Fibrations in Symplectic Topology

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Abstract. Every symplectic form on a $2n$-dimensional manifold is locally the Cartesian product of n area forms. This local product structure has global implications in symplectic topology. After briefly reviewing the most important achievements in symplectic topology of the past 4 years, the talk will discuss several different situations in which one can see this influence: for example, the use of fibered mappings in the construction of efficient symplectic embeddings of fat ellipsoids into small balls, and the theory of Hamiltonian fibrations (work of Lalonde, Polterovich, Salamon and the speaker). The most spectacular example is Donaldson's recent work, showing that every compact symplectic manifold admits a symplectic Lefschetz pencil.

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1 Introduction

In this talk I will give an overview of what has been achieved in symplectic topology in the past 4 years and then will discuss the relevance of symplectic fibrations. First, I will review some basic facts.

A symplectic manifold (M, ω) is a pair consisting of a smooth $2n$-dimensional manifold M together with a closed 2-form ω that is nondegenerate, i.e. the top power ω^n never vanishes. By Darboux's theorem such a form ω can always be expressed locally as the sum

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

Thus the only invariants of a symplectic manifold are global. The other essential feature of symplectic geometry is its connection with dynamics. Every function $H : M \to \mathbf{R}$ has a symplectic gradient X_H, which is the vector field defined by the equation $\omega(X_H, \cdot) = dH$. Because ω is closed, the flow of X_H is a family ϕ_t^H of

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symplectomorphisms, i.e. diffeomorphisms that preserve the symplectic structure. Thus $(\phi_t^H)^*(\omega) = \omega$ for all t.

One can think that these flows are built into the local structure of a symplectic manifold. Any (local) hypersurface Q in (M, ω) is a regular level set $H = const$ of some function H. Since $dH(X_H) = \omega(X_H, X_H) = 0$, the vector field X_H is tangent to Q and so induces a flow on it. The corresponding flow lines are independent of the choice of H and so give rise to a 1-dimensional foliation on Q called the *characteristic foliation*. As we shall see in §4.1, these foliations give rise to a good theory of symplectic connections on symplectic fibrations.

Another more global consequence is that each symplectic manifold gives rise to an interesting infinite-dimensional group, namely the *group of symplectomorphisms* $\mathrm{Symp}(M, \omega)$. Its identity component contains a connected subgroup of finite codimension, called the *Hamiltonian subgroup* $\mathrm{Ham}(M, \omega)$. This consists of all symplectomorphisms that are the time-1 map of some Hamiltonian flow ϕ_t^H, where here one allows the Hamiltonian $H_t : M \to \mathbf{R}$ to depend on time $t \in [0, 1]$. I shall say more about these groups in §2.6 and §4.2 below.

A basic theme in symplectic topology is that properties that hold locally are often valid more globally. One example is Darboux's theorem. Here the local statement is that there is a unique symplectic structure at a point (i.e. on the tangent space), and this extends to the fact that there is a unique structure in a neighborhood of each point. Another example is Arnold's conjecture. The local statement here is that when M is compact every Hamiltonian symplectomorphism that is sufficiently close to the identity in the C^1-topology has at least $\sum_i \dim H^i(M, \mathbf{R})$ distinct fixed points, provided that these are all nondegenerate.[2] The global statement is that this remains true for all elements of $\mathrm{Ham}(M, \omega)$. This has now been proved: see §2.2.

One further example of this phenomenon that I want to mention here concerns the fact that a symplectic form ω is a local product: by Darboux's theorem ω can always be expressed locally as the Cartesian product of n area forms $dx \wedge dy$ on $\mathbf{R}^2$. Observe that a general symplectomorphism does not preserve this local product structure. For example, the linear map

$$L : (x_1, y_1, x_2, y_2) \mapsto (x_1 + x_2, y_1, x_2, y_2 - y_1)$$

preserves ω but neither preserves nor interchanges its individual summands $dx_i \wedge dy_i, i = 1, 2$. Nevertheless, I hope to show in this talk that the existence of this local product structure is reflected globally in various important ways, both in the "semi-local" properties that are discussed in §3 and in the theory of symplectic fibrations that is presented in §4. The best evidence is, of course, Donaldson's theorem on the existence of symplectic Lefschetz pencils that is discussed in §2.3 below.

[2] A fixed point x of ϕ is said to be *nondegenerate* if the graph of ϕ in $M \times M$ intersects the diagonal transversally at the point (x, x). There are other versions of Arnold's conjecture that allow degenerate fixed points and/or make homotopy theoretic rather than homological estimates of the number of fixed points, but these have not yet been established in full generality.

1.1 Analytic techniques in symplectic topology

Until Donaldson's recent work, there were two main sources of analytic techniques in symplectic geometry, variational methods (that relate to the above mentioned flows) and elliptic methods. These have been combined to create powerful tools such as Floer theory. Since Hofer who is one of the main exponents of the variational method is also talking at this I.C.M. I will not say anything more about this here, and will concentrate on more purely elliptic methods that exploit the close relation of symplectic geometry with complex geometry.

One important kind of symplectic manifold is a *Kähler manifold.* This is a complex manifold M that admits a Riemannian metric g that is well adapted to the induced almost complex structure J on the tangent bundle TM.[3] One way of expressing the Kähler condition is that the bilinear form ω defined by

$$\omega(v, w) = g(Jv, w)$$

is skew-symmetric and closed. Since the nondegeneracy of g implies that of ω, the form ω is symplectic. As a kind of converse, observe that a symplectic manifold always supports an almost complex structure J on the tangent bundle TM that is compatible with ω in the sense that the bilinear form g defined by the above equation is a positive definite inner product. In fact, for any symplectic manifold M there is a contractible set $\mathcal{J}(\omega)$ of such almost complex structures. In most cases, these will not be integrable. It was Gromov who first realised (in 1985) how to use these almost complex structures to get information about the underlying symplectic structure: see [G1], [G2].

Gromov's fundamental idea was to look at spaces of *J-holomorphic curves* in (M, ω, J). These are maps u from a Riemann surface (Σ, j) to the almost complex manifold (M, J) that satisfy the generalized Cauchy–Riemann equation

$$du \circ j = J \circ du.$$

If J is integrable, u is a (parametrized) holomorphic curve of the usual kind. Even if J is not integrable, these curves behave very much as one would expect, basically because every almost complex structure on a 2-manifold is integrable. In particular, the ellipticity of the Cauchy–Riemann equation implies that the set $\mathcal{M}(A, J)$ of all such curves that represent the homology class $A \in H_2(M; \mathbf{Z})$ is a finite-dimensional manifold for generic J in $\mathcal{J}(\omega)$. The other essential ingredient comes from the existence of the symplectic form ω. This gives an *a priori* bound to the energy (or $W^{1,2}$-norm) of the elements in $\mathcal{M}(A, J)$, which in turn implies that this space has a well-behaved compactification. Hence it makes sense to try to count the number of these curves that intersect certain homology classes in M. In general, one gets a finite number that is independent of J. This gives rise to symplectic invariants, that in various contexts are called Gromov invariants, Gromov–Witten invariants, or Gromov–Taubes invariants and so on. Many foundational results in symplectic topology can be proved using J-holomorphic curves,

[3] An almost complex structure is an automorphism of the tangent bundle TM with square equal to $-\mathrm{Id}$. If it is induced from an underlying complex structure on M it is said to be *integrable.*

for example the nonsqueezing theorem that we discuss below. They are also an essential ingredient in symplectic versions of Floer homology.

2 Recent advances

In this section I will list some of the most significant advances in symplectic geometry of the past 4 years. I will be very brief (and in particular do not attempt to give full references) since in many cases other people will be giving talks on these subjects at this I.C.M.

2.1 Taubes–Seiberg–Witten theory

A few months after the Seiberg–Witten equations were first formulated in Fall 1993, Taubes realised that the methods used by Witten to calculate the associated invariants for Kähler manifolds could be adapted to the symplectic case. This was the first time that methods of gauge theory were found to interact significantly with symplectic geometry. His first results [T1,2] from Spring 1994 established a structure theorem for the Seiberg–Witten invariants of symplectic 4-manifolds, that implied in particular that they do not vanish. He then wrote a series of deep papers that showed that these invariants coincide with a certain kind of Gromov invariant that counts J-holomorphic curves in an appropriate way: see [T3–6] and also Ionel–Parker [IP1].

This has opened the door to the construction of many interesting examples of symplectic 4-manifolds as well as to a much better understanding of the relation of smooth 4-manifolds to symplectic ones: cf. the I.C.M. talks of Taubes and Fintushel–Stern. For example, Taubes gave the first examples of manifolds that satisfy the necessary topological preconditions for being symplectic (namely they support an almost complex stucture and also have a cohomology class $a \in H^2(M, \mathbf{R})$ whose top power does not vanish) but nevertheless have no symplectic structure. One such example is the connected sum $\mathbf{C}P^2 \# \mathbf{C}P^2 \# \mathbf{C}P^2$ of three copies of the projective plane, which cannot be symplectic because its Seiberg–Witten invariants vanish. Another consequence is a proof that there is only one symplectic structure on the complex projective plane (up to rescaling) (see [G1] and [T2]) and a complete classification of symplectic structures on blow-ups of rational and ruled surfaces. This last is a combination of work by Li–Liu [LL1,LL2], Ohta–Ono [OO] and Liu [Liu] on Seiberg–Witten theory for symplectic manifolds with $b^+ = 1$, work by Lalonde–McDuff [LM4] classifying symplectic structures on ruled surfaces and work on blow-ups by McDuff [Mc1] and Biran [Bi].

2.2 General Gromov–Witten invariants

The theory of J-holomorphic curves outlined above was unsatisfactory for many years because there was a basic technical problem (the "multiply-covered curve problem") that meant that it worked only in a very restricted class of manifolds. In 1994 Kontsevich suggested a way to get around this difficulty using the concept of stable maps and other ideas from algebraic geometry, and subsequently several teams have made this a reality. Among them are Fukaya–Ono [FO], Li–Tian [LiT], Liu–Tian [LiuT], Ruan [R], and Siebert [Sieb], who have all completed substantial papers on this subject in the past two years. (See also Hofer–Salamon [HS].) This important foundational work shows that methods that one might think are intrinsically algebraic can be extended to the smooth symplectic context. Another consequence is a proof that the nondegenerate case of Arnold's conjecture holds on all symplectic manifolds: see [FO], [LiuT].

One of the recent insights that has come from string theory and quantum physics is that Gromov–Witten invariants have very interesting formal properties: for example they give rise to a deformation of the cup product on the cohomology ring of a symplectic manifold. This is known as quantum cohomology: see Ruan–Tian [RT]. These invariants have been also used to solve long-standing problems in enumerative geometry and have many other applications: cf. the I.C.M. talks by Vafa and Ruan.

2.3 DONALDSON THEORY

In the past two years Donaldson has developed a completely new way to use the existence of almost complex structures on symplectic manifolds, taking the manifold (M, J) to be not the target space but rather the domain of the maps considered. He has developed a theory of "almost holomorphic" sections of certain "almost ample" line bundles that imitates the usual theory in the Kähler case so faithfully that he can prove that every closed symplectic manifold admits a symplectic Lefschetz pencil: see [D1,D2] and also the Bourbaki seminar [Sik]. I will state a version of the theorem here because of its relevance to the theme of this talk. For a much fuller discussion, see Donaldson's I.C.M. talk.

THEOREM 2.1 *Let (M, ω) be a closed symplectic manifold such that the cohomology class $[\omega]$ is integral. Then for each sufficiently large k there is a symplectic submanifold B_k of codimension 4 and a smooth map $p : M - B_k \to \mathbf{C}P^1$ that has only finitely many singular points. Each fiber of p is symplectically embedded except at its singular points, and near these p has the form $(z_1, \dots, z_n) \mapsto \sum_i z_i^2$ in suitable local coordinates $(z_1, \dots, z_n) \in \mathbf{C}^n$. Finally, p extends smoothly to the blow-up $\widetilde{M}$ of M along B_k.*

The induced map $p : \widetilde{M} \to \mathbf{C}P^1$ is usually called a Lefschetz fibration. It is constructed so that its general fiber F_k represents the Poincaré dual $\mathrm{PD}(k[\omega])$ of a suitably large integral multiple of the symplectic cohomology class $[\omega]$. Auroux [Au] has shown that for sufficiently large k the codimension-2 symplectic submanifold F_k is unique up to isotopy. Similarly, it can be shown that the whole structure of the Lefschetz pencil is unique up to isotopy for sufficiently large k. Moreover the symplectic form on such a pencil is determined up to deformation[4] by the symplectic form on the fiber F_k. Hence, in principle, the classification of symplectic $2n$-manifolds can be reduced to that of symplectic $(2n-2)$-manifolds, and hence to the complicated world of symplectic 4-manifolds. This, in turn, is

[4]Two closed symplectic manifolds $(V, \omega_V), (W, \omega_W)$ are said to be *deformation equivalent* if there is a diffeomorphism $\phi : V \to W$ and a family of not necessarily cohomologous symplectic forms $\omega_t, t \in [0, 1]$ on V such that $\omega_0 = \omega_V, \omega_1 = \phi^*(\omega_W)$.

reduced to data concerning Riemann surfaces. Many very interesting questions arise here, and I refer you to the I.C.M. talks by Donaldson and Fintushel–Stern for further discussion. One important point is that it is not known whether the classification of symplectic 4-manifolds is more complicated than that of smooth 4-manifolds. For example, I do not know any example of a smooth 4-manifold that supports two symplectic structures which are not deformation equivalent.

2.4 CONTACT GEOMETRY

Contact geometry is the odd-dimensional analog of symplectic geometry. It is now particularly well understood in dimension 3 because there are two ways to get geometric information about a contact 3-manifold M. One can reduce to 2-dimensions by looking at the intersection of the contact structure with families of surfaces in M, an approach pioneered by Eliashberg [E3] and Giroux [Gi], and one can also use elliptic techniques in the 4-dimensional symplectization $M \times \mathbf{R}$. For new developments in this area I refer you to the I.C.M. talks by Eliashberg and Hofer.

2.5 HOFER GEOMETRY

Hofer [H] pointed out in 1990 that the group of Hamiltonian symplectomorphisms carries a biinvariant metric, that is now called the Hofer metric. There have been significant advances in understanding the properties of this metric and its geometric and dynamic implications, notably by Bialy–Polterovich [BP], a series of papers by Polterovich (see [P]) and Lalonde–McDuff [LM1–3]. In particular, the papers [LM1–3] develop a new elliptic approach to Hofer geometry, and show that the energy-capacity inequality that is basic to the whole theory is equivalent to the nonsqueezing theorem discussed in §3 below. There also is an interesting connection between the Hofer length of an element in $\pi_1(\mathrm{Ham}(M))$ and properties of the associated symplectic fibration over S^2 with fiber M: see [P] §7, and §4.2 below. For further details, see the I.C.M. talk by Polterovich [P].

2.6 THE TOPOLOGY OF THE GROUP OF SYMPLECTOMORPHISMS

There has been quite a bit of recent progress in understanding the relations between the groups

$$\mathrm{Ham}(M,\omega) \hookrightarrow \mathrm{Symp}(M,\omega) \hookrightarrow \mathrm{Diff}(M)$$

for closed symplectic manifolds (M,ω). Observe that the inclusion $\mathrm{Ham}(M,\omega) \to \mathrm{Symp}(M,\omega)$ induces an isomorphism on all homotopy groups except for π_0 and π_1. As far as concerns π_0, the Hamiltonian group is path-connected by definition, while $\mathrm{Symp}(M,\omega)$ often is not. As for π_1, there is an exact sequence

$$0 \to \pi_1(\mathrm{Ham}(M,\omega)) \to \pi_1(\mathrm{Symp}(M,\omega)) \to \Gamma_\omega \to 0,$$

where Γ_ω is a countable subgroup of $H^1(M,\mathbf{R})$ that is called the *Flux group*: see [MS] or [LMP1]. It is not hard to show that $\mathrm{Ham}(M,\omega)$ coincides with the identity component of $\mathrm{Symp}(M,\omega)$ in the case when $b_1(M) = \mathrm{rk}\, H^1(M,\mathbf{R}) = 0$, in particular if M itself is simply connected.

Perhaps the most surprising recent result is that of the stability of Hamiltonian loops, i.e if $\{\phi_t\}_{t\in[0,1]}$ represents an element of $\pi_1(\mathrm{Ham}(M,\omega))$ then any perturbation $\{\phi'_t\}$ of the loop $\{\phi_t\}$ that preserves some nearby symplectic form ω' represents an element of $\pi_1(\mathrm{Symp}(M,\omega'))$ that lies in the image of $\pi_1(\mathrm{Ham}(M,\omega'))$: see Lalonde–McDuff–Polterovich [LMP2]. Another way of saying this is that if $\phi \in \pi_1(\mathrm{Symp}(M,\omega))$ and $\phi' \in \pi_1(\mathrm{Symp}(M,\omega'))$ map to the same element of $\pi_1(\mathrm{Diff}(M))$ and if ϕ maps to zero in Γ_ω then ϕ' must map to zero in $\Gamma_{\omega'}$. It follows fairly easily that the Flux subgroup Γ_ω never has more than $b_1(M)$ generators. It is still not known whether it is always discrete. This would be the case if and only if the group $\mathrm{Ham}(M,\omega)$ is closed in $\mathrm{Symp}(M,\omega)$ with respect to the C^1-topology: see [LMP1].

Otherwise the theory is at the stage of computing interesting examples. Seidel [Seid1] has found a very nice construction that shows that for many symplectic 4-manifolds that contain a Lagrangian 2-sphere[5] the map $\pi_0(\mathrm{Symp}(M,\omega)) \to \pi_0(\mathrm{Diff}(M))$ is not injective. This work is based on an analysis of the Floer homology of the generalized Dehn twists that occur as monodromy in Lefschetz fibrations: see Donaldson's I.C.M. talk. Seidel has also shown that when M is the product of two projective spaces $\mathbf{C}P^m \times \mathbf{C}P^n$, where $m \le n$, the map $\pi_k(\mathrm{Symp}(M,\omega)) \to \pi_k(\mathrm{Diff}(M))$ is not surjective for odd $k \le 2n-1$. Many of the above results are proved by considering properties of appropriate fibrations: see §4.2 below.

We end this section by mentioning an example where the rational cohomology of the groups $\mathrm{Symp}(M,\omega)$ has been fully worked out. Here (M,ω) is the product $S^2 \times S^2$ equipped with the symplectic form $\omega^\lambda = (1+\lambda)\sigma_0 \oplus \sigma_1$, where $\sigma_i, i = 0,1$, is an area form on S^2 of total area 1. Let $G^\lambda, \lambda \ge 0$ denote the corresponding group of symplectomorphisms. Gromov showed in [G1] that, when $\lambda = 0$, G^λ is deformation equivalent to the extension of the Lie group $\mathrm{SO}(3) \times \mathrm{SO}(3)$ by the involution that interchanges the two factors. Abreu showed in [Ab] that when $0 < \lambda \le 1$ the group G^λ no longer has the homotopy type of a Lie group since its rational cohomology ring has an even-dimensional generator. Abreu–McDuff [AM] have completed this calculation, showing that when $k-1 < \lambda \le k$

$$H^*(G^\lambda, \mathbf{Q}) = \Lambda(x_1, x_3, x'_3) \otimes S(x_{4k}),$$

where x_i, x'_i denote generators in dimension i, Λ is an exterior algebra and S is a polynomial algebra. One can give a meaning to the "limit" of these groups G^λ as $\lambda \to \infty$ and show that this is homotopy equivalent to the group $\mathcal{D}$ of fiberwise orientation-preserving diffeomorphisms of the trivial fibration $S^2 \times S^2 \to S^2$. Since $\mathrm{Diff}(S^2)$ is homotopy equivalent to the Lie group $\mathrm{O}(3)$, the group $\mathcal{D}$ is homotopy equivalent to a group $\mathcal{D}'$ that fits in the exact sequence

$$0 \to \mathrm{Map}(S^2, \mathrm{SO}(3)) \to \mathcal{D}' \to \mathrm{SO}(3) \to 0.$$

The cohomology ring of $\mathcal{D}$ is isomorphic to $\Lambda(x_1, x_3, x'_3)$ and restricts onto this part of $H^*(G^\lambda)$, while the "jumping generator" x_{4k} dies in the limit.

[5] i.e. a sphere on which the symplectic form vanishes.

2.7 SYMPLECTIC FIBRATIONS

A unifying theme that is relevant to several of the areas mentioned above is that of symplectic fibration. This concept occurs in symplectic topology in several closely related variants, but one essential ingredient is a fibration (possibly local and/or singular) with a family of cohomologous symplectic forms on its fibers. Moreover, these fiberwise forms should be induced by the ambient symplectic form, if there is one. (A precise definition is given in §4.)

I pointed out in various places above that the proofs use properties of symplectic fibrations. It is also worth noting that the use of (local) fibrations is ubiquitous in 4-dimensional symplectic topology. This is obvious in so far as Donaldson's theory goes. However, this remark applies also to the kinds of symplectic surgeries that have been recently developed and explored. For instance, almost all the new examples of symplectic 4-manifolds are constructed using the fiber connect sum (see Gompf [Go], and McCarthy–Wolfson [MW]) which exploits the local fibered structure of a symplectic manifold near a symplectic submanifold with trivial normal bundle. This construction is also known as the symplectic sum. It has good formal properties: see for example Ionel–Parker [IP2] and McDuff–Symington [MSy]. Other symplectic surgeries developed by Luttinger [Lu], Eliashberg and Polterovich [EP] and Symington [Sym] also use the canonical local fibered structure of a symplectic manifold near a symplectic or Lagrangian submanifold.

As another example, observe that the knot surgeries used by Fintushel and Stern in [FS] to construct a family X_K of homotopy $K3$-surfaces are only known to yield symplectic manifolds when the knot K is fibered. To some extent this is a matter of expedience: the presence of a suitable fibration allows one to construct a symplectic form out of forms on the base and the fibers. However, Donaldson's theorem shows that fibrations are intrinsic to the structure of symplectic manifolds, and it is quite possible that it will eventually be shown that Fintushel and Stern's manifolds X_K are symplectic if and only if the knot K is fibered.

3 SYMPLECTIC RIGIDITY

In this section we will discuss "semi-local" symplectic topology, which I take to mean properties of open subsets of Euclidean space and of the symplectomorphisms between them. To emphasize that we are dealing with the standard symplectic form here, I will denote it by $\omega_0 = \sum_i dx_i \wedge dy_i$. We will begin with a discussion of Gromov's nonsqueezing theorem, which is the basis of all symplectic topology, and then in §3.2 will talk about some more specialised problems concerning symplectic embeddings.

3.1 THE NONSQUEEZING THEOREM

Gromov's nonsqueezing theorem [G1] answers the question of when a ball can be symplectically embedded in a cylinder. To emphasize the relation with fibrations we will think of the cylinder

$$Z^{2n}(\lambda) = B^2(\lambda) \times \mathbf{R}^{2n-2}$$

as the inverse image of the 2-disc $B^2(\lambda)$ of radius λ by the projection

$$p : \mathbf{R}^{2n} \to \mathbf{R}^2, \qquad (x_1, \dots, y_n) \mapsto (x_1, y_1).$$

Then, if $B^{2n}(r)$ denotes the (closed) standard ball of radius r in Euclidean space $\mathbf{R}^{2n}$ the nonsqueezing theorem can be stated as follows.

Theorem 3.1 *For all (local) symplectomorphisms ϕ of $\mathbf{R}^{2n}$*

$$\text{area}\,\big(p \circ \phi(B^{2n}(r))\big) \geq \pi r^2.$$

In other words, it is impossible to embed a standard ball of radius r into the cylinder $Z^{2n}(\lambda)$ of radius λ when $\lambda < r$.

This property of symplectomorphisms is fundamental. Indeed it characterises symplectomorphisms in the following sense. Suppose that ψ is a diffeomorphism such that

$$\text{area}\,\big(p \circ L \circ \psi(B^{2n}(x; r))\big) \geq \pi r^2$$

for all linear symplectomorphisms L and all sufficiently small balls $B^{2n}(x; r)$ in $\mathbf{R}^{2n}$. Then $\psi^*(\omega_0) = \pm\omega_0$. If in addition ψ is orientation preserving we must have the $+$ sign when n is odd. Applying this to the diffeomorphism $\psi \times \mathrm{Id}$ of $\mathbf{R}^{2n} \times \mathbf{R}^2$, one also can characterize symplectomorphisms in this way when n is even. This is the essential ingredient of the proof by Eliashberg [E1] (see also Ekeland–Hofer [EH]) that the group of symplectomorphisms is C^0-closed in the group of diffeomorphisms.[6] As Gromov pointed out in his 1986 ICM talk [G2], without this there would be no interesting theory of symplectic topology. This result is also the foundation of the theory of symplectic measurements such as the Gromov width of sets[7] and the Hofer norm on the group of Hamiltonian symplectomorphisms that is discussed in Polterovich's talk [P].

I will consider two aspects of this theorem in more detail below. Firstly, if one thinks of it as a statement about symplectic embeddings, the question obviously arises as to what other symplectic embeddings are possible between standard objects such as ellipsoids and polydiscs. Secondly, one can view this theorem as a fact about the trivial fibration

$$p : Z^{2n}(\lambda) \to B^2(\lambda),$$

and ask whether general symplectic fibrations have similar properties.

To end this section, I'd like to say one more thing concerning the relation of the C^0 (or uniform) topology to the symplectic world. Using the above ideas it is possible to define the notion of a *symplectic homeomorphism* between two smooth symplectic manifolds, though very little is known about the properties of such maps. For example, as in [EH] one can define the notion of a symplectic capacity such as the Gromov width and then say that a homeomorphism is symplectic if it preserves the capacity of sufficiently small open sets. Here I want to mention

[6] The C^0-topology is the topology of uniform convergence on compact subsets.

[7] The Gromov width $w_G(U)$ of an open subset U of (M^{2n}, ω) is defined to be the supremum of the numbers πr^2 such that the ball $B^{2n}(r)$ embeds symplectically in U.

a slightly different question. Colin [C] has recently shown that contact structures are C^0-stable in dimension 3 though not in higher dimensions. In other words, two plane fields ξ, ξ' on a closed 3-manifold that are sufficiently C^0-close and that both satisfy the contact condition are isotopic through a family of contact plane fields.

QUESTION 3.2 *Is there a symplectic analog of this result?*

It is not even clear what is the appropriate notion of "C^0-close" in this context. In the contact case the condition for a hyperplane field ξ to be contact involves the first derivative of the defining form α. In other words, if $\xi = \ker \alpha$ and the manifold has dimension $2n+1$ then one requires that $\alpha \wedge d\alpha^n \neq 0$. It follows that one can get a sensible C^0-topology by using the C^0-topology on the defining forms α (which is, of course, equivalent to using the C^0-topology on the plane fields themselves). However, any two symplectic forms ω and ω' that are cohomologous and sufficiently C^0-close may be joined by the symplectic isotopy $t\omega + (1-t)\omega', t \in [0,1]$, and so are diffeomorphic by Moser's theorem. Hence this is not the right analog. The question is whether there is an intrinsic C^0 notion of a symplectic structure for which the above stability result would hold at least in dimension 4. One might, for example, say that two symplectic structures are ε-close on a compact domain K if

$$|w_G(U, \omega) - w_G(U, \omega')| \leq \varepsilon$$

for all open subsets $U \subset K$, where w_G is the Gromov width defined above. It is not known what the consequences of such a definition would be.

This raises the whole question of what a symplectic structure "really is". I do not think that it is just a structure that allows certain analytic techniques (such as those of Gromov, Taubes and Donaldson) to work. As the nonsqueezing theorem shows there is a geometric flavor to the theory that does not seem to be captured this way. I would argue that one important geometric element is the presence of the local characteristic foliations mentioned in §1 and that another is the local product structure. An idea of what one might expect is suggested by the Eliashberg–Thurston [ET] paper on confoliations, where the authors work out the relation between foliations and contact structures and show that an essential ingredient of a contact structure is a "positive twist" condition.

3.2 SYMPLECTIC EMBEDDINGS AND FOLDING

Let us write $E(a_1, \ldots, a_n)$ for the ellipsoid

$$E(a_1, \ldots, a_n) = \{z \in \mathbf{R}^{2n} \ : \ \sum_i \frac{x_i^2 + y_i^2}{a_i} \leq 1\}.$$

It is well known that every ellipsoid in $\mathbf{R}^{2n}$ is linearly symplectomorphic to one of the form $E(a_1, \ldots, a_n)$, where $a_1 \leq \ldots \leq a_n$. Consider the question of when $E(a_1, \ldots, a_n)$ embeds symplectically into the unit ball $B^{2n}(1) = E(1, \ldots, 1)$. Floer, Hofer and Wysocki [FHW] looked at the 4-dimensional case and showed using symplectic homology that when the ellipsoid is "round", that is when the ratio a_2/a_1 is ≤ 2, there is such an embedding only if the ellipsoid is a subset of the ball. Recently Schlenk [Sch] extended this result to higher dimensions, basing his argument on Ekeland–Hofer capacities.

THEOREM 3.3 *If $a_n \leq 2a_1$ then $E(a_1,\ldots,a_n)$ embeds symplectically in $B^{2n}(1)$ only if $a_n \leq 1$.*

He has also shown that this result is sharp in the sense that as soon as $a_n > 2a_1$ it is possible to construct symplectic embeddings of $E(a_1,\ldots,a_n)$ into a ball with radius r, where $r^2 < a_n$. To be precise, he proved:

THEOREM 3.4 *Given any $\nu > \varepsilon > 0$ there is a symplectic embedding*

$$E(1,\ldots,1,2+2\nu) \hookrightarrow E(2+\nu+\varepsilon,\ldots,2+\nu+\varepsilon) = B^{2n}(\sqrt{2+\nu+\varepsilon}).$$

The proof constructs explicit embeddings by a technique known as symplectic folding. This is based on an idea of Traynor [Tr], who realised that in these embedding questions it is useful to think of a ball or ellipsoid as fibered over the 2-disc $E(a_1)$ via the projection

$$p : E(a_1,\ldots,a_n) \to E(a_1).$$

Observe that the fiber of p at a point $x \in E(a_1)$ is simply the ellipsoid $E(a_2',\ldots a_n')$ where $a_i' = a_i(a_1 - |x|^2)/a_1$. The idea is to construct embeddings of $E(a_1,\ldots,a_n)$ into $\mathbf{R}^{2n} = \mathbf{R}^2 \times \mathbf{R}^{2n-2}$ of the form $f \times g$ where $f : E(r_1) \to \mathbf{R}^2$ is area-preserving and $g : E(a_2,\ldots a_n) \to \mathbf{R}^{2n-2}$ is symplectic. In doing this one just has to control the image of $f \times g$ on the "partial product" $E(a_1,\ldots,a_n)$. This technique was developed further by Lalonde–McDuff [LM1], who incorporated the idea of folding.

For simplicity, we explain this in the case $n = 2$. The idea is that what is really important about the fibration $p : E(a_1,a_2) \to E(a_1)$ is:

(i) the fact that the subset B_c of the base $E(a_1)$ over which the fiber has area $\geq c$ is connected;
(ii) the fact that the fibers are nested, i.e. if we identify the fibers with subsets of $\mathbf{R}^2$, then fibers of equal area are identical and lie inside the fibers of greater area; and
(iii) the precise function $A(c) = \operatorname{area} B_c$.

It is shown in [LM2] that any other smoothly triangulable set T_Y of $\mathbf{R}^4$ that fibers over a smoothly triangulable set Y in $\mathbf{R}^2$ of area πa_1 and has properties (i), (ii) and the same function $A(c)$ is equivalent to the ellipsoid $E(a_1,a_2)$ in the following sense: for any $\varepsilon > 0$, one can symplectically embed $E(a_1,a_2)$ into an ε-neighborhood of T and also symplectically embed T into an ε-neighborhood of $E(a_1,a_2)$. These embeddings are also fibered, i.e. of the form $(z,w) \mapsto f(z) \times g(w)$. In particular we can take Y to be a set consisting of two rectangles of total area a_1 joined by a line segment I, and then map T_Y by embeddings into the product space $\mathbf{R}^2 \times \mathbf{R}^2$ that are fibered over each rectangle and "folded" over the interval

I. The set in T_Y that lies over I is a product $I \times F$ and the folding map has the form

$$I \times F \ \to\ U \times \mathbf{R}^2 \subset \mathbf{R} \times \mathbf{R} \times \mathbf{R}^2, \quad (t, x) \mapsto (t, H(t, x), \phi_t(x)).$$

The minimum amount of room needed to make this fold (i.e. the minimum area of U) is closely related to the Hofer norm of the embedding $\phi_1 \circ \phi_0^{-1} : \phi_0(F) \to \mathbf{R}^2$. In this construction one can see the relevance of local symplectic fibrations and the close connection between embedding problems and Hofer geometry that was exhibited in [LM1].

Here is a problem suggested by Schlenk [Sch]. Define $s(a)$ for $a \geq 1$ to be the infimum of the numbers s such that there is a symplectic embedding of the ellipsoid $E(1, a)$ into the ball $E(s, s)$. Schlenk has shown that as $a \to \infty$ the image of $E(1, a)$ fills up an arbitrarily large percentage of the volume of the ball. Thus $s(a)^2/a$ converges to 1 as $a \to \infty$.

QUESTION 3.5 *Find sharp estimates for $s(a)$, in particular as $a \searrow 2$.*

By Theorem 3.3 $s(a) = a$ for $a \leq 2$, but otherwise this function is unknown. Schlenk has made some computer calculations of the best upper bound for $s(a)$ that can be obtained by (multiple) folding but it is not clear whether his estimate is even asymptotically sharp as $a \searrow 2$. To improve this estimate one would need a new way to construct symplectic embeddings. It would be interesting to know if there is another way to construct such embeddings that is not so closely tied to the local product structure as is the method of folding.

Here is another embedding problem that involves understanding the interaction of an embedded ball with a fibration. The nonsqueezing theorem gives an obstruction for a ball B to embed in a cylinder. But when this obstruction vanishes we do not yet know much about the space of all symplectic embeddings ϕ of the ball into the cylinder, except that it is path-connected when $n = 2$. Consider the slicing of the cylinder $Z^{2n}(1)$ by the flat discs $D_x = B^2(1) \times \{x\}, x \in \mathbf{R}^{2n-2}$, that intersect the boundary $\partial Z^{2n}(1)$ along the leaves of its characteristic foliation. Each disc D_x has an area form given by the restriction of the standard symplectic form ω_0.

QUESTION 3.6 *Find a lower bound for*

$$c_r = \min_{\phi} \max_{x} \ \operatorname{area} \phi(B) \cap D_x,$$

where ϕ varies over all symplectic embeddings of the ball B of radius r. In particular, does $\lim_{r \to 1} c_r/\pi r^2 = 1$?

Polterovich pointed out[8] that the ratio $c_r/\pi r^2 \to 0$ as $r \to 0$. One can see this by beginning with a slicing (or foliation) of $\mathbf{R}^{2n}$ by parallel isotropic 2-planes (i.e. planes on which ω_0 vanishes) and then slightly perturbing it to a slicing by parallel symplectic planes whose intersections with the standard ball $B = B^{2n}(r)$

[8]Private communication

have ω_0-area $\le \varepsilon\pi r^2$ for some ε. There is a symplectomorphism ψ of $\mathbf{R}^{2n}$ that takes the slicing $\mathbf{R}^2 \times \{x\}$ to this new one, and, provided that r is sufficiently small, we can arrange that the restriction $\psi^{-1}|_B$ extends to a symplectomorphism ϕ of $\mathbf{R}^{2n}$ with support in the cylinder $Z^{2n}(1)$. Hence, for these r, we find $c_r/\pi r^2 \le \varepsilon$. On the other hand, we showed in [LM3] that when $r = 1$ any embedding ϕ of the unit ball B into the cylinder $B^2(1) \times \mathbf{R}^{2n-2}$ must intersect the boundary of the cylinder in a set that contains some flat circle $\partial B^2(1) \times \{x\}$. Hence one could say that $c_1 = \pi$.

4 Symplectic fibrations

We will begin by describing the general theory of (nonsingular) symplectic fibrations that originated in work of Guillemin, Lerman and Sternberg [GLS], and then will discuss some of the recent results about their structure.

4.1 Symplectic connections and Hamiltonian fibrations

A (nonsingular) fibration $p : P \to B$ is said to be *symplectic* if its fiber is a symplectic manifold (M, ω) and the structural group of the fibration is $\mathrm{Symp}(M, \omega)$. It follows that every fiber $M_b = p^{-1}(b)$ carries a well defined symplectic form ω_b. However, neither the base B nor the total space P need have a symplectic form. (In fact, here we may take the base to be any CW complex.)

There is an especially nice theory when all spaces involved are manifolds and p is smooth. (In this case we will say that the fibration is smooth.) A 2-form τ on P that restricts to ω_b on each fiber M_b is called a *connection 2-form*. Note that τ need not be either closed or nondegenerate. Nevertheless, the fact that it is nondegenerate on the fibers implies that its restriction to the inverse image $p^{-1}(\gamma)$ of any smooth path $\gamma : [0, 1] \to B$ in the base has a one-dimensional kernel[9] that is everywhere transverse to the fibers. Hence the integral lines of this kernel are horizontal lifts of γ that define parallel translation of the fibers along γ. It is easy to see from this description that parallel translation preserves the symplectic forms on the fibers precisely when the restriction of τ to any submanifold of the form $p^{-1}(\gamma)$ is closed. Thus one needs

$$d\tau(v_1, v_2, \cdot) = 0$$

whenever the vectors v_1, v_2 are vertical, i.e. tangent to a fiber. In this case the connection form τ is said to be *symplectic*.

It is not hard to see that every symplectic fibration has a symplectic connection τ. However, one cannot always choose τ to be closed. For example, if $p : S^3 \to S^2$ is the Hopf map, the composite map $S^3 \times S^1 \to S^3 \xrightarrow{p} S^2$ can be given the structure of a symplectic fibration, but clearly does not support a closed connection 2-form.

[9]The kernel is spanned by vectors v such that $\tau(v, w) = 0$ for all vectors w tangent to $p^{-1}(\gamma)$. This is a generalization of the characteristic foliation on a hypersurface in the sense that if τ were a symplectic form on P then this kernel would consist precisely of the vectors tangent to the characteristic foliation of τ on the hypersurface $p^{-1}(\gamma)$.

Thurston showed in [Th] that there is a closed connection form if and only if there is a cohomology class $a \in H^2(P, \mathbf{R})$ that restricts to the symplectic class $[\omega_b]$ in each fiber. There are some obvious situations in which such a class a always exists, for example if $[\omega_b]$ is the first Chern class of the tangent bundle to the fibers.[10] In [GLS] Guillemin, Lerman and Sternberg prove that if the manifold M is simply connected every symplectic fibration with fiber M supports a closed connection 2-form. They give a beautiful construction of this form (that they call the coupling form) from the curvature of a symplectic connection on P. This result was extended by McDuff–Salamon, who prove the following result in [MS].

THEOREM 4.1 *Suppose that $M \to P \to B$ is a smooth symplectic fibration with fiber (M, ω). Then the following conditions are equivalent:*
(i) The structural group of the fibration can be reduced to $\mathrm{Ham}(M, \omega)$*;*
(ii) The fibration is symplectically trivial over the 1-skeleton of B and supports a closed connection 2-form.

NOTE One needs to assume triviality over the 1-skeleton of B because the group $\mathrm{Ham}(M, \omega)$ is path-connected. It should be possible to define a subgroup H of $\mathrm{Symp}(M, \omega)$ such that fibrations with structural group H are precisely those with closed connection 2-form: see [LMP3]. This group H would have to be disconnected and have identity component equal to $\mathrm{Ham}(M, \omega)$.

DEFINITION 4.2 A smooth symplectic fibration $p : M \to B$ is said to be *Hamiltonian* if it satisfies one of the equivalent conditions in the above theorem. A symplectic form Ω on the total space of a symplectic fibration $p : P \to B$ is said to be *compatible with* the fibration if restricts to ω_b on each fiber M_b of p.

PROPOSITION 4.3 *Let $p : P \to B$ be a Hamiltonian fibration and suppose that B has a symplectic form σ_B. Then there is a symplectic form Ω on P that is compatible with p and is unique up to deformation.*

PROOF: Take $\Omega = \tau + \kappa p^*(\sigma_B)$, where τ is some closed connection form and $\kappa > 0$ is sufficiently large. For more details see [Th] (or [MS]).

4.2 THE TOPOLOGY OF SYMPLECTIC FIBRATIONS

One way to construct symplectic fibrations is to start with an element $\phi \in \pi_k(\mathrm{Symp}(M, \omega))$ and use it as a clutching function to construct a bundle over S^{k+1}:

$$p : P_\phi = \left(D_+^{k+1} \times M\right) \cup_\phi \left(D_-^{k+1} \times M\right) \;\to\; S^{k+1}.$$

When $k > 1$ the resulting fibrations are Hamiltonian, but this may not be so when $k = 1$ since $\pi_1(\mathrm{Ham}(M, \omega))$ is often different from $\pi_1(\mathrm{Symp}(M, \omega))$: see § 2.6. Since S^2 is symplectic, it follows from Proposition 4.3 above that the loop ϕ is Hamiltonian precisely when the total space P_ϕ carries a symplectic form Ω that is compatible with the fibration $p : P_\phi \to S^2$.

[10] Note that this bundle has a well defined complex structure since the space $\mathcal{J}(\omega_b)$ of fiberwise compatible almost complex structures is contractible for all $b \in B$.

Seidel pointed out in [Seid2] that every element of $\pi_1(\mathrm{Ham}(M,\omega))$ gives rise to an automorphism of the quantum cohomology of M (cf. § 2.2). By interpreting this automorphism in terms of the geometry of the bundle $P_\phi \to S^2$, Lalonde–McDuff–Polterovich showed in [LMP2] that the Leray spectral sequence for the rational cohomology of the total space P_ϕ degenerates. To do this it is enough to show that every rational homology class $\alpha \in H_*(M,\mathbf{Q})$ is the intersection with $[M]$ of a homology class $\widetilde{\alpha} \in H_{*+2}(P)$. Roughly speaking, one constructs $\widetilde{\alpha}$ as the set of points in P that lie on a suitable family of J-holomorphic sections of $p: P \to S^2$ that intersect a cycle representing α.

This argument generalizes significantly, for example to Hamiltonian fibrations over any sphere: see [LMP3].

QUESTION 4.4 *If $(M,\omega) \to P \to B$ is a fibration with structural group* $\mathrm{Ham}(M,\omega)$*, is the rational cohomology $H^*(P,\mathbf{Q})$ of P isomorphic as a vector space to $H^*(B;\mathbf{Q}) \otimes H^*(M,\mathbf{Q})$?*

The answer is known to be "yes" when the hard Lefschetz theorem holds for $H^*(M,\mathbf{Q})$: see [B]. However, it is "no" if one drops the Hamiltonian condition. For example, the Kodaira–Thurston manifold in [Th] that is symplectic but nonKähler is the total space of a symplectic fibration $X \to T^2$ with fiber T^2. Here

$$X = T^2 \times S^1 \times [0,1]/\sim, \quad (x,y,s,0) \sim (x, x+y, s, 1),$$

and it is easily seen that $b_1(X) = 3$ rather than 4.

The story concerning the multiplicative structure of $H^*(P,\mathbf{Q})$ is more complicated. Here one can consider both the standard cup product and also versions of the quantum (or deformed) cup product. Seidel exploits properties of the quantum product in his work on $\mathrm{Symp}(\mathbf{C}P^m \times \mathbf{C}P^n)$ that was mentioned in 2.6 above. He also pointed out[11] that if (M,ω) admits no J-holomorphic spheres at all and if P is a fibration over S^2 then $H^*(P,\mathbf{Q})$ is isomorphic as a ring (under cup product) with the product of the rings $H^*(S^2,\mathbf{Q})$ and $H^*(M,\mathbf{Q})$. The following generalization looks very plausible, but the full details of the proof are not yet worked out: see [Mc2]. We say that the quantum product is trivial if it equals the usual cup product.

CLAIM 4.5 *Let $(M,\omega) \to P \to S^2$ be a fibration with structural group* $\mathrm{Ham}(M,\omega)$*, and suppose that the quantum product on M is trivial. Then $H^*(P,\mathbf{Q})$ is isomorphic as a ring (under cup product) with the product of the rings $H^*(S^2,\mathbf{Q})$ and $H^*(M,\mathbf{Q})$.*

If the quantum product on M is nontrivial, no general statement about the ring structure of $H^*(P)$ has yet been found. Nor is it yet clear what happens with bases other than S^2.

In view of Donaldson's work mentioned in §2.3 above, it would be interesting to have an answer to the following question.

[11] Private communication

QUESTION 4.6 *To what extent do these results carry over to Lefschetz (i.e. singular) fibrations?*

In the algebraic case there is a good understanding of the cohomology of Lefschetz pencils: see Looijenga [L] for example. However, this is closely related to the fact that the hard Lefschetz theorem holds for algebraic manifolds, and so it is not clear what, if anything, will carry over to the symplectic case.

Finally, we remark that these ideas allow one to decide when the nonsqueezing theorem holds for the fibration $P \to S^2$: see [Mc2]. By this we mean the following. Let (P, Ω) be a symplectic $(2n+2)$-dimensional manifold such that Ω is compatible with the fibration $P \to S^2$, and define the *area* of (P, Ω) to be the number A such that

$$\frac{1}{(n+1)!}\int_P \Omega^{n+1} \;=\; \frac{A}{n!}\int_M \omega^n.$$

Thus, if the fibration $P \to S^2$ is symplectically trivial so that (P, Ω) is the product $(S^2 \times M, \sigma \oplus \omega)$, A is simply the area of the base (S^2, σ). Then we will say that *the nonsqueezing theorem holds for the fibration* $p : (P, \Omega) \to S^2$ if the area A constrains the size of the balls that embed into (P, Ω), i.e. if $\pi r^2 \le A$ whenever $B^{2n+2}(r)$ embeds symplectically in (P, Ω). By considering the case when P is $\mathbf{C}P^2$ blown up at a point, it is not hard to see that some condition is needed in order for the nonsqueezing theorem to hold. It looks very likely that by studying properties of J-holomorphic sections one can establish the following claim: see [Mc2].

CLAIM 4.7 *Let* $p : P \to S^2$ *be a symplectic fibration whose fiber* (M, ω) *has trivial quantum product, and let* Ω *be a symplectic form on* P *compatible with* p*. Then the nonsqueezing theorem holds for the fibration* $p : (P, \Omega) \to S^2$.

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