Geometry of Infinitesimal Group Schemes

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We consider affine group schemes G over a field k of characteristic $p > 0$. Equivalently, we consider finitely generated commutative k-algebras $k[G]$ (the coordinate algebra of G) endowed with the structure of a Hopf algebra. The group scheme G is said to be *finite* if $k[G]$ is finite dimensional (over k) and a finite group scheme is said to be *infinitesimal* if the (finite dimensional) algebra $k[G]$ is local. A *rational* $\mathbf{G}$*-module* is a k-vector space endowed with the structure of a comodule for the Hopf algebra $k[G]$. The abelian category of rational G-modules has enough injectives, so that $Ext^i_G(M,N)$ is well defined for any pair of rational G-modules M, N and any non-negative integer i. Unlike the situation in characteristic 0, this category has many non-trivial extensions reflected by the cohomology groups we study.

We sketch recent results concerning the cohomology algebras $H^*(G,k)$ and the $H^*(G,k)$-modules $Ext^*_G(M,M)$ for infinitesimal group schemes G and finite dimensional rational G-modules M. These results, obtained with Andrei Suslin and others, are inspired by analogous results for finite groups. Indeed, we anticipate but have yet to realize a common generalization to the context of finite group schemes of our results and those for finite groups established by D. Quillen [Q1], J. Carlson [C], G. Avrunin and L. Scott [A-S], and others. Although there is considerable parallelism between the contexts of finite groups and infinitesimal group schemes, new techniques have been required to work with infinitesimal group schemes. Since the geometry first occuring in the context of finite groups occurs more naturally and with more structure in these recent developments, we expect these developments to offer new insights into the representation theory of finite groups.

The most natural examples of infinitesimal group schemes arise as *Frobenius kernels* of affine algebraic groups G over k (i.e., affine group schemes whose coordinate algebras are reduced). Recall that the Frobenius map

$$F : G \to G^{(1)}$$

of an affine group scheme is associated to the natural map $k[G]^{(1)} \to k[G]$ of k-algebras. (For any k-vector space V and any positive integer r, the r-th Frobenius twist $V^{(r)}$ is the k-vector space obtained by base change by the p^r-th power map $k \to k$.) The r-th Frobenius kernel of G, denoted $G_{(r)}$, is defined to be the kernel of the r-th iterate of the Frobenius map, $ker\{F^r : G \to G^{(r)}\}$; thus,

$$k[G_{(r)}] = k[G]/(X^{p^r}; X \in \mathcal{M}_e)$$

where $\mathcal{M}_e \subset k[G]$ is the maximal ideal at the identity of G.

If M is an irreducible rational G-module for an algebraic group G, then $M^{(r)}$ is again irreducible; moreover, for $r \neq s$, $M^{(r)}$ is not isomorphic to $M^{(s)}$. It is easy to see that a rational G-module N is the r-th twist of some rational G-module M if and only if $G_{(r)}$ acts trivially on N. Thus, much of the representation theory of an algebraic group G is lost when rational G-modules are viewed by restriction as $G_{(r)}$-modules. On the other hand, in favorable cases the category of rational G-modules is equivalent to the category locally finite modules for the hyperalgebra of G, the ind-object $\{G_{(r)}, r \geq 0\}$ (see, for example, [CPS]).

The special case of the 1st infinitesimal kernel $G_{(1)}$ of an algebraic group G is a familiar object. The (k-linear) dual $k[G_{(1)}]^{\#}$ of the coordinate algebra of $G_{(1)}$ is naturally isomorphic to the *restricted enveloping algebra* of the p-restricted Lie algebra $g = Lie(G)$. Thus, the category of rational $G_{(1)}$-modules is naturally isomorphic to the category of restricted g-modules. The results we describe below are natural generalizations and refinements of results earlier obtained by the author and Brian Parshall for p-restricted Lie algebras (see, for example, [FP1], [FP2], [FP3], [FP4]).

Throughout our discussion, unless otherwise specified, k will denote an arbitrary (but fixed) field of characteristic $p > 0$ and the finite group schemes we consider will be finite over k.

§1. Finite Generation and Strict Polynomial Functors

The following theorem proved by the author and Andrei Suslin is fundamental in its own right and aspects of its proof play a key role in further developments. This result, valid for an arbitrary finite group scheme, is a common generalization of the finite generation of the cohomology of finite groups proved by L. Evens [E] and B. Venkov [V], and the finite generation of restricted Lie algebra cohomology (cf. [FP1].

THEOREM 1.1 [F-S]. *Let G be a finite group scheme over k and let M be a finite dimensional rational G-module. Then $H^*(G,k)$ is a finitely generated k-algebra and $H^*(G,M)$ is a finite $H^*(G,k)$-module.*

After base extension via some finite field extension K/k, G_K as a finte group scheme over K is a semi-direct product of a finite group by an infinitesimal group scheme. Using classical results about finite generation of cohomology of finite groups (cf. [E]) and the fact that any infinitesimal group scheme G admits an embedding in some $GL_{n(r)}$, we find that Theorem 1.1 is implied by the following more concrete assertion.

THEOREM 1.2 [F-S]. *For any $n > 1, r \geq 1$, there exist rational cohomology classes*

$$e_r \in H^{2p^{r-1}}(GL_n, gl_n^{(r)})$$

which restrict non-trivially to

$$H^{2p^{r-1}}(GL_{n(1)}, gl_n^{(r)}) = H^{2p^{r-1}}(GL_{n(1)}, k) \otimes gl_n^{(r)},$$

where gl_n denotes the adjoint representation of the algebraic general linear group GL_n. Moreover, these classes e_r induce a GL_n-equivariant map of k-algebras

$$\phi : \bigotimes_{i=1}^{r} S^*((gl_n^{(r)})^{\#}[2p^{i-1}]) \to H^*(GL_{n(r)}, k),$$

where $S^*((gl_n^{(r)})^{\#}[2p^{i-1}])$ denotes the symmetric algebra on the vector space $gl_n^{(r)\#}$ placed in degree $2p^{i-1}$, with the property that $H^*(GL_{n(r)}, k)$ is thereby a finite module over $\bigotimes_{i=1}^{r} S^*((gl_n^{(r)})^{\#}[2p^{i-1}])$.

We may interpret e_1 as the group extension associated to the general linear group over the ring $W_2(k)$ of Witt vectors of length 2 over k:

$$1 \to gl_n \to GL_{n,W_2(k)} \to GL_{n,k} \to 1,$$

where $GL_{n,k}$ denotes the algebraic general linear group GL_n over k (with k made explicit). Alternatively, we can view $e_1 \in Ext^2_{GL_n}(I_n^{(1)}, I_n^{(1)})$ as the extension of rational GL_n-modules

$$0 \to I_n^{(1)} \to S^p(I_n) \to \Gamma^p(I_n) \to I_n^{(1)} \to 0 \qquad (1.2.1)$$

where I_n denotes the canonical n-dimensional representation of GL_n, $S^p(I_n)$ denotes the p-th symmetric power of I_n defined as the coinvariants under the action of the symmetric group Σ_p on the p-th tensor power $I_n^{\otimes p}$, and $\Gamma^p(I_n)$ denotes the p-th divided power of I_n defined as the invariants of Σ_p on $I_n^{\otimes p}$. It would be of considerable interest to give an explicit description for e_r for $r \geq 2$; even for e_2, this is a considerable challenge, for we require an extension of $I_n^{(2)}$ by itself of length $2p$.

The core of the proof of Theorem 2 utilizes standard complexes, the exact Koszul complex and the DeRham complex whose cohomology is known by a theorem of P. Cartier [Ca]. Our strategy is taken from V. Franjou, J. Lannes, and L. Schwartz (cf. [FLS]). It appears to be essential to first work "stably with respect to n" rather than work directly with rational GL_n-modules.

Indeed, we introduce the concept of a *strict polynomial functor* on finite dimensional k-vector spaces and our computations of Ext-groups occur in this abelian category $\mathcal{P} = \mathcal{P}_k$ (with enough projective and injective objects). There is a natural transformation

$$\mathcal{P} \to \mathcal{F}$$

from $\mathcal{P}$ to the category $\mathcal{F}$ of all functors from finite dimensional k-vector spaces to k-vector spaces. If k is a finite field, this "forgetful" natural transformation is not faithful. If $F \in \mathcal{F}$, then the difference functor $\Delta(F)$ is defined by $\Delta(F)(V) = ker\{F(V \oplus k) \to F(V)\}$. A functor $F \in \mathcal{F}$ is said to be polynomial if $\Delta^N(F) = 0$ for $N >> 0$; each strict polynomial functor when viewed in $\mathcal{F}$ is a polynomial functor. There is a well defined formulation of the degree of $P \in \mathcal{P}$ which has the property that this is greater than or equal to the degree of P when viewed as a polynomial functor in $\mathcal{F}$. One very useful property of $\mathcal{P}$ is that it splits as a direct sum of categories $\mathcal{P}_d$ of strict polynomial functors homogeneous of degree d.

The extension (1.2.1) arises from the extension of strict polynomial functors of degree p

$$0 \to I^{(1)} \to S^p \to \Gamma^p \to I^{(1)} \to 0$$

by evaluation on the vector space k^n. We prove that Ext_{GL_n}-groups can be computed as Ext-groups in the category of strict polynomial functors. Indeed, in a recent paper with V. Franjou and A. Scorichenko and A. Suslin, we prove the following theorem (a weak version of which was proved independently by N. Kuhn [K]). This theorem incorporates earlier results of the author and A. Suslin [A-S] as well as W. Dwyer's stability theorem [D] for the cohomology of the finite groups $GL_n(\mathbb{F}_q)$.

Theorem 1.3 [FFSS]. *Set k equal to the finite field $\mathbb{F}_q$ for q a power of p. Let $\mathcal{P}_{\mathbb{F}_q}$ denote the category of strict polynomial functors on finite dimensional $\mathbb{F}_q$-vector spaces and let $\mathcal{F}_{\mathbb{F}_q}$ denote the category of polynomial functors from finite dimensional $\mathbb{F}_q$-vector spaces to $\mathbb{F}_q$-vector spaces. For $P, Q \in \mathcal{P}_{\mathbb{F}_q}$ of degree d, there is a natural commutative diagram of Ext-groups*

$$\begin{array}{ccc} Ext^i_{\mathcal{P}_{\mathbb{F}_q}}(P^{(r)}, Q^{(r)}) & \longrightarrow & Ext^i_{GL_{n,\mathbb{F}_q}}(P^{(r)}(\mathbb{F}_q^n), Q^{(r)}(\mathbb{F}_q^n)) \\ \downarrow & & \downarrow \\ Ext^i_{\mathcal{F}_{\mathbb{F}_q}}(P, Q) & \longrightarrow & Ext^i_{GL_n(\mathbb{F}_q)}(P(\mathbb{F}_q^n), Q(\mathbb{F}_q^n)) \end{array}$$

which satisfies the following:

(a.) The upper horizontal arrow is an isomorphism provided that $n \geq dp^r$. (This is valid with $\mathbb{F}_q$ replaced by an arbitary field k of characteristic p.)

(b.) $Ext^i_{\mathcal{P}_{\mathbb{F}_q}}(P^{(r)}, Q^{(r)}) \equiv Ext^i_{\mathcal{P}_k}(P^{(r)}, Q^{(r)}) \otimes_{\mathbb{F}_q} k$ for any field extension $k/\mathbb{F}_q$.

(c.) The lower horizontal arrow is an isomorphism for $n >> i, d$.

(d.) The left vertical map is an isomorphism for $r \geq log_p(\frac{i+1}{2})$ provided that $q \geq d$.

In proving part (c.) of Theorem 1.3 in [FFSS,App.1], A. Suslin verifies a conjecture of S. Betley and T. Pirashvili asserting that "stable K-theory equals topological Hochschild homology" for finite functors on $\mathbb{F}_q$-vector spaces.

We say that a sequence $A^0, A^1, \ldots, A^n, \ldots$ of functors (respectively, strict polynomial functors) is *exponential* (resp., exponential strict polynomial) if

$$A^0(k) = k, \quad A^n(V \oplus W) \equiv \bigoplus_{m=0}^{n} A^m(V) \otimes A^{n-m}(W).$$

Examples of exponential strict polynomial functors are the identity functor, the symmetric power, the divided power, the tensor power and their Frobenius twists. The following proposition, which first arose in the context of additive functors in the work of T. Pirashvili [P], appears to distill an essential feature of Ext-groups in categories of functors which does not hold for Ext-groups for rational G-modules. This property (and the injectivity of symmetric functors in the category $\mathcal{P}$) much facilitates computations.

Proposition 1.4 [FFSS]. *Let A^* be an exponential strict polynomial functor and let B, C be strict polynomial functors. Then we have a natural isomorphism*

$$Ext^*_{\mathcal{P}}(A^n, B \otimes C) \simeq \bigoplus_{m=0}^{n} Ext^*_{\mathcal{P}}(A^m, B) \otimes Ext^*_{\mathcal{P}}(A^{n-m}, C).$$

Proposition 1.4 also holds if the category $\mathcal{P}$ of strict polynomial functors is replaced by the category $\mathcal{F}$. However, computations are made much easier in $\mathcal{P}$ because the splitting $\mathcal{P} = \oplus_{d \geq 0} \mathcal{P}_d$ implies that $Ext^*_{\mathcal{P}}(P, Q) = 0$ whenever P, Q are homogeneous strict polynomial functors of different degree.

For the computation of $Ext^*_{\mathcal{P}}(I^{(r)}, I^{(r)})$ needed to prove Theorem 2, we merely require the special case of Proposition 1.4 in which A is linear (i.e., $A^n = 0$ for $n > 1$): as a Hopf algebra,

$$Ext^*_{\mathcal{P}}(I^{(r)}, I^{(r)}) \simeq (k[t]/t^{p^r})^{\#},$$

generated as an algebra by the classes $e_r, e_{r-1}^{(1)}, \dots, e_1^{(r-1)}$ each of which has p-th power equal to 0. The generality of Proposition 1.4 is employed in [FFSS] to give the complete calculation of the tri-graded Hopf algebras

$$Ext^*_{\mathcal{P}}(\Gamma^{*(j)}, S^{*(r)}), \quad Ext^*_{\mathcal{P}}(\Gamma^{*(j)}, \Lambda^{*(r)}), \quad Ext^*_{\mathcal{P}}(\Gamma^{*(j)}, \Gamma^{*(r)}),$$

$$Ext^*_{\mathcal{P}}(\Lambda^{*(j)}, S^{*(r)}), \quad Ext^*_{\mathcal{P}}(\Lambda^{*(j)}, \Lambda^{*(r)}), \quad Ext^*_{\mathcal{P}}(S^{*(j)}, S^{*(r)}),$$

as well as complete calculations of the corresponding $Ext_{\mathcal{F}_{\mathbb{F}_q}}$ tri-graded Hopf algebras.

§2. $H^*(G, k)$ and 1-parameter Subgroups

For simple algebraic groups G, the author and B. Parshall [FP2], [FP4] computed $H^*(G_{(1)}, k)$ provided that the Coexter number $h(G)$ of G satisfies $h(G) < 3p - 1$ (except in type G_2, in which case the bound was $h(G) < 4p - 1$). This bound has been improved to $h(G) < p$ by H. Andersen and J. Jantzen [A-J]. The answer is intriguingly geometric: the algebra $H^*(G_{(1)}, k)$ is concentrated in even degrees and is isomorphic to the coordinate algebra of the nilpotent cone $\mathcal{N} \subset g = Lie(G)$. For "small" p, a precise determination of $H^*(G_{(1)}, k)$ appears to be quite difficult. Our model for the "computation" of $H^*(G, k)$ for an infinitesimal group scheme G is D. Quillen's identification [Q1] of the maximal ideal spectrum $SpecH^{ev}(\pi, k)$ of the cohomology of a finite group π as the colimit of the linear varieties $E \otimes_{\mathbf{F}_p} k$ indexed by the category of elementary abelian p-subgroups $E \subset \pi$.

We shall frequently use $|G|$ to denote the affine scheme associated to the commutative k-algebra $H^{ev}(G,k)$. In other words, $|G| = SpecH^{ev}(G,k)$. Following work of the author and B. Parshall, J. Jantzen [J] proved for any infinitesimal group scheme G of height 1 that the natural map $|G| \to g = Lie(G)$ has image the closed subvariety of p-nilpotent elements $X \in g$ (i.e., elements X such that $X^{[p]} = 0$).

In this section, we describe work of the author, Christopher Bendel, and Andrei Suslin which identifies the affine scheme $|G|$ up to universal finite homeomorphism for any infinitesimal group scheme G. Our identification is in terms of the scheme of "1-parameter subgroups" G. Recall that the infinitesimal group scheme G is said to have *height* r provided that r is the least integer for which G can be embedded as a closed subgroup of some $GL_{n(r)}$. By an abuse of notation, we call a homomorphism $\mathbb{G}_{a(r)} \to G$ an *infinitesimal of height* r *1-parameter subgroup* of G. (We use the notation $\mathbb{G}_a$ to denote the additive group whose coordinate algebra is the polynomial ring in 1 variable). We verify that the functor on finite commutative k-algebras which sends the algebra A to the set of infinitesimal of height r 1-parameter subgroups of $G \otimes A$ over A is representable by an affine scheme $V_r(G)$:

$$Hom_{k-alg}(k[V_r(G)], A) = Hom_{A-group\ schemes}(\mathbb{G}_{a(r)} \otimes A, G \otimes A).$$

Here, $G \otimes A$ is the A-group scheme obtained from G over k by base change via $k \to A$. Clearly, $V_r(G) = V_r(G_{(r)})$.

For $G = GL_n$, $V_r(GL_n)$ is the closed reduced subscheme of $gl_n^{\times r}$ consisting of r-tuples of p-nilpotent, pairwise commuting matrices. A similar description applies for G equal to symplectic, orthogonal, and special linear algebraic groups and various closed subgroups of these groups [SFB1]. An explicit description of $V_r(G)$ is lacking for any arbitrary algebraic group G.

The following determination of $H^*(\mathbb{G}_{a(r)}, k)$ by E. Cline, B. Parshall, L. Scott, and W. van der Kallen is fundamental.

Theorem 2.1 [CPSK].

1. *Assume that $p \neq 2$. Then the cohomology algebra $H^*(\mathbb{G}_a, k)$ is a tensor product of a polynomial algebra $k[x_1, x_2, ...]$ in generators x_i of degree 2 and an exterior algebra $\Lambda(\lambda_1, \lambda_2, ...)$ in generators λ_i of degree one. If $p = 2$, then $H^*(\mathbb{G}_a, k) = k[\lambda_1, \lambda_2, ...]$ is a polynomial algebra in generators λ_i of degree 1; in this case, we set $x_i = \lambda_i^2$.*
2. *Let $F : \mathbb{G}_a \to \mathbb{G}_a$ denote the Frobenius endomorphism, then $F^*(x_i) = x_{i+1}$, $F^*(\lambda_i) = \lambda_{i+1}$.*
3. *Let s be an element of k and use the same notation s for the endomorphism (multiplication by s) of $\mathbb{G}_a$. Then $s^*(x_i) = s^{p^i} x_i$, $s^*(\lambda_i) = s^{p^{i-1}} \lambda_i$.*
4. *Restriction of x_i and λ_i to $\mathbb{G}_{a(r)}$ is trivial for $i > r$. Denoting the restrictions of x_i and λ_i (for $i \leq r$) to $\mathbb{G}_{a(r)}$ by the same letter we have*

$$\begin{aligned} H^*(\mathbb{G}_{a(r)}, k) &= k[x_1, ..., x_r] \otimes \Lambda(\lambda_1, ..., \lambda_r) && p \neq 2 \\ H^*(\mathbb{G}_{a(r)}, k) &= k[\lambda_1, ..., \lambda_r] && p = 2. \end{aligned}$$

The class $x_r \in H^2(\mathbb{G}_{a(r)}, k)$ plays a special role for us, as can be seen both in the following proposition and in Theorem 3.1.

PROPOSITION 2.2 [SFB1]. *For any affine group scheme G, there is a natural homomorphism of graded commutative k-algebras*

$$\psi : H^{ev}(G,k) \to k[V_r(G)]$$

which multiplies degrees by $\frac{p^r}{2}$. For an element $a \in H^{2n}(G,k)$, $\psi(a)$ is the coefficient of x_r^n in the image of a under the composition

$$H^*(G,k) \to H^*(G,k) \otimes k[V_r(G)] = H^*(G \otimes k[V_r(G)], k[V_r(G)])$$

$$\xrightarrow{u^*} H^*(\mathbb{G}_{a(r)} \otimes k[V_r(G)], k[V_r(G)]) = H^*(\mathbb{G}_{a(r)}, k) \otimes k[V_r(G)],$$

where $u : \mathbb{G}_{a(r)} \otimes k[V_r(G)] \to G \otimes k[V_r(G)]$ is the universal infinitesimal of height r 1-parameter subgroup of G.

Alternatively, the map of schemes $\Psi : V_r(G) \to SpecH^{ev}(G,k)$ is obtained by sending a K-point of $V_r(G)$ coresponding to a 1-parameter subgroup $\nu : \mathbb{G}_{a(r)} \otimes K \to G \otimes K$ to the K-point of $SpecH^{ev}(G,k)$ corresponding to

$$eval_{x_1=1} \circ \epsilon_{K*} \circ \nu^* : H^{ev}(G,K) \to H^{ev}(\mathbb{G}_{a(r)},K) \to H^{ev}(\mathbb{G}_{a(1)},K) \to K.$$

Here, $\epsilon_* : H^*(\mathbb{G}_{a(r)},k) \to H^*(\mathbb{G}_{a(1)},k)$ is induced by the coalgebra map

$$\epsilon : k[\mathbb{G}_{a(r)}] = k[t]/t^{p^r} \to k[s]/s^p = k[\mathbb{G}_{a(1)}] \tag{2.2.1}$$

defined to be the k-linear map sending t^i to 0 if i is not divisible by p^{r-1} and to s^j if $i = jp^{r-1}$. (Note that ϵ_* sends $x_i \in H^2(\mathbb{G}_{a(r)},k)$ to $x_1 \in H^2(\mathbb{G}_{a(1)},k)$ if $i = r$ and to 0 otherwise.)

The following "geometric description" of $H^*(G,k)$ is the assertion that the homomorphism ψ of Proposition 2.2 is an isomorphism modulo nilpotents.

THEOREM 2.3 [SFB2]. *Let G be an infinitesimal group scheme of height $\leq r$. Then the kernel of the natural homomorphism*

$$\psi : H^{ev}(G,k) \to k[V_r(G)]$$

is nilpotent and its image contains all p^r-th powers of $k[V_r(G)]$.

In particular, the associated map of affine schemes

$$\psi : V_r(G) \to |G|$$

is a finite universal homeomorphism.

The proof of Theorem 2.3 splits naturally into two parts. We first prove surjectivity modulo nilpotents as stated in the following theorem.

THEOREM 2.4 [SFB1]. *The homomorphism ϕ of Theorem 1.2 factors as the composition*

$$j^* \circ \overline{\phi} : \bigotimes_{i=1}^{r} S^*(gl_n^{(r)\#}[2p^{i-1}]) \to k[V_r(GL_n)] \to H^*(GL_{n(r)}, k),$$

where j denotes the natural closed embedding $V_r(GL_n) \subset gl_n^{\times r}$. Moreover, the composition

$$\psi \circ \overline{\phi} : k[V_r(GL_n)] \to H^*(GL_{n(r)}, k) \to k[V_r(GL_n)]$$

equals F^r, the r-th iterate of Frobenius.

Theorem 2.4 provides surjectivity modulo nilpotents for any closed subgroup $G \subset GL_{n(r)}$ by the surjectivity of $k[V_r(GL_n)] \to k[V_r(G)]$ and the naturality of ψ.

The proof of Theorem 2.4 entails the study of characteristic classes

$$e_r(j)(G, V) \in Ext_G^{2j}(V^{(r)}, V^{(r)})$$

associated to a rational representation $G \to GL(V)$ and the universal class

$$e_r(j) = \frac{(e_1^{(r-1)})^{j_0}(e_2^{(r-2)})^{j_1} \ldots e_r^{j_{r-1}}}{(j_0!)(j_1!)\cdots(j_{r-1}!)} \in Ext_{\mathcal{P}_{\mathbb{F}_p}}^{2j}(I^{(r)}, I^{(r)}).$$

In particular, we determine $e_r(j)(\mathbb{G}_{a(r)} \otimes A, V_{\underline{\alpha}})$, where $V_{\underline{\alpha}}$ is the free A-module A^n made into a rational $\mathbb{G}_{a(r)} \otimes A$-module via $\underline{\alpha} : \mathbb{G}_{a(r)} \otimes A \to GL_n \otimes A$ (given by an r-tuple $\alpha_0, \ldots, \alpha_{r-1}$ of p-nilpotent, pairwise commuting matrices in $GL_n(A)$). This determination involves a careful study of coproducts to reduce the problem of identifying these characteristic classes to the special case in which $\underline{\alpha}$ consists of a single non-zero p-nilpotent matrix. These coproducts are in turn identified by investigating characteristic classes for the special case $r = 2$.

The assertion of injectivity modulo nilpotents in Theorem 2.3 is a consequence of the following detection theorem. The generality of this statement is useful when considering the algebras $Ext_G^*(M, M)$ as we do in the next section.

THEOREM 2.5 [SFB2]. *Let G be an infinitesimal group scheme of height $\leq r$ and let Λ be an associative, unital, rational G-algebra. Then $z \in H^n(G, \Lambda)$ is nilpotent if and only if for every field extension K/k and every 1-parameter subgroup $\nu : \mathbb{G}_{a(r)} \otimes K \to G \otimes K$, the class $\nu^*(z_K) \in H^n(\mathbb{G}_{a(r)} \otimes K, \Lambda_K)$ is nilpotent.*

For G unipotent, Theorem 2.5 is proved in a manner similar to that employed by D. Quillen to prove his theorem that the cohomology modulo nilpotents of a finite group is detected on elementary abelian subgroups [Q2]. Namely, analogues of the Quillen-Venkov Lemma [Q-V] and J.-P. Serre's cohomological characterization of elementary abelian p-groups [S] are proved. In contrast to the context of finite groups in which a transfer argument permits reduction to nilpotent groups (i.e., to p-Sylow subgroups), we require a new strategy to extend this detection theorem to arbitrary infinitesimal group schemes. We develop a generalization of a spectral sequence of H. Andersen and J. Jantzen [A-J] in order to relate the cohomology of G/B with coefficients in the cohomology of B to the cohomology of G, where $B \subset G$ is a Borel subgroup.

§3. Geometry for G-modules

As in the case of finite groups, we investigate $Ext^*_G(M,M)$ as a $H^{ev}(G,k)$-module, where G is an infinitesimal group scheme and M a finite dimensional rational G-module. The techniques discussed in the previous section enable us to prove the following analogue of "Carlson's Conjecture", a result proved by G. Avrunin and L. Scott for $k\pi$-modules for a finite group π [A-S]. The proof of Avrunin and Scott involved the study of representations of abelian Lie algebras with trivial restriction. Their result was generalized to an arbitrary finite dimensional restricted Lie algebra by the author and B. Parshall [FP3].

Theorem 3.1 [SFB2]. *Let G be an infinitesimal group scheme of height $\leq r$ and let M be a finite dimensional G-module. Define the closed subscheme*

$$|G|_M \ \subset \ |G| = SpecH^{ev}(G,k)$$

to be the the reduced closed subscheme defined by the radical of the annihilator ideal of $Ext^*_G(M,M)$. Define the closed subscheme

$$V_r(G)_M \ \subset \ V_r(G)$$

to be the reduced closed subscheme whose K-points for any field extension K/k are those 1-parameter subgroups $\nu : \mathbb{G}_{a(r)} \otimes K \to G \otimes K$ with the property that $\epsilon_{K*} \circ \nu^*(M)$ is a projective $\mathbb{G}_{a(1)}$-module (where ϵ is given in (2.1.1)). Then the finite universal homeomorphism

$$\Phi : V_r(G) \to |G|$$

of Theorem 2.3 satisfies

$$\Phi^{-1}(|G|_M) = V_r(G)_M.$$

Observe that $|G|_M$ is essentially cohomological in nature whereas $V_r(G)_M$ is defined without reference to cohomology. Properties of $V_r(G)_M$ (and thus of $|G|_M$) can often be verified easily.

Proposition 3.2. *(cf. [SFB2]) Let G be an infinitesimal group scheme and let $\mathcal{G}(G)$ denote the reduced Green ring of isomorphism classes of finite dimensional rational G-modules modulo projectives. Then sending M to $V_r(G)_M$ determines a function*

$$\Theta : \mathcal{G}(G) \to \{reduced,\ closed,\ conical\ subschemes\ of\ V_r(G)\}$$

such that

(a.) Θ *is surjective.*
(b.) If $\Theta(M) = pt$, *then M is projective.*
(c.) $\Theta(M \oplus N) = \Theta(M) \cup \Theta(N)$.
(d.) $\Theta(M \otimes N) = \Theta(M) \cap \Theta(N)$.

§4. Speculations Concerning Local Type

We briefly mention a possible formulation of the "local type" of finite dimensional rational G-modules M for infinitesimal group schemes G.

Proposition 4.1. *Let G be an infinitesimal group scheme of height r and let M be a rational G-module of dimension m. Then M determines a conjugacy class of p-nilpotent matrices in $GL_m(k[V_r(G)])$ associated to the rational $G_{a(1)} \otimes k[V_r(G)]$-module $\epsilon_{k[V_r(G)]*} \circ u^*(M \otimes k[V_r(G)])$, where $u : \mathbb{G}_{a(r)} \otimes k[V_r(G)] \to G \otimes k[V_r(G)]$ is the universal infinitesimal of height r 1-parameter subgroup of G and $\epsilon : k[\mathbb{G}_{a(r)}] \to k[\mathbb{G}_{a(1)}]$ is given in (2.2.1).*

The following formulation of local type is one possible "numerical invariant" which we can derive from the matrix of Proposition 4.1.

Definition 4.2. *Let G be an infinitesimal group scheme of height r. For each prime ideal x of $k[V_r(G)]$, let $\nu_x : \mathbb{G}_{a(r)} \otimes k(x) \to G \otimes k(x)$ be the 1-parameter subgroup associated to the residue homomorphism $k[V_r(G)] \to k(x)$, where $k(x) = frac\{k[V_r(G)]/x\}$. For a finite dimensional rational G-module M, we define the local type of M to be the lower semi-continuous function*

$$t_M : V_r(G) \to \mathbb{N}^p,$$

where $t_M(x) = (t_{M,p}(x), \ldots, t_{M,1}(x))$ with $t_{M,i}(x)$ equal to the number of Jordan blocks of size i of $\epsilon_{k(x)*}(\nu_x^*(M \otimes k(x)))$ as a rational $\mathbb{G}_{a(1)} \otimes k(x)$-module.

Thus, the points of $V_r(G)_M$ are those points $x \in V_r(G)$ such that $t_M(x) \neq (\frac{m}{p}, 0, \ldots 0)$.

We conclude with two questions, even partial answers to which would be of considerable interest.

Question 4.3. *Describe in module-thoeretic terms the condition on a pair of rational G-modules M, N that implies $t_M = t_N$.*

Question 4.4. *Characterize those functions $t : V_r(G) \to \mathbb{N}^{p-1}$ for which there exists some finite dimensional rational G-module M with $t = t_M$.*

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