Birational Automorphisms of Higher-Dimensional Algebraic Varieties

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Abstract. The present survey covers the known results on the groups of birational automorphisms, rationality problem and birational classification for Fano fibrations.

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0. Birational geometry starts with M.Nöther's paper [45] on Cremona transformations. The problems of birational geometry of algebraic varieties, that is, birational classification, the rationality and unirationality problems, structure of the group of birational automorphisms, formed a subject of exclusive attention for the Italian classics, including C.Segre, Castelnuovo, Enriques, Comessatti, B.Segre, Fano, Morin, Predonzan and many others, see, for instance, [59]. Italian geometers, first of all — G.Fano, laid the foundation of the modern birational geometry, outlined solutions to certain hard problems, gave surprisingly exact forecasts and suggested some crucial ideas.

The modern period of birational geometry started with Yu.I.Manin's papers on geometry of surfaces over non-closed fields [38,39]. The breakthrough into higher dimensions was made in 1970 in the papers of V.A.Iskovskikh and Yu.I.Manin [29] and H.Clemens and Ph.Griffiths [7], where the Lüroth problem got its negative solution (both the techniques and the final results of these papers were absolutely independent of each other). In [29] Iskovskikh and Manin, using certain classical ideas of Nöther and Fano, developed a new method of study of birational correspondences between algebraic varieties (which have no nontrivial differential-geometric birational invariants) — the *method of maximal singularities.* The results which were obtained by means of this method in 70s were summed up 15 years ago in [24,25]. Since that day, a considerable progress has been made in the field. It is worth noting that, although we have now new approaches and concepts [34,35], this method up to this day is the most effective tool in birational geometry. The contemporary state of the theory form the subject of the present survey.

1. The aim of birational geometry is birational classification of algebraic varieties. In the most general sense, for two given varieties V, V' we should be able to say, whether their function fields $k(V)$ and $k(V')$ are isomorphic, and if yes, how such

an isomorphism can be obtained (here k is an algebraically closed field of characteristic zero; the principal case is $k = \mathbf{C}$). We may understand the classification problem also as the problem of investigating birational geometry of the given variety V, that is, those geometric properties which are independent of the concrete model of the field $k(V)$.

Birational geometry is most interesting, rich and also hard to study for *Fano fibrations* $\pi: V \to S$, the generic fiber F_η of which is a Fano variety over the non-closed field $k(S)$, that is, the canonical class K_{F_η} is negative. In other words, the fiber F_t over a point $t \in S$ of general position is a Fano variety over the field k. For this class of objects we can specify the following particular cases in the general problem of birational classification.

(1) Describe all the structures of a Fano fibration on the given variety V (in the *birational* sense). In many cases this problem can be transformed into the following question: is a birational map $\chi: V - - \to V'$ between two given Fano fibrations $\pi: V \to S$ and $\pi': V' \to S'$ fiber-wise?

(2) Compute the group of birational automorphisms $\mathrm{Bir}\, V = \mathrm{Aut}\, k(V)$. If $V = \mathbf{P}^m$, we get the *m-dimensional Cremona group.*

(3) The *rationality problem*: whether V is birational to $\mathbf{P}^m$ (in algebraic terms: whether $k(V)$ is a purely transcendent extension $k(t_1, \ldots, t_m)$ of the field of constants)?

The problem of birational classification naturally generalizes to *rational correspondences*: for two given algebraic varieties V, V' describe the set of rational (p,q)-correspondences between them, $p, q \geq 1$. For $V' = \mathbf{P}^m, p = 1$ we get the classical *unirationality problem* (whether the field $k(V)$ can be embedded in $k(t_1, \ldots, t_m)$?). Unfortunately, today we have got no methods, which could make it possible to study the subject, only direct constructions of unirationality of the type of B.Segre [64], U.Morin [44] and A.Predonzan [46], see also the modern papers [9,41].

2. Fano fibrations satisfy the classical *termination condition for canonical adjunction*, the importance of which was understood by the Italian classics: for any divisor D the linear system $|D + nK_V|$ is empty for n sufficiently high. The *threshold of canonical adjunction*

$$c(V, D) = \sup\{ \frac{b}{a} \,\Big|\, a, b \in \mathbf{Z}_+ \backslash \{0\}, |aD + bK_V| \neq \emptyset \}.$$

is a quantitative characteristic of termination. To study a birational map $\chi: V - - \to V'$, we compare the corresponding thresholds on V and V': let $|D'|$ be a linear system of divisors on V', free in codimension 1, and $|D| = |D(\chi)| = (\chi^{-1})_* |D'|$ be its proper inverse image on V, then we get two numbers $c(V, D)$ and $c(V', D')$. In a certain natural sense the threshold $c(V, D)$ characterizes the "complexity" or "size" of the linear system $|D|$. Decreasing the threshold by means of an "elementary" birational map $\tau: V_1 - - \to V$, where V_1 is, generally speaking, another model of the field $k(V)$, we "simplify" the system $|D|$ and thus the map χ itself: $c(V_1, D(\chi \circ \tau)) < c(V, D(\chi))$. This is the general idea of simplification (in the traditional terminology, *untwisting*) of a birational map.

Definition 1. (i) A Fano fibration $\pi: V \to S$ is said to be *birationally rigid*, if for any V', D', χ' there exists a birational automorphism of the generic fiber $\chi^* \in \mathrm{Bir}\, F_\eta \subset \mathrm{Bir}\, V$ such that the composition $\chi \circ \chi^*: V - - \to V'$ satisfies the monotonicity condition: $c(V, D) \leq c(V', D')$.

(ii) A Fano fibration is said to be *birationally superrigid*, if the monotonicity condition is always true (i.e. we can take $\chi^* = \mathrm{id}$.)

The property of being (super)rigid characterizes birational geometry of a variety in an exhaustive way.

Proposition 1. *Assume that the Fano fibration $\pi: V \to S$ satisfies the following condition: for any divisor D and the induced divisor D_η on the generic fiber F_η the thresholds $c(V, D)$ and $c(F_\eta, D_\eta)$ coincide, and, moreover, that* $\mathrm{Pic}\, F_\eta \cong \mathbf{Z}$. *Assume the Fano fibration $\pi: V \to S$ to be birationally rigid. Then any birational map $\chi: V - - \to V'$, where $\pi': V' \to S'$ is a Fano fibration of the same dimension, is fiber-wise, that is, $\pi' \circ \chi = \alpha \circ \pi$ for some (dominant rational) map of the base $\alpha: S - - \to S'$.*

Corollary 1. *In the assumptions of Proposition 1 the variety V is non-rational. Any birational automorphism $\chi \in \mathrm{Bir}\, V$ is fiber-wise.*

Corollary 2. *Birationally rigid Fano variety V with* $\mathrm{Pic}\, V \cong \mathbf{Z}$ *cannot be fibered (by a rational map) into rationally connected varieties over a positive-dimensional base.*

Corollary 3. *For a birationally superrigid smooth Fano variety V with* $\mathrm{Pic}\, V \cong \mathbf{Z}$ *the groups of birational and biregular automorphisms coincide,* $\mathrm{Bir}\, V = \mathrm{Aut}\, V$.

3. Fix a smooth (or with $\mathbf{Q}$-factorial terminal singularities) Fano fibration $\pi: V \to S$. Let $\chi: V - - \to V'$ be a birational map onto another Fano fibration. Assume that the monotonicity condition does not hold: $n = c(V, D) > c(V', D')$.

Proposition-Definition 2. *There exists a geometric (that is, realizable by a prime Weil divisor on a certain projective model of the function field) discrete valuation $\nu: k(V) \to \mathbf{Z}$, which satisfies the* Nöther-Fano(-Manin-Iskovskikh) inequality

$$\nu(|D|) > na(\nu, V),$$

where $a(\cdot)$ is the discrepancy. These valuations are called maximal singularities *of the map χ or the system $|D|$. If ν is of the form $\nu_B = \mathrm{mult}_B$, where $B \subset V$ is an irreducible cycle of codimension≥ 2, then B is said to be a* maximal cycle. *Otherwise, ν is said to be* infinitely near.

The general scheme of arguments which prove (super)rigidity looks as follows. It turns out that (in all the cases that can be succefully studied by this method) the maximal singularities are an exceptional phenomenon. Only very special cycles $B \subset V$ can appear as maximal (in many cases they do not occur at all), and if there is no maximal cycle, there is no infinitely near maximal singularities, either. Exclusion of the infinitely near case is based upon the following key

Proposition 3. *Let $D_{1,2} \in |D|$ be general divisors and the centre B of ν on V be of codimension≥ 3, and assume that B is not contained in the singular locus of V. Let $Z = (D_1 \bullet D_2)$ be the algebraic cycle of their scheme-theoretic* *intersection (it is an effective cycle of codimension 2). Then*

$$\operatorname{mult}_B Z \geq 4\frac{\nu(|D|)^2}{a(\nu,V)^2} > 4n^2.$$

This very inequality makes the essence of the *test class* method of Iskovskikh-Manin, which was developed in [29]. Gradually [48-55] it was discovered that this fact has a very general character.

After all the potentially maximal cycles B have been detected, for each of them one constructs an "untwisting" automorphism $\tau_B \in \operatorname{Bir} V$. Taking the composition $\chi \circ \tau_B$, we simplify the map, that is, decrease the adjunction threshold $n(\chi) = c(V, D(\chi))$. After a finite number of steps the composition $\chi \circ \tau_{B_1} \circ \ldots \tau_{B_N}$ satisfies the monotonicity condition. Simultaneously we get a copresentation of the group $\operatorname{Bir} V$ (generators and relations). If maximal singularities do not occur at all, we conclude that V is birationally superrigid.

4. Here is the list of Fano varieties, birational geometry of which has been successfully studied by means of the method of maximal singularities.

1) Double spaces $V \to \mathbf{P}^M$ of index 1, branched over a hypersurface $W_{2M} \subset \mathbf{P}^M$ of degree $2M$. The hypersurface can contain a singular point $x \in W$ of general position of multiplicity $2m$, $m \leq M-2$, $M \geq 3$.

2) Double quadrics $V \to Q \subset \mathbf{P}^{M+1}$ of index 1, branched over a divisor, which is cut out on Q by a hypersurface $W_{2(M-1)}$ of degree $2(M-1)$. In dimensions ≥ 4 the branch divisor is smooth, in dimension 3 it may contain a non-degenerate double point.

3) Hypersurfaces $V = V_M \subset \mathbf{P}^M$ of degree M, $M \geq 4$. For $M = 4$ V is either smooth or is allowed to have exactly one non-degenerate double point $x \in V$, lying on exactly 24 distinct lines on V. For $M = 5$ V is arbitrary smooth, for $M \geq 6$ V is general in the following sense: for any point $x \in V$ and any system $(z_1, \ldots, z_M)$ of affine coordinates on $\mathbf{P}^M$ with the origin at x the sequence of polynomials $(q_1, \ldots, q_{M-1})$ makes a regular sequence, where $f = q_1 + \ldots + q_{M-1} + q_M$ is the equation of V with respect to z_*, q_i are homogeneous degree i.

4) Double Veronese cone $V \to W \subset \mathbf{P}^6$ of dimension three, that is, W is the cone over the Veronese surface in $\mathbf{P}^5$, and the non-singular branch divisor is cut out by a cubic, not passing through the vertex.

5) General complete intersections $V_{2\cdot 3} = Q_2 \cap Q_3 \subset \mathbf{P}^5$ (the normal bundle of any line $L \subset V$ is $\mathcal{N}_{L/V} \cong \mathcal{O}_L \oplus \mathcal{O}_L(-1)$ and there is no plane $P \subset \mathbf{P}^5$ such that $P \cap V$ consists of three lines passing through a point).

6) General (in particular, quasismooth) hypersurfaces in the weighted projective space $V_d \subset \mathbf{P}(1, a_1, a_2, a_3, a_4)$, $d = a_1 + \ldots + a_4$, which are $\mathbf{Q}$-factorial Fano threefolds with terminal singularities. There are 95 families of these varieties [16], starting from $V_4 \subset \mathbf{P}^4$ and ending by $V_{66} \subset \mathbf{P}(1,5,6,22,33)$. From this list we should exclude the quartic V_4 (which is already present in 3)) and $V_6 \subset \mathbf{P}(1,1,1,1,3)$, which is just the double space (class 1)).

Theorem 1. *A) The following Fano varieties are birationally superrigid:*
— all the members of the class 1) ([24] for smooth 3-folds, [49] for smooth double spaces of dimension ≥ 4, [53] for the singular case)*;*

— all the members of 2) of dimension ≥ 4 [49]*;*
— all the smooth members of 3) ([29] for the quartic, [48] for the quintic, [55] for the rest of the cases) *and 4)* [24,33].
Their groups of birational and biregular automorphisms coincide. For a general member of the class 3) it is trivial, of the classes 1), 2) and 4) it is $\mathbf{Z}/2\mathbf{Z}$*.*

B) All the rest of Fano varieties from the list above are birationally rigid. For each of them there is the exact sequence

$$1 \to B(V) \to \operatorname{Bir} V \to \operatorname{Aut} V \to 1,$$

where $B(V)$ *is the untwisting subgroup, that is,* χ^* *from Definition 1 can be taken from this subgroup. More exactly:*
— for three-dimensional double quadrics (class 2), $M = 3$*)* $B(V)$ *is the free product of the one-dimensional family of involutions* τ_L*, associated with the lines* $L \subset V$ *(i.e., irreducible rational curves with* $(L \cdot K_V) = -1$*), which do not lie in the branch divisor* ([24] for the smooth and [18] for the singular case)*;*
— for the singular quartics $B(V)$ *is the free product of 25 involutions* $\tau_i, i = 0, \ldots, 24$*, where* τ_0 *is the reflection from the double point* x *and* τ_i *is the reflection from* x *in the fibers of the elliptic fibration, generated by the net of planes containing* $L_i \ni x$ *(*[50]*);*
— for $V = V_{2\cdot 3}$ *the subgroup* $B(V)$ *is an "almost free" product of two one-dimensional families of involutions* α_L*, for all the lines* $L \subset V$*, and* β_Y*, for all the irreducible conics* $Y \subset V$ *such that the plane* $P(Y) \supset Y$ *lies in* Q_2*. There is a finite number of relations* $(\alpha_{L_1}\alpha_{L_2}\alpha_{L_3})^2 = 1$*, where the lines* L_i *lie in the same plane* (the proof was started in [24] and completed in [51], for a complete exposition see [31])*;*
— for the weighted hypersurfaces (class 6)) $B(V)$ *is generated by a finite number of involutions, associated with the terminal singular points* [12]*. For some of these varieties there are no birational involutions at all, so that they are actually superrigid.*

5. Here is the list of Fano fibrations over a non-trivial base, birational geometry of which has been succefully studied by the method of maximal singularities.

1) Standard conic bundles $\pi: V \to S$, $\dim V \geq 3$, with a big discriminant divisor $D \subset S$: $|D + 4K_S| \neq \emptyset$.

2) Smooth threefolds $\pi: V \to \mathbf{P}^1$, fibered into del Pezzo surfaces, $\operatorname{Pic} V = \mathbf{Z}K_V \oplus \mathbf{Z}F$, where F is the class of a fiber, F_η is a del Pezzo surface of degree $d = 1, 2, 3$ over the non-closed field $k(t)$, satisfying the K^2-*condition*: the numerical class $MK_V^2 - f$ in not effective for any $M \in \mathbf{Z}$, $f \in A^2(V)$ is the class of a line in a fiber (for $d = 3$ it is also assumed that if F_t is a singular fiber, then it has exactly one singular point lying on exactly six lines on F_t).

3) General smooth 4-folds $\pi: V \to \mathbf{P}^1$, fibered into quartic threefolds, satisfying the K^2-condition: $MK_V^2 - f$ is not effective, where f is the class of a hyperplane section of a fiber, $M \in \mathbf{Z}$.

4) Smooth higher-dimensional varieties $\pi: V \to \mathbf{P}^1$, fibered into double spaces of index 1 (class 1) from Sec. 4 above), satisfying the K^2-condition.

5) Certain varieties with a pencil of double quadrics of index 1 (class 2) from Sec. 4), satisfying the K^2-condition.

6) The general double cone $V \to Q_2 \subset \mathbf{P}^4$, where Q_2 is the non-degenerate quadric cone and the branch divisor is cut out by a quartic, which does not pass through the vertex. The variety V has two obvious pencils of del Pezzo surfaces of degree 2, induced by the pencils of planes on Q_2.

Theorem 2. *A) Any birational map of a variety from the class 1) above onto another conic bundle is fiber-wise* [61,62].

B) Fano fibrations from the class 2) for $d = 1$ and from the classes 3)-5) are superrigid. For a general variety from the class 3) $\mathrm{Bir}\, V$ *is a trivial group, otherwise (for a general variety) it is isomorphic to* $\mathbf{Z}/2\mathbf{Z}$ [54,56].

C) Fano fibrations from the class 2) for $d = 2, 3$ are birationally rigid. The following exact sequence holds

$$1 \to \mathrm{Bir}\, F_\eta \to \mathrm{Bir}\, V \to G \to 1,$$

where G is a finite, generically trivial group of fiber-wise birational automorphisms. (See [54]. The group $\mathrm{Bir}\, F_\eta$ was described by Yu.I.Manin [38-40]. It is generated by involutions, associated to sections of π for $d = 2$, and for $d = 3$ — to sections and bisections of π.)

D) Varieties of the type 6) are birationally rigid as Fano varieties. For any pencil $|\Lambda|$ of rational surfaces on V there is a birational automorphism χ^*, which transforms $|\Lambda|$ into one of the two "default" pencils. The group $\mathrm{Bir}\, V$ *is generated by the subgroups* $\mathrm{Bir}\, F_{\eta_i}$*, $i = 1, 2$ (F_{η_i} are the generic fibers of the "default" pencils). Their intersection* $\mathrm{Bir}\, F_{\eta_1} \cap \mathrm{Bir}\, F_{\eta_2}$ *is generated by a finite number of involutions* [19].

6. Conjectures and open problems.

1) Let $V_{m_1 \cdot \ldots \cdot m_k} \subset \mathbf{P}^{m_1 + \ldots + m_k}$ be a Fano complete intersection of index 1 with sufficiently mild singularities. Then V is birationally (super)rigid.

2) By analogy with the weighted 3-fold hypersurfaces, we should expect that 1) is true for the weighted case, either.

3) The rigidity facts about Fano fibrations can be looked at as a realization of the following informal principle:

> if a Fano fibration is "sufficiently twisted" over the base, then birational geometry of V reduces to birational geometry of the generic fiber F_η.

It seems that this principle holds in a much more general situation than A)-C) of Theorem 2. For instance, if $V \hookrightarrow \mathbf{P}(\mathcal{E})$, where $\mathcal{E}$ is a locally free sheaf on S of rank $m_1 + \ldots + m_k + 1$ and the generic fiber F_η is a complete intersection $V_{m_1 \cdot \ldots \cdot m_k} \subset \mathbf{P}_\eta^{m_1 + \ldots + m_k}$, then "sufficient twistedness" over the base implies that the Fano fibration $\pi \colon V \to S$ is birationally (super)rigid. As in 2) above, this statement should be true for the weighted case, too.

4) Hypersurfaces $V_m \subset \mathbf{P}^M$ of index $M + 1 - m \geq 2$ obviously have a lot of structures of a Fano fibration. It seems natural to suggest that all these structures come from the "natural" ones, the fibers of which are Fano complete intersections in linear subspaces of $\mathbf{P}^M$. For instance, linear systems $|\Lambda_i|$, $i = 1, \ldots, k$, cut out on V by hypersurfaces of degrees $m_1, \ldots, m_k$, where $m + m_1 + \ldots + m_k \leq M$, determine a structure of a Fano fibration

$$\pi = (\pi_1, \ldots, \pi_k) \colon V - - \to \mathbf{P}^{n_1} \times \ldots \times \mathbf{P}^{n_k},$$

$n_i = \dim|\Lambda_i|$. Another example: for a quartic $V = V_4 \subset \mathbf{P}^M$ of dimension ≥ 4 we suggest that all the structures of a fibration into rational surfaces come from the linear projections from the planes $P \subset V$ and V can not be fibered into rational curves (by a rational map). The general cubic $V = V_3 \subset \mathbf{P}^M$, $M \geq 5$, is non-rational. The coincidence $\mathrm{Bir}\, V = \mathrm{Aut}\, V$ is very likely to be true for all the hypersurfaces of degree 4 and higher, at least for general ones (for ceratin special smooth quartics non-trivial birational automorphisms do exist, but their construction only confirms that they represent an exceptional phenomenon). Similarly for complete intersections.

5) Computation of the Cremona group $\mathrm{Bir}\, \mathbf{P}^n$, even for $n = 3$, and of the group $\mathrm{Bir}\, V_3$ for the higher-dimensional cubic still remains an open problem, seeming to be inaccessible for the modern techniques. In [25] a complete description of the group $\mathrm{Bir}\, V_2$ for the double space $\mathbf{P}^3$ of index 2 (branched over a quartic) was announced. Unfortunately, it also remains an open problem (although the fact itself seems to be true).

6) We have got no rationality criterion for threefolds. The crucial problem here is to prove the well-known (conjectural) Iskovskikh-Shokurov rationality criterion for conic bundles, see [28].

7) It is important to study the structure of infinitely near maximal singularities. There is a conjecture that if a linear system $|D|$ has a maximal singularity ν, it also has another maximal singularity μ (satisfying the same Nöther-Fano inequality), which can be realized as a weighted blow up. In dimension three this conjecture describes all the extremal contractions to smooth points.

8) Up to this day we are unable to prove non-unirationality otherwise but by producing differential forms. On the other hand, the general quartics in $\mathbf{P}^4$, speaking not of double spaces or general hypersurfaces and complete intersections of a small index and high dimension, seem to be non-unirational. Recently some new direct constructions of unirationality appeared [9,41].

7. The prospects of birational classification.

The well-known achievements of the minimal model program (or Mori theory) [8,32,36,47,42,43,57,65,66,70] generated some hope to convert the three-dimensional birational geometry from a collection of separate results and constructions into a regular theory. The corresponding concept of factorization of birational maps between (three-dimensional) Mori fiber spaces was developed by Sarkisov [63] and got the name Sarkisov program [58]. It was exhaustively substantiated by A.Corti [10], see also [11]. After it had been proved that any birational map between Mori fiber spaces can be factorized into a chain of elementary links, it was natural to apply this general theory to certain families of three-folds, in order to re-think on a higher level the classical results of the method of maximal singularities. As an object for this experiment the above-mentioned 95 families [16] of weighted hypersurfaces were chosen [12]. However, the results turned out to be rather unexpected: *all* the discovered elementary links were just *involutions* of the classical type, which (so far) permits no explanation from the Mori-theoretic viewpoint. On the other hand, now we have got Mori-theoretic analogs of the crucial technical means of the classical method (A.Corti's techniques of "reduction to log canonical surfaces", which in dimension three can replace the old techniques of counting multiplicities, although the latter still seems more transparent and natural). All in all, the result of [12] turned out to be much more in the spirit of the method of maximal singularities than the modern concepts.

Sarkisov program is different from the classical approach by its essentially "dynamical" viewpoint: simplifying (untwisting) a birational map, we replace the initial model by a new one, whereas the traditional approach makes use of automorphisms (the model is always the same). For the weighted hypersurfaces the dynamical viewpoint turned out to be useless. Of course, it goes without saying that in the general case (for instance, for the projective space $\mathbf{P}^3$) it is impossible to reduce all the ampleness of birational geometry to a single model. This can be seen even in the two-dimensional case. However, in spite of all the perfection of two-dimensional birational geometry, which can be looked at as an ideal object of realization of Sarkisov program, there is still a feeling of dissatisfaction. For instance, the modern proof of the Nöther theorem on Cremona transformations *formally* makes use of *all* the minimal rational surfaces, whereas essentially only three models are of real use: the very $\mathbf{P}^2$, $\mathbf{F}_1$ and $\mathbf{P}^1 \times \mathbf{P}^1$. This example and all the higher dimensional ones suggest that the modern concept of a minimal model is *too fine* for the rough purposes of birational classification. Sometimes (and even in the "majority" of cases) the minimal model is unique (rigidity phenomenon). But then we have no need in the dynamical viewpoint! In other cases we need some new, essentially more rough approach to the problem of choice of a suitable model for a given field of rational functions.

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