p-Adic Hodge Theory in the Semi-Stable Reduction Case

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Abstract. We survey the statement and the proof (by K. Kato and the author) of the semi-stable conjecture of Fontaine-Jannsen on p-adic étale cohomology and crystalline cohomology, generalizing it to truncated simplicial schemes. Thanks to the alteration of de Jong, this generalization especially implies that the p-adic étale cohomology of any proper variety (which may have singularity) is potentially semi-stable.

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§1. Introduction.

The p-adic Hodge theory is an analogue of the Hodge theory for a variety X over a p-adic field K (a complete discrete valuation field of mixed characteristic $(0, p)$ with perfect residue field) and it compares p-adic étale cohomology with the action of the Galois group and de Rham cohomology with some additional structures (depending on how good the reduction of the variety is). In the semi-stable reduction case, it was formulated by J.-M. Fontaine and U. Jannsen [Fo3], [Fo4] as a conjecture (called the semi-stable conjecture or $C_{\rm st}$ for short), and it asserts that p-adic étale cohomology with the action of the Galois group and de Rham cohomology with the Hodge filtration and certain additional structures coming from log crystalline cohomology of the special fiber can be constructed from each other. (See §2 for more details).

The conjecture was studied by many people such as J.-M. Fontaine, W. Messing, S. Bloch, K. Kato, G. Faltings and O. Hyodo (cf. [Fo-M], [Bl-K], [Fa1], [Fa2], [H], [H-K], [K]), and finally it was completely solved by the author [T1]. It was also proved by G. Faltings by a different method [Fa4] afterwards together with its generalization to relative cohomology, non-constant coefficients and an open variety. A new proof was also given by W. Niziol using K-theory at least in the good reduction case [Ni]. Also a theory for p-torsion cohomology in the semi-stable reduction case was established by G. Faltings and C. Breuil ([Fa3], [Br2], [Br3]) by generalizing the theory of Fontaine-Laffaille in the good reduction case [Fo-L].

In these notes, I give a survey of the statement and the proof [K], [T1] of $C_{\rm st}$, generalizing it to truncated simplicial schemes (an analogue of [D2]). Thanks to the alteration of de Jong [dJ], this generalization implies that the p-adic étale cohomology of any proper variety (which may have singularity) is potentially semi-stable. The details of the proof for simplicial schemes will be given elsewhere. Unlike [K], [T1], here we use the log version of the syntomic and the syntomic-étale sites [Fo-M] to define log syntomic cohomology [Br3] and to construct the map from log syntomic cohomology to p-adic étale cohomology since it is easily generalized to simplicial log schemes.

NOTATION: Let K be a complete discrete valuation field of mixed characteristic $(0,p)$ whose residue field k is perfect. Let W be the ring of Witt-vectors with coefficients in k and let K_0 denote the field of fractions of W. Let $\overline{K}$ be an algebraic closure of K and set $G_K := \mathrm{Gal}(\overline{K}/K)$. We choose and fix a uniformizer π of K. For a ring, a scheme or a log scheme over W, we denote its reduction mod p^n by the subscript n.

§2. THE SEMI-STABLE CONJECTURE.

(2.1) We first recall the theory of semi-stable representations by Fontaine ([Fo1], [Fo2], [Fo3], [Fo4]) briefly. We need the rings $B_{\mathrm{st}} \subset B_{\mathrm{dR}}$ associated to K, which have the following structures and properties. The ring B_{dR} is a complete discrete valuation field with residue field $\hat{\overline{K}}$ endowed with an action of G_K. B_{dR} is filtered by the discrete valuation and it contains $\mathbb{Q}_p(r)$ $(r \in \mathbb{Z})$ and $\overline{K}$. We have $B_{\mathrm{dR}}^{G_K} = K$ and the image of a non-zero element of $\mathbb{Q}_p(1)$ is a uniformizer of B_{dR}. B_{st} is a G_K-stable subring of B_{dR} containing $\mathbb{Q}_p(r)$ $(r \in \mathbb{Z})$ and K_0 and endowed, additionally, with the Frobenius φ and the monodromy operator N satisfying $N\varphi = p\varphi N$. The natural homomorphism $B_{\mathrm{st}} \otimes_{K_0} K \to B_{\mathrm{dR}}$ is injective, $B_{\mathrm{st}}^{G_K} = K_0$ and $Fil^r B_{\mathrm{dR}} \cap B_{\mathrm{st}}^{\varphi=p^r,\, N=0} = \mathbb{Q}_p(r)$ $(r \in \mathbb{Z})$. The ring B_{st} with the actions of G_K, φ, N is independent of π, but the embedding $B_{\mathrm{st}} \hookrightarrow B_{\mathrm{dR}}$ depends on it.

By a p-adic representation of G_K, we mean a $\mathbb{Q}_p$-vector space of finite dimension endowed with a continuous linear action of G_K, and, by a filtered (φ, N)-module, a K_0-vector space of finite dimension D endowed with a semi-linear automorphism φ, a linear endomorphism N and an exhaustive and separated filtration Fil^i $(i \in \mathbb{Z})$ on $D_K := D \otimes_{K_0} K$ such that $N\varphi = p\varphi N$. Then, to a p-adic representation V, one can associate a filtered (φ, N)-module $D_{\mathrm{st}}(V) := (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{G_K}$ functorially. We have $\dim_{K_0} D_{\mathrm{st}}(V) \leq \dim_{\mathbb{Q}_p} V$ and we say V is *semi-stable* if the equality holds. The restriction of D_{st} to the category of semi-stable representations is fully faithful and exact; its quasi-inverse is given by $V_{\mathrm{st}}(D) := Fil^0(B_{\mathrm{dR}} \otimes_K D_K) \cap (B_{\mathrm{st}} \otimes_{K_0} D)^{\varphi=1,\, N=0}$.

(2.2) Let X be a scheme over O_K isomorphic to the finite base change of a proper semi-stable scheme. Then, using log crystalline cohomology, one can give a canonical (φ, N)-module structure D^q on $H^q_{\mathrm{dR}}(X_K/K)$ (depending on π) (see §3). We have the following theorem conjectured by J.-M. Fontaine and U. Jannsen ([T1], [Fa4], see also [Fo-M], [Fa2], [K-M], [K]) .

THEOREM 2.2.1. (C_{st}). *With the notation and the assumption as above, the p-adic representation $H^q_{\mathrm{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is semi-stable, and there exists a canonical isomorphism of filtered (φ, N)-modules $D_{\mathrm{st}}(H^q_{\mathrm{ét}}(X_{\overline{K}}, \mathbb{Q}_p)) \cong D^q$ functorial on X and* *compatible with the product structures, Chern classes of vector bundles on X_K and cycle classes of cycles on X_K.*

We need more arguments than [T1] for the compatibility with Chern classes and cycle classes, which will be given elsewhere.

Our generalization to truncated simplicial schemes is the following:

Theorem 2.2.2. (=Theorem 7.1.1). *Let m be a non-negative integer and let $X^\cdot$ be an m-truncated simplicial scheme such that each X^i $(0 \le i \le m)$ satisfies the assumption on X above. Then $\tilde{H}^q_{\text{ét}}(X^\cdot_{\overline{K}}, \mathbb{Q}_p)$ is semi-stable and there exists a canonical isomorphism of filtered (φ, N)-modules $D_{\text{st}}(\tilde{H}^q_{\text{ét}}(X^\cdot_{\overline{K}}, \mathbb{Q}_p)) \cong \tilde{H}^q_{\text{log-crys}}(Y^\cdot/W)$. (See §6 for the definition of $\tilde{H}^q_{\text{ét}}$ and $\tilde{H}^q_{\text{log-crys}}$.)*

Thanks to the compactification theorem of Nagata [Na], the alteration of de Jong [dJ] and cohomological descent [SD] (cf. [D2]), we obtain the following corollary (cf. (6.4.1)).

Corollary 2.2.3. *For any proper scheme X_K over K, $H^q_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is potentially semi-stable.*

§3. Log crystalline cohomology and de Rham cohomology.

We will survey the log crystalline cohomology defined and studied in [H-K] briefly (see also [T1] §4).

(3.1) Let S denote Spec (O_K) endowed with the log structure defined by the closed point. Let $i_{n,\pi}\colon S_n \to E_n$ be the PD-envelope of the exact closed immersion of S_n into an affine line Spec $(W_n[T])$ with the log structure on the origin defined by sending T to the chosen π. We have $\Gamma(E_n, \mathcal{O}_{E_n}) \cong W[T, T^{me}/m!\ (m \ge 1)] \otimes W_n$, which we denote by R_{E_n}, where $e := [K : K_0]$. Put $R_E := \mathbb{Q} \otimes \varprojlim_n R_{E_n}$. Let $i_{n,0}\colon \underline{W}_n \hookrightarrow E_n$ be the exact closed immersion defined by the ideal generated by T, $T^{me}/m!$ $(m \ge 1)$. $\underline{W}_n$ and E_n have canonical liftings of Frobenius compatible with $i_{n,0}$ defined by $T \mapsto T^p$.

(3.2) Let X be a smooth fine saturated log scheme over S whose underlying scheme is proper over O_K. Let Y denote the special fiber of X (as a log scheme) and assume its underlying scheme is reduced. We consider three kinds of crystalline cohomology $H^*_{\text{crys}}(Y/\underline{W}_n)$, $H^*_{\text{crys}}(X_n/E_n)$, $H^*_{\text{crys}}(X_n/S_n) \cong H^*_{\text{dR}}(X_n/S_n)$, which we write without crys in the following. The first two are endowed with φ, N satisfying $N\varphi = p\varphi N$ compatible with the pull-back by $\{i_{n,0}\}$. We denote the $\varprojlim_n$ of these cohomology groups by the symbols without the subscript n.

Theorem 3.2.1. (Hyodo-Kato [H-K] §5, cf. [T1] §4.4). *There exists a unique K_0-linear section $s\colon H^q(Y/\underline{W})_{\mathbb{Q}} \to H^q(X/E)_{\mathbb{Q}}$ compatible with φ of the pull-back by $\{i_{0,n}\}$. It is also compatible with N and induces an isomorphism*

$$R_E \otimes_{K_0} H^q(Y/\underline{W})_{\mathbb{Q}} \xrightarrow{\sim} H^q(X/E)_{\mathbb{Q}}. \tag{3.2.2}$$

Furthermore the composite with the pull-back by $\{i_{\pi,n}\}$ induces an isomorphism

$$\rho_\pi\colon K \otimes_{K_0} H^q(Y/\underline{W})_{\mathbb{Q}} \xrightarrow{\sim} H^q(X/S)_{\mathbb{Q}} \cong H^q_{\text{dR}}(X_K/K). \tag{3.2.3}$$

Thus $H^*_{\mathrm{dR}}(X_K/K)$ is endowed naturally with a $(\varphi,\, N)$-module structure.

§4. LOG SYNTOMIC COHOMOLOGY.

We will survey the log version [Br1], [Br3] of the theory of syntomic cohomology [Fo-M]. See [Br1], [Br3] for details except the proof of Proposition 4.4.1 in the case $r \geq p$.

(4.1) To make the theory compatible with the theory of the log syntomic-étale site in §5, we change the topology slightly; we define the big and the small syntomic site X_{SYN}, X_{syn} of a fine log scheme X using syntomic morphisms $f\colon Y \to Z$ of fine log schemes in the sense of Kato [K] (2.5) such that the underlying morphisms of schemes of f are locally quasi-finite and that the cokernels of $(f^*M_Z)^{\mathrm{gp}} \to M_Y^{\mathrm{gp}}$ are torsion, which we will call *strictly syntomic* morphisms in these notes. Every proof in [Br1], [Br3] still works for this modified syntomic site. The big syntomic topos is functorial, but the small syntomic topos is not. However the small syntomic site is functorial as a topology in the sense of Artin [A], and it is sufficient in our application. For an exact nilimmersion, the direct image functors of the big and the small syntomic topos are exact.

(4.2) For a fine log scheme with a quasi-coherent PD-ideal (T, I, γ) such that $n\mathcal{O}_T = 0$ for some positive integer n and a fine log scheme X over T, we have the big crystalline site with syntomic topology $(X/T, I, \gamma)_{\mathrm{CRYS,\,SYN}}$ (or $(X/T)_{\mathrm{CRYS,\,SYN}}$ for short), and we have a commutative diagram of topos:

$$
\begin{array}{ccccc}
(X/T)^{\sim}_{\mathrm{CRYS,\,SYN}} & \xrightarrow{\alpha} & (X/T)^{\sim}_{\mathrm{CRYS}} & \xrightarrow{\beta} & (X/T)^{\sim}_{\mathrm{crys}} \\
\big\downarrow{\scriptstyle U_{X/T,\,\mathrm{SYN}}} & & \big\downarrow{\scriptstyle U_{X/T}} & & \big\downarrow{\scriptstyle u_{X/T}} \\
X^{\sim}_{\mathrm{SYN}} & \longrightarrow & X^{\sim}_{\mathrm{\acute{E}T}} & \longrightarrow & X^{\sim}_{\mathrm{\acute{e}t}}.
\end{array} \tag{4.2.1}
$$

It is easy to see that β_* is exact.

PROPOSITION 4.2.2. ([Br1] §3). *We have* $RU_{X/T,\,\mathrm{SYN}*}J^{[r]}_{X/T} = U_{X/T,\,\mathrm{SYN}*}J^{[r]}_{X/T}$ *and* $R\alpha_*J^{[r]}_{X/T} = J^{[r]}_{X/T}$.

(4.3) Let us return to the situation in (3.2). We denote by $\mathcal{O}^{\mathrm{crys}}_n$ and $J^{[r]}_n$ the restriction of $U_{X_n/W_n,\,\mathrm{SYN}*}\mathcal{O}_{X_n/W_n}$ and $U_{X_n/W_n,\,\mathrm{SYN}*}J^{[r]}_{X_n/W_n}$ to $(X_n)_{\mathrm{syn}}$ and also their direct images in $(X_m)_{\mathrm{syn}}$ $(m \geq n)$. Here W_n is endowed with the trivial log structure. By Proposition 4.2.2, the cohomology of these sheaves give us $H^*_{\mathrm{crys}}(X_n/W_n, \mathcal{O}_{X_n/W_n})$ and $H^*_{\mathrm{crys}}(X_n/W_n, J^{[r]}_{X_n/W_n})$, and we have $\Gamma(Y, \mathcal{O}^{\mathrm{crys}}_n) = \Gamma_{\mathrm{crys}}(Y/W_n, \mathcal{O}_{Y/W_n}) = \Gamma_{\mathrm{crys}}(Y_1/W_n, \mathcal{O}_{Y_1/W_n})$. By the last equality, $\mathcal{O}^{\mathrm{crys}}_n$ is naturally endowed with the Frobenius endomorphism φ. $\mathcal{O}^{\mathrm{crys}}_n$ and $J^{[r]}_n$ are flat over $\mathbb{Z}/p^n\mathbb{Z}$ and $\mathcal{O}^{\mathrm{crys}}_{n+1} \otimes \mathbb{Z}/p^n\mathbb{Z} \cong \mathcal{O}^{\mathrm{crys}}_n$, $J^{[r]}_{n+1} \otimes \mathbb{Z}/p^n\mathbb{Z} \cong J^{[r]}_n$ ([Br3] 3.1.4).

(4.4) We see easily $\varphi(J^{[r]}_n) \subset p^r\mathcal{O}^{\mathrm{crys}}_n$ if $r \leq p-1$. However, this is false in general and we use the following modification of $J^{[r]}_n$: $J^{<r>}_n := \{x \in J^{[r]}_{n+s} | \varphi(x) \in p^r\mathcal{O}^{\mathrm{crys}}_{n+s}\}/p^n$ $(s \geq r)$. The right hand side is independent of s, $J^{<r>}_n$ is flat over

$\mathbb{Z}/p^n\mathbb{Z}$ and $J_{n+1}^{<r>} \otimes \mathbb{Z}/p^n\mathbb{Z} \cong J_n^{<r>}$. Define $\varphi_r\colon J_n^{<r>} \to \mathcal{O}_n^{\mathrm{crys}}$ by setting $\varphi_r(x \mod p^n) = y \mod p^n$ for $x \in J_{n+r}^{[r]}$, $y \in \mathcal{O}_{n+r}^{\mathrm{crys}}$ such that $\varphi(x) = p^r y$. Set $S_n^r := \mathrm{Ker}(1-\varphi_r\colon J_n^{<r>} \to \mathcal{O}_n^{\mathrm{crys}})$.

Proposition 4.4.1. (cf. [Fo-M] III 1.1, [Br3] 3.1.4). *The following sequence is exact for $r \geq 0$: $0 \longrightarrow S_n^r \longrightarrow J_n^{<r>} \xrightarrow{1-\varphi_r} \mathcal{O}_n^{\mathrm{crys}} \longrightarrow 0$.*

We have a natural product structure $S_n^r \otimes S_n^{r'} \to S_n^{r+r'}$. The presheaf $Y \mapsto \Gamma(Y, M_Y^{\mathrm{gp}})$ on $(X_n)_{\mathrm{syn}}$ is a sheaf, and we denote it by M_n^{gp}. We have a symbol map $M_{n+1}^{\mathrm{gp}} \to S_n^1[1]$ (in the derived category) (cf. [Fo-M] III 6.3).

(4.5) We define $H^q(X, S_n^r)$ to be $H^q((X_{n+s})_{\mathrm{syn}}, S_n^r)$ $(s \geq r)$ and $H^q(\overline{X}, S_n^r)$ to be the inductive limit of $H^q(X', S_n^r)$, where $X' = X \times_S S'$ with S' the log scheme associated to a finite extension K' of K contained in $\overline{K}$. We write the $\mathbb{Q}\otimes \varprojlim_n$ of these cohomology groups by the same symbols with S_n^r replaced by $S_{\mathbb{Q}_p}^r$ (cf. [Fo-M] III 1.2).

§5. The log syntomic-étale site.

We will give a log version of the theory of the syntomic-étale site in [Fo-M]. In this section, by a formal scheme, we mean a locally noetherian formal scheme locally of finite type over Spf (W).

(5.1) The notion of log structure is easily extended to formal schemes. We say that a morphism $f\colon \mathfrak{X} \to \mathfrak{Y}$ of fine log formal schemes is étale, smooth, syntomic, strictly syntomic and an exact closed immersion if, for every integer $n \geq 1$, its reduction mod p^n is étale, smooth, ... respectively. We say f is *étale on the generic fibers*, if étale locally on the underlying formal scheme of $\mathfrak{X}$, f has a factorization $\mathfrak{X} \xrightarrow{i} \mathfrak{Z} \xrightarrow{g} \mathfrak{Y}$ with $\mathfrak{Z}$ affine, i an exact closed immersion and g smooth such that $K_0 \otimes_W \Gamma(\mathfrak{X}, \mathcal{I}/\mathcal{I}^2) \to K_0 \otimes_W \Gamma(\mathfrak{X}, i^*\Omega^1_{\mathfrak{Z}/\mathfrak{Y}})$ is an isomorphism, where $\mathcal{I}$ is the ideal of $\mathcal{O}_{\mathfrak{Z}}$ defining $\mathfrak{X}$. We say f is *syntomic-étale* if it is strictly syntomic and étale on the generic fibers. For a fine log scheme $\mathfrak{X}$, we define the small syntomic-étale site $\mathfrak{X}_{\mathrm{sé}}$ using syntomic-étale morphisms. We define $X_{\mathrm{sé}}$ similarly for a fine log formal scheme X over W. These sites are functorial only as topologies in the sense of Artin [A].

(5.2) Let us return to the situation in (3.2). Let $\hat{X}$ denote the p-adic completion of X. Then we have the following commutative diagram of topos, where the subscript ét denotes the étale site of the underlying scheme or formal scheme.

$$\begin{tikzcd}
\hat{X}_{\mathrm{sé}}^{\sim} \arrow[r, "i_{\mathrm{sé}}"] \arrow[d, "\hat{\varepsilon}"] & X_{\mathrm{sé}}^{\sim} \arrow[d, "\varepsilon"] & (X_K)_{\mathrm{sé}}^{\sim} \arrow[l, "j_{\mathrm{sé}}"'] \arrow[d, "\varepsilon_K"] \\
Y_{\mathrm{ét}}^{\sim} = \hat{X}_{\mathrm{ét}}^{\sim} \arrow[r, "i_{\mathrm{ét}}"] & X_{\mathrm{ét}}^{\sim} & (X_K)_{\mathrm{ét}}^{\sim} \arrow[l, "j_{\mathrm{ét}}"']
\end{tikzcd}$$

Lemma 5.2.1. (cf. [Fo-M] III 4.1). *The direct image functor $i_{n*}\colon (X_n)_{\mathrm{syn}}^{\sim} \to \hat{X}_{\mathrm{sé}}^{\sim}$ is exact for any integer $n \geq 1$.*

Here we need the additional condition on log structures in the definition of strictly syntomic morphisms. We also denote by the same letter the direct image of S_n^r on $\hat{X}_{\text{sé}}$, whose cohomology coincides with $H^*(X, S_n^r)$.

PROPOSITION 5.2.2. (cf. [Fo-M] III 4.4). *The functor $\mathcal{F} \mapsto (i_{\text{sé}}^*\mathcal{F}, j_{\text{sé}}^*\mathcal{F}, i_{\text{sé}}^*\mathcal{F} \to i_{\text{sé}}^* j_{\text{sé}*} j_{\text{sé}}^*\mathcal{F})$ from the category of sheaves on $X_{\text{sé}}$ to the category of triples $(\mathcal{G}, \mathcal{H}, \mathcal{G} \to i_{\text{sé}}^* j_{\text{sé}*}\mathcal{H})$ where $\mathcal{G}$ (resp. $\mathcal{H}$) are sheaves on $\hat{X}_{\text{sé}}$ (resp. $(X_K)_{\text{sé}}$) is an equivalence of categories.*

Using A_{crys} of a sufficiently small $Y \in \text{Ob}(X_{\text{sé}})$ and the exact sequence [T1] A3.26 for this A_{crys}, we can construct a natural homomorphism $S_n^r \to i_{\text{sé}}^* j_{\text{sé}*} j'_* \mathbb{Z}/p^n\mathbb{Z}(r)'$ compatible with the product structures and with the symbol maps $M_{n+1}^{\text{gp}} \to S_n^1[1]$, $\mathcal{O}_{X_{\text{triv}}}^* \to \mathbb{Z}/p^n\mathbb{Z}(1)[1]$. Here $\mathbb{Z}/p^n\mathbb{Z}(r)' = (p^a a!)^{-1}\mathbb{Z}_p(r)/p^n$ ($r = (p-1)a + b$, $a, b \in \mathbb{Z}$, $0 \le b < p-1$) (cf. [Fo-M] III §5), X_{triv} is the locus on X_K where the log structure is trivial, and j' denotes $(X_{\text{triv}})^\sim_{\text{ét}} \to (X_K)^\sim_{\text{sé}}$. By Proposition 5.2.2, we can glue S_n^r and $j'_*\mathbb{Z}/p^n\mathbb{Z}(r)'$ by this homomorphism and obtain a sheaf $\mathcal{S}_n^r$ on $X_{\text{sé}}$.

PROPOSITION 5.2.3. *The base change morphism $i_{\text{ét}}^* R\varepsilon_* \to R\hat{\varepsilon}_* i_{\text{sé}}^*$ is an isomorphism.*

This is stated in [K-M] in the case of schemes, but its proof is not sufficient. We need to prove that, for an injective sheaf on $X_{\text{sé}}$, $i_{\text{sé}}^*\mathcal{F}$ is $R\hat{\varepsilon}_*$-acyclic and the étale sheafification of the pre-sheaf pull-back of $\mathcal{F}$ on $\hat{X}_{\text{sé}}$ is a syntomic-étale sheaf. From this proposition and the proper base change theorem for étale cohomology, we obtain a homomorphism $H^q(X, S_n^r) \to H^q_{\text{ét}}(X_{\text{triv}}, \mathbb{Z}/p^n\mathbb{Z}(r)')$. Taking the inductive limit with respect to the finite base changes of X, we obtain a homomorphism

$$H^q(\overline{X}, S_n^r) \longrightarrow H^q_{\text{ét}}((X_{\text{triv}})_{\overline{K}}, \mathbb{Z}/p^n\mathbb{Z}(r)') \tag{5.2.4}$$

compatible with the action of G_K. We can prove the following theorem by modifying the argument in [T1] slightly and using Proposition 4.4.1.

THEOREM 5.2.5. *If X is isomorphic to the finite base change of a proper semi-stable scheme endowed with the log structure defined by the special fiber. Then, for $q \le r$, there exists $N \ge 0$ such that the kernel and the cokernel of* (5.2.4) *are killed by p^N for every $n \ge 1$.*

(5.2.4) was proven to be an isomorphism if $r \le p-2$ by M. Kurihara and K. Kato before [T1]. In fact, we can prove the theorem without the assumption on X.

§6. COHOMOLOGY OF TRUNCATED SIMPLICIAL TOPOS.

(6.1) We can relax the condition (b) in the definition of D-topos in [SD] (1.2.1) as follows. For a bifibered category E over a $\mathcal{U}$-small category D whose fibers are $\mathcal{U}$-topos, $\underline{\Gamma}(E)$ is a $\mathcal{U}$-topos (cf. [SD] (1.2.12)), and, for a functor $f\colon D' \to D$ with D' $\mathcal{U}$-small, $f^*\colon \underline{\Gamma}(E) \to \underline{\Gamma}(D' \times_D E)$ has a left and a right adjoints (cf. [SD] (1.2.9)).

For a D-functor $\varphi_*: E \to E'$ induced by a cartesian D^0-functor $\psi: T' \to T$ of fibered categories over D° whose fibers are topologies in the sense of Artin [A] such that ψ_i $(i \in \mathrm{Ob}(D))$ and $m^*: T_i \to T_j$, $m^*: T'_i \to T'_j$ $(m: i \to j \in \mathrm{Mor}(D))$ are morphisms of topologies, one can calculate $R^+\underline{\Gamma}(\varphi_*)$ "fiber by fiber" (cf. [SD] (1.3.12)). These facts are necessary to apply [SD] to the syntomic and the syntomic-étale sites.

(6.2) Let Δ (resp. $\Delta[m]$) denotes the category of the ordered sets $[n] = \{0, \dots, n\}$ (resp. such that $n \leq m$) and the increasing maps. For a ringed topos $(S, \mathcal{O}_S)$, we can define the triangulated functor $R^+\varepsilon_{\cdot *}: D^+(\underline{\Gamma}(S \times \Delta[m]), \mathcal{O}_S) \to D^+(S, \mathcal{O}_S)$ by associating to $\Delta[m] \to C^+(S, \mathcal{O}_S); [n] \mapsto K^{\cdot n}$ the simple complex associated to $K^{\cdot 0} \to K^{\cdot 1} \to \dots \to K^{\cdot m} \to 0 \to \dots$ with $d^n = \sum_{0 \leq i \leq n+1} (-1)^i \partial^i$ $(n \leq m)$ (cf. [SD] (2.3.9)). By the filtration bête with respect to the second index, the above functor factors through $D^+F(S, \mathcal{O}_S)$ the derived category of filtered complexes (cf. [SD] (2.5.3)), and we have a spectral sequence

$$(6.2.1) \quad E_1^{a,b} = H^b(K^{\cdot a}) \quad (\text{if } 0 \leq a \leq m),\ 0 \quad (\text{otherwise}) \Longrightarrow H^{a+b}(R^+\varepsilon_{\cdot *}K^{\cdot\cdot})$$

For $K^\cdot \in D^+(\underline{\Gamma}(S \times \Delta), \mathcal{O}_S)$, we have a natural morphism $R^+\varepsilon_*(K^\cdot) \to R^+\varepsilon_{\cdot *}(L^+ i_m^* K^\cdot)$, which is a quasi-isomorphism in degree $\leq m-1$ if $H^q(K^\cdot) = 0$ $(q < 0)$. Here ε_* is as in [SD] (2.1.1) and i_m^* denotes the functor $\underline{\Gamma}(S \times \Delta) \to \underline{\Gamma}(S \times \Delta[m])$: the composite with $\Delta[m] \to \Delta$.

(6.3) Now we will discuss the simplicial version of (3.2). Let $X^\cdot \to S$ be an m-truncated simplicial fine log scheme whose components satisfy the assumption on X in (3.2). We denote by $R\Gamma^\cdot(Y^\cdot/\underline{W}_n)$ the derived direct image of the structure sheaf under the morphism of topos $(Y^\cdot/\underline{W}_n)^\sim_{\mathrm{crys}} \to \underline{\Gamma}((\mathrm{Sets}) \times \Delta[m])$ defined by taking global sections on each component. We define $R\Gamma^\cdot(X^\cdot_n/S_n)$ and $R\Gamma^\cdot(X^\cdot_n/E_n)$ similarly. Then, by generalizing the argument in [H-K], we can define φ and N satisfying $N\varphi = p\varphi N$ on the first and the last complexes and show, with the notation of [H-K](4.11), (4.12), that φ on $\mathbb{Q} \otimes \{R\Gamma^\cdot(Y^\cdot/\underline{W}_n)\}_n$ is an automorphism and that there exists a unique section compatible with φ of the pull-back by $\{i_{n,0}\}$ (3.1) :

$$(6.3.1) \qquad \mathbb{Q} \otimes \{R_{E_n} \otimes_{W_n} \{R\Gamma^\cdot(Y^\cdot/\underline{W}_n)\}_n \xrightarrow{\sim} \mathbb{Q} \otimes \{R\Gamma^\cdot(X^\cdot_n/E_n)\}_n$$

(cf. Theorem 3.2.1). (We can avoid to use an embedding system in the proof of [H-K] (2.24) by generalizing [Br3] (2.2.1.1), (2.2.1.2) and using the argument of [T2] 3.8.)

Define $\tilde{H}^q(Y^\cdot/\underline{W}_n)$ to be $H^q(R^+\varepsilon_{\cdot *}R\Gamma^\cdot(Y^\cdot/\underline{W}_n))$. Then, from (6.2.1), we obtain a spectral sequence converging to this cohomology such that $E_1^{a,b} = H^b(Y^a/\underline{W}_n)$ if $0 \leq a \leq m$ and 0 otherwise, which we denote by $\tilde{E}(Y^\cdot/\underline{W}_n)$. We define the cohomology $\tilde{H}^q$ and the spectral sequence $\tilde{E}$ in the same way for the other two complexes. Then φ and N and the isomorphism (6.3.1) for the complexes on $\underline{\Gamma}((\mathrm{Sets}) \times \Delta[m])$ induce those for the corresponding spectral sequences.

We define $\tilde{H}^q(Y^\cdot/\underline{W})_{\mathbb{Q}}$, $\tilde{E}(Y^\cdot/\underline{W})_{\mathbb{Q}}$ by taking $\mathbb{Q} \otimes \varprojlim_n$. Note that all terms of the spectral sequences $\tilde{E}(Y^\cdot/\underline{W}_n)$ are finitely generated over W_n. Then, every term of $\tilde{E}(Y^{\cdot}/\underline{W})_{\mathbb{Q}}$ is a (φ, N)-module of finite dimension. We define $\tilde{H}^q(X^{\cdot}/S)_{\mathbb{Q}}$ and $\tilde{E}(X^{\cdot}/S)_{\mathbb{Q}}$ in the same way. Then these are isomorphic to $\tilde{H}^q(X^{\cdot}_K, \Omega^{\cdot}_{X^{\cdot}_K})$ and $\tilde{E}(X^{\cdot}_K, \Omega^{\cdot}_{X^{\cdot}_K})$ (defined similarly as above) and hence $E_1^{a,b}$ and E_∞^c of the spectral sequence have the Hodge filtrations induced by $\sigma_{\geq i}\Omega^{\cdot}_{X^{\cdot}_K}$. We endow $E_r^{a,b}$ $(r \geq 2)$ with the filtration induced by that on $E_1^{a,b}$ as a sub-quotient. As in Theorem 3.2.1, (6.3.1) and the pull-back by $\{i_{n,\pi}\}$ (3.1) induce an isomorphism

$$\rho_\pi : K \otimes_{K_0} \tilde{E}(Y^{\cdot}/\underline{W})_{\mathbb{Q}} \xrightarrow{\sim} \tilde{E}(X^{\cdot}_K, \Omega^{\cdot}_{X^{\cdot}_K}). \tag{6.3.2}$$

Especially, each term of $\tilde{E}(Y^{\cdot}/\underline{W})_{\mathbb{Q}}$ has a natural filtered (φ, N)-module structure.

(6.4) For an m-truncated simplicial K-scheme $X^{\cdot}$, we define $R\Gamma^{\cdot}_{\text{ét}}(X^{\cdot}, \mathbb{Z}/p^n\mathbb{Z}(r))$, $\tilde{H}^q_{\text{ét}}(X^{\cdot}, \mathbb{Z}/p^n\mathbb{Z}(r))$ and $\tilde{E}_{\text{ét}}(X^{\cdot}, \mathbb{Z}/p^n\mathbb{Z}(r))$ in the same way as (6.3). If $X^{\cdot} \to X$ is an m-truncated proper hypercovering, by cohomological descent ([SD] (3.3.3), (4.3.2)) and the remark in the end of (6.2), we have an isomorphism for $q \leq m-1$:

$$H^q_{\text{ét}}(X, \mathbb{Z}/p^n\mathbb{Z}(r)) \cong H^q_{\text{ét}}(\operatorname{cosk}_m(X^{\cdot}), \mathbb{Z}/p^n\mathbb{Z}(r)) \cong \tilde{H}^q_{\text{ét}}(X^{\cdot}, \mathbb{Z}/p^n\mathbb{Z}(r)). \tag{6.4.1}$$

§7. p-adic Hodge theory for truncated simplicial proper semi-stable schemes.

(7.1) Let $X^{\cdot} \to S$ be an m-truncated simplicial fine log formal scheme over S such that X^i $(i \leq m)$ is isomorphic to the finite base change of a proper semi-stable scheme endowed with the log structure defined by the special fiber. The result of (6.3) is applicable to $X^{\cdot}$. Set $\tilde{H}^q_{\text{ét}}(X^{\cdot}_{\overline{K}}, \mathbb{Q}_p) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n \tilde{H}^q_{\text{ét}}(X^{\cdot}_{\overline{K}}, \mathbb{Z}/p^n\mathbb{Z})$. In this section, we give an outline of the proof of the following theorem.

THEOREM 7.1.1. *The étale cohomology $\tilde{H}^q_{\text{ét}}(X^{\cdot}_{\overline{K}}, \mathbb{Q}_p)$ with the action of G_K is semi-stable and there exists a canonical isomorphism of filtered (φ, N)-modules: $D_{\text{st}}(\tilde{H}^q(X^{\cdot}_{\overline{K}}, \mathbb{Q}_p)) \cong \tilde{H}^q(Y^{\cdot}/\underline{W})_{\mathbb{Q}}$.*

(7.2) We can define the simplicial analogue $\tilde{H}^q(\overline{X}^{\cdot}, S_n^r)$, $\tilde{H}^q(\overline{X}^{\cdot}, S_{\mathbb{Q}_p}^r)$ of $H^q(\overline{X}, S_n^r)$ and $H^q(\overline{X}, S_{\mathbb{Q}_p}^r)$, and a spectral sequence $\tilde{E}(\overline{X}^{\cdot}, S_n^r)$ converging to $\tilde{H}^q(\overline{X}^{\cdot}, S_n^r)$ in the same way as (6.3). Here we use (6.1). Let d be the maximum of $\dim(X^i_K)$ $(0 \leq i \leq m)$. Then, for $r \geq 2d$, by §5, we have a canonical morphism $\tilde{E}(\overline{X}^{\cdot}, S_n^r) \to \tilde{E}_{\text{ét}}(X^{\cdot}_{\overline{K}}, \mathbb{Z}/p^n\mathbb{Z}(r)')$ whose kernel and cokernel are killed by a p^N independent of $n \geq 1$. Hence $\tilde{E}(\overline{X}^{\cdot}, S_{\mathbb{Q}_p}^r) := \mathbb{Q} \otimes \varprojlim_n \tilde{E}(\overline{X}^{\cdot}, S_n^r)$ is well-defined and $\tilde{E}(\overline{X}^{\cdot}, S_{\mathbb{Q}_p}^r) \cong \tilde{E}_{\text{ét}}(X^{\cdot}_{\overline{K}}, \mathbb{Q}_p(r))$.

(7.3) On the other hand, similarly as [K], [T1] §4, we have natural morphisms of spectral sequences

$$\tilde{E}(\overline{X}^{\cdot}, S_n^r) \longrightarrow \tilde{E}(\overline{X}^{\cdot}_n/W_n) \longrightarrow \tilde{E}(\overline{X}^{\cdot}_n/E_n) \cong \Gamma(\overline{S}_n/E_n) \otimes_{R_{E_n}} \tilde{E}(X^{\cdot}_n/E_n)$$

and, using (6.3.1) and [K] (3.7), we obtain a morphism $\tilde{E}(\overline{X}^{\cdot}, S_{\mathbb{Q}_p}^r) \to B_{\text{st}}^+ \otimes_{K_0} \tilde{E}(Y^{\cdot}/\underline{W})_{\mathbb{Q}}$ whose image is contained in the part where $N = 0$ and $\varphi = p^r$.

(7.4) Using $\mathbb{Q}_p(-r) \hookrightarrow B_{\mathrm{st}}$, we obtain from (7.2) and (7.3) a morphism of spectral sequences

$$B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} \tilde{E}_{\text{ét}}(X^{\cdot}_{\overline{K}}, \mathbb{Q}_p) \longrightarrow B_{\mathrm{st}} \otimes_{K_0} \tilde{E}(Y^{\cdot}/\underline{W})_{\mathbb{Q}} \tag{7.4.1}$$

compatible with the action of G_K, φ and N. The E_1-term of (7.4.1) is nothing but the comparison map of C_{st} obtained by using (5.2.4) ([K] §6, [T1] §4). Hence (7.4.1) is an isomorphism. Now it remains to prove that this induces a filtered isomorphism after $B_{\mathrm{dR}} \otimes_{B_{\mathrm{st}}}$.

(7.5) A morphism between admissible filtered (φ, N)-modules is strictly compatible with the filtrations and the kernel and the cokernel are again admissible. Hence, by using [D1] (1.3.13), (1.3.16) and the fact that the $E_1^{a,b}$-term of $\tilde{E}(Y^{\cdot}/\underline{W})_{\mathbb{Q}}$ is admissible, we see, by induction on r, that $E_r^{a,b}$ are admissible and $d_r^{a,b}$ are strictly compatible with the filtrations. By [D1] (1.3.17), the filtration on $E_\infty^{a,b}$ coincides with the filtration induced by that on $\tilde{H}^{a+b}(Y^{\cdot}/\underline{W})_{\mathbb{Q}}$, and, by [D2] (7.2.8), the Hodge spectral sequence for $\tilde{H}^*(X^{\cdot}_K, \Omega^{\cdot}_{X^{\cdot}_K})$ degenerates. From the last fact, we obtain an isomorphism

$$\tilde{H}^q(\overline{X}^{\cdot}/S, J^{[r]})_{\mathbb{Q}} \cong Fil^r(B^+_{\mathrm{dR}} \otimes_K \tilde{H}^q(X^{\cdot}_K, \Omega^{\cdot}_{X^{\cdot}_K}))) \tag{7.5.1}$$

in the same way as [T1] 4.7. This implies that the E_∞^c-term of (7.4.1) is compatible with the filtrations (cf. [T1] 4.8.5) after $B_{\mathrm{dR}} \otimes_{B_{\mathrm{st}}}$. Now the claim follows from the fact that (7.4.1) is a filtered isomorphism in the E_1-term after $B_{\mathrm{dR}} \otimes_{B_{\mathrm{st}}}$.

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