String Theory and Duality

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Abstract. String Duality is the statement that one kind of string theory compactified on one space is equivalent in some sense to another string theory compactified on a second space. This draws a connection between two quite different spaces. Mirror symmetry is an example of this. Here we discuss mirror symmetry and another "heterotic/type II" duality which relates vector bundles on a K3 surface to a Calabi–Yau threefold.

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1 Introduction

Superstring theory does not currently have a complete definition. What we have instead are a set of incomplete definitions each of which fill in some of the unknown aspects of the other partial definitions. Naturally two questions immediately arise given this state of affairs:

1. Is each partial definition consistent with the others?

2. How completely do the partial definitions combine to define string theory?

Neither of these questions has yet to be answered and indeed both questions appear to be quite deep. The first of these concerns the subject of "string duality". Let us list the set of known manifestations of string theory each of which leads to a partial definition:

1. Type I open superstring

2. Type IIA superstring

3. Type IIB superstring

4. $E_8 \times E_8$ heterotic string

5. $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic string

6. Eleven-dimensional supergravity (or "M-theory")

The first five of these "theories" describe a string, which is closed in all cases except the first, propagating in flat ten-dimensional Minkowski space $\mathbb{R}^{9,1}$. The last theory is more like that of a membrane propagating in eleven-dimensional Minkowski space $\mathbb{R}^{10,1}$. (Note that many people like to think of string theory as a manifestation of M-theory rather than the other way around.)

Instead of using a completely flat Minkowski space, one may try to "compactify" these string theories by replacing the Minkowski space by $X \times M$, where X is some compact $(10-d)$-dimensional manifold (or $(11-d)$-dimensional in the case of M-theory) and $M \cong \mathbb{R}^{d-1,1}$. So long as all length scales of X are large with respect to any natural length scale intrinsic to the string theory, we can see that $X \times M$ may approximate the original flat Minkowski space. This is called the "large radius limit" of X. One of the most fascinating aspects of string theory is that frequently we may also make sense of compactifications when X is small, or contains a small subspace in some sense. An extreme case of this is when X is singular. In particular, X need not be a manifold in general.

The key ingredient to be able to analyze string theories on general spaces, X, is *supersymmetry.* For our purposes we may simply regard a supersymmetry as a spinor representation of the orthogonal group of the Minkowski space in which the string theory lives. In general a theory may have more than one supersymmetry in which case the letter "N" is commonly used to denote this number. In the above theories the type I and heterotic strings together with M-theory each have $N=1$ while the type II strings correspond to $N=2$.

Upon compactification, the value of N will change depending upon the global holonomy of the Levi-Cevita connection of the tangent bundle of X. The new supersymmetries of M are constructed from the components of the old spinor representations of the original Minkowski space which are invariant under this holonomy. We will give some examples of this process shortly.

The general rule is that the more supersymmetry one has, the more tightly constrained the string theory is and the easier it is to analyze away from the large radius limit. Note that this rule really depends upon the total number of components of all the supersymmetries and so a large d has the same effect as a large N (since M has d dimensions and so its spinor representation would have a large dimension).

As well as constraining the string theory so that it may be more easily analyzed, supersymmetry can be regarded as a coarse classification of compactifications. A knowledge of d and N provides a great deal of information about the resulting system. Almost every possibility for d and N is worth at least one long lecture in itself. We will deal with the case $d=4$ and $N=2$ about which probably the most is known at this present time.

The principle of duality can now be stated as follows. Given a specific string and its compactification on X can one find another string theory compactified on another space, Y, such that the "physics" in the uncompactified space, M, is isomorphic between the two compactifications? This is important if our first question of this introduction is to be answered. In particular it should always be true for any pair of string theories in our list unless there is a good reason for a "failure" of one of the strings in some sense. We will see an example of this below.

A mathematical analysis of duality requires a precise definition of the physics of a compactification. This is not yet known in generality. What we do know is a set of objects which are determined by the physics, such as moduli spaces, partition functions, correlation functions, BPS soliton spectra etc., which may be compared to find necessary conditions for duality.

The most basic object one may study to identify the physics of two dual theories is their moduli spaces. Roughly speaking this should correspond to the moduli spaces of X and Y although one always requires "extra data" beyond this. It is the extra data which leads to the mathematical richness of the subject. Clearly if two theories are to be identified, one must be able to identify their moduli spaces point by point. This will be the focus of this talk.

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2 String Data

In order to be able to describe the moduli space of each string theory we are required to give the necessary data which goes into constructing each one. Unfortunately, we do not have anywhere near enough space to describe the origin of what follows. We refer to [1, 12, 19] for more details.

The theories which yield $d = 4$ and $N = 2$ in which we will interested are specified by the following

- The type IIA string is compactified on a Calabi–Yau threefold X (which has SU(3) holonomy). The following data specifies the theory.
  1. A Ricci-flat metric on X.
  2. A B-field $\in H^2(X, \mathbb{R}/\mathbb{Z})$.
  3. A Ramond-Ramond (RR) field $\in H^{\mathrm{odd}}(X, \mathbb{R}/\mathbb{Z})$.
  4. A dilaton+axion, $\Phi \in \mathbb{C}$.
- The type IIB string is compactified on a Calabi–Yau threefold Y (which also has SU(3) holonomy). The following data specifies the theory.
  1. A Ricci-flat metric on Y.
  2. A B-field $\in H^2(Y, \mathbb{R}/\mathbb{Z})$.
  3. A Ramond-Ramond (RR) field $\in H^{\mathrm{even}}(Y, \mathbb{R}/\mathbb{Z})$.
  4. A dilaton+axion, $\Phi \in \mathbb{C}$.
- The $E_8 \times E_8$ heterotic string is compactified on a product of a K3 surface, Z, and an elliptic curve, E_H. This product has SU(2) holonomy. The following data specifies the theory.
  1. A Ricci-flat metric on $Z \times E_H$.
  2. A B-field $\in H^2(Z \times E_H, \mathbb{R}/\mathbb{Z})$.

3. A vector bundle $V \to (Z \times E_H)$ with a connection satisfying the Yang–Mills equations and whose structure group $\subseteq E_8 \times E_8$. The respective characteristic classes in H^4 for V and the tangent bundle of $Z \times E_H$ are fixed to be equal.
4. A dilaton+axion, $\Phi \in \mathbb{C}$.

In each case we can only expect the data to provide a faithful coordinate system in some limit. This is a consequence of the the fact that we only really have a partial definition of each string theory. A sufficient condition for faithfulness is that the target space is large — i.e., all minimal cycles have a large volume, *and* $|\Phi| \gg 1$. Beyond this we may expect "quantum corrections". In general the global structure of the moduli space can be quite incompatible with this parameterization — it is only reliable near some boundary.

On general holonomy arguments (see, for example, [2,10]) one can argue that the moduli space factorizes locally

$$\mathscr{M} \cong \mathscr{M}_H \times \mathscr{M}_V, \tag{1}$$

where (at least at smooth points) $\mathscr{M}_H$ is a quaternionic Kähler manifold and $\mathscr{M}_V$ is a special Kähler manifold. We refer the reader to [16] for the definition of a special Kähler manifold. These restricted holonomy types are expected to remain *exact* after quantum corrections have been taken into account.

We may now organize the above parameters into how they span $\mathscr{M}_H$ and $\mathscr{M}_V$. First we note that Yau's theorem [28] tells us that the Ricci-flat metric on a Calabi–Yau manifold is uniquely determined by a choice of complex structure and by fixing the cohomology class of the Kähler form, $J \in H^2(\bullet, \mathbb{R})$. We may combine J and B to form the "complexified Kähler form" $B + iJ \in H^2(\bullet, \mathbb{C}/\mathbb{Z})$. We then organize as follows

- The Type IIA string: $\mathscr{M}_V$ is parametrized by the complexified Kähler form of X. $H^{\mathrm{odd}}(X, \mathbb{R}/\mathbb{Z}) \cong H^3(X, \mathbb{R}/\mathbb{Z})$ is the *intermediate Jacobian* of X and is thus an abelian variety. We then expect a factorization $\mathscr{M}_H \cong \mathbb{C} \times \mathscr{M}'_H$, where Φ is the coordinate along the $\mathbb{C}$ factor. Finally we have a fibration $\mathscr{M}'_H \to \mathscr{M}_{\mathrm{cx}}(X)$ with fibre given by the intermediate Jacobian, and $\mathscr{M}_{\mathrm{cx}}(X)$ is the moduli space of complex structures on X.

- The Type IIB string: $\mathscr{M}_V$ is now parametrized by the complex structure of Y. $H^{\mathrm{even}}(Y, \mathbb{R}/\mathbb{Z}) \cong H^0(Y, \mathbb{R}/\mathbb{Z}) \oplus H^2(Y, \mathbb{R}/\mathbb{Z}) \oplus H^4(Y, \mathbb{R}/\mathbb{Z}) \oplus H^6(Y, \mathbb{R}/\mathbb{Z})$ may be viewed as an abelian variety. We again expect a factorization $\mathscr{M}_H \cong \mathbb{C} \times \mathscr{M}'_H$, where Φ is the coordinate along the $\mathbb{C}$ factor. Finally we have a fibration $\mathscr{M}'_H \to \mathscr{M}_{\mathrm{Kf}}(Y)$ with fibre given by the RR fields, and $\mathscr{M}_{\mathrm{Kf}}(Y)$ is the moduli space of the complexified Kähler form of Y.

- The $E_8 \times E_8$ heterotic string: Let us first assume that the bundle $V \to (Z \times E_H)$ factorizes as $(V_Z \to Z) \times (V_E \to E_H)$. Thus the structure group of V_Z times the structure group of V_E is a subgroup of $E_8 \times E_8$. We now expect $\mathscr{M}_V$ to factorize as $\mathbb{C} \times \mathscr{M}'_V$, where Φ is the coordinate along the $\mathbb{C}$ factor (see [15] for a more precise statement). $\mathscr{M}'_V$ is then the total moduli space of $V_E \to E_H$ including deformations of the complex structure and the complexified Kähler form of E_H. $\mathscr{M}_H$ is the total moduli space of the fibration $V_Z \to Z$ including deformations of the Ricci-flat metric of Z.

Again we emphasize that the above statements are approximate and only valid when the target space is large and $|\Phi| \gg 1$. They should be exact only at the boundary of the moduli space corresponding to these limits. It is important to see that factorization of the moduli space will restrict the way that the quantum corrections may act. For example, in the type IIA string the dilaton, Φ, lives in $\mathscr{M}_H$. This means that $\mathscr{M}_V$ cannot be subject to corrections related to having a finite $|\Phi|$. Equally, the Kähler form parameter governs the size of X and so $\mathscr{M}_H$ will not be subject to corrections due to finite size.

It is this property that some parts of the moduli space can be free from quantum corrections and that the interpretation of this part can vary from string theory to string theory which lies at the heart of the power of string duality. If two theories are simultaneously exact at some point in the moduli space then we may address the first question in our introduction. If at every point in the moduli space some theory (perhaps as yet unknown) is in some sense exact then we may address the second question.

3 Mirror Duality

Mirror symmetry as first suggested in [9, 20] was a duality between "conformal field theories". We may make a different version of mirror symmetry, a little more in the spirit of "full" string theories, by proposing the following [4]:

Definition 1 *The pair (X,Y) of Calabi–Yau threefolds is said to be a* mirror *pair if and only if the type IIA string compactified on X is physically equivalent to the type IIB string compactified on Y.*

Of course, this definition is mathematically somewhat unsatisfying as it depends on physics. However, it encompasses previous definitions of mirror symmetry. We also assume the following

Proposition 1 *If (X,Y) is a mirror pair then so is (Y,X).*

While this proposal is obvious from the old definitions it is not completely clear that we may establish it rigorously using the above definition.

Applying this to the moduli space description in the previous section we immediately see that, ignoring quantum corrections, $\mathscr{M}_{\mathrm{Kf}}(X)$ should be identified with $\mathscr{M}_{\mathrm{cx}}(Y)$ and equally $\mathscr{M}_{\mathrm{Kf}}(Y)$ should be identified with $\mathscr{M}_{\mathrm{cx}}(X)$. We know that $\mathscr{M}_V$ is unaffected by Φ corrections and we expect $\mathscr{M}_{\mathrm{cx}}(Y)$ to be *exact* since it is also unaffected by size corrections.

We expect that $\mathscr{M}_{\mathrm{Kf}}(X)$ be affected by size corrections. Similarly, given proposition 1, $\mathscr{M}_{\mathrm{cx}}(X)$ is exact and $\mathscr{M}_{\mathrm{Kf}}(Y)$ will suffer from size corrections. We will use the notation $\mathscr{Q}$ to refer to a fully corrected moduli space. Thus $\mathscr{Q}_{\mathrm{Kf}}(X) \cong \mathscr{Q}_{\mathrm{cx}}(Y) \cong \mathscr{M}_{\mathrm{cx}}(Y)$ but $\mathscr{Q}_{\mathrm{Kf}}(X) \ncong \mathscr{M}_{\mathrm{Kf}}(X)$.

The corrections to $\mathscr{M}_{\mathrm{Kf}}(X)$ take the form of "world-sheet" instantons and were studied in detail in celebrated work of Candelas et al [8]. In particular, the assertion that $\mathscr{Q}_{\mathrm{Kf}}(X) \cong \mathscr{Q}_{\mathrm{cx}}(Y)$ allows one to count the numbers of rational curves on X. Subsequently a great deal of work (see for example [18, 23, 24, 26]) has been done which has made this curve counting much more rigorous.

As well as $\mathscr{M}_{\mathrm{Kf}}$ and $\mathscr{M}_{\mathrm{cx}}$, it is instructive to look at the abelian fibres of $\mathscr{M}_H$ in the context of mirror symmetry. The effect of equating $\mathscr{M}_{\mathrm{cx}}(X)$ with $\mathscr{M}_{\mathrm{Kf}}(Y)$ is to equate

$$H^3(X,\mathbb{Z}) \sim H^0(Y,\mathbb{Z}) \oplus H^2(Y,\mathbb{Z}) \oplus H^4(Y,\mathbb{Z}) \oplus H^6(Y,\mathbb{Z}), \tag{2}$$

but that we expect this correspondence to make sense only if Y is very large. Note that by going around closed loops in $\mathscr{M}_{\mathrm{cx}}(X)$ we expect to have an action on $H^3(X,\mathbb{Z})$ induced by monodromy. If we were to take (2) to be literally true then we have to say the same thing about the action of closed loops in $\mathscr{M}_{\mathrm{Kf}}(X)$ acting on the even integral cycles in Y. That is to say, we would be claiming that if one begins with, say, a point representing an element of $H^0(Y,\mathbb{Z})$ we could smoothly shrink Y down to some small size and then smoothly let it reëxpand in some inequivalent way such that our point had magically transformed itself into, say, a 2-cycle! Clearly this does not happen in classical geometry.

The suggestion therefore [3, 7] is that quantum corrections should be applied to the notion of integral cycles so that, in the context of stringy geometry, 0-cycles *can* turn into 2-cycles when Y is small. Thus the notion of dimensionality must be uncertain for small cycles.

Of central importance to the study of mirror pairs is being in a region of moduli space where the quantum corrections are small. That is we require Y to be large. This amounts to a specification of the Kähler form on Y and must therefore specify some condition on the complex structure of X. This was analyzed by Morrison:

PROPOSITION 2 *If Y is at its large radius limit then X is at a degeneration of complex structure corresponding to maximal unipotent monodromy.*

We refer the reader to [25] for an exact statement of this. The idea is that X degenerates such that a variation of mixed Hodge Structures around this point leads to monodromy compatible with (2).

The point we wish to emphasize here is that when X is very large then the complex structure of Y is restricted to be very near a particular point in $\mathscr{M}_{\mathrm{cx}}(X)$. We only really expect mirror symmetry to be "classically" true at this degeneration. Close to this degeneration we may measure quantum perturbations leading to such effects as counting rational curves. A long way from this degeneration mirror symmetry is much more obscure from the point of view of classical geometry.

It is possible to have a Calabi–Yau threefold, X, whose moduli space $\mathscr{M}_{\mathrm{cx}}(X)$ contains no points of maximal unipotency. In this case, its mirror, Y, can have no large radius limit. Since clearly any classical Calabi–Yau threefold may be taken to be any size, Y cannot have an interpretation as a Calabi–Yau threefold. This is the sense in which duality can sometimes break down.

4 Heterotic/Type IIA Duality

Having discussed mirror duality between the type IIA and the type IIB string we will now try to repeat the above analysis for the duality between the type IIA and the $E_8 \times E_8$ heterotic string. This duality was first suggested in [14, 22] following the key work of [21, 27].

In this case $\mathscr{M}_V$ is currently fairly well-understood (see, for example [2] and references therein). Here we will discuss $\mathscr{M}_H$ which provides a much richer structure.

First let us discuss the quantum corrections. On the heterotic side, $\mathscr{M}_H$ contains the deformations of Z as well as the vector bundle over it. Note that in the the case of K3 surfaces we may not factorize the moduli space of Ricci-flat metrics into a moduli space of complex structures and the Kähler cone. This follows from the fact that given a fixed Ricci-flat metric, we have an S^2 of complex structures. The size of the K3 surface is a parameter of $\mathscr{M}_H$ and so we expect $\mathscr{M}_H$ to suffer from quantum corrections due to size effects for the heterotic string.

We also know that on the type IIA side, the dilaton is contained in $\mathscr{M}_H$. Thus we expect $\mathscr{M}_H$ to suffer from corrections due to Φ for the type IIA string. We managed to evade worrying about such effects in our discussion of mirror symmetry but here we are not so lucky.

Let us now attempt to find the place in the moduli space where we may ignore the quantum effects both due to Φ and due to size. To do this we require the following:

Proposition 3 *If a type IIA string compactified on a Calabi–Yau threefold X is dual to a heterotic string compactified on a factorized bundle over a product of a K3 surface, Z, and an elliptic curve E_H, then X must be in the form of an elliptic fibration $\pi_F : X \to \Sigma$ with a section and a K3 fibration $\pi_A : X \to B$. Here Σ is a birationally ruled surface and $B \cong \mathbb{P}^1$.*

Note that these fibrations may contain degenerate fibres. We refer to [2] for details.

Let us now assume that Z is in the form of an elliptic fibration over B with a section. Given this, we claim the following:

Proposition 4 *The limit of large Z automatically ensures that $\Phi \to \infty$ for the type IIA string. In this limit, X also undergoes a degeneration to $X_1 \cup_{Z_*} X_2$, where X_1 and X_2 are each elliptic fibrations over a birationally ruled surface and are each fibrations over $B \cong \mathbb{P}^1$ with generic fibre given by a rational elliptic surface (RES). $Z_* = X_1 \cap X_2$ is isomorphic to Z as a complex variety.*

We refer to [6, 17] for a proof.

Recall that a RES is a complex surface given by $\mathbb{P}^2$ blown up at nine points given by the intersection of two cubic curves. In a sense, for elliptic fibrations a RES is "half of a K3 surface". This degeneration is viewed as each K3 fibre of the fibration $\pi_A : X \to B$ breaking up into two RES's.

This degeneration therefore provides the analogue of the "maximally unipotent" degeneration in the case of mirror symmetry. There are important differences however. Note that while the maximally unipotent degeneration of mirror symmetry essentially corresponds to a point in the moduli space of complex structures, the degeneration given by proposition 4 is not rigid — it corresponds a family of dual theories. In the case of mirror symmetry, by taking Y to be large we needed to fix a point in $\mathscr{M}_{\mathrm{Kf}}(Y)$ and thus $\mathscr{M}_{\mathrm{cx}}(X)$. Here we need to take the K3 surface Z to its large radius limit but this does *not* fix a point in $\mathscr{M}_H$. We may still vary the complex structure of Z (subject only to the constraint that it be an elliptic fibration with a section) and we may still vary the bundle V_Z.

We should therefore be able to see the moduli space of complex structures on the elliptic K3 surface, Z, and the moduli space of the vector bundle V_Z *exactly* from this degeneration of X. The correspondence $Z \cong Z_* = X_1 \cap X_2$ tells us how the moduli space of Z can be seen from the moduli space of the degenerated X. The moduli space of the vector bundle is a little more interesting.

V_Z may be split into a sum of two bundles $V_{Z,1}$ and $V_{Z,2}$ each of which has a structure group $\subseteq E_8$. We will identify $V_{Z,1}$ from a curve $C_1 \subset Z_*$ and $V_{Z,2}$ from $C_2 \subset Z_*$. C_1 and C_2 will form the *spectral curves* of their respective bundles in the sense of [13].

Let us consider a single RES fibre Q_b of the fibration $X_1 \to B$, where $b \in B$. Q_b is itself an elliptic fibration $\pi_Q : Q_b \to \mathbb{P}^1$. The section of the elliptic fibration $\pi_F : X \to \Sigma$ determines a distinguished section $\sigma_0 \subset Q_b$. Blowing this down gives a Del Pezzo surface with 240 lines $\sigma_1, \dots, \sigma_{240}$.

We then have

PROPOSITION 5 *The fibre of the branched cover $C_1 \to B$ is given by the set of points $\{\sigma_i \cap Z_*; i = 1, \dots, 240\}$,*

with an analogous construction for C_2. We refer to [5, 11] for details.

We also have the data from the abelian fibre of $\mathscr{M}_H$ corresponding to the RR fields. In the case of heterotic/type IIA duality we have [5]

PROPOSITION 6

$$\Lambda_0 \cong H^1(C_1, \mathbb{Z}) \oplus H^1(C_2, \mathbb{Z}) \oplus H^2_T(Z, \mathbb{Z}),$$

where Λ_0 is the sublattice of $H^3(X, \mathbb{Z})$ invariant under monodromy around the degeneration of proposition 4 and $H^2_T(Z, \mathbb{Z})$ is the lattice of transcendental 2-cocycles in Z.

Thus the RR-fields of the type IIA string map to the Jacobians of C_1 and C_2, required to specify the bundle data, and to the B-field on Z.

Proposition 6 should embody much of the spirit of the duality between the type IIA string and the $E_8 \times E_8$ heterotic string in a similar way that equation (2) embodies mirror symmetry. In particular Λ_0 is not invariant under monodromy around any loop in the moduli space and so the notion of what constitutes the E_8-bundles and what constitutes the K3 surface Z should be blurred in general — just as the notion of 0-cycles and 2-cycles is blurred in mirror symmetry.

The analysis of the moduli space $\mathscr{M}_H$ is very much in its infancy. In this talk we have not even mentioned how to compute quantum corrections — the above discussion was purely for the exact classical limit. There appear to be many adventures yet to be encountered in bringing the understanding of heterotic/type IIA duality to the same level as that of mirror symmetry.

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