Barsotti-Tate Groups and Crystals

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At the international congress of 1970 in Nice, A. Grothendieck gave a lecture with the title "Groupes de Barsotti-Tate et cristaux". Grothendieck's lecture describes the crystalline Dieudonné module functor for Barsotti-Tate (or p-divisible) groups over schemes of characteristic p. In this lecture we will see that crystalline Dieudonné module theory has progressed quite a bit since then. We have results on faithfulness, equivalence and faithfulness up to isogeny, there are applications to questions concerning abelian schemes, and we can use the theory to predict results on higher crystalline cohomology groups.

Another purpose of these lecture notes is to explain results on extensions of homomorphisms of p-divisible groups and applications that were obtained by the author.

1. Generalities

We fix a prime number p. Let S denote a scheme of characteristic p. For the definition of a p-divisible group over S we refer to Grothedieck's exposé [1]. One way to obtain a p-divisible group over S is to consider the p-divisible group associated to an abelian scheme A over S. This is basically the system $G = \{G(n)\}$ of finite locally free group schemes $G(n) := A[p^n]$ over S. It is usually denoted $A[p^\infty]$.

The crystalline Dieudonné module functor is a functor

$$\mathbb{D} : {BT_S}^\circ \longrightarrow DC_S.$$

Here BT_S stands for the category of p-divisible groups over S and DC_S stands for the category of Dieudonné crystals over S.

We would like to indicate the meaning of the term "Dieudonné crystal over S". Suppose that $S = \mathrm{Spec}(A)$ is affine. Let $J \to \mathbb{Z}_p[\{x_\alpha\}] \to A$ be a surjection of a polynomial ring over $\mathbb{Z}_p$ onto A with kernel J. Note that $p \in J$. We would like to have "divided powers" on J. This we achieve by formally adding $f^n/n!$ for every $f \in J$; we obtain a new ring D. Up to torsion one can think of this as a subring of $\mathbb{Z}_p[\{x_\alpha\}] \otimes \mathbb{Q}$. Let $\hat{D}$ denote the p-adic completion of D. There is a module of continuous differentials $\hat{\Omega}^1_{\hat{D}}$ and a differential

$$d : \hat{D} \longrightarrow \hat{\Omega}^1_{\hat{D}}.$$

Furthermore, there is still a surjection $\hat{D} \to A$ and the Frobenius endomorphism of A lifts to an endomorphism $\sigma : \hat{D} \to \hat{D}$ (for example by mapping x_α to x_α^p).

In this situation, a Dieudonné crystal over $\operatorname{Spec} A$ is given by a *crystalline Dieudonné module* over $(\hat{D}, \mathrm{d}, \sigma)$. Such a Dieudonné module is a quadruple (M, ∇, F, V) where

a) M is a finite locally free $\hat{D}$ module,
b) $\nabla : M \to M \otimes \hat{\Omega}^1_{\hat{D}}$ is a p-nilpotent connection over the differential d, and
c) $F : M \otimes_\sigma \hat{D} \to M$ and $V : M \to M \otimes_\sigma \hat{D}$ are linear maps, horizontal (for ∇) and satisfy $FV = p$ and $VF = p$.

It turns out that this notion is independent of our choices in the construction of $\hat{D}$ and functorial in A. Thus we obtain a category DC_S for every scheme of characteristic p. We obtain the category of *nondegenerate F-crystals* (see [10]) if we consider *(∇, F)-modules over $\hat{D}$*: triples (M, ∇, F), with M, ∇ and F as above such that the kernel and cokernel of F are annihilated by a power of p. If we write FC_S for the category of nondegenerate F-crystals then there is a forgetful functor $DC_S \to FC_S$. This functor is fully faithful in almost all situations and certainly fully faithful up to isogeny.

For a construction of the functor $\mathbb{D}$ we refer to [11], [12] and [13]. The functor $\mathbb{D}$ turns $\mathbb{G}_m[p^\infty]$ into the Dieudonné module $(\hat{D}, \mathrm{d}, p, 1)$ (i.e., $\nabla = \mathrm{d}, F = p, V = 1$) and it turns $\mathbb{Q}_p/\mathbb{Z}_p$ into the module $(\hat{D}, \mathrm{d}, 1, p)$.

It is clear that the definition of Dieudonné crystals (and F-crystals) given above is rather hard to work with; in fact a lot of work has been done to describe the category (for special $S = \operatorname{Spec}(A)$) in terms of more suitable rings $\hat{D}$. We will see an example of this below.

2. Properties of $\mathbb{D}$

Quite a lot is known due to work of Berthelot, Bloch, Kato, Messing and the author. Here we just list the strongest results that are known to the author. At the moment of writing these notes, the results of (vi)–(ix) have not yet been published.

(i) $\mathbb{D}$ is an equivalence over a perfect field; this follows from the classical Dieudonné theory, as was mentioned in Grothendieck's lecture.
(ii) $\mathbb{D}$ is provably faithful whenever S is reasonable; for example if S is reduced, or if S is Noetherian. The author does not know of a single counter example.
(iii) $\mathbb{D}$ is fully faithful on schemes having locally a p-basis, see [2].
(iv) $\mathbb{D}$ is an equivalence on regular schemes of finite type over a field with a finite p-basis, see [3].
(v) $\mathbb{D}$ is fully faithful up to isogeny over schemes of finite type over a field with a finite p-basis, see [3].
(vi) The finite p-basis hypothesis may be removed from the two last statements, see [4].
(vii) $\mathbb{D}$ is fully faithful in certain cases where S is a local complete intersection. For example if S is of finite type over a field and a l.c.i., and more generally if S is excellent and all of its complete local rings are complete intersections, see [4] (compare also [2]).
(viii) $\mathbb{D}$ is fully faithful up to isogeny over an excellent local ring, see [4].

(ix) $\mathbb{D}$ is essentially surjective up to isogeny over a surface, see [5].

Perhaps the functor $\mathbb{D}$ is an equivalence up to isogeny over schemes of finite type over a field? There are also some negative results:

(x) $\mathbb{D}$ is not fully faithfull in general. This fails even over the ring $\mathbb{F}_p[x,y]/(x^2,xy,y^2)$. See [2, 4.4.1].

(xi) For the experts we mention that the crystalline Dieudonné module functor on the category of *truncated* Barsotti-Tate groups is not fully faithful over $\mathbb{F}_p[t]/(t^p)$. This answers a quetion of [2, 4.4.3] in the negative.

3. Extending homomorphisms, an application of Dieudonné modules

Let G and H be p-divisible groups over a discrete valuation ring R with field of fractions K. Consider the map

$$\mathrm{Hom}(G,H) \longrightarrow \mathrm{Hom}(G_K,H_K). \tag{1}$$

In [6] Tate proved that (1) is a bijection when the characteristic of K is zero: any homomorphism between the generic fibres extends to a homomorphism over R.

In the introduction of exposé IX in SGA 7 (by A. Grothendieck, M. Raynaud and D. Rim) it was mentioned as a problem whether the same holds when the characteristic of K is p. In this case, set $S = \operatorname{Spec} R$ and $\eta = \operatorname{Spec} K$. By the results mentioned in the previous section, we can translate this, using $\mathbb{D}$, into a question on Dieudonné crystals: Is the natural restriction functor $DC_S \to DC_\eta$ fully faithful? This follows from the following stronger theorem.

Theorem 1. [7, Theorem 1.1] Assume R has a p-basis. The restriction functor on nondegenerate F-crystals $FC_S \to FC_\eta$ is fully faithful.

In the following section we will try to explain what kind of mathematics goes into the proof of this theorem. In the rest of this section we indicate a few corollaries of the result.

Theorem 2. Let R be an integrally closed, Noetherian, integral domain, with field of fractions K. Let G and H be p-divisible groups over R. A homomorphism $f : G \otimes_R K \to H \otimes_R K$ extends uniquely to a homomorphism $G \to H$.

This occurs in Tate's paper [6], with the additional assumption that K has characteristic 0. The reduction to the case where R is a complete discrete valuation ring is in [6, page 181]. The theorem then follows from Theorem 1, see [7, Introduction].

Theorem 3. [7, Theorem 2.5] Let A be an abelian variety over the discretely valued field K with valuation ring R. Let $G = A[p^\infty]$ be the associated p-divisible group. Then A has good reduction over R if and only if G has good reduction over R. Similarly for semi-stable reduction.

Of course one has to define carefully the significance of the terms "good reduction" and "semi-stable reduction" for p-divisible groups. For this see [7], compare with SGA 7 exposé IX.

Theorem 4. [7, Theorem 2.6] Let F be a field finitely generated over $\mathbb{F}_p$. Let A and B be abelian varieties over F. The natural map

$$\mathrm{Hom}(A,B) \otimes_{\mathbb{Z}} \mathbb{Z}_p \longrightarrow \mathrm{Hom}(A[p^\infty], B[p^\infty])$$

is bijective.

The case of a finite field was done by Tate [9]. The corresponding result where one replaces the p-divisible group by the ℓ-adic Tate module ($\ell \neq p$) has been known for some time now, see [14], [15] and references therein.

4. Power series and F-crystals

In this section we will try to explain what kind of algebra is used in [7] to prove Theorem 1.

Consider the ring $\Omega = \mathbb{Z}_p[[t]]$ together with the derivation $\frac{\mathrm{d}}{\mathrm{d}t}$ and "Frobenius" map $\sigma : t \mapsto t^p$. This is an example of a ring simpler than the ring $\hat{D}$ of Section 2 which can still be used to describe F-crystals over $\operatorname{Spec} \mathbb{F}_p[[t]]$. Recall that a (θ, F)-module over Ω is a triple (M, θ, F), see Section 1. Thus $\theta : M \to M$ is additive and satisfies $\theta(fm) = f\theta(m) + \frac{\mathrm{d}f}{\mathrm{d}t} m$ and F can be seen as a σ-linear map $M \to M$. The horizontality of F means that $\theta(F(m)) = pt^{p-1}F(\theta(m))$.

Our goal is to study systems of equations of the type (with $s \in \mathbb{N}$)

$$\left\{ \begin{array}{rcl} \theta(m) & = & 0 \\ F(m) & = & p^s m \end{array} \right. \tag{2}$$

Here m will be an element of $M \otimes_\Omega \Gamma$, where $\Omega \subset \Gamma$ is an extension of rings such that the derivation $\frac{\mathrm{d}}{\mathrm{d}t}$ and the "Frobenius" map σ extend to Γ. The extensions of σ and $\frac{\mathrm{d}}{\mathrm{d}t}$ will be denoted by the same symbols, and they will induce extensions $F = F \otimes \sigma$ and $\theta = \theta \otimes 1 + 1 \otimes \frac{\mathrm{d}}{\mathrm{d}t}$ on $M \otimes \Gamma$.

The question that has to be answered is of the form: Is any solution m to (2) of the form $m_0 \otimes 1$, where $m_0 \in M$. Of course this is going to depend on the ring Γ.

The specific ring in question is the following: Γ is the ring of formal Laurent series

$$f = \sum_{n \in \mathbb{Z}} a_n t^n,$$

such that $a_n \in \mathbb{Z}_p$ and such that $a_n \to 0$ as $n \to -\infty$. It is obvious how to extend σ and $\frac{\mathrm{d}}{\mathrm{d}t}$. Another description of Γ is that it is the p-adic completion of the localization of Ω at the prime ideal (p).

Thus we have to prove that any solution m to (2) does not have terms with negative exponents in its expansion with repect to some basis of M. The idea is to proceed in two steps: (a) one proves that any solution m is at least (rigid) analytic in some annulus $\eta < |t| < 1$, and (b) using horizontality prove that m extends to an analytic section over the whole disc $|t| < 1$.

More precisely, one defines a subring $\Gamma_c \subset \Gamma$ of elements

$$f = \sum_{n \in \mathbb{Z}} a_n t^n,$$

such that $\exists \eta > 1 : |a_n| \eta^{-n} \to 0$ for $n \to -\infty$. Each element of Γ_c can be thought of as a rigid analytic function on some small annulus as above. The idea to use the ring Γ_c was first introduced by U. Zannier, who solved the case $\mathrm{rk}\, M = 2$. However, rings like it had already occurred in the context of Monsky-Washnitzer cohomology and overconvergent F-crystals, see next section.

The first step (a) is the harder of the two. Here we use the ring $\Gamma_{1,c}$ consisting of expressions

$$f = \sum_{\alpha \in \mathbb{Z}[1/p]} a_\alpha t^\alpha$$

with $a_\alpha \to 0$ for $\alpha \to -\infty$ and with a certain convergence condition as in the definition of Γ_c above. Note that σ does extend to $\Gamma_{1,c}$, whereas θ does not. In some sense the main new phenomenon observed in [7] is that there is always a filtration

$$M \otimes \Gamma_{1,c} = M_a \supset \ldots \supset M_1 \supset 0,$$

such that the submodules M_i are F-stable and such that on each quotient M_i/M_{i-1} the map F has pure slope s_i with $s_1 > s_2 > \ldots > s_a$. Roughly speaking F has pure slope $s \in \mathbb{Q}$ if for any element $m \in M$ the sequence of elements $F^n(m)$ become divisible by p^{ns-C} for some constant C, and $\det(F) = p^{\dim(M)s}(unit)$. This in some sense means that all "eigenvalues" of F have p-adic valuation s. For the experts we remark here that this filtration is opposite to the "usual" slope filtration on the module $M \otimes_\Omega \Gamma$.

Having proved this one can deduce step (a): any solution m of (2) lies in $M \otimes \Gamma_c$. To finish, i.e., to do step (b), one applies Dwork's trick which says that the connection on M is isomorphic to the trivial connection over the rigid analytic disc.

5. Overconvergent F-crystals

Overconvergent F-crystals are supposed to be the p-adic analogue of lisse ℓ-adic sheaves. They have been introduced by Berthelot, see [8] for example. Presently, there are more questions then answers concerning these crystals. In this section we recall the semi-stable reduction conjecture for these objects; such a conjecture occurs in work of R. Crew, N. Tsuzuki and others. It is the p-adic analog of the phenomenon of quasi-unipotent monodromy and the nilpotent orbit theorem for variations of Hodge structure.

Suppose that $\mathcal{E}$ is a nondegenerate F-crystal over $\mathbb{P}^1_{\mathbb{F}_p} \setminus \{0\}$. Then $\mathcal{E}$ will give rise to a (θ, F)-module $M(\mathcal{E}) = (M, \theta, F)$ over the ring Γ described in the previous section. Let us say that $\mathcal{E}$ is an overconvergent F-crystal on $\mathbb{P}^1 \setminus \{0\}$ if $M \cong N \otimes_{\Gamma_c} \Gamma$ for some (θ, F)-module (N, θ, F) over Γ_c. This definition is not the same as the correct definition (see [8]), but undoubtedly it is equivalent.

Of course this definition is too specialized. We leave it to the reader to formulate the meaning of overconvergence when $\mathcal{E}$ is a nondegenerate F-crystal over a smooth affine curve X over a field k of characteristic p. (Of course there will be an "overconvergence" condition at each "missing point" of X.)

Next, let us try to explain what it means for $\mathcal{E}$ over $\mathbb{P}^1_{\mathbb{F}_p} \setminus \{0\}$ to have semi-stable reduction at $t = 0$. This means that there should exist a finite free $\mathbb{Z}_p[[t]]$-module N, a connection $\theta : N \to (1/t)N$ with at worst a logarithmic pole and a horizontal

σ-linear map $F : N \to N$, all of this such that $N \otimes \Gamma \cong M(\mathcal{E})$. In more technical terms: $\mathcal{E}$ should extend to a (nondegenerate) log-F-crystal over $\mathbb{P}^1$.

Again this is too special. Let $\mathcal{E}$ be an F-crystal over a smooth curve X over a field of characteristic p. Then there is a natural notion of semi-stable reduction of $\mathcal{E}$ at each point x of a projective completion $\overline{X}$ of X. The conjecture can now be formulated as follows:

Conjecture. For any overconvergent F-crystal $\mathcal{E}$ over the curve X there exists a finite morphism of curves $\pi : Y \to X$ such that $\pi^*\mathcal{E}$ has semi-stable reduction at every point of a projective completion $\overline{Y}$ of Y.

The evidence for this conjecture is slender; it has been proved for unit root crystals by N. Tsuzuki. Assuming the conjecture one can prove finiteness for the rigid cohomology of $\mathcal{E}$ over X.

There are several natural generalizations of these notions to the case of varieties of higher dimension. For example one could define a nondegenerate F-crystal over a variety X to be overconvergent if its pullback to every curve mapping to X is overconvergent. (This is not the current definition, see [8].) Then one can ask whether every such overconvergent F-crystal pulls back to a log-F-crystal on $\overline{Y}$, where Y is an alteration of X and $Y \subset \overline{Y}$ is a nice smooth compactification of Y (as in [16]).

For all of these questions and much more on p-adic cohomology we refer the reader to work of P. Berthelot, N. Tsuzuki, R. Crew, G. Christol, Z. Mebkhout, J. Etesse, B. Le Stum, B. Chiarellotto and others.

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