Mirror Symmetry and Toric Geometry

Victor V. Batyrev

Abstract. A brief survey of some recent progress towards a mathematical understanding of Mirror Symmetry is given. Using toric geometry, we can express Mirror Symmetry via an elementary duality of special polyhedra.

1991 Mathematics Subject Classification: Primary 14J32, 52B20, 81T60; Secondary 14J45, 33C50, 81T30.
Keywords and Phrases: Toric Varieties, Calabi-Yau Manifolds, Generalized Hypergeometric Functions, Rational Curves.

Introduction

Mirror Symmetry is a remarkable discovery by physicists who suggested that the partition functions of two physical theories obtained from two *different* Calabi-Yau manifolds V and V^* can be *identified* [59]. So far mathematicians couldn't find any appropriate language for a rigorous formulation of this identification (we refer the reader to Kontsevich's talk [50] for the most general conceptual framework that could help to find such a languange). Without knowing a mathematical reason for Mirror Symmetry it simply remains for one to believe in its existence. This belief is supported by many computational experiments followed by attempts to find rigorous mathematical explanations of their results.

In this talk we shall give a brief survey of some recent progress, based on toric geometry, towards a mathematical understanding of Mirror Symmetry. Loosely speaking, toric geometry provides some kind of "Platonic" approach to Mirror Symmetry, because it replaces the highly nontrivial duality between some mathematical objects, which we still don't completely know, by an elementary polar duality of special convex polyhedra. Of course, such a simplification can't reflect the whole nature of Mirror Symmetry, but it helps to form our intuition and find reasonable mathematical tests for this duality.

1 Polar Duality of Reflexive Polyhedra

Let M be a free abelian group of rank d, $N = Hom(M, \mathbf{Z})$ the dual group, and $\langle \cdot, \cdot \rangle : M \times N \to \mathbf{Z}$ the natural nondegenerate pairing. We denote by $M_{\mathbf{R}}$ (resp. by $N_{\mathbf{R}}$) the scalar extension $M \otimes_{\mathbf{Z}} \mathbf{R}$ (resp. $M \otimes_{\mathbf{Z}} \mathbf{R}$).

Definition 1.1 [6] A convex d-dimensional polyhedron $\Delta \subset M_{\mathbf{R}}$ is called *reflexive* if the following conditions are satisfied:

(i) all vertices of Δ belong to the lattice $M \subset M_{\mathbf{R}}$;

(ii) the zero vector $0 \in M$ belongs to the interior of Δ;

(iii) all vertices of the polar polyhedron

$$\Delta^* := \{ b \in N_{\mathbf{R}} \, : \, \langle a, b \rangle \geq -1 \ \ \forall a \in \Delta \}$$

belong to the dual lattice $N \subset N_{\mathbf{R}}$.

If $\Delta \subset M_{\mathbf{R}}$ is a reflexive polyhedron, then $\Delta^* \subset N_{\mathbf{R}}$ is again a reflexive polyhedron and $(\Delta^*)^* = \Delta$. So we obtain a natural involution $\Delta \leftrightarrow \Delta^*$ on the set of all d-dimensional reflexive polyhedra. This involution plays a crucial role in our approach to Mirror Symmetry. In the case $d = 3$, the involution $\Delta \leftrightarrow \Delta^*$ provides an interpretation of Arnold's Strange Duality [1, 30, 31, 32, 48].

Toric geometry, or theory of toric varieties, establishes remarkable relations between mathematical objects in convex geometry, e.g. convex cones and polyhedra, and algebraic varieties (see [24, 25, 34, 35, 62]). Toric varieties $\mathbf{P}_{\Delta}$ associated with reflexive polyhedra Δ are Fano varieties with at worst Gorenstein canonical singularities. Let $T_M = \mathit{Spec}\, \mathbf{C}[M]$ be the algebraic torus with lattice of characters M. Denote by $Z_f \subset T_M$ the affine hypersurface in T_M defined by the equation

$$f(x_1, \ldots, x_d) = \sum_{m \in \Delta \cap M} a_m x^m = 0,$$

where the set $\{a_m\}_{m \in \Delta \cap M}$ consists of generically choosen complex numbers. Then the projective closure of Z_f in $\mathbf{P}_{\Delta}$ is a normal irreducible variety $\overline{Z}_f$ having trivial canonical class. If we repeat the same procedure with the polar reflexive polyhedron Δ^*, then in the dual torus $T_N := \mathit{Spec}\, \mathbf{C}[N]$ we obtain another affine hypersurface $Z_g \subset T_N$ defined by an equation

$$g(y_1, \ldots, y_d) = \sum_{n \in \Delta^* \cap N} b_n y^n = 0$$

We denote by $\overline{Z}_g \subset \mathbf{P}_{\Delta^*}$ the projective compactification of Z_g in $\mathbf{P}_{\Delta^*}$. The pair $(\overline{Z}_f, \overline{Z}_g)$ is conjectured to be *mirror symmetric* [6]. If $d = 4$, then $\overline{Z}_f$ (reps. $\overline{Z}_g$) is birational to a smooth Calabi-Yau 3-fold $\widehat{Z_f}$ (resp. $\widehat{Z_g}$) and one has the equations

$$h^{1,1}(\widehat{Z_f}) = h^{2,1}(\widehat{Z_g}), \ \ h^{1,1}(\widehat{Z_g}) = h^{2,1}(\widehat{Z_f}),$$

which admit an interpretation by means of a Monomial-Divisor Mirror Map [2]. It is known that the volume of a reflexive polyhedron can be estimated by a constant depending only on d [5]. Consequently there exist only finitely many d-dimensional reflexive polyhedra Δ up to $GL(M)$-isomorphism. Some results towards a classification of reflexive polyhedra of dimension $d \leq 4$ were obtained by Kreuzer and Skarke [68, 53, 54]. It turned out that all examples of Calabi-Yau 3-folds constructed by physicists from hypersurfaces in 7555 different weighted projective spaces can be obtained from 4-dimensional reflexive polyhedra [23].

Moreover, all moduli spaces of Calabi-Yau hypersurfaces in 4-dimensional toric varieties can be connected into a web using a series of simple transformations [3, 4]. The latter confirms a conjecture of M. Reid on the connectedness of the moduli space of Calabi-Yau 3-folds [64].

The polar duality for reflexive polyhedra can be extended to a more general duality for reflexive Gorenstein cones in [11]. This generalization allowed us to express in the same way not only the construction for mirrors of Calabi-Yau complete intersections in Gorenstein toric Fano varieties [20, 56], but also the construction for mirrors of rigid Calabi-Yau 3-folds [22].

2 Topological Mirror Symmetry Test and Stringy Hodge Numbers

If two smooth Calabi-Yau $(d-1)$-folds (V, V^*) form a mirror pair, then the Hodge numbers of V and V^* are related by the equalities

$$h^{p,q}(V) = h^{d-1-p,q}(V^*), \;\; 0 \leq p, q \leq d-1,$$

which are known as a simplest *topological Mirror Symmetry test.* A formulation of this test for projective Calabi-Yau hypersurfaces $\overline{Z}_f \subset \mathbf{P}_\Delta$ and $\overline{Z}_g \subset \mathbf{P}_{\Delta^*}$ turns out to be rather nontrivial, because these hypersurfaces are usually singular. Moreover, we can't expect that a projective *smooth* birational model $\widehat{Z_f}$ of $\overline{Z}_f$ having trivial canonical class always exists if $d \geq 5$. On the other hand, it was observed in [12] that Betti and Hodge numbers of such birational models are uniquely determined. This observation supported the idea of *stringy Hodge numbers* for singular Calabi-Yau varieties which we proposed in [9]. Denote by $E(W; u, v)$ the E-polynomial of a complex quasi-projective variety W. It is defined by the formula

$$E(W; u, v) := \sum_{p,q} e^{p,q}(W) u^p v^q,$$

where $e^{p,q}(W) := \sum_{k \geq 0} (-1)^k h^{p,q}(H^k_c(W, \mathbf{C}))$ is the Hodge-Deligne number of W (see [26]).

Definition 2.1 [13] Let X be a normal quasi-projective variety over $\mathbf{C}$ with at worst Gorenstein canonical singularities, $\rho : Y \to X$ a resolution of singularities whose exceptional locus $D \subset Y$ is a normal crossing divisor with components $D_1, \ldots, D_r$, and $K_Y = \rho^* K_X + \sum_{i=1}^r a_i D_i$. We set $I = \{1, \ldots, r\}$ and define the *stringy E-function* of X by the formula

$$E_{\mathrm{st}}(X; u, v) := \sum_{J \subset I} E(D_J^\circ; u, v) \prod_{j \in J} \frac{uv - 1}{(uv)^{a_j + 1} - 1},$$

where

$$D_J^\circ := \{x \in X \; : \; x \in D_j \Leftrightarrow j \in J\}.$$

If X is projective and $E_{\mathrm{st}}(X; u, v)$ is a polynomial, then we define *stringy Hodge numbers* $h^{p,q}_{\mathrm{st}}(X)$ by the formula

$$E_{\mathrm{st}}(X; u, v) := \sum_{p,q} (-1)^{p+q} h^{p,q}_{\mathrm{st}}(X) u^p v^q.$$

It is important to remark that the above definition doesn't depend on the choice of a resolution ρ [13]. A proof of this independence uses a variant of a non-archimedean integration proposed by Kontsevich [52] and developed by Denef and Loeser [28]. Using some ideas from [27], we can prove the following:

Theorem 2.2 [10] *Let Δ and Δ^* be two dual to each other reflexive polyhedra of arbitrary dimension d. Then the stringy E-functions of the corresponding projective Calabi-Yau hypersurfaces $\overline{Z}_f \subset \mathbf{P}_\Delta$ and $\overline{Z}_g \subset \mathbf{P}_{\Delta^*}$ satisfy the duality*

$$E_{\rm st}(\overline{Z}_f; u, v) = (-u)^{d-1} E_{\rm st}(\overline{Z}_g; u^{-1}, v),$$

i.e., stringy Hodge numbers of $\overline{Z}_f$ and $\overline{Z}_g$ satisfy the topological Mirror Symmetry test:

$$h^{p,q}_{\rm st}(\overline{Z}_f) = h^{d-1-p,q}_{\rm st}(\overline{Z}_g) \quad (0 \leq p, q \leq d-1).$$

We remark that the last result holds true for all Calabi-Yau complete intersections in Gorenstein toric Fano varieties and agrees with the duality for reflexive Gorenstein cones.

Let $X := V/G$ be a quotient of a smooth Calabi-Yau manifold V modulo a regular action of a finite group G. It was shown in [17] that the *stringy Euler number*

$$e_{\rm st}(X) := \lim_{u,v\to 1} E_{\rm st}(X; u, v) = \sum_{J \subset I} e(D_J^\circ) \prod_{j \in J} \frac{1}{a_j + 1}$$

coincides with the orbifold physicists' Euler number $e(V, G)$ defined by Dixon-Harvey-Vafa-Witten formula [29]:

$$e(V, G) := \frac{1}{|G|} \sum_{gh=hg} e(V^g \cap V^h),$$

where

$$V^g \cap V^h := \{x \in V \ : \ gx = x \,\&\, hx = x\}.$$

This formula is closely related to the so-called McKay correspondence [65].

3 Counting Rational Curves and GKZ-hypergeometric Functions

Let $\partial\Delta$ be the boundary of a reflexive polyhedron $\Delta \subset M_{\mathbf{R}}$, $\{m_1, \ldots, m_r\} := \partial\Delta \cap M$, and $\{a_{m_1}, \ldots, a_{m_r}\}$ the set of coefficients in equations $f(x) = 1 - \sum_{i=1}^r a_{m_i} x^{m_i} = 0$ defining affine Calabi-Yau hypersurfaces $Z_f \subset T_M$. In [7] it was shown that the power series

$$\Phi(a_{m_1}, \ldots, a_{m_r}) = \sum \frac{(k_1 + \ldots + k_r)!}{k_1! \cdots k_r!} a_{m_1}^{k_1} \cdots a_{m_r}^{k_r},$$

where $(k_1, \ldots, k_r) \in \mathbf{Z}^r_{\geq 0}$ runs over all nonnegative integral solutions to the equation $k_1 m_1 + \cdots + k_r m_r = 0$, admits an interpretation as a period of a regular differential $(d-1)$-form $\omega \in H^0(\overline{Z}_f, \Omega^{d-1}_{\overline{Z}_f})$ and satisfies the holonomic differential system introduced by Gelfand, Kapranov and Zelevinsky [36], i.e. Φ is a generalized GKZ-hypergeometric function. If $\Delta \subset \mathbf{R}^4$ is the convex hull of vectors $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$, $(-1,-1,-1,-1)$, then the corresponding series Φ has the form

$$\Phi_0(z) = \sum_{k \geq 0} \frac{(5k)!}{(k!)^5} z^k,$$

where $z = a_1 a_2 a_3 a_4 a_5$. The function $\Phi_0(z)$ satisfies the differential equation $L\Phi(z) = 0$, where

$$L := \left(z\frac{d}{dz}\right)^4 - 5z\left(5z\frac{d}{dz}+1\right)\left(5z\frac{d}{dz}+2\right)\left(5z\frac{d}{dz}+3\right)\left(5z\frac{d}{dz}+4\right),$$

and can completed to a natural basis $\{\Phi_0, \Phi_1, \Phi_2, \Phi_3\}$ of its solutions. Using Mirror Symmetry, Candelas, de la Ossa, Green and Parkes, in the famous paper [21], predicted that the formal power series expansion of the function

$$\mathcal{F}(q) = \frac{5}{2}\left(\frac{\Phi_1}{\Phi_0}\frac{\Phi_2}{\Phi_0} - \frac{\Phi_3}{\Phi_0}\right)$$

with respect to the variable $q = q(z) := \exp(\Phi_1/\Phi_0)$ coincides with the power series

$$F(q) := \frac{5}{2}(\log q)^3 + \sum_{j>0} K_j q^j,$$

where

$$K_j = \sum_{k|j} n_{j/k} k^{-3}$$

and n_i is the "number of rational curves" of degree i on a generic Calabi-Yau quintic 3-fold in $\mathbf{P}^4$. A mathematical verification of this exciting prediction of Mirror Symmetry demanded a lot effort by many mathematicians. As a first step one needed a rigorous mathematical definition for the "number of rational curves". Such a definition has been obtained in terms of Gromov-Witten classes introduced and investigated by Kontsevich-Manin [49], Ruan-Tian [66], and Li-Tian [57]. The second step was the idea of Kontsevich concerning an equivariant Bott's localization formula with respect to torus action on the moduli spaces of stable maps of $\mathbf{P}_1$ to $\mathbf{P}_4$ [51]. The crucial remarkable progress was obtained by Givental who succeeded in identifying solutions of quantum differential equations obtained from equivariant Gromov-Witten classes with the GKZ-hypergeometric periods of mirrors [38, 39, 40, 41]. Detailed expositions of Givental's ideas are contained in [19, 63]. Another complete mathematical proof of this famous prediction of Mirror Symmetry was obtained in 1997 by Lian, Liu and Yau in [58] using so-called *linear gauge σ-models* associated with toric varieties (see Morrison-Plesser [60]).

It was observed in [8] that GKZ-hypergeometric functions allow to make analogous predictions for the "number of rational curves" in arbitrary Calabi-Yau complete intersections in toric varieties. Many of such predictions related to GKZ-hypergeometric functions were investigated by Hosono, Klemm, Lian, Theisen and

Yau in [43, 44, 45, 46]. The most general framework for the study of resonant GKZ-hypergeometric systems associated with reflexive Gorenstein cones was developed by Stienstra [68].

4 Further developements

It is interesting to analyse possibilities for extending toric methods beyond the class of Calabi-Yau complete intersections in Gorenstein toric Fano varieties. A natural class for testing Mirror Symmetry consists of Calabi-Yau complete intersections in homogeneous manifolds, e.g. in Grassmanians, in partial flag manifolds etc. A general construction of mirrors for this class of Calabi-Yau manifolds has been proposed in [14, 15, 16]. An interesting generalization of Givental's technique for complete intersections in homogeneous spaces was obtained by Kim [47].

Another interesting direction is related to the celebrated Strominger-Yau-Zaslow interpretation of Mirror Symmetry as a T-duality using special Lagrangian torus fibrations [69] (see also [61, 42]). Recently some topological torus fibrations on Calabi-Yau hypersurfaces in toric varieties were constructed by Zharkov [70] using methods from [37]. These fibrations agree with some predictions of Leung and Vafa [55].

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Victor Batyrev
Mathematisches Institut
Universität Tübingen
72076 Tübingen, Germany
batyrev@bastau.mathematik.
uni-tuebingen.de