Generic Large Cardinals: New Axioms for Mathematics?

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Abstract. This article discusses various attempts at strengthening the axioms for mathematics, *Zermelo-Fraenkel Set Theory with the Axiom of Choice* It focuses on a relatively recent collection of axioms, *generic large cardinals*, their success at settling well known independent problems and their relations to other strengthenings of ZFC, such as large cardinals.

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Introduction. While the standard axiomatization of mathematics *Zermelo-Fraenkel Set Theory with the Axiom of Choice* (ZFC) has been extremely successful in resolving the foundational issues that arose at the turn of the century, it has some shortcomings. These shortcomings are largely due to its inability to settle various natural problems.

Most prominent among these problems are Hilbert's 1^{st} problem (the Continuum Hypothesis), and issues having to do with the use of the Axiom of Choice. The development of Forcing, in the early 1960's, led to independence results in most areas of mathematics that have a strong infinitary character, particularly including measure theory and other parts of analysis, infinite group theory, topology and combinatorics.

This paper surveys some of these independence results and the attempts at finding new axiom systems to settle these questions. It will focus on a technique that arose naturally in relating large cardinals with combinatorial and descriptive set theoretic properties of sets of size (roughly) the continuum. This technique generated plausible properties of the universe. Taken as axioms they settle most of the important independent statements of mathematics.

Without further explanation, the first few uncountable cardinals are $\aleph_1, \aleph_2, \ldots, \aleph_n, \ldots$ and the first uncountable limit cardinal and its successor are $\aleph_\omega$ and $\aleph_{\omega+1}$. The natural numbers will be denoted alternately as $\boldsymbol{N}$ or more commonly ω, the first limit ordinal. The cardinality of the real numbers will be referred to as $\boldsymbol{c}$, and the cardinality of the power set of a set X as 2^X. In particular, $2^\omega = \boldsymbol{c}$. If λ is a cardinal, $n \in \boldsymbol{N}$, then λ^{+n} will be the n^{th} cardinal past λ. Lapsing into the jargon of subfield, I will refer to the mathematical universe as V. (Due to space limitations the author has not attempted to credit appropriate authors, particularly for well-known results.)

INDEPENDENCE RESULTS. Gödel's theorems ([5]) show that any consistent axiom system $\mathcal{A}$ sufficiently strong to encompass elementary number theory and sufficiently concrete to be recognized as an axiom system (i.e. $\mathcal{A}$ is recursively enumerable) must be *incomplete*. This means that there are statements φ such that there are examples of mathematical structures satisfying the axiomatization $\mathcal{A}$ that satisfy φ and examples of structures that satisfy $\mathcal{A}$ and satisfy the negation of φ. (A simple analagous situation is that the property of being *abelian* is independent of *Group Theory* because there are examples of abelian and non-abelian groups.) Further, Gödel gave a uniform method of producing such a φ: it is a number-theoretic statement equivalent to the consistency of $\mathcal{A}$.

After the shock of this result wears off the question arises as to whether there are statements of "ordinary mathematics" that are independent of the standard axioms of set theory. On one level the answer is clearly affirmative: Matijasevič ([14]), using results of Davis, Putnam and Robinson ([2]), showed that every recursively enumerable set of natural numbers is the range of a diophantine polynomial (of several variables) applied to the natural numbers. (This gave a solution to Hilbert's 10^{th} problem. ([7])) Since the collection of inconsistencies of a recursively enumerable axiom system $\mathcal{A}$ can be coded canonically as a recursively enumerable set of natural numbers, the consistency of $\mathcal{A}$ is equivalent to the non-existence of a natural number solution to a particular diophantine equation. If we fix $\mathcal{A}$ to be our (consistent) axiom system, such as ZFC (or ZFC with large cardinals) we find that there is a diophantine equation such that the (non-)existence of an integer solution to this diophantine equation is independent of $\mathcal{A}$.

Mathematical problems that arose from motivations outside mathematical logic itself eventually were seen to be independent. The most famous of these is Hilbert's 1^{st} problem: the Continuum Hypothesis. The Continuum Hypothesis (or CH) is the statement that the real numbers have cardinality the first uncountable cardinal. Equivalently $\boldsymbol{c} = \aleph_1$. Another equivalent statement is that every infinite subset of the real numbers is either countable or has cardinality $\boldsymbol{c}$.

Gödel ([6]) discovered a canonical example of the axioms of ZFC, called the *Constructible Universe*, L. The idea behind this example is that it is built using only concrete operations, with the only non-constructive elements being the infinite ordinals in the domain of these functions. Gödel showed that if the Zermelo-Fraenkel axioms hold, then the Continuum Hypothesis held in L along with the controversial *Axiom of Choice*. Hence Gödel showed that if the Zermelo-Fraenkel axioms are consistent, then they are consistent with the Continuum Hypothesis and the Axiom of Choice.

An important breakthrough came with the advent of *Forcing* in 1963, in a paper of Cohen ([1]). In this paper, Cohen gave a general method of building new examples of ZF from old ones. (In some ways the method is analogous to adding an algebraic element to a field.) Cohen used this method to show that the Axiom of Choice and the Continuum Hypothesis are independent of ZF.

Forcing, as developed by Solovay and others, became a primary tool for showing independence results. Among the most prominent statements shown to be independent of ZFC:

- *Most statements of infinitary cardinal arithmetic such as the Generalized*

Continuum Hypothesis and the Singular Cardinals Hypothesis.

• *The existence of a* Suslin line*, a complete linear ordering with no uncountable collection of disjoint open intervals that is* NOT *isomorphic to the real line.* After this came an extensive body of work showing independence results in many parts of point-set topology.

• *The independence of the existence of a non-Lebesgue measurable set from* ZF + The Axiom of Countable Choice*.* This shows that the existence of a non-measurable set is inherently tied up with the use of a non-constructive uncountable set existence principle.

• *The existence of a non-free Whitehead group.* This result and related techniques led to a plethora of independence results in abelian groups and homological algebra.

• *The existence of a discontinuous homomorphism between Banach Algebras.*

• *Many infinitary combinatorial principles, particularly in infinitary Ramsey Theory.*

• *The existence of a locally finite group action on a measure space X with a unique invariant mean (positive linear functional of norm 1).*

• *The existence of a paradoxical decomposition of the sphere S^2 constructed using $\mathcal{G}_\delta$ and $\mathcal{F}_\sigma$ subsets of $\boldsymbol{R}^5$ and the operations of complement and projection.*

Is there a meaningful way of settling these questions? Is there anything more to say after they have been shown to be independent of ZFC?

A potential response is to suggest that whatever process led to the acceptance of ZFC as an axiomatization for mathematics (despite its controversial beginnings) may lead to other assumptions that settle, or partially settle most of the problems we are interested in.

THE AXIOM V=L. Jensen ([8, 9]) realized that Gödel's Constructible Universe had a "fine structure" that made it amenable to the kind of close study that settles the types of problems mentioned above. Moreover, he discovered a technique, that when applied with suitable cleverness, appears to answer essentially any question about L. As part of this work, he discovered various combinatorial principles such as $\Box_\kappa$ and $\diamond_\kappa$ that are highly applicable in domains beyond L.

While the axiom of constructibility is very effective, most people working in set theory reject it as inappropriate. This is primarily because the axiom saying "every set is constructible" is viewed as *restrictive* and thus does not account for all of the possible behavior of sets or other mathematical objects.

Further, in the constructible universe there are "pathologies" such as easily constructible paradoxical decompositions of the sphere.

DETERMINACY AXIOMS. The *Axiom of Determinacy*, proposed by Mycielski and Steinhaus ([13]) is a nonconstructive existence principle that contradicts the Axiom of Choice. It makes sense however, to assert it in limited domains such as the collection of Projective Sets or in the smallest model of ZF containing all of the real numbers. These assertions do not ostensibly contradict the Axiom of Choice for the class of all sets.

Given a set A contained in the unit interval $[0, 1]$ one can associate a game G_A where players I, II alternate playing a sequence of digits $n_0, n_1, n_2, \ldots$. (Each $n_i \in \{0, 1, \ldots, 9\}$.) The resulting play yields a number a in the unit interval whose decimal expansion is $a = .n_0n_1n_2\dots$. We declare player II the winner if $a \in A$. The assertion that A is determined is the assertion that either player I or player II has a winning strategy in G_A. A collection $\mathbf{\Gamma}$ of subsets of $\boldsymbol{R}$ is said to be determined iff every element $A \in \mathbf{\Gamma}$ is determined.

Martin [11] showed that all Borel sets are determined. However, in L there is a subset of the real line that is the projection of a Borel set in the plane that is not determined using strategies in L. Hence one can go no further in ZFC.

Why are determinacy axioms attractive? Asserting determinacy for reasonably robust classes $\mathbf{\Gamma}$ implies that every element of $\mathbf{\Gamma}$ is nicely regular, e.g. is Lebesgue measurable, has the Property of Baire and uniformization holds in the relevant guise. So, for example, asserting determinacy for projective sets implies that there is no paradoxical decomposition using projective sets. (Projective sets are the subsets of $\boldsymbol{R}^n$ constructed from Borel sets in higher dimensions using the operations of projection and complement.)

The drawbacks of determinacy are twofold. First off, it says nothing about sets that are not in its domain. For example, while determinacy in $L(\boldsymbol{R})$ tells you that there is no Suslin line in $L(\boldsymbol{R})$, it says nothing about the *actual* existence of a Suslin line. Secondly, there appears to be no extrinsic motivating heuristic for determinacy. Its appeal and force lie in its effectiveness and the body of coherent, predictable consequences.

LARGE CARDINALS. The other main source of new axioms for the mathematical universe is a collection of ideas called *large cardinals*. These axioms were generated by intuitions about "higher infinities", sets whose relation to smaller sets were roughly similar to the relation between $\boldsymbol{N}$ and finite sets.

Another motivation for large cardinals is the idea of *reflection*: the set formation process has no natural stopping point, for at such a point we would simply take the union of all sets constructed and form a new set. Hence any property that holds in the mathematical universe should hold of many set-approximations of the mathematical universe. Moreover, since this is a property of the universe, there should be many sets that, in turn, have this property relative to smaller sets, etc. The sets that have the reflection properties relative to smaller sets are the large cardinals.

Eventually large cardinal axioms came to be stated more or less uniformly as the existence of certain kinds of symmetries. Technically these are elementary embeddings j from the universe V to transitive classes M. (An embedding is *elementary* iff for all properties ϕ and all $a_1, \dots a_n$, if ϕ holds of $a_1, \dots, a_n$, then ϕ holds of $j(a_1), \dots j(a_n)$. So, e.g., if X is a manifold, $j(X)$ is a manifold.)

These axioms vary in strength according to where j sends ordinals and the closure of the class M. (We can classify M according to the least cardinality of a set X such that $X \notin M$. A theorem of Kunen proves that there always is such a set.) An important ordinal is the smallest ordinal moved by j, called the *critical point* of j, or $crit(j)$.

A well-known example of such an axiom was proposed by Ulam; the axiom of a *Measurable Cardinal*. Ulam formulated this as the statement that there is a set K and a countably additive 2-valued measure defined on all subsets of K. Using ultraproducts, this can be stated in modern language as the existence of a non-trivial elementary embedding of V to some transitive model M with critical point κ.

The notion of supercompact and huge cardinals can also be stated as the existence of measures on sets with certain additional structure. The statement in terms of elementary embeddings is more conceptual:

Definition. A cardinal κ is *λ-supercompact* iff there is an elementary embedding $j : V \to M$, where M is a transitive class and M contains every λ sequence of ordinals. κ is supercompact iff κ is λ-supercompact for all λ.

A cardinal κ is *n-huge* iff there is an elementary embedding $j : V \to M$ with critical point κ such that M is closed under $j^n(\kappa)$-sequences.

For each elementary embedding there is an ideal object in the target model M (or system of ideal objects, in more sophisticated set ups) that determine the nature of the embedding. In particular, it determines the closure of M. Each element ι of M determines a measure and with respect to this measure every property of the ideal object holds at almost every point in the measure space determined by ι. In particular, if $S \subset \iota = crit(j)$ is stationary, then for almost every $\alpha < \iota, S \cap \alpha$ is stationary. If ι is taken to be the ideal point, then the ultraproduct of V by the measure determined by ι yields the model M.

By focusing on the ideal points one can see the reflection implied by the elementary embedding. An important example of such reflection is the statement that if κ is supercompact and $\lambda > \kappa$ is a regular cardinal then every stationary subset of λ reflects to an ordinal of cofinality less than κ. This property, while useful in its own right as a construction principle, contradicts $\Box$.

Large cardinals are also significant in that many of the combinatorial properties of $\boldsymbol{N}$ hold at large cardinals. For example Rowbottom's Theorem, a direct analogue of Ramsey's theorem, states that if κ is measurable then every partition of the finite subsets of κ into less than κ colors has a homogeneous set of size κ. Baumgartner and Hajnal showed that strong partition properties hold at the cardinal successor of ω. Recent results of Hajnal and the author show that analogous partition properties hold at the successor of a measurable cardinal.

Results of Ulam (and later Tarski and Keisler) showed that large cardinals, such as measurable cardinals, must be inacessibly larger than most ordinary mathematical objects, such as the real numbers $\boldsymbol{c}$. (Recent results of Gitik and Shelah show that if $\mathcal{I}$ is a countably complete ideal on a cardinal such as $\boldsymbol{c}$ (or $P(\boldsymbol{c})$) then $P(\boldsymbol{c})/\mathcal{I}$ does not have a dense countable set; the least possible density is $\aleph_1$.)

Gödel suggested that large cardinal assumptions may eventually be a route to settling the continuum hypothesis. This hope was dashed however by a theorem of Levy and Solovay ([10]) that showed that "small forcing" does not affect large cardinals. In particular the Continuum Hypothesis is independent of of any large cardinal assumption. This theorem and the apparent remoteness of these cardinals to ordinary sets is a major drawback of large cardinal assumptions.

Large cardinals do have a coherent motivating heuristic and independent affirming intuitions. They have also proved essential for relative consistency results, such as the failure of the singular cardinals hypothesis. (e.g. Jensen's *Covering Lemma* showed that large cardinals were strictly necessary.)

GRAND UNIFICATION. In the 70's and early 80's large cardinal axioms and determinacy axioms were viewed as competing attempts at extending the axioms ZFC. Martin and Harrington had showed various connections between some of the weaker versions of the two systems of axioms, but the exact relationships weren't clear.

An important breakthrough came in 1984 ([3]), when it was realized that large cardinal axioms implied the existence of large cardinal type embeddings, where the embedding $j : V \to M$ was definable not in V, but in a forcing extension of V. These elementary embeddings have critical point $\aleph_1$, and thus the embeddings are immediately relevant to "small" sets such as the real numbers. Moreover, this discovery uncovered a new class of relatively weak large cardinals, the *Woodin cardinals*. (Named after the person who isolated the definition.)

Following on the heels of this discovery, Martin and Steel ([12]) showed that determinacy for the class of projective sets follows from the existence of sufficiently many Woodin cardinals. Woodin ([16]), using the generic large cardinal embeddings, showed that determinacy held for all sets in $L(\boldsymbol{R})$. In particular, all of the consequences of determinacy follow from large cardinals.

More recent work has exactly fixed many of the relations between large cardinal and determinacy axioms, often showing that a particular large cardinal axiom implies determinacy of a class of sets $\boldsymbol{\Gamma}$, which in turn implies the consistency of a slightly weaker large cardinal axiom.

This close relationship has become a major feature of the contemporary study of other extensions of ZFC. By and large they are all known to either follow from, or be equiconsistent with large cardinal axioms. This is viewed by many people as being suggestive that the various alternative axiom systems suggested are simply different aspects of the same phenomenon, hence confirming large cardinal axioms.

Despite this type of confirmation and large cardinals' role of calibrating the consistency strength of most independent propositions of ZFC, it remains frustrating that they cannot actually settle important problems such as the Continuum Hypothesis.

GENERIC LARGE CARDINALS. Generic large cardinals are a marriage of large cardinals and forcing. The axioms assert the existence of an elementary embedding $j : V \to M$, where M is a transitive model, where j is definable in a forcing extension of the universe $V[G]$. These embeddings can be viewed as *virtual* versions of large cardinal embeddings, whose specifics are revealed by forcing with the appropriate partial ordering. (This technique was first used by Solovay. Jech and Prikry, realizing its interest, isolated the notion of a *precipitous* ideal.)

The advantage of generic large cardinals is that the critical point of j can be a "small" cardinal such as $\aleph_1$. With some limitations this allows these cardinals to have similar reflection and resemblance properties as posited by large cardinal axioms on highly inacessible cardinals. Moreover, it allows one to state "symmetry principles" that can hold in a generic extension of the universe. By and large the motivational principles used to generate large cardinals can be restated to apply to generic large cardinal axioms, virtually verbatim.

The current study of generic large cardinal axioms now breaks into three parts: their consequences as axioms, showing their consistency relative to large cardinals and showing that they imply the existence of inner models with large cardinals. Many research programs in the area combine one or more of these parts. In a typical example, relative consistency results of properties of $\aleph_2$ can be shown by first establishing that they follow from a generic large cardinal property and then showing that the property is consistent relative to a conventional large cardinal. They can be used in the other direction as well; an archetypical result in the area, shown by Solovay, is that the existence of a real-valued measurable cardinal implies the existence of an inner model with a 2-valued measurable cardinal. This was done by first showing that a real-valued measurable cardinal implied the existence of a generic large cardinal, which in turn implied the existence of an inner model with a measurable cardinal.

The parameters involved in determining a generic large cardinal are expanded to include the nature (in particular the density or saturation) of the partial ordering $\boldsymbol{P}$ involved in the forcing. Analogously to large cardinals, the transitive model M typically contains an ideal object, ι, whose existence implies the closure of the model M. Rather than determining a measure, this ideal object determines an ideal $\mathcal{I}$ in the ground model V on any set Z such that $\iota \in j(Z)$. Most of the relevant properties of $\boldsymbol{P}$ (particularly the stronger properties such as saturation) are inherited by the Boolean algebra $P(Z)/\mathcal{I}$, and hence we primarily discuss the saturation and density properties of $\mathcal{I}$ (or more properly $P(Z)/\mathcal{I}$.) We will refer to embeddings as generically huge, or generically λ-supercompact if the closure of M corresponds to the analogous large cardinal property. To simplify statements of theorems, we will often neglect the optimal hypothesis.

The first result is that if there is a generic huge embedding such that $j(\boldsymbol{c}) = 2^{\boldsymbol{c}}$, defined in the simplest possible forcing extension, then the continuum hypothesis holds and there is a Suslin line:

•(Foreman) *Suppose that there is a normal and fine $\aleph_1$-dense ideal on the collection of subsets of $2^{\boldsymbol{c}}$ of cardinality $\boldsymbol{c}$. Then the continuum hypothesis holds and there is a Suslin line. (Woodin has reduced the hypothesis of the first assertion to the existence of an $\aleph_1$-dense ideal on $\aleph_2$.)*

To extend this to the GCH, there are several possible axioms, one that stresses the resemblance between successor cardinals is the hypothesis of the following theorem:

•(Foreman) *Suppose that for all regular λ, $n \in \boldsymbol{N}$ there is a generic huge embedding sending $\aleph_{k+1}$ to λ^{+k} $(k \leq n)$. Then the Continuum Hypothesis implies the Generalized Continuum Hypothesis.*

Just as large cardinals imply stationary set reflection, generic large cardinals do as well. Magidor showed (in a different guise) that if for all n, $\aleph_n$ is generically supercompact by $\aleph_{n-1}$-closed forcing then every stationary subset of $\aleph_{\omega+1}$ reflects. Since Jensen's $\Box$ implies the existence of non-reflecting stationary sets, generic embeddings imply the failure of $\Box$. However, there are variations of $\Box$, that while strictly weaker, are nearly as useful. The strongest of these is $\Box_{\kappa,\omega}$. The following theorem shows that it is possible to have some of the best of $\Box$ and stationary set reflection.

•(Cummings, Foreman, Magidor) *Suppose that there is an example of set theory with infinitely many supercompact cardinals. Then there is an example of* *set theory where every stationary subset of* $\aleph_{\omega+1}$ *reflects and where* $\square_{\kappa,\omega}$ *holds.*

The proof of this theorem uses generic supercompactness in a subtle way. Magidor and the author showed that generic supercompactness by countably closed forcing is incompatible with Weak Square ([4]). Instead, in this proof, each $\aleph_n$ is generically supercompact by a closed forcing notion in a stationary set preserving extension of V.

As one might expect, generic large cardinals have implications for other topics in the theory of singular cardinals, such as the "PCF" theory developed by Shelah. For example, if there is a generic huge embedding, sending $\aleph_1$ to $\aleph_{\omega+1}$, then there is no "Good Scale" in the sense of the PCF theory. The flow goes the other way as well; using PCF theory one can show that there is no "generic ω-huge cardinal", an analogue to a result of Kunen for ordinary large cardinals.

Generic large cardinals have similar effects on Ramsey Theory as large cardinals:

•(Foreman, Hajnal) *Suppose that there is an* $\aleph_1$*-dense ideal on* $\aleph_2$*. Then the partition property* $\omega_2 \to (\omega_1^2+1,\alpha)$ *holds for all* $\alpha < \omega_2$*.*

Generic large cardinal axioms have other combinatorial consequences. For example the existence of generic huge embeddings with simple forcing notions imply that every graph on $\aleph_n$ with infinite chromatic number has subgraphs of all smaller infinite chromatic numbers (and these subgraphs have the same finite subgraphs as the original graph.)

One can postulate other properties of the forcing $\boldsymbol{P}$. Suppose that κ is a regular cardinal. Say that $\boldsymbol{P}$ is *κ-tame* if $\boldsymbol{P}$ is a regular subalgebra of the partial ordering for adding a Cohen subset of a cardinal less than κ followed by a product of κ-closed and strongly κ-c.c. partial orderings. Mitchell showed that it is consistent for $\aleph_2$ to be generically weakly compact by an $\aleph_1$-tame partial ordering. Abraham improved this to two consecutive cardinals.

•(Cummings, Foreman) *Suppose that it is consistent for there to be infinitely many supercompact cardinals. Then it is consistent that for all* $n \geq 2$*,* $\aleph_n$ *is generically weakly compact by an* $\aleph_{n-1}$*-tame* $\boldsymbol{P}$*. Moreover, this implies that for all* $n \geq 2$*, there is no Aronszajn tree on* $\aleph_n$*.*

These have applications in other parts of mathematics where infinitary combinatorics plays a role. As an example we consider the case of a vector space X over a field F, with a symmetric bilinear form ϕ. If we choose a basis $\{x_\alpha : \alpha < \kappa\}$ for X and let $X_\alpha = \mathrm{span}\{x_\beta : \beta < \alpha\}$ we can consider $\Gamma(X,\phi) = \{\alpha : X = X_\alpha \oplus X_\alpha^\perp\}$. This set is invariant under isomorphism modulo the non-stationary ideal on κ. (This is called the Γ-*invariant.*) It makes sense to ask which sets can arise this way.

•(Foreman, Spinas) *Suppose that* $\aleph_2$ *is generically weakly compact by an* $\aleph_1$*-tame partial ordering. Then there is a subset of* $\aleph_2$ *that is not the* Γ*-invariant of any* (X,F,ϕ)*.*

In addition to the role of generic large cardinal axioms in the unification of the axiom systems of large cardinals and determinacy, Woodin has shown directly that they imply determinacy:

•(Woodin) *The axiom of determinacy in* $L(\boldsymbol{R})$ *is equiconsistent with "ZFC + there is an* $\aleph_1$*-dense ideal on* $\aleph_1$*."*

There are many open problems about which generic large cardinals can be shown to be consistent from large cardinals. However much progress has been made. A partial listing of such results includes:

•(Woodin, improving results of Kunen, Laver and Magidor) *Let $n \in \boldsymbol{N}$. Assuming the consistency of an almost-huge cardinal, it is consistent that there is an $\aleph_n$-complete, $\aleph_n$-dense ideal on $\aleph_n$.*

•(Foreman) *Assuming the consistency of a huge cardinal, it is consistent that for all regular κ, there is a κ^+-saturated ideal on κ.*

•(Foreman) *Assuming the consistency of a 2-huge cardinal, then for all n, it is consistent that there is a generic 2-huge embedding with critical point $\aleph_n$.*

•(Foreman) *Assuming the existence of a huge cardinal, it is consistent that there is a countably complete, uniform $\aleph_1$-dense ideal on $\aleph_2$.*

•(Steel-Van Wesep from determinacy assumptions, Foreman, Magidor and Shelah from large cardinal assumptions with Shelah proving the optimal theorem) *Assuming the consistency of a Woodin cardinal, it is consistent that the non-stationary ideal on $\aleph_1$ is $\aleph_2$-saturated.*

It is also possible to show that the generic large cardinal axioms form a hierarchy in consistency strength. A typical theorem includes:

•(Foreman) *Let $n > 1$. Suppose that there is a generic n-huge embedding by the partial ordering $Col(\omega, \aleph_1)$. Then it is consistent to have a generic (n-1)-huge embedding with partial ordering $Col(\omega, \aleph_1)$.*

Further it is possible, in certain cases to show from generic embeddings that large cardinals are consistent. For example:

•(Steel) *Suppose that there is a saturated ideal on $\aleph_1$ and a measurable cardinal, then there is an inner model with a Woodin cardinal.*

Using naive technology one can show that the existence of certain generic elementary embeddings imply inner models with huge cardinals. Using this fact, one can find strong Chang's conjecture principles of the $\aleph_n$'s that lie strictly between a huge cardinal and a 2-huge cardinal.

With the exception of the results mentioned in the next section, generic large cardinals give a coherent theory that settles most of the classical independent statements of mathematics. Many are known to be consistent relative to conventional large cardinals. Are all principles generated this way consistent? Are they consistent with each other? It turns out that there are non-trivial restrictions on the saturation properties of various natural ideals.

Most prominent among these are the results of Shelah, and Shelah and Gitik. Shelah's theorem states that if $\mathcal{I}$ is a saturated ideal on κ^+, then the collection of ordinals of cofinality different from the cofinality κ is an element of $\mathcal{I}$; in particular, if $\kappa > \omega$, the non-stationary ideal on κ^+ is not saturated. Shelah and Gitik showed that the non-stationary ideal on the successor of a singular cardinal κ is not saturated, even when restricted to the points having cofinality equal to the cofinality of κ. The following theorem extends work of Burke and Matsubara.

•(Foreman, Magidor) *Suppose that $\kappa < \lambda, \aleph_1 < \lambda$. Then the non-stationary ideal on $P_\kappa(\lambda)$ is not λ^+ saturated.*

Finally it is possible to show that the limitations on the closure of the target model M for a generic elementary embedding are roughly similar as they are for conventional large cardinals.

- *There is no $\aleph_\omega$-saturated ideal on the subsets of $\aleph_\omega$ of order type $\aleph_\omega$.*

MARTIN'S MAXIMUM AND P-MAX. In [3], Magidor, Shelah and the author formulated a principle called *Martin's Maximum* and showed that it implied that the non-stationary ideal on $\aleph_1$ is $\aleph_2$-saturated, and the singular cardinals hypothesis holds.

Woodin showed (assuming a mild large cardinal hypothesis) that if the non-stationary ideal on $\aleph_1$ is $\aleph_2$-saturated then there is a fairly concrete surjection $\rho : \boldsymbol{R} \to \aleph_2$. Further, he developed a canonical theory "P-max" to describe the sets of hereditary cardinality $\aleph_1$, and showed that this theory is canonical and robust in many ways. Further it has a close connection with Martin's Maximum and its variants such as MM^+ and MM^{++}.

As of this writing, this theory appears to be particular to $\aleph_1$, as the results in the previous section (and others) show that it is inconsistent for the non-stationary ideal on $\aleph_1$ to be saturated and have an $\aleph_1$-preserving generic elementary embedding with critical point $\aleph_2$.

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