The Abelian Defect Group Conjecture

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Abstract. Let G be a finite group and k an algebraically closed field of characteristic $p > 0$. If B is a block of the group algebra kG with defect group D, the Brauer correspondent of B is a block b of $kN_G(D)$. When D is abelian, the blocks B and b, although they are rarely isomorphic or even Morita equivalent, seem to be very closely related. For example, Alperin's Weight Conjecture predicts that they should have the same number of simple modules. Broué's Abelian Defect Group Conjecture gives a more precise prediction of the relationship between B and b: their module categories should have equivalent derived categories. In this article we survey this conjecture, some of its consequences, and some of the recent progress that has been made in verifying it in special cases.

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1 Notation and terminology

Throughout this article, G will denote a finite group.

We shall be dealing with the characteristic p representation theory of G, where p is a prime. We shall use three coefficient rings. The ring $\mathcal{O}$ will be a complete discrete valuation ring with residue field k of characteristic p and field of fractions K of characteristic zero. Since we shall not be concerned with rationality questions, we shall assume that these coefficient rings are all 'large enough' in that they contain enough roots of unity.

As well as the group algebras $\mathcal{O}G$, kG and KG, we shall be concerned with various direct factors. We shall choose our notation so that if we denote an $\mathcal{O}$-algebra by $\mathcal{O}A$, we shall use the notation kA and KA for $\mathcal{O}A \otimes_{\mathcal{O}} k$ and $\mathcal{O}A \otimes_{\mathcal{O}} K$ respectively. It is well-known that the natural surjection $\mathcal{O}G \to kG$ induces a bijection between the primitive central idempotents of $\mathcal{O}G$ and kG, so that if

$$\mathcal{O}G \cong \mathcal{O}A_1 \times \cdots \times \mathcal{O}A_n,$$

where $\mathcal{O}A_1, \ldots, \mathcal{O}A_n$ are the blocks (i.e., the minimal direct factors) of $\mathcal{O}G$, then

$$kG \cong kA_1 \times \cdots \times kA_n,$$

where $kA_1, \ldots, kA_n$ are the blocks of kG.

Here we shall only be concerned with finitely-generated modules, and if R is any ring, we shall denote the category of finitely-generated right R-modules by $\mathrm{mod}(R)$. By a 'module' for a ring R we shall always mean a right module.

2 Local representation theory and Alperin's Weight Conjecture

There are two major themes running through much of the modular representation theory of general finite groups that relate the representation theory of a group G to that of smaller groups.

The first of these is Clifford theory, which relates the representation theory of a group G with a normal subgroup N to that of N and G/N. This is an area that has been studied systematically and intensively, but we shall not say much about it here.

The second theme is sometimes known as local representation theory, which describes the relationship between the representation theory of G and of *local subgroups*: normalizers of non-trivial p-subgroups in G. This relationship has been exploited in a more ad hoc way than Clifford theory; this is partly because, as we shall see, the precise relationship is unclear or at best conjectural.

One classical example of a theorem of local representation theory is Brauer's First Main Theorem. Recall that the *defect group* of a block kA of kG is a minimal subgroup D of G such that every kA-module is a direct summand of a module induced from kD. The defect group is always a p-subgroup and is determined uniquely up to conjugacy in G.

THEOREM 2.1 (BRAUER'S FIRST MAIN THEOREM) *If D is a p-subgroup of G, there is a natural bijection between the blocks of kG with defect group D and the blocks of $kN_G(D)$ with defect group D.*

The block of $kN_G(D)$ corresponding to a block kA of kG with defect group D is called the *Brauer correspondent* of kA. The *principal block* of kG (i.e., the unique block that is not contained in the augmentation ideal of kG) has a Sylow p-subgroup P of G as its defect group, and its Brauer correspondent is the principal block of $kN_G(P)$.

The most famous example of a general conjecture in local representation theory is the following, due to Alperin [A], known as Alperin's Weight Conjecture, which has inspired a great deal of interest since it was formulated in the 1980s.

CONJECTURE 2.2 (ALPERIN'S WEIGHT CONJECTURE) *The number of isomorphism classes of simple kG-modules is equal to the number of pairs (Q,S), where Q runs over a set of representatives of conjugacy classes of p-subgroups of G and, for each Q, S runs over a set of representatives of isomorphism classes of simple projective $k[N_G(Q)/Q]$-modules.*

If we ignore the simple kG-modules that are projective, this conjecture claims that the number of non-projective simple kG-modules is equal to the number of pairs (Q,S) where Q is a non-trivial p-subgroup of G. In other words, it claims that the number of non-projective simple kG-modules is 'locally determined' (i.e., determined by local subgroups) in a precise fashion.

There is a more precise 'blockwise' version [A] of Alperin's Weight Conjecture, dealing with the number of simple modules for a single block of kG in terms of suitable blocks of local subgroups. We shall not state the general conjecture here, but only the special case for a block with abelian defect group, which has a particularly simple form.

CONJECTURE 2.3 *Let kA be a block of kG with an abelian defect group D. Then kA and its Brauer correspondent have the same number of isomorphism classes of simple modules.*

For principal blocks, this has the following special case.

CONJECTURE 2.4 *Suppose G has an abelian Sylow p-subgroup P. Then the principal blocks of kG and $kN_G(P)$ have the same number of isomorphism classes of simple modules.*

3 BROUÉ'S ABELIAN DEFECT GROUP CONJECTURE

Since Alperin's Weight Conjecture reduces, for a block with abelian defect group, to the claim that the block and its Brauer correspondent have the same number of simple modules, it is natural to wonder whether there is some structural relationship between the two blocks that explains this. It is certainly not true in general that the blocks are isomorphic or even Morita equivalent. Broué [B] conjectured such a relationship in terms of derived categories.

There are now several accessible introductions to the theory of derived categories, such as the one contained in Weibel's book [W]. If R is a noetherian ring, we shall denote by $D^b(R)$ the bounded derived category of $\text{mod}(R)$. Recall that the objects of $D^b(R)$ are the chain complexes of finitely-generated R-modules with only finitely many non-zero terms. As usual we shall think of $\text{mod}(R)$ as embedded in $D^b(R)$ by identifying an R-module M with the complex whose only non-zero term is M in degree zero. The morphisms of $D^b(R)$ are obtained from the chain maps by formally adjoining inverses to all chain maps that induce isomorphisms in homology. Recall finally that $D^b(R)$ has the structure of a 'triangulated category': in particular, for each object X of $D^b(R)$ we can form an object $X[n]$ for $n \in \mathbb{Z}$ by shifting the complex X to the left by n places.

CONJECTURE 3.1 (BROUÉ'S ABELIAN DEFECT GROUP CONJECTURE) *Let $\mathcal{O}A$ be a block of $\mathcal{O}G$ with abelian defect group D and let $\mathcal{O}B$ be its Brauer correspondent (hence a block of $\mathcal{O}N_G(D)$). Then $D^b(\mathcal{O}A)$ and $D^b(\mathcal{O}B)$ are equivalent as triangulated categories.*

If R and S are noetherian rings such that $D^b(R)$ and $D^b(S)$ are equivalent as triangulated categories, we say that R and S are *derived equivalent.* Derived equivalence is clearly implied by Morita equivalence, but the converse is not true.

We have stated the conjecture over $\mathcal{O}$; it is not hard to prove that this implies the corresponding statement over k.

The Grothendieck group $K_0(\mathcal{T})$ of a triangulated category $\mathcal{T}$ can be defined [G] in a similar way to that of an abelian category, and if $\mathcal{T} = D^b(R)$ for some finite-dimensional k-algebra R, then $K_0(\mathcal{T})$ is a free abelian group whose rank is equal to the number of isomorphism classes of simple R-modules. Hence the Abelian Defect Group Conjecture implies the blockwise version of the Weight Conjecture for blocks with abelian defect group. An important open problem is to formulate a generalization of Broué's conjecture that would imply the Weight Conjecture for a general block.

4 PROVING DERIVED EQUIVALENCE

Given two rings R and S, how does one go about proving that they are derived equivalent? I shall assume that either R and S are both finite-dimensional k-algebras or they are both $\mathcal{O}$-free $\mathcal{O}$-algebras of finite rank over $\mathcal{O}$, although everything that follows applies much more generally.

Most of the classical theory of Morita equivalence has generalizations to derived equivalence.

Recall first that R and S have equivalent module categories if and only if S is isomorphic to the endomorphism algebra of a finitely-generated projective generator for R. This has the following analogue [R1] for derived equivalence.

THEOREM 4.1 *R and S are derived equivalent if and only if S is isomorphic to the endomorphism algebra, in $D^b(R)$, of an object T such that*

(i) T is a bounded complex of finitely-generated projective R-modules,

(ii) $\mathrm{Hom}_{D^b(R)}(T, T[i]) = 0$ for $i \neq 0$, and

(iii) If X is an object of $D^b(R)$ such that $\mathrm{Hom}_{D^b(R)}(T, X[i]) = 0$ for all $i \in \mathbb{Z}$, then $X \cong 0$.

An object T satisfying conditions (i) to (iii) of the theorem is called a (one-sided) *tilting complex*. Condition (iii) has equivalent forms that are easier to check directly in practice.

Another well-known criterion for R and S to be Morita equivalent is that there should be an R-S-bimodule X and an S-R-bimodule Y (which is in fact isomorphic to $\mathrm{Hom}_R(X, R)$) such that X and Y are finitely-generated and projective as right modules and as left modules (but not usually projective as bimodules) and such that $X \otimes_S Y \cong R$ and $Y \otimes_R X \cong S$ as bimodules. Then the functor $? \otimes_R X$ is an equivalence of module categories. This also has an analogy for derived categories, first proved in [R2] but with a better subsequent proof by Keller [K].

THEOREM 4.2 *R and S are derived equivalent if and only if there is a bounded complex X of R-S-bimodules and a bounded complex $Y = \mathrm{Hom}_R(X, R)$ of S-R-bimodules such that*

(i) All the terms of X and Y are finitely-generated and projective as left modules and as right modules,

(ii) As a complex of R-bimodules, $X \otimes_S Y \cong R \oplus C$ for some acyclic complex C, and

(iii) As a complex of S-bimodules, $Y \otimes_R X \cong S \oplus C'$ for some acyclic complex C'.

A complex X that satisfies the conditions of the theorem is called a *two-sided tilting complex*. If we forget the left action of S on Y, then Y becomes a one-sided tilting complex for R.

If X is a two-sided tilting complex, then the functor

$$? \otimes_R X : D^b(R) \longrightarrow D^b(S)$$

is an equivalence of triangulated categories.

As we shall see in Section 6, a two-sided tilting complex seems to be the natural object to seek in order to prove the Abelian Defect Group Conjecture, although in small examples it is easier to do calculations with one-sided tilting complexes.

5 Character-theoretic consequences

If $\mathcal{O}A$ and $\mathcal{O}B$ are derived equivalent blocks and X is a two-sided tilting complex, then it is easy to check that $X \otimes_{\mathcal{O}} K$ is also a two-sided tilting complex for the semisimple algebras KA and KB.

The Grothendieck group of $D^b(KA)$ can be naturally identified with the group $K_0(KA)$ of generalized characters of KA, so X induces an isomorphism

$$\theta : K_0(KA) \cong K_0(KB).$$

The indecomposable objects of $D^b(R)$ for a semisimple K-algebra R are all of the form $M[i]$ for some irreducible R-module M and some integer i. It follows that θ maps each irreducible character χ of KA to $\pm\phi$ for some irreducible character ϕ of KB: in other words, θ is an isometry. Since the functors $? \otimes_R X$ and $? \otimes_S \mathrm{Hom}_R(X, R)$ take projective modules to complexes of projective modules, it follows that θ restricts to an isomorphism

$$\theta_p : K_{0,p}(KA) \longrightarrow K_{0,p}(KB)$$

between the subgroups of the groups of generalized characters generated by the characters of projective modules for $\mathcal{O}A$ and $\mathcal{O}B$. Such an isometry is called a *perfect isometry* by Broué [B] and can be characterized in terms of arithmetic properties of character values.

A consequence of the Abelian Defect Group Conjecture is therefore the following weaker character-theoretic conjecture, which is however still strong enough to imply the Weight Conjecture for blocks with abelian defect group.

Conjecture 5.1 *If $\mathcal{O}A$ is a block of $\mathcal{O}G$ with abelian defect group and with Brauer correspondent $\mathcal{O}B$, then there is a perfect isometry*

$$K_0(KA) \longrightarrow K_0(KB).$$

It is easier to perform calculations with characters than with derived categories, and so it is no surprise that this weaker conjecture has been verified in many more cases than the Abelian Defect Group Conjecture. One of the most impressive examples is the following, proved by Fong and Harris [FH] using the classification of finite simple groups. In fact, they proved an even stronger character-theoretic statement.

THEOREM 5.2 (FONG, HARRIS) *If $p = 2$ and G has an abelian Sylow p-subgroup P, there is a perfect isometry between the principal blocks of $\mathcal{O}G$ and $\mathcal{O}N_G(P)$.*

6 SPLENDID EQUIVALENCES

Here we shall briefly summarize some of the main results of [R3], giving some extra conditions that the two-sided tilting complexes predicted by the Abelian Defect Group Conjecture are expected to satisfy. We shall restrict our attention to the case of principal blocks, although Harris [H] and Puig [P] have given generalizations to non-principal blocks.

Suppose then that G has an abelian Sylow p-subgroup and that $\mathcal{O}A$ and $\mathcal{O}B$ are the principal blocks of $\mathcal{O}G$ and $\mathcal{O}N_G(P)$ respectively. We can consider a two-sided tilting complex X for $\mathcal{O}A$ and $\mathcal{O}B$ as a complex of $\mathcal{O}[G \times N_G(P)]$-modules. We say that X is a *splendid* tilting complex if it satisfies the following conditions, where as before $Y = \mathrm{Hom}_{\mathcal{O}A}(X, \mathcal{O}A)$.

- X is a complex of $\mathcal{O}[G \times N_G(P)]$-modules whose restrictions to $\mathcal{O}[P \times P]$ are permutation modules of the form $\mathcal{O}\Omega$, where the point stabilizers of Ω are conjugate to subgroups of the diagonal embedding of P in $P \times P$.
- $X \otimes_{\mathcal{O}B} Y \cong \mathcal{O}A \oplus C$, where C is a contractible complex of $\mathcal{O}A$-bimodules.
- $Y \otimes_{\mathcal{O}A} X \cong \mathcal{O}B \oplus C'$, where C' is a contractible complex of $\mathcal{O}B$-bimodules.

A derived equivalence induced by a splendid tilting complex is called a *splendid equivalence*.

Of course, we can make a similar definition over k. The second and third conditions are of course stronger than the conditions in the definition of a two-sided tilting complex, where C and C' were only required to be acyclic. Known examples suggest that Broué's Abelian Defect Group Conjecture should still be true if we require the derived equivalences it predicts to be splendid.

The main property that motivates the introduction of the idea of splendid equivalence is given in the next theorem [R3].

THEOREM 6.1 *If G has an abelian Sylow p-subgroup P and there is a splendid equivalence between the principal blocks of $\mathcal{O}G$ and $N = \mathcal{O}N_G(P)$, then for each subgroup $Q \leq P$ there is a splendid equivalence between the principal blocks of $\mathcal{O}C_G(Q)$ and $\mathcal{O}C_N(Q)$.*

In fact, a more precise statement can be made about the relationship between the two perfect isometries induced by the splendid equivalences: a splendid equivalence induces what Broué [B] calls an 'isotypy': a compatible family of perfect isometries between principal blocks of $\mathcal{O}C_G(Q)$ and $\mathcal{O}C_N(Q)$, one for each subgroup $Q \leq P$.

7 RECENT PROGRESS IN VERIFYING THE ABELIAN DEFECT GROUP CONJECTURE

A complete proof of the conjecture still seems a long way off. For several years after Broué formulated the conjecture, it could only be proved for fairly simple blocks, such as those with cyclic defect group, where a lot was known about the precise structure of the blocks. However, in the last few years there has been significant progress in developing techniques to verify it for particular groups.

The most complex infinite family of examples for which the conjecture has been verified is given by the following theorem [C].

THEOREM 7.1 (CHUANG) *The Abelian Defect Group Conjecture is true for all blocks of symmetric groups whose defect group has order p^2. Moreover, the derived equivalence is splendid.*

In particular, Chuang's theorem proves the conjecture for all blocks of the symmetric group S_n if $n < 3p$.

Consider, for simplicity, the principal block of kG, where G has an abelian Sylow p-subgroup P. The main obstacle to performing calculations to verify the Abelian Defect Group Conjecture in this case has been that the precise structure of the projective kG-modules is hard to calculate for all but the simplest examples, so it is hard to calculate one-sided tilting complexes. In contrast, the structure of projective $kN_G(P)$-modules is relatively easy to understand. In as yet unpublished work, Okuyama has introduced an ingenious technique, based on a theorem of Linckelmann [L], that allows him to verify the conjecture for several groups G without knowing the precise structure of the projective kG-modules. In fact, as a byproduct of his verifications, it is possible to calculate the structure of these modules.

Here are a few examples of the cases that Okuyama has settled.

THEOREM 7.2 (OKUYAMA, 1997) *For $p = 3$, the Abelian Defect Group Conjecture is true for the principal blocks of the groups $M_{11}, M_{21}, M_{22}, M_{23}$ and HS.*

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