Meager Forking and m-Independence

Ludomir Newelski

Abstract. We describe meager forking, m-independence and related notions of geometric model theory relevant for Vaught's conjecture and more generally for classifying countable models of a superstable theory.

1991 Mathematics Subject Classification: 03C45
Keywords and Phrases: Vaught's conjecture, superstable theory

0 Introduction

Throughout, $T = T^{eq}$ is a complete theory in a countable first-order language L and we work within a large saturated model $\mathfrak{C}$ of T (a monster model). Until section 5 we assume that T is stable. Often we assume that T is small, i.e. $S_n(\emptyset)$ is countable for every $n < \omega$. The general references are [Bu5, Pi].

The main motivation here is Vaught's conjecture for superstable theories. Vaught's conjecture says that if T has $< 2^{\aleph_0}$ countable models, then T has countably many of them. If T is not small, then T has $2^{\aleph_0}$ countable models. So the assumptions that T is small or even that T has $< 2^{\aleph_0}$ countable models appear naturally in many theorems in this paper. Thus far Vaught's conjecture is proved for ω-stable theories [SHM] and superstable theories of finite U-rank [Bu4]. The main tools of Shelah in [SHM] are forking of types and forking independence. These tools are combinatorial in nature. Forking is also the main tool in [Sh]. [Bu4, Ne1, Ne3] indicate that in order to approach Vaught's conjecture for superstable theories we may need some new ideas and tools, of more geometric and algebraic character.

In a series of papers I introduced meager forking, m-independence and other notions intended for a fine analysis of countable models. Meager forking relates forking to the topological structure of the space of types. It is used to show that the topological character of forking is related to the geometry of forking. An important problem arising in the context of Vaught's conjecture is to describe the ways in which a type in a superstable theory may be non-isolated, and also to describe the sets of stationarizations of such a type. Here m-independence and the calculus of traces of types are useful. Apart from their relevance to Vaught's conjecture, these notions may be important for model theory in general. Indeed, in a small stable theory m-independence is the strongest natural notion of independence (on finite tuples) refining forking independence. So there is a hope that with sharper tools we can better describe countable models. The theory of m-independence is in many ways parallel to the theory of forking independence of Shelah [Sh]. Also, restricted to $*$-algebraic tuples, m-independence may be defined in an arbitrary (small) theory T.

Usually, $a, b, c, \ldots$ denote finite tuples and $A, B, C, \ldots$ finite sets of elements of $\mathfrak{C}$. $x, y, z, \ldots$ denote finite tuples of variables.

1 MEAGER FORKING AND MEAGER TYPES

Assume $s(x)$ is a (possibly incomplete) type over $\mathfrak{C}$. $[s]$ denotes the class of types in variables x containing $s(x)$. $s(\mathfrak{C})$ denotes the set of tuples from $\mathfrak{C}$ realizing s. We define the trace of s over A as the set $Tr_A(s) = \{tp(a/acl(A)) : a \in s(\mathfrak{C})\}$, a closed subset of $S(acl(A))$. In particular, for $p \in S(A)$, $Tr_A(p)$ is the set of stationarizations of p over A.

Assume P is a closed subset of $S(acl(A))$. We say that forking is meager on P if for every formula $\varphi(x)$ forking over A, the set $Tr_A(\varphi) \cap P$ is nowhere dense in P (equivalently: for every finite $B \supset A$, the set of types $r \in P$ with a forking extension in $S(acl(B))$ is meager in P). For $p \in S(A)$ we say that forking is meager on p, if forking is meager on $Tr_A(p)$.

Assume r is a stationary regular type. We say that $\varphi(x) \in L(A)$ is an r-formula (over A) if

- every type in $S(acl(A)) \cap [\varphi]$ is either hereditarily orthogonal to r or regular non-orthogonal to r,
- the set $P_\varphi = \{p \in S(acl(A)) \cap [\varphi] : p \not\perp r\}$ is closed and non-empty,
- r-weight 0 is definable on φ, that is whenever $a \in \varphi(\mathfrak{C})$ and $w_r(a/Ac) = 0$, then for some formula $\psi(x, y)$ over $acl(A)$, true of (a, c), if $\psi(a', c')$ holds, then $w_r(a'/Ac') = 0$.

If $P_\varphi = \{p\}$ is a singleton, then we say that p is strongly regular. Strongly regular types were an essential ingredient in describing countable models of an ω-stable theory in [SHM].

For a stationary regular type $r \in S(B)$, forking induces a closure operator cl on $r(\mathfrak{C})$ defined by $a \in cl(X)$ iff $a \not\perp X(B)$, where $\{a\} \cup X \subseteq r(\mathfrak{C})$. cl is a (combinatorial) pregeometry on $r(\mathfrak{C})$ (this is in fact equivalent to regularity of r), which we call the forking geometry on r. We say that r is [locally] modular, if this geometry is [locally] modular. We say that r is non-trivial, if this geometry is non-trivial [Pi].

Locally modular regular types are important in geometric model theory. If r is non-trivial and locally modular, then the associated geometry is either affine or projective over some division ring [Hr1]. By [HS], in a superstable T, for any non-trivial regular type r, r-formulas exist.

DEFINITION 1 ([NE5]) *We say that a regular stationary type r is meager if for some (equivalently: any) r-formula φ, forking is meager on P_φ.*

For instance, every properly weakly minimal non-trivial type is meager.

THEOREM 1 ([NE5]) *Every meager type is non-trivial and locally modular.*

This theorem improves [Bu1, LP]. It shows that the topological character of forking on a regular type is relevant to its geometric properties. Hrushovski and Shelah proved in [HS] that in a superstable theory without the omitting types order property

($*$) every regular type is either locally modular or non-orthogonal to a strongly regular type.

So in this case either the forking geometry on a type is nice or the situation is similar to the ω-stable case. Hrushovski [Hr2] gave an example of a regular type in a superstable theory, for which ($*$) fails.

QUESTION 1 *Does ($*$) hold in any superstable theory with $< 2^{\aleph_0}$ countable models ?*

Following [Ta] we say that a regular type $p \in S(A)$ is eventually strongly non-isolated (esn), if some non-forking extension p' of p over a finite $A' \supset A$ is strongly non-isolated, that is, for every finite $B \supset A'$, p' is almost orthogonal to any isolated type in $S(B)$. Also we say that p is almost strongly regular (asr), via $\varphi \in p$, if φ is a p-formula over A and $P_\varphi = Tr_A(p)$. Since by Theorem 1 every meager type is locally modular, the following characterization of non-trivial esn types is relevant for Question 1.

THEOREM 2 ([NE7]) *Assume T is small superstable and p is a non-trivial regular type. Then p is esn iff (1) or (2) below holds. Moreover, (1) and (2) are mutually exclusive.*
(1) p is non-orthogonal to an almost strongly regular non-isolated type.
(2) p is meager.

2 $\mathcal{M}$-RANK AND M-INDEPENDENCE

In this section T is small and stable. For $p \in S(A)$, $Tr_A(p)$ is either finite or homeomorphic to the Cantor set. We measure traces of types by comparing topologically traces of their various extensions. This is done by means of $\mathcal{M}$-rank and m-independence.

Assume $q \in S(B)$ is a non-forking extension of $p \in S(A)$ ($A \subseteq B$). Then $Tr_A(q)$ is a closed subset of $Tr_A(p)$ and either is open in $Tr_A(p)$ or is nowhere dense in $Tr_A(p)$. In the former case we call q an m-free and in the latter a meager extension of p. So q is an m-free extension of p iff q is isolated in the set of non-forking extensions of p in $S(B)$.

DEFINITION 2 ([NE3]) *The rank function $\mathcal{M}$ is the minimal function defined on the set of all complete types over finite sets, with values in $Ord \cup \{\infty\}$, such that for every $\alpha \in Ord$ we have*
$\mathcal{M}(p) \geq \alpha + 1$ iff $\mathcal{M}(q) \geq \alpha$ for some meager non-forking extension q of p.
$\mathcal{M}(a/A)$ abbreviates $\mathcal{M}(tp(a/A))$. We say that T is m-stable if $\mathcal{M}(p) < \infty$ for every p.

DEFINITION 3 ([NE8]) *We say that a is m-independent from B over A (symbolically: $a\overset{m}{\perp}B(A)$) if $tp(a/A\cup B)$ is an m-free extension of $tp(a/A)$.*

m-independence has similar properties as forking independence.

PROPOSITION 1 ([NE3, NE8]) *(1)(symmetry) If $a\overset{m}{\perp}b(A)$, then $b\overset{m}{\perp}a(A)$.*
(2) (transitivity) $a\overset{m}{\perp}B\cup C(A)$ iff $a\overset{m}{\perp}B(A)$ and $a\overset{m}{\perp}C(A\cup B)$.
(3) $a\overset{m}{\perp}B(A)$ is invariant under automorphisms of $\mathfrak{C}$ and under changes of enumerations of a, A, B.
(4)(acl-triviality) If $B\subseteq acl(A)$, then $a\overset{m}{\perp}B(A)$.
(5) In a small theory, $\overset{m}{\perp}$ has an extension property, i.e. every type $p\in S(A)$ has an m-free extension over any finite $B\supset A$.

THEOREM 3 ([NE10]) *In a small stable theory m-independence is the strongest notion of independence on finite tuples and finite sets of elements of $\mathfrak{C}$, which refines forking independence and has the properties exhibited in Proposition 1.*

In a small stable theory, in the following Lascar-style inequalities

$$\text{(L)}\quad \mathcal{M}(a/Ab)+\mathcal{M}(b/A)\leq\mathcal{M}(ab/A)\leq\mathcal{M}(a/Ab)\oplus\mathcal{M}(b/A)$$

the right side is always true, while the left side holds if $a\perp b(A)$ (that is, if a, b are forking-independent over A).

In a small superstable theory $\mathcal{M}$-rank may be used to find meager types [Ne6] (similarly as U-rank considerations lead to regular types [Ls]). To find many such types we need types of large (infinite, but $<\infty$) $\mathcal{M}$-rank, to begin with. Unfortunately, no such types are known in a small stable theory.

CONJECTURE 1 ([NE7, THE $\mathcal{M}$-GAP CONJECTURE]) *In a small stable theory there is no type p with $\omega\leq\mathcal{M}(p)<\infty$.*

This conjecture is true for superstable theories under the few models assumption.

THEOREM 4 ([NE5, NE7]) *If T is superstable with $<2^{\aleph_0}$ countable models, then T is m-stable. Moreover, for every type p, $\mathcal{M}(p)$ is finite and $\leq U(p)$.*

The proof of this theorem relies on the construction of some meager types and the analysis of traces of some types in the associated meager groups (defined below). The special case of theorem 4, where T is weakly minimal and $U(p)=1$, was conjectured by Saffe and proved in [Ne1]. It was decisive in the proof of Vaught's conjecture for weakly minimal theories [Bu3, Ne1].

Using the notions of $\mathcal{M}$-rank and m-independence we get the following description of traces of types.

THEOREM 5 ([NE8, THE TRACE THEOREM]) *If T is superstable with $<2^{\aleph_0}$ countable models, then for every $p\in S(A)$ there is a formula $\varphi(x)$ (usually not in p) with $Tr_A(\varphi)=Tr_A(p)$. In particular, if p is regular and forking is meager on p, then p is isolated and meager.*

[Ne8, Theorem 2.6] contains more information on traces of meager types.

Regarding Theorem 3 we should mention that there is a notion of independence intermediate between $\overset{m}{\perp}$ and $\perp$. Namely, assume again $q \in S(B)$ is a non-forking extension of $p \in S(A)$. On $Tr_A(p)$ there is a natural probabilistic Haar measure, invariant under $Aut(\mathfrak{C}/A)$. We say that q is a μ-free extension of p if $Tr_A(q)$ has positive measure in $Tr_A(p)$. This leads to the notion of μ-independence $\overset{\mu}{\perp}$ (implicitly used in [LS]), having the properties from Proposition 1 [Ne8]. Also, $\overset{m}{\perp} \Rightarrow \overset{\mu}{\perp} \Rightarrow \perp$.

Tanovic proved that m-independence and μ-independence are equal in an m-stable theory [Ne8], and I proved there that they are equal in an m-normal theory (defined below). In particular, by Theorem 4 we could say that in a superstable theory with $< 2^{\aleph_0}$ countable models, "measure equals category". No theory is known in which these two notions of independence differ.

3 THE $\mathcal{M}$-GAP CONJECTURE AND M-NORMAL THEORIES

In this section we assume T is small stable. In an attempt to refute the $\mathcal{M}$-gap conjecture I constructed in [Ne8] small weakly minimal groups with types of various $\mathcal{M}$-ranks. However the traces of types in these groups are not complicated, they are just translates of traces of some generic subgroups. This leads to the definition of an m-normal theory.

DEFINITION 4 ([NE8]) *T is m-normal if for every finite $A \subseteq B$ and $a \in \mathfrak{C}$, for some $E \in FE(A)$, the set $Tr_A(a/B) \cap [E(x,a)]$ has finitely many conjugates over Aa.*

The idea underlying this definition is that in an m-normal theory, locally $Tr_A(a/B)$ can be almost recovered from Aa alone. This corresponds to the condition $Cb(a/A) \subseteq acl(a)$, defining 1-based theories.

There is an evident analogy between the theory of m-independence and the theory of forking independence: meager forking, $\mathcal{M}$-rank, meager types, m-stability correspond to forking, U-rank, regular types, superstability. (Unfortunately in the theory of m-independence there is no good counterpart of the notion of a stationary type.) m-normality corresponds to 1-basedness. In order to justify this we need to introduce $*$-finite tuples, which play for m-independence a role similar to imaginaries in forking.

DEFINITION 5 ([NE8]) *(1) A $*$-finite tuple is a tuple $a_I = \langle a_i, i \in I\rangle$ of elements of $\mathfrak{C}$ (with the index set I countable), such that $a_I \subseteq dcl(a)$ for some finite tuple a of elements of $\mathfrak{C}$. Moreover, we say that a_I is $*$-algebraic over A if $a_I \subseteq acl(A)$.*
(2) $S_I(A)$ denotes the space of complete types over A, in variables $x_I = \langle x_i, i \in I\rangle$. If a_I is $*$-finite [$*$-algebraic over A], then we call $tp(a_I/A)$ $*$-finite [$*$-algebraic].

EXAMPLE 1 Let $p = tp(a/A) \in S(A)$. Then $a^* = \langle a/E : E \in FE(A)\rangle$ is a $*$-finite $*$-algebraic over A tuple naming $tp(a/acl(A))$ over A.

EXAMPLE 2 Let $G \subseteq \mathfrak{C}$ be a group definable over A and let $G_n, n < \omega$, be a sequence of A-definable subgroups of finite index in G with $G^0 = \cap_n G_n$ (G^0 is the connected component of G). Then an element a/G^0 of G/G^0 may be regarded as a $*$-finite $*$-algebraic over A tuple $\langle a/G_n, n<\omega\rangle$. So G/G^0 is a $*$-finite $*$-algebraic group.

The definitions of forking, traces of types, $\mathcal{M}$-rank and m-independence work also for $*$-finite tuples and $*$-finite types. From now on we let $a,b,c,\dots$ denote $*$-finite tuples and $A,B,C,\dots$ finite sets of $*$-finite tuples of elements of $\mathfrak{C}$. Finite tuples or sets of elements of $\mathfrak{C}$ will be called standard.

Most importantly, in the new set-up Proposition 1 remains valid, also (L) holds in the same way as for standard tuples. Theorem 4 is true, except that for a $*$-finite type p, $\mathcal{M}(p)$ may be larger than $U(p)$. Unfortunately Theorem 5 does not hold for $*$-finite types. The change of the set-up does not affect the value of the $\mathcal{M}$-ranks of standard types. Also, in Example 1, $\mathcal{M}(a/A)=\mathcal{M}(a^*/A)$. This is an important point, showing that $*$-algebraic tuples are the backbone of $\mathcal{M}$-rank and m-independence. If $p\in S_I(A)$ is $*$-algebraic, then there is a natural correspondence between $Tr_A(p)$ and $p(\mathfrak{C})$, inducing on $p(\mathfrak{C})$ a compact topology.

The next theorem explains the definition of an m-normal theory with the help of $*$-algebraic tuples, making it similar to the definition of a 1-based theory using imaginaries.

THEOREM 6 ([NE7, NE12]) *T is m-normal iff for every finite A and a,b $*$-algebraic over A, there is a $c\in acl_A(a)\cap acl_A(b)$ with $a\overset{m}{\perp}b(Ac)$.*

Here $c\in acl_A(a)$ means that c has finitely many Aa-conjugates. For an infinite set I of $*$-finite tuples, $c\in acl(I)$ means that $c\in acl(I_0)$ for some finite $I_0\subset I$.

Buechler characterized 1-based theories among superstable theories of finite rank as those where every U-rank 1 type is locally modular [Bu2]. This explains the geometric importance of 1-basedness. In the case of m-normality we can give a similar description. Since $*$-algebraic types are the backbone of m-independence, this description refers to some geometries on $*$-algebraic types of $\mathcal{M}$-rank 1.

Assume $p\in S_I(A)$ is $*$-algebraic, of $\mathcal{M}$-rank 1. We say that $I\subseteq p(\mathfrak{C})$ is a flat Morley sequence in p if I is countably infinite, m-independent over A and dense in $p(\mathfrak{C})$ (by [Ne10], such an I is unique up to $Aut(\mathfrak{C}/A)$). Now acl_A induces a pregeometry on $p(\mathfrak{C})$ (just like acl induces the forking geometry on a U-rank 1 type). We say that p is locally modular if for some flat Morley sequence I in p, the localized acl_{AI}-geometry on $p(\mathfrak{C})$ is modular.

We define the notion of [almost] m-orthogonality analogously to the corresponding definition in the theory of forking. We say that T has weak m-coordinatization if every $*$-algebraic type of $\mathcal{M}$-rank >0 is m-nonorthogonal to a $*$-algebraic type of $\mathcal{M}$-rank 1. We say that T has full m-coordinatization if for every A and a $*$-algebraic over A with $\mathcal{M}(a/A)>0$, there is some $b\in acl_A(a)$ with $\mathcal{M}(b/A)=1$.

The next three theorems justify our interest in m-normal theories.

THEOREM 7 ([NE12]) *Assume T is small, of finite $\mathcal{M}$-rank. Then the following are equivalent.*
(1) T is m-normal.
(2) T has full m-coordinatization and every $*$-algebraic $\mathcal{M}$-rank 1 type is locally modular.

(3) T has weak m-coordinatization and every $*$-algebraic $\mathcal{M}$-rank 1 type is locally modular.

THEOREM 8 ([NE8, NE12]) *In an m-normal theory there is no type p with $\omega \leq \mathcal{M}(p) < \infty$.*

So for m-normal theories the $\mathcal{M}$-gap conjecture is true. The small weakly minimal groups referred to at the beginning of this section are m-normal. I know no small theory, which is not m-normal.

THEOREM 9 ([NE11, NE12]) *If T is superstable with $< 2^{\aleph_0}$ countable models, then T is m-normal.*

Regarded as properties of m-independence, Theorems 4,7 and 9 correspond to the result from [CHL] saying that every $\aleph_0$-stable $\aleph_0$-categorical theory has finite Morley rank and is 1-based. [Ne11] contains more information on $*$-algebraic types of $\mathcal{M}$-rank 1 in superstable theories with $< 2^{\aleph_0}$ countable models.

4 MEAGER GROUPS

Meager groups are some definable groups of standard elements of $\mathfrak{C}$. First we shall define however the notion of a $*$-finite group. We say that G is a $*$-finite group if G is a type-definable group consisting of uniformly $*$-finite tuples, that is for some finite set A and a tuple $f_I = \langle f_i, i \in I \rangle$ of A-definable functions, $G = \{f_I(a) : a \in X\}$ for some set $X \subseteq \mathfrak{C}$ type-definable over A. $\mathcal{G} \subseteq S_I(acl(A))$ denotes the set of generic types of G. For $B \supseteq A$ we say that $a_I \in G$ is m-generic over B (and $tp(a_I/B)$ is m-generic) if $a_I \overset{m}{\perp} B(A)$, $tp(a_I/acl(A)) \in \mathcal{G}$ and $Tr_A(a_I/B)$ is open in $\mathcal{G}$. We define $\mathcal{M}(G)$ as $\mathcal{M}(p)$ for any m-generic type p of elements of G. Also there is a natural group structure on $\mathcal{G}$, given by independent multiplication of types [Ne2]. G is called $*$-algebraic if elements of G are $*$-algebraic over A (the group from Example 2 is a good example here).

Now assume $G \subseteq \mathfrak{C}$ is an A-definable regular abelian group in a stable theory. As above, $\mathcal{G} \subseteq S(acl(A))$ denotes the set of generic types of G. Let $p \in \mathcal{G}$ be the generic type of G^0, the connected component of G. Notice that G is a p-formula and $\mathcal{G} = P_G$. So p is meager iff forking is meager on $\mathcal{G}$. In this case we call G a meager group.

By [Hr1], for any locally modular regular type q there is a regular group non-orthogonal to q, so every meager type is non-orthogonal to a meager group. We will say more on such groups.

Assume G is a locally modular regular abelian group definable over A. Let $\mathcal{G}m$ denote the set of modular types in $\mathcal{G}$ (so $p \in \mathcal{G}m$ and $\mathcal{G}m$ is a subgroup of $\mathcal{G}$). Let Gm (the modular component of G) be the subgroup of G generated by the realizations of types in $\mathcal{G}m$. In a small theory, $\mathcal{G}m$ is closed in $\mathcal{G}$ and $\mathcal{G} \setminus \mathcal{G}m$ is open in $S(acl(A))$ [Ne5].

THEOREM 10 ([NE5, NE7]) *Assume T is superstable with $< 2^{\aleph_0}$ countable models and $G \subseteq \mathfrak{C}$ is a locally modular regular abelian group definable over $\emptyset$. Then:*
(1) G is meager iff $[G : Gm] = [\mathcal{G} : \mathcal{G}m]$ is infinite iff $\mathcal{G}m$ is nowhere dense in $\mathcal{G}$.

(2) If G is meager, then $\mathcal{M}(G) = \mathcal{M}(Gm) + 1$.
(3)(generalized Saffe's condition) If G is meager and $a \in G$ is generic over A, then exactly one of the following conditions holds:
(a) $Tr_\emptyset(a/A)$ is open in $\mathcal{G}$ (i.e. a is m-generic over A and $tp(a/A)$ is isolated).
(b) $Tr_\emptyset(a/A)$ is contained in finitely many cosets of $\mathcal{G}m$ (so it is nowhere dense and $tp(a/A)$ is non-isolated).

Also, with every locally modular group G we associate a division ring F_G of definable pseudo-endomorphisms of G^0, and forking dependence on G^0 is essentially the linear dependence over F_G [Hr1]. Now if G is meager, then F_G is a locally finite field and every element of F_G is definable over $acl(\emptyset)$ [Lo, Ne5].

Using the above ideas we can prove Vaught's conjecture for some superstable theories of infinite rank. For instance, we have the following theorem.

THEOREM 11 ([NE9]) *Assume $T = Th(G)$, where G is a meager group of U-rank ω and $\mathcal{M}$-rank 1, with F_G being a prime field. Then Vaught's conjecture is true for T.*

The proof of this theorem uses also ideas from [Bu3] and from [Ne3, Ne4] on describing models piece-by-piece. This leads to some "relative Vaught's conjecture" results, which consist in the following.

Suppose $\Phi(x)$ is a countable disjunction of formulas in T. Then we can consider the restricted (many-sorted) theory $T\lceil\Phi = Th(\Phi(\mathfrak{C}))$. Proving Vaught's conjecture for T *relative* to Φ means proving Vaught's conjecture for T under the assumption of Vaught's conjecture for $T\lceil\Phi$. [Ne9, Ne13] contain some results of this form. $T = Th(G)$ for some meager group G there and $\Phi(x)$ is a disjunction of formulas such that $\Phi(G) = G^- = \{a \in G : a \text{ is non-generic}\}$, or $\Phi(G) = Gm$.

5 A GENERALIZATION

As mentioned in section 3, $*$-algebraic tuples are the backbone of m-independence. Definition 3 (of m-independence) makes sense in an arbitrary theory if a is $*$-algebraic over A. m-independence restricted to $*$-algebraic tuples has all the properties from Proposition 1 (but smallness is needed to get (5)). Then (1)-(5) from Proposition 1 imply (L), which for $*$-algebraic tuples holds fully (because when a, b are $*$-algebraic over A, then $a \perp b(A)$). Also, Theorems 7 and 8 hold for an arbitrary small theory (or even just for a theory, where $*$-algebraic tuples satisfy conditions (1)-(5) from Proposition 1) [Ne12]. This suggests a possibility of applying m-independence in an unstable context.

Hrushovski and Pillay prove in [HP] that every 1-based group is abelian-by-finite. In [Ne12] I develop a theory of $*$-algebraic groups in a small m-normal theory parallel in some respects to [HP].

THEOREM 12 ([NE12]) *Assume G is a $*$-algebraic group type-definable over $\emptyset$, in a small m-normal theory. Assume $a \in G$ and $p = tp(a/A)$. Then $p(G)$ is a finite union of cosets of subgroups of G definable over parameters algebraic over $\emptyset$. Also, G is abelian-by-finite.*

Any $*$-algebraic group is a topological profinite group. It would be interesting to extract the topological content of Theorem 12. Since the group G/G^0 from Example 2 is $*$-algebraic, we get the following surprising corollary.

COROLLARY 1 *Assume G is a (standard) group interpretable in a superstable theory with $< 2^{\aleph_0}$ countable models. Then G/G^0 is abelian-by-finite.*

QUESTION 2 *Is any $*$-algebraic group interpretable in a small (stable) theory abelian-by-finite ?*

Regarding this question we should mention that by the results from [Ba], if G is a standard group interpretable in a superstable theory, then G/G^0 is solvable-by-finite, and if additionally $\mathcal{M}(G/G^0) = 1$, then G/G^0 is abelian-by-finite.

REFERENCES

[Ba] A.Baudisch, *On superstable groups*, Journal of London Mathematical Society (2) 42(1990), 452-464.

[Bu1] S.Buechler, *The geometry of weakly minimal types*, Journal of Symbolic Logic 50(1985), 1044-1053.

[Bu2] S.Buechler, *Locally modular theories of finite rank*, Annals of Pure and Applied Logic 30(1986), 83-95.

[Bu3] S.Buechler, *Classification of small weakly minimal sets I*, in: CLASSIFICATION THEORY, PROCEEDINGS, CHICAGO 1985, ed. J.T.Baldwin, Springer 1987, 32-71.

[Bu4] S.Buechler, *Vaught's conjecture for superstable theories of finite rank*, Ann.Pure Appl.Logic, to appear.

[Bu5] S.Buechler, ESSENTIAL STABILITY THEORY, Springer 1996.

[CHL] G.Cherlin, L.Harrington, A.Lachlan, *$\aleph_0$-categorical $\aleph_0$-stable structures*, Ann.Pure Appl.Logic 28(1986), 103-135.

[Hr1] E.Hrushovski, *Locally modular regular types*, in: CLASSIFICATION THEORY, PROCEEDINGS, CHICAGO 1985, ed. J.T.Baldwin, Springer 1987, 132-164.

[Hr2] E.Hrushovski, *A new strongly minimal set*, Ann.Pure Appl.Logic 62(1993), 147-166.

[HP] E.Hrushovski, A.Pillay, *Weakly normal groups*, in: LOGIC COLLOQUIUM'85, ed. Paris Logic Group, North Holland 1987, 233-244.

[HS] E.Hrushovski, S.Shelah, *A dichotomy theorem for regular types*, Ann.Pure Appl.Logic 45(1989), 157-169.

[Ls] D.Lascar, *Relation entre le rang U et le poids*, Fundamenta Mathematicae 121(1984), 117-123.

[LS] M.C.Laskowski, S.Shelah, *Forcing isomorphism II*, J.Symb.Logic 61(1996), 1305-1320.

[Lo] J.Loveys, *Abelian groups with modular generic*, J.Symb.Logic 56(1991), 250-259.

[LP] L.F.Low, A.Pillay, *Superstable theories with few countable models*, Archive for Mathematical Logic 31(1992), 457-465.

[Ne1] L.Newelski, *A proof of Saffe's conjecture*, Fund.Math. 134(1990), 143-155.

[Ne2] L.Newelski, *On type-definable subgroups of a stable group*, Notre Dame Journal of Formal Logic 32(1991), 173-187.

[Ne3] L.Newelski, *A model and its subset*, J.Symb.Logic 57(1992), 644-658.

[Ne4] L.Newelski, *Scott analysis of pseudo-types*, J.Symb.Logic 58(1993), 648-663.

[Ne5] L.Newelski, *Meager forking*, Ann.Pure Appl.Logic 70(1994), 141-175.

[Ne6] L.Newelski, *$\mathcal{M}$-rank and meager types*, Fund.Math. 146(1995), 121-139.

[Ne7] L.Newelski, *$\mathcal{M}$-rank and meager groups*, Fund.Math. 150(1996), 149-171.

[Ne8] L.Newelski, *$\mathcal{M}$-gap conjecture and m-normal theories*, Israel Journal of Mathematics, to appear.

[Ne9] L.Newelski, *Vaught's conjecture for some meager groups*, Israel J.Math., to appear.

[Ne10] L.Newelski, *Flat Morley sequences*, J.Symb.Logic, to appear.

[Ne11] L.Newelski, *Geometry of $*$-finite types*, J.Symb.Logic, to appear.

[Ne12] L.Newelski, *m-normal theories*, preprint.

[Ne13] L.Newelski, *On countable meager groups*, preprint.

[Pi] A.Pillay, GEOMETRIC STABILITY THEORY, Oxford 1996.

[Sh] S.Shelah, CLASSIFICATION THEORY, 2nd ed., North Holland 1990.

[SHM] S.Shelah, L.Harrington, M.Makkai, *A proof of Vaught's conjecture for $\aleph_0$-stable theories*, Israel J.Math. 49(1984), 259-278.

[Ta] P.Tanovic, *Fundamental order and the number of countable models*, Ph.D. thesis, McGill University, December 1993.

Instytut Matematyczny
Uniwersytetu Wrocławskiego
pl.Grunwaldzki 2/4
50-384 WROCŁAW, Poland
e-mail: newelski@math.uni.wroc.pl