Cohomology of Moduli Spaces of Stable Curves

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Abstract. We report on recent progress towards the determination of the rational cohomology of the moduli spaces of stable curves.

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The moduli space of smooth n-pointed genus g curves, denoted $\mathcal{M}_{g,n}$, parametrizes isomorphism classes of objects of the form $(C; p_1, \dots, p_n)$, where C is a smooth genus g curve and $p_1, \dots, p_n$ are distinct points of C, provided that $2g-2+n>0$. It has been known for a long time that $\mathcal{M}_{g,n}$ is a quasi-projective variety (cf. [24] for $n=0$); it is also known, since the work of Deligne, Mumford and Knudsen [6][22][26] that $\mathcal{M}_{g,n}$ is connected and that, although in general non complete, it admits a projective compactification $\overline{\mathcal{M}}_{g,n}$. We wish to describe some recent advances towards the determination of the rational cohomology of $\overline{\mathcal{M}}_{g,n}$, especially in low degree or in low genus. Everything will take place over the complex numbers.

1. Natural Classes

The points of $\overline{\mathcal{M}}_{g,n}$ correspond to isomorphism classes of *stable* n-pointed genus g curves; we recall what these are. Let C be a connected complete curve whose singularities are, at worst, nodes, and let $p_1, \dots, p_n$ be *smooth* points of C. The *graph* Γ associated to these data consists, first of all, of a set $V = V(\Gamma)$ of *vertices* and a set $L = L(\Gamma)$ of *half-edges*. The set V is just the set of components of the normalization N of C, while L is the set of all points of N mapping to a node or to one of the p_i. The elements of L mapping to nodes come in pairs, the *edges* of the graph, while the remaining ones are called *legs*. For any $v \in V$, we let g_v be the genus of the corresponding component of N, L_v the set of half-edges incident to v, and l_v its cardinality. In addition, the numbering of the p_i yields a numbering of the legs.

The (arithmetic) genus of C can be read off from its graph, and is nothing but the sum of the g_v plus the number of edges minus the number of vertices plus one. The graph will be said to be *stable* if $2g_v - 2 + l_v > 0$ for any vertex v. One says that $(C; p_1, \dots, p_n)$ is a stable n-pointed genus g curve if its graph is stable; it is easy to see that this is the same as saying that $(C; p_1, \dots, p_n)$ has a finite automorphism group. Occasionally, it will be useful to consider stable curves whose marked points are indexed by an arbitrary finite set I, rather than by a set of the form $\{1, \dots, n\}$; we will refer to these as I-pointed curves and shall denote the corresponding moduli space by $\overline{\mathcal{M}}_{g,I}$.

Although in general not smooth, $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ are orbifolds; in particular, their rational cohomology satisfies Poincaré duality, and the Hodge structure on $H^k(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$ is pure of weight k, for any k.

We will consider two basic types of morphisms between moduli spaces of curves. The first,

$$\pi : \overline{\mathcal{M}}_{g,I\cup\{j\}} \to \overline{\mathcal{M}}_{g,I}\ ,$$

simply consists in forgetting about the point labelled by j and passing to the stable model, i.e., roughly speaking, contracting to points all the components that fail to pass the test $2g_v - 2 + l_v > 0$. This morphism has canonical sections σ_i, $i \in I$; the section σ_i associates to any I-pointed curve a new $I \cup \{j\}$-pointed curve obtained by attaching a smooth rational "tail" at the point labelled by i and labelling i and j two distinct points of the tail.

These sections enter, in two different ways, in the construction of cohomology classes on $\overline{\mathcal{M}}_{g,I}$. First of all, we may pull back via σ_i the Chern class of the relative dualizing sheaf of π; the resulting class is usually denoted ψ_i. Next denote by D_i the divisor on $\overline{\mathcal{M}}_{g,I\cup\{j\}}$ traced out by σ_i; then, following [27] and [1], we set

$$\kappa_a = \pi_*(c_1(\omega_\pi(\textstyle\sum D_i))^{a+1})$$

for any non-negative integer a. While the ψ_i are degree two classes, κ_a has degree $2a$.

Further classes can be constructed via the second basic type of map between moduli spaces. For any genus g, I-pointed graph Γ with vertex set V the morphism

$$\xi_\Gamma : X_\Gamma = \prod_{v\in V} \overline{\mathcal{M}}_{g_v,L_v} \to \overline{\mathcal{M}}_{g,I}$$

is obtained by identifying pairs of points corresponding to edges. Observe that stability implies that, for any one of the factors of the left-hand side, either $g_v < g$ or $g_v = g$ and $|L_v| < |I|$. This makes it possible to recursively define *natural*, or *tautological*, classes on $\overline{\mathcal{M}}_{g,I}$. Such a class is simply one that belongs to the subring of $H^*(\overline{\mathcal{M}}_{g,I},\mathbb{Q})$ generated by the κ_a, the ψ_i, and the pushforwards of natural classes via all the morphisms ξ_Γ (or, equivalently, via all the morphisms ξ_Γ where Γ is a graph with only one edge). Notice that all natural classes are algebraic. The image of the morphism ξ_Γ is the closure of the locus of curves whose graph is Γ; its codimension equals the number of edges of Γ. The orbifold fundamental class of this locus, defined as the pushforward via ξ_Γ of the orbifold fundamental class of $\prod_{v\in V}\overline{\mathcal{M}}_{g_v,L_v}$, divided by the order of the automorphism group of Γ, and denoted δ_Γ, is obviously a natural class. The graphs Γ with one edge, which correspond to classes δ_Γ of degree two, come in two kinds. There is the graph with one edge and one vertex of genus $g-1$ (provided g is positive), which we denote by Γ_{irr}, and there are the graphs with one edge and two vertices of genera a and $b = g - a$; if A is the subset of I indexing the legs attached to the genus a vertex we denote such a grap by $\Gamma_{a,A}$. Notice that $\Gamma_{a,A} = \Gamma_{g-a,I\setminus A}$ and that $|A| \geq 2$ if $a = 0$. For brevity, we set $\delta_{irr} = \delta_{\Gamma_{irr}}$, $\delta_{a,A} = \delta_{\Gamma_{a,A}}$.

Two questions now arise. The first is, how far is the cohomology ring of $\overline{\mathcal{M}}_{g,n}$ from the subring of natural classes. The second is, what is the structure of the latter. It is certainly not the case that the natural classes exhaust the cohomology of $\overline{\mathcal{M}}_{g,n}$, except in special cases. In fact, it is known that $H^{11}(\overline{\mathcal{M}}_{1,11},\mathbb{Q})$ is not zero, and Pikaart [29] has shown that this can be used to construct nonzero odd-degree cohomology classes in higher genus as well. On the other hand, to my knowledge, nobody has yet produced an even-dimensional cohomology class on some moduli space $\overline{\mathcal{M}}_{g,n}$ which is not natural. For what we know, then, although the evidence in favour of this is very weak, it might still be possible that the even-dimensional cohomology of $\overline{\mathcal{M}}_{g,n}$ is entirely made up of natural classes or, more modestly, that this is true for $H^k(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$ provided k is even and small enough relative to g.

2. Low Degree

Much of what we know about the cohomology of $\mathcal{M}_{g,n}$ for general g and n is due to Harer. In a series of papers [13][16][17] he essentially answered for $H^k(\mathcal{M}_{g,n},\mathbb{Q})$, $k=2,3,4$, the analogues of the questions we asked in section 1, the easier case of H^1 having been settled before [25]. In this context a natural class is simply a polynomial in the κ_a and the ψ_i. What turns out to be the case is that $H^k(\mathcal{M}_{g,n},\mathbb{Q})$ vanishes for $g\geq 1$ when $k=1$ and for $g\geq 9$ ($g\geq 6$ for $n=0$) when $k=3$, while $H^2(\mathcal{M}_{g,n},\mathbb{Q})$ is freely generated by κ_1 and the ψ_i for $g\geq 3$. As for $H^4(\mathcal{M}_{g,n},\mathbb{Q})$, what Harer shows is that it is freely generated by κ_2 and κ_1^2 for $g\geq 10$ and $n=0$. In proving these results, Harer uses geometric topology and Teichmüller theory. He uses the same ingredients in another paper [15] to give a bound on (in effect, to compute) the cohomological dimension for constructible sheaves of $\mathcal{M}_{g,n}$, for any g and n. The bound is a direct consequence of the construction of a cellular decomposition of $\mathcal{M}_{g,n}$ by means of Strebel differentials. It would be very interesting to give a proof of this result via algebraic geometry or, alternatively, by producing an exhaustion function on $\mathcal{M}_{g,n}$ with appropriate convexity properties.

A direct calculation of the first, second, third, and fifth rational cohomology groups of $\overline{\mathcal{M}}_{g,n}$ has been carried out in [2]. The results are the following. First of all, $H^k(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$ vanishes for $k=1,3,5$ and for all g and n. Secondly, $H^2(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$ is generated by κ_1, the ψ_i and the fundamental classes of the components of the boundary $\partial\mathcal{M}_{g,n}=\overline{\mathcal{M}}_{g,n}\setminus\mathcal{M}_{g,n}$, freely for $g\geq 3$, and modulo explicit relations otherwise; in particular, $H^2(\overline{\mathcal{M}}_{2,n},\mathbb{Q})$ is freely generated by the ψ_i and the fundamental classes of the components of the boundary, while $H^2(\overline{\mathcal{M}}_{1,n},\mathbb{Q})$ is freely generated by the fundamental classes of the components of the boundary. It should be observed that it is known that $\overline{\mathcal{M}}_{g,n}$ is always simply connected (cf. for instance [5]).

The method of proof is entirely algebro-geometric, except for the fact that Harer's bound on the cohomological dimension of $\overline{\mathcal{M}}_{g,n}$ is used; this is one of the reasons why it would be important to give an algebro-geometric proof of Harer's result. We now outline the argument. Harer's bound on the cohomological dimension of $\mathcal{M}_{g,n}$ is that this does not exceed $n-3$ for $g=0$, $4g-5$ for $n=0$, and $4g-4+n$ otherwise. Poincaré duality and the exact sequence of compactly supported cohomology for the inclusion of $\partial\mathcal{M}_{g,n}$ in $\overline{\mathcal{M}}_{g,n}$ immediately show that

$$H^k(\overline{\mathcal{M}}_{g,n},\mathbb{Q})\to H^k(\partial\mathcal{M}_{g,n},\mathbb{Q})\quad \text{is injective for } k\leq d(g,n),$$

where

$$d(g,n) = \begin{cases} n-4 & \text{if } g = 0, \\ 2g-2 & \text{if } n = 0, \\ 2g-3+n & \text{if } g > 0, n > 0. \end{cases}$$

The idea is to use this Lefschetz-type remark to compute $H^k(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ inductively on g and n. Before we can do this, however, we need a further remark. The components of $\partial\mathcal{M}_{g,n}$ are precisely the images of the morphisms ξ_Γ, where Γ runs through all graphs with only one edge. We denote by X the disjoint union of the spaces X_Γ such that Γ has one edge, and by ξ the obvious map from X to $\overline{\mathcal{M}}_{g,n}$. Since X is an orbifold, $H^k(X, \mathbb{Q})$ has a Hodge structure of weight k, and the kernel of $\xi^* : H^k(\partial\mathcal{M}_{g,n}, \mathbb{Q}) \to H^k(X, \mathbb{Q})$ is $W_{k-1}H^k(\partial\mathcal{M}_{g,n}, \mathbb{Q})$. As morphisms of mixed Hodge structures are strictly compatible with the filtrations, a class in $H^k(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ maps to zero in $H^k(X, \mathbb{Q})$ only if it maps to a class in $H^k(\partial\mathcal{M}_{g,n}, \mathbb{Q})$ which comes from $W_{k-1}H^k(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$, that is, since the Hodge structure on $H^k(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ is pure of weight k, only if it maps to zero. The conclusion is that

$$\xi^* : H^k(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \to \bigoplus_{\Gamma \text{ has one edge}} H^k(X_\Gamma, \mathbb{Q}) \quad \text{is injective for } k \leq d(g,n).$$

It is now straightforward to prove the vanishing of $H^k(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ for $k = 1, 3, 5$, by induction on g and n. In fact, using Künneth (and, for $k \geq 3$, the vanishing of H^h for h less than k and odd) we see that the right-hand side of the above inclusion is a direct sum of H^k of moduli spaces $\overline{\mathcal{M}}_{g',n'}$ such that either $g' < g$ or $g' = g$ and $n' < n$. This reduces us to checking directly a finite number of cases in low genus. For instance, when $k = 1$, the moduli spaces to be examined are just $\overline{\mathcal{M}}_{0,3}$, $\overline{\mathcal{M}}_{0,4}$ and $\overline{\mathcal{M}}_{1,1}$; since the first is a point and the remaining two are isomorphic to the projective line, we are done in this case. We'll return to the initial cases of the induction for $k = 3, 5$ in the next section.

It should be remarked that the argument outlined above works just as well in higher odd degree. For instance if, as I suspect, $H^k(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ vanishes for all g and n and for $k = 7$, or for $k = 7, 9$, then to prove this it would suffice to do "by hand" the finite number of cases when $k > d(g,n)$.

The induction step is a little more involved for $H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$. Suppose $d(g,n) \geq 2$; we wish to show that $\alpha \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ is a natural class. For any graph Γ with one edge let α_Γ be the pullback of α to X_Γ. By induction hypothesis we know that each α_Γ is a natural class; on the other hand, for any two graphs Γ and Γ', the classes α_Γ and $\alpha_{\Gamma'}$ pull back to the same class on the fiber product of X_Γ and $X_{\Gamma'}$. The idea, roughly speaking, is to try and show that these compatibility conditions force $\xi^*(\alpha)$ to lie in the image under ξ^* of the natural classes on $H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$; once this is done, we conclude by the injectivity of ξ^*. One key ingredient in making all this work is that we have a very good control on how the natural classes pull back under the maps ξ_Γ, or on how they intersect [1][7].

An important step in the proof, which is also interesting per se, is the following. Let $\varphi : \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$ be the morphism that one obtains by identifying the points labelled by $n+1$ and $n+2$ (this is nothing but $\xi_{\Gamma_{irr}}$). Then

$$\varphi^* : H^k(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \to H^k(\overline{\mathcal{M}}_{g-1,n+2}, \mathbb{Q}) \quad \text{is injective for } k \leq \min(2g-2, g+5).$$

As a toy example, we prove this for $k = 2$, $g = 3$, $n = 0$. We need to show that, if $x \in H^2(\overline{\mathcal{M}}_3, \mathbb{Q})$ pulls back to zero under φ, then it pulls back to zero via all the morphisms ξ_Γ such that Γ is a graph with one edge. In the case at hand there is only one such graph beyond Γ_{irr}, namely the graph Γ with a vertex of genus 2 and one of genus 1. Look at the commutative diagram

$$\begin{array}{ccc}
\overline{\mathcal{M}}_{1,\{i,j,h\}} \times \overline{\mathcal{M}}_{1,\{l\}} & \xrightarrow{\eta} & \overline{\mathcal{M}}_{2,2} \\
{\scriptstyle \varphi' \times \mathbf{1}} \big\downarrow & & {\scriptstyle \varphi} \big\downarrow \\
\overline{\mathcal{M}}_{2,\{h\}} \times \overline{\mathcal{M}}_{1,\{l\}} & \xrightarrow{\xi_\Gamma} & \overline{\mathcal{M}}_3
\end{array}$$

where φ' is the analogue of φ and η consists in identifying the points labelled by h and l. Then, by the vanishing of H^1, the second cohomology group of the lower left corner is just $H^2(\overline{\mathcal{M}}_{2,\{h\}}, \mathbb{Q}) \oplus H^2(\overline{\mathcal{M}}_{1,\{l\}}, \mathbb{Q})$, so we can write $\xi_\Gamma^*(x) = (y, z)$. Since $\varphi(x) = 0$, $(\varphi'(y), z)$ vanishes, showing in particular that $z = 0$. A similar argument shows that y vanishes as well, finishing the proof.

In a certain sense, the injectivity of φ^* in high enough genus can be viewed as a partial analogue, in our context, of the stability results of Harer and Ivanov [14][20][18] for the cohomology of $\mathcal{M}_{g,n}$.

In principle, one could try to treat higher even degree cohomology groups of $\overline{\mathcal{M}}_{g,n}$ along the same lines as those followed for H^2. Let us look at H^4, for instance. The initial cases of the induction are no problem at all. In performing induction, however, aside from the greater complication of the linear algebra involved, a further problem arises. The method of calculating $H^k(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ we have outlined requires, at each stage of the induction, that we have complete control on all the relations satisfied by the natural classes in $H^k(\overline{\mathcal{M}}_{g',n'}, \mathbb{Q})$ for $g' < g$ or for $g' = g$, $n' < n$. When $k = 2$ it is a relatively easy matter to find them. Already for $k = 4$, however, it is not at all clear what precisely these relations are; new and unexpected ones in low genus have recently been discovered [10][28][3], but there may well be more. It is a very interesting problem to find all relations satisfied by the natural classes in $H^4(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ and, in perspective, in higher even degree as well.

It would also be of considerable interest to give a proof of the induction step in even degree which is not as computational, but based on a more conceptual understanding of why natural classes on the components of the boundary which match on "intersections" of these patch together to yield a natural class on the whole space.

3. Low Genus

Keel [21] has determined the cohomology ring of $\overline{\mathcal{M}}_{0,n}$, for any $n \geq 3$, in terms of generators and relations. The ring in question is generated by the classes δ_Γ, where Γ runs through all stable graphs with one edge, which in this case are all of the form $\Gamma_{0,A}$, where A is a subset of $\{1,\ldots,n\}$ such that $2 \leq |A| \leq n-2$. The relations are generated by a set of linear ones and a set of quadratic ones. The linear relations are that, for any set $\{i,j,h,k\}$ of distinct indices,

$$\sum_{\substack{A\ni i,j\\ A\not\ni h,k}} \delta_{0,A} = \sum_{\substack{A\ni i,h\\ A\not\ni j,k}} \delta_{0,A} = \sum_{\substack{A\ni i,k\\ A\not\ni j,h}} \delta_{0,A}\,.$$

The quadratic relations say that $\delta_{0,A}\cdot\delta_{0,B} = 0$ unless $A\cap B=\emptyset$, $A\cap B^c=\emptyset$, $A^c\cap B=\emptyset$, or $A^c\cap B^c=\emptyset$.

In higher genus our knowledge is far less complete. Getzler [9][11] has found a generating function for the Serre characteristics (and also for their $\mathbb{S}_n$-equivariant versions) of the moduli spaces $\overline{\mathcal{M}}_{1,n}$ and the Serre characteristic of $\overline{\mathcal{M}}_{2,n}$ for $n\leq 3$ (again, $\mathbb{S}_n$-equivariant or not). The Serre characteristic of a quasi-projective variety is defined as the Euler characteristic of its compactly supported cohomology in the Grothendieck group of mixed Hodge structures; it is important to notice that, since $\overline{\mathcal{M}}_{g,n}$ is an orbifold, and hence the Hodge structure on its k-th cohomology group is pure of weight k for any k, the Serre characteristic of $\overline{\mathcal{M}}_{g,n}$ determines the Hodge numbers. As an example of the results one obtains, the non-zero Hodge numbers of $\overline{\mathcal{M}}_{2,2}$ turn out to be $h^{0,0}=h^{5,5}=1$, $h^{1,1}=h^{4,4}=6$, and $h^{2,2}=h^{3,3}=14$. Bini, Gaiffi and Polito [4] have found a generating function for the Euler characteristics of the spaces $\overline{\mathcal{M}}_{2,n}$, and Harer has found a generating function for the Euler characteristics $\chi(\overline{\mathcal{M}}_{g,n})$; the formula given in [4] is a closed expression in a single, recursively computable, power series (the series $A(t)$ below).

The arguments used to obtain all these results have a common basis. Let I be a finite set and let Γ be a stable genus g, I-pointed graph. We denote by $\mathcal{M}(\Gamma)$ the moduli space of those stable, genus g, I-pointed curves whose graph is Γ. This is a locally closed subspace of $\overline{\mathcal{M}}_{g,I}$ which is nothing but the image of $\prod_{v\in V(\Gamma)}\mathcal{M}_{g_v,L_v}$ under the map ξ_Γ. In fact, $\mathcal{M}(\Gamma)$ is the quotient of $\prod_{v\in V(\Gamma)}\mathcal{M}_{g_v,L_v}$ modulo the automorphism group of Γ; as such, it is an orbifold. The $\mathcal{M}(\Gamma)$ give a stratification of $\overline{\mathcal{M}}_{g,I}$, the *topological stratification.* Now suppose, for instance, that we want to calculate the Euler characteristic of $\overline{\mathcal{M}}_{g,n}$. This is just the sum of the characteristics of the open strata in the topological stratification, since these satisfy Poincaré duality. Thus it suffices to know the Euler characteristics of the open moduli spaces $\mathcal{M}_{g',n'}$ for $g'<g$ or for $g'=g$, $n'\leq n$, and of certain quotients of their products. The proper setup for systematcally exploiting this phenomenon is the one of modular operads [12]. Here, however, we content ourselves with sketching the argument of [4] for the spaces $\overline{\mathcal{M}}_{1,n}$.

The top stratum of $\overline{\mathcal{M}}_{g,n}$ is $\mathcal{M}_{g,n}$. Its Euler characteristic could be calculated using the methods of [19], but for $g\leq 2$ it can be computed in an elementary way. Look for instance at $\pi:\mathcal{M}_{1,n+1}\to\mathcal{M}_{1,n}$. Since any automorphism of a smooth genus 1 curve fixing five or more points is the identity, for $n\geq 5$ the fiber is a smooth genus 1 curve minus n points; by the multiplicativity of Euler characteriscs in fibrations $\chi(\mathcal{M}_{1,n+1}) = -n\chi(\mathcal{M}_{1,n})$. When $n\leq 4$ this has to be modified a bit to take into account the fact that π is no longer a fibration, but some of the fibers are quotients of a smooth genus 1 curve minus n points modulo a finite group. At any rate, it is straightforward to compute $\chi(\mathcal{M}_{1,n})$ inductively on n starting from $\chi(\mathcal{M}_{1,1}) = 1$; it turns out to be 1 for $n = 2$, 0 for $n = 3, 4$, and $(-1)^n(n-1)!/12$ for $n \geq 5$.

The goal is to compute the generating function $K_g(t) = \sum_n \chi(\overline{\mathcal{M}}_{g,n})\frac{t^n}{n!}$ (for $g = 1$). For any fixed g, the genus g stable graphs fall into a finite number of different patterns, the contribution of each of which to the generating function is handled separately. In genus 1 there are just two patterns: some graphs contain a genus 1 vertex, to which a finite number of trees are attached, while the remaining ones contain a "necklace" of edges, again with trees attached.

Let us look at a graph Γ of the first kind. It has no automorphisms. If we sever the edges stemming from the genus 1 vertex, we are left with a graph consisting of a genus 1 vertex with m legs, and stable graphs of genus zero $\Gamma_1, \dots, \Gamma_h$, $h \leq m$, where Γ_i is $(k_i + 1)$-pointed and $n = m - h + \sum k_i$. Thus $\chi(\mathcal{M}(\Gamma)) = \chi(\mathcal{M}_{1,m}) \prod_i \chi(\mathcal{M}(\Gamma_i))$. Now set

$$A(t) = t + \sum_{n\geq 2} \sum_G \chi(\mathcal{M}(G))\frac{t^n}{n!}\,,$$

where the inner sum runs through all genus zero stable $(n+1)$-pointed graphs. The contribution of the graphs we are considering to the generating function K_1 is then $\sum_n \chi(\mathcal{M}_{1,n})\frac{A^n}{n!}$. The same considerations show that $A = t + \sum_{n\geq 2} \chi(\mathcal{M}_{0,n+1})\frac{A^n}{n!}$; since $\chi(\mathcal{M}_{0,n+1})$ can be easily calculated (it equals $(-1)^n(n-2)!$) this relation makes it possible to recursively compute the coefficients of A. Since the characteristics $\chi(\mathcal{M}_{1,n})$ are known, the contribution of the graphs of the first kind to K_1 can be calculated to any given order. The contribution of the graphs of the second kind can be evaluated by similar means; the only new fact is that, when the necklace consists of a single edge, or of two edges, there is an order two automorphisms (reversing orientation of the edge, or interchanging the two edges). For instance, the contribution coming from a graph falling in the first of these two subcases is of the form $\chi(\mathcal{M}_{0,m}/\mathbb{S}_2)\prod_i \chi(\mathcal{M}(\Gamma_i))$, where $\Gamma_1, \dots, \Gamma_h$ are stable genus zero graphs and $h \leq m-2$. If $m \geq 5$, $\mathcal{M}_{0,m} \to \mathcal{M}_{0,m}/\mathbb{S}_2$ is unramified, so $\chi(\mathcal{M}_{0,m}/\mathbb{S}_2) = \frac{1}{2}\chi(\mathcal{M}_{0,m})$, while $\chi(\mathcal{M}_{0,m}/\mathbb{S}_2) = \chi(\mathcal{M}_{0,m})$ for $m = 3, 4$. Putting everything together gives the final result

$$\sum_n \chi(\overline{\mathcal{M}}_{1,n})\frac{t^n}{n!} = \frac{19}{12}A + \frac{23}{24}A^2 + \frac{5}{18}A^3 + \frac{1}{24}A^4 - \frac{1}{12}\log(1+A) - \frac{1}{2}\log(1-\log(1+A))\,.$$

For Serre characteristics, the strategy is similar. That the characteristic of $\overline{\mathcal{M}}_{g,n}$ can be expressed in terms of those of the strata in the topological stratification follows from the fact that there is a spectal sequence abutting to $H^\bullet(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ whose E_2 term is

$$E_2^{p,q} = \bigoplus_{\Gamma \text{ has } -q \text{ edges}} H_c^{p+q}(\mathcal{M}(\Gamma), \mathbb{Q})\,.$$

What seems really hard, in this approach, is computing Serre characteristics of open strata, in particular those of the open moduli spaces $\mathcal{M}_{g,n}$. The naive method we used for Euler characteristics cannot be employed since Serre characteristics do not behave multiplicatively in fibrations. To treat the cases $g = 1$ [8] and $g = 2$, $n \leq 3$ [11], Getzler uses a subtle argument whose strategy is to reduce, via the Leray spectral sequence for $\mathcal{M}_{1,n} \to \mathcal{M}_{1,1}$ (resp., for $\mathcal{M}_{2,n} \to \mathcal{M}_2$) and other technology, to calculating the cohomology of certain mixed Hodge modules on $\mathcal{M}_{1,1}$ (resp., on $\mathcal{M}_2$), which is then handled via Eichler-Shimura theory (resp., via Faltings' Eichler spectral sequence). It is not clear how much farther these methods can be pushed.

Virtually all the initial cases of the induction described in section 2 and leading to the determination of the low-dimensional cohomology of $\overline{\mathcal{M}}_{g,n}$ are covered by the results of [21], [9] and [11], but for most of them simple direct proofs are also available. The only cases that escape are those of $\overline{\mathcal{M}}_3$ and $\overline{\mathcal{M}}_{3,1}$, which are needed to trigger the induction for H^5. These can be deduced, via a variant of the arguments of section 2, from the results of [23], where the Poincaré polynomials of $\mathcal{M}_3$ and $\mathcal{M}_{3,1}$ are determined.

It remains to observe that the results of [10], [11] and [3] completely describe not only the additive structure of the cohomology of $\overline{\mathcal{M}}_{1,n}$ and $\overline{\mathcal{M}}_{2,m}$ for $n \leq 4$ and $m \leq 3$, but the multiplicative structure as well.

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