Tilting Theory and Quasitilted Algebras

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Introduction

Tilting theory is a central topic in the representation theory of artin algebras, with origins in work of Bernstein–Gelfand–Ponomarev from the early seventies. There has been extensive interaction with various research directions in representation theory, as well as in other branches of algebra. In this paper we survey the development of tilting theory, and in particular we discuss quasitilted algebras, a recent outgrowth of tilting theory.

We consider for simplicity finite dimensional algebras over an algebraically closed field k, and we will often just say that Λ is an algebra. We deal with the category $\operatorname{mod}\Lambda$ of finitely generated Λ-modules. A Λ-module T of projective dimension at most one is a tilting module if $\operatorname{Ext}^1_\Lambda(T,T)=0$ and there is an exact sequence $0\to\Lambda\to T_0\to T_1\to 0$, where T_0 and T_1 are summands of finite direct sums of copies of T. In Section 1 we give some basic properties of such tilting modules. This includes associated torsion pairs together with induced equivalences of subcategories of $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Gamma$ belonging to the torsion pairs, where Γ is the endomorphism algebra $\operatorname{End}_\Lambda(T)^{\mathrm{op}}$ [BB, HRi]. We also discuss predecessors of the theory [BGP, APR].

The material included in Section 1 was developed around 1980. In Section 2 we treat three main lines of further developments. The first two go via a generalization to tilting modules of finite projective dimension [M, H1]. One direction is concerned with the correspondence between tilting modules and a certain type of subcategories of $\operatorname{mod}\Lambda$ [AR]. The second line of development goes via the discovery of the connection with derived categories [H1]. In the third direction a tilting theory with respect to torsion pairs in abelian categories is developed [HRS].

When Λ is hereditary, the algebras $\Gamma=\operatorname{End}_\Lambda(T)^{\mathrm{op}}$, where T is a tilting Λ-module, are by definition the tilted algebras. This is an important class of finite dimensional algebras, since many questions about arbitrary algebras can be reduced to questions on tilted algebras. As a by-product of the general theory of tilting with respect to torsion pairs, the quasitilted algebras are introduced in [HRS], as a generalization of tilted algebras. Central properties of this class of algebras, which also contains the canonical algebras of Ringel [Rin1], are discussed in Section 4.

The quasitilted algebras are defined in terms of tilting objects in hereditary abelian $k-$categories $\mathcal{H}$ with finite dimensional homomorphism and extension spaces over the algebraically closed field k. The last two sections are devoted to investigating such categories $\mathcal{H}$ with tilting objects, mainly motivated by wanting to obtain information on quasitilted algebras. In Section 5 we deal with the noetherian case. It is proved in [L] that the noetherian $\mathcal{H}$ are exactly $\operatorname{mod} H$ for a finite dimensional hereditary k-algebra H and the categories $\operatorname{coh} \mathcal{X}$ of coherent sheaves on weighted projective lines introduced in [GL]. We also investigate the relationship with the problem of when the Grothendieck group of $\mathcal{H}$ is free abelian of finite rank [RV2].

In Section 6 we deal with the question of what the hereditary abelian categories with tilting object $\mathcal{H}$ look like in general. It is conjectured that $\mathcal{H}$ (connected) must be derived equivalent to one of the categories $\operatorname{mod} H$ or $\operatorname{coh} \mathcal{X}$ above, and we prove this conjecture when there is at least one simple object and also when there is a directing object [HRe3, HRe2]. We end with a discussion of related problems about quasitilted algebras.

Due to limited space, several important results and developments related to tilting theory and quasitilted algebras are not included. For additional references we refer to the bibliography in the cited papers.

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1 Classical tilting theory and historical predecessors

Let Λ be a finite dimensional algebra. In this section T is a tilting Λ-module of projective dimension at most one. We shall investigate various subcategories associated with T, along with induced equivalences of subcategories of $\operatorname{mod} \Lambda$ and $\operatorname{mod} \operatorname{End}_\Lambda(T)^{\mathrm{op}}$.

The subcategory $\mathcal{T} = \operatorname{Fac} T$ of $\operatorname{mod} \Lambda$ plays an important role in the theory, where the objects of $\operatorname{Fac} T$ are the factors of finite direct sums of copies of T. Under our assumptions, one can show that $\mathcal{T}$ is equal to $\{ C; \operatorname{Ext}^1_\Lambda(T, C) = 0 \}$, and the category $\mathcal{T}$ is a *torsion class* in $\operatorname{mod} \Lambda$, that is, $\mathcal{T}$ is closed under factor modules and extensions. Associated with $\mathcal{T}$ is the *torsion free* class $\mathcal{F} = \{ C; \operatorname{Hom}_\Lambda(T, C) = 0 \}$, and $(\mathcal{T}, \mathcal{F})$ is a *torsion pair* associated with T. Dually, when U is a cotilting module of injective dimension at most one, that is, the dual $D(U)$ of U is a tilting module of projective dimension at most one in $\operatorname{mod} \Lambda^{\mathrm{op}}$, where D denotes the duality $\operatorname{Hom}_k(\ , k)$, there is associated with U the torsion free class $\mathcal{Y} = \operatorname{Sub} U = \{ C; \operatorname{Ext}^1_\Lambda(C, U) = 0 \}$. The objects of $\operatorname{Sub} U$ are submodules of finite direct sums of copies of U. Then there is an associated torsion pair $(\mathcal{X}, \mathcal{Y})$, where $\mathcal{X} = \{ C; \operatorname{Hom}_\Lambda(C, U) = 0 \}$.

A basic feature of tilting theory is the interplay between $\operatorname{mod} \Lambda$ and $\operatorname{mod} \Gamma$, when $\Gamma = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$. When T is a tilting Λ-module, T is also a tilting Γ^{op}-module, and hence $D(T)$ is a cotilting Γ-module. If $(\mathcal{T}, \mathcal{F})$ denotes the torsion pair in $\operatorname{mod} \Lambda$ associated with T and $(\mathcal{X}, \mathcal{Y})$ the torsion pair in $\operatorname{mod} \Gamma$ associated with $U = D(T)$, there are induced equivalences of categories $\operatorname{Hom}_\Lambda(T, \ ) \colon \mathcal{T} \to \mathcal{Y}$ and $\operatorname{Ext}^1_\Lambda(T, \ ) \colon \mathcal{F} \to \mathcal{X}$. This gives the possibility of transforming information between $\operatorname{mod} \Lambda$ and $\operatorname{mod} \Gamma$ in case one of the module categories is better known than the other one. This point of view has been particularly successful when one of the algebras, say Λ, is hereditary, so that Γ is a tilted algebra. Then the torsion theory $(\mathcal{X}, \mathcal{Y})$ *splits*, that is, each indecomposable object in $\operatorname{mod} \Gamma$ is in $\mathcal{X}$ or in $\mathcal{Y}$.

It is an important property of a tilting Λ-module T that $\mathrm{Hom}_\Lambda(T,\ )$ induces an equivalence between $\mathcal{T}$ and $\mathcal{Y}$. If conversely there is an equivalence between subcategories of $\mathrm{mod}\,\Lambda$ and $\mathrm{mod}\,\Gamma$, for two algebras Λ and Γ, one can ask if there is an associated tilting module T such that $\mathrm{Hom}_\Lambda(T,\ )$ (or $\mathrm{Ext}^1_\Lambda(T,\ )$) induces the given equivalence. Actually, the origin of tilting theory comes from [BGP], through the occurrence of an interesting equivalence of subcategories for two module categories, in the setting of representations of quivers. This equivalence was interpreted module theoretically in [APR] as $\mathrm{Hom}_\Lambda(T,\ )$ for a special type of what is now called a tilting module T, and extended to more general settings. Further generalizations were made in [BB, HRi], leading to the foundations of the classical tilting theory, with basic setup as discussed above.

2 Three lines of further developments

We discuss three not entirely independent directions of further developments. The first two go via a generalization of tilting and cotilting modules, dropping the requirements that the projective (or injective) dimension is at most one.

A module T in $\mathrm{mod}\,\Lambda$ for an algebra Λ is a tilting module if $\mathrm{pd}_\Lambda T$ (the projective dimension of T) is finite, $\mathrm{Ext}^i_\Lambda(T,T)=0$ for $i>0$, and there is an exact sequence of Λ-modules $0\to\Lambda\to T_0\to T_1\to\cdots\to T_r\to 0$ with each T_i a summand of a finite direct sum of copies of T (see [M, H1]). A cotilting module is defined dually. Let $T=T^{(1)}\oplus\cdots\oplus T^{(m)}$ be a direct sum of indecomposable modules. We say that the tilting module T is basic if the $T^{(i)}$ are pairwise nonisomorphic, and in this case m is the rank of the Grothendieck group $K_0(\mathrm{mod}\,\Lambda)$.

The first direction we deal with is only concerned with modules over Λ. We have already seen that associated with a tilting module T of projective dimension at most one is the subcategory $\mathcal{T}=\{\,C;\mathrm{Ext}^1_\Lambda(T,C)=0\,\}$ of $\mathrm{mod}\,\Lambda$, and more generally we associate $\mathcal{T}=\{\,C;\mathrm{Ext}^i_\Lambda(T,C)=0 \text{ for } i>0\,\}$ with an arbitrary tilting module T, and dually $\mathcal{Y}=\{\,B;\mathrm{Ext}^i_\Lambda(B,U)=0 \text{ for } i>0\,\}$ with a cotilting module U. In order to formulate the crucial properties of $\mathcal{T}$ and $\mathcal{Y}$ we recall some important terminology. A full subcategory $\mathcal{C}$ of $\mathrm{mod}\,\Lambda$ is *covariantly finite* in $\mathrm{mod}\,\Lambda$ if for each X in $\mathrm{mod}\,\Lambda$ there is a map $g\colon X\to C$ with C in $\mathcal{C}$ such that for any map $h\colon X\to C'$ with C' in $\mathcal{C}$, there is a map $t\colon C\to C'$ with $tg=h$ [ASm1]. Further, $\mathcal{C}$ is *coresolving* if it is closed under extensions and cokernels of monomorphisms. The notions of *contravariantly finite* and *resolving* subcategories are defined dually. For simplicity we only give the main results for finite global dimension, in which case the notions of tilting and cotilting module coincide [AR]. The case of projective (or injective) dimension at most one was already done in [ASm2].

Theorem 1. *Let Λ be an algebra of finite global dimension, and let T be in $\mathrm{mod}\,\Lambda$.*

(a) The assignment $T\mapsto\mathcal{T}=\{\,C;\mathrm{Ext}^i_\Lambda(T,C)=0 \text{ for } i>0\}$ induces a one-one correspondence between basic tilting modules and covariantly finite coresolving subcategories of $\mathrm{mod}\,\Lambda$. The module T is reconstructed via taking Ext-projective objects in $\mathcal{T}$.

(b) *The assignment $U \mapsto \mathcal{Y} = \{\, C; \operatorname{Ext}^i(C,U) = 0$ for $i > 0 \,\}$ induces a one-one correspondence between basic (co-)tilting modules and contravariantly finite resolving subcategories of* $\operatorname{mod}\Lambda$. *The module T is reconstructed via taking* Ext*-injective objects in $\mathcal{Y}$.*

The other two directions are concerned with the interplay between Λ and $\Gamma = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$, where T is a tilting module, including induced equivalences between subcategories. A major breakthrough was the discovery of the connection with derived categories [H1]. We cite the following [H1, CPS].

Theorem 2. *Let T be a tilting module over an algebra Λ. The derived functor $R\operatorname{Hom}(T,\ )$ induced by $\operatorname{Hom}_\Lambda(T,\ )\colon \operatorname{mod}\Lambda \to \operatorname{mod}\Gamma$ gives an equivalence $D^b(\Lambda) \to D^b(\Gamma)$ between the bounded derived categories for Λ and Γ if (and only if) T is a tilting module.*

Subsequently, dealing with the category of coherent sheaves on a weighted projective space, a similar result was obtained in [GL, Ba], introducing a notion of tilting sheaf analogous to the notion of tilting module. In this formulation, a previous result from [Be] on establishing a derived equivalence between the category $\operatorname{coh}\mathbb{P}^n$ of coherent sheaves on the n-dimensional projective space and some finite dimensional algebras, was incorporated in this setting, the crucial sheaves in [Be] being interpreted as special cases of tilting sheaves.

Through a further generalization of tilting modules to tilting complexes, a Morita theory for derived categories was developed in [Ric] in order to describe exactly when two algebras are derived equivalent.

The third direction has its starting point in the theory of tilting (or cotilting) modules of projective (or injective) dimension at most one, with a strong influence of the associated equivalence of derived categories in this setting [HRS]. The crucial basis for generalization is the torsion pair $(\mathcal{T},\mathcal{F})$ associated with a tilting module, where it is known that $\mathcal{T}$ contains all injective modules. We consider torsion pairs $(\mathcal{T},\mathcal{F})$ in $\operatorname{mod}\Lambda$ where $\mathcal{T}$ contains all injective modules (equivalently, $\mathcal{T}$ is a cogenerator), but which do not necessarily come from a tilting module. We call them *tilting torsion pairs*. Then we "tilt" with respect to the torsion pair $(\mathcal{T},\mathcal{F})$ to obtain an abelian category, which is equivalent to $\operatorname{mod}\Gamma$ with $\Gamma = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$ when $(\mathcal{T},\mathcal{F})$ is induced by a tilting module T. The idea is to perform the construction inside the bounded derived category $D^b(\Lambda)$. Let more generally $\mathcal{A}$ be an abelian category with a torsion pair $(\mathcal{T},\mathcal{F})$. There is an abelian category $\mathcal{B} \subset D^b(\mathcal{A})$ with torsion pair $(\mathcal{F}[1],\mathcal{T})$, and we have the following [HRS].

Theorem 3. *If $(\mathcal{T},\mathcal{F})$ is a tilting torsion pair (that is, $\mathcal{T}$ is a cogenerator for $\mathcal{A}$), and either $\mathcal{A}$ has enough injectives or $\mathcal{B}$ has enough projectives, there is induced a triangle equivalence between $D^b(\mathcal{A})$ and $D^b(\mathcal{B})$.*

In order for the new category $\mathcal{B}$ to be equivalent to $\operatorname{mod}\Gamma$ for some algebra Γ, we need that the torsion pair is induced by what we call a tilting object T in $\mathcal{A}$, generalizing the notion of tilting module of projective dimension at most one. Motivated by the fact that the endomorphism algebras $\operatorname{End}_\Lambda(T)^{\mathrm{op}}$ play a main role when T is a tilting module over a hereditary algebra Λ we introduce the more general class of algebras $\mathrm{End}_{\mathcal{H}}(T)^{\mathrm{op}}$, called *quasitilted* algebras, when T is a tilting object in a hereditary abelian k-category with finite dimensional homomorphism and extension spaces [HRS]. Note that $\mathcal{A}$ is said to be hereditary if the Yoneda $\mathrm{Ext}^2(\ ,\ )$ is zero, and in this case T is a tilting object if $\mathrm{Ext}^1_{\mathcal{A}}(T,T) = 0$ and $\mathrm{Hom}_{\mathcal{A}}(T,X) = 0 = \mathrm{Ext}^1_{\mathcal{A}}(T,X)$ implies $X = 0$ [H2].

3 External connections

Tilting theory has played, and continues to play, a central role in the representation theory of algebras. Many questions about arbitrary algebras can be reduced to a problem about tilted algebras, where the theory is much more developed. For example, there is a useful criterion for finite representation type based on a class of tilted algebras. In addition, there are connections and interrelationships with most of the main topics and directions in representation theory. There are connections with relative homological algebra, as the concepts can be formulated in a relative setting [ASo], with stable equivalence [TW], with the generalized Nakayama conjecture and the finitistic dimension conjecture [BS, HU] and with Koszul algebras [GRS]. The connection with derived categories opened up new interesting directions. There are also interrelationships with other parts of algebra, which we discuss in this section.

A characteristic feature of finite dimensional algebras is the wealth of examples of various types available. For example, there are numerous nontrivial examples of derived equivalences, of interest in other areas where such equivalences occur.

The study of many classes of algebras has been motivated by which types of algebras are interesting in other fields. One such example is the *quasihereditary algebras*. Associated with a quasihereditary algebra is a canonical subcategory of modules $\mathcal{C}$ having a so-called $\triangle$-filtration (and also one with modules having ∇-filtration). As a beautiful illustration of Theorem 1 it was proved that $\mathcal{C}$ is contravariantly finite and resolving, and hence has an associated tilting module [Rin2]. This special tilting module associated with a quasihereditary algebra now plays an important role in the representation theory of algebraic groups, where by abuse of terminology, the word tilting module is used for an indecomposable summand of this particular tilting module [D].

There has also been a fruitful interplay between tilting theory and the theory of maximal Cohen–Macaulay modules over a complete local noetherian Cohen–Macaulay ring R. Here the dualizing module ω is the analogue of a cotilting module. Actually, the definition of a cotilting module for an algebra can be rephrased in such a way that ω becomes a cotilting module [AR]. Then the category $\mathcal{Y} = \{C : \mathrm{Ext}^i_\Lambda(C,U) = 0 \text{ for } i > 0\}$ associated with a cotilting module U is the category $\mathrm{MCM}(R)$ of maximal Cohen–Macaulay modules over R. The theory of (maximal) Cohen–Macaulay approximations expresses amongst other things that the category $\mathcal{C} = \mathrm{MCM}(R)$ is contravariantly finite resolving [ABu], and the well known duality $\mathrm{Hom}_R(\ ,\omega)\colon \mathrm{MCM}(R) \to \mathrm{MCM}(R)$ corresponds to a similar one for algebras. Here there was mutual interplay between the developments within finite dimensional algebras and higher dimensional theory [ABr, ASm1, ASm2, ABu, AR]. In particular, the work on Cohen-Macaulay approximations in [ABu] influenced the work on tilting and cotilting modules and their associated subcategories in [AR], where the point of view of the dualizing module being a cotilting module was stressed. Accordingly, the dualizing module and the special tilting (or equivalently, cotilting) module for quasihereditary algebras are special cases of the same common framework, hence also maximal Cohen–Macaulay modules and modules with $\triangle$-filtrations. Also dualizing complexes from algebraic geometry are similar to tilting complexes. Other connections with algebraic geometry via derived equivalence were discussed in Section 2.

After the description of derived equivalences via tilting complexes in [Ric], there has been a lot of activity on this topic in the representation theory of finite groups (see [Br]). The general theory of tilting with respect to torsion pairs has been applied to abstract blowing down in [V].

4 Quasitilted algebras

In this section we give some main results on quasitilted algebras. The type of questions investigated for this class of algebras illustrates the kind of information one is usually looking for about algebras in general. In particular, since quasitilted algebras generalize tilted algebras, established properties of tilted algebras serve as a guideline, as well as the properties of another important class of quasitilted algebras: the canonical algebras of Ringel [Rin1].

We start by giving some interesting and useful characterizations of quasitilted algebras [HRS].

Theorem 4. *The following are equivalent for an algebra Λ.*

(a) Λ *is quasitilted.*

(b) $\operatorname{gl.dim} \Lambda \leq 2$ *and for each indecomposable Λ-module C we have* $\operatorname{pd}_\Lambda C \leq 1$ *or* $\operatorname{id}_\Lambda C \leq 1$, *where* $\operatorname{id}_\Lambda C$ *denotes the injective dimension of C.*

(c) If there is a sequence $X \to \cdots \to P$ of nonzero maps between indecomposable Λ-modules and P is projective, then $\operatorname{pd}_\Lambda X \leq 1$.

An interesting feature of the quasitilted algebras is that they contain the canonical algebras. The canonical k -algebras are special triangular matrix algebras of the form

$$H[M] = \begin{pmatrix} k & 0 \\ M & H \end{pmatrix},$$

called one-point extension of H by M, where H is hereditary and M is an H-module. The AR-quiver of an algebra is built from information given by almost split sequences, and tubes are important types of components occurring (see [ARS]). The canonical algebras provide examples of algebras with families of tubes of arbitrary type $(n_1, \dots, n_t)$, where the n_i are greater than one. In addition, there is a curious trisection of the indecomposable modules into subcategories $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$, where $\mathcal{Q}$ consists of what is called a sincere family of standard stable tubes, $\mathrm{Hom}(\mathcal{R},\mathcal{Q}) = 0 = \mathrm{Hom}(\mathcal{Q},\mathcal{P}) = \mathrm{Hom}(\mathcal{R},\mathcal{P})$, and any map $f\colon P \to R$ with P in $\mathcal{P}$ and R in $\mathcal{R}$ factors through any tube in $\mathcal{Q}$ [Rin1].

A natural related question is to investigate when a one-point extension $H[M]$ of a hereditary algebra H is quasitilted. We give the following result in this direction [HRS].

Theorem 5. *Let H be a indecomposable tame hereditary algebra, and let M be a nonzero regular module in $\mathrm{mod}\, H$. Then $H[M]$ is quasitilted if and only if M is quasisimple (that is, M is indecomposable and the middle term of the almost split sequence with M on the right is indecomposable).*

A central question for algebras is to describe the structure of the AR-quiver. For tilted algebras there is such a description, and also for canonical algebras, but there is yet no general description for quasitilted algebras. Of the information available, we cite the following (see [CH, CS, HRe1]).

Theorem 6. *(a) A quasitilted algebra has a preprojective component.*

(b) No component of the AR-quiver of a quasitilted non-tilted algebra Λ contains both a projective and an injective module.

Interesting open questions are whether the regular components for quasitilted non-tilted algebras are always tubes or of the form $\mathbb{Z}A_\infty$ (see [ARS]), and whether there is only one preprojective component.

A lot of effort in the representation theory of algebras has been given to classification of algebras of finite or tame representation type. For the quasitilted algebras the ones of finite type are already tilted [HRS], and there is a description for the tame quasitilted algebras [S].

5 Noetherian hereditary categories

Let throughout the rest of the paper $\mathcal{H}$ denote a hereditary abelian k-category with finite dimensional homomorphism and extension spaces. Since the quasitilted algebras are defined as endomorphism algebras of tilting objects in such hereditary k-categories, it is a central problem, in connection with understanding the whole class of quasitilted algebras, to classify the possible $\mathcal{H}$ which have a tilting object. In this section we discuss the noetherian case.

If $\mathcal{H}$ has a tilting object, then $\mathcal{H}$ has almost split sequences, and the Grothendieck group $K_0(\mathcal{H})$ is free abelian of finite rank [HRS]. We consider a natural class of categories $\mathcal{H}$ where $K_0(\mathcal{H})$ being free abelian of finite rank implies the existence of a tilting object [RV2].

A first example of a desired $\mathcal{H}$ with tilting object, which is not equivalent to $\mathrm{mod}\, H$ for a hereditary algebra H, is the category $\mathrm{coh}\, \mathbb{P}^1(k)$ of coherent sheaves on the projective line. More generally, there is introduced in [GL] the category $\mathrm{coh}\, \mathcal{X}$ of coherent sheaves on a weighted projective line $\mathcal{X}$. It was shown in [GL] that the canonical algebras could be realized as endomorphism algebras of particular tilting sheaves in $\mathrm{coh}\, \mathcal{X}$. This work was used to give an alternative approach to studying the module theory for canonical algebras. The following gives a complete description of the noetherian $\mathcal{H}$ with tilting object [L].

THEOREM 7. *The* $\operatorname{coh}\mathcal{X}$ *and the* $\operatorname{mod} H$ *where* H *is a hereditary algebra constitute all connected noetherian hereditary* $\mathcal{H}$ *with tilting object.*

The category $\operatorname{coh}\mathbb{P}^1(k)$ has an alternative description as the quotient category of the finitely generated $\mathbb{Z}$-graded $k[X,Y]$-modules modulo those of finite length. More generally, there is an interesting source of hereditary categories $\mathcal{H}_S$ (containing the $\operatorname{coh}\mathcal{X}$) arising from two-dimensional $\mathbb{Z}$-graded isolated singularities S, finitely generated as a module over the center (see [RV2]). Interpreting $\mathcal{H}_S$ as the category of coherent modules over a sheaf of hereditary orders with center a nonsingular projective curve X, we have the following [RV2].

THEOREM 8. *Let* $S = k + S_1 + S_2 + \cdots$ *be a* $\mathbb{Z}$*-graded two-dimensional isolated singularity, with each* S_i *finite dimensional, and* S *finitely generated as a module over its center. Let* $\mathcal{H}_S$ *be the quotient category of finitely generated* $\mathbb{Z}$*-graded* S*-modules with degree zero maps, modulo the full subcategory of objects of finite length. Then the following are equivalent.*

(a) $K_0(\mathcal{H}_S) \simeq \mathbb{Z}^n$ *for some* n.

(b) The projective curve X *is a finite product of copies of* $\mathbb{P}^1(k)$.

(c) $\mathcal{H}_S$ *has a tilting object.*

(d) $\mathcal{H}_S$ *is equivalent to some* $\operatorname{coh}\mathcal{X}$.

Possible choices for S with $K_0(\mathcal{H}_S) \simeq \mathbb{Z}^n$ for some n are two-dimensional $\mathbb{Z}$-graded Cohen–Macaulay isolated singularities of finite (graded) representation type, a complete classification of which is given in [RV1]. Other examples are $S = k[X,Y,Z]/(X^i + Y^j + Z^t)$, where i, j, t are pairwise relatively prime positive integers, and then $K_0(\mathcal{H}_S) \simeq \mathbb{Z}^{i+j+t-1}$ (see [GL]). The noetherian categories $\mathcal{H}_S$ with $K_0(\mathcal{H}_S) \simeq \mathbb{Z}^n$ form in a sense a bridge between some isolated Cohen–Macaulay two-dimensional singularities and a class of finite dimensional algebras, providing an additional connection between the areas.

6 HEREDITARY CATEGORIES WITH TILTING OBJECTS

We have seen in the previous section that the hereditary categories $\mathcal{H}$ with tilting object can be described in the noetherian case. In this section we discuss what can be said in general.

Since $\mathcal{H}$ is hereditary, the bounded derived category $D^b(\mathcal{H})$ has a simple description, as the indecomposable objects in this case are isomorphic to stalk complexes. When $\mathcal{H}$ has a tilting object, any hereditary abelian k-category $\mathcal{H}'$ derived equivalent to $\mathcal{H}$ also has a tilting object (and finite dimensional homomorphism and extension spaces) [HRe2]. Hence we obtain new hereditary categories $\mathcal{H}$ with tilting object by describing those in the same derived equivalence class as $\operatorname{coh}\mathcal{X}$ and $\operatorname{mod} H$ (see [LS, H2]).

An interesting open problem is whether there are more hereditary categories $\mathcal{H}$ with tilting object than those derived equivalent to $\operatorname{mod} H$ or $\operatorname{coh}\mathcal{X}$, or formulated differently, to finite dimensional hereditary or canonical algebras. We have the following information [HRe3].

Theorem 9. *Let $\mathcal{H}$ be a connected hereditary abelian k-category with tilting object. If $\mathcal{H}$ has some simple object, then $\mathcal{H}$ is derived equivalent to a hereditary or to a canonical algebra.*

Since every noetherian object has a simple quotient, it is also sufficient to require the existence of some noetherian object. If all objects are noetherian, the result follows from [L].

For hereditary algebras each indecomposable projective module is directing, that is does not lie on a nontrivial cycle of nonisomorphisms. We also have the following [HRe2].

Theorem 10. *Let $\mathcal{H}$ be a connected hereditary abelian k-category with tilting object. If $\mathcal{H}$ has some directing object, then $\mathcal{H}$ is derived equivalent to a finite dimensional hereditary k-algebra.*

The following provides further information along these lines [S].

Theorem 11. *If $\mathcal{H}$ is a connected hereditary abelian k-category with tilting object T such that $\operatorname{End}_{\mathcal{H}}(T)^{\mathrm{op}}$ is a tame algebra, then $\mathcal{H}$ is derived equivalent to a hereditary or to a canonical algebra.*

An important feature of hereditary categories $\mathcal{H}$ playing an essential role in the proof of Theorem 9, but also of more general interest, is the following result from [HRe3].

Theorem 12. *Let $\mathcal{H}$ be a connected hereditary abelian k-category with tilting object. Then for each exceptional object E in $\mathcal{H}$ (that is, $\operatorname{Ext}^1_{\mathcal{H}}(E,E) = 0$ and $\operatorname{End}_{\mathcal{H}}(E) \simeq k$) which is of infinite length and in $\operatorname{Fac} T$ for a tilting object T, the perpendicular category $E^{\perp}$ is equivalent to $\operatorname{mod} H$ for a finite dimensional hereditary k-algebra H.*

It is a consequence of Theorem 12 that any quasitilted algebra is derived equivalent to some one-point extension algebra $H[M]$ of a hereditary algebra H (see also [H2]). Hence a thorough investigation of such algebras $H[M]$ would also shed light on the problem of describing the $\mathcal{H}$ with tilting object.

We mention some open problems about quasitilted algebras, which would be answered if it is proved there are no more hereditary categories $\mathcal{H}$ with tilting object than those discussed above. A trisection for the canonical algebras was discussed in Section 4, and this trisection property characterizes a larger class of quasitilted algebras [LP]. Weakening the requirements on the middle part, other classes of quasitilted algebras can be characterized in such terms [LS, PR]. It is not known if this is the case for the quasitilted algebras. Another problem is formulated in terms of Hochschild cohomology. Is there some quasitilted algebra Λ with $H^1(\Lambda) \neq 0$ and $H^2(\Lambda) \neq 0$ [H3]? Denote by $\mathcal{D}$ the indecomposable Λ-modules C such that C and all its predecessors with respect to paths of nonzero maps have projective dimension at most one, and by $\mathcal{C}$ the indecomposable Λ-modules C such that C and all its successors with respect to paths of nonzero maps have injective dimension at most one. Is $\mathcal{C} \cap \mathcal{D}$ not empty for a quasitilted algebra Λ [HRS]?

A natural enlargement of the class of quasitilted algebras is the class of algebras derived equivalent to some hereditary category $\mathcal{H}$ with tilting object (or equivalently to a quasitilted algebra). It would be interesting to find a homological characterization of these algebras, called piecewise hereditary algebras (see [HRe3]).

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