibits the sought-for "reason" as the fact that certain given subspaces of given dual vector spaces are orthogonal complements of each other.

For any quadratic operad $\mathcal{P}$ there is a natural morphism of dg-operads $\mathbf{D}(\mathcal{P}) \to \mathcal{P}^!$, and $\mathcal{P}$ is called Koszul if this is a quasiisomorphism. Thus, Theorem 2.9(b) implies that the operads $\mathcal{A}s$, $\mathcal{C}om$ and $\mathcal{L}ie$ are Koszul. Similarly to Koszul quadratic algebras of Priddy [27], Koszul operads possess many nice properties allowing one to calculate the homological invariants in an elementary way.

A generalization of the theory of quadratic and Koszul operads to the case when $\mathcal{P}$ is not necessarily generated by $\mathcal{P}(2)$, was developed by E. Getzler [8]. In this case, the meaning of "quadratic" is that all the relations follow from those involving only the simplest instances of superposition of the generators. It was proved in [8] that in this more general sense, the operads $H_\bullet(\overline{\mathcal{M}}_0)$ and $\mathcal{G}$ from Theorems 2.5 and 2.8 are quadratic, Koszul and dual to each other.

4. Cyclic and modular operads.

The algebro-geometric examples of operads from §2B in fact possess more than just an operad structure. First, the division of the $n+1$ marked points (or boundary components) inth n inputs and one output is artificial procedure. So even though, say, $\overline{\mathcal{M}}_0(n)$ is the n-ary part of an operad, we have an action of $S_{n+1} = \mathrm{Aut}\{0, ..., n\}$ on it. Motivated by this, E. Getzler and the author [9] called a cyclic operad an operad $\mathcal{P}$ together with S_{n+1}-action on each $\mathcal{P}_n$ which is compatible with compositions in the following sense. Denote by τ_n the cycle $(0, 1, ..., n) \in S_{n+1}$. Then it is required that $\tau_1(1) = 1$, and that

$$\tau_{m+n-1}(p(1, ..., 1, q)) = (\tau_n q)(\tau_m(p), 1, ..., 1), \quad p \in \mathcal{P}(m), q \in \mathcal{P}(n).$$

So all the examples from $\S 2B$ are cyclic operads in this sense.

It is not obvious that this concept should have any meaning from the point of view of §2A, but it does. Let $\mathcal{P}$ be a linear (or dg-) operad. It was shown in [9] (generalizing some observations of M. Kontsevich), that a cyclic structure on $\mathcal{P}$ is precisely the data necessary to meaningfully speak about *invariant scalar products* on $\mathcal{P}$-algebras. For example, a scalar product B on a Lie algebra is called invariant if $B([x, y], z) = B(x, [y, z])$, and similarly for the other types of algebras from Example 2.2. This indicates (and this is indeed the case) that these operads are cyclic. The operads from Theorems 2.5 and 2.8 are cyclic too. For a cyclic operad $\mathcal{P}$ it is notationally convenient to denote the S_n-module $\mathcal{P}(n-1)$ by $\mathcal{P}((n))$, thereby emphasizing the symmetry between the inputs and the output. We will also call a cyclic $\mathcal{P}$-algebra a pair (A, g) consisting of an algebra and an invariant scalar product. Theorems 2.5 and 2.7 have an even nicer formulation in terms of cyclic operads (see [24] for the detailed proof of (a)).

Theorem 4.1. *(a) A finite dimensional (graded) cyclic algebra over $H_\bullet(\overline{\mathcal{M}}_0)$ is the same as a formal germ of a potential Frobenius (super-)manifold in the sense of [5, 24].*
(b) For any smooth projective variety V the intersection pairing g on $H_\bullet(V, \mathbf{C})$ makes it into a cyclic $H_\bullet(\overline{\mathcal{M}})$-algebra.

If, in §2B, we consider moduli spaces of curves of arbitrary genus, then there is still another structure present: two marked points (or boundary components) *of the same curve* can be glued together, producing a curve of genus higher by 1 and number of marked points (or boundary components) less by 2. This structure was axiomatized in [10] under the name "modular operad".

Explicitly, a modular operad $\mathcal{P}$ (in a monoidal category $\mathcal{C}$) is a collection of objects $\mathcal{P}((g, n))$ given for $n, g \geq 0$ such that $2g - 2 + n > 0$ (the number g is called genus), with S_n acting on $\mathcal{P}((g, n))$ and the following data:

(4.2) A structure of a cyclic operad on the collection of $\mathcal{P}((n)) = \coprod_g \mathcal{P}((g, n))$ so that the genus of any superposition is equal to the sum of the genera of the elements involved. (For categories from (1.4) we should understand $\coprod$ as the direct sum). We can speak of $\mathcal{P}((g, I))$ for $|I| = n$, via the S_n-action.

(4.3) The contraction maps $\xi_{i,j} : \mathcal{P}(\!(g, I)\!) \to \mathcal{P}(\!(g+1, I-\{i,j\})\!)$ given for any finite set I, $i \neq j \in I$, satisfying natural equivariance and coherence conditions ([10], §3).

A "flow chart" for a modular operad $\mathcal{P}$ is given by a *stable n-graph*, that it, a connected graph G with some number n of external legs (edges not terminating in a vertex) which are numbered by $1, ..., n$, plus an assignment, to any vertex v, of a number $g(v) \geq 0$ so that $2g(v) - 2 + n(v) > 0$, where $n(v)$ is the valence of v. The (total) genus of a stable n-graph G is defined as $\sum_v g(v)$ plus the first Betti number of G. Let $\Gamma(\!(g,n)\!)$ be the set of isomorphism classes of stable n-graphs of genus g. For $G \in \Gamma(\!(g,n)\!)$ set $\mathcal{P}(\!(G)\!) = \bigotimes_{v \in \mathrm{Vert}(G)} \mathcal{P}(g(v), \mathrm{Ed}(v))$, where $\mathrm{Ed}(v)$ is the set of (half-)edges issuing from v. Then a modular operad structure on $\mathcal{P}$ gives the superposition map $\mathcal{P}(\!(G)\!) \to \mathcal{P}(\!(g,n)\!)$.

In the dg-framework there is a related concept of a *twisted modular operad* where we require superposition maps of the form $\mathcal{P}(\!(G)\!) \otimes \mathrm{Det}(\mathbf{C}^{\mathrm{Ed}(G)}) \to \mathcal{P}(\!(g,n)\!)$, where $\mathrm{Ed}(G)$ is the set of all edges of G and $\mathrm{Det}(V) = \Lambda^{\dim(V)}(V)[\dim(V)]$. As was shown in [10], the following generalization of the cobar-duality to modular operads encompasses Kontsevich's graph complexes in full generality.

THEOREM 4.4. *For a modular dg-operad $\mathcal{P}$ the collection of*

$$F(\mathcal{P})(\!(g,n)\!) = \bigoplus_{G \in \Gamma(\!(g,n)\!)} \left(\mathrm{Det}(\mathbf{C}^{\mathrm{Ed}(G)}) \otimes \bigotimes_{v \in \mathrm{Vert}(G)} \mathcal{P}(\!(g,v)\!)^* \right)_{\mathrm{Aut}(G)}$$

has natural differentials and composition maps which make it into a twisted modular dg-operad $F(\mathcal{P})$ called the Feynman transform of $\mathcal{P}$.
(b) The functor $\mathcal{P}$ takes quasiisomorphisms to quasiisomorphisms and gives an equivalence between the derived categories of modular dg-operads and twisted modular dg-operads.

The inverse to F is constructed similarly to F but with a different determinantal twist.

EXAMPLE 4.5. Let $\mathcal{P} = \mathcal{A}s$ is the associative operad considered as a modular operad, i.e., $\mathcal{A}s(\!(g,n)\!) = 0, g > 0$, $\mathcal{A}s(\!(0,n)\!) = \mathcal{A}s(n-1)$. Then

$$(F\mathcal{A}s)(\chi, 0) = \bigoplus_{2g-2+n=\chi} C_\bullet(|M_{g,n}|/S_n, \mathbf{C}),$$

where $|M_{g,n}|$ is the coarse moduli space of smooth curves of genus g with n punctures, and $C_\bullet$ is the chain complex with respect to Penner's cell decomposition labelled by "fat graphs", i.e., graphs with a cyclic order on each $\mathrm{Ed}(v)$. The reason is that $\mathcal{A}s(\!(0,n)\!) = \mathcal{A}s(n-1)$, as an S_n-module, can be identified with the vector space spanned by all cyclic orders on $\{1, ..., n\}$.

One of the main results of [10] is the determination of the Euler characteristics of the $F(\mathcal{P})(\!(g,n)\!)$ (as elements of the representation ring of S_n) in terms of those of $\mathcal{P}(\!(g,n)\!)$. The set of $\chi(\mathcal{P}(\!(g,n)\!))$ is encoded into a formal power series $C_\mathcal{P}(h, p_1, p_2, ...)$ of infinitely many variables, and $C_{F(\mathcal{P})}$ is identified with a certain formal Fourier transform of $C_\mathcal{P}$ with respect to a Gaussian measure on $\mathbf{R}^\infty$. In the case when $\mathcal{P} = \mathcal{A}s$, the infinite-dimensional integral in the Fourier transform can be calculated explicitly by separating the variables, and we have the following theorem.

Theorem 4.4. *The series*

$$\Psi(h) = \sum_{\chi=1}^{\infty} h^{\chi} \sum_{2g-2+n=\chi} e(|M_{g,n}|/S_n),$$

where e is the topological Euler characteristic, is calculated as follows:

$$\Psi(h) = \sum_{n,l=1}^{\infty} \frac{\mu(l)}{l} \Psi_n(h^l), \quad \text{where}$$

$$\Psi_n(h) = \sum_{k=1}^{\infty} \frac{\zeta(-k)}{-k} \alpha_n^{-k} + (\alpha_n + \frac{1}{2}) \ln(n h^n \alpha_n) - \alpha_n + \frac{1}{n h^n} - c(n)/2n,$$

$$\alpha_n = \alpha_n(h) = \frac{1}{n} \sum_{d|n} \frac{\phi(d)}{h^{n/d}}, \quad c_n = \frac{1}{2}(1 + (-1)^n)$$

and ϕ, μ, ζ are respectively, the Euler, Möbius and Riemann zeta functions.

Recall [13] that the *orbifold* Euler characteristic of $M_{g,1}$ is equal to the rational number $\zeta(1-2g)$.

5. Operads and curvature invariants.

The operadic point of view turned out to be useful even in such "classical" parts of geometry as the theory of characteristic classes. Let M be a complex manifold and $T = TM$ its tangent bundle. Then the $\mathcal{E}_M(n) = \{\mathrm{Hom}(T^{\otimes n}, T)\}$ form an operad in the category of holomorphic vector bundles on M and hence $\mathbf{E}_M = \{\mathbf{E}_M(n) = H^\bullet(M, \mathcal{E}_M(n))\}$ (holomorphic cohomology) is a graded operad. On the other hand, the curvature of any Hermitian metric h on M defines a Dolbeault cohomology class $\alpha_M \in H^1(M, \mathrm{Hom}(T^{\otimes 2}, T)) = \mathbf{E}_M(2)^1$ (the Atiyah class). This class is symmetric with respect to the S_2-action on $\mathbf{E}_M(2)$ (when h is Kähler, even the curvature form is symmetric). Consider the desuspension $\Sigma^{-1}(\alpha_M)$ (see §2) which is an element of $\Sigma^{-1}(\mathbf{E}_M)(2)^0$ anti-symmetric with respect to S_2. The following fact, inspired by [29, 18], was proved in [14].

Theorem 5.1. *The element $\Sigma^{-1}(\alpha_M)$ satisfies the Jacobi identity in the operad $\Sigma^{-1}(\mathbf{E}_M)$, i.e., it defines a morphism of operads $\mathcal{L}ie \to \Sigma^{-1}(\mathbf{E}_M)$.*

One can say that algebraic geometry is based on the operad $\mathcal{C}om$, governing commutative associative algebras. It is a natural idea to develop some generalized geometries based on more general linear operads $\mathcal{P}$. For example, for $\mathcal{P} = \mathcal{A}s$, we get "noncommutative geometry" based on associative but not necessarily commutative algebras, and several important approaches to such geometry have been developed [4, 7, 23, 28]. The case of a general $\mathcal{P}$ presents of course, even more difficulties, but Theorem 5.1 suggests the following heuristic principle which is confirmed whenever $\mathcal{P}$-geometry can be given sense:

Pre-theorem 5.2. *Let $\mathcal{P}$ be a Koszul operad. Then, curvature invariants of a "space" in $\mathcal{P}$-geometry satisfy the constraints of the Koszul dual operad $\mathcal{P}^!$.*

Example 5.3: Formal $\mathcal{P}$-geometry. We can always speak about $D^n_{\mathcal{P}}$, the *formal n-disk* in $\mathcal{P}$-geometry, i.e., the object corresponding to the completion of the free $\mathcal{P}$-algebra on n generators, cf. [17]. Being infinitesimal, it does not by itself possess global curvature invariants. However, the question becomes interesting for group structures on $D^n_{\mathcal{P}}$, i.e., n-dimensional formal groups over $\mathcal{P}$. Such groups were studied by M. Lazard [21] who, in a work pre-dating the modern concept of an operad, developed an analog of Lie theory for them. A modern interpretation of his theory [6, 11] revealed that the analog of a Lie algebra for a formal group over $\mathcal{P}$ is in fact a $\mathcal{P}^!$-algebra.

Example 5.4: Semiformal $\mathcal{A}s$-geometry. In [15], the author developed a formalism of "noncommutative formal neighborhoods" (called NC-thickenings) of a smooth algebraic variety M. They are ringed spaces $X = (M, \mathcal{O}_X)$ where $\mathcal{O}_X$ is a sheaf of noncommutative rings with $\mathcal{O}_X/[\mathcal{O}_X, \mathcal{O}_X] = \mathcal{O}_M$ and such that its completion at any point is isomorphic to $\mathbf{C}\langle\langle x_1, ..., x_n\rangle\rangle$, the algebra of noncommutative formal power series.

Theorem 5.5. *Let M be a smooth algebraic variety. Then any NC-thickening X of M has a characteristic class $\alpha_X^- \in H^1(M, \Omega^2_M \otimes T)$. The sum $\alpha_X = \alpha_M + \alpha_X^- \in H^1(M, \mathrm{Hom}(T^{\otimes 2}, T))$ is such that $\Sigma^{-1}\alpha_X \in \Sigma^{-1}(\mathbf{E}_M)(2)^0$ is an associative element, i.e., gives rise to a morphism of operads $\mathcal{A}s \to \Sigma^{-1}\mathbf{E}_M$. Moreover, α_X is an A_∞-element in the sense of Stasheff [31].*

References

[1] K. Behrend, Gromov-Witten invariants in algebraic geometry, Invent. Math. 127 (1997), 607-617.

[2] K. Behrend, Y.I. Manin, Stacks of stable maps and Gromov-Witten invariants, Duke Math. J. 85 (1996), 1-60.

[3] F.R. Cohen, Cohomology of C_{n+1}-spaces, Lecture Notes in Math. 533, p. 207-351, Springer, 1977.

[4] A. Connes, Noncommutative Geometry, Academic Press, 1996.

[5] B. Dubrovin, Geometry of 2D topological field theories, Lecture Notes in Math. 1620, p. 120-348, Springer, 1996.

[6] B. Fresse, Lie theory of formal groups over an operad, preprint IRMA, Strasbourg 1996.

[7] I.M. Gelfand, V.S. Retakh, Quasideterminants I, Selecta Math. 3 (1997), 517-546.

[8] E. Getzler, Operads and moduli spaces of Riemann surfaces, in: "The moduli space of curves" (R. Dijkgraaf et al. Eds.), 199-230, Birkhäuser, Boston, 1995.

[9] E. Getzler, M. Kapranov, Cyclic operads and cyclic homology, in "Geometry, Topology and Physics for R. Bott" (S.T. Yau, Ed.) p.167-201, Intern. Press, Cambridge MA 1995.

[10] E. Getzler, M. Kapranov, Modular operads, Compositio Math. 110 (1998), 65-126.

[11] V. Ginzburg, M. Kapranov, Koszul duality for operads, Duke Math. J. 76 (1994), 203-272.
[12] J. Harer, The cohomology of the moduli space of curves, Lect. Notes in Math. 1337, p. 138-221, Springer, 1988.
[13] J. Harer, D. Zagier, The Euler characteristic of the moduli space of curves, Invent. Math. 85 (1986), 457-485.
[14] M. Kapranov, Rozansky-Witten invariants via Atiyah classes, preprint alg-geom/9704009, Compositio Math., to appear.
[15] M. Kapranov, Noncommutative geometry based on commutator expansions, preprint math.AG/9802041, Crelle's J., to appear.
[16] A.N. Kolmogoroff, On the representation of continuous functions of several variables as superpositions of functions of smaller number of variables, Soviet. Math. Dokl. 108 (1956), 179-182.
[17] M. Kontsevich, Formal noncommutative symplectic geometry, in "The Gelfand Mathematical Seminars" (L. Corwin et al. Eds.) p. 173-187, Birkhäuser, Boston, 1993.
[18] M. Kontsevich, Rozansky-Witten invariants via formal geometry, preprint dg-ga/9704009, Compositio Math. to appear.
[19] M. Kontsevich, Y.I. Manin, Gromov-Witten classes, quantum cohomology and enumerative geometry, Comm. Math. Phys. 164 (1994), 525-562.
[20] I. Kriz, J. P. May, Operads, Algebras, Modules and Motives, Asterisque, 233, 1985.
[21] M. Lazard, Lois de groupe et analyseurs, Ann. ENS, 72 (1955), 299-400.
[22] J.-L. Loday, La renaissance des opérades, Sém. Bourbaki, Exp. 792, 1994-5.
[23] Y.I. Manin, Topics in Noncommutative Geometry, Princeton Univ. Press, 1992.
[24] Y.I. Manin, Frobenius manifolds, quantum cohomology and moduli spaces, preprint math.QA/9801006.
[25] J.P. May, Geometry of Iterated Moduli Spaces, Lecture Notes in Math. 271, Springer, 1972.
[26] R. Penner, Perturbative series and the moduli space of punctured surfaces, J. Diff. Geom. 27 (1988), 35-53.
[27] S. Priddy, Koszul resolutions, Trans. AMS, 152 (1970), 39-60.
[28] A. Rosenberg, Noncommutative Algebraic Geometry and Representations of Quantized Algebras, Kluwer Publ. 1995.
[29] L. Rozansky, E. Witten, Hyperkähler geometry and invariants of 3-manifolds, Selecta Math. 3 (1997), 401-458.
[30] G. Segal, Geometric aspects of quantum field theory, Proc. ICM-90, p. 1387-1396, Springer, 1991.
[31] J.D. Stasheff, Homotopy associativity of H-spaces, Trans. AMS, 108 (1963), 275-312.

M. Kapranov
Department of Mathematics
Northwestern University
Evanston IL 60208 USA