Documenta Mathematica

Journal der
Deutschen Mathematiker-Vereinigung
Gegründet 1996
Extra Volume

A Collection of Manuscripts
Written in Honour of

Andrei A. Suslin

on the Occasion of His Sixtieth Birthday

Editors:
I. Fesenko, E. Friedlander,
A. Merkurjev, U. Rehmann

Documenta Mathematica, Journal der Deutschen Mathematiker-Vereinigung, veröffentlicht Forschungsarbeiten aus allen mathematischen Gebieten und wird in traditioneller Weise referiert. Es wird indiziert durch Mathematical Reviews, Science Citation Index Expanded, Zentralblatt für Mathematik.

Artikel können als TEX-Dateien per E-Mail bei einem der Herausgeber eingereicht werden. Hinweise für die Vorbereitung der Artikel können unter der unten angegebenen WWW-Adresse gefunden werden.

Documenta Mathematica, Journal der Deutschen Mathematiker-Vereinigung, publishes research manuscripts out of all mathematical fields and is refereed in the traditional manner. It is indexed in Mathematical Reviews, Science Citation Index Expanded, Zentralblatt für Mathematik.

Manuscripts should be submitted as TEX -files by e-mail to one of the editors. Hints for manuscript preparation can be found under the following web address.

`http://www.math.uni-bielefeld.de/documenta`

ISBN 978-3-936609-48-6 ISSN 1431-0635 (Print) ISSN 1431-0643 (Internet)

SPARC
Leading Edge

Documenta Mathematica is a Leading Edge Partner of SPARC, the Scholarly Publishing and Academic Resource Coalition of the Association of Research Libraries (ARL), Washington DC, USA.

Address of Technical Managing Editor: Ulf Rehmann, Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, D-33501 Bielefeld,

DOCUMENTA MATHEMATICA

EXTRA VOLUME: ANDREI A. SUSLIN'S SIXTIETH BIRTHDAY, 2010

Preface

Over four decades, Andrei Suslin has conducted inspirational research at St. Petersburg University (LOMI) and Northwestern University. Andrei's impact on algebraic K-theory, motivic cohomology, central simple algebras, cohomology of groups, and representation theory have fundamentally changed these subjects. Many of the best results in these areas are due to Andrei, many more were achieved using his ideas and guidance. Andrei's influence extends beyond his published achievements, for he has been most generous in sharing his ideas and insights. With great admiration, this volume of Documenta Mathematica is dedicated to him.

St. Petersburg memories, Sasha Merkurjev

The Boarding School # 45 was a unique special place. It collected talented pupils in the North-West region of the Soviet Union. It was the only way into mathematics for many people living outside of big cities. Suslin taught at this school during 3 years when he was an undergraduate student. His style made a tremendous impact on me that I have never experienced later. Not only on me – for example, I just recently met my class-mate Sasha Koldobskiy (he is professor at the University of Missouri) and he shares the same feelings. Needless to say that already at that time I decided to study algebra. Such early decisions were not exceptional: Nikita Karpenko asked me to be his advisor when he was a 9^{th} year student at the School # 45.

Andrei's passion for mathematics and his systematic approach were a model for us. We saw him reading algebra books like Bourbaki commutative algebra in a bus or metro. During short breaks between lessons he draw complicated diagrams in the notebook (standard thin 2 kopeks notebooks where Andrei used to record all his math) – that time Andrei was working on a problem in finite geometry and combinatorics. I guess that work was not successful and at the beginning of the senior year Andrei realized that he has nothing yet done for the diploma work to be completed in 9 months. That is how he turned to Serre's conjecture concerning modules over polynomial rings.

During boring meetings we had to sit at, Andrei would ask me to give him problems to solve from recent mathematical olympiads, and often my list ended before the meeting was over. Andrei was a winner of the International Mathematical Olympiad in 1967.

The "olympiad spirit" has an interesting consequence: Andrei considers every mathematical problem as a personal challenge. That is why there are not so many Suslin's conjectures: by making a conjecture Andrei admits that he failed to prove it himself.

Andrei's impact of mathematicians has been tremendous, not only his own graduate students but on many others fortunate to be around him. I remember spontaneous seminars (for many hours) Andrei started when people randomly get together in his room at LOMI. I remember his lectures on the foundations of motivic cohomology in the late 80's, when it was rather an improvisation at the board than lectures. Two of Andrei's graduate students, Vanya Panin and Serge Yagunov, are organizers of this birthday celebration; other people who can call Andrei an informal advisor include Sasha Smirnov, Sasha Nenashev, myself, ... During these seminars Andrei generously shared his ideas. (Markus Rost is another personality of this type.)

Immediately after his graduation, Andrei was hired as an assistant professor at the University (so he has never been a graduate student). He worked on Serre's conjecture and tried to hide from the rest of the university world – at least he did not propose themes for students' work, and I was not able to get him as thesis advisor.

Andrei liked to work at night – this habit comes from the time when he lived in an apartment shared by several families (with one bathroom and kitchen), so he could only work in the kitchen after midnight.

The most funny story about Andrei (unfortunately not for publishing) is that once he was a member of the Congress of the Young Communist League (he was the only doctor of sciences in the country younger 28) and he was given a speech to read about Brezhnev helping him to prove Serre's Conjecture. As an exchange he was promised a separate apartment but it did not work out.

Perspective of a friend and colleague, Eric Friedlander

Andrei has been my close friend for many years. We first met in Oberwolfach in the late 1970's. Andrei's English was perfect; not only did he speak and understand the language, but he understood subtle nuances which astonished me. We talked mathematics, but also about many other matters. This was the time his mathematical legend was already being established.

Perhaps few remember that Andrei was an "all Leningrad" gymnast. This showed when he lectured, for he seemed more poised at the blackboard. Some of us have never learned, despite much trying, to imitate his style of speaking slowly, writing very large symbols on the blackboard, all the while conveying elegantly and efficiently the essence of his mathematics.

A few years later, Andrei and I both visited University of Paris 7. An early memory of that year followed Andrei's talk and gold medal at the College de France. We wandered around Paris at 7:30pm looking for dinner. All restaurants were empty, but all were reserved for the night, just as had been the case of restaurants in the USSR. One morning Andrei called me to say that during the night he proved the Quillen-Lichtenbaum Conjecture for algebraically closed fields of positive characteristic and asked if I would photocopy his manuscript at IHES. Andrei stood at the entrance of the peripherique on the fringe of

Paris, handing through my car window his coffee-stained manuscript as the car briefly paused before quickly merging into traffic. What did this Russian to American exchange look like to an observer? When he first talked about this result in a Paris seminar, the audience broke tradition to give him an ovation. The 1986 ICM in Berkeley was the "Mathematical Congress of Absent Russians". The world mathematical community eagerly anticipated the remarkable, almost mythical creators of so much new mathematics. Sadly, Andrei was among those not allowed to attend, but I was given a manuscript of his plenary address. This manuscript consisted of page after page of new results on algebraic K-theory. After spending time with Andrei in Paris, I had the privilege of visiting the Suslin family in their St. Petersburg apartment; my achievement was explaining the colloquial English in a popular cartoon series, not quite equal to Andrei's explanations of mathematical lectures given in Russian which we attended in Novosibirsk. Food memories include the delicious "Russian salad" and the rich soup of cepes (from the woods near the Suslin dacha) prepared by Olga Suslina. A measure of time passing has been watching Andrei's daughters Olga and Maria grow from young girls to successful adults with children of their own.

Andrei visited M.I.T. and the University of Chicago in the early 1990's. To my overwhelming delight and benefit, Andrei decided to join the Northwestern faculty in 1995. A frequent image which comes to mine is of Andrei pacing outside my office ignoring whatever weather Chicago was throwing us, while I stayed warm and dry by scribbling on a blackboard. The best of those times was our extended effort to prove finite generation of certain cohomology rings; this was a question that I had thought about for years, and the most important step I took towards its solution was to consult Andrei. Vladimir Voevodsky was briefly our colleague at Northwestern. Indeed, a few years earlier, I had arranged for Andrei to meet Vladimir, recognizing that their different styles and powerful mathematical talents could be blended together in a very fruitful manner.

So many mathematicians over the years have benefited from Andrei's insights and confidence. If someone mentioned a result, then typically Andrei would say he is sure it is right. On the other hand, should he need the result he would produce his own proof – typically improving the statement as well as the proof – or find a counter-example. With me, perhaps Andrei was a bit more relaxed for he would occasionally tell me something was nonsense and even occasionally admit after extended discussion that he was wrong. Those interactions are among my best memories of our days together at Northwestern. Andrei's generosity extended to looking after me on the ski slopes, willingness to drive to the airport at an awful hour, and other matters of daily life. Our friendship has been the most remarkable aspect of my mathematical career.

I. Fesenko, E. Friedlander, A. Merkurjev, U. Rehmann

Infinitesimal Deformations and the ℓ-Invariant

To Andrei Alexandrovich Suslin, for his 60th birthday

Denis Benois

Received: November 11, 2009
Revised: January 22, 2010

Abstract. We give a formula for the generalized Greenberg's ℓ-invariant which was constructed in [Ben2] in terms of derivatives of eigenvalues of Frobenius.

2000 Mathematics Subject Classification: 11R23, 11F80, 11S25, 11G40, 14F30
Keywords and Phrases: p-adic representation, (φ,Γ)-module, L-function

Introduction

0.1. Let M be a pure motive over $\mathbb{Q}$ with coefficients in a number field E. Assume that the L-function $L(M,s)$ is well defined. Fixinig an embedding $\iota\,:\,E\hookrightarrow\mathbb{C}$ we can consider it as a complex-valued Dirichlet series $L(M,s)=\sum\limits_{n=0}^{\infty}a_nn^{-s}$ which converges for $s\gg 0$ and is expected to admit a meromorphic continuation to $\mathbb{C}$ with a functional equation of the form

$$\Gamma(M,s)\,L(M,s)\;=\;\varepsilon(M,s)\,\Gamma(M^*(1),-s)\,L(M^*(1),-s)$$

where $\Gamma(M,s)$ is the product of some Γ-factors and the ε-factor has the form $\varepsilon(M,s)=ab^s$.

Assume that M is critical and that $L(M,0)\neq 0$. Fix a finite place $\lambda|p$ of E and assume that the λ-adic realization M_λ of M is semistable in the sense of Fontaine [Fo3]. The (φ,N)-module $\mathbf{D}_{\mathrm{st}}(M_\lambda)$ associated to M_λ is a finite dimensional E_λ-vector space equipped with an exhaustive decreasing filtration $\mathrm{Fil}^i\mathbf{D}_{\mathrm{st}}(M_\lambda)$, a E_λ-linear bijective frobenius $\varphi\,:\,\mathbf{D}_{\mathrm{st}}(M_\lambda)\to\mathbf{D}_{\mathrm{st}}(M_\lambda)$ and a nilpotent monodromy operator N such that $N\,\varphi\,=\,p\,\varphi\,N$. We say that a (φ,N)-submodule D of $\mathbf{D}_{\mathrm{st}}(M_\lambda)$ is regular if

$$\mathbf{D}_{\mathrm{st}}(M_\lambda)\;=\;D\oplus\mathrm{Fil}^0\mathbf{D}_{\mathrm{st}}(M_\lambda)$$

as E_λ-vector spaces. The theory of Perrin-Riou [PR] suggests that to any regular D one can associate a p-adic L-function $L_p(M,D,s)$ interpolating rational parts of special values of $L(M,s)$. In particular, the interpolation formula at $s=0$ should have the form

$$L_p(M,D,0) \,=\, \mathcal{E}(M,D)\,\frac{L(M,0)}{\Omega_\infty(M)}$$

where $\Omega_\infty(M)$ is the Deligne period of M and $\mathcal{E}(M,D)$ is a certain product of Euler-like factors. Therefore one can expect that $L_p(M,D,0)=0$ if and only if $\mathcal{E}(M,D)=0$ and in this case one says that $L_p(M,D,s)$ has a trivial zero at $s=0$.

0.2. According to the conjectures of Bloch and Kato [BK], the E_λ-adic representation M_λ should have the following properties:

C1) The Selmer groups $H^1_f(M_\lambda)$ and $H^1_f(M^*_\lambda(1))$ are zero.
C2) $H^0(M_\lambda)\,=\,H^0(M^*_\lambda(1))\,=\,0$ where we write H^* for the global Galois cohomology.
Moreover one expects that
C3) $\varphi\,:\,\mathbf{D}_{\mathrm{st}}(M_\lambda)\to\mathbf{D}_{\mathrm{st}}(M_\lambda)$ is semisimple (semisimplicity conjecture).
We also make the following assumption which is a direct generalization of the hypothesis **U)** from [G].
C4) The (φ,Γ)-module $\mathbf{D}^{\dagger}_{\mathrm{rig}}(M_\lambda)$ has no saturated subquotients of the form $U_{m,n}$ where $U_{m,n}$ is the unique crystalline (φ,Γ)-module sitting in a non split exact sequence

$$0\to\mathcal{R}_L(|x|x^m)\to U_{m,n}\to\mathcal{R}_L(x^{-n})\to 0,\quad L=E_\lambda$$

(see §1 for unexplained notations).

In [Ben2], we extended the theory of Greenberg [G] to L-adic pseudo geometric representations which are semistable at p and satisfy **C1-4)**. Namely to any regular $D\subset\mathbf{D}_{\mathrm{st}}(V)$ of a reasonably behaved representation V we associated an integer $e\geqslant 0$ and an element $\mathcal{L}(V,D)\in L$ which can be seen as a vast generalization of the $\mathcal{L}$-invariants constructed in [Mr] and [G]. If $V=M_\lambda$ we set $\mathcal{L}(M,D)=\mathcal{L}(M_\lambda,D)$. A natural formulation of the trivial zero conjecture states as follows:

Conjecture. $L_p(M,D,s)$ has a zero of order e at $s=0$ and

$$\lim_{s\to 0}\frac{L_p(M,D,s)}{s^e}\,=\,\mathcal{E}^+(M,D)\,\mathcal{L}(M^*(1),D^*)\,\frac{L(M,0)}{\Omega_\infty(M)}, \tag{0.1}$$

where $\mathcal{E}^+(M,D)$ is the subproduct of $\mathcal{E}(M,D)$ obtained by "excluding zero factors" and $D^*\,=\,\mathrm{Hom}(\mathbf{D}_{\mathrm{st}}(V)/D,\mathbf{D}_{\mathrm{st}}(L(1)))$ is the dual regular module (see [Ben2] for more details). We refer to this statement as Greenberg's conjecture because if M_λ is ordinary at p it coincides with the conjecture formulated in [G], p.166. Remark that if M_λ is crystalline at p, Greenberg's conjecture is compatible with Perrin-Riou's theory of p-adic L-functions [Ben3].

0.3. Consider the motive M_f attached to a normalized newform $f = \sum_{n=1}^{\infty} a_n q^n$ of weight $2k$ on $\Gamma_0(Np)$ with $(N,p) = 1$. The complex L-function of M_f is $L(f,s) = \sum_{n=1}^{\infty} a_n n^{-s}$. The twisted motive $M_f(k)$ is critical. The eigenvalues of φ acting on $\mathbf{D}_{\mathrm{st}}(M_{f,\lambda}(k))$ are $\alpha = p^{-k}a_p$ and $\beta = p^{1-k}a_p$ with $v_p(a_p) = k-1$. The unique regular submodule of $\mathbf{D}_{\mathrm{st}}(M_{f,k}(k))$ is $D = E_\lambda d$ where $\varphi(d) = \alpha\, d$ and $L_p(M_f(k), D, s) = L_p(f, s+k)$ where $L_p(f,s)$ is the classical p-adic L-function associated to a_p via the theory of modular symbols [Mn], [AV]. If $a_p = p^{k-1}$, the function $L_p(f,s)$ vanishes at $s = k$. In this case several constructions of the $\mathcal{L}$-invariant based on different ideas were proposed (see [Co1], [Tm], [Mr], [O], [Br]). Thanks to the work of many people it is known that they are all equal and we refer to [Cz3] and [BDI] for further information. As $M_f(k)$ is self-dual (i.e. $M_f(k) \simeq M_f^*(1-k)$) one has $\mathcal{L}(M_f^*(1-k), D^*) = \mathcal{L}(M_f(k), D)$ (see also section 0.4 below). Moreover it is not difficult to prove that $\mathcal{L}(M_f(k), D)$ coincides with the $\mathcal{L}$-invariant of Fontaine-Mazur $\mathcal{L}_{\mathrm{FM}}(f)$ [Mr] ([Ben2], Proposition 2.3.7) and (0.1) takes the form of the Mazur-Tate-Teitelbaum conjecture

$$L_p'(f,k) = \mathcal{L}(f)\, \frac{L(f,k)}{\Omega_\infty(f)}$$

where we write $\mathcal{L}(f)$ for an unspecified $\mathcal{L}$-invariant and $\Omega_\infty(f)$ for the Shimura period of f [MTT]. This conjecture was first proved by Greenberg and Stevens in the weight two case [GS1] [GS2]. In the unpublished note [St], Stevens generalized this approach to the higher weights. Other proofs were found by Kato, Kurihara and Tsuji (unpublished but see [Cz2]), Orton [O], Emerton [E] and by Bertolini, Darmon and Iovita [BDI]. The approach of Greenberg and Stevens is based on the study of families of modular forms and their p-adic L-functions. Namely, Hida (in the ordinary case) and Coleman [Co1] (in general) constructed an analytic family $f_x = \sum_{n=1}^{\infty} a_n(x) q^n$ of p-adic modular forms for $x \in \mathbb{C}_p$ passing through f with $f = f_{2k}$. Next, Panchishkin [Pa] and independently Stevens (unpublished) constructed a two-variable p-adic L-function L-function $L_p(x,s)$ satisfying the following properties:

- $L_p(2k,s) = L_p(f,s)$.
- $L_p(x, x-s) = -\langle N \rangle^{s-x} L_p(x,s)$.
- $L_p(x,k) = (1 - p^{k-1}a_p(x)^{-1})\, L^*(x)$ where $L_p^*(x)$ is a p-adic analytic function such that $L_p^*(2k) = L(f,k)/\Omega_\infty(f)$.

From these properties it follows easily that

$$L_p'(f,k) = -2\, d \log a_p(2k)\, \frac{L(f,k)}{\Omega_\infty(f)},$$

where $d\log a_p(x)=a_p(x)^{-1}\dfrac{da_p(x)}{dx}$. Thus the Mazur-Tate-Teitelbaum conjecture is equivalent to the assertion that

$$\mathcal{L}(f)=-2\,d\log a_p(2k). \tag{0.2}$$

This formula was first proved for weight two by Greenberg and Stevens. In the higher weight case several proofs of (0.2) have been proposed:
1. By Stevens [St], working with Coleman's $\mathcal{L}$-invariant $\mathcal{L}_{\mathrm{C}}(f)$ defined in [Co1].
2. By Colmez [Cz5], working with the Fontaine-Mazur's $\mathcal{L}$-invariant $\mathcal{L}_{\mathrm{FM}}(f)$ defined in [Mr].
3. By Colmez [Cz6], working with Breuil's $\mathcal{L}$-invariant $\mathcal{L}_{\mathrm{Br}}(f)$ defined in [Br].
4. By Bertolini, Darmon and Iovita [BDI], working with Teitelbaum's $\mathcal{L}$-invariant $\mathcal{L}_{\mathrm{T}}(f)$ [Tm] and Orton's $\mathcal{L}$-invariant $\mathcal{L}_{\mathrm{O}}(f)$ [O].

0.4. In this paper, working with the $\mathcal{L}$-invariant defined in [Ben2] we generalize (0.2) to some infinitesimal deformations of pseudo geometric representations. Our result is purely algebraic and is a direct generalization of Theorem 2.3.4 of [GS2] using the cohomology of (φ,Γ)-modules instead Galois cohomology. Let V be a pseudo-geometric representation with coefficients in $L/\mathbb{Q}_p$ which satisfies **C1-4)**. Fix a regular submodule D. In view of (0.1) it is convenient to set

$$\ell(V,D)=\mathcal{L}(V^*(1),D^*).$$

Suppose that $e=1$. Conjecturally this means that the p-adic L-function has a simple trivial zero. Then either $D^{\varphi=p^{-1}}$ or $(D^*)^{\varphi=p^{-1}}$ has dimension 1 over L. To fix ideas, assume that $\dim_L D^{\varphi=p^{-1}}=1$. Otherwise, as one expects a functional equation relating $L_p(M,D,s)$ and $L_p(M^*(1),D^*,-s)$ one can consider $V^*(1)$ and D^* instead V and D. We distinguish two cases. In each case one can express $\ell(V,D)$ directly in terms of V and D.

• *The crystalline case:* $D^{\varphi=p^{-1}}\cap N\left(\mathbf{D}_{\mathrm{st}}(V)^{\varphi=1}\right)=\{0\}$. Let $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$ be the (φ,Γ)-module over the Robba ring $\mathcal{R}_L$ associated to V [Ber1], [Cz1]. Set $D_{-1}=(1-p^{-1}\varphi^{-1})D$ and $D_0=D$. The two step filtration $D_{-1}\subset D_0\subset \mathbf{D}_{\mathrm{st}}(V)$ induces a filtration

$$F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)\subset F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)\subset \mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$$

such that $\mathrm{gr}_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)\simeq \mathcal{R}_L(\delta)$ is the (φ,Γ)-module of rank 1 associated to a character $\delta:\mathbb{Q}_p^*\to L^*$ of the form $\delta(x)=|x|x^m$ with $m\geqslant 1$. The cohomology of (φ,Γ)-modules of rank 1 is studied in details in [Cz4]. Let $\eta:\mathbb{Q}_p^*\to L^*$ be a continuous character. Colmez proved that $H^1(\mathcal{R}_L(\eta))$ is a one dimensional L-vector space except for $\eta(x)=|x|x^m$ with $m\geqslant 1$ and $\eta(x)=x^{-n}$ with $n\leqslant 0$. In the exceptional cases $H^1(\mathcal{R}_L(\eta))$ has dimension 2 and can be canonically decomposed into direct sum of one dimensional subspaces

$$H^1(\mathcal{R}_L(\eta))\simeq H^1_f(\mathcal{R}_L(\eta))\oplus H^1_c(\mathcal{R}_L(\eta)),\quad \eta(x)=|x|x^m \text{ or } \eta(x)=x^{-n} \tag{0.3}$$

([Ben2], Theorem 1.5.7). The condition **C1)** implies that

$$H^1(V) \simeq \bigoplus_{l \in S} \frac{H^1(\mathbb{Q}_l, V)}{H^1_f(\mathbb{Q}_l, V)} \tag{0.4}$$

for a finite set of primes S. This isomorphism defines a one dimensional subspace $H^1(D,V)$ of $H^1(V)$ together with an injective localisation map $\kappa_D : H^1(D,V) \to H^1(\mathcal{R}_L(\delta))$. Then $\ell(V,D)$ is the slope of $\mathrm{Im}(\kappa_D)$ with respect to the decomposition of $H^1(\mathcal{R}_L(\delta))$ into direct sum (0.3). Let

$$0 \to V \to V_x \to L \to 0$$

be an extension in the category of global Galois representations such that $\mathrm{cl}(x) \in H^1(D,V)$ is non zero. We equip $\mathbf{D}^\dagger_{\mathrm{rig}}(V_x)$ with a canonical filtration

$$\{0\} \subset F_{-1}\mathbf{D}^\dagger_{\mathrm{rig}}(V_x) \subset F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V_x) \subset F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V_x) \subset \mathbf{D}^\dagger_{\mathrm{rig}}(V_x)$$

such that $F_i\mathbf{D}^\dagger_{\mathrm{rig}}(V_x) = F_i\mathbf{D}^\dagger_{\mathrm{rig}}(V)$ for $i = -1, 0$ and $\mathrm{gr}_1\mathbf{D}^\dagger_{\mathrm{rig}}(V_x) \simeq \mathcal{R}_L$. Let $V_{A,x}$ be an infinitesimal deformation of V_x over $A = L[T]/(T^2)$ endowed with a filtration $F_i\mathbf{D}^\dagger_{\mathrm{rig}}(V_{A,x})$ such that $F_i\mathbf{D}^\dagger_{\mathrm{rig}}(V) = F_i\mathbf{D}^\dagger_{\mathrm{rig}}(V_{A,x}) \otimes_A L$. Write

$$\mathrm{gr}_0\mathbf{D}^\dagger_{\mathrm{rig}}(V_{A,x}) \simeq \mathcal{R}_A(\delta_{A,x}), \qquad \mathrm{gr}_1\mathbf{D}^\dagger_{\mathrm{rig}}(V_{A,x}) \simeq \mathcal{R}_A(\psi_{A,x})$$

with $\delta_{A,x}$, $\psi_{A,x} : \mathbb{Q}_p^* \to A^*$.

Theorem 1. *Assume that* $\left.\dfrac{d(\delta_{A,x}\psi^{-1}_{A,x})(u)}{dT}\right|_{T=0} \neq 0$ *for* $u \equiv 1 \pmod{p^2}$. *Then*

$$\ell(V,D) = -\log(u)\left.\frac{d\log(\delta_{A,x}\psi^{-1}_{A,x})(p)}{d\log(\delta_{A,x}\psi^{-1}_{A,x})(u)}\right|_{T=0}$$

(note that the right hand side does not depend on the choice of u).

• *The semistable case:* $D^{\varphi=p^{-1}} \subset N\left(\mathbf{D}_{\mathrm{st}}(V)^{\varphi=1}\right)$. Set $D_{-1} = (1-p^{-1}\varphi^{-1})D$, $D_0 = D$ and $D_1 = N^{-1}(D^{\varphi=p^{-1}}) \cap \mathbf{D}_{\mathrm{st}}(V)^{\varphi=1}$. The filtration

$$D_{-1} \subset D_0 \subset D_1 \subset \mathbf{D}_{\mathrm{st}}(V)$$

induces a filtration

$$F_{-1}\mathbf{D}^\dagger_{\mathrm{rig}}(V) \subset F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V) \subset F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V) \subset \mathbf{D}^\dagger_{\mathrm{rig}}(V)$$

Then $\mathrm{gr}_0\mathbf{D}^\dagger_{\mathrm{rig}}(V) \simeq \mathcal{R}_L(\delta)$ and $\mathrm{gr}_1\mathbf{D}^\dagger_{\mathrm{rig}}(V) \simeq \mathcal{R}_L(\psi)$ where the characters δ and ψ are such that $\delta(x) = |x|x^m$ and $\psi(x) = x^{-n}$ for some $m \geqslant 1$ and $n \geqslant 0$. Set $M = F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V)/F_{-1}\mathbf{D}^\dagger_{\mathrm{rig}}(V)$ and consider the map $\kappa_D : H^1(M) \to H^1(\mathcal{R}_L(\psi))$ induced by the projection $M \to \mathcal{R}_L(\psi)$. The image of κ_D is a one dimensional L-subspace of $H^1(\mathcal{R}_L(\psi))$ and $\ell(V,D)$ is the slope of $\mathrm{Im}(\kappa_D)$ with respect to (0.3).
Assume that V_A is an infinitesimal deformation of V equipped with a filtration $F_i\mathbf{D}^\dagger_{\mathrm{rig}}(V_A)$ such that $F_i\mathbf{D}^\dagger_{\mathrm{rig}}(V) = F_i\mathbf{D}^\dagger_{\mathrm{rig}}(V_A) \otimes_A L$. Write $\mathrm{gr}_0\mathbf{D}^\dagger_{\mathrm{rig}}(V_A) \simeq \mathcal{R}_A(\delta_A)$ and $\mathrm{gr}_1\mathbf{D}^\dagger_{\mathrm{rig}}(V_A) \simeq \mathcal{R}_A(\psi_A)$.

THEOREM 2. *Assume that*

$$\left.\frac{d(\delta_A\psi_A^{-1})(u)}{dT}\right|_{T=0} \neq 0 \ \textit{for } u\equiv 1 \pmod{p^2}. \tag{0.5}$$

Then

$$\ell(V,D)=-\log(u)\left.\frac{d\log(\delta_A\psi_A^{-1})\,(p)}{d\log(\delta_A\psi_A^{-1})(u)}\right|_{T=0}.$$

Remark that in the semistable case $\ell(V,D)=\mathcal{L}(V,D)$.

For classical modular forms the existence of deformations having the above properties follows from the theory of Coleman-Mazur [CM] together with deep results of Saito and Kisin [Sa], [Ki]. Applying Theorem 2 to the representation $M_{f,\lambda}(k)$ we obtain a new proof of (0.2) with the Fontaine-Mazur $\mathcal{L}$-invariant. Remark that the local parameter T corresponds to the weight of a p-adic modular form and (0.5) holds automatically. In the general case the existence of deformations satisfying the above conditions should follow from properties of eigenvarieties of reductive groups [BC].

The formulations of Theorems 1 and 2 look very similar and the proof is essentially the same in the both cases. The main difference is that in the crystalline case the ℓ-invariant is global and contains information about the localisation map $H^1(V)\to H^1(\mathbb{Q}_p,V)$. In the proof of Theorem 1 we consider V_x as a representation of the local Galois group but the construction of V_x depends on the isomorphism (0.4). In the semistable case the definition of $\ell(V,D)$ is purely local and the hypothesis **C1-2)** can be omitted. However **C1-2)** are essential for the formulation of Greenberg conjecture because (0.1) is meaningless if $L(M,0)=0$. One can compare our results with Hida's paper [Hi] where the case of ordinary representations over totally real ground field is studued.

Here goes the organization of this paper. The §1 contains some background material. In section 1.1 we review the theory of (φ,Γ)-modules and in section 1.2 recall the definition of the ℓ-invariant following [Ben2]. The crystalline and semistable cases of trivial zeros are treated in §2 and §3 respectively. I would like to thank Pierre Parent for several very valuable discussions which helped me with the formulation of Theorem 1 and the referee for pointing out several inaccuracies in the first version of this paper.

It is a great pleasure to dedicate this paper to Andrei Alexandrovich Suslin on the occasion of his 60th birthday.

§1. The ℓ-invariant

1.1. (φ,Γ)-modules. ([Fo1], [Ber1], [Cz1])

1.1.1. Let p be a prime number. Fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and set $G_{\mathbb{Q}_p}=\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. We denote by $\mathbb{C}_p$ the p-adic completion of $\overline{\mathbb{Q}}_p$ and write $|\cdot|$ for the absolute value on $\mathbb{C}_p$ normalized by $|p|=1/p$. For any $0\leqslant r<1$ set

$$B(r,1)=\{z\in\mathbb{C}_p\mid p^{-1/r}\leqslant |z|<1\}.$$

Let $\chi\,:\,G_{\mathbb{Q}_p}\to\mathbb{Z}_p^*$ denote the cyclotomic character. Set $H_{\mathbb{Q}_p}=\ker(\chi)$ and $\Gamma=G_{\mathbb{Q}_p}/H_{\mathbb{Q}_p}$. The character χ will be often considered as an isomorphism $\chi\,:\,\Gamma\xrightarrow{\sim}\mathbb{Z}_p^*$. Let L be a finite extension of $\mathbb{Q}_p$. For any $0\leqslant r<1$ we denote by $\mathbf{B}^{\dagger,r}_{\mathrm{rig},L}$ the ring of p-adic functions $f(\pi)=\sum\limits_{k\in\mathbb{Z}}a_k\pi^k$ $(a_k\in L)$ which are holomorphic on the annulus $B(r,1)$. The Robba ring over L is defined as $\mathcal{R}_L=\bigcup\limits_r\mathbf{B}^{\dagger,r}_{\mathrm{rig},L}$. Recall that $\mathcal{R}_L$ is equipped with commuting, L-linear, continuous actions of Γ and a frobenius φ which are defined by

$$\begin{aligned}&\gamma(f(\pi))=f((1+\pi)^{\chi(\gamma)}-1), \qquad \gamma\in\Gamma,\\ &\varphi(f(\pi))=f((1+\pi)^p-1).\end{aligned}$$

Set $t=\log(1+\pi)=\sum\limits_{n=1}^{\infty}(-1)^{n-1}\dfrac{\pi^n}{n}$. Remark that $\gamma(t)=\chi(\gamma)\,t$ and $\varphi(t)=p\,t$.
A finitely generated free $\mathcal{R}_L$-module $\mathbf{D}$ is said to be a (φ,Γ)-module if it is equipped with commuting semilinear actions of Γ and φ and such that $\mathcal{R}_L\varphi(\mathbf{D})=\mathbf{D}$. The last condition means simply that $\varphi(e_1),\ldots,\varphi(e_d)$ is a basis of $\mathbf{D}$ if $e_1,\ldots,e_d$ is.
Let $\delta\,:\,\mathbb{Q}_p^*\to L^*$ be a continuous character. We will write $\mathcal{R}_L(\delta)$ for the (φ,Γ)-module $\mathcal{R}_Le_\delta$ of rank 1 defined by

$$\varphi(e_\delta)=\delta(p)\,e_\delta,\qquad \gamma(e_\delta)=\delta(\chi(\gamma))\,e_\delta,\quad \gamma\in\Gamma.$$

For any $\mathbf{D}$ we let $\mathbf{D}(\chi)$ denote the φ-module $\mathbf{D}$ endowed with the action of Γ twisted by the cyclotomic character χ.
Fix a topological generator $\gamma\in\Gamma$. For any (φ,Γ)-module $\mathbf{D}$ we denote by $C_{\varphi,\gamma}(\mathbf{D})$ the complex

$$0\to\mathbf{D}\xrightarrow{f}\mathbf{D}\oplus\mathbf{D}\xrightarrow{g}\mathbf{D}\to 0$$

with $f(x)=((\varphi-1)x,(\gamma-1)x)$ and $g(y,z)=(\gamma-1)y-(\varphi-1)z$ ([H1], [Cz4]). We shall write $H^*(\mathbf{D})$ for the cohomology of $C_{\varphi,\gamma}(\mathbf{D})$. The main properties of these groups are the following

1) Long cohomology sequence. A short exact sequence of (φ,Γ)-modules

$$0\to \mathbf{D}'\to \mathbf{D}\to \mathbf{D}''\to 0$$

gives rise to an exact sequence

$$0\to H^0(\mathbf{D}')\to H^0(\mathbf{D})\to H^0(\mathbf{D})\xrightarrow{\Delta^0} H^1(\mathbf{D}')\to\cdots\to H^2(\mathbf{D}'')\to 0.$$

2) Euler-Poincaré characteristic. $H^i(\mathbf{D})$ are finite dimensional L-vector spaces and

$$\chi(\mathbf{D})=\sum_{i=0}^{2}(-1)^i\dim_L H^i(\mathbf{D})\ =\ -\mathrm{rg}(\mathbf{D}).$$

(see [H1] and [Li]).

3) Computation of the Brauer group. The map

$$\mathrm{cl}(x)\mapsto -\left(1-\frac{1}{p}\right)^{-1}(\log\chi(\gamma))^{-1}\mathrm{res}(xdt)$$

is well defined and induces an isomorphism $\mathrm{inv}\,:\,H^2(\mathcal{R}_L(\chi))\xrightarrow{\sim} L$ (see [H2] [Ben1] and [Li]).

4) The cup-products. Let $\mathbf{D}$ and $\mathbf{M}$ be two (φ,Γ)-modules. For all i and j such that $i+j\leqslant 2$ define a bilinear map

$$\cup\,:\,H^i(\mathbf{D})\times H^j(\mathbf{M})\to H^{i+j}(\mathbf{D}\otimes\mathbf{M})$$

by

$$\begin{aligned}
&\mathrm{cl}(x)\cup\mathrm{cl}(y)=\mathrm{cl}(x\otimes y)\qquad \text{if } i=j=0,\\
&\mathrm{cl}(x)\cup\mathrm{cl}(y_1,y_2)=\mathrm{cl}(x\otimes y_1,x\otimes y_2)\qquad \text{if } i=0,\ j=1,\\
&\mathrm{cl}(x_1,x_2)\cup\mathrm{cl}(y_1,y_2)=\mathrm{cl}(x_2\otimes\gamma(y_1)-x_1\otimes\varphi(y_2))\qquad \text{if } i=1,\ j=1,\\
&\mathrm{cl}(x)\cup\mathrm{cl}(y)=\mathrm{cl}(x\otimes y)\qquad \text{if } i=0,\ j=2.
\end{aligned}$$

These maps commute with connecting homomorphisms in the usual sense.

5) Duality. Let $\mathbf{D}^*=\mathrm{Hom}_{\mathcal{R}_L}(\mathbf{D},\mathcal{R}_L)$. For $i=0,1,2$ the cup product

$$H^i(\mathbf{D})\times H^{2-i}(\mathbf{D}^*(\chi))\xrightarrow{\cup} H^2(\mathcal{R}_L(\chi))\simeq L \tag{1.1}$$

is a perfect pairing ([H2], [Li]).

1.1.2. Recall that a filtered (φ,N)-module with coefficients in L is a finite dimensional L-vector space M equipped with an exhaustive decreasing filtration Fil^iM, a linear bijective map $\varphi\,:\,M\to M$ and a nilpotent operator $N\,:\,M\to M$ such that $\varphi N=p\,\varphi N$. Filtered (φ,N)-modules form a $\otimes$-category which we denote by $\mathbf{MF}^{\varphi,N}$. A filtered (φ,N)-module M is said to be a Dieudonné module if $N = 0$ on M. Filtered Dieudonné modules form a full subcategory $\mathbf{MF}^{\varphi}$ of $\mathbf{MF}^{\varphi,N}$. It is not difficult to see that the series $\log(\varphi(\pi)/\pi^p)$ and $\log(\gamma(\pi)/\pi)$ ($\gamma \in \Gamma$) converge in $\mathcal{R}_L$. Let $\log \pi$ be a transcendental element over the field of fractions of $\mathcal{R}_L$ equipped with actions of φ and Γ given by

$$\varphi(\log \pi) = p \log \pi + \log\left(\frac{\varphi(\pi)}{\pi^p}\right), \qquad \gamma(\log \pi) = \log \pi + \log\left(\frac{\gamma(\pi)}{\pi}\right).$$

Thus the ring $\mathcal{R}_{L,\log} = \mathcal{R}_L[\log \pi]$ is equipped with natural actions of φ and Γ and the monodromy operator $N = -\left(1-\frac{1}{p}\right)^{-1} \frac{d}{d\log \pi}$. For any (φ,Γ)-module $\mathbf{D}$ set

$$\mathcal{D}_{\mathrm{st}}(\mathbf{D}) = (\mathbf{D} \otimes_{\mathcal{R}_L} \mathcal{R}_{L,\log}[1/t])^{\Gamma}$$

with $t = \log(1+\pi)$. Then $\mathcal{D}_{\mathrm{st}}(\mathbf{D})$ is a finite dimensional L-vector space equipped with natural actions of φ and N such that $N\varphi = p\,\varphi N$. Moreover, it is equipped with a canonical exhaustive decreasing filtration $\mathrm{Fil}^i\mathcal{D}_{\mathrm{st}}(\mathbf{D})$ which is induced by the embeddings $\iota_n : \mathbf{B}^{\dagger,r}_{\mathrm{rig},L} \hookrightarrow L_{\infty}[[t]]$, $n \gg 0$ constructed in [Ber1] (see [Ber2] for more details). Set

$$\mathcal{D}_{\mathrm{cris}}(\mathbf{D}) = \mathcal{D}_{\mathrm{st}}(\mathbf{D})^{N=0} = (\mathbf{D}[1/t])^{\Gamma}.$$

Then

$$\dim_L \mathcal{D}_{\mathrm{cris}}(\mathbf{D}) \leqslant \dim_L \mathcal{D}_{\mathrm{st}}(\mathbf{D}) \leqslant \mathrm{rg}(\mathbf{D})$$

and one says that $\mathbf{D}$ is semistable (resp. crystalline) if $\dim_L \mathcal{D}_{\mathrm{cris}}(\mathbf{D}) = \mathrm{rg}(\mathbf{D})$ (resp. if $\dim_L \mathcal{D}_{\mathrm{st}}(\mathbf{D}) = \mathrm{rg}(\mathbf{D})$). If $\mathbf{D}$ is semistable, the jumps of the filtration $\mathrm{Fil}^i\mathbf{D}_{\mathrm{st}}(\mathbf{D})$ are called the Hodge-Tate weights of $\mathbf{D}$ and the tangent space of $\mathbf{D}$ is defined as $t_{\mathbf{D}}(L) = \mathcal{D}_{\mathrm{st}}(\mathbf{D})/\mathrm{Fil}^0\mathcal{D}_{\mathrm{st}}(\mathbf{D})$.
We let denote by $\mathbf{M}^{\varphi,\Gamma}_{\mathrm{pst}}$ and $\mathbf{M}^{\varphi,\Gamma}_{\mathrm{cris}}$ the categories of semistable and crystalline representations respectively. In [Ber2] Berger proved that the functors

$$(1.2) \qquad \mathcal{D}_{\mathrm{st}} : \mathbf{M}^{\varphi,\Gamma}_{\mathrm{pst}} \to \mathbf{MF}^{\varphi,N}, \qquad \mathcal{D}_{\mathrm{cris}} : \mathbf{M}^{\varphi,\Gamma}_{\mathrm{cris}} \to \mathbf{MF}^{\varphi}$$

are equivalences of $\otimes$-categories.

1.1.3. As usually, $H^1(\mathbf{D})$ can be interpreted in terms of extensions. Namely, to any cocycle $\alpha = (a,b) \in Z^1(C_{\varphi,\gamma}(\mathbf{D}))$ one associates the extension

$$0 \to \mathbf{D} \to \mathbf{D}_{\alpha} \to \mathcal{R}_L \to 0$$

such that $\mathbf{D}_{\alpha} = \mathbf{D} \oplus \mathcal{R}_L e$ with $\varphi(e) = e + a$ and $\gamma(e) = e + b$. This defines a canonical isomorphism

$$H^1(\mathbf{D}) \simeq \mathrm{Ext}^1(\mathcal{R}_L, \mathbf{D}).$$

We say that $\mathrm{cl}(\alpha)\in H^1(\mathbf{D})$ is crystalline if $\dim_L \mathcal{D}_{\mathrm{cris}}(\mathbf{D}_\alpha)=\dim_L \mathcal{D}_{\mathrm{cris}}(\mathbf{D})+1$ and define

$$H^1_f(\mathbf{D})=\{\mathrm{cl}(\alpha)\in H^1(\mathbf{D})\mid \mathrm{cl}(\alpha) \text{ is crystalline}\}.$$

It is easy to see that $H^1_f(\mathbf{D})$ is a subspace of $H^1(\mathbf{D})$. If $\mathbf{D}$ is semistable (even potentially semistable), one has

$$\begin{aligned} &H^0(\mathbf{D})=\mathrm{Fil}^0\mathcal{D}_{\mathrm{st}}(\mathbf{D})^{\varphi=1,N=0},\\ &\dim_L H^1_f(\mathbf{D})=\dim_L t_{\mathbf{D}}(L)+\dim_L H^0(\mathbf{D}) \end{aligned}\tag{1.3}$$

(see [Ben2], Proposition 1.4.4 and Corollary 1.4.5). Moreover, $H^1_f(\mathbf{D})$ and $H^1_f(\mathbf{D}^*(\chi))$ are orthogonal complements to each other under duality (1.1) ([Ben2], Corollary 1.4.10).

1.1.4. Let $\mathbf{D}$ be semistable (φ,Γ)-module of rank d. Assume that $\mathcal{D}_{\mathrm{st}}(\mathbf{D})^{\varphi=1}=\mathcal{D}_{\mathrm{st}}(\mathbf{D})$ and that the all Hodge-Tate weights of $\mathbf{D}$ are $\geqslant 0$. Since $N\varphi=p\varphi N$ this implies that $N=0$ on $\mathcal{D}_{\mathrm{st}}(\mathbf{D})$ and $\mathbf{D}$ is crystalline. The results of this section are proved in [Ben2] (see Proposition 1.5.9 and section 1.5.10). The canonical map $\mathbf{D}^\Gamma\to\mathcal{D}_{\mathrm{cris}}(\mathbf{D})$ is an isomorphism and therefore $H^0(\mathbf{D})\simeq\mathcal{D}_{\mathrm{cris}}(\mathbf{D})=\mathbf{D}^\Gamma$ has dimension d over L. The Euler-Poincaré characteristic formula gives

$$\dim_L H^1(\mathbf{D})=d+\dim_L H^0(\mathbf{D})+\dim_L H^0(\mathbf{D}^*(\chi))=2d.$$

On the other hand $\dim_L H^1_f(\mathbf{D})=d$ by (1.3). The group $H^1(\mathbf{D})$ has the following explicit description. The map

$$\begin{aligned} &i_{\mathbf{D}}\,:\,\mathcal{D}_{\mathrm{cris}}(\mathbf{D})\oplus\mathcal{D}_{\mathrm{cris}}(\mathbf{D})\to H^1(\mathbf{D}),\\ &i_{\mathbf{D}}(x,y)=\mathrm{cl}(-x,\,\log\chi(\gamma)\,y) \end{aligned}$$

is an isomorphism. (Remark that the sign -1 and $\log\chi(\gamma)$ are normalizing factors.) We let denote $i_{\mathbf{D},f}$ and $i_{\mathbf{D},c}$ the restrictions of $i_{\mathbf{D}}$ on the first and second summand respectively. Then $\mathrm{Im}(i_{\mathbf{D},f})=H^1_f(\mathbf{D})$ and we set $H^1_c(\mathbf{D})=\mathrm{Im}(i_{\mathbf{D},c})$. Thus we have a canonical decomposition

$$H^1(\mathbf{D})\simeq H^1_f(\mathbf{D})\oplus H^1_c(\mathbf{D})$$

([Ben2], Proposition 1.5.9).

Now consider the dual module $\mathbf{D}^*(\chi)$. It is crystalline, $\mathcal{D}_{\mathrm{cris}}(\mathbf{D}^*(\chi))^{\varphi=p^{-1}}=\mathcal{D}_{\mathrm{cris}}(\mathbf{D}^*(\chi))$ and the all Hodge-Tate weights of $\mathbf{D}^*(\chi)$ are $\leqslant 0$. Let

$$[\ ,\ ]_{\mathbf{D}}\,:\,\mathcal{D}_{\mathrm{cris}}(\mathbf{D}^*(\chi))\times\mathcal{D}_{\mathrm{cris}}(\mathbf{D})\to L$$

denote the canonical pairing. Define

$$i_{\mathbf{D}^*(\chi)}\,:\,\mathcal{D}_{\mathrm{cris}}(\mathbf{D}^*(\chi))\oplus\mathcal{D}_{\mathrm{cris}}(\mathbf{D}^*(\chi))\to H^1(\mathbf{D}^*(\chi))$$

by

$$i_{\mathbf{D}^*(\chi)}(\alpha,\beta)\cup i_{\mathbf{D}}(x,y)=[\beta,x]_{\mathbf{D}}-[\alpha,y]_{\mathbf{D}}.$$

As before, let $i_{\mathbf{D}*(\chi),\,f}$ and $i_{\mathbf{D}^*(\chi),\,c}$ denote the restrictions of $i_{\mathbf{D}}$ on the first and second summand respectively. From $H^1_f(\mathbf{D}^*(\chi))=H^1_f(\mathbf{D})^\perp$ it follows that $\mathrm{Im}(i_{\mathbf{D}^*(\chi),\,f})=H^1_f(\mathbf{D}^*(\chi))$ and we set $H^1_c(\mathbf{D}^*(\chi))=\mathrm{Im}(i_{\mathbf{D}^*(\chi),\,c})$.

Write ∂ for the differential operator $(1+\pi)\dfrac{d}{d\pi}$.

Proposition 1.1.5. *Let $\mathcal{R}_L(|x|x^m)$ be the (φ,Γ)-module $\mathcal{R}_L e_\delta$ associated to the character $\delta(x)=|x|x^m$ $(m\geqslant 1)$. Then*

i) $\mathcal{D}_{\mathrm{cris}}(\mathcal{R}_L(|x|x^m))$ is the one-dimensional L-vector space generated by $t^{-m}e_\delta$. Moreover $\mathcal{D}_{\mathrm{cris}}(\mathcal{R}_L(|x|x^m))=\mathcal{D}_{\mathrm{cris}}(\mathcal{R}_L(|x|x^m))^{\varphi=p^{-1}}$ and the unique Hodge-Tate weight of $\mathcal{R}_L(|x|x^m)$ is $-m$.

ii) $H^0(\mathcal{R}_L(|x|x^m))=0$ and $H^1(\mathcal{R}_L(|x|x^m))$ is the two-dimensional L-vector space generated by $\alpha_m^*=-\left(1-\dfrac{1}{p}\right)\mathrm{cl}(\alpha_m)$ and $\beta_m^*=\left(1-\dfrac{1}{p}\right)\log\chi(\gamma)\,\mathrm{cl}(\beta_m)$ where

$$\alpha_m=\frac{(-1)^{m-1}}{(m-1)!}\,\partial^{m-1}\left(\frac{1}{\pi}+\frac{1}{2},a\right)e_\delta$$

with $a\in\mathcal{R}_L^+=\mathcal{R}_L\cap L[[\pi]]$ such that $(1-\varphi)\,a=(1-\chi(\gamma)\gamma)\left(\dfrac{1}{\pi}+\dfrac{1}{2}\right)$ and

$$\beta_m=\frac{(-1)^{m-1}}{(m-1)!}\,\partial^{m-1}\left(b,\frac{1}{\pi}\right)e_\delta$$

with $b\in\mathcal{R}_L$ such that $(1-\varphi)\left(\dfrac{1}{\pi}\right)=(1-\chi(\gamma)\gamma)\,b$. Moreover $i_{m,f}(1)=\alpha_m^*$ and $i_{m,c}(1)=\beta_m^*$ where i_m denotes the map i defined in 1.1.4 for $\mathcal{R}_L(|x|x^m)$. In particular, $H^1_f(\mathcal{R}_L(|x|x^m))$ is generated by α_m^* and $H^1_c(\mathcal{R}_L(|x|x^m))$ is generated by β_m^*.

iii) Let $x=\mathrm{cl}(u,v)\in H^1(\mathcal{R}_L(|x|x^m))$. Then

$$x=a\,\mathrm{cl}(\alpha_m)+b\,\mathrm{cl}(\beta_m)$$

with $a=\mathrm{res}(ut^{m-1}dt)$ and $b=\mathrm{res}(vt^{m-1}dt)$.

iv) The map

$$\begin{aligned}&\mathrm{Res}_m\,:\,\mathcal{R}_L(|x|x^m)\to L,\\&\mathrm{Res}_m(\alpha)=-\left(1-\frac{1}{p}\right)^{-1}(\log\chi(\gamma))^{-1}\mathrm{res}\left(\alpha t^{m-1}dt\right)\end{aligned}$$

induces an isomorphism $\mathrm{inv}_m\,:\,H^2(\mathcal{R}_L(|x|x^m))\simeq L$. Moreover

$$\mathrm{inv}_m(\omega_m)=1\qquad\text{where }\ \omega_m=(-1)^m\left(1-\frac{1}{p}\right)\frac{\log\chi(\gamma)}{(m-1)!}\,\mathrm{cl}\left(\partial^{m-1}(1/\pi)\right)$$

Proof. The assertions i) and ii) are proved in [Cz4], sections 2.3-2.5 and [Ben2], Theorem 1.5.7 and (16). The assertions iii) and iv) are proved in [Ben2], Proposition 1.5.4 iii) Corollary 1.5.5.

1.1.6. In [Fo1], Fontaine worked out a general approach to the classification of p-adic representations in terms of (φ,Γ)-modules. Thanks to the work of Cherbonnier-Colmez [CC] and Kedlaya [Ke] this approach allows to construct an equivalence

$$\mathbf{D}^{\dagger}_{\mathrm{rig}} : \mathbf{Rep}_L(G_{\mathbb{Q}_p}) \to \mathbf{M}^{\varphi,\Gamma}_{\text{ét}}$$

between the category of L-adic representations of $G_{\mathbb{Q}_p}$ and the category $\mathbf{M}^{\varphi,\Gamma}_{\text{ét}}$ of étale (φ,Γ)-modules in the sense of [Ke]. If V is a L-adic representation of $G_{\mathbb{Q}_p}$, define

$$\mathbf{D}_{\mathrm{st}}(V)=\mathcal{D}_{\mathrm{st}}(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)), \qquad \mathbf{D}_{\mathrm{cris}}(V)=\mathcal{D}_{\mathrm{cris}}(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)).$$

Then $\mathbf{D}_{\mathrm{st}}$ and $\mathbf{D}_{\mathrm{cris}}$ are canonically isomorphic to classical Fontaine's functors [Fo2], [Fo3] defined using the rings $\mathbf{B}_{\mathrm{st}}$ and $\mathbf{B}_{\mathrm{cris}}$ ([Ber1], Theorem 0.2). The continuous Galois cohomology $H^*(\mathbb{Q}_p,V)=H^*_{\mathrm{cont}}(G_{\mathbb{Q}_p},V)$ is functorially isomorphic to $H^*(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))$ ([H1], [Li]). and under this isomorphism

$$H^1_f(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)) \simeq H^1_f(\mathbb{Q}_p,V)$$

where $H^1_f(\mathbb{Q}_p,V)=\ker(H^1(\mathbb{Q}_p,V)\to H^1(\mathbb{Q}_p,V\otimes \mathbf{B}_{\mathrm{cris}}))$ is H^1_f of Bloch and Kato [BK].

1.2. THE ℓ-INVARIANT.

1.2.1. The results of this section are proved in [Ben2], 2.1-2.2. Denote by $\overline{\mathbb{Q}}^{(S)}/\mathbb{Q}$ the maximal Galois extension of $\mathbb{Q}$ unramified outside $S\cup\{\infty\}$ and set $G_S=\mathrm{Gal}(\overline{\mathbb{Q}}^{(S)}/\mathbb{Q})$. If V is a L-adic representation of G_S we write $H^*(V)$ for the continuous cohomology of G_S with coefficients in V. If V is potentially semistable at p, set

$$H^1_f(\mathbb{Q}_l,V)=\begin{cases} \ker(H^1(\mathbb{Q}_l,V)\to H^1(\mathbb{Q}_l^{\mathrm{nr}},V) & \text{if } l\neq p,\\ H^1_f(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)) & \text{if } l=p.\end{cases}$$

The Selmer group of Bloch and Kato is defined by

$$H^1_f(V)=\ker\left(H^1(V)\to \bigoplus_{l\in S}\frac{H^1(\mathbb{Q}_l,V)}{H^1_f(\mathbb{Q}_l,V)}\right).$$

Assume that V satisfies the condition **C1-4)** of 0.2.
The Poitou-Tate exact sequence together with **C1)** gives an isomorphism

$$H^1(V)\simeq \bigoplus_{l\in S}\frac{H^1(\mathbb{Q}_l,V)}{H^1_f(\mathbb{Q}_l,V)}. \tag{1.4}$$

Recall that a (φ, N)-submodule D of $\mathbf{D}_{\mathrm{st}}(V)$ is said to be regular if the canonical projection $D \to t_V(L)$ is an isomorphism. To any regular D we associate a filtration on $\mathbf{D}_{\mathrm{st}}(V)$

$$\{0\} \subset D_{-1} \subset D_0 \subset D_1 \subset \mathbf{D}_{\mathrm{st}}(V)$$

setting

$$D_i = \begin{cases} (1-p^{-1}\varphi^{-1})\, D + N(D^{\varphi=1}) & \text{if } i=-1, \\ D & \text{if } i=0, \\ D + \mathbf{D}_{\mathrm{st}}(V)^{\varphi=1} \cap N^{-1}(D^{\varphi=p^{-1}}) & \text{if } i=1. \end{cases}$$

By (1.2) this filtration induces a filtration on $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$ by saturated (φ, Γ)-submodules

$$\{0\} \subset F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V) \subset F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V) \subset F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V) \subset \mathbf{D}^{\dagger}_{\mathrm{rig}}(V).$$

Set $W = F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)/F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$. In [Ben2], Proposition 2.1.7 we proved that

$$W \simeq W_0 \oplus W_1 \oplus M, \tag{1.5}$$

where W_0 and W_1 are direct summands of $\mathrm{gr}_0\left(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)\right)$ and $\mathrm{gr}_1\left(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)\right)$ of ranks $\dim_L H^0(W^*(\chi))$ and $\dim_L H^0(W)$ respectively. Moreover M seats in a non split exact sequence

$$0 \to M_0 \xrightarrow{f} M \xrightarrow{g} M_1 \to 0$$

with $\mathrm{rg}(M_0) = \mathrm{rg}(M_1)$, $\mathrm{gr}_0\left(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)\right) = M_0 \oplus W_0$ and $\mathrm{gr}_1\left(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)\right) = M_1 \oplus W_1$. Set

$$e = \mathrm{rg}(W_0) + \mathrm{rg}(W_1) + \mathrm{rg}(M_0).$$

Generalizing [G] we expect that the p-adic L-function $L_p(V, D, s)$ has a zero of order e at $s = 0$.

If $W_0 = 0$, the main construction of [Ben2] associates to V and D an element $\mathcal{L}(V, D) \in L$ which can be viewed as a generalization of Greenberg's $\mathcal{L}$-invariant to semistable representations. Now assume that $W_1 = 0$. Let $D^* = \mathrm{Hom}(\mathbf{D}_{\mathrm{st}}(V)/D, \mathbf{D}_{\mathrm{st}}(\mathbb{Q}_p(1)))$ be the dual regular space. As the decompositions (1.5) for the pairs (V, D) and $(V^*(1), D^*)$ are dual to each other, one can define

$$\ell(V, D) = \mathcal{L}(V^*(1), D^*).$$

In this paper we do not review the construction of the $\mathcal{L}$-invariant but give a direct description of $\ell(V, D)$ in terms of V and D in two important particular cases.

1.2.2. THE CRYSTALLINE CASE: $W = W_0$ (see [Ben2], 2.2.6-2.2.7 and 2.3.3). In this case W is crystalline, $W_1 = M = 0$ and $F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V) = F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V)$. From the decomposition (1.5) it is not difficult to obtain the following description of $H^1_f(\mathbb{Q}_p, V)$ in the spirit of Greenberg's local conditions:

$$H^1_f(\mathbb{Q}_p, V) \;=\; \ker \left(H^1(F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V)) \to \frac{H^1(W)}{H^1_f(W)} \right). \tag{1.6}$$

Let $H^1(D,V)$ denote the inverse image of $H^1(F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V))/H^1_f(\mathbb{Q}_p,V)$ under the isomorphism (1.4). Thus one has a commutative diagram

$$\begin{tikzcd} H^1(D,V) \arrow[r] \arrow[dr] & H^1(F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V)) \arrow[d] \\ & H^1(\mathbf{D}^\dagger_{\mathrm{rig}}(V)) \end{tikzcd} \tag{1.7}$$

where the vertical map is injective ([Ben2], section 2.2.1). From (1.6) it follows that the composition map

$$\kappa_D \;:\; H^1(D,V) \to H^1(F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V)) \to H^1(W)$$

is injective. By construction, $\mathcal{D}_{\mathrm{cris}}(W) = D/D_{-1} = D^{\varphi = p^{-1}}$. As D is regular, the Hodge-Tate weights of W are $\leqslant 0$. Thus one has a decomposition

$$i_W \;:\; \mathcal{D}_{\mathrm{cris}}\,(W) \oplus \mathcal{D}_{\mathrm{cris}}\,(W) \simeq H^1_f\,(W) \oplus H^1_c\,(W) \simeq H^1\,(W)\,.$$

Denote by $p_{D,f}$ and $p_{D,c}$ the projection of $H^1\,(W)$ on the first and the second direct summand respectively. We have a diagram

$$\begin{tikzcd} & \mathcal{D}_{\mathrm{cris}}(W) \\ H^1(D,V) \arrow[ur, "\rho_{D,f}"] \arrow[r, "\kappa_D"] \arrow[dr, "\rho_{D,c}"'] & H^1(W) \arrow[u, "p_{D,f}"'] \arrow[d, "p_{D,c}"] \\ & \mathcal{D}_{\mathrm{cris}}(W) \end{tikzcd}$$

where $\rho_{D,c}$ is an isomorphism. Then

$$\ell(V,D) = \det\nolimits_L \left(\rho_{D,f} \circ \rho_{D,c}^{-1} \mid \mathcal{D}_{\mathrm{cris}}(W) \right).$$

1.2.3. The semistable case: $W=M$ (see [Ben2], 2.2.3-2.2.4 and 2.3.3). In this case W is semistable , $W_0=W_1=0$ and

$$H^1_f(\mathbb{Q}_p,V)=\ker\left(H^1(F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V))\to H^1(M_1)\right). \tag{1.8}$$

Let $H^1(D,V)$ be the inverse image of $H^1(F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V))/H^1_f(\mathbb{Q}_p,V)$ under the isomorphism (1.4). Consider the exact sequence

$$\begin{tikzcd}
H^1(M_0) \arrow[r, "h_1(f)"] & H^1(M) \arrow[r, "h_1(g)"] & H^1(M_1) \arrow[r, "\Delta^1"] & H^2(M_0) \arrow[r] & 0\,. \\
& H^1(D,V) \arrow[u, "\kappa_D"] \arrow[ur, "\bar\kappa_D"'] & & &
\end{tikzcd}$$

By (1.8), the map $\bar\kappa_D$ is injective and it is not difficult to prove that the image of $H^1(D,V)$ in $H^1(M_1)$ coincides with $\mathrm{Im}(h_1(g))$ ([Ben2], section 2.2.3). Thus in the semistable case the position of $H^1(D,V)$ in $H^1(M_1)$ is completely determined by the the restriction of V on the decomposition group at p. By construction, $\mathcal{D}_{\mathrm{st}}(M_1)=D_1/D$ where $(D_1/D)^{\varphi=1}=D_1/D$ and the Hodge-Tate weights of M_1 are $\geqslant 0$. Again, one has an isomorphism

$$i_{M_1}\,:\,\mathcal{D}_{\mathrm{cris}}\left(M_1\right)\oplus\mathcal{D}_{\mathrm{cris}}\left(M_1\right)\simeq H^1_f\left(M_1\right)\oplus H^1_c\left(M_1\right)\simeq H^1\left(M_1\right)$$

which allows to construct a diagram

$$\begin{tikzcd}
& \mathcal{D}_{\mathrm{st}}(M_1) \\
\mathrm{Im}(h_1(g)) \arrow[ur, "\rho_{D,f}"] \arrow[r, "\kappa_D"] \arrow[dr, "\rho_{D,c}"'] & H^1(M_1) \arrow[u, "p_{D,f}"'] \arrow[d, "p_{D,c}"] \\
& \mathcal{D}_{\mathrm{st}}(M_1).
\end{tikzcd}$$

Then

$$\ell(V,D)=\mathcal{L}(V,D)=\det\nolimits_L\left(\rho_{D,f}\circ\rho^{-1}_{D,c}\mid \mathcal{D}_{\mathrm{st}}(M_1)\right). \tag{1.9}$$

From (1.5) it is clear that if $e=1$ then either $W=W_0$ with $\mathrm{rg}(W_0)=1$ or $W=M$ with $\mathrm{rg}(M_0)=\mathrm{rg}(M_1)=1$. We consider these cases separately in the rest of the paper.

§2. The crystalline case

2.1. Let $A = L[T]/(T^2)$ and let V_A be a free finitely generated A-module equipped with a A-linear action of G_S. One says that V_A is an infinitesimal deformation of a p-adic representation V if $V \simeq V_A \otimes_A L$. Write $\mathcal{R}_A = A \otimes_L \mathcal{R}_L$ and extend the actions of φ and Γ to $\mathcal{R}_A$ by linearity. A (φ,Γ)-module over $\mathcal{R}_A$ is a free finitely generated $\mathcal{R}_A$-module $\mathbf{D}_A$ equipped with commuting semilinear actions of φ and Γ and such that $\mathcal{R}_A\varphi(\mathbf{D}_A) = \mathbf{D}_A$. We say that $\mathbf{D}_A$ is an infinitesimal deformation of a (φ,Γ)-module $\mathbf{D}$ over $\mathcal{R}_L$ if $\mathbf{D} = \mathbf{D}_A \otimes_A L$.

2.2. Let V be a p-adic representation of G_S which satisfies the conditions **C1-4)** and such that $W = W_0$. Moreover we assume that $\mathrm{rg}(W) = 1$. Thus W is a crystalline (φ,Γ)-module of rank 1 with $\mathcal{D}_{\mathrm{cris}}(W) = \mathcal{D}_{\mathrm{cris}}(W)^{\varphi=p^{-1}}$ and such that $\mathrm{Fil}^0\mathcal{D}_{\mathrm{cris}}(W) = 0$. This implies that

$$W \simeq \mathcal{R}_L(\delta) \quad \text{with} \quad \delta(x) = |x|x^m, \quad m \geqslant 1. \tag{2.1}$$

(see for example [Ben2], Proposition 1.5.8). Note that the Hodge-Tate weight of W is $-m$. The L-vector space $H^1(D,V)$ is one dimensional. Fix a basis $\mathrm{cl}(x) \in H^1(D,V)$. We can associate to $\mathrm{cl}(x)$ a non trivial extension

$$0 \to V \to V_x \to L \to 0.$$

This gives an exact sequence of (φ,Γ)-modules

$$0 \to \mathbf{D}^\dagger_{\mathrm{rig}}(V) \to \mathbf{D}^\dagger_{\mathrm{rig}}(V_x) \to \mathcal{R}_L \to 0.$$

From (1.7) it follows that there exists an extension in the category of (φ,Γ)-modules

$$0 \to F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V) \to \mathbf{D}_x \to \mathcal{R}_L \to 0$$

which is inserted in a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V) \arrow[r] \arrow[d] & \mathbf{D}_x \arrow[r] \arrow[d] & \mathcal{R}_L \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \mathbf{D}^\dagger_{\mathrm{rig}}(V) \arrow[r] & \mathbf{D}^\dagger_{\mathrm{rig}}(V_x) \arrow[r] & \mathcal{R}_L \arrow[r] & 0.
\end{tikzcd}$$

Define a filtration

$$\{0\} \subset F_{-1}\mathbf{D}^\dagger_{\mathrm{rig}}(V_x) \subset F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V_x) \subset F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V_x) \subset \mathbf{D}^\dagger_{\mathrm{rig}}(V_x)$$

by $F_i\mathbf{D}^\dagger_{\mathrm{rig}}(V_x) = F_i\mathbf{D}^\dagger_{\mathrm{rig}}(V)$ for $i = -1, 0$ and $F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V_x) = \mathbf{D}_x$. Set

$$W_x = F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V_x)/F_{-1}\mathbf{D}^\dagger_{\mathrm{rig}}(V_x).$$

Thus one has a diagram

$$\begin{tikzcd} 0 \arrow[r] & F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V) \arrow[r] \arrow[d] & \mathbf{D}_x \arrow[r] \arrow[d] & \mathcal{R}_L \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & W \arrow[r] & W_x \arrow[r] & \mathcal{R}_L \arrow[r] & 0. \end{tikzcd}$$

2.3. Let $V_{A,x}$ be an infinitesimal deformation of V_x. Assume that $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{A,x})$ is equipped with a filtration by saturated (φ,Γ)-modules over $\mathcal{R}_A$:

$$\{0\} \subset F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{A,x}) \subset F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{A,x}) \subset F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{A,x}) \subset \mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{A,x})$$

such that $F_i\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{A,x})\otimes_A L \simeq F_i\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_x)$ for all i. The quotients $\mathrm{gr}_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{A,x})$ and $\mathrm{gr}_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{A,x})$ are (φ,Γ)-modules of rank 1 over $\mathcal{R}_A$ and by [BC], Proposition 2.3.1 there exists unique characters $\delta_{A,x},\psi_{A,x}\ :\ \mathbb{Q}_p^* \to A^*$ such that $\mathrm{gr}_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{A,x}) \simeq \mathcal{R}_A(\delta_{A,x})$ and $\mathrm{gr}_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{A,x}) \simeq \mathcal{R}_A(\psi_{A,x})$. It is clear that $\delta_{A,x} \pmod T = \delta$ and $\psi_{A,x} \pmod T = 1$. One has a diagram

$$\begin{tikzcd} 0 \arrow[r] & F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_A) \arrow[r] \arrow[d] & F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{A,x}) \arrow[r] \arrow[d] & \mathcal{R}_A(\psi_A) \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & W_A \arrow[r] & W_{A,x} \arrow[r] & \mathcal{R}_A(\psi_A) \arrow[r] & 0 \end{tikzcd}$$

with $W_A = \mathrm{gr}_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{A,x})$ and $W_{A,x} = F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{A,x})/F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{A,x})$. Assume that

$$\left.\frac{d(\delta_{A,x}\psi_{A,x}^{-1})(u)}{dT}\right|_{T=0} \neq 0, \qquad u \equiv 1 \pmod{p^2}$$

(as the multiplicative group $1+p^2\mathbb{Z}_p$ is procyclic it is enough to assume that this holds for $u = 1+p^2$.)

Theorem 1. *Let $V_{A,x}$ be an infinitesimal deformation of V_x which satisfies the above conditions. Then*

$$\ell(V,D) = -\log\chi(\gamma)\left.\frac{d\log(\delta_{A,x}\psi_{A,x}^{-1})(p)}{d\log(\delta_{A,x}\psi_{A,x}^{-1})(\chi(\gamma))}\right|_{T=0}.$$

This theorem will be proved in section 2.5. We start with an auxiliary result which plays a key role in the proof. Set $\delta(x) = |x|x^m$ $(m \geqslant 1)$ and fix a character $\delta_A\ :\ \mathbb{Q}_p^* \to A^*$ such that $\delta_A \pmod T = \delta$. Consider the exact sequence

$$0 \to \mathcal{R}_L(\delta) \to \mathcal{R}_A(\delta_A) \to \mathcal{R}_L(\delta) \to 0$$

and denote by B^i_δ the connecting maps $H^i(\mathcal{R}_L(\delta)) \to H^{i+1}(\mathcal{R}_L(\delta))$.

PROPOSITION 2.4. *One has*

$$\operatorname{inv}_m\left(\mathrm{B}^1_\delta(\alpha^*_m)\right) = (\log\chi(\gamma))^{-1} d\log\delta_A(\chi(\gamma))\big|_{T=0},$$
$$\operatorname{inv}_m\left(\mathrm{B}^1_\delta(\beta^*_m)\right) = d\log\delta_A(p)\big|_{T=0}.$$

Proof. a) Recall that

$$\alpha^*_m = -\left(1-\frac{1}{p}\right)\frac{(-1)^{m-1}}{(m-1)!}\,\mathrm{cl}\left(\partial^{m-1}\left(\frac{1}{\pi}+\frac{1}{2},a\right)e_\delta\right).$$

Let $e_{A,\delta}$ be a generator of $\mathcal{R}_A(\delta_A)$ such that $e_\delta = e_{A,\delta} \pmod{T}$. Directly from the definition of the connecting map

$$\begin{aligned}\mathrm{B}^1_\delta(\alpha^*_m) \;=\; -\left(1-\frac{1}{p}\right)\frac{(-1)^{m-1}}{(m-1)!}\,\mathrm{cl}\left(\frac{1}{T}\left((\gamma-1)\left(\partial^{m-1}\left(\frac{1}{\pi}+\frac{1}{2}\right)e_{A,\delta}\right)-\right.\right.\\ \left.\left.-\;(\varphi-1)\left(\partial^{m-1}(a)e_{A,\delta}\right)\right)\right).\end{aligned}$$

Write

$$\begin{aligned}(\gamma-1)\left(\partial^{m-1}\left(\frac{1}{\pi}+\frac{1}{2}\right)e_{A,\delta}\right)-(\varphi-1)\left(\partial^{m-1}(a)e_{A,\delta}\right)=\\ =\left(\chi(\gamma)^{-m}\delta_A(\chi(\gamma))-1\right)\partial^{m-1}\left(\frac{1}{\pi}+\frac{1}{2}\right)e_{A,\delta}+z\end{aligned}$$

where

$$z=\left(\gamma-\chi(\gamma)^{-m}\right)\partial^{m-1}\left(\frac{1}{\pi}+\frac{1}{2}\right)\delta_A(\chi(\gamma))e_{A,\delta}\;-\;\left(\delta_A(p)\varphi-1\right)\partial^{m-1}(a)e_{A,\delta}.$$

Since $\delta_A(\chi(\gamma))\equiv\chi(\gamma)^m \pmod{T}$, from the definition of a it follows that $z\equiv 0 \pmod{T}$. On the other hand, as $a\in\mathcal{R}_L^+$ and

$$\left(\gamma-\chi(\gamma)^{-m}\right)\partial^{m-1}\left(\frac{1}{\pi}+\frac{1}{2}\right)\in\mathcal{R}_L^+$$

we obtain that $z/T\in\mathcal{R}_L^+e_\delta$. Thus the class of z/T in $H^2(\mathcal{R}_L(\delta))$ is zero. On the other hand, writing δ_A in the form

$$\delta_A(u)=u^m+T\frac{d\delta_A(u)}{dT}\bigg|_{T=0}$$

one finds that

$$\frac{\chi(\gamma)^{-m}\delta_A(\chi(\gamma))-1}{T}\;=\;d\log\delta_A(\chi(\gamma))\big|_{T=0}$$

and the first formula follows from Proposition 1.1.5 iv).
b) By the definition of B_δ^1

$$\mathrm{B}_\delta^1(\beta_m^*) = \left(1-\frac{1}{p}\right)\frac{(-1)^{m-1}\log\chi(\gamma)}{(m-1)!}\,\mathrm{cl}\left(\frac{1}{T}\left((\gamma-1)\left(\partial^{m-1}(b)e_{A,\delta}\right)-\right.\right.$$
$$\left.\left.-\ (\varphi-1)\ \left(\partial^{m-1}\big(1/\pi\big)e_{A,\delta}\right)\right)\right).$$

As

$$\delta_A(p)\ \left(\varphi-\delta(p)^{-1}\right)\ \left(\partial^{m-1}\big(1/\pi\big)\right)\ =\ \frac{\delta_A(p)}{\delta(p)}\ \left(\delta(\chi(\gamma))\,\gamma-1\right)\,\partial^{m-1}(b)$$

we can write

$$(\gamma-1)\left(\partial^{m-1}(b)e_{A,\delta}\right)-(\varphi-1)\ \left(\partial^{m-1}\big(1/\pi\big)e_{A,\delta}\right)=$$
$$=-\left(\delta(p)^{-1}\delta_A(p)-1\right)\partial^{m-1}\big(1/\pi\big)+w$$

where

$$w=\left(\delta_A(\chi(\gamma))\,\gamma-1\right)\ (\partial^{m-1}b)\,e_{A,\delta}\ +\ \frac{\delta_A(p)}{\delta(p)}\ \left(\delta(\chi(\gamma))\,\gamma-1\right)\ (\partial^{m-1}b)e_{A,\delta}.$$

Remark that

$$\frac{\delta(p)^{-1}\delta_A(p)-1}{T}\ =\ -d\log\delta_A(p)\big|_{T=0}$$

On the other hand

$$\mathrm{res}\left(\partial^{m-1}(b)\,t^{m-1}dt\right)=0$$

(see [Ben2], proof of Corollary 1.5.6). As $\mathrm{res}\left((\chi(\gamma)^m\gamma-1)\,\partial^{m-1}(b)\,t^{m-1}dt\right)=0$, this implies that $\mathrm{res}\left(\gamma(\partial^{m-1}b)\,t^{m-1}dt\right)=0$ and we obtain that $\mathrm{Res}_m(w)=0$. Thus

$$\mathrm{inv}_m(\mathrm{B}_\delta^1(\beta_m^*))\ =\ -d\mathrm{log}\,\delta_A(p)\big|_{T=0}\mathrm{Res}_m(\omega_m)=d\mathrm{log}\,\delta_A(p)\big|_{T=0}$$

and the Proposition is proved.

2.5. We pass to the proof of Theorem 1. By Proposition 1.1.5, $H^1(W)$ is a two dimensional L-vector space generated by α_m^* and β_m^*. One has a commutative diagram with exact rows

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & W \arrow[r] \arrow[d] & W_x \arrow[r] \arrow[d] & \mathcal{R}_L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & W_A \arrow[r] \arrow[d] & W_{A,x} \arrow[r] \arrow[d] & \mathcal{R}_A(\psi_{A,x}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & W \arrow[r] \arrow[d] & W_x \arrow[r] \arrow[d] & \mathcal{R}_L \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Twisting the middle row by $\psi_{A,x}^{-1}$ and taking into account that $\psi_{A,x} \equiv 1 \pmod{T}$ we obtain

(2.2)
$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & W \arrow[r] \arrow[d] & W_x \arrow[r] \arrow[d] & \mathcal{R}_L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & W_A(\psi_{A,x}^{-1}) \arrow[r] \arrow[d] & W_{A,x}(\psi_{A,x}^{-1}) \arrow[r] \arrow[d] & \mathcal{R}_A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & W \arrow[r] \arrow[d] & W_x \arrow[r] \arrow[d] & \mathcal{R}_L \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The connecting map $\Delta^0 : H^0(\mathcal{R}_L) \to H^1(W)$ sends 1 to $y = \kappa_D(\mathrm{cl}(x))$ and we can write

$$y = a\,\alpha_m^* + b\,\beta_m^*$$

with $a, b \in L$. Directly from the definition of the ℓ-invariant one has

$$\ell(V,D) = b^{-1}a. \tag{2.3}$$

The diagram (2.2) gives rise to a commutative diagram

$$\begin{tikzcd}
H^0(\mathcal{R}_L) \arrow[r, "\Delta^0"] \arrow[d, "\mathrm{B}^0"] & H^1(W) \arrow[d, "\mathrm{B}^1_W"] \\
H^1(\mathcal{R}_L) \arrow[r, "\Delta^1"] & H^2(W).
\end{tikzcd}$$

Since the rightmost vertical row of (2.2) splits, the connecting map B^0 is zero and

$$a\mathrm{B}^1_W(\alpha_m^*) + b\mathrm{B}^1_W(\beta_m^*) = \mathrm{B}^1_W(y) = 0.$$

As $W_A(\psi_{A,x}^{-1}) \simeq \mathcal{R}_A(\delta_{A,x}\psi_{A,x}^{-1})$, Proposition 2.4 gives

$$\mathrm{inv}_m(\mathrm{B}^1_W(\alpha_m^*)) = (\log(\chi(\gamma))^{-1} d\log(\delta_{A,x}\psi_{A,x}^{-1})(\chi(\gamma))\big|_{T=0},$$
$$\mathrm{inv}_m(\mathrm{B}^1_W(\beta_m^*)) = d\log(\delta_{A,x}\psi_{A,x}^{-1})(p)\big|_{T=0}.$$

Together with (2.3) this gives the Theorem.

§3. The semistable case

3.1. In this section we assume that V is a p-adic representation which satisfies the conditions **C1-4)** and such that $W = M$. Thus one has an exact sequence

$$0 \to M_0 \xrightarrow{f} W \xrightarrow{g} M_1 \to 0 \tag{3.1}$$

where M_0 and M_1 are such that $e = \mathrm{rg}(M_0) = \mathrm{rg}(M_1)$. We will assume that $e = 1$. Then

$$\begin{aligned} M_0 &= \mathcal{R}_L e_\delta \simeq \mathcal{R}_L(\delta), &\quad \delta(x) &= |x|x^m, \quad m \geqslant 1,\\ M_1 &= \mathcal{R}_L e_\psi \simeq \mathcal{R}_L(\psi), &\quad \psi(x) &= x^{-n}, \quad n \geqslant 0 \end{aligned}$$

(see for example [Ben2], Lemma 1.5.2 and Proposition 1.5.8). Thus

$$\{0\} \subset F_{-1}\mathbf{D}^\dagger_{\mathrm{rig}}(V) \subset F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V) \subset F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V) \subset \mathbf{D}^\dagger_{\mathrm{rig}}(V)$$

with $\mathrm{gr}_0\mathbf{D}^\dagger_{\mathrm{rig}}(V) \simeq \mathcal{R}_L(\delta)$ and $\mathrm{gr}_1\mathbf{D}^\dagger_{\mathrm{rig}}(V) \simeq \mathcal{R}_L(\psi)$. Assume that V_A is an infinitesimal deformation of V and that $\mathbf{D}^\dagger_{\mathrm{rig}}(V_A)$ is equipped with a filtration by saturated (φ,Γ)-modules over $\mathcal{R}_A$

$$\{0\} \subset F_{-1}\mathbf{D}^\dagger_{\mathrm{rig}}(V_A) \subset F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V_A) \subset F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V_A) \subset \mathbf{D}^\dagger_{\mathrm{rig}}(V_A)$$

such that

$$F_i\mathbf{D}^\dagger_{\mathrm{rig}}(V_A) \otimes_L A \simeq F_i\mathbf{D}^\dagger_{\mathrm{rig}}(V), \qquad -1 \leqslant i \leqslant 1.$$

Then

$$\mathrm{gr}_0\mathbf{D}^\dagger_{\mathrm{rig}}(V_A) \simeq \mathcal{R}_A(\delta_A), \qquad \mathrm{gr}_i\mathbf{D}^\dagger_{\mathrm{rig}}(V_A) \simeq \mathcal{R}_A(\psi_A),$$

where $\delta_A,\, \psi_A\, :\, \mathbb{Q}_p^* \to A^*$ are such that $\delta_A \pmod{T} = \delta$ and $\psi_A \pmod{T} = \psi$. As before, assume that

$$\left.\frac{d(\delta_A\psi_A^{-1})(u)}{dT}\right|_{T=0} \neq 0, \qquad u \equiv 1 \pmod{p^2}.$$

Theorem 2. *Let V_A be an infinitesimal deformation of V which satisfies the above conditions. Then*

$$\ell(V,D) \;=\; -\log\chi(\gamma)\,\left.\frac{d\log(\delta_A\psi_A^{-1})(p)}{d\log(\delta_A\psi_A^{-1})(\chi(\gamma))}\right|_{T=0} \tag{3.2}$$

3.2. Proof of Theorem 2. The classes $x_n^* = -\mathrm{cl}(t^n e_\psi, 0)$ and $y_n^* = \log\chi(\gamma)\,\mathrm{cl}(0, t^n e_\psi)$ form a basis of $H^1(M_1)$ and $H^1_f(M_1)$ is generated by x_n^* (see section 1.1.4). Consider the long cohomology sequence associated to (3.1):

$$\cdots \to H^1(M_0) \xrightarrow{h_1(f)} H^1(W) \xrightarrow{h_1(g)} H^1(M_1) \xrightarrow{\Delta^1} H^2(M_0) \to \cdots.$$

We can also consider the dual sequence $0 \to M_1^*(\chi) \to W^*(\chi) \to M_0^*(\chi) \to 0$ and write

$$\cdots \to H^0(M_0^*(\chi)) \xrightarrow{\Delta_*^0} H^1(M_1^*(\chi)) \to H^1(W^*(\chi)) \to H^1(M_0^*(\chi)) \to \cdots .$$

As $M_0^*(\chi) = \mathcal{R}_L e_{\delta^{-1}\chi}$ is isomorphic to $\mathcal{R}_L(x^{1-m})$, the cohomology $H^0(M_0^*(\chi))$ is the one dimensional L-vector space generated by $\xi = t^{m-1} e_{\delta^{-1}\chi}$. Write

$$\Delta_*^0(\xi) \;=\; a\,\alpha_{n+1}^* + b\,\beta_{n+1}^*,$$

where α_{n+1}^*, β_{n+1}^* is the canonical basis of $H^1(M_1^*(\chi)) \simeq \mathcal{R}_L(|x|x^{n+1})$. From the duality it follows that $\mathrm{Im}(\Delta_*^0)$ is orthogonal to $\ker(\Delta^1)$ under the pairing

$$H^1(\mathcal{R}_L(|x|x^{n+1})) \times H^1(\mathcal{R}_L(x^{-n})) \xrightarrow{\cup} L$$

Since

$$\alpha_{n+1}^* \cup x_n^* = \beta_{n+1}^* \cup y_n^* = 0, \quad \alpha_{n+1}^* \cup y_n^* = -1, \quad \beta_{n+1}^* \cup x_n^* = 1$$

(see Proposition 1.1.5 ii), we obtain that $\mathrm{Im}(h_1(g)) = \ker(\Delta^1)$ is generated by $ax_n^* + by_n^*$. By the definition of the $\mathcal{L}$-invariant

$$\mathcal{L}(V,D) = b^{-1}a. \tag{3.3}$$

Set $W_A = F_1\mathbf{D}_{\mathrm{rig}}^{\dagger}(V)/F_{-1}\mathbf{D}_{\mathrm{rig}}^{\dagger}(V)$. One has a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{R}_L(\psi^{-1}\chi) \arrow[r] \arrow[d] & W^*(\chi) \arrow[r] \arrow[d] & \mathcal{R}_L(\delta^{-1}\chi) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{R}_A(\psi_A^{-1}\chi) \arrow[r] \arrow[d] & W_A^*(\chi) \arrow[r] \arrow[d] & \mathcal{R}_A(\delta_A^{-1}\chi) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{R}_L(\psi^{-1}\chi) \arrow[r] \arrow[d] & W^*(\chi) \arrow[r] \arrow[d] & \mathcal{R}_L(\delta^{-1}\chi) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Now the theorem can be proved either by twisting this diagram by $\delta_A\chi^{-1}$ and applying the argument used in the proof of Theorem 2.3 or by the following direct computation. One has an anticommutative square

$$\begin{tikzcd}
H^0(\mathcal{R}_L(\delta^{-1}\chi)) \arrow[r, "\Delta_*^0"] \arrow[d, "\mathrm{B}^0_{\delta^{-1}\chi}"] & H^1(\mathcal{R}_L(\psi^{-1}\chi)) \arrow[d, "\mathrm{B}^1_{\psi^{-1}\chi}"] \\
H^1(\mathcal{R}_L(\delta^{-1}\chi)) \arrow[r, "\Delta_*^1"] & H^2(\mathcal{R}_L(\psi^{-1}\chi)).
\end{tikzcd}$$

Thus

$$\mathrm{B}^1_{\psi^{-1}\chi}\Delta^0_*(\xi) \;=\; -\Delta^1_*\mathrm{B}^0_{\delta^{-1}\chi}(\xi). \tag{3.4}$$

From Proposition 2.3 it follows that

(3.5)
$$\mathrm{inv}_{n+1}\big(\mathrm{B}^1_{\psi^{-1}\chi}\Delta^0_*(\xi)\big) \;=\; a\,\mathrm{inv}_{n+1}\big(\mathrm{B}^1_{\psi^{-1}\chi}(\alpha^*_{n+1})\big) \;+\; b\,\big(\mathrm{B}^1_{\psi^{-1}\chi}(\beta^*_{n+1})\big) \;=$$
$$= -a\log(\chi(\gamma))^{-1}d\log\psi_A(\chi(\gamma))\big|_{T=0} \;-\; b\,d\log\psi_A(p)\big|_{T=0}.$$

Fix a generator $e_{A,\delta^{-1}\chi}$ of $\mathcal{R}_A(\delta_A^{-1}\chi)$. We can assume that $e_{A,\delta^{-1}\chi}$ is a lifting of $e_{\delta^{-1}\chi}$ and set $\xi_A = t^{m-1}e_{A,\delta^{-1}\chi}$. Directly by the definition of the connecting map

$$\begin{aligned}
\mathrm{B}^0_{\delta^{-1}\chi}(\xi) \;&=\frac{1}{T}\,\mathrm{cl}((\varphi-1)\,\xi_A,(\gamma-1)\,\xi_A) \;=\\
&=\;\frac{1}{T}\mathrm{cl}((p^{m-1}\delta_A^{-1}(p)-1)\,\xi_A,(\chi(\gamma)^m\delta_A^{-1}(\chi(\gamma))-1)\,\xi_A) \;=\\
&=\;-\mathrm{cl}\big(d\log\delta_A(p)\,\xi, d\log\delta_A(\chi(\gamma))\,\xi\big)\big|_{T=0}.
\end{aligned}$$

Let $\hat{\xi}$ be a lifting of ξ in $W^*(\chi)$. Then

$$\Delta^1_*\mathrm{B}^0_{\delta^{-1}\chi}(\xi) \;=\; -\mathrm{cl}\big(d\log\delta_A(p)\,(\gamma-1)\,\hat{\xi} - d\log\delta_A(\chi(\gamma))\,(\varphi-1)\,\hat{\xi}\big)\big|_{T=0}.$$

On the other hand, $\Delta^0_*(\xi) = \mathrm{cl}((\varphi-1)\,\hat{\xi},(\gamma-1)\,\hat{\xi})$ and by Proposition 1.1.5 iii)

$$\begin{aligned}
\mathrm{res}\big((\varphi-1)\,(\hat{\xi})\,t^n dt\big) \;&=\; \left(1-\frac{1}{p}\right)\,a,\\
\mathrm{res}\big((\gamma-1)\,(\hat{\xi})\,t^n dt\big) \;&=\; \log(\chi(\gamma))\,\left(1-\frac{1}{p}\right)\,b.
\end{aligned}$$

Thus,

(3.6)
$$\mathrm{inv}_{n+1}\big(\Delta^1_*\mathrm{B}^0_{\delta^{-1}\chi}(\xi)\big) \;=$$
$$= b\,d\log\delta_A(p)\big|_{T=0} \;+\; a\log(\chi(\gamma))^{-1}d\log\delta_A(\chi(\gamma))\big|_{T=0}.$$

From (3.4), (3.5) and (3.6) we obtain that

$$a\,(\log\chi(\gamma))^{-1}d\mathrm{log}(\delta_A\psi_A^{-1})(\chi(\gamma))\big|_{T=0} = \;-b\,d\log(\delta_A\psi_A^{-1})\,(p)\big|_{T=0}.$$

Together with (3.3) this prove the theorem.

3.4. REMARK. It would be interesting to generalize Theorems 1 and 2 to the case $e>1$. For this one should first understand what kind of filtrations on

$\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$ appears naturally if V comes from automorphic forms [BC].

3.5. Modular forms. Let f be a normalized newform of weight $x_0 = 2k$ which is split multiplicative at p. Let $V = M_{f,\lambda}$ be the λ-adic representation associated to f by Deligne [D]. The structure of $\mathbf{D}_{\mathrm{st}}(V)$ is well known (see for example [Cz2]) Namely, $\mathbf{D}_{\mathrm{st}}(V) = Ld_1 + Ld_2$ with $N(d_2) = d_1$, $N(d_1) = 0$, $\varphi(d_2) = p^k d_2$ and $\varphi(d_1) = p^{k-1}d_1$. Thus $\mathbf{D}_{\mathrm{st}}(V(k)) = Ld_1^{(k)} + Ld_2^{(k)}$ with $\varphi(d_2^{(k)}) = d_2^{(k)}$, $\varphi(d_1^{(k)}) = p^{-1}d_1^{(k)}$ and $D = \mathbf{D}_{\mathrm{cris}}(V(k)) = Ld_1^{(k)}$ is the unique regular subspace of $\mathbf{D}_{\mathrm{st}}(V(k))$. It is clear that $D_{-1} = 0$, $D_1 = \mathbf{D}_{\mathrm{st}}(V(k))$ and for the associated filtration on $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V(k))$ we have $F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V(k)) = (D \otimes \mathcal{R}_L[1/t]) \cap \mathbf{D}^{\dagger}_{\mathrm{rig}}(V(k))$, $F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V(k)) = \mathbf{D}^{\dagger}_{\mathrm{rig}}(V(k))$. In [Ben2], Proposition 2.2.6 it is proved that $\mathcal{L}(V(k),D)$ coincides with the $\mathcal{L}$-invariant of Fontaine-Mazur $\mathcal{L}_{\mathrm{FM}}(f)$.

In [Co2], Coleman constructed an analytic family of overconvergent modular forms $f_x = \sum\limits_{n=1}^{\infty} a_n(x)q^n$ on an affinoid disk U containing $2k$ which satisfies the following conditions

• For any $x \in \mathbb{N} \cap U$ the form f_x is classical.

• $f_{x_0} = f$.

Moreover, one can interpolate the p-adic representations associated to classical forms f_x $(x \in \mathbb{N} \cap U)$ and construct a two dimensional representation $\mathcal{V}$ of $G_{\mathbb{Q}}$ over the Tate algebra $\mathcal{O}(U)$ of U such that

• For any integer $x \in \mathbb{N}$ in U the Galois representation $\mathcal{V}_x$ obtained by specialization of $\mathcal{V}$ at x is isomorphic to the λ-adic representation associated to f_x [CM]. In particular, it is semistable with the Hodge-Tate weights $(0, x-1)$ [Fa]. By continuity this implies that for all $x \in U$ the Hodge-Tate-Sen weights of $\mathcal{V}_x$ are $(0, x-1)$.

• $\wedge^2\mathcal{V}_x \simeq L_x\left(\chi^{1-2k}\langle\chi\rangle^{2k-x}\right)$ where as usually $\langle\chi\rangle$ denotes the projection of χ and L_x is the field of coefficients of $\mathcal{V}_x$.

• $\left(\mathbf{B}_{\mathrm{cris}}^{\varphi=a_p(x)}\hat{\otimes}\mathcal{V}\right)^{G_{\mathbb{Q}_p}}$ is locally free of rank 1 on U [Sa], [Ki].

Let $\mathcal{O}_{x_0}$ denote the local ring of U at x_0 and let $A = \mathcal{O}_{x_0}/(T^2)$ where $T = x - x_0$ is a local parameter at x_0. Then $V_A = \mathcal{V} \otimes_{\mathcal{O}(U)} \mathcal{O}_{x_0}$ of $V = \mathcal{V}_{x_0}$ is an infinitesimal deformation of $V = \mathcal{V}_{x_0}$. It is not difficult to see that

$$F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_A) \;=\; \mathcal{R}_A \otimes_L \mathcal{D}_{\mathrm{cris}}(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_A))^{\varphi=a_p(x)}$$

is a saturated (φ,Γ)-submodule of $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_A)$ ([BC], Lemma 2.5.2 iii)). We see immediately that $F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_A) \simeq \mathcal{R}_A(\delta_A)$ where $\delta_A(u) = 1$ for $u \in \mathbb{Z}_p^*$ and $\delta_A(p) = a_p(2k) + a_p'(2k)T \pmod{T^2}$ with $a_p(2k) = p^{k-1}$. Set $F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_A) = \mathbf{D}^{\dagger}_{\mathrm{rig}}(V_A)$. As

$$\langle\chi(\gamma)\rangle = \exp\bigl((2k-x)\log\chi(\gamma)\bigr) = 1 - (\log\chi(\gamma))T \pmod{T^2}$$

we obtain that

$$(\psi_A\delta_A)(p) = 1, \qquad (\psi_A\delta_A)(\chi(\gamma)) = 1 - (\log\chi(\gamma))T \pmod{T^2}$$

Thus $\psi_A(\chi(\gamma)) = 1 - \log\chi(\gamma)\,T \pmod{T^2}$ and $d\log\psi_A(\chi(\gamma))\big|_{T=0} = -\log\chi(\gamma)$. Twisting V_A by χ^k we obtain an infinitesimal deformation $V_A(k)$ of $V(k)$. The formula (3.2) writes

$$\mathcal{L}(V(k), D) = -2\,d\log a_p(2k).$$

In particular we obtain that $\mathcal{L}_{\mathrm{FM}}(f) = -2\,d\log a_p(2k)$. The first direct proof of this result was done in [Cz5] using Galois cohomology computations inside the rings of p-adic periods. Remark that in [Cz6], Colmez used the theory of (φ,Γ)-modules to prove this formula with Breuil's $\mathcal{L}$-invariant. His approach is based on the local Langlands correspondence for two-dimensional trianguline representations.

References

[AV] Y. Amice and J. Vélu, *Distributions p-adiques associées aux séries de Hecke*, Astérisque **24-25** (1975), 119-131.

[Ben1] D. Benois, *Iwasawa theory of crystalline representations*, Duke Math. J. **104** (2000), no. 2, 211-267.

[Ben2] D. Benois, *A generalization of Greenberg's $\mathcal{L}$-invariant*, Preprint available on arXiv:0906.2857 (2009).

[Ben3] D. Benois, *Trivial zeros of Perrin-Riou's L-functions*, Preprint available on arXiv:0906.2862 (2009).

[Ber1] L. Berger, *Représentations p-adiques et équations différentielles*, Invent. Math. **148** (2002), no. 2, 219-284.

[Ber2] L. Berger, *Equations différentielles p-adiques et (φ, N)-modules filtrés*, Astérisque **319** (2008), 13-38.

[BK] S. Bloch, K. Kato, *L-functions and Tamagawa numbers of motives*, Grothendieck Festschrift, vol. 1 (1990), 333-400.

[BC] J. Bellaïche et G. Chenevier, *p-adic families of Galois representations and higher rank Selmer groups*, to appear in "Astérisque".

[BDI] M. Bertolini, H. Darmon, A. Iovita, *Families of automorphic forms on definite quaternion algebras and Teitelbaum's conjecture*, Astérisque (to appear).

[Br] C. Breuil, *Invariant $\mathcal{L}$ et série spéciale p-adique*, Ann. Sci ENS **37** (2004), 559-610.

[Co1] R. Coleman, *A p-adic Shimura isomorphism and p-adic periods of modular forms*, Contemp. Math. **165** (1994), 21-51.

[Co2] R. Coleman, *p-adic Banch spaces and families of modular forms*, Invent. Math. **127** (1997), 417-479.

[CM] R. Coleman and B. Mazur, *The eigencurve*, Galois representations in Arithmetic Algebraic geometry (Durham 1996),(A.J. Scholl and R.L. Taylor eds.) London Math. Soc. Lecture Notes Ser., vol. 254, Cambridge Univ. Press, 1998, 1-113.

[Cz1] P. Colmez, *Les conjectures de monodromie p-adiques*, Séminaire Bourbaki 2001/02, Astérisque **290** (2003), 53-101.

[Cz2] P. Colmez, *La conjecture de Birch et Swinnerton-Dyer p-adique*, Séminaire Bourbaki 2002/03, Astérisque **294** (2004), 251-319.

[Cz3] P. Colmez, *Zéros supplémentaires de fonctions L p-adiques de formes modulaires*, Algebra and Number theory, Hindustan book agency 2005, 193-210.

[Cz4] P. Colmez, *Représentations triangulines de dimension* 2, Astérisque **319** (2008), 213-258.

[Cz5] P. Colmez, *Invariants $\mathcal{L}$ et dérivées de valeurs propres de Frobenius*, Preprint (2003).

[Cz6] P. Colmez, *Série principale unitaire pour* $\mathrm{GL}_2(\mathbb{Q}_p)$ *et représentations triangulines de dimension* 2, Preprint (2005).

[CC] F. Cherbonnier and P. Colmez, *Représentations p-adiques surconvergentes*, Invent. Math. **133** (1998), 581-611.

[D] P. Deligne, *Formes modulaires et représentations l-adiques, Sém. Bourbaki 1968/69, exp.343*, Lecture Notes in Math. **179** (1971), 139-172.

[E] M. Emerton, *p-adic L-functions and unitary completions of representations of p-adic reductive groups*, Duke Math. J. **130** (2005), 353-392.

[Fa] G. Faltings, *Hodge-Tate structures and modular forms*, Math. Ann. **278** (1987), 133-149.

[Fo1] J.-M. Fontaine, *Représentations p-adiques des corps locaux,* The Grothendieck Festschrift*, vol. II, Birkhäuser, Boston* (1991), 249-309.

[Fo2] J.-M. Fontaine, *Le corps des périodes p-adiques*, Astérisque **223** (1994), 59-102.

[Fo3] J.-M. Fontaine, *Représentations p-adiques semi-stables*, Astérisque **223** (1994), 113-184.

[G] R. Greenberg, *Trivial zeros of p-adic L-functions*, Contemp. Math. **165** (1994), 149-174.

[GS1] R. Greenberg and G.Stevens, *p-adic L-functions and p-adic periods of modular forms*, Invent. Math. **111** (1993), 407-447.

[GS2] R. Greenberg and G.Stevens, *On the conjecture of Mazur, Tate and Teitelbaum*, Contemp. Math. **165** (1994), 183-211.

[H1] L. Herr, *Sur la cohomologie galoisienne des corps p-adiques*, Bull. Soc. math. France **126** (1998), 563-600.

[H2] L. Herr, *Une approche nouvelle de la dualité locale de Tate*, Math. Annalen **320** (2001), 307-337.

[Hi] H. Hida, *$\mathcal{L}$-invariant of p-adic L-functions, Conference of L-functions, World Sci. Publ., Hackensack, N.J., 2007*, 17-53.

[Ke] K. Kedlaya, *A p-adic monodromy theorem*, Ann. of Math. **160** (2004), 94-184.

[Ki] M. Kisin, *Overconvergent modular forms and the Fontaine-Mazur conjecture*, Invent. Math. **153** (2003), 373-454.

[Li] R. Liu, *Cohomology and Duality for* (φ, Γ)*-modules over the Robba ring*, Int. Math. Research Notices **3** (2007), 32 pages.

[Mn] Y. Manin, *Periods of cusp forms and p-adic Hecke series*, Math. USSR Sbornik **92** (1973), 371-393.

[Mr] B. Mazur, *On monodromy invariants occuring in global arithmetic and Fontaine's theory*, Contemp. Math. **165** (1994), 1-20.

[MTT] B. Mazur, J.Tate, J. Teitelbaum, *On p-adic analogues of the conjectures of Burch and Swinnerton-Dyer*, Invent. Math. **84** (1986), 1-48.

[O] L. Orton, *An elementary proof of a weak exceptional zero conjecture*, Canad. J. Math. **56** (2004), no. 2, 373-405.

[Pa] A. Panchishkin, *Two variable p-adic L-functions attached to eigenfamilies of positive slope*, Invent. Math. **154** (2003), 551-615.

[PR] B. Perrin-Riou, *Fonctions L p-adiques des représentations p-adiques*, Astérisque **229** (1995).

[Sa] T. Saito, *Modular forms and p-adic Hodge theory*, Invent. Math. **129** (1997), 607-620.

[St] G. Stevens, *Coleman* $\mathcal{L}$*-invariant and families of modular forms*, Preprint.

[Tm] J. Teitelbaum, *Values of p-adic L-functions and a p-adic Poisson kernel*, Invent. Math. **101** (1990), no. 2, 395-410.

Institut de Mathématiques
Université Bordeaux I
351 cours de la Libération
33405 Talence Cedex
France
denis.benois@math.u-bordeaux1.fr

Motivically Functorial Coniveau Spectral Sequences; Direct Summands of Cohomology of Function Fields

Mikhail V. Bondarko

Received: September 4, 2009
Revised: June 7, 2010

Abstract. The goal of this paper is to prove that coniveau spectral sequences are motivically functorial for all cohomology theories that could be factorized through motives. To this end the motif of a smooth variety over a countable field k is decomposed (in the sense of Postnikov towers) into twisted (co)motives of its points; this is generalized to arbitrary Voevodsky's motives. In order to study the functoriality of this construction, we use the formalism of weight structures (introduced in the previous paper). We also develop this formalism (for general triangulated categories) further, and relate it with a new notion of a *nice duality* (pairing) of (two distinct) triangulated categories; this piece of homological algebra could be interesting for itself.

We construct a certain *Gersten* weight structure for a triangulated category of *comotives* that contains DM_{gm}^{eff} as well as (co)motives of function fields over k. It turns out that the corresponding *weight spectral sequences* generalize the classical coniveau ones (to cohomology of arbitrary motives). When a cohomological functor is represented by a $Y \in \operatorname{Obj} DM_{-}^{eff}$, the corresponding coniveau spectral sequences can be expressed in terms of the (homotopy) t-truncations of Y; this extends to motives the seminal coniveau spectral sequence computations of Bloch and Ogus.

We also obtain that the comotif of a smooth connected semi-local scheme is a direct summand of the comotif of its generic point; comotives of function fields contain twisted comotives of their residue fields (for all geometric valuations). Hence similar results hold for any cohomology of (semi-local) schemes mentioned.

2010 Mathematics Subject Classification: 14F42, 14C35, 18G40, 19E15, 14F20, 14C25, 14C35.
Keywords and Phrases: Motives, coniveau, weight structure, t-structure, triangulated category, semi-local scheme, cohomology.

Contents

Introduction

Let k be our perfect base field.

We recall two very important statements concerning coniveau spectral sequences. The first one is the calculation of E_2 of the coniveau spectral sequence for cohomological theories that satisfy certain conditions; see [5] and [8]. It was proved by Voevodsky that these conditions are fulfilled by any theory H represented by a motivic complex C (i.e. an object of DM_-^{eff}; see [25]); then the E_2-terms of the spectral sequence could be calculated in terms of the (homotopy t-structure) cohomology of C. This result implies the second one: H-cohomology of a smooth connected semi-local scheme (in the sense of §4.4 of [26]) injects into the cohomology of its generic point; the latter statement was extended to all (smooth connected) primitive schemes by M. Walker.

The main goal of the present paper is to construct (motivically) functorial coniveau spectral sequences converging to cohomology of arbitrary motives; there should exist a description of these spectral sequences (starting from E_2) that is similar to the description for the case of cohomology of smooth varieties (mentioned above).

A related objective is to clarify the nature of the injectivity result mentioned; it turned our that (in the case of a countable k) the cohomology of a smooth connected semi-local (more generally, primitive) scheme is actually a direct summand of the cohomology of its generic point. Moreover, the (twisted) cohomology of a residue field of a function field K/k (for any geometric valuation of K) is a direct summand of the cohomology of K. We actually prove more in §4.3.

Our main homological algebra tool is the theory of *weight structures* (in triangulated categories; we usually denote a weight structure by w) introduced in the previous paper [6]. In this article we develop it further; this part of the paper could be interesting also to readers not acquainted with motives (and could be read independently from the rest of the paper). In particular, we study *nice dualities* (certain pairings) of (two distinct) triangulated categories; it seems that this subject was not previously considered in the literature at all. This allows us to generalize the concept of *adjacent* weight and t-structures (t) in a triangulated category (developed in §4.4 of [6]): we introduce the notion of *orthogonal structures* in (two possibly distinct) triangulated categories. If Φ is a nice duality of triangulated $\underline{C}, \underline{D}$, $X \in Obj\underline{C}$, $Y \in Obj\underline{D}$, t is orthogonal to w, then the spectral sequence S converging to $\Phi(X,Y)$ that comes from the t-truncations of Y is naturally isomorphic (starting from E_2) to the *weight spectral sequence* T for the functor $\Phi(-,Y)$. T comes from *weight truncations* of X (note that those generalize stupid truncations for complexes). Our approach yields an abstract alternative to the method of comparing similar spectral sequences using filtered complexes (developed by Deligne and Paranjape, and used in [22], [11], and [6]). Note also that we relate t-truncations in $\underline{D}$ with *virtual t-truncations* of cohomological functors on $\underline{C}$. Virtual t-truncations for cohomological functors are defined for any $(\underline{C}, w)$ (we do not need any triangulated 'categories of functors' or t-structures for them here); this notion was introduced in §2.5 of [6] and is studied further in the current paper.

Now, we explain why we really need a certain new category of *comotives* (containing Voevodsky's DM_{gm}^{eff}), and so the theory of adjacent structures (i.e. orthogonal structures in the case $\underline{C} = \underline{D}$, $\Phi = \underline{C}(-,-)$) is not sufficient for our purposes. It was already proved in [6] that weight structures provide a powerful tool for constructing spectral sequences; they also relate the cohomology of objects of triangulated categories with t-structures adjacent to them. Unfortunately, a weight structure corresponding to coniveau spectral sequences cannot exist on $DM_{-}^{eff} \supset DM_{gm}^{eff}$ since these categories do not contain (any) motives for function fields over k (as well as motives of other schemes not of finite type over k; still cf. Remark 4.5.4(5)). Yet these motives should generate the *heart* of this weight structure (since the objects of this heart should corepresent covariant exact functors from the category of homotopy invariant sheaves with transfers to Ab).

So, we need a category that would contain certain homotopy limits of objects of DM_{gm}^{eff}. We succeed in constructing a triangulated category $\mathfrak{D}$ (of *comotives*) that allows us to reach the objectives listed. Unfortunately, in order to control morphisms between homotopy limits mentioned we have to assume k to be countable. In this case there exists a large enough triangulated category $\mathfrak{D}_s$ ($DM_{gm}^{eff} \subset \mathfrak{D}_s \subset \mathfrak{D}$) endowed with a certain *Gersten weight structure* w; its heart is 'generated' by comotives of function fields. w is (left) orthogonal to the homotopy t-structure on DM_{-}^{eff} and (so) is closely connected with coniveau spectral sequences and Gersten resolutions for sheaves. Note still: we need k to be countable only in order to construct the Gersten weight structure. So

those readers who would just want to have a category that contains reasonable homotopy limits of geometric motives (including comotives of function fields and of smooth semi-local schemes), and consider cohomology theories for this category, may freely ignore this restriction. Moreover, for an arbitrary k one can still pass to a countable homotopy limit in the Gysin distinguished triangle (as in Proposition 3.6.1). Yet for an uncountable k countable homotopy limits don't seem to be interesting; in particular, they definitely do not allow to construct a Gersten weight structure (in this case).
So, we consider a certain triangulated category $\mathfrak{D} \supset DM_{gm}^{eff}$ that (roughly!) 'consists of' (covariant) homological functors $DM_{gm}^{eff} \to Ab$. In particular, objects of $\mathfrak{D}$ define covariant functors $SmVar \to Ab$ (whereas another 'big' motivic category DM_{-}^{eff} defined by Voevodsky is constructed from certain sheaves i.e. contravariant functors $SmVar \to Ab$; this is also true for all motivic homotopy categories of Voevodsky and Morel). Besides, DM_{gm}^{eff} yields a family of (weak) cocompact cogenerators for $\mathfrak{D}$. This is why we call objects of $\mathfrak{D}$ comotives. Yet note that the embedding $DM_{gm}^{eff} \to \mathfrak{D}$ is covariant (actually, we invert the arrows in the corresponding 'category of functors' in order to make the Yoneda embedding functor covariant), as well as the functor that sends a smooth scheme U (not necessarily of finite type over k) to its comotif (which coincides with its motif if U is a smooth variety).
We also recall the Chow weight structure w'_{Chow} introduced in [6]; the corresponding *Chow-weight* spectral sequences are isomorphic to the classical (i.e. Deligne's) weight spectral sequences when the latter are defined. w'_{Chow} could be naturally extended to a weight structure w_{Chow} for $\mathfrak{D}$. We always have a natural comparison morphism from the Chow-weight spectral sequence for (H, X) to the corresponding coniveau one; it is an isomorphism for any birational cohomology theory. We consider the category of birational comotives $\mathfrak{D}_{bir}$ i.e. the localization of $\mathfrak{D}$ by $\mathfrak{D}(1)$ (that contains the category of birational geometric motives introduced in [15]; though some of the results of this unpublished preprint are erroneous, this makes no difference for the current paper). It turns our that w and w_{Chow} induce the same weight structure w'_{bir} on $\mathfrak{D}_{bir}$. Conversely, starting from w'_{bir} one can 'glue' (from *slices*) the weight structures induced by w and w_{Chow} on $\mathfrak{D}/\mathfrak{D}(n)$ for all $n > 0$. Moreover, these structures belong to an interesting family of weight structures indexed by a single integral parameter! It could be interesting to consider other members of this family. We relate briefly these observations with those of A. Beilinson (in [3] he proposed a 'geometric' characterization of the conjectural motivic t-structure).
Now we describe the connection of our results with related results of F. Deglise (see [9], [10], and [11]; note that the two latter papers are not published at the moment yet). He considers a certain category of pro-motives whose objects are naive inverse limits of objects of DM_{gm}^{eff} (this category is not triangulated, though it is *pro-triangulated* in a certain sense). This approach allows to obtain (in a universal way) classical coniveau spectral sequences for cohomology of motives of smooth varieties; Deglise also proves their relation with the homotopy t-truncations for cohomology represented by an object of DM_{-}^{eff}. Yet for

cohomology theories not coming from motivic complexes, this method does not seem to extend to (spectral sequences for cohomology of) arbitrary motives; motivic functoriality is not obvious also. Moreover, Deglise didn't prove that the pro-motif of a (smooth connected) semi-local scheme is a direct summand of the pro-motif of its generic point (though this is true, at least in the case of a countable k). We will tell much more about our strategy and on the relation of our results with those of Deglise in §1.5 below. Note also that our methods are much more convenient for studying functoriality (of coniveau spectral sequences) than the methods applied by M. Rost in the related context of cycle modules (see [24] and §4 of [10]).

The author would like to indicate the interdependencies of the parts of this text (in order to simplify reading for those who are not interested in all of it). Those readers who are not (very much) interested in (coniveau) spectral sequences, may avoid most of section 2 and read only §§2.1 –2.2 (Remark 2.2.2 could also be ignored). Moreover, in order to prove our direct summands results (i.e. Theorem 4.2.1, Corollary 4.2.2, and Proposition 4.3.1) one needs only a small portion of the theory of weight structures; so a reader very reluctant to study this theory may try to derive them from the results of §3 'by hand' without reading §2 at all. Still, for motivic functoriality of coniveau spectral sequences and filtrations (see Proposition 4.4.1 and Remark 4.4.2) one needs more of weight structures. On the other hand, those readers who are more interested in the (general) theory of triangulated categories may restrict their attention to §§1.1– 1.2 and §2; yet note that the rest of the paper describes in detail an important (and quite non-trivial) example of a weight structure which is orthogonal to a t-structure with respect to a nice duality (of triangulated categories). Moreover, much of section §4 could also be extended to a general setting of a triangulated category satisfying properties similar to those listed in Proposition 3.1.1; yet the author chose not to do this in order to make the paper somewhat less abstract.

Now we list the contents of the paper. More details could be found at the beginnings of sections.

We start §1 with the recollection of t-structures, idempotent completions, and Postnikov towers for triangulated categories. We describe a method for extending cohomological functors from a full triangulated subcategory to the whole $\underline{C}$ (after H. Krause). Next we recall some results and definitions for Voevodsky's motives (this includes certain properties of Tate twists for motives and cohomological functors). Lastly, we define pro-motives (following Deglise) and compare them with our triangulated category $\mathfrak{D}$ of comotives. This allows to explain our strategy step by step.

§2 is dedicated to weight structures. First we remind the basics of this theory (developed in §[6]). Next we recall that a cohomological functor H from an (arbitrary triangulated category) $\underline{C}$ endowed with a weight structure w could be 'truncated' as if it belonged to some triangulated category of functors (from $\underline{C}$) that is endowed with a t-structure; we call the corresponding pieces of H its *virtual t-truncations*. We recall the notion of a weight spectral sequence (introduces in ibid.). We prove that the derived exact couple for a weight spectral sequence could be described in terms of virtual t-truncations. Next we introduce the definition a *(nice) duality* $\Phi : \underline{C}^{op} \times \underline{D} \to \underline{A}$ (here $\underline{D}$ is triangulated, $\underline{A}$ is abelian), and of *orthogonal* weight and t-structures (with respect to Φ). If w is orthogonal to t, then the virtual t-truncations (corresponding to w) of functors of the type $\Phi(-,Y)$, $Y \in Obj\underline{D}$, are exactly the functors 'represented via Φ' by the actual t-truncations of Y (corresponding to t). Hence if w and t are orthogonal with respect to a nice duality, the weight spectral sequence converging to $\Phi(X,Y)$ (for $X \in Obj\underline{C}$, $Y \in Obj\underline{D}$) is naturally isomorphic (starting from E_2) to the one coming from t-truncations of Y. We also mention some alternatives and predecessors of our results. Lastly we compare weight decompositions, virtual t-truncations, and weight spectral sequences corresponding to distinct weight structures (in possibly distinct triangulated categories).

In §3 we describe the main properties of $\mathfrak{D} \supset DM^{eff}_{gm}$. The exact choice of $\mathfrak{D}$ is not important for most of this paper; so we just list the main properties of $\mathfrak{D}$ (and its certain *enhancement* $\mathfrak{D}'$) in §3.1. We construct $\mathfrak{D}$ using the formalism of differential graded modules in §5 later. Next we define comotives for (certain) schemes and ind-schemes of infinite type over k (we call them pro-schemes). We recall the notion of a primitive scheme. All (smooth) semi-local pro-schemes are primitive; primitive schemes have all nice 'motivic' properties of semi-local pro-schemes. We prove that there are no $\mathfrak{D}$-morphisms of positive degrees between comotives of primitive schemes (and also between certain Tate twists of those). In §3.6 we prove that the Gysin distinguished triangle for motives of smooth varieties (in DM^{eff}_{gm}) could be naturally extended to comotives of pro-schemes. This allows to construct certain Postnikov towers for comotives of pro-schemes; these towers are closely related with classical coniveau spectral sequences for cohomology.

§4 is central in this paper. We introduce a certain *Gersten weight structure* for a certain triangulated category $\mathfrak{D}_s$ ($DM^{eff}_{gm} \subset \mathfrak{D}_s \subset \mathfrak{D}$). We prove that Postnikov towers constructed in §3.6 are actually *weight Postnikov towers* with respect to w. We deduce our (interesting) results on direct summands of comotives of function fields. We translate these results to cohomology in the obvious way.

Next we prove that weight spectral sequences for the cohomology of X (corresponding to the Gersten weight structure) are naturally isomorphic (starting from E_2) to the classical coniveau spectral sequences if X is the motif of a smooth variety; so we call these spectral sequence coniveau ones in the general case also. We also prove that the Gersten weight structure w (on $\mathfrak{D}_s$) is orthogonal to the homotopy t-structure t on DM^{eff}_{-} (with respect to a certain Φ). It follows that for an arbitrary $X \in ObjDM^s$, for a cohomology theory represented by $Y \in ObjDM^{eff}_{-}$ (any choice of) the coniveau spectral sequence that converges to $\Phi(X,Y)$ could be described in terms of the t-truncations of Y (starting from E_2).

We also define coniveau spectral sequences for cohomology of motives over uncountable base fields as the limits of the corresponding coniveau spectral

sequences over countable perfect subfields of definition. This definition is compatible with the classical one; so we establish motivic functoriality of coniveau spectral sequences in this case also.
We also prove that the *Chow weight structure* for DM_{gm}^{eff} (introduced in §6 of [6]) could be extended to a weight structure w_{Chow} on $\mathfrak{D}$. The corresponding *Chow-weight* spectral sequences are isomorphic to the classical (i.e. Deligne's) ones when the latter are defined (this was proved in [6] and [7]). We compare coniveau spectral sequences with Chow-weight ones: we always have a comparison morphism; it is an isomorphism for a *birational* cohomology theory. We consider the category of birational comotives $\mathfrak{D}_{bir}$ i.e. the localization of $\mathfrak{D}$ by $\mathfrak{D}(1)$. w and w_{Chow} induce the same weight structure w'_{bir} on $\mathfrak{D}_{bir}$; one almost can glue w and w_{Chow} from copies of w'_{bir} (one may say that these weight structures could almost be glued from the same slices with distinct shifts).
§5 is dedicated to the construction of $\mathfrak{D}$ and the proof of its properties. We apply the formalism of differential graded categories, modules over them, and of the corresponding derived categories. A reader not interested in these details may skip (most of) this section. In fact, the author is not sure that there exists only one $\mathfrak{D}$ suitable for our purposes; yet the choice of $\mathfrak{D}$ does not affect cohomology of (comotives of) pro-schemes and of Voevodsky's motives.
We also explain how the differential graded modules formalism can be used to define base change (extension and restriction of scalars) for comotives. This allows to extend our results on direct summands of comotives (and cohomology) of function fields to pro-schemes obtained from them via base change. We also define tensoring of comotives by motives (in particular, this yields Tate twist for $\mathfrak{D}$), as well as a certain cointernal Hom (i.e. the corresponding left adjoint functor).
§6 is dedicated to properties of comotives that are not (directly) related with the main results of the paper; we also make several comments. We recall the definition of the additive category $\mathfrak{D}^{gen}$ of generic motives (studied in [9]). We prove that the exact conservative *weight complex* functor corresponding to w (that exists by the general theory of weight structures) could be modified to an exact conservative $WC : \mathfrak{D}_s \to K^b(\mathfrak{D}^{gen})$. Next we prove that a cofunctor $\underline{Hw} \to Ab$ is representable by a homotopy invariant sheaf with transfers whenever is converts all products into direct sums.
We also note that our theory could be easily extended to (co)motives with coefficients in an arbitrary ring. Next we note (after B. Kahn) that reasonable motives of pro-schemes with compact support do exist in DM_{-}^{eff}; this observation could be used for the construction of an alternative model for $\mathfrak{D}$. Lastly we describe which parts of our argument do not work (and which do work) in the case of an uncountable k.
A caution: the notion of a weight structure is quite a general formalism for triangulated categories. In particular, one triangulated category can support several distinct weight structures (note that there is a similar situation with t-structures). In fact, we construct an example for such a situation in this paper (certainly, much simpler examples exist): we define the Gersten weight

structure w for $\mathfrak{D}_s$ and a Chow weight structure w_{Chow} for $\mathfrak{D}$. Moreover, we show in §4.9 that these weight structures are compatible with certain weight structures defined on the localizations $\mathfrak{D}/\mathfrak{D}(n)$ (for all $n > 0$). These two series of weight structures are definitely distinct: note that w yields coniveau spectral sequences, whereas w_{Chow} yields Chow-weight spectral sequences, that generalize Deligne's weight spectral sequences for étale and mixed Hodge cohomology (see [6] and [7]). Also, the weight complex functor constructed in [7] and [6] is quite distinct from the one considered in §6.1 below (even the targets of the functors mentioned are completely distinct).

The author is deeply grateful to prof. F. Deglise, prof. B. Kahn, prof. M. Rovinsky, prof. A. Suslin, prof. V. Voevodsky, and to the referee for their interesting remarks. The author gratefully acknowledges the support from Deligne 2004 Balzan prize in mathematics. The work is also supported by RFBR (grants no. 08-01-00777a and 10-01-00287a).

Notation. For a category C, $A, B \in ObjC$, we denote by $C(A, B)$ the set of A-morphisms from A into B.

For categories C, D we write $C \subset D$ if C is a full subcategory of D.

For additive C, D we denote by $\mathrm{AddFun}(C, D)$ the category of additive functors from C to D (we will ignore set-theoretic difficulties here since they do not affect our arguments seriously).

Ab is the category of abelian groups. For an additive B we will denote by B^* the category $\mathrm{AddFun}(B, Ab)$ and by B_* the category $\mathrm{AddFun}(B^{op}, Ab)$. Note that both of these are abelian. Besides, Yoneda's lemma gives full embeddings of B into B_* and of B^{op} into B^* (these send $X \in ObjB$ to $X_* = B(-, X)$ and to $X^* = B(X, -)$, respectively).

For a category C, $X, Y \in ObjC$, we say that X is a *retract* of Y if id_X could be factorized through Y. Note that when C is triangulated or abelian then X is a retract of Y if and only if X is its direct summand. For any $D \subset C$ the subcategory D is called *Karoubi-closed* in C if it contains all retracts of its objects in C. We will call the smallest Karoubi-closed subcategory of C containing D the *Karoubization* of D in C; sometimes we will use the same term for the class of objects of the Karoubization of a full subcategory of C (corresponding to some subclass of $ObjC$).

For a category C we denote by C^{op} its opposite category.

For an additive $\underline{C}$ an object $X \in Obj\underline{C}$ is called cocompact if $\underline{C}(\prod_{i\in I} Y_i, X) = \bigoplus_{i\in I} \underline{C}(Y_i, X)$ for any set I and any $Y_i \in Obj\underline{C}$ such that the product exists (here we don't need to demand all products to exist, though they actually will exist below).

For $X, Y \in Obj\underline{C}$ we will write $X \perp Y$ if $\underline{C}(X, Y) = \{0\}$. For $D, E \subset Obj\underline{C}$ we will write $D \perp E$ if $X \perp Y$ for all $X \in D$, $Y \in E$. For $D \subset \underline{C}$ we will denote by $D^{\perp}$ the class

$$\{Y \in Obj\underline{C} : \ X \perp Y \ \forall X \in D\}.$$

Sometimes we will denote by $D^{\perp}$ the corresponding full subcategory of $\underline{C}$. Dually, ${}^{\perp}D$ is the class $\{Y \in Obj\underline{C} : \ Y \perp X \ \forall X \in D\}$. This convention is opposite to the one of §9.1 of [21].

In this paper all complexes will be cohomological i.e. the degree of all differentials is $+1$; respectively, we will use cohomological notation for their terms.

For an additive category B we denote by $C(B)$ the category of (unbounded) complexes over it. $K(B)$ will denote the homotopy category of complexes. If B is also abelian, we will denote by $D(B)$ the derived category of B. We will also need certain bounded analogues of these categories (i.e. $C^b(B)$, $K^b(B)$, $D^-(B)$).

$\underline{C}$ and $\underline{D}$ will usually denote some triangulated categories. We will use the term 'exact functor' for a functor of triangulated categories (i.e. for a for a functor that preserves the structures of triangulated categories).

$\underline{A}$ will usually denote some abelian category. We will call a covariant additive functor $\underline{C} \to \underline{A}$ for an abelian $\underline{A}$ *homological* if it converts distinguished triangles into long exact sequences; homological functors $\underline{C}^{op} \to \underline{A}$ will be called *cohomological* when considered as contravariant functors $\underline{C} \to \underline{A}$.

$H : \underline{C}^{op} \to \underline{A}$ will always be additive; it will usually be cohomological.

For $f \in \underline{C}(X,Y)$, $X, Y \in Obj\underline{C}$, we will call the third vertex of (any) distinguished triangle $X \xrightarrow{f} Y \to Z$ a cone of f. Note that different choices of cones are connected by non-unique isomorphisms, cf. IV.1.7 of [13]. Besides, in $C(B)$ we have canonical cones of morphisms (see section §III.3 of ibid.).

We will often specify a distinguished triangle by two of its morphisms.

When dealing with triangulated categories we (mostly) use conventions and auxiliary statements of [13]. For a set of objects $C_i \in Obj\underline{C}$, $i \in I$, we will denote by $\langle C_i \rangle$ the smallest strictly full triangulated subcategory containing all C_i; for $D \subset \underline{C}$ we will write $\langle D \rangle$ instead of $\langle Obj D \rangle$.

We will say that C_i generate $\underline{C}$ if $\underline{C}$ equals $\langle C_i \rangle$. We will say that C_i *weakly cogenerate* $\underline{C}$ if for $X \in Obj\underline{C}$ we have $\underline{C}(X, C_i[j]) = \{0\}\ \forall i \in I,\ j \in \mathbb{Z} \implies X = 0$ (i.e. if ${}^{\perp}\{C_i[j]\}$ contains only zero objects).

We will call a partially ordered set L a (filtered) *projective system* if for any $x, y \in L$ there exists some maximum i.e. a $z \in L$ such that $z \geq x$ and $z \geq y$. By abuse of notation, we will identify L with the following category D: $Obj D = L$; $D(l', l)$ is empty whenever $l' < l$, and consists of a single morphism otherwise; the composition of morphisms is the only one possible. If L is a projective system, C is some category, $X : L \to C$ is a covariant functor, we will denote $X(l)$ for $l \in L$ by X_l. We will write $Y = \varprojlim_{l \in L} X_l$ for the limit of this functor; we will call it the inverse limit of X_l. We will denote the colimit of a contravariant functor $Y : L \to C$ by $\varinjlim_{l \in L} Y_l$ and call it the direct limit. Besides, we will sometimes call the categorical image of L with respect to such an Y an *inductive system*.

Below I, L will often be projective systems; we will usually require I to be countable.

A subsystem L' of L is a partially ordered subset in which maximums exist (we will also consider the corresponding full subcategory of L). We will call L' unbounded in L if for any $l \in L$ there exists an $l' \in L'$ such that $l' \geq l$.

k will be our perfect base field. Below we will usually demand k to be countable. Note: this yields that for any variety the set of its closed (or open) subschemes is countable.

We also list central definitions and main notation of this paper.

First we list the main (general) homological algebra definitions. t-structures, t-truncations, and Postnikov towers in triangulated categories are defined in §1.1; weight structures, weight decompositions, weight truncations, weight Postnikov towers, and weight complexes are considered in §2.1; virtual t-truncations and nice exact complexes of functors are defined in §2.3; weight spectral sequences are studied in §2.4; (nice) dualities and orthogonal weight and t-structures are defined in Definition 2.5.1; right and left weight-exact functors are defined in Definition 2.7.1.

Now we list notation (and some definitions) for motives. $DM_{gm}^{eff} \subset DM_{-}^{eff}$, HI and the homotopy t-structure for DM_{gm}^{eff} are defined in §1.3; Tate twists are considered in §1.4; $\mathfrak{D}^{naive}$ is defined in §1.5; comotives ($\mathfrak{D}$ and $\mathfrak{D}'$) are defined in §3.1; in §3.2 we discuss pro-schemes and their comotives; in §3.3 we recall the definition of a primitive scheme; in §4.1 we define the Gersten weight structure w on a certain triangulated $\mathfrak{D}_s$; we consider w_{Chow} in §4.7; $\mathfrak{D}_{bir}$ and w'_{bir} are defined in §4.9; several differential graded constructions (including extension and restriction of scalars for comotives) are considered in §5; we define $\mathfrak{D}^{gen}$ and $WC : \mathfrak{D}_s \to K^b(\mathfrak{D}^{gen})$ in §6.1.

1 Some preliminaries on triangulated categories and motives

§1.1 we recall the notion of a t-structure (and introduce some notation for it), recall the notion of an idempotent completion of an additive category; we also recall that any small abelian category could be faithfully embedded into Ab (a well-known result by Mitchell).

In §1.2 we describe (following H. Krause) a natural method for extending cohomological functors from a full triangulated $\underline{C}' \subset \underline{C}$ to $\underline{C}$.

In §1.3 we recall some definitions and results of Voevodsky.

In §1.4 we recall the notion of a Tate twist; we study the properties of Tate twists for motives and homotopy invariant sheaves.

In §1.5 we define pro-motives (following [9] and [10]). These are not necessary for our main result; yet they allow to explain our methods step by step. We also describe in detail the relation of our constructions and results with those of Deglise.

1.1 t-structures, Postnikov towers, idempotent completions, and an embedding theorem of Mitchell

To fix the notation we recall the definition of a t-structure.

Definition 1.1.1. A pair of subclasses $\underline{C}^{t\geq 0}, \underline{C}^{t\leq 0} \subset Obj\underline{C}$ for a triangulated category $\underline{C}$ will be said to define a t-structure t if $(\underline{C}^{t\geq 0}, \underline{C}^{t\leq 0})$ satisfy the following conditions:

(i) $\underline{C}^{t\geq 0}, \underline{C}^{t\leq 0}$ are strict i.e. contain all objects of $\underline{C}$ isomorphic to their elements.
(ii) $\underline{C}^{t\geq 0} \subset \underline{C}^{t\geq 0}[1]$, $\underline{C}^{t\leq 0}[1] \subset \underline{C}^{t\leq 0}$.
(iii) **Orthogonality**. $\underline{C}^{t\leq 0}[1] \perp \underline{C}^{t\geq 0}$.
(iv) t-**decomposition**. For any $X \in Obj\underline{C}$ there exists a distinguished triangle

$$A \to X \to B[-1] \to A[1] \tag{1}$$

such that $A \in \underline{C}^{t\leq 0}, B \in \underline{C}^{t\geq 0}$.

We will need some more notation for t-structures.

Definition 1.1.2. 1. A category $\underline{Ht}$ whose objects are $\underline{C}^{t=0} = \underline{C}^{t\geq 0} \cap \underline{C}^{t\leq 0}$, $\underline{Ht}(X,Y) = \underline{C}(X,Y)$ for $X,Y \in \underline{C}^{t=0}$, will be called the *heart* of t. Recall (cf. Theorem 1.3.6 of [2]) that $\underline{Ht}$ is abelian (short exact sequences in $\underline{Ht}$ come from distinguished triangles in $\underline{C}$).
2. $\underline{C}^{t\geq l}$ (resp. $\underline{C}^{t\leq l}$) will denote $\underline{C}^{t\geq 0}[-l]$ (resp. $\underline{C}^{t\leq 0}[-l]$).

Remark 1.1.3. 1. The axiomatics of t-structures is self-dual: if $\underline{D} = \underline{C}^{op}$ (so $Obj\underline{C} = Obj\underline{D}$) then one can define the (opposite) weight structure t' on $\underline{D}$ by taking $\underline{D}^{t'\leq 0} = \underline{C}^{t\geq 0}$ and $\underline{D}^{t'\geq 0} = \underline{C}^{t\leq 0}$; see part (iii) of Examples 1.3.2 in [2].
2. Recall (cf. Lemma IV.4.5 in [13]) that (1) defines additive functors $\underline{C} \to \underline{C}^{t\leq 0} : X \to A$ and $C \to \underline{C}^{t\geq 0} : X \to B$. We will denote A, B by $X^{t\leq 0}$ and $X^{t\geq 1}$, respectively.
3. (1) will be called the *t-decomposition* of X. If $X = Y[i]$ for some $Y \in Obj\underline{C}$, $i \in \mathbb{Z}$, then we will denote A by $Y^{t\leq i}$ (it belongs to $\underline{C}^{t\leq 0}$) and B by $Y^{t\geq i+1}$ (it belongs to $\underline{C}^{t\geq 0}$), respectively. Sometimes we will denote $Y^{t^{\leq} i}[-i]$ by $t_{\leq i}Y$; $t_{\geq i+1}Y = Y^{t^{\geq} i+1}[-i-1]$. Objects of the type $Y^{t^{\leq} i}[j]$ and $Y^{t^{\geq} i}[j]$ (for $i,j \in \mathbb{Z}$) will be called *t-truncations of* Y.
4. We denote by $X^{t=i}$ the i-th cohomology of X with respect to t i.e. $(Y^{t\leq i})^{t\geq 0}$ (cf. part 10 of §IV.4 of [13]).
5. The following statements are obvious (and well-known): $\underline{C}^{t\leq 0} = {}^{\perp}\underline{C}^{t\geq 1}$; $\underline{C}^{t\geq 0} = \underline{C}^{t\leq -1\perp}$.

Now we recall the notion of idempotent completion.

Definition 1.1.4. An additive category B is said to be *idempotent complete* if for any $X \in ObjB$ and any idempotent $p \in B(X,X)$ there exists a decomposition $X = Y \bigoplus Z$ such that $p = i \circ j$, where i is the inclusion $Y \to Y \bigoplus Z$, j is the projection $Y \bigoplus Z \to Y$.

Recall that any additive B can be canonically idempotent completed. Its idempotent completion is (by definition) the category B' whose objects are (X,p) for $X \in ObjB$ and $p \in B(X,X) :\ p^2 = p$; we define

$$A'((X,p),(X',p')) = \{f \in B(X,X') :\ p'f = fp = f\}.$$

It can be easily checked that this category is additive and idempotent complete, and for any idempotent complete $C \supset B$ we have a natural full embedding $B' \to C$.
The main result of [1] (Theorem 1.5) states that an idempotent completion of a triangulated category $\underline{C}$ has a natural triangulation (with distinguished triangles being all retracts of distinguished triangles of $\underline{C}$).
Below we will need the notion of a Postnikov tower in a triangulated category several times (cf. §IV2 of [13])).

Definition 1.1.5. Let $\underline{C}$ be a triangulated category.
1. Let $l \leq m \in \mathbb{Z}$.
We will call a bounded Postnikov tower for $X \in Obj\underline{C}$ the following data: a sequence of $\underline{C}$-morphisms $(0 =)Y_l \to Y_{l+1} \to \cdots \to Y_m = X$ along with distinguished triangles

$$Y_i \to Y_{i+1} \to X_i \tag{2}$$

for some $X_i \in Obj\underline{C}$; here $l \leq i < m$.
2. An unbounded Postnikov tower for X is a collection of Y_i for $i \in \mathbb{Z}$ that is equipped (for all $i \in \mathbb{Z}$) with: connecting arrows $Y_i \to Y_{i+1}$ (for $i \in \mathbb{Z}$), morphisms $Y_i \to X$ such that all the corresponding triangles commute, and distinguished triangles (2).
In both cases, we will denote $X_{-p}[p]$ by X^p; we will call X^p the *factors* of out Postnikov tower.

Remark 1.1.6. 1. Composing (and shifting) arrows from triangles (2) for two subsequent i one can construct a complex whose terms are X^p (it is easily seen that this is a complex indeed, cf. Proposition 2.2.2 of [6]). This observation will be important for us below when we will consider certain weight complex functors.
2. Certainly, a bounded Postnikov tower could be easily completed to an unbounded one. For example, one could take $Y_i = 0$ for $i < l$, $Y_i = X$ for $i > m$; then $X^i = 0$ if $i < l$ or $i \geq m$.
Lastly, we recall the following (well-known) result.

Proposition 1.1.7. *For any small abelian category $\underline{A}$ there exists an exact faithful functor $\underline{A} \to Ab$.*

Proof. By the Freyd-Mitchell's embedding theorem, any small $\underline{A}$ could be fully faithfully embedded into $R - \text{mod}$ for some (associative unital) ring R. It remains to apply the forgetful functor $R - \text{mod} \to Ab$. □

Remark 1.1.8. 1. We will need this statement below in order to assume that objects of $\underline{A}$ 'have elements'; this will considerably simplify diagram chase. Note that we can assume the existence of elements for a not necessarily small $\underline{A}$ in the case when a reasoning deals only with a finite number of objects of $\underline{A}$ at a time.
2. In the proof it suffices to have a faithful embedding $\underline{A} \to R - \text{mod}$; this weaker assertion was also proved by Mitchell.

1.2 Extending cohomological functors from a triangulated subcategory

We describe a method for extending cohomological functors from a full triangulated $\underline{C}' \subset \underline{C}$ to $\underline{C}$ (after H. Krause). Note that below we will apply some of the results of [17] in the dual form. The construction requires $\underline{C}'$ to be skeletally small i.e. there should exist a (proper) subset $D \subset Obj\underline{C}'$ such that any object of $\underline{C}'$ is isomorphic to some element of D. For simplicity, we will sometimes (when writing sums over $Obj\underline{C}'$) assume that $Obj\underline{C}'$ is a set itself. Since the distinction between small and skeletally small categories will not affect our arguments and results, we will ignore it in the rest of the paper.

If $\underline{A}$ is an abelian category, then $\operatorname{AddFun}(\underline{C}'^{op}, \underline{A})$ is abelian also; complexes in it are exact whenever they are exact componentwisely.

Suppose that $\underline{A}$ satisfies AB5 i.e. it is closed with respect to all small coproducts, and filtered direct limits of exact sequences in $\underline{A}$ are exact.

Let $H' \in \operatorname{AddFun}(\underline{C}'^{op}, \underline{A})$ be an additive functor (it will usually be cohomological).

Proposition 1.2.1. *I Let $\underline{A}, H'$ be fixed.*

1. There exists an extension of H' to an additive functor $H : \underline{C} \to \underline{A}$. It is cohomological whenever H is. The correspondence $H' \to H$ defines an additive functor $\operatorname{AddFun}(\underline{C}'^{op}, \underline{A}) \to \operatorname{AddFun}(\underline{C}^{op}, \underline{A})$.

2. Moreover, suppose that in $\underline{C}$ we have a projective system X_l, $l \in L$, equipped with a compatible system of morphisms $X \to X_l$, such that the latter system for any $Y \in Obj\underline{C}'$ induces an isomorphism $\underline{C}(X, Y) \cong \varinjlim \underline{C}(X_l, Y)$. Then we have $H(X) \cong \varinjlim H(X_l)$.

II Let $X \in Obj\underline{C}$ be fixed.

1. One can choose a family of $X_l \in Obj\underline{C}$ and $f_l \in \underline{C}(X, X_l)$ such that (f_l) induce a surjection $\oplus H'(X_l) \to H(X)$ for any $H', \underline{A}$, and H as in assertion I1.

2. Let $F' \xrightarrow{f'} G' \xrightarrow{g'} H'$ be a (three-term) complex in $\operatorname{AddFun}(\underline{C}'^{op}, \underline{A})$ *that is exact in the middle; suppose that H' is cohomological. Then the complex $F \xrightarrow{f} G \xrightarrow{g} H$ (here F, G, H, f, g are the corresponding extensions) is exact in the middle also.*

Proof. I1. Following §1.2 of [17] (and dualizing it), we consider the abelian category $C = \underline{C}'^* = \operatorname{AddFun}(\underline{C}', Ab)$ (this is Mod $\underline{C}'^{op}$ in the notation of Krause). The definition easily implies that direct limits in C are exactly componentwise direct limits of functors. We have the Yoneda's functor $i' : \underline{C}^{op} \to C$ that sends $X \in Obj\underline{C}$ to the functor $X^* = (Y \mapsto \underline{C}(X, Y),\ Y \in Obj\underline{C}')$; it is obviously cohomological. We denote by i the restriction of i' to $\underline{C}'$ (i is opposite to a full embedding).

By Lemma 2.2 of [17] (applied to the category $\underline{C}'^{op}$) we obtain that there exists an exact functor $G : C \to \underline{A}$ that preserves all small coproducts and satisfies $G \circ i = H'$. It is constructed in the following way: if for $X \in Obj\underline{C}$ we have an exact sequence (in C)

$$\oplus_{j\in J} X_j^* \to \oplus_{l\in L} X_l^* \to X^* \to 0 \tag{3}$$

for $X_j, X_l \in C'$, then we set

$$G(X) = \operatorname{Coker} \oplus_{j\in J} H'(X_j) \to \oplus_{l\in L} H'(X_l). \tag{4}$$

We define $H = G \circ i'$; it was proved in loc.cit. that we obtain a well-defined functor this way. As was also proved in loc.cit., the correspondence $H' \mapsto H$ yields a functor; H is cohomological if H' is.
2. The proof of loc.cit. shows (and mentions) that G respects (small) filtered inverse limits. Now note that our assertions imply: $X^* = \varinjlim X_l^*$ in C.
II 1. This is immediate from (4).
2. Note that the assertion is obviously valid if $X \in Obj\underline{C}'$. We reduce the general statement to this case.
Applying Yoneda's lemma to (3) is we obtain (canonically) some morphisms $f_l : X \to X_l$ for all $l \in L$ and $g_{lj} : X_l \to X_j$ for all $l \in L$, $j \in J$, such that: for any $l \in L$ almost all g_{lj} are 0; for any $j \in J$ almost all g_{lj} is 0; for any $j \in J$ we have $\sum_{l\in L} g_{lj} \circ f_l = 0$.
Now, by Proposition 1.1.7, we may assume that $\underline{A} = Ab$ (see Remark 1.1.8). We should check: if for $a \in G(X)$ we have $g_*(a) = 0$, then $a = f_*(b)$ for some $b \in F(X)$.
Using additivity of $\underline{C}'$ and $\underline{C}$, we can gather finite sets of X_l and X_j into single objects. Hence we can assume that $a = G(f_{l_0})(c)$ for some $c \in G(X_l)$ ($= G'(X_l)$), $l_0 \in L$ and that $g_*(c) \in H(g_{l_0 j_0})(H(X_{j_0}))$ for some $j_0 \in J$, whereas $g_{l_0 j_0} \circ f_{l_0} = 0$. We complete $X_{l_0} \to X_{j_0}$ to a distinguished triangle $Y \xrightarrow{\alpha} X_{l_0} \xrightarrow{g_{l_0 j_0}} X_{j_0}$; we can assume that $B \in Obj\underline{C}'$. We obtain that f_{l_0} could be presented as $\alpha \circ \beta$ for some $\beta \in \underline{C}(X, Y)$. Since H' is cohomological, we obtain that $H(\alpha)(g_*(c)) = 0$. Since $Y \in Obj\underline{C}$, the complex $F(Y) \to G(Y) \to H(Y)$ is exact in the middle; hence $G(\alpha)(c) = f_*(d)$ for some $d \in F(Y)$. Then we can take $b = F(\beta)(d)$.

□

1.3 Some definitions of Voevodsky: reminder

We use much notation from [25]. We recall (some of) it here for the convenience of the reader, and introduce some notation of our own.
$Var \supset SmVar \supset SmPrVar$ will denote the class of all varieties over k, resp. of smooth varieties, resp. of smooth projective varieties.
We recall that for categories of geometric origin (in particular, for $SmCor$ defined below) the addition of objects is defined via the disjoint union of varieties operation.
We define the category $SmCor$ of smooth correspondences. $ObjSmCor = SmVar$, $SmCor(X, Y) = \bigoplus_U \mathbb{Z}$ for all integral closed $U \subset X \times Y$ that are finite over X and dominant over a connected component of X; the composition

of correspondences is defined in the usual way via intersections (yet, we do not need to consider correspondences up to an equivalence relation).

We will write $\dots \to X^{i-1} \to X^i \to X^{i+1} \to \dots$, for $X^l \in SmVar$, for the corresponding complex over $SmCor$.

$PreShv(SmCor)$ will denote the (abelian) category of additive cofunctors $SmCor \to Ab$; its objects are usually called *presheaves with transfers*.

$Shv(SmCor) = Shv(SmCor)_{Nis} \subset PreShv(SmCor)$ is the abelian category of additive cofunctors $SmCor \to Ab$ that are sheaves in the Nisnevich topology (when restricted to the category of smooth varieties); these sheaves are usually called *sheaves with transfers*.

$D^-(Shv(SmCor))$ will be the bounded above derived category of $Shv(SmCor)$.

For $Y \in SmVar$ (more generally, for $Y \in Var$, see §4.1 of [25]) we consider $L(Y) = SmCor(-, Y) \in Shv(SmCor)$. For a bounded complex $X = (X^i)$ (as above) we will denote by $L(X)$ the complex $\dots \to L(X^{i-1}) \to L(X^i) \to L(X^{i+1}) \to \dots \in C^b(Shv(SmCor))$.

$S \in Shv(SmCor)$ is called homotopy invariant if for any $X \in SmVar$ the projection $\mathbb{A}^1 \times X \to X$ gives an isomorphism $S(X) \to S(\mathbb{A}^1 \times X)$. We will denote the category of homotopy invariant sheaves (with transfers) by HI; it is an exact abelian subcategory of $SmCor$ by Proposition 3.1.13 of [25].

$DM_-^{eff} \subset D^-(Shv(SmCor))$ is the full subcategory of complexes whose cohomology sheaves are homotopy invariant; it is triangulated by loc.cit. We will need the *homotopy t*-structure on DM_-^{eff}: it is the restriction of the canonical t-structure on $D^-(Shv(SmCor))$ to DM_-^{eff}. Below (when dealing with DM_-^{eff}) we will denote it by just by t. We have $\underline{Ht} = HI$.

We recall the following results of [25].

Proposition 1.3.1. *1. There exists an exact functor $RC : D^-(Shv(SmCor)) \to DM_-^{eff}$ right adjoint to the embedding $DM_-^{eff} \to D^-(Shv(SmCor))$.*

2. $DM_-^{eff}(M_{gm}(Y)[-i], F) = \mathbb{H}^i(F)(Y)$ (the i-th Nisnevich hypercohomology of F computed in Y) for any $Y \in SmVar$.

3. Denote $RC \circ L$ by M_{gm}. Then the corresponding functor $K^b(SmCor) \to DM_-^{eff}$ could be described as a certain localization of $K^b(SmCor)$.

Proof. See §3 of [25]. □

Remark 1.3.2. 1. In [25] (Definition 2.1.1) the triangulated category DM_{gm}^{eff} (of *effective geometric motives*) was defined as the idempotent completion of a certain localization of $K^b(SmCor)$. This definition is compatible with a *differential graded enhancement* for DM_{gm}^{eff}; cf. §5.3 below. Yet in Theorem 3.2.6 of [25] it was shown that DM_{gm}^{eff} is isomorphic to the idempotent completion of (the categorical image) $M_{gm}(C^b(SmCor))$; this description of DM_{gm}^{eff} will be sufficient for us till §5.

2. In fact, RC could be described in terms of so-called Suslin complexes (see loc.cit.). We will not need this below. Instead, we will just note that RC sends $D^-(Shv(SmCor))^{t\le 0}$ to $DM_-^{eff\,t\le 0}$.

1.4 Some properties of Tate twists

Tate twisting in $DM_-^{eff} \supset DM_{gm}^{eff}$ is given by tensoring by the object $\mathbb{Z}(1)$ (it is often denoted just by $-(1)$). Tate twist has several descriptions and nice properties. We will only need a few of them; our main source is §3.2 of [25]; a more detailed exposition could be found in [20] (see §§8–9).

In order to calculate the tensor product of $X, Y \in Obj DM_-^{eff}$ one should take any preimages X', Y' of X, Y in $Obj D^-(Shv(SmCor))$ with respect to RC (for example, one could take $X' = X$, $Y' = Y$); next one should resolve X, Y by direct sums of $L(Z_i)$ for $Z_i \in SmVar$; lastly one should tensor these resolutions using the identity $L(Z) \otimes L(T) = L(Z \times T)$ for $Z, T \in SmVar$, and apply RC to the result. This tensor product is compatible with the natural tensor product for $K^b(SmCor)$.

We note that any object $D^-(Shv(SmCor))^{t\le 0}$ has a resolution concentrated in negative degrees (the canonical resolution of the beginning of §3.2 of [25]). It follows that $DM_-^{eff\,t\le 0} \otimes DM_-^{eff\,t\le 0} \subset DM_-^{eff\,t\le 0}$ (see Remark 1.3.2(2); in fact, there is an equality since $\mathbb{Z} \in Obj HI$).

Next, we denote $\mathbb{A}^1 \setminus \{0\}$ by G_m. The morphisms $pt \to G_m \to pt$ (the point is mapped to 1 in G_m) induce a splitting $M_{gm}(G_m) = \mathbb{Z} \oplus \mathbb{Z}(1)[1]$ for a certain (*Tate*) motif $\mathbb{Z}(1)$; see Definition 3.1 of [20]. For $X \in Obj DM_-^{eff}$ we denote $X \otimes \mathbb{Z}(1)$ by $X(1)$.

One could also present $\mathbb{Z}(1)$ as $\mathrm{Cone}(pt \to G_m)[-1]$; hence the Tate twist functor $X \mapsto X(1)$ is compatible with the functor $- \otimes (\mathrm{Cone}(pt \to G_m)[-1])$ on $C^b(SmCor)$ via M_{gm}. We also obtain that $DM_-^{eff\,t\le 0}(1) \subset DM_-^{eff\,t\le 1}$.

Now we define certain twists for functors.

Definition 1.4.1. For an $G \in \mathrm{AddFun}(DM_{gm}^{eff}, Ab)$, $n \ge 0$, we define $G_{-n}(X) = G(X(n)[n])$.

Note that this definition is compatible with those of §3.1 of [26]. Indeed, for $X \in SmVar$ we have $G_{-1}(M_{gm}(X)) = G(M_{gm}(X \times G_m))/G(M_{gm}(X)) = \mathrm{Ker}(G(M_{gm}(X \times G_m)) \to G(M_{gm}(X)))$ (with respect to natural morphisms $X \times pt \to X \times G_m \to X \times pt$); G_{-n} for larger n could be defined by iterating $-_{-1}$.

Below we will extend this definition to (co)motives of pro-schemes.

For $F \in Obj DM_-^{eff}$ we will denote by F_* the functor $X \mapsto DM_-^{eff}(X, F) : DM_{gm}^{eff} \to Ab$.

Proposition 1.4.2. *Let $X \in SmVar$, $n \ge 0$, $i \in \mathbb{Z}$.*

1. For any $F \in Obj DM_-^{eff}$ we have: $F_{*-n}(M_{gm}(X)[-i])$ is a retract of $\mathbb{H}^i(F)(X \times G_m^{\times n})$ (which can be described explicitly).

2. There exists a t-exact functor $T_n : DM_-^{eff} \to DM_-^{eff}$ such that for any $F \in Obj DM_-^{eff}$ we have $F_{*-n} \cong (T_n(F))_*$.

Proof. 1. Proposition 1.3.1 along with our description of $\mathbb{Z}(1)$ yields the result.
2. For F represented by a complex of $F^i \in Obj Shv(SmCor)$ ($i \in \mathbb{Z}$) we define $T_n(F)$ as the complex of $T_n(F^i)$, where $T_n : PreShv(SmCor) \to PreShv(SmCor)$ is defined similarly to $-_{-n}$ in Definition 1.4.1. $T_n(F^i)$ are sheaves since $T_n(F_i)(X)$, $X \in SmVar$, is a functorial retract of $F_i(X \times G_m^n)$. In order to check that we actually obtain a well-defined a t-exact functor this way, it suffices to note that the restriction of T_n to $Shv(SmCor)$ is an exact functor by Proposition 3.4.3 of [9].
Now, it suffices to check that T_n defined satisfies the assertion for $n = 1$. In this case the statement follows easily from Proposition 4.34 of [26] (note that it is not important whether we consider Zariski or Nisnevich topology by Theorem 5.7 of ibid.).

□

1.5 Pro-motives vs. comotives; the description of our strategy

Below we will embed DM_{gm}^{eff} into a certain triangulated category $\mathfrak{D}$ of *comotives*. Its construction (and computations in it) is rather complicated; in fact, the author is not sure whether the main properties of $\mathfrak{D}$ (described below) specify it up to an isomorphism. So, before working with co-motives we will (following F. Deglise) describe a simpler category of *pro-motives*. The latter is not needed for our main results (so the reader may skip this subsection); yet the comparison of the categories mentioned would clarify the nature of our methods.
Following §3.1 of [9], we define the category $\mathfrak{D}^{naive}$ as the additive category of naive i.e. formal (filtered) pro-objects of DM_{gm}^{eff}. This means that for any $X : L \to DM_{gm}^{eff}$, $Y : J \to DM_{gm}^{eff}$ we define

$$\mathfrak{D}^{naive}(\varprojlim_{l\in L} X_l, \varprojlim_{j\in J} Y_j) = \varprojlim_{j\in J} (\varinjlim_{l\in L} DM_{gm}^{eff}(X_l, Y_j)). \tag{5}$$

The main disadvantage of $\mathfrak{D}^{naive}$ is that it is not triangulated. Still, one has the obvious shift for it; following Deglise, one can define pro-distinguished triangles as (filtered) inverse limits of distinguished triangles in DM_{gm}^{eff}. This allows to construct a certain motivic coniveau exact couple for a motif of a smooth variety in §4.2 of [10] (see also §5.3 of [9]). This construction is parallel to the classical construction of coniveau spectral sequences (see §1 of [8]). One starts with certain 'geometric' Postnikov towers in DM_{gm}^{eff} (Deglise calls them *triangulated exact couples*). For $Z \in SmVar$ we consider filtrations $\varnothing = Z_{d+1} \subset Z_d \subset Z_{d-1} \subset \cdots \subset Z_0 = Z$; Z_i is everywhere of codimension $\geq i$ in Z for all i. Then we have a system of distinguished triangles relating $M_{gm}(Z \setminus Z_i)$ and $M_{gm}(Z \setminus Z_i \to Z \setminus Z_{i+1})$; this yields a Postnikov tower. Then one passes to the inverse limit of these towers in $\mathfrak{D}^{naive}$ (here the connecting morphisms are induced by the corresponding open embeddings). Lastly, the functorial form of the Gysin distinguished triangle for motives allows Deglise to identify $X_i = \varprojlim(M_{gm}(Z \setminus Z_i \to Z \setminus Z_{i+1}))$ with the product of shifted Tate twists of pro-motives of all points of Z of codimension i. Using the results of see §5.2 of [9] (the relation of pro-motives with cycle modules of M. Rost, see [24]) one can also compute the morphisms that connect X^i with X^{i+1}.

Next, for any cohomological $H : DM_{gm}^{eff} \to \underline{A}$, where $\underline{A}$ is an abelian category satisfying AB5, one can extend H to $\mathfrak{D}^{naive}$ via the corresponding direct limits. Applying H to the motivic coniveau exact couple one gets the classical coniveau spectral sequence (that converges to the H-cohomology of Z). This allows to extend the seminal results of §6 of [5] to a comprehensive description of the coniveau spectral sequence in the case when H is represented by $Y \in ObjDM_{-}^{eff}$ (in terms of the homotopy t-truncations of Y; see Theorem 6.4 of [11]).

Now suppose that one wants to apply a similar procedure for an arbitrary $X \in ObjDM_{gm}^{eff}$; say, $X = M_{gm}(Z^1 \xrightarrow{f} Z^2)$ for $Z^1, Z^2 \in SmVar$, $f \in SmCor(Z^1, Z^2)$. One would expect that the desired exact couple for X could be constructed from those for Z^j, $j = 1, 2$. This is indeed the case when f satisfies certain codimension restrictions; cf. §7.4 of [6]. Yet for a general f it seems to be quite difficult to relate the filtrations of distinct Z^j (by the corresponding Z_i^j). On the other hand, the formalism of weight structures and weight spectral sequences (developed in [6]) allows to 'glue' certain *weight* Postnikov towers for objects of a triangulated categories equipped with a weight structure; see Remark 4.1.2(3) below.

So, we construct a certain triangulated category $\mathfrak{D}$ that is somewhat similar to $\mathfrak{D}^{naive}$. Certainly, we want distinguished triangles in $\mathfrak{D}$ to be compatible with inverse limits that come from 'geometry'. A well-known recipe for this is: one should consider some category $\mathfrak{D}'$ where (certain) cones of morphisms are functorial and pass to (inverse) limits in $\mathfrak{D}'$; $\mathfrak{D}$ should be a localization of $\mathfrak{D}'$. In fact, $\mathfrak{D}'$ constructed in §5.3 below could be endowed with a certain (Quillen) model structure such that $\mathfrak{D}$ is its homotopy category. We will never use this fact below; yet we will sometimes call inverse limits coming from $\mathfrak{D}'$ homotopy limits (in $\mathfrak{D}$).

Now, in Proposition 4.3.1 below we will prove that cohomological functors $H : DM_{gm}^{eff} \to \underline{A}$ could be extended to $\mathfrak{D}$ in a way that is compatible with homotopy limits (those coming from $\mathfrak{D}'$). So one may say that objects of $\mathfrak{D}$ have the same cohomology as those of $\mathfrak{D}^{naive}$. On the other hand, we have to pay the price for $\mathfrak{D}$ being triangulated: (5) does not compute morphisms between homotopy limits in $\mathfrak{D}$. The 'difference' could be described in terms of certain higher projective limits (of the corresponding morphism groups in DM_{gm}^{eff}).

Unfortunately, the author does not know how to control the corresponding $\varprojlim^2$ (and higher ones) in the general case; this does not allow to construct a weight structure on a sufficiently large triangulated subcategory of $\mathfrak{D}$ if k

is uncountable (yet see §6.5, especially the last paragraph of it). In the case of a countable k only $\varprojlim^1$ is non-zero. In this case the morphisms between homotopy limits in $\mathfrak{D}$ are expressed by the formula (28) below. This allows to prove that there are no morphisms of positive degrees between certain Tate twists of comotives of function fields (over k). This immediately yields that one can construct a certain weight structure on the triangulated subcategory $\mathfrak{D}_s$ of $\mathfrak{D}$ generated by products of Tate twists of comotives of function fields (in fact, we also idempotent complete $\mathfrak{D}_s$). Now, in order to prove that $\mathfrak{D}_s$ contains DM^{eff}_{gm} it suffices to prove that the motif of any smooth variety X belongs to $\mathfrak{D}_s$. To this end it clearly suffices to decompose $M_{gm}(X)$ into a Postnikov tower whose factors are products of Tate twists of comotives of function fields. So, we lift the motivic coniveau exact couple (constructed in [10]) from $\mathfrak{D}^{naive}$ to $\mathfrak{D}$. Since cones in $\mathfrak{D}'$ are compatible with inverse limits, we can construct a tower whose terms are the homotopy limits of the corresponding terms of the geometric towers mentioned. In fact, this could be done for an uncountable k also; the difficulty is to identify the analogues of X_i in $\mathfrak{D}$. If k is countable, the homotopy limits corresponding to our tower are countable also. Hence (by an easy well-known result) the isomorphism classes of these homotopy limits could be computed in terms of the corresponding objects and morphisms in DM^{eff}_{gm}. This means: it suffices to compute X^i in $\mathfrak{D}^{naive}$ (as was done in [10]); this yields the result needed. Note that we cannot (completely) compute the $\mathfrak{D}$-morphisms $X^i \to X^{i+1}$; yet we know how they act on cohomology.

The most interesting application of the results described is the following one. We prove that there are no positive $\mathfrak{D}$-morphisms between (certain) Tate twists of comotives of smooth semi-local schemes (or *primitive schemes*, see below); this generalizes the corresponding result for function fields. It follows that these twists belong to the *heart* of the weight structure on $\mathfrak{D}_s$ mentioned. Therefore comotives of (connected) primitive schemes are retracts of comotives of their generic points. Hence the same is true for the cohomology of the comotives mentioned and also for the corresponding pro-motives. Also, the comotif of a function field contains as retracts twisted comotives of its residue fields (for all geometric valuations); this also implies the corresponding results for cohomology and pro-motives.

Remark 1.5.1. In fact, Deglise mostly considers pro-objects for Voevodsky's DM_{gm} and of DM^{eff}_{-}; yet the distinctions are not important since the full embeddings $DM^{eff}_{gm} \to DM_{gm}$ and $DM^{eff}_{gm} \to DM^{eff}_{-}$ obviously extend to full embedding of the corresponding categories of pro-objects. Still, the embeddings mentioned allow Deglise to extend several nice results for Voevodsky's motives to pro-motives.

2. One of the advantages of the results of Deglise is that he never requires k to be countable. Besides, our construction of weight Postnikov towers mentioned heavily relies on the functoriality of the Gysin distinguished triangle for motives (proved in [10]; see also Proposition 2.4.5 of [9]).

2 Weight structures: reminder, truncations, weight spectral sequences, and duality with t-structures

In §2.1 we recall basic definitions of the theory of weight structures (it was developed in [6]; the concept was also independently introduced in [23]). Note here that weight structures (usually denoted by w) are natural counterparts of t-structures. Weight structures yield weight truncations; those (vastly) generalize stupid truncations in $K(B)$: in particular, they are not canonical, yet any morphism of objects could be extended (non-canonically) to a morphism of their weight truncations. We recall several properties of weight structures in §2.2.

We recall *virtual t-truncations* for a (cohomological) functor $H : \underline{C} \to \underline{A}$ (for $\underline{C}$ endowed with a weight structure) in §2.3 (these truncations are defined in terms of weight truncations). Virtual t-truncations were introduced in §2.5 of [6]; they yield a way to present H (canonically) as an extension of a cohomological functor that is positive in a certain sense by a 'negative' one (as if H belonged to some triangulated category of functors $\underline{C} \to \underline{A}$ endowed with a t-structure). We study this notion further here, and prove that virtual t-truncations for a cohomological H could be characterized up to a unique isomorphism by their properties (see Theorem 2.3.1(III4)). In order to give some characterization also for the 'dimension shift' (connecting the positive and the negative virtual t-truncations of H), we introduce the notion of a *nice (strongly exact)* complex of functors. We prove that complexes of representable functors coming from distinguished triangles in $\underline{C}$ are nice, as well as those complexes that could be obtained from nice strongly exact complexes of functors $\underline{C}' \to \underline{A}$ for some small triangulated $\underline{C}' \subset \underline{C}$ (via the extension procedure given by Proposition 1.2.1).

In §2.4 we consider weight spectral sequences (introduced in §§2.3–2.4 of [6]). We prove that the derived exact couple for the weight spectral sequence $T(H)$ (for $H : \underline{C} \to \underline{A}$) could be naturally described in terms of virtual t-truncations of H. So, one can express $T(H)$ starting from E_2 (as well as the corresponding filtration of H^*) in these terms also. This is an important result, since the basic definition of $T(H)$ is given in terms of *weight Postnikov towers* for objects of $\underline{C}$, whereas the latter are not canonical. In particular, this result yields canonical functorial spectral sequences in classical situations (considered by Deligne; cf. Remark 2.4.3 of [6]; note that we do not need rational coefficients here).

In §2.5 we introduce the definition a *(nice) duality* $\Phi : \underline{C}^{op} \times \underline{D} \to \underline{A}$, and of (left) *orthogonal* weight and t-structures (with respect to Φ). The latter definition generalizes the notion of *adjacent* structures introduced in §4.4 of [6] (this is the case $\underline{C} = \underline{D}$, $\underline{A} = Ab$, $\Phi = \underline{C}(-,))$. If w is orthogonal to t then the virtual t-truncations (corresponding to w) of functors of the type $\Phi(-,Y)$, $Y \in Obj\underline{D}$, are exactly the functors 'represented via Φ' by the actual t-truncations of Y (corresponding to t). We also prove that (nice) dualities could be extended from $\underline{C}'$ to $\underline{C}$ (using Proposition 1.2.1). Note here that (to the knowledge of the author) this paper is the first one which considers 'pairings' of triangulated categories.

In §2.6 we prove: if w and t are orthogonal with respect to a nice duality, the weight spectral sequence converging to $\Phi(X,Y)$ (for $X \in Obj\underline{C}$, $Y \in Obj\underline{D}$) is naturally isomorphic (starting from E_2) to the one coming from t-truncations of Y. Moreover even when the duality is not nice, all E_r^{pq} for $r \geq 2$ and the filtrations corresponding to these spectral sequences are still canonically isomorphic. Here niceness of a duality (defined in §2.5) is a somewhat technical condition (defined in terms of nice complexes of functors). Niceness generalizes to pairings ($\underline{C} \times \underline{D} \to \underline{A}$) the axiom TR3 (of triangulated categories: any commutative square in $\underline{C}$ could be completed to a morphism of distinguished triangles; note that this axiom could be described in terms of the functor $\underline{C}(-,-) : \underline{C} \times \underline{C} \to Ab$). We also discuss some alternatives and predecessors of our methods and results.
In §2.7 we compare weight decompositions, virtual t-truncations, and weight spectral sequences corresponding to distinct weight structures (in possibly distinct triangulated categories, connected by an exact functor).

2.1 Weight structures: basic definitions

We recall the definition of a weight structure (see [6]; in [23] D. Pauksztello introduced weight structures independently and called them co-t-structures).

Definition 2.1.1 (Definition of a weight structure). A pair of subclasses $\underline{C}^{w\leq 0}, \underline{C}^{w\geq 0} \subset Obj\underline{C}$ for a triangulated category $\underline{C}$ will be said to define a weight structure w for $\underline{C}$ if they satisfy the following conditions:
(i) $\underline{C}^{w\geq 0}, \underline{C}^{w\leq 0}$ are additive and Karoubi-closed (i.e. contain all retracts of their objects that belong to $Obj\underline{C}$).
(ii) **"Semi-invariance" with respect to translations.**
$\underline{C}^{w\geq 0} \subset \underline{C}^{w\geq 0}[1]$; $\underline{C}^{w\leq 0}[1] \subset \underline{C}^{w\leq 0}$.
(iii) **Orthogonality.**
$\underline{C}^{w\geq 0} \perp \underline{C}^{w\leq 0}[1]$.
(iv) **Weight decomposition**.
For any $X \in Obj\underline{C}$ there exists a distinguished triangle

$$B[-1] \to X \to A \xrightarrow{f} B \tag{6}$$

such that $A \in \underline{C}^{w\leq 0}, B \in \underline{C}^{w\geq 0}$.

A simple example of a category with a weight structure is $K(B)$ for any additive B: positive objects are complexes that are homotopy equivalent to those concentrated in positive degrees; negative objects are complexes that are homotopy equivalent to those concentrated in negative degrees. Here one could also consider the subcategories of complexes that are bounded from above, below, or from both sides.
The triangle (6) will be called a *weight decomposition* of X. A weight decomposition is (almost) never unique; still we will sometimes denote any pair (A,B) as in (6) by $X^{w\leq 0}$ and $X^{w\geq 1}$. Besides, we will call objects of the type

$(X[i])^{w\le 0}[j]$ and $(X[i])^{w\ge 0}[j]$ (for $i,j\in\mathbb{Z}$) *weight truncations* of X. A shift of the distinguished triangle (6) by $[i]$ for any $i\in\mathbb{Z},\ X\in Obj\underline{C}$ (as well as any its rotation) will sometimes be called a *shifted weight decomposition*.
In $K(B)$ (shifted) weight decompositions come from stupid truncations of complexes.
We will also need the following definitions and notation.

Definition 2.1.2. Let $X\in Obj\underline{C}$.

1. The category $\underline{Hw}\subset\underline{C}$ whose objects are $\underline{C}^{w=0}=\underline{C}^{w\ge 0}\cap\underline{C}^{w\le 0}$, $\underline{Hw}(Z,T)=\underline{C}(Z,T)$ for $Z,T\in\underline{C}^{w=0}$, will be called the *heart* of the weight structure w.
2. $\underline{C}^{w\ge l}$ (resp. $\underline{C}^{w\le l}$, resp. $\underline{C}^{w=l}$) will denote $\underline{C}^{w\ge 0}[-l]$ (resp. $\underline{C}^{w\le 0}[-l]$, resp. $\underline{C}^{w=0}[-l]$).
3. We denote $\underline{C}^{w\ge l}\cap\underline{C}^{w\le i}$ by $\underline{C}^{[l,i]}$.
4. $X^{w\le l}$ (resp. $X^{w\ge l}$) will denote $(X[l])^{w\le 0}$ (resp. $(X[l-1])^{w\ge 1}$).
5. $w_{\le i}X$ (resp. $w_{\ge i}X$) will denote $X^{w\le i}[-i]$ (resp. $X^{w\ge i}[-i]$).
6. w will be called *non-degenerate* if
$$\cap_l\underline{C}^{w\ge l}=\cap_l\underline{C}^{w\le l}=\{0\}.$$
7. We consider $\underline{C}^b=(\cup_{i\in\mathbb{Z}}\underline{C}^{w\le i})\cap(\cup_{i\in\mathbb{Z}}\underline{C}^{w\ge i})$ and call it the class of *bounded* objects of $\underline{C}$.

   For $X\in\underline{C}^b$ we will usually take $w_{\le i}X=0$ for i small enough, $w_{\ge i}X=0$ for i large enough.

   We will also denote by $\underline{C}^b$ the corresponding full subcategory of $\underline{C}$.
8. We will say that $(\underline{C},w)$ is bounded if $\underline{C}^b=\underline{C}$.
9. We will call a Postnikov tower for X (see Definition 1.1.5) a *weight Postnikov tower* if all Y_i are some choices for $w_{\ge 1-i}X$. In this case we will call the complex whose terms are X^p (see Remark 1.1.6) a *weight complex* for X.

   We will call a weight Postnikov tower for X *negative* if $X\in\underline{C}^{w\le 0}$ and we choose $w_{\ge j}X$ to be 0 for all $j>0$ here.
10. $D\subset Obj\underline{C}$ will be called extension-stable if for any distinguished triangle $A\to B\to C$ in $\underline{C}$ we have: $A,C\in D\implies B\in D$.

    We will also say that the corresponding full subcategory is extension-stable.
11. $D\subset Obj\underline{C}$ will be called *negative* if for any $i>0$ we have $D\perp D[i]$.

Remark 2.1.3. 1. One could also dualize our definition of a weight Postnikov tower i.e. build a tower from $w_{\leq l}X$ instead of $w_{\geq l}X$. Our definition of a weight Postnikov tower is more convenient for our purposes since in §3.6 below we will consider $Y_i = j(Z_0 \setminus Z_i)$ instead of $= j(Z_0 \setminus Z_i \to Z_0)[-1]$. Yet this does not make much difference; see §1.5 of [6] and Theorem 2.2.1(12) below. In particular, our definition of the weight complex for X coincides with Definition 2.2.1 of ibid. Note also, that Definition 1.5.8 of ibid (of a weight Postnikov tower) contained both 'our' part of the data and the dual part.
2. Weight Postnikov towers for objects of $\underline{C}$ are far from being unique; their morphisms (provided by Theorem 2.2.1(15) below) are not unique also (cf. Remark 1.5.9 of [6]). Yet the corresponding weight spectral sequences for cohomology are unique and functorial starting from E_2; see Theorem 2.4.2 of ibid. and Theorem 2.4.2 below for more detail. In particular, all possible choices of a weight complex for X are homotopy equivalent (see Theorem 3.2.2(II) and Remark 3.1.7(3) in [6]).

2.2 Basic properties of weight structures

Now we list some basic properties of notions defined. In the theorem below we will assume that $\underline{C}$ is endowed with a fixed weight structure w everywhere except in assertions 18 – 20.

Theorem 2.2.1. *1. The axiomatics of weight structures is self-dual: if $\underline{D} = \underline{C}^{op}$ (so $Obj\underline{C} = Obj\underline{D}$) then one can define the (opposite) weight structure w' on $\underline{D}$ by taking $\underline{D}^{w'\leq 0} = \underline{C}^{w\geq 0}$ and $\underline{D}^{w'\geq 0} = \underline{C}^{w\leq 0}$.*

2. *We have*
$$\underline{C}^{w\leq 0} = \underline{C}^{w\geq 1\perp} \tag{7}$$
and
$$\underline{C}^{w\geq 0} = {}^{\perp}\underline{C}^{w\leq -1}. \tag{8}$$

3. *For any $i \in \mathbb{Z}$, $X \in Obj\underline{C}$ we have a distinguished triangle $w_{\geq i+1}X \to X \to w_{\leq i}X$ (given by a shifted weight decomposition).*

4. *$\underline{C}^{w\leq 0}$, $\underline{C}^{w\geq 0}$, and $\underline{C}^{w=0}$ are extension-stable.*

5. *All $\underline{C}^{w\leq i}$ are closed with respect to arbitrary (small) direct products (those, which exist in $\underline{C}$); all $\underline{C}^{w\geq i}$ and $\underline{C}^{w=i}$ are additive.*

6. *For any weight decomposition of $X \in \underline{C}^{w\geq 0}$ (see (6)) we have $A \in \underline{C}^{w=0}$.*

7. *If $A \to B \to C \to A[1]$ is a distinguished triangle and $A, C \in \underline{C}^{w=0}$, then $B \cong A \oplus C$.*

8. *If we have a distinguished triangle $A \to B \to C$ for $B \in \underline{C}^{w=0}$, $C \in \underline{C}^{w\leq -1}$ then $A \cong B \bigoplus C[-1]$.*

9. *If $X \in \underline{C}^{w=0}$, $X[-1] \to A \xrightarrow{f} B$ is a weight decomposition (of $X[-1]$), then $B \in \underline{C}^{w=0}$; $B \cong A \bigoplus X$.*

10. *Let $l \leq m \in \mathbb{Z}$, $X, X' \in Obj\underline{C}$; let weight decompositions of $X[m]$ and $X'[l]$ be fixed. Then any morphism $g : X \to X'$ can be completed to a morphism of distinguished triangles*

$$\begin{tikzcd}
w_{\geq m+1}X \arrow[r] \arrow[d, "a"] & X \arrow[r, "c"] \arrow[d, "g"] & w_{\leq m}X \arrow[d, "b"] \\
w_{\geq l+1}X' \arrow[r] & X' \arrow[r, "d"] & w_{\leq l}X'
\end{tikzcd} \tag{9}$$

This completion is unique if $l < m$.

11. *Consider some completion of a commutative triangle $w_{\geq m+1}X \to w_{\geq l+1}X \to X$ (that is uniquely determined by the morphisms $w_{\geq m+1}X \to X$ and $w_{\geq l+1}X \to X$ coming from the corresponding shifted weight decompositions; see the previous assertion) to an octahedral diagram:*

$$\begin{tikzcd}
w_{\leq l}X \arrow[dd, "{[1]}"] \arrow[dr, "{[1]}"] & & X \arrow[ll] \\
& w_{\geq l+1}X \arrow[ur] \arrow[dl] & \\
w_{[l+1,m]}X \arrow[rr, "{[1]}"] & & w_{\geq m+1}X \arrow[uu] \arrow[ul]
\end{tikzcd}$$

$$\begin{tikzcd}
w_{\leq l}X \arrow[dd, "{[1]}"] & & X \arrow[ll] \arrow[dl] \\
& w_{\leq m}X \arrow[ul] \arrow[dr, "{[1]}"] & \\
w_{[l+1,m]}X \arrow[rr, "{[1]}"] \arrow[ur] & & w_{\geq m+1}X \arrow[uu]
\end{tikzcd}$$

Then $w_{[l+1,m]}X \in \underline{C}^{[l+1,m]}$; all the distinguished triangles of this octahedron are shifted weight decompositions.

12. *For $X, X' \in Obj\underline{C}$, $l, l', m, m' \in \mathbb{Z}$, $l < m$, $l' < m'$, $l > l'$, $m > m'$, consider two octahedral diagrams: (11) and a similar one corresponding to the commutative triangle $w_{\geq m+1}X \to w_{\geq l+1}X \to X$ and $w_{\geq m'+1}X' \to w_{\geq l'+1}X \to X$ (i.e. we fix some choices of these diagrams). Then any $g \in \underline{C}(X, X')$ could be uniquely extended to a morphism of these diagrams. The corresponding morphism $h : w_{[l+1,m]}X \to w_{[l'+1,m']}X'$ is characterized uniquely by any of the following conditions:*

(i) there exists a $\underline{C}$-morphism i that makes the squares

$$\begin{tikzcd} w_{\geq l+1}X \arrow[r] \arrow[d, "i"] & X \arrow[d, "g"] \\ w_{\geq l'+1}X' \arrow[r] & X' \end{tikzcd} \tag{10}$$

and

$$\begin{tikzcd} w_{\geq l+1}X \arrow[r] \arrow[d, "i"] & w_{[l+1,m]}X \arrow[d, "h"] \\ w_{\geq l'+1}X' \arrow[r] & w_{[l'+1,m']}X' \end{tikzcd} \tag{11}$$

commutative.

(ii) there exists a $\underline{C}$-morphism j that makes the squares

$$\begin{tikzcd} X \arrow[r] \arrow[d, "g"] & w_{\leq m}X \arrow[d, "j"] \\ X' \arrow[r] & w_{\leq m'}X' \end{tikzcd} \tag{12}$$

and

$$\begin{tikzcd} w_{[l+1,m]}X \arrow[r] \arrow[d, "h"] & w_{\leq m}X \arrow[d, "j"] \\ w_{[l'+1,m']}X' \arrow[r] & w_{\leq m'}X' \end{tikzcd} \tag{13}$$

commutative.

13. *For any choice of $w_{\geq i}X$ there exists a weight Postnikov tower for X (see Definition 2.1.2(9)). For any weight Postnikov tower we have* $\mathrm{Cone}(Y_i \to X) \in \underline{C}^{w\leq -i}$*; $X^i \in \underline{C}^{w=0}$.*

14. *Conversely, any bounded Postnikov tower (for X) with $X^i \in \underline{C}^{w=0}$ is a weight Postnikov tower for it.*

15. *For $X, X' \in Obj\underline{C}$ and arbitrary weight Postnikov towers for them, any $g \in \underline{C}(X, X')$ can be extended to a morphism of Postnikov towers (i.e. there exist morphisms $Y_i \to Y'_i$, $X^i \to X'^i$, such that the corresponding squares commute).*

16. *For $X, X' \in \underline{C}^{w\leq 0}$, suppose that $f \in \underline{C}(X, X')$ can be extended to a morphism of (some of) their negative Postnikov towers that establishes an isomorphism $X^0 \to X'^0$. Suppose also that $X' \in \underline{C}^{w=0}$. Then f yields a projection of X onto X' (i.e. X' is a retract of X via f).*

17. *$\underline{C}^b$ is a Karoubi-closed triangulated subcategory of $\underline{C}$. w induces a non-degenerate weight structure for it, whose heart equals $\underline{Hw}$.*

18. *For a triangulated idempotent complete $\underline{C}$ let $D \subset Obj\underline{C}$ be negative. Then there exists a unique weight structure w on the Karoubization T of $\langle D\rangle$ in $\underline{C}$ such that $D \subset T^{w=0}$. Its heart is the Karoubization of the closure of D in $\underline{C}$ with respect to (finite) direct sums.*

19. *For the weight structure mentioned in the previous assertion, $T^{w\leq 0}$ is the Karoubization of the smallest extension-stable subclass of $Obj\underline{C}$ containing $\cup_{i\geq 0}D[i]$; $T^{w\geq 0}$ is the Karoubization of the smallest extension-stable subclass of $Obj\underline{C}$ containing $\cup_{i\leq 0}D[i]$.*

20. *For the weight structure mentioned in two previous assertions we also have*
$$T^{w\leq 0} = (\cup_{i<0}D[i])^{\perp};\ T^{w\geq 0} = {}^{\perp}(\cup_{i>0}D[i]).$$

Proof. 1. Obvious; cf. Remark 1.1.3 of [6] (and Remark 1.1.2 of ibid. for more detail).

2. These are parts 1 and 2 of Proposition 1.3.3 of ibid.

3. Obvious (since $[i]$ is exact up to change of signs of morphisms); cf. Remark 1.2.2 of ibid.

4. This is part 3 of Proposition 1.3.3 of ibid.

5. Obvious from the definition and parts 4 of loc.cit.

6. This is part 6 of Proposition 1.3.3 of ibid.

7. This is part 7 of loc.cit.

8. It suffices to note that $\underline{C}(B, C) = 0$, hence the triangle splits.

9. This is part 8 of loc.cit.

10. This is Lemma 1.5.1 of ibid.

11. The only non-trivial statement here is that $w_{[l+1,m]}X \in \underline{C}^{[l+1,m]}$ (it easily implies: the left hand side of the lower cap in (11) also yields a shifted weight decomposition). (11) yields distinguished triangles: $T_1 = (w_{\geq l+1}X \to w_{[l+1,m]}X \to w_{\geq m+1}X[1])$ and $T_2 = (w_{\leq l}X \to w_{[l+1,m]}X[1] \to w_{\leq m}X[1])$. Hence assertion 4 yields the result.

12. By assertion 10, g extends uniquely to a morphism of the following distinguished triangles: from $T_3 = (w_{\geq m+1}X \to X \to w_{\leq m}X)$ to $T_3' = (w_{\geq m'+1}X' \to X' \to w_{\leq m'}X)$, and from $T_4 = (w_{\geq l+1}X \to X \to w_{\leq l}X)$ to $T_4' = (w_{\geq l'+1}X' \to X' \to w_{\leq l'}X)$; next we also obtain a unique morphism from T_1 (as defined in the proof of the previous assertion) to its analogue T_1'. Putting all of this together: we obtain unique morphisms of all of the vertices of our octahedra, which are compatible with all the edges of the octahedra expect (possibly) those that belong to T_2 (as

defined above). We also obtain that there exists unique i and h that complete (10) and (11) to commutative squares.

Now, the morphism $w_{\le l}X \to w_{[l+1,m]}X$ could be decomposed into the composition of morphisms belonging to T_1 and T_3. Hence in order to verify that we have actually constructed a morphism of octahedral diagrams, it remains to verify the commutativity of the squares

$$\begin{CD} w_{\le m}X @>>> w_{\le l}X \\ @VgVV @VVjV \\ w_{\le m'}X' @>>> w_{\le l'}X' \end{CD} \tag{14}$$

and (13) i.e. we should check that the two possible compositions of arrows for each of the squares are equal. Now, assertion 10 implies: the compositions in question for (14) both equal the only morphism q that makes the square

$$\begin{CD} X @>>> w_{\le m}X \\ @VgVV @VVqV \\ X' @>>> w_{\le l'}X' \end{CD}$$

commutative. Similarly, the compositions for (13) both equal the only morphism r that makes the square

$$\begin{CD} w_{\ge l+1}X @>>> w_{[l+1,m]}X \\ @VVV @VVrV \\ X' @>>> w_{\le m'}X' \end{CD}$$

commutative. Here we use the part of the octahedral axiom that says that the square

$$\begin{CD} w_{\ge l+1}X @>>> w_{[l+1,m]}X \\ @VVV @VVV \\ X @>>> w_{\le m}X \end{CD}$$

is commutative (as well as the corresponding square for (X', l', m')).

Lastly, as we have already noted, the condition (i) characterizes h uniquely; for similar (actually, exactly dual) reasons the same is true for (ii). Since the morphism $w_{[l+1,m]}X \to w_{[l'+1,m']}X'$ coming from the morphism of the octahedra constructed satisfies both of these conditions, it is characterized by any of them uniquely.

13. Immediate from part 2 of (Proposition 1.5.6) of loc.cit (and also from assertion 11).

14. Immediate from Remark 1.5.9(2) of ibid.

15. Immediate from part 1 (of Remark 1.5.9) of loc.cit.

16. It suffices to prove that $\operatorname{Cone} f \in \underline{C}^{w\le -1}$. Indeed, then the distinguished triangle $X \xrightarrow{f} X' \to \operatorname{Cone} f$ necessarily splits.

    We complete the commutative triangle $X^{w\le -1} \to X'^{w\le -1} \to X^0 (= X'^0)$ to an octahedral diagram. Then we obtain $\operatorname{Cone} f \cong \operatorname{Cone}(X^{w\le -1} \to X'^{w\le -1})[1]$; hence $\operatorname{Cone} f \in \underline{C}^{w\le -1}$ indeed.

17. This is Proposition 1.3.6 of ibid.

18. By Theorem 4.3.2(II1) of ibid., there exists a unique weight structure on $\langle D\rangle$ such that $D \subset \langle D\rangle^{w=0}$. Next, Proposition 5.2.2 of ibid. yields that w can be extended to the whole T; along with part Theorem 4.3.2(II2) of loc.cit. it also allows to calculate $T^{w=0}$ in this case.

19. Immediate from Proposition 5.2.2 of ibid. and the description of $\langle H\rangle^{w\le 0}$ and $\langle H\rangle^{w\ge 0}$ in the proof of Theorem 4.3.2(II1) of ibid.

20. If $X \in T^{w\le 0}$ then the orthogonality condition for w immediately yields: $Y \perp X$ for any $Y \in \cup_{i<0} D[i]$.

    Conversely, suppose that for some $X \in ObjT$ we have $Y \perp X$ for all $Y \in \cup_{i<0} D[i]$. Then $Y \perp X$ also for all Y belonging to the smallest extension-stable subclass of $Obj\underline{C}$ containing $\cup_{i<0} D[i]$. Hence this is also true for all $Y \in T^{w\ge 1}$ (see the previous assertion). Hence (7) yields: $X \in T^{w\le 0}$. We obtain the first part of the assertion.

    The second part of the assertion is dual to the first one (and easy from (8)).

$\square$

Remark 2.2.2. 1. In the notation of assertion 10, for any a (resp. b) such that the left (resp. right) hand square in (9) commutes there exists some b (resp. some a) that makes (9) a morphism of distinguished triangles (this is just axiom TR3 of triangulated categories). Hence for $l < m$ the left (resp. right) hand side of (9) characterizes a (resp. b) uniquely.

2. Assertions 10 and 12 yield mighty tools for proving that a construction described in terms of weight decompositions is functorial (in a certain sense). In particular, the proofs of functoriality of weight filtration and virtual t-truncations for cohomology (we will consider these notions below) in [6] were based on assertion 10.

   Now we explain what kind of functoriality could be obtained using assertion loc.cit. Actually, such an argument was already used in the proof of assertion 12.

   In the notation of assertion 10 we will say that a and b are compatible with g (with respect to the corresponding weight decompositions). Now suppose that for some $X'' \in Obj\underline{C}$, some $n \le l$, $g' \in \underline{C}(X', X'')$, and a distinguished triangle $w_{\geq n+1}X'' \to X' \to w_{\leq n}X'$ we have morphisms $a' : w_{\geq l+1}X' \to w_{\geq n+1}X''$ and $b' : w_{\leq l}X' \to w_{\leq n}X''$ compatible with g'. Then $a' \circ a$ and $b' \circ b$ are compatible with $g' \circ g$ (with respect to the corresponding weight decompositions)! Moreover, if $n < m$ then $(a' \circ a, b' \circ b)$ is exactly the (unique!) pair of morphisms compatible with $g' \circ g$.

3. In the notation of assertion 12 we will (also) say that $h : w_{[l+1,m]}X \to w_{[l'+1,m']}X'$ is compatible with g. Note that h is uniquely characterized by (i) (or (ii)) of loc.cit.; hence in order to characterize it uniquely it suffices to fix g and all the rows in (10) and (11) (or in (12) and (13)). Besides, we obtain that h is functorial in a certain sense (cf. the reasoning above).

4. Assertion 11 immediately implies: for any $l < m$ the class of all possible $w_{\leq l}X$ coincides with the class of possible $w_{\leq l}(w_{\leq m}X)$, whereas the class of possible $w_{\geq m}X$ coincides with those of $w_{\geq m}(w_{\geq l}X)$.

   Besides, assertion 11 also allows to construct weight Postnikov towers (cf. §1.5 of [6]). Hence $w_{[i,i]}X$ is just $X^i[-i]$ (for any $i \in \mathbb{Z},\ X \in Obj\underline{C}$), and a weight complex for any $w_{[l+1,m]}X$ can be assumed to be the corresponding stupid truncation of the weight complex of X.

5. Assertions 10 and 15 will be generalized in §2.7 below to the situation when there are two distinct weight structures; this will also clarify the proofs of these statements. Besides, note that our remarks on functoriality are also actual for this setting.

   Some of the proofs in §2.7 may also help to understand the concept of virtual t-truncations (that we will start to study just now) better.

2.3 Virtual t-truncations of (cohomological) functors

Till the end of the section $\underline{C}$ will be endowed with a fixed weight structure w; $H : \underline{C} \to \underline{A}$ ($\underline{A}$ is an abelian category) will be a contravariant (usually, cohomological) functor. We will not consider covariant (homological) functors here; yet certainly, dualization is absolutely no problem.
Now we recall the results of §2.5 of [6] and develop the theory further.

Theorem 2.3.1. *Let $H : \underline{C} \to \underline{A}$ be a contravariant functor, $k \in \mathbb{Z}$, $j > 0$.*
I The assignments $H_1 = H_1^{kj} : X \to \operatorname{Im}(H(w_{\leq k}X) \to H(w_{\leq k+j}X))$ and $H_2 = H_2^{kj} : X \to \operatorname{Im}(H(w_{\geq k}X) \to H(w_{\geq k+j}X))$ define contravariant functors $\underline{C} \to \underline{A}$ that do not depend (up to a canonical isomorphism) from the choice of weight decompositions. We have natural transformations $H_1 \to H \to H_2$.
II Let $k' \in \mathbb{Z}$, $j' > 0$. Then there exist the following natural isomorphisms.
1. $(H_1^{kj})_1^{k'j'} \cong H_1^{\min(k,k'),\max(k+j,k'+j')-\min(k,k')}$.
2. $(H_2^{kj})_2^{k'j'} \cong H_2^{\min(k,k'),\max(k+j,k'+j')-\min(k,k')}$.

3. $(H_1^{kj})_2^{k'j'} \cong (H_2^{k'j'})_1^{kj} \cong \operatorname{Im}(H(w_{[k,k']}X) \to H(w_{[k+j,k'+j']}X))$. *Here the last term is defined using the connection morphism* $w_{[k+j,k'+j']}X \to w_{[k,k']}X$ *that is compatible with* id_X *in the sense of Remark 2.2.2(3); the last isomorphism is functorial in the sense described in loc.cit.*
III Let H *be cohomological,* $j = 1$*; let* k *be fixed.*
1. H_l $(l = 1, 2)$ *are also cohomological; the transformations* $H_1 \to H \to H_2$ *extend canonically to a long exact sequence of functors*

$$\cdots \to H_2 \circ [1] \to H_1 \to H \to H_2 \to H_1 \circ [-1] \to \dots \tag{15}$$

(i.e. the sequence is exact when applied to any $X \in Obj\underline{C}$*).*
2. $H_1 \cong H$ *whenever* H *vanishes on* $\underline{C}^{w \geq k+1}$*.*
3. $H \cong H_2$ *whenever* H *vanishes on* $\underline{C}^{w \leq k}$*.*
4. Let $H' \xrightarrow{f} H \xrightarrow{g} H''$ *be a (three-term) complex of functors exact in the middle such that:*
(i) H', H'' *are cohomological.*
(ii) for any $X \in Obj\underline{C}$ *we have* $\operatorname{Coker} g(X) \cong \operatorname{Ker} f(X[-1])$ *(we do not fix these isomorphisms).*
(iii) H' *vanishes on* $\underline{C}^{w \geq k+1}$*;* H'' *vanishes on* $\underline{C}^{w \leq k}$*.*
Then $H' \xrightarrow{f} H$ *is canonically isomorphic to* $H_1 \to H$*;* $H \xrightarrow{g} H''$ *is canonically isomorphic to* $H \to H_2$*.*

Proof. I This is Proposition 2.5.1(III1) of [6].
II Easily follows from Theorem 2.2.1, parts 11 and 12; see Remark 2.2.2.
III1. This is Proposition 2.5.1(III2) of [6].
2. If H vanishes on $\underline{C}^{w \geq k+1}$ then for any X we have $w_{\geq k+1}X = 0$; hence H_2 vanishes. Therefore in the long exact sequence $\cdots \to H_2(X[1]) \to H_1 \to H \to H_2(X) \to \dots$ given by assertion II1 we have $H_2(X[1]) \cong 0 \cong H_2(X)$; we obtain $H_1 \cong H$.
Conversely, suppose that $H_1 \cong H$. Let $X \in Obj\underline{C}^{w \geq k+1}$; we can assume that $w_{\leq k}X = 0$. Then we have $H(X) \cong H_1(X) = \operatorname{Im} H(w_{\leq k}X) \to H(w_{\leq k+1}X)) = 0$.
3. It suffices to apply assertion II1 to the dual functor $\underline{C}^{op} \to \underline{A}^{op}$; note that the axiomatics of abelian categories, triangulated categories, and weight structures are self-dual (see Remark 1.1.3(1) and Theorem 2.2.1(1)).
4. We should check that in the diagram

$$\begin{tikzcd} H_1' \arrow[r, "g"] \arrow[d, "h"] & H_1 \arrow[d] \\ H' \arrow[r] & H \end{tikzcd}$$

g and h are isomorphisms. Then $g \circ h^{-1}$ will yield the first isomorphism desired, whereas dualization will yield the remaining half of the statement.
Now, assertion III2 yields that g in isomorphism.

Next, for an $X \in Obj\underline{C}$ we choose some weight decompositions for $X[k]$ and $X[k+1]$ and consider the diagram

$$\begin{array}{ccccccc} H''((w_{\le k}X)[1]) & \longrightarrow & H'(w_{\le k}X) & \xrightarrow{l} & H(w_{\le k}X) & \longrightarrow & H''(w_{\le k}X) \\ & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \\ H''((w_{\le k+1}X)[1]) & \longrightarrow & H'(w_{\le k+1}X) & \xrightarrow{m} & H(w_{\le k+1}X) & \longrightarrow & H''(w_{\le k+1}X). \end{array}$$

By our assumptions, $H''((w_{\le k}X)[1]) \cong H''(w_{\le k}X) \cong H''((w_{\le k+1}X)[1]) \cong 0$; hence l is an isomorphism and m is a monomorphism. Hence the induced map $\operatorname{Im} a \to \operatorname{Im} b$ is an isomorphism; so h is an isomorphism (since its application to any $X \in Obj\underline{C}$ is an isomorphism).

□

Definition 2.3.2. [virtual t-truncations of H]
Let $k, m \in \mathbb{Z}$. For a (co)homological H we will call H_l^{k1}, $l = 1, 2$, $k \in \mathbb{Z}$, *virtual t-truncations* of H. We will often denote them simply by H_l; in this case we will assume $k = 0$ unless k is specified explicitly.
We denote the following functors $\underline{C} \to \underline{A}$: H_1^{k1}, $H_2^{k-1,1}$, $(H_2^{m1})_1^{k1}$, and $X \mapsto (H_1^{01})_2^{-11}(X[k])$ by $\tau_{\le k}H$, $\tau_{\ge k}H$, $\tau_{[m+1,k]}H$, and $H^{\tau=k}$, respectively. Note that all of these functors are cohomological if H is.

Remark 2.3.3. 1. Note that H often lies in a certain triangulated 'category of functors' $\underline{D}$ (whose objects are certain cohomological functors $\underline{C} \to \underline{A}$). We will axiomatize this below by introducing the notion of a duality $\Phi : \underline{C}^{op} \times \underline{D} \to \underline{A}$: if Φ is a duality then for any $Y \in Obj\underline{D}$ we have a cohomological functor $\Phi(-, Y) : \underline{C} \to \underline{A}$. It is also often the case when the virtual t-truncations defined are compatible with actual t-truncations with respect to some t-structure t on $\underline{D}$ (see below). Still, it is very amusing that these t-truncated functors as well as their transformations corresponding to t-decompositions (see Definition 1.1.1) can be described without specifying any $\underline{D}$ and Φ!
2. Below we will need an explicit description of the connecting morphisms in (15). We give it here (following the proof of Proposition 2.5.1 of [6]).
The transformation $H_1 \to H$ (resp. $H \to H_2$) for any k, j can be calculated by applying H to any possible choice either of $X \to w_{\le k}X$ or of $X \to w_{\le k+j}X$ (resp. of $w_{\ge k}X \to X$ or of $w_{\ge k+j}X \to X$) that comes from any possible choice the corresponding weight decomposition. The transformation $H_2 \to H_1 \circ [-1]$ for $j = 1$ is given by applying H to any possible choice either of the morphism $w_{\le k+1}X \to w_{\ge k+2}X[1]$ or of the morphism $w_{\le k}X \to w_{\ge k+1}X[1]$ that comes from any possible choice of a weight decomposition of $X[k]$.

Here we use the following trivial observation: for $\underline{A}$-morphisms $X_1 \xrightarrow{f_1} Y_1$ and $X_2 \xrightarrow{f_2} Y_2$ any $g : X_1 \to X_2$ (resp. $h : Y_1 \to Y_2$) is compatible with at most one morphism $i : \operatorname{Im} f_1 \to \operatorname{Im} f_2$; if such an i exists, we will say that it is induced by g (resp. by h). Certainly, here f_1 could be equal to id_{X_1} or f_2 could be equal to id_{X_2}.

3. For any k, j, and any $\underline{C}$-morphism $g : X \to Y$ the morphism $H_1(X) \to H_1(Y)$ (resp. $H_2(X) \to H_2(Y)$) is induced by any choice of either of the morphism $w_{\le k}X \to w_{\le k}Y$ or of $w_{\le k+j}X \to w_{\le k+j}Y$ (resp. of the morphism $w_{\ge k}X \to w_{\ge k}Y$ or of $w_{\ge k+j}X \to w_{\ge k+j}Y$) that is compatible with g with respect to the corresponding weight decomposition (in the sense of Remark 2.2.2(2)); see the proof of Proposition 2.5.1 of [6].

We would like to extend assertion III4 of Theorem 2.3.1 to a statement on a (canonical) isomorphism of long exact sequences of functors. To this end we need the following definition.

Definition 2.3.4. 1. We will call a sequence of functors $C = \dots \to H'' \circ [1] \stackrel{[1](h)}{\to} H' \stackrel{f}{\to} H \stackrel{g}{\to} H'' \stackrel{h}{\to} H' \circ [-1] \to \dots$ of contravariant functors $\underline{C} \to Ab$ a *strongly exact complex* if H', H, H'' are cohomological and $C(X)$ is a long exact sequence for any $X \in Obj\underline{C}$; here $[1](h)$ is the transformation induced by h.
2. We will also say that a strongly exact complex C is *nice* in H if the following condition is fulfilled:
For any distinguished triangle $T = A \stackrel{l}{\to} B \stackrel{m}{\to} C \stackrel{n}{\to} A[1]$ in $\underline{C}$ the natural morphism p:

$$\operatorname{Ker}((H'(A)\bigoplus H(B)\bigoplus H''(C)) \xrightarrow{\begin{pmatrix} f(A) & -H(l) & 0\\ 0 & g(B) & -H''(m)\\ -H'([-1](n)) & 0 & h(C)\end{pmatrix}} (H(A)\bigoplus H''(B)\bigoplus H'(C[-1]))) \stackrel{p}{\to} \operatorname{Ker}((H'(A)\bigoplus H(B)) \xrightarrow{f(A)\oplus -H(l)} H(A)) \text{ is epimorphic.} \tag{16}$$

Now we describe the connection of (16) with truncated realizations; our arguments will also somewhat clarify the meaning of this condition.

Theorem 2.3.5. *1. Let C be a strongly exact complex of functors that is nice in H; let $H' \stackrel{f}{\to} H \stackrel{g}{\to} H''$ (a 'piece' of C) satisfy the conditions of assertion III4 of Theorem 2.3.1. Then C is canonically isomorphic to (15).*
2. Let $X \to Y \to Z$ be a distinguished triangle in $\underline{C}$. Then $C = \dots \to \underline{C}(-, X) \to \underline{C}(-, Y) \to \underline{C}(-, Z) \to \dots$ is a strongly exact complex of functors $\underline{C} \to Ab$; it is nice in $\underline{C}(-, Y)$.
3. Let there exist a (skeletally) small full triangulated $\underline{C}' \subset \underline{C}$ such that the restriction of a strongly exact complex C to $\underline{C}'$ is nice in H. For $D \in Obj\underline{C}$ we consider the projective system $L(D)$ whose elements are (E, i) : $E \in Obj\underline{C}'$, $i \in \underline{C}(D, E)$; we set $(E, i) \ge (E', i')$ if $(E, i) = (E' \bigoplus E'', i' \oplus i'')$ for some $(E'', i'') \in L(D)$.
Suppose that for any $D \in \underline{C}$ and for $G = H'$ and $G = H$ we have

$$\varinjlim_{L(D)} (\operatorname{Im} G(i) : G(E) \to G(D)) = G(D); \tag{17}$$

here we also assume that these limits exist. Then C is nice on $\underline{C}$ also.
4. Let $\underline{C}' \subset \underline{C}$ be a (skeletally) small triangulated subcategory, let $\underline{A}$ satisfy AB5. Let $C' = \dots \to H' \to H \to H'' \to \dots$ be a strongly exact complex of functors $\underline{C}' \to \underline{A}$. We extend all its terms from $\underline{C}'$ to $\underline{C}$ by the method of Proposition 1.2.1 and denote the complex obtained by C; we carry on the notation for the terms and arrows from C' to C. Then C is a strongly exact complex also (and its terms are cohomological functors).
It is nice in H whenever C' is.

Proof. 1. It suffices to check that the isomorphism provided by Theorem 2.3.1(III4) is compatible with the coboundaries if (16) is fulfilled. We can assume $\underline{A} = Ab$; see Remark 1.1.8. Then (16) transfers into: for any $(x, y): x \in H'(A),\ y \in H(B),\ f(A)(x) = H(l)(y)$ there exists a

$$z \in H''(C) \text{ such that } g(B)(y) = H''(z) \text{ and } H([-1](n))(x) = h(C)(z). \quad (18)$$

We should prove: if the images of $x \in H_2(X)$ and of $y \in H''(X)$ in $H_2''(X)$ coincide, $w \in H_1(X[-1])$ and $t = H(X)(y) \in H'(X[-1])$ are their coboundaries, then w and t come from some (single) $u \in H_1'(X[-1])$.
We lift x to some $x' \in H(w_{\geq k+1}X)$. Then (16) (if we substitute $w_{\geq k+1}$ for A and X for B in it) implies the existence of some $v \in H'((w_{\leq k}X)[-1])$ whose image in $H'(X[-1])$ (resp. in $H(w_{\leq k}X[-1])$) coincides with t (resp. with the coboundary of x'). Hence we can take u being the image of v (in $H_1'(X[-1])$).
2. Since the bi-functor $\underline{C}(-,-)$ is (co)homological with respect to both arguments, C is a strongly exact complex indeed. It remains to note: (16) in this case just means that any commutative square can be completed to a morphism of distinguished triangles; so it follows from the corresponding axiom (TR3) of triangulated categories.
3. First suppose that $\underline{A} = Ab$ (or any other abelian category equipped with an exact faithful functor $\underline{A} \to Ab$ that respects small direct limits; note that below we will only need $\underline{A} = Ab$). Then we should check (18).
Now note: it suffices to prove that there exist $A', B' \in Obj\underline{C}',\ l' \in \underline{C}(A', B')$, $\alpha \in \underline{C}(A, A'),\ \beta \in \underline{C}(B, B'),\ x' \in H'(A'),\ g' \in H(B')$ such that:

$$x = H'(\alpha)(x'),\ y = H(\beta)(y'),\ l' \circ \alpha = \beta \circ l,\ f(A')(x') = H(l')(y'). \quad (19)$$

Indeed, denote $C' = \operatorname{Cone}(l')$; denote by γ some element of $\underline{C}(C, C')$ that completes

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' \end{array}$$

to a morphism of triangles. Let $z' \in H''(C')$ be some element satisfying the obvious analogue of (18). Then $h = H''(\gamma)(h')$ is easily seen to satisfy (18).
Now we construct $A', B', \dots$ as desired. Note that in this case the assumption (17) is equivalent to: for any $t \in G(D)$ there exist $E \in Obj\underline{C}'$, $s \in G(D)$, and

$r \in \underline{C}(D,E)$, such that $t = G(r)(s)$ (since $\underline{C}'$ is additive). So, we can choose $A' \in Obj\underline{C}'$, $\alpha \in \underline{C}(A,A')$, $x' \in H'(A')$ such that $x = H'(\alpha)(x')$. We complete $q = \alpha \oplus l \in \underline{C}(A, A' \bigoplus B)$ to a distinguished triangle $A \to A' \bigoplus B \overset{p=p_1 \oplus p_2}{\to} D$. Since $H(q)((-H'(f(A')(x'), y)) = 0$, there exists an $s \in H(D)$ such that $H(p)(s) = (-H'(f(A')(x'), y)$ (recall that H is cohomological on $\underline{C}$). So, we have $H(p_2)(s) = y$, $-H(p_1)(s) = f(A')(X')$, $p_2 \circ l = -p_1 \circ \alpha$.
D fits for B' if it lies in $Obj\underline{C}'$. In the general case using (17) again, we choose $B' \in Obj\underline{C}'$, $\delta \in \underline{C}(D, B')$, $g' \in H(Y)$, such that $s = H(\delta)(g')$. Then it is easily seen that taking $l' = -\delta \circ p_1$, $\beta = \delta \circ p_2$, we complete the choice of a set of data satisfying (19).
This argument can be modified to work for a general $\underline{A}$. To this end we separate those parts of the reasoning where we used the fact that H is cohomological from those where we deal with limits; this allows us to 'work as if $\underline{A} = Ab$'.
We denote $\operatorname{Ker}(H'(A) \bigoplus H(B)) \to H(A))$ (with respect to the morphism in (16) by $S(A,B)$, and $\operatorname{Ker}(H'(A) \bigoplus H(B) \bigoplus H''(C)) \to H(A) \bigoplus H''(B) \bigoplus H'(C[-1])$ by $T(A,B,C)$.
Then we have a commutative diagram

$$\begin{array}{ccc}
\varinjlim(\operatorname{Im}(T(A',B',C') \to T(A,B,C))) & \xrightarrow{t'} & \varinjlim(\operatorname{Im}(S(A',B') \to S(A,B))) \\
\downarrow & & \downarrow{\scriptstyle i} \\
T(A,B,C) & \xrightarrow{t} & S(A,B)
\end{array}$$

here the first direct limit above is taken with respect to morphisms of triangles $(A \to B \to C) \to (A' \to B' \to C')$ for $A', B', C' \in Obj\underline{C}'$ (the ordering is similar to those of (17)); the second limit is taken similarly with respect to morphisms $(A \to B) \to (A' \to B')$ for $A', B' \in Obj\underline{C}'$. Since the restriction of C to $\underline{C}'$ is nice in H, for all A', B', C' the morphism $T(A', B', C') \to S(A', B')$ is epimorphic; hence t' is epimorphic. Therefore, it suffices to prove that i is epimorphic.
Now let us fix $A' = A_0$ and $\alpha = \alpha_0$. We use the notation introduced above; denote the preimage of $\operatorname{Im}(H'(\alpha) : H'(A') \to H'(A))$ with respect to the natural morphism $S(A,B) \to H'(A)$ by J. Then J equals $\operatorname{Im}(H'(A') \times H(D) \to S(A,B))$. Indeed, here we can apply Proposition 1.1.7 (see Remark 1.1.8) and then apply the reasoning 'with elements' used above.
In any $\underline{A}$ we obtain: since $\Phi(D,Y) = \varinjlim(\operatorname{Im}(\Phi(B',Y) \to \Phi(D,Y)))$, we obtain that $G = \varinjlim(\operatorname{Im}(S(A_0, B', X, Y) \to S(A, B, X, Y)))$. Here we use the following fact (valid in any abelian $\underline{A}$): if $J_i \subset J' \in Obj\underline{A}$, $\varinjlim J_i = J$ (for some projective system), $u : J' \to J$ is an $\underline{A}$-epimorphism, then $\varinjlim u(J_i) = J$.
Now, passing to the limit with respect to (A_0, α_0) (using (17)) finishes the proof.
4. C is a complex indeed since the extension procedure is functorial.
By Proposition 1.2.1(I1), all the terms of C are cohomological on $\underline{C}$. Also, part II2 of loc.cit. immediately implies that C is exact (i.e. $C(X)$ is exact for any $X \in Obj\underline{C}$). Hence C is a strongly exact complex.

Obviously, if C is nice in H then C' also is.
Conversely, let C' be nice in H. Then Proposition 1.2.1(II1) implies that H' and H satisfy (17) (for all D). Hence C is nice in H by assertion 3.

□

2.4 Weight spectral sequences and filtrations; relation with virtual t-truncations

Definition 2.4.1. For an arbitrary $(\underline{C}, w)$ let $H : \underline{C} \to \underline{A}$ be a cohomological functor ($\underline{A}$ is any abelian category).
We define $W^i(H) : \underline{C} \to \underline{A}$ as $X \to \operatorname{Im}(H(w_{\leq i}X) \to H(X))$.

By Proposition 2.1.2(2) of [6], $W^i(H)(X)$ does not depend on the the choice of the weight decomposition of $X[i]$; it also defines a (canonical) subfunctor of $H(X)$.
Now recall that Postnikov towers yield spectral sequences for cohomology. We will denote $H(X[-i])$ by $H^i(X)$ (for $X \in Obj\underline{C}$). We will also use the notation of Definition 2.3.2.

Theorem 2.4.2. *Let $k, m \in \mathbb{Z}$.*
I1. For any weight Postnikov tower for X (see Definition 2.1.2(9)) there exists a spectral sequence $T = T(H, X)$ with $E_1^{pq}(T) = H^q(X^{-p})$ such that the map $E_1^{pq} \to E_1^{p+1q}$ is induced by the morphism $X^{-p-1} \to X^{-p}$ (coming from the tower). We have $T(H, X) \implies H^{p+q}(X)$ for any $X \in \underline{C}^b$.
One can construct it using the following exact couple: $E_1^{pq} = H^q(X^{-p})$, $D_1^{pq} = H^q(X^{w \geq 1-p})$.
2. T is (covariantly) functorial in H; it is contravariantly $\underline{C}$-functorial in X starting from E_2.
3. Denote the step of filtration given by $(E_1^{l,m-l} : l \geq -k)$ on $H^m(X)$ by $F^{-k}H^m(X)$. Then $F^{-k}H^m(X) = (W^k H^m)(X)$.
II The derived exact couple for $T(H, X)$ can be naturally calculated in terms of virtual t-truncations of H in the following way: $E_2^{pq} \cong E_2'^{pq} = (H^q)^{\tau=-p}(X)$, $D_2^{pq} = D_2'^{pq} = (\tau_{\geq q}H)(X[1-p])$; the connecting morphisms of the couple $((E_2', D_2'))$ come from (15).
III1. $F^{-k}H^m(X) = \operatorname{Im}((\tau_{\leq k}H^m)(X) \to H^m(X))$ (with respect to the connecting morphism mentioned in Theorem 2.3.1(I)).
2. For any $r \geq 2$, $p, q \in \mathbb{Z}$ there exists a functorial isomorphism $E_r^{pq} \cong (F^p(\tau_{[-p+2-r,-p+r-2]}H)^q)^p / F^{p+1}(\tau_{[-p+2-r,-p+r-2]}H)^q)^p$.

Proof. I This is Theorem 2.4.2 of [6]; see also Remark 2.4.1 of ibid. for the discussion of exact couples.
In fact, assertion 1 follows easily from well known properties of Postnikov towers and of related spectral sequences.
II Since virtual t-truncations are functorial, the exact couple (D_2', E_2') is functorial also.

The definitions of the derived exact couple and of the virtual t-truncations imply immediately that D_2^{pq} and their connecting maps are exactly $D_2'^{pq}$ (and their connecting morphisms) specified in the assertion.
It remains to compare E_2 with E_2', and also the connecting maps of exact couples starting and ending in E_2 with those for E_2'. It suffices to consider $p = q = 0$. Our strategy is the following one. First we construct an isomorphism $E_2^{00} \to E_2'^{00}$; our construction depends on some choices. Then we prove that the isomorphism constructed is actually natural (in particular, it does not depend on the choices made). Lastly we verify that the isomorphisms of the terms of the exact couples constructed is compatible with the connecting morphisms of these couples. Note that in this (last) part of the argument we can make those choices (of certain weight decompositions) that we like.
By the definition of the derived exact couple we have: E_2^{00} is the 0-th cohomology of the complex $(H(X^{-j}))$ (for any choice of the weight complex (X^i)). $E_2'^{00}$ is the image of $H(k)$ where $k \in \underline{C}(w_{[0,1]}X, w_{[-1,0]}X)$ is any morphism that is compatible with id_X with respect to the corresponding weight decompositions (see see Theorem 2.3.1(II3) and Remark 2.2.2(3)). So, we should compare a subfactor of $H(X^0)$ with a subobject of $H(w_{[0,1]}X)$.
Now suppose that we are given an octahedral diagram containing a commutative triangle $w_{[1,1]}X \to w_{[0,1]}X \to w_{[-1,1]}X$ (see Theorem 2.2.1(11)). We could obtain it as follows: fix some $w_{[-1,1]}X$; then choose certain $w_{[0,1]}X = w_{\geq 0}(w_{[-1,1]}X)$ and $w_{[1,1]}X = w_{\geq 1}(w_{[-1,1]}X)$ (see Remark 2.2.2(4)). For any possible completion of the commutative triangle $w_{[1,1]}X \to w_{[0,1]}X \to w_{[-1,1]}X$ to an octahedral diagram, the remaining vertices of the octahedron are certain $w_{[-1,0]}X$, $w_{[0,0]}X = X^0$, and $w_{[-1,-1]}X = X^{-1}[1]$ (by Theorem 2.2.1(11)). We obtain morphisms $w_{[0,1]}X \xrightarrow{i} X^0 \xrightarrow{j} w_{[-1,0]}X$ such that $k = j \circ i$. Moreover, $\operatorname{Im}(H(X^1) \to H(X^0)) = \operatorname{Ker} H(i)$. Hence $H(i)$ induces some monomorphism $\alpha : H(X^0)/\operatorname{Im}(H(X^1) \to H(X^0))$ to $H(w_{[0,1]}X)$. Besides, $\operatorname{Ker}(H(X^0) \to H(X^{-1})) = \operatorname{Im} H(j)$; therefore the restriction of α to $\alpha^{-1}(\operatorname{Im} H(k))$ yields an isomorphism $\beta : E_2^{00} \to E_2'^{00}$.
Now we verify that the isomorphism constructed is natural.
Note that it actually depends only on $w_{[0,1]}X \xrightarrow{i} X^0$ and $\operatorname{Im} H(k)$ (we used the remaining data only in order to verify that we actually obtain an isomorphism). So, suppose that we have $X' \in Obj\underline{C}$, $g \in \underline{C}(X, X')$, and some choice of $w_{\geq 0}X'$, $w_{\geq 1}X'$, and $w_{\geq 2}X'$. We have canonical connecting morphisms $w_{\geq 0}X' \to w_{\geq 1}X' \to w_{\geq 2}X'$ that are compatible with $id_{X'}$ with respect to the morphisms $w_{\geq l}X' \to X'$ $(l = 0, 1, 2)$. Applying Theorem 2.2.1(11), we obtain a choice of $w_{[0,1]}X' \xrightarrow{i'} X'^0$. We also fix some choice of $H(k')$ (in order to do this we fix some choice of $w_{\leq -1}X$ and of $w_{[-1,0]}X$). Note that all of these choices are necessarily compatible with some choice of the isomorphism $\beta' : E_2^{00}(X') \to E_2'^{00}(X')$ constructed as above (see 2.2.2(2)).
Now we choose some morphisms $g_l : w_{\geq l}X \to w_{\geq l}X'$, for $-1 \leq l \leq 2$, compatible with g (see Remark 2.2.2(2)). These choices could be extended to some morphisms $a : w_{[0,1]}X \to w_{[0,1]}X'$ and $b : X^0 \to X'^0$ (by extending morphisms

of arrows to morphism of distinguished triangles).
Now we verify the commutativity of the diagram

$$\begin{array}{ccc} w_{[0,1]}X & \xrightarrow{i} & X^0 \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\ w_{[0,1]}X' & \xrightarrow{i'} & X'^0 \end{array}$$

It follows from Theorem 2.2.1(10) applied to the morphism $g_0 : w_{\geq 0}X \to w_{\geq 0}X'$, $l = 1$, $m = 2$ (since both $b \circ i$ and $i' \circ a$ are compatible with g_0). Moreover, Remark 2.2.2(3) yields that $H(a)$ sends $H(k)$ to $H(k')$. We obtain a commutative diagram

$$\begin{array}{ccc} E_2^{00} & \xrightarrow{\beta} & E_2'^{00} \\ \downarrow & & \downarrow \\ E_2^{00}(H, X') & \xrightarrow{\beta'} & E_2'^{00}(H, X') \end{array}$$

Since $E_2^{00}(H,-)$ and $E_2'^{00}(H,-)$ are $\underline{C}^{op}$-functorial (and the vertical arrows in the diagram are exactly those that yield this functoriality; see Remark 2.3.3(3)), we obtain the naturality in question.
Now it remains to prove that the isomorphisms of terms of exact couples constructed above is compatible with the (two remaining) connecting morphisms of these couples.
First consider the morphisms $E_2^{00} \to D_2^{10}$. Recall (by the definition of the derived exact couple) that it is induced by any morphism $w_{\geq 0}X \to X^0$ that extends to a weight decomposition of $w_{\geq 0}X$ (here we consider E_2^{00} as a subfactor of $H(X^0)$). On the other hand, the morphism $E_2'^{00} \to D_2'^{10} = \operatorname{Im}(H(w_{\geq -1}X) \to H(w_{\geq 0}X))$ is induced by any possible choice of a morphism $w_{\geq 0}X \to w_{[0,1]}X$ that yields a weight decomposition of $w_{\geq 0}X[1]$ (by Remark 2.3.3(2); see also Remark 2.2.2(3)). Hence it suffices to note that the triangle $w_{\geq 0}X \to w_{[0,1]}X \xrightarrow{i} X^0$ is necessarily commutative by Remark 2.2.2.
It remains consider the morphism $D_2^{1,-1} \to E_2^{00}$. It is induced by the morphism $X^0 \to w_{\geq 1}X$ (that yields a weight decomposition of $w_{\geq 0}X$). The morphism $D_2'^{1,-1}(= \operatorname{Im}(H(w_{\geq 1}X)[1]) \to H(w_{\geq 2}X)[1])) \to E_2'^{00}$ is induced by the morphism $w_{[0,1]}X \to w_{\geq 2}X[1]$. Hence it suffices to construct a commutative square

$$\begin{array}{ccc} w_{[0,1]}X & \xrightarrow{i} & X^0 \\ \downarrow & & \downarrow \\ w_{\geq 2}X[1] & \longrightarrow & w_{\geq 1}X[1] \end{array}$$

By applying Theorem 2.2.1(11) to the commutative triangle $w_{\geq 2}X \to w_{\geq 1}X \to w_{\geq 0}X$ we obtain that there exists such a commutative square with a certain i_0

instead of i. Note that (by loc.cit.) i_0 yields a weight decomposition of $w_{[0,1]}X$. It suffices to verify that we may take i_0 for i i.e. that i_0 could be completed to an octahedral diagram one of whose faces yields some choice of the commutative triangle $w_{[1,1]}X \to w_{[0,1]}X \to w_{[-1,1]}X$. We take $w_{[1,1]}X = \operatorname{Cone} i_0[-1]$, choose some $w_{[-1,1]}X$ (coming from the same $w_{\leq 1}X$ as $w_{[0,1]}X$). By Remark 2.2.2(2) we obtain a unique commutative triangle $w_{[1,1]}X \to w_{[0,1]}X \to w_{[-1,1]}X$ that is compatible with $id_{w_{\leq 1}X}$ respect to the corresponding weight decompositions. It remains to apply Theorem 2.2.1(11).

III We can assume $k = m = 0$.

1. In the notation of Theorem 2.3.1 we consider the morphism of spectral sequences $M : T(H_1, X) \to T(H, X)$ (induced by $H_1 \to H$). Part II of loc.cit. implies: M is an isomorphism on E_2^{pq} for $p \geq -k$ and $E_2^{pq}(T(H_1, X)) = 0$ otherwise. The assertion follows immediately.

2. Similarly to the the previous reasoning, we have natural isomorphisms: $E_2^{pq}(T(\tau_{[2-r,r-2]}H, X) \cong E_2^{pq}(T(H, X))$ for $2-r \leq p \leq r-2$ and $= 0$ otherwise. It easily follows that $E_\infty^{pq}(T(\tau_{[2-r,r-2]}H, X) \cong E_r^{pq}(T(\tau_{[-p+2-r,-p+r-2]}H, X)$. The result follows immediately.

□

Remark 2.4.3. 1. The dual of assertion II is: if we consider the alternative exact couple for our weight spectral sequence (see Remark 2.1.3) then the derived exact couple can also be described in terms of virtual t-truncations (in a way that is dual in an appropriate sense to that of Theorem 2.4.2).

2. Possibly, at least a part of (assertion II of) the theorem could be proved by studying the functoriality of the derived exact couple (and applying Theorem 2.3.5(1)).

2.5 Dualities of triangulated categories; orthogonal weight and t-structures

Let $\underline{C}, \underline{D}$ be triangulated categories. We study certain pairings of triangulated categories $\underline{C}^{op} \times \underline{D} \to \underline{A}$. In the following definition we consider a general $\underline{A}$, yet below we will mainly need $\underline{A} = Ab$.

Definition 2.5.1. 1. We will call a (covariant) bi-functor $\Phi : \underline{C}^{op} \times \underline{D} \to \underline{A}$ a *duality* if it is bi-additive, homological with respect to both arguments; and is equipped with a (bi)natural transformation $\Phi(X, Y) \cong \Phi(X[1], Y[1])$.

2. We will say that Φ is *nice* if for any distinguished triangle $X \to Y \to Z$ the corresponding (strongly exact) complex of functors

$$\cdots \to \Phi(-, X) \to \Phi(-, Y) \to \Phi(-, Z) \xrightarrow{f} \Phi([-1](-), X) \to \dots \qquad (20)$$

is nice in $\Phi(-, Y)$ (see Definition 2.3.4); here f is obtained from the natural morphism $\Phi(-, Z) \to \Phi(-, X[1])$ by applying the (bi)natural transformation mentioned above.

3. Suppose that $\underline{C}$ is endowed with a weight structure w, $\underline{D}$ is endowed with a t-structure t. Then we will say that w is (left) *orthogonal* to t with respect to Φ if the following *orthogonality condition* is fulfilled:

$$\Phi(X,Y)=0 \text{ if: } X\in \underline{C}^{w\le 0} \text{ and } Y\in \underline{D}^{t\ge 1}, \text{ or } X\in \underline{C}^{w\ge 0} \text{ and } Y\in \underline{D}^{t\le -1}. \tag{21}$$

4. If w is defined on $\underline{C}^{op}$, t is defined on $\underline{D}^{op}$, w is left orthogonal to t (with respect to some duality); then we will say that the corresponding opposite weight structure on $\underline{C}$ is *right orthogonal* to the opposite t-structure for $\underline{D}$.

Remark 2.5.2. 1. The axioms of Φ immediately imply that (20) is a strongly exact complex of functors indeed (whether Φ is nice or not).
2. Certainly, if Φ is nice then (20) is nice at any term (since we can 'rotate' distinguished triangles in $\underline{D}$).

First we prove a statement that will simplify checking the orthogonality of weight and t-structures.

Proposition 2.5.3. *Let $\Phi:\underline{C}^{op}\times\underline{D}\to\underline{A}$ be some duality; let $(\underline{C},w)$ be bounded. Then w is (left) orthogonal to t whenever there exists a $D\subset\underline{C}^{w=0}$ such that any object of $\underline{C}^{w=0}$ is a retract of a finite direct sum of elements of D and*

$$\Phi(X,Y)=0\ \forall\ X\in D,\ Y\in\underline{D}^{t\ge 1}\cup\underline{D}^{t\le -1}. \tag{22}$$

Proof. If w is is left orthogonal to t, then (22) for $D=\underline{C}^{w=0}$ follows immediately from the orthogonality condition.
Conversely, let D satisfy the assumptions of our assertion. Hence we have: $\Phi(X,Y)=0$ if $X\in D[i],\ i\ge 0, Y\in\underline{D}^{t\ge 1}$, or if $X\in D[i],\ i\le 0, Y\in\underline{D}^{t\le -1}$.
Now suppose that for some $E,F\subset Obj\underline{C}$ we have: any object of $\underline{C}^{w\le 0}$ is a retract of an object of E, any object of $\underline{C}^{w\ge 0}$ is a retract of an object of F. Then it obviously suffices to check that $\Phi(X,Y)=0$ if either $X\in E$ and $Y\in\underline{D}^{t\ge 1}$ or $X\in F$ and $Y\in\underline{D}^{t\le -1}$.
Now by Theorem 2.2.1(19), we can take E being the smallest extension-stable subcategory of $\underline{C}$ containing $D[i]$, $i\ge 0$; and F being the smallest extension-stable subcategory of $\underline{C}$ containing $D[i]$, $i\le 0$. To conclude the proof it remains to note that for a distinguished triangle $X\to Y\to Z$ in $\underline{C}$, $O\in Obj\underline{D}$ we have: $\Phi(X,O)=0=\Phi(Z,O)\implies\Phi(Y,O)=0$. $\square$

When (weight and t-) structures are orthogonal, virtual t-truncations of $\Phi(-,Y)$ are given by t-truncations in $\underline{D}$. We use the notation of Definition 2.3.2.

Proposition 2.5.4. *1. Let t be orthogonal to w with respect to Φ, $k\in\mathbb{Z}$. For $Y\in Obj\underline{D}$ denote the functor $\Phi(-,Y):\underline{C}\to\underline{A}$ by H. Then we have an isomorphism of complexes $(\tau_{\le k}H\to H\to\tau_{\ge k}H)\cong(\Phi(-,t_{\le k}Y)\to H\to\Phi(-,t_{\ge k+1}Y))$ (where the connecting maps of the second complex are induced by t-truncations); this isomorphism is natural in Y.*

2. Suppose also that Φ is nice. Then the (strongly exact) complex of functors that sends X to

$$\cdots \to \Phi(X, t_{\leq k}Y) \to \Phi(X,Y) \to \Phi(X, t_{\geq k+1}Y) \to \Phi(X[-1], t_{\leq k}Y) \to \dots \tag{23}$$

(constructed as in the definition of a nice duality) is naturally isomorphic to (15).

Proof. 1. Since t and w orthogonal, $\Phi(-, t_{\leq k}Y)$ vanishes on $\underline{C}^{w \geq k+1}$, whereas $\Phi(-, t_{\geq k+1}Y)$ vanishes on $\underline{C}^{w \leq k}$. Moreover, (23) yields that $H' = \Phi(-, t_{\leq k}Y)$ and $H'' = \Phi(-, t_{\geq k+1}Y)$ also satisfy the condition (iii) of Theorem 2.3.1(III4). Hence the theorem yields the claim.
2. Immediate from the previous assertion and Theorem 2.3.5(1).

□

Remark 2.5.5. Note that we actually need quite a partial case of the 'niceness condition' for Φ in order to prove assertion 2. Hence here (and so, in all the applications below) we will not need the niceness condition in its full generality. Possibly, the corresponding partial case of the condition is weaker than the whole assertion; yet checking it does not seem to be much easier.
Also, it seems quite possible that for an arbitrary (not necessarily nice) duality there exists some isomorphism of (15) with (23) if we modify the boundary maps of the second complex. Yet there seems to be no way to choose such a modification canonically.

'Natural' dualities are nice; we will justify this thesis now.

Proposition 2.5.6. *1. If $\underline{A} = Ab$, $\underline{D} = \underline{C}$, then $\Phi : (X,Y) \mapsto \underline{C}(X,Y)$ is a nice duality.*
2. For some duality $\Phi : \underline{C}^{op} \times \underline{D} \to \underline{A}$ let there exist a (skeletally) small full triangulated $\underline{C}' \subset \underline{C}$ such that: the restriction of Φ to $\underline{C}'^{op} \times \underline{D}$ is a nice duality (of $\underline{C}'$ with $\underline{D}$); for any $X \in Obj\underline{D}$ the functor $G = \Phi(-,X)$, $\underline{C}^{op} \to \underline{A}$, satisfies (17). Then Φ is nice also.
3. For $\underline{D}$, $\underline{C}' \subset \underline{C}$ as above, $\underline{A}$ satisfying AB5, let $\Phi' : \underline{C}'^{op} \times \underline{D} \to \underline{A}$ be a duality. For any $Y \in Obj\underline{D}$ we extend the functor $\Phi'(-,Y)$ from $\underline{C}'$ to $\underline{C}$ by the method of Proposition 1.2.1; we denote the functor obtained by $\Phi(-,Y)$. Then the corresponding bi-functor Φ is a duality ($\underline{C}^{op} \times \underline{D} \to \underline{A}$). It is nice whenever Φ' is.

Proof. Immediate from parts 2–4 of Theorem 2.3.5.

□

Remark 2.5.7. 1. Proposition 2.5.6(1) yields an important family of nice dualities; this case was thoroughly studied in [6] (in sections 4 and 7). We will say that w is left (resp. right) *adjacent* to t if it is left (resp. right) orthogonal to it with respect to $\Phi(X,Y) = \underline{C}(X,Y)$. Note that for w left (resp. right) adjacent to t with respect to this definition we necessarily have $\underline{C}^{w \leq 0} = \underline{C}^{t \leq 0}$ (resp.

$\underline{C}^{w\geq 0} = \underline{C}^{t\geq 0}$) by Theorem 2.2.1(2) and Remark 1.1.3(2); so this definition is actually compatible with Definition 4.4.1 of [6].
One can generalize this family as in §8.3 of ibid.: for $\underline{A} = Ab$ and an exact $F : \underline{D} \to \underline{C}$ we define $\Phi(X,Y) = \underline{C}(X, F(Y))$. Certainly, one could also dualize this construction (in a certain sense) and consider $F : \underline{C} \to \underline{D}$ and $\Phi(X,Y) = \underline{C}(F(X), Y)$.
2. Another (general) family of dualities is mentioned in Remark 6.4.1(2) of ibid. All the families of dualities mentioned can be expanded using part 3 of the proposition.
3. It is also easy to construct a duality that is not nice. To this end one can start with $\underline{C} = \underline{D}$, $\Phi = \underline{C}(-,-)$ and then modify the choice of distinguished triangles in $\underline{D}$ (without changing the shift in $\underline{D}$, and changing nothing in $\underline{C}$) in a way that would not affect the properties of functors to be cohomological. The simplest way to do this is to proclaim a triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ to be distinguished in $\underline{D}$ if $X \xrightarrow{-f} Y \xrightarrow{-g} Z \xrightarrow{-h} X[1]$ is distinguished in $\underline{C}$. Certainly, such a modification is not very 'serious'; in particular, one can 'fix the problem' by multiplying the isomorphism $\Phi(X,Y) \cong \Phi(X[1], Y[1])$ by -1.
The author does not know whether any duality can be made nice by modifying the choice of the class of distinguished triangles (in $\underline{D}$), or by modifying the isomorphism mentioned. Note also that the question whether there exists a $\underline{D}$ for which such a modification can change the 'equivalence class' of triangulations is well-known to be open.

2.6 Comparison of weight spectral sequences with those coming from (orthogonal) t-truncations

Now we describe the relation of weight spectral sequences with orthogonal structures.

Theorem 2.6.1. *Let w for $\underline{C}$ and t for $\underline{D}$ be orthogonal with respect to a duality Φ; let $i, j \in \mathbb{Z}$, $X \in Obj\underline{C}$, $Y \in Obj\underline{D}$.*

1. *Consider the spectral sequence S coming from the following exact couple: $D_2^{pq}(S) = \Phi(X, Y^{t\geq q}[p-1])$, $E_2^{pq}(S) = \Phi(X, Y^{t=q}[p])$ (we start S from E_2). It naturally converges to $\Phi(X, Y[p+q])$ if $X \in \underline{C}^b$.*

2. *Let T be the weight spectral sequence given by Theorem 2.4.2 for the functor $H : Z \mapsto \Phi(Z, Y)$. Then for all $r \geq 2$ we have natural isomorphisms $E_r^{pq}(T(H,X)) \cong E_r^{pq}(S)$. There is also an equality $F^{-k}H^m(X) = \operatorname{Im}(\Phi(X, t_{\leq k}Y[m]) \to H^m(X))$ (here we use the notation of part I4 of loc.cit.) compatible with this isomorphism.*

3. *Suppose that Φ is also nice. Then the isomorphism mentioned in the previous assertion extends naturally to the isomorphism of of T with S (starting from E_2).*

4. *Let $\dots \to X^{-j-1} \to X^{-j} \to X^{1-j} \to \dots$ denote an arbitrary choice of the weight complex for X. Then we have a functorial isomorphism*

$$\Phi(X, Y^{t=i}[j]) \cong$$
$$(\operatorname{Ker}(\Phi(X^{-j}, Y[i]) \to \Phi(X^{-1-j}, Y[i]))/\operatorname{Im}(\Phi(X^{1-j}, Y[i]) \to \Phi(X^{-j}, Y[i])). \tag{24}$$

Proof. 1. The theory of t-structures easily yields: $Y^{t\geq q}$ and $Y^{t=q}$ can be functorially organized into a certain Postnikov tower for Y. Hence the usual results on spectral sequences coming from Postnikov towers (see §IV2, Exercise 2, of [13]) yield the assertion easily.

2. Immediate from Proposition 2.5.4(1) and Theorem 2.4.2(III). Note that the latter assertion does not use the 'dimension shift' in (15).

3. Proposition 2.5.4(2) and Theorem 2.4.2(II) imply: there is a natural isomorphism of the derived exact couple for T with the exact couple of S ('at level 2'). The result follows immediately.

4. This is just assertion 2 for E_2-terms.

□

Remark 2.6.2. 1. So, we justified parts 4 and 5 of Remark 4.4.3 of [6].

2. Note that the spectral sequence denoted by S in (Remark 4.4.3(4) and §6.4 of) ibid. started from E_1; so it differed from our S and T by a certain shift of indices.

3. So, we developed an 'abstract triangulated alternative' to the method of comparing similar spectral sequences that was developed by Deligne and Paranjape. The latter method used filtered complexes; it was applied in [22], [11], and in §6.4 of [6]. The disadvantage of this approach is that one needs extra information in order to construct the corresponding filtered complexes; this makes difficult to study the naturality of the isomorphism constructed. Moreover, in some cases the complexes required cannot exist at all; this is the case for the *spherical weight structure* and its adjacent Postnikov t-structure for $\underline{C} = \underline{D} = SH$ (the topological stable homotopy category; see §4.6 of [6]; yet in this case one can compare the corresponding spectral sequences using topology).

4. One could modify our reasoning to prove a version of the theorem that does not mention weight and t-structures. To this end instead of considering a weight Postnikov tower for X and the Postnikov tower coming from t-truncations of Y one should just compare spectral sequences coming from some Postnikov towers for X and Y in the case when these Postnikov towers satisfy those 'orthogonality' conditions (with respect to a (nice) duality Φ) that are implied by the orthogonality of structures

condition in our situation. Yet it seems difficult to obtain the naturality of the isomorphisms in Theorem 2.6.1(3) using this approach.

5. Even more generally, it suffices to have an inductive system of Postnikov towers in $\underline{D}$ and a projective system of Postnikov towers in $\underline{C}$ such that the orthogonality conditions required are satisfied in the (double) limit. Then the comparison statements for the double limits of the corresponding spectral sequences are valid also. A very partial (yet rather important) example of a reasoning of this sort is described in §7.4 of [6]. Besides, this approach could possibly yield the comparison result of §6 of [11] (even without assuming k to be countable as we do here).

6. A simple (yet important) case of (24) is: for any $i \in \mathbb{Z}$

$$X \in \underline{C}^{w=i} \implies \forall Y \in Obj\underline{D} \text{ we have } \Phi(X,Y) \cong \Phi(X, Y^{t=i}). \qquad (25)$$

2.7 'Change of weight structures'; comparing weight spectral sequences

Now we compare weight decompositions, virtual t-truncations, and weight spectral sequences corresponding to distinct weight structures. In order make our results more general (and to apply them below) we will assume that these structures are defined on distinct triangulated categories; yet the case when both are defined on $\underline{C}$ is also interesting.

So, till the end of the section we will assume: $\underline{C}, \underline{D}$ are triangulated categories endowed with weight structures w and v, respectively; $F : \underline{C} \to \underline{D}$ is an exact functor.

Definition 2.7.1. 1. We will say that F is *right weight-exact* if $F(\underline{C}^{w\ge 0}) \subset \underline{D}^{v\ge 0}$.
2. If F is fully faithful and right weight-exact, we will say that v *dominates* w.
3. We will say that F is *left weight-exact* if $F(\underline{C}^{w\le 0}) \subset \underline{D}^{v\le 0}$.
4. F will be called *weight-exact* if it is both right and left weight-exact.
We will say that w *induces* v (via F) if F is a weight-exact localization functor.

Proposition 2.7.2. *Let F be a right weight-exact functor; let $l \ge m \in \mathbb{Z}$, $X \in Obj\underline{D}$, $X' \in Obj\underline{C}$, $g \in \underline{D}(F(X'), X)$.*
1. Let weight decompositions of $X[m]$ with respect to v and $X'[l]$ with respect to w be fixed. Then g can be completed to a morphism of distinguished triangles

$$\begin{tikzcd} F(w_{\ge l+1}X') \arrow[r] \arrow[d, "a"] & F(X') \arrow[r] \arrow[d, "g"] & F(w_{\le l}X') \arrow[d, "b"] \\ v_{\ge m+1}X \arrow[r] & X \arrow[r] & v_{\le m}X \end{tikzcd} \qquad (26)$$

This completion is unique if $l > m$.

2. For arbitrary weight Postnikov towers $Po_v(X)$ for X (with respect to v) and Po_wX' for X' (with respect to w), g can be extended to a morphism $F_*(Po_wX') \to Po_v(X)$.
3. Let $H : \underline{D} \to \underline{A}$ be any functor, $k \in \mathbb{Z}$, $j > 0$. Denote $H \circ F$ by G. Then (26) allows to extend $H(g)$ naturally to a diagram

$$\begin{array}{ccccc} H_1^v(X) & \longrightarrow & H(X) & \longrightarrow & H_2^v(X) \\ \downarrow & & \downarrow{\scriptstyle H(g)} & & \downarrow \\ G_1^w(X') & \longrightarrow & G(X') & \longrightarrow & G_2^w(X') \end{array}$$

here we add the weight structure chosen as an index to the notation of Theorem 2.3.1(I).

Proof. 1. Since F is right weight-exact, $\underline{D}(F(w_{\geq n+1}X'), v_{\leq m}X) = \{0\}$ for any $n \geq m$. Hence the composition morphism $F(w_{\geq l+1}X') \to v_{\leq m}X$ is zero; if $l > m$ then $\underline{D}(F(w_{\geq l+1}X'), (v_{\leq m}X)[-1]) = \{0\}$. The result follows easily; see Proposition 1.1.9 of [2].
2. Assertion 1 (in the case $l = m$) yields that there exists a system of morphisms $f_i \in \underline{D}(F(w_{\geq i}X'), v_{\geq i}X)$ compatible with g; we fix such a system. Applying the same assertion for any pair of $l, m :\ l > m$, we obtain that f_l is compatible with f_m (here we use arguments similar to those described in Remark 2.2.2). Finally, since any commutative square can be extended to a morphism of the corresponding distinguished triangles (an axiom of triangulated categories), we obtain that we can complete (uniquely up to a non-canonical isomorphism) the data chosen to a morphism of Postnikov towers (i.e. choose a compatible system of morphisms $F(X'^i) \to X^i$).
3. Easy from assertion 1; note that for any commutative square in $\underline{A}$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle h} & & \downarrow \\ Z & \xrightarrow{g} & T \end{array}$$

if we fix the rows then the morphism $g \circ h : X \to T$ completely determines the morphism $\operatorname{Im} f \to \operatorname{Im} g$ induced by h.

□

We easily obtain a comparison morphism of weight spectral sequences.

Proposition 2.7.3. *I Let F, X', G be as in the previous proposition; suppose also that H is cohomological. Set $X = F(X')$, $g = id_X$.*
1. There exists some comparison morphism of the corresponding weight spectral sequences $M : T_v(H, X) \to T_w(G, X')$. Moreover, this morphism is unique and additively functorial (in g) starting from E_2.
2. Let there exist $D \subset \underline{C}^{w=0}$ such that any $Y \in \underline{C}^{w=0}$ is a retract of some $Z \in D$, and that for any $Z \in D$ there exists a choice of $Z^{w\geq 1}$ such that

$E_2^{pq}T_v(H, F(Z^{w\ge 1})) = \{0\}$ *for all* $p, q \in \mathbb{Z}$. *Then (any choice of)* M *yields an isomorphism of the spectral sequence functors starting from* E_2.
3. Let $\underline{E}$ *be a triangulated category endowed with a weight structure* u, $F' : \underline{D} \to \underline{E}$ *a right weight-exact functor; suppose that* $H = E \circ F'$ *for some cohomological functor* $E : \underline{E} \to \underline{A}$. *Then we have the following associativity property for comparison of weight spectral sequences: the composition of* M *with (any choice of) a comparison morphisms* $M' : T_u(E, F'(X)) \to T_v(H, X)$ *constructed as in assertion 1, starting from* E_2 *is canonically isomorphic to (any choice of a similarly constructed) comparison morphism* $T_u(E, F'(X)) \to T_w(G, X')$.
II Let H, X', X, G *be as above, but suppose that* $F : \underline{C} \to \underline{D}$ *is left weight-exact. Then a method dual to the one for assertion I1 yields a transformation* $N : T_w(G, X') \to T_v(H, X)$; *this construction satisfies the duals for all properties of* M *described in assertion I.*

Proof. I 1. In order to construct some comparison morphism, it suffices to construct a morphism of the corresponding exact couples that is compatible with id_X. Hence it suffices to use Proposition 2.7.2(2) to obtain a morphism of the corresponding Postnikov towers, and then apply H to it.
Theorem 2.4.2(II) yields that weight spectral sequences could be described in terms of the corresponding virtual t-truncations. Hence Proposition 2.7.2(3) implies all the functoriality properties of M listed.
2. It suffices to prove that M is an isomorphism on $E_2^{**}T_w(G, Y)$ for all $Y \in Obj\underline{C}$.
Since $D \subset \underline{C}^{w\ge 0}$, this assertion is true for any $Y \in D$. Since $Z \mapsto E_2(T(G, Z))$ is a cohomological functor for any weight structure (see Theorem 2.4.2 and the remark at Definition 2.3.2), the assertion is also true for any $Y \in Obj\underline{C}^b$. To conclude it suffices to note that for any H, any $Y \in Obj\underline{C}$, any finite 'piece' of $E_2^{**}T_w(G, Y)$ coincides with the corresponding piece of $E_2^{**}T_w(G, w_{[i,j]}Y)$ (for any choice of $w_{[i,j]}Y$) if i is small enough and j is large enough, and this isomorphism is compatible with M.
3. We recall that comparison morphisms for weight spectral sequences were constructed using Proposition 2.7.2(1). It easily follows that $M' \circ M$ is one of the possible choices for a comparison morphism $T_u(E, F' \circ F(X)) \to T_w(G, X')$. It suffices to apply assertion I1 to conclude that this fixed choice of a comparison morphism coincides with any other possible choice starting from E_2.
II We obtain the assertion from assertion I immediately by dualization (see Theorem 2.2.1(1)); here one should consider the duals of $\underline{C}$, $\underline{D}$, and $\underline{A}$ (and also 'dualize' the connecting functors). □

Remark 2.7.4. M is an isomorphism (starting from E_2) also if: there exists a localization of $\underline{D}$ such that H factorizes through it, v induces a weight structure v' on it, w induces a weight structure on the categorical image of $\underline{C}$ that coincides with the restriction of v' to it (since both weight spectral sequences are isomorphic to the spectral sequence corresponding to this new weight structure).

Yet this conditions are somewhat restrictive since weight structures do not 'descend' to localizations in general (since for an exact $F' : \underline{C} \to \underline{E}$ the classes $F'_*(\underline{C}^{w\geq 1})$ and $F'_*(\underline{C}^{w\leq 0})$ are not necessarily orthogonal in $\underline{E}$).

In order to simplify checking right and left weight-exactness of functors, we will need the following easy statement.

Lemma 2.7.5. *Let w be bounded.*
1. An exact $J : \underline{C} \to \underline{D}$ is a right weight-exact whenever there exists a $D \subset \underline{C}^{w=0}$ such that any $Y \in \underline{C}^{w=0}$ is a retract of some $X \in D$, and that for any $X \in D$ we have $J(Y) \in \underline{D}^{v\geq 0}$.
2. An exact $J : \underline{C} \to \underline{D}$ is a left weight-exact whenever there exists a $D \subset \underline{C}^{w=0}$ such that any $Y \in \underline{C}^{w=0}$ is a retract of some $X \in D$, and that for any $X \in D$ we have $J(Y) \in \underline{D}^{v\leq 0}$.

Proof. It suffices to prove assertion 1, since assertion 2 is exactly its dual.
If J is right weight-exact functor, then we can take $D = \underline{C}^{w=0}$
Now we prove the converse statement. Since $\underline{D}^{v\geq 0}$ is Karoubi-closed and extension-stable in $\underline{D}$, Theorem 2.2.1(19) yields that $J(\underline{C}^{w\geq 0})$ indeed belongs to $\underline{D}^{v\geq 0}$.

□

3 Categories of comotives (main properties)

We embed DM_{gm}^{eff} into a certain big triangulated motivic category $\mathfrak{D}$; we will call it objects *comotives*. We will need several properties of $\mathfrak{D}$; yet we will never use its description directly. For this reason in §3.1 we only list the main properties of $\mathfrak{D}$.

In §3.2 we associate certain comotives to (disjoint unions of) 'infinite intersections' of smooth varieties over k (we call those pro-schemes). We also introduce certain Tate twists for these comotives.

In §3.3 we recall the definition of a primitive scheme (note that in the case of a finite k we call a scheme primitive whenever it is smooth semi-local). The main motivic property of primitive schemes (proved by M. Walker) is: $F(S)$ injects into $F(S_0)$ if S is primitive connected, S_0 is its generic point, and F is a homotopy invariant presheaf with transfers.

In §3.4 we study the relation of (comotives of) primitive schemes with the homotopy t-structure for DM_-^{eff}.

In §3.5 we prove that there are no $\mathfrak{D}$-morphisms of positive degrees between comotives of primitive schemes (and also certain Tate twists of those); this is also true for products of comotives mentioned.

In §3.6 we prove that one can pass to countable homotopy limits in Gysin distinguished triangles; this yields Gysin distinguished triangles for comotives of pro-schemes. This allows to construct certain Postnikov towers for comotives of pro-schemes (and their Tate twists), whose factors are twisted products of comotives of function fields (over k). The construction of the tower is parallel to the classical construction of coniveau spectral sequences (see §1 of [8]).

3.1 Comotives: an 'axiomatic description'

We will define $\mathfrak{D}$ below as the derived category of differential graded functors $J \to B(Ab)$; here J yields a differential graded enhancement of DM_{gm}^{eff} (cf. [4], [19], or [7]), $B(Ab)$ is the differential graded category of complexes over Ab. We will also need some category $\mathfrak{D}'$ that projects to $\mathfrak{D}$ (a certain model of $\mathfrak{D}$). Derived categories of differential graded functors were studied in detail in [12] and [16]. We will define and study them in §5 below; now we will only list their properties that are needed for the proofs of main statements.
Below we will also need certain (filtered) inverse limits several times. $\mathfrak{D}$ is a triangulated category; so it is no wonder that there are no nice limits in it. So we consider a certain additive $\mathfrak{D}'$ endowed with an additive functor $p : \mathfrak{D}' \to \mathfrak{D}$. We will call (the images of) inverse limits from $\mathfrak{D}'$ homotopy limits in $\mathfrak{D}$.
The relation of $\mathfrak{D}'$ with $\mathfrak{D}$ is similar to the relation of $C(\underline{A})$ with $D(\underline{A})$. In particular, $\mathfrak{D}'$ is closed with respect to all (small filtered) inverse limits; we have functorial cones of morphisms in $\mathfrak{D}'$ that are compatible with inverse limits.
We will need some conventions and definitions introduced in Notation; in particular, I, L will be projective systems; I is countable.

Proposition 3.1.1. *1. There exists a triangulated category $\mathfrak{D} \supset DM_{gm}^{eff}$; all objects of DM_{gm}^{eff} are cocompact in $\mathfrak{D}$.*

2. *There exists an additive category $\mathfrak{D}'$ closed with respect to arbitrary (small filtered) inverse limits, and an additive functor $p : \mathfrak{D}' \to \mathfrak{D}$ that preserves (small) products. Moreover, p is surjective on objects.*

3. *$\mathfrak{D}'$ is endowed with a certain invertible shift functor* $[1]$ *that is compatible with the shift on $\mathfrak{D}$ and respects inverse limits.*

4. *There is a functorial cone of morphisms in $\mathfrak{D}'$ defined; it is compatible with* $[1]$ *and respects inverse limits.*

5. *Any triangle of the form $X \xrightarrow{f} Y \to \operatorname{Cone}(f) \to X[1]$ in $\mathfrak{D}'$ becomes distinguished in $\mathfrak{D}$.*

6. *The composition functor $M_{gm} : C^b(SmCor) \to DM_{gm}^{eff} \to \mathfrak{D}$ can be canonically factorized through an additive functor $j : C^b(SmCor) \to \mathfrak{D}'$. Shifts and cones of morphisms in $C^b(SmCor)$ are compatible with those in $\mathfrak{D}'$ via j.*

7. *For any $X \in M_{gm}(C^b(SmCor)) \subset Obj\mathfrak{D}$, any $Y : L \to \mathfrak{D}'$ we have $\mathfrak{D}(p(\varprojlim_{l\in L} Y_l), X) = \varinjlim_{l\in L} \mathfrak{D}(p(Y_l), X)$.*

8. *DM_{gm}^{eff} weakly cogenerates $\mathfrak{D}$ (i.e. we have ${}^{\perp}DM_{gm}^{eff} = \{0\}$, see Notation).*

9. *Let a sequence $i_n \in I$, $n > 0$, be increasing (i.e. $i_{n+1} > i_n$ for any $n > 0$) unbounded (see Notation). Then for all functors $X : I \to \mathfrak{D}'$, we have functorial distinguished triangles in $\mathfrak{D}$:*

$$p(\varprojlim_{i\in I} X_i) \to p(\prod X_{i_n}) \xrightarrow{e} p(\prod X_{i_n}); \tag{27}$$

e is the product of $id_{X_{i_n}} \oplus -\phi_n : X_{i_{n+1}} \to X_{i_n}$; here ϕ_n are the morphisms coming from I via X.

10. *There exists a differential graded enhancement for $\mathfrak{D}$; see §5.1 below.*

Remark 3.1.2. 1. Since below we will prove some statements for $\mathfrak{D}$ only using its 'axiomatics' (i.e. the properties listed in Proposition 3.1.1), these results would also be valid in any other category that fulfills these properties. This could be useful, since the author is not sure at all that all possible $\mathfrak{D}$ are isomorphic.
2. Moreover, one could modify the axiomatics of $\mathfrak{D}$ and consider instead a category that would only contain the triangulated subcategory of DM_{gm}^{eff} generated by motives of smooth varieties of dimension $\le n$ (for a fixed $n > 0$). Our results and arguments below can be easily carried over to this setting (with minor modifications; it is also useful here to weaken condition 8 in the Proposition). This makes sense since these 'geometric pieces' of DM_{gm}^{eff} are self-dual with respect to Poincare duality (at least, if $\operatorname{char} k = 0$); see §6.4 below. See also Remark 4.5.2(2).
Alternatively, we can weaken the condition for the functor $DM_{gm}^{eff} \to \mathfrak{D}$ to be a full embedding. For example, it could be interesting to consider the version of $\mathfrak{D}$ for which this functor kills $DM_{gm}^{eff}(n)$ (for some fixed $n > 0$).
Lastly note that we do not really need condition 2 in its full generality (below).

Now we derive some consequences from the axiomatics listed.

Corollary 3.1.3. *1. For any $Z \in Obj DM_{gm}^{eff} \subset Obj\mathfrak{D}$, any $X : L \to \mathfrak{D}'$ we have $\mathfrak{D}(p(\varprojlim_{l\in L} X_l), Z) = \varinjlim_{l\in L} \mathfrak{D}(p(X_l), Z)$.*

2. *For any $T \in Obj\mathfrak{D}$, all functors $Y : I \to \mathfrak{D}'$ we have functorial short exact sequences*

$$\{0\} \to \varprojlim{}^1 \mathfrak{D}(T, p(Y_i)[-1]) \to \mathfrak{D}(T, p(\varprojlim Y_i)) \to \varprojlim \mathfrak{D}(T, p(Y_i)) \to \{0\};$$

here $\varprojlim^1$ is the (first) derived functor of $\varprojlim = \varprojlim_I$.

3. *For all functors $X : L \to C^b(SmCor)$, $Y : I \to C^b(SmCor)$, we have functorial short exact sequences*

$$\begin{aligned}\{0\} \to \varprojlim{}^1_{i\in I}(\varinjlim_{l\in L} \mathfrak{D}(M_{gm}(X_l), M_{gm}(Y_i)[-1])) \to \\ \mathfrak{D}(p(\varprojlim_{l\in L} j(X_l)), p(\varprojlim_{i\in I} j(Y_i))) \to \\ \varprojlim_{i\in I}(\varinjlim_{l\in L} \mathfrak{D}(M_{gm}(X_l), M_{gm}(Y_i))) \to \{0\}.\end{aligned} \tag{28}$$

4. *$\mathfrak{D}$ is idempotent complete.*

Proof. 1. If $Z \in M_{gm}(C^b(SmCor))$, then the assertion is exactly Proposition 3.1.1(7).

It remains to note that any $Z \in ObjDM_{gm}^{eff}$ is a retract of some object coming from $C^b(SmCor)$.

2. Since inverse limits and their derived functors do not change when we replace a projective system by any unbounded subsystem, we can assume that L consists of some i_n as in (27).

Now, (27) yields a long exact sequence

$$\cdots \to \prod_{i\in I} \mathfrak{D}(T, p(Y_i)[-1]) \xrightarrow{f} \prod_{i\in I} \mathfrak{D}(T, p(Y_i)[-1]) \to \mathfrak{D}(T, p(\varprojlim_{i\in I} Y_i)) \\ \to \prod_{i\in I} \mathfrak{D}(T, p(Y_i)) \xrightarrow{g} \prod_{i\in I} \mathfrak{D}(T, p(Y_i)) \to \dots,$$

here f and g are induced by e in (27).

It is easily seen that $\operatorname{Ker} g \cong \varprojlim \mathfrak{D}(T, M_{gm}(Y_m))$.

Lastly, Remark A.3.6 of [21] allows to identify $\operatorname{Coker} f$ with $\varprojlim^1 \mathfrak{D}(T, M_{gm}(Y_m)[-1])$.

3. Immediate from the previous assertions.

4. Since $\mathfrak{D}'$ is closed with respect to all inverse limits, it is closed with respect to all (small) products. Then Proposition 3.1.1(2) yields that $\mathfrak{D}$ is also closed with respect to all products. Now, Remark 1.6.9 of [21] yields the result (in fact, the proof uses only countable products).

□

We will often call the objects of $\mathfrak{D}$ *comotives.*

3.2 Pro-schemes and their comotives

Now we have certain inverse limits for objects (coming from) $C^b(SmCor)$; this allows to define (reasonable) comotives for certain schemes that are not (necessarily) of finite type over k (and for their disjoint unions). We also define certain Tate twists of those.

We will call certain ind-schemes over k *pro-schemes.* An ind-scheme V/k is a pro-scheme if it is a countable disjoint union of schemes, such that each of them is a projective limit of smooth varieties of dimension $\leq c_V$ for some fixed $c_V \geq 0$ (in the category of schemes) with connecting morphisms being open dense embeddings. One may say that a pro-scheme is a countable disjoint union of countable intersections of smooth varieties. Note that (the spectrum of) any function field over k is a pro-scheme; any smooth k-variety is a pro-scheme also.

We have the operation of countable disjoint union for pro-schemes of bounded dimension.

Now, we would like to present a (not necessarily connected) pro-scheme V as projective limits of smooth varieties V_i. This is easy if V is connected (cf. Lemma 3.2.9 of [9]). In the general case one should allow (formally) zero morphisms between connected components of V_i (for distinct i). So we consider a new category $SmVar'$ containing the category of all smooth varieties as a (non-full!) subcategory. We take $ObjSmVar' = SmVar$; for any smooth connected varieties $X, Y \in SmVar$ we have $SmVar'(X,Y) = Mor_{Var}(X,Y) \cup \{0\}$; the composition of a zero morphism with any other one is zero; $SmVar'(\sqcup_i X_i, \sqcup_j Y_j) = \sqcup_{i,j} SmVar'(X_i, Y_j)$. $SmVar'$ can be embedded into $SmCor$ (certainly, this embedding is not full).

We will write $V = \varprojlim V_i$ (this is not possible in the category of ind-schemes, but works in $\mathrm{Pro}-SmVar'$). Note that the set of connected components of V is the inductive limit of the corresponding sets for V_i.

Now, for any pro-scheme $V = \varprojlim V_i$, any $s \geq 0$, we introduce the following notation: $M_{gm}(V)(s) = p(\varprojlim(j(V_i)(s))) \in Obj\mathfrak{D}$ (see Proposition 3.1.1); we will denote $M_{gm}(V)(0)$ by $M_{gm}(V)$ and call $M_{gm}(V)$ the comotif of V. This notation should be considered as formal i.e. we do not define Tate twists on $\mathfrak{D}$ (till §5.4.3).

Obviously, if $V \in SmVar$, its comotif (and its twists) coincides with its motif (and its twists), so we can use the same notation for them.

If $\underline{A}$ is a category closed with respect to filtered direct limits, $H' : DM_{gm}^{eff} \to \underline{A}$ is a functor, we can (formally) extend it to co-motives in question; we set:

$$H(M_{gm}(V)(s)[n]) = \varinjlim H'(M_{gm}(V_i)(s)[n]). \tag{29}$$

Remark 3.2.1. 1. For a general H' this notation should be considered as formal. Yet in the case $H' = (-,Y) : \mathfrak{D} \to Ab$, $Y \in ObjDM_{gm}^{eff} \subset Obj\mathfrak{D}$, we have $H(M_{gm}(V)(i)[n]) = \mathfrak{D}(M_{gm}(V)(i)[n], X)$; see Corollary 3.1.3(1), i.e. (29) yields the value of a well-defined functor $\mathfrak{D} \to Ab$ at $M_{gm}(V)(s)[n]$. We will only need H' of this sort till §4.3.

More generally, there exists such an H if $\underline{A}$ satisfies AB5 and H' is cohomological; we will call the corresponding H an *extended cohomology theory*, see Remark 4.3.2 below.

2. Let V^j be a countable set of pro-schemes (of bounded dimensions). Then $M_{gm}(\sqcup V^j) = \prod M_{gm}(V^j)$ by Proposition 3.1.1(2).

Besides, for any H' as in (29) we have $H(M_{gm}(\sqcup V^j)(s)[n]) = \bigoplus H(M_{gm}(V^j)(s)[n])$.

Below we will need some conventions for pro-schemes.

For pro-schemes $U = \varprojlim U_i$ and $V = \varprojlim V_j$ we will call an element of $\varprojlim_{i \in I}(\varinjlim_{j \in J} SmCor(U_i, V_j))$ an open embedding if it can be obtained as a double limit of open embeddings $U_i \to V_j$ (as varieties). If $U = V \setminus W$ for some pro-scheme W, we will say that W is a closed sub-pro-scheme of V. Note that in this case any connected component of W is a closed subscheme of some

connected component of V; yet some components of V could contain an infinite set of connected components of W.
For $V = \sqcup V^j$, V^j are connected pro-schemes, we will call the maximum of the transcendence degrees of function fields of V^j the dimension of V (note that this is finite). We will say that a sub-pro-scheme $U = \sqcup U^m$, U^m are connected, is everywhere of codimension r (resp. $\geq r$, for some fixed $r \geq 0$) in $V = \sqcup V^j$ if for every induced embedding $U^m \to V^j$ the difference of their dimensions (defined as above) is r (resp. $\geq r$).
We will call the inverse limit of the sets of points of V_i of a fixed codimension $s \geq 0$ the set of points of V of codimension s (note that any element of this set indeed defines a point of some connected component of V).

3.3 Primitive schemes: reminder

In [29] M. Walker proved that primitive schemes in the case of an infinite k have 'motivic' properties similar to those of smooth semi-local schemes (in the sense of §4.4 of [26]). Since we don't want to discriminate the case of a finite k, we will modify slightly the standard definition of primitive schemes.

Definition 3.3.1. If k is infinite then a (pro-)scheme is called primitive if all of its connected components are affine and their coordinate rings R_j satisfy the following primitivity criterion: for any $n > 0$ every polynomial in $R_j[X_1, \dots, X_n]$ whose coefficients generate R_j as an ideal over itself, represents an R_j-unit.
If k is finite, then we will call a pro-scheme primitive whenever all of its connected components are semi-local (in the sense of §4.4 of [26]).

Remark 3.3.2. Recall that in the case of infinite k all semi-local k-algebras satisfy the primitivity criterion (see Example 2.1 of [29]).

Below we will mostly use the following basic property of primitive schemes.

Proposition 3.3.3. *Let S be a primitive pro-scheme, let S_0 be the collection of all of its generic points; F is a homotopy invariant presheaf with transfers. Then $F(S) \subset F(S_0)$; here we define F on pro-schemes as in (29).*

Proof. We can assume that S is connected (so it is a smooth primitive scheme). Hence in the case of infinite k our assertion follows from Theorem 4.19 of [29]. Now, if k is finite, then S_0 is semi-local (by our convention); so we may apply Corollary 4.18 of [26] instead.

□

3.4 Basic motivic properties of primitive schemes

We will call a primitive pro-scheme just a primitive scheme. We prove certain motivic properties of primitive schemes (in the form in which we will need them below).

PROPOSITION 3.4.1. *For $F \in ObjDM_-^{eff}$ we define $H'(-) = DM_-^{eff}(-,F)$ on DM_{gm}^{eff}; we also define $H(M_{gm}(V)(i)[n])$ as in (29). Let S be a primitive scheme, $m \geq 0$, $i \in \mathbb{Z}$.*
1. Let $F \in DM_-^{eff\,t\leq -1}$ (t is the homotopy t-structure, that we considered in §1.3). Then $H(M_{gm}(S)(m)[m]) = \{0\}$.
2. More generally, for any $F \in ObjDM_-^{eff}$ we have $H([M_{gm}(S)(m)[m]) \cong F^0_{-m}(S)$ where $F^0 = F^{t=0}$, F^0_{-m} is the m-th Tate twist of F^0 (see Definition 1.4.1).

Proof. 1. We consider the homotopy invariant presheaf with transfers F_{-m} : $X \mapsto DM_-^{eff}(M_{gm}(X)(m)[m],F)$. We should prove that $F_{-m}(S) = 0$ (here we extend F_{-m} to pro-schemes in the usual way i.e. as in (29)).
(29) also yields that $F_{-m}(\sqcup S_i) = \bigoplus F_{-m}(S_i)$. Hence by Proposition 3.3.3, it suffices to consider the case of S being (the spectrum of) a function field over k. Since F_{-m} is represented by an object of $DM_-^{eff\,t\leq -1}$ (see Proposition 1.4.2(2)), it suffices to note that any field is a Henselian scheme i.e. a point in the Nisnevich topology.
2. By Proposition 1.4.2, for any $X \in SmVar$ we have $M_{gm}(X)(m)[m] \perp DM_-^{eff\,t\geq 1}$. Hence we can assume $F \in DM_-^{eff\,t\leq 0}$.
Next, using assertion 1, we can easily reduce the situation to the case $F = F^{t=0} \in ObjHI$ (by considering the t-decomposition of $F[-1]$). In this case the statement is immediate from Proposition 1.4.2(1).

□

LEMMA 3.4.2. *Let $U \to U'$ be an open dense embedding of smooth varieties.*
1. We have $\mathrm{Cone}(M_{gm}(U) \to M_{gm}(U')) \in DM_-^{eff\,t\leq -1}$.
2. Let S be primitive. Then for any $n, m, i \geq 0$ the map

$$\mathfrak{D}(M_{gm}(S)(m)[m], M_{gm}(U)(n)[n+i]) \to \mathfrak{D}(M_{gm}(S)(m)[m], M_{gm}(U')(n)[n+i])$$

is surjective.

Proof. 1. We denote $\mathrm{Cone}(M_{gm}(U) \to M_{gm}(U')) \in DM_-^{eff\,t\leq -1}$ by C. Obviously, $C \in DM_-^{eff\,t\leq 0}$. Let H denote $C^{t=0}$ ($H \in ObjHI$). By Corollary 4.19 of [26], we have $H(U) \subset H(U')$. Next, from the long exact sequence $\{0\}(= DM_-^{eff}(M_{gm}(U)[1],H)) \to DM_-^{eff}(C,H) \to DM_-^{eff}(U',H) \to DM_-^{eff}(U,H) \to \dots$ we obtain $C \perp H$. Then the long exact sequence $\dots \to DM_-^{eff}(C^{t\leq -1}[2],H) \to DM_-^{eff}(H,H) \to DM_-^{eff}(C,H) \to \dots$ yields $H = 0$.
2. It suffices to check that $M_{gm}(S)(m)[m] \perp C(n)[n+i]$. Since $M_{gm}(U)(n)[n]$ is canonically a retract of $M_{gm}(U \times G_m^n)$, we can assume that $n = 0$.
Now the claim follows immediately from assertion 1 and Proposition 3.4.1(1).
□

3.5 On morphisms between comotives of primitive schemes

We will need the fact that certain 'positive' morphism groups are zero.
Let $n, m, \geq 0$, $i > 0$, $Y = \varprojlim Y_l$ $(l \in L)$, be any pro-scheme, X be a primitive scheme.

Proposition 3.5.1. *1. The natural homomorphism*

$$\mathfrak{D}(M_{gm}(X)(m)[m], M_{gm}(Y)[n](n)) \to \\ \to \varprojlim_l (\varinjlim_{X \subset Z, Z \in SmVar} DM^{eff}_{gm}(Z(m)[m], M_{gm}(Y_l)(n)[n]))$$

is surjective.

2. $M_{gm}(X)(m)[m] \perp M_{gm}(Y)[n+i](n)$.

Proof. Note first that by the definition of the Tate twist (1), it can be lifted to $C^b(SmCor)$.

1. This is immediate from the short exact sequence (28).

2. By Remark 3.2.1(2), we may suppose that Y is connected. Then we apply (28) again. The corresponding $\varprojlim$-term is zero by Proposition 3.4.1(1). Lastly, the surjectivity proved in Lemma 3.4.2(2) yields that the corresponding $\varprojlim^1$-term is zero. Indeed, the groups $\mathfrak{D}(M_{gm}(X)(m)[m], M_{gm}(Y_l)[n+i-1](n))$ obviously satisfy the Mittag-Leffler condition; see §A.3 of [21].

   In fact, one could easily deduce the assertion from the results of the previous subsection and (27) directly (we do not need much of the theory of higher limits in this paper).

□

Remark 3.5.2. In fact, this statement, as well as all other properties of (primitive) pro-schemes that we need, are also true for not necessary countable disjoint unions of (primitive) pro-schemes. This observation could be used to extend the main results of the paper to a somewhat larger category; yet such an extension does not seem to be important.

3.6 The Gysin distinguished triangle for pro-schemes; 'Gersten' Postnikov towers for comotives of pro-schemes

We prove that we can pass to countable homotopy limits in Gysin distinguished triangles.

Proposition 3.6.1. *Let Z, X be pro-schemes, Z a closed subscheme of X (everywhere) of codimension r. Then for any $s \geq 0$ the natural morphism $M_{gm}(X \setminus Z)(s) \to M_{gm}(X)(s)$ extends to a distinguished triangle (in $\mathfrak{D}$): $M_{gm}(X \setminus Z)(s) \to M_{gm}(X)(s) \to M_{gm}(Z)(r+s)[2r]$.*

Proof. First assume $s = 0$.
We can assume $X = \varprojlim X_i$, $Z = \varprojlim Z_i$ for $i \in I$, where $X_i, Z_i \in SmVar$, Z_i is closed everywhere of codimension r in X_i for all $i \in I$.
We take $Y_i = j(X_i \setminus Z_i \to X_i)$, $Y = p(\varprojlim_{i\in I} Y_i)$. By parts 4 and 5 of Proposition 3.1.1 we have a distinguished triangle $M_{gm}(X \setminus Z) \to M_{gm}(X) \to Y$.
It remains to prove that $Y \cong M_{gm}(Z)(r)[2r]$. Proposition 2.4.5 of [9] (a functorial form of the Gysin distinguished triangle for Voevodsky's motives) yields that $p(Y_i) \cong M_{gm}(Z_i)(r)[2r]$; moreover, the connecting morphisms $p(Y_i) \to p(Y_{i+1})$ are obtained from the corresponding morphisms $M_{gm}(Z_i) \to M_{gm}(Z_{i+1})$ by tensoring by $\mathbb{Z}(r)[2r]$. It remains to recall: by Proposition 3.1.1(9), the isomorphism class of a homotopy limit in $\mathfrak{D}$ can be completely described in terms of (objects and morphisms) of $\mathfrak{D}$ (i.e. we don't have to consider the lifts of objects and morphisms to $\mathfrak{D}'$). This yields the result.
Now, since $M_{gm}(X \times G_m) = M_{gm}(X) \bigoplus M_{gm}(X)(1)[1]$ for any $X \in SmVar$ (hence this is also true for pro-schemes), the assertion for the case $s = 0$ yields the general case easily. $\square$

Now we will construct a certain Postnikov tower $Po(X)$ for X being the (twisted) comotif of a pro-scheme Z that will be related to the coniveau spectral sequences (for cohomology) of Z; our method was described in §1.5 above. Note that we consider the general case of an arbitrary pro-scheme Z (since in this paper pro-schemes play an important role); yet this case is not much distinct from the (partial) case of $Z \in SmVar$.

Corollary 3.6.2. *We denote the dimension of Z by d (recall the conventions of §3.2).*
For all $i \geq 0$ we denote by Z^i the set of points of Z of codimension i.
For any $s \geq 0$ there exists a Postnikov tower for $X = M_{gm}(Z)(s)[s]$ such that $l = 0$, $m = d+1$, $X_i \cong \prod_{z\in Z^i} M_{gm}(z)(i+s)[2i+s]$.

Proof. As above, it suffices to prove the statement for $s = 0$. Since any product of distinguished triangles is distinguished, we can assume Z to be connected.
We consider a projective system L whose elements are sequences of closed subschemes $\varnothing = Z_{d+1} \subset Z_d \subset Z_{d-1} \subset \cdots \subset Z_0$. Here $Z_0 \in SmVar$, $Z_l \in Var$ for $l > 0$, Z is open in Z_0 (see §3.2; Z_0 is connected; in the case when $Z \in SmVar$ we only take $Z_0 = Z$); for all $j > 0$ we have: Z_j is everywhere of codimension $\geq j$ in Z_0; all irreducible components of all Z_j are everywhere of codimension $\geq j$ in Z_0; and Z_{j+1} contains the singular locus of Z_j (for $j \leq d$). The ordering in L is given by open embeddings of varieties $U_j = Z_0 \setminus Z_j$ for $j > 0$. For $l \in L$ we will denote the corresponding sequence by $\varnothing = Z^l_{d+1} \subset Z^l_d \subset Z^l_{d-1} \subset \cdots \subset Z^l_0$. Note that L is countable!
By the previous proposition, for any j we have a distinguished triangle $M_{gm}(\varprojlim(Z^l_0 \setminus Z^l_j)) \to M_{gm}(\varprojlim(Z^l_0 \setminus Z^l_{j+1})) \to M_{gm}(\varprojlim(Z^l_j \setminus Z^l_{j+1})(j)[2j])$.
It remains to compute the last term; we fix some j.
We have $\varprojlim_{l\in L'}(Z^l_j \setminus Z^l_{j+1})) \cong \prod_{z\in Z^i} M_{gm}(z)$. Indeed, for all $l \in L$ the variety $Z^l_j \setminus Z^l_{j+1}$ is the disjoint union of some locally closed smooth subschemes of

Z_0^l of codimension j; for any $z_0 \in Z^j$ for $l \in L$ large enough z_0 is contained in $Z_j^l \setminus Z_{j+1}^l$ as an open sub-pro-scheme, and the inverse limit of connected components of $Z_j^l \setminus Z_{j+1}^l$ containing z_0 is exactly z_0. Now, we can apply the functor $X \mapsto M_{gm}(X)(j)[2j]$ to this isomorphism. We obtain $M_{gm}(\varprojlim(Z_j^l \setminus Z_{j+1}^l)(j)[2j]) \cong \prod_{z \in Z^i} M_{gm}(z)(i)$. This yields the result.

$\square$

Remark 3.6.3. 1. Alternatively, one could construct $Po(X)$ for the (twisted) comotif of a pro-scheme $T = \varprojlim T^l$ as the inverse limit of the Postnikov towers for T^l (constructed as above yet with fixed $Z_0^l = T^l$); certainly, to this end one should pass to the limit in $\mathfrak{D}'$. It is easily seen that one would get the same tower this way.

2. Certainly, if we shift a Postnikov tower for $M_{gm}(Z)(s)[s]$ by $[j]$ for some $j \in \mathbb{Z}$, we obtain a Postnikov tower for $M_{gm}(Z)(s)[s+j]$. We didn't formulate assertion 2 for these shifts only because we wanted X^p to belong to $\mathfrak{D}_s^{w=0}$ (see Proposition 4.1.1 below).

3. Since the calculation of X^i used Proposition 3.1.1(9), our method cannot describe connecting morphisms between them (in $\mathfrak{D}$). Yet one can calculate the 'images' of the connecting morphisms in $\mathfrak{D}^{naive}$; see §1.5 and §6.1.

4 Main motivic results

The results of the previous section combined with those of §2.2 allow us to construct (in §4.1) a certain *Gersten weight structure* w on a certain triangulated $\mathfrak{D}_s$: $DM_{gm}^{eff} \subset \mathfrak{D}_s \subset \mathfrak{D}$. Its main property is that comotives of function fields over k (and their products) belong to $\underline{Hw}$. It follows immediately that the Postnikov tower $Po(X)$ provided by Corollary 3.6.2 is a *weight Postnikov tower* with respect to w. Using this, in §4.2 we prove: if S is a primitive scheme, S_0 is its dense sub-pro-scheme, then $M_{gm}(S)$ is a direct summand of $M_{gm}(S_0)$; $M_{gm}(K)$ (for a function field K/k) contains (as retracts) comotives of primitive schemes whose generic point is K, as well as twisted comotives of residue fields of K (for all geometric valuations).

In §4.3 we (easily) translate these results to cohomology; in particular, the cohomology of (the spectrum of) K contains direct summands corresponding to the cohomology of primitive schemes whose generic point is K, as well as twisted cohomology of residue fields of K. Here one can consider any cohomology theory $H : \mathfrak{D}_s \to \underline{A}$; one can obtain such an H by extending to $\mathfrak{D}_s$ any cohomological $H' : DM_{gm}^{eff} \to \underline{A}$ if $\underline{A}$ satisfies AB5 (by means of Proposition 1.2.1). Note: in this case the cohomology of pro-schemes mentioned is calculated in the 'usual' way.

In §4.4 we consider weight spectral sequences corresponding to (the Gersten weight structure) w. We observe that these spectral sequences generalize naturally the classical coniveau spectral sequences. Besides, for a fixed $H : \mathfrak{D}_s \to \underline{A}$ our (generalized) coniveau spectral sequence converging to $H^*(X)$ (where X

could be a motif or just an object of $\mathfrak{D}_s$) is $\mathfrak{D}_s$-functorial in X (i.e. it is motivically functorial for objects of DM_{gm}^{eff}); this fact is non-trivial even when restricted to motives of smooth varieties.

In §4.5 we prove that there exists a nice duality $\mathfrak{D}^{op} \times DM_-^{eff} \to Ab$ (extending the bi-functor $DM_-^{eff}(-,-) : DM_{gm}^{eff\,op} \times DM_-^{eff} \to Ab$); the Gersten weight structure w (on $\mathfrak{D}_s$) is left orthogonal to the homotopy t-structure t on DM_-^{eff} with respect to it. This allows to apply Theorem 2.6.1: in the case when H comes from $Y \in ObjDM_-^{eff}$ we prove the isomorphism (starting from E_2) of (the coniveau) $T(H,X)$ with the spectral sequence corresponding to the t-truncations of Y. We describe $ObjDM_{gm}^{eff} \cap \mathfrak{D}_s^{w\le i}$ in terms of t (for DM_-^{eff}). We also note that our results allow to describe torsion motivic cohomology in terms of (torsion) étale cohomology (see Remark 4.5.4(4)).

In §4.6 we define the coniveau spectral sequence (starting from E_2) for cohomology of a motif X over a not (necessarily) countable perfect base field l as the limit of the corresponding coniveau spectral sequences over countable perfect subfields of definition for X. This definition is compatible with the classical one (for X being the motif of a smooth variety); so we obtain motivic functoriality of classical coniveau spectral sequences over a general base field.

In §4.7 we prove that the Chow weight structure for DM_{gm}^{eff} (introduced in §6 of [6]) could be extended to $\mathfrak{D}$ (certainly, the corresponding weight structure w_{Chow} differs from w). We will call the corresponding weight spectral sequences *Chow-weight* ones; note that they are isomorphic to classical (i.e. Deligne's) weight spectral sequences when the latter are defined.

In §4.8 we use the results §2.7 to compare coniveau spectral sequences with Chow-weight ones. We always have a comparison morphism; it is an isomorphism if H is a *birational* cohomology theory.

In §4.9 we consider the category of birational comotives $\mathfrak{D}_{bir}$ (a certain 'completion' of birational motives of [15]) i.e. the localization of $\mathfrak{D}$ by $\mathfrak{D}(1)$. It turns our that w and w_{Chow} induce the same weight structure w'_{bir} on $\mathfrak{D}_{bir}$. Conversely, starting from w'_{bir} one can glue 'from slices' the weight structures induced by w and w_{Chow} on $\mathfrak{D}/\mathfrak{D}(n)$ for all $n > 0$. Furthermore, these structures belong to an interesting family of weight structures indexed by a single integral parameter; other terms of this family could be also interesting!

4.1 The Gersten weight structure for $\mathfrak{D}_s \supset DM_{gm}^{eff}$

Now we describe the main weight structure of this paper. Unfortunately, the author does not know whether it is possible to define the Gersten weight structure (see below) on the whole $\mathfrak{D}$. Yet for our purposes it is quite sufficient to define the corresponding weight structure on a certain triangulated subcategory $\mathfrak{D}_s \subset \mathfrak{D}$ containing DM_{gm}^{eff} (and comotives of all pro-schemes).

In order to make the choice of $\mathfrak{D}_s \subset \mathfrak{D}$ compatible with extensions of scalars, we bound certain dimensions of objects of $\underline{Hw}$.

We will denote by H the full subcategory of $\mathfrak{D}$ whose objects are all countable products $\prod_{l\in L} M_{gm}(K_l)(n_l)[n_l]$; here K_l are (the spectra of) function fields over k, $n_l \geq 0$; we assume that the transcendence degrees of K_l/k and n_l are bounded.

Proposition 4.1.1. *1. Let $\mathfrak{D}_s$ be the Karoubi-closure of $\langle H\rangle$ in $\mathfrak{D}$. Then $\underline{C} = \mathfrak{D}_s$ can be endowed with a unique weight structure w such that $\underline{Hw}$ contains H.*
2. $\underline{Hw}$ is the idempotent completion of H.
3. $\mathfrak{D}_s$ contains DM_{gm}^{eff} as well as all $M_{gm}(Z)(l)$ for Z being a pro-scheme, $l \geq 0$.
4. For any primitive S, $i \geq 0$, we have $M_{gm}(S)(i)[i] \in \mathfrak{D}_s^{w=0}$.
5. Let Z be a pro-scheme, $s \geq 0$. Then $M_{gm}(Z)(s)[s] \in \mathfrak{D}_s^{w\leq 0}$; the Postnikov tower for $M_{gm}(Z)(s)[s]$ given by Corollary 3.6.2 is a weight Postnikov tower for it.

Proof. 1. By Proposition 3.5.1(2), H is negative (since any object of H is a finite sum of $M_{gm}(X_i)(m_i)$ for some primitive pro-schemes X_i, $m_i \in \mathbb{Z}$). Besides, $\mathfrak{D}$ is idempotent complete (see Corollary 3.1.3(4)); hence $\mathfrak{D}_s$ and $\mathfrak{D}_s^{w=0}$ also are. Hence we can apply Theorem 2.2.1(18) (for $D = H$).
2. Also immediate from Theorem 2.2.1(18).
3. $M_{gm}(Z)(l) \in Obj\mathfrak{D}_s$ by Corollary 3.6.2; in particular, this is true for $Z \in SmVar$. It remains to note that DM_{gm}^{eff} is the Karoubization of $\langle M_{gm}(U) : U \in SmVar\rangle$ in $\mathfrak{D}$.
4. It suffices to note that $M_{gm}(S)(i)[i]$ belongs both to $\mathfrak{D}_s^{w\leq 0}$ and to $\mathfrak{D}_s^{w\geq 0}$ by Theorem 2.2.1(20). Here we use Proposition 3.5.1(2) again.
5. We have $X^i \in \mathfrak{D}_s^{w=0}$. Hence Theorem 2.2.1(14) yields the result. Note here that we have $Y_0 = 0$ in the notation of Definition 2.1.2(9).

$\square$

We will call w the *Gersten* weight structure, since it is closely connected with Gersten resolutions of cohomology (cf. §4.5 below). By default, below w will denote the Gersten weight structure.

Remark 4.1.2. 1. $\underline{Hw}$ is idempotent complete since $\mathfrak{D}_s$ is.
2. In fact, one could easily prove similar statements for $\underline{C}$ being just $\langle H\rangle$ (instead of its Karoubization in $\mathfrak{D}$). Certainly, for this version of $\underline{C}$ we will only have $\underline{C} \supset M_{gm}(K^b(SmCor))$.
Besides, note that for any function field K'/k, any $r \geq 0$, there exists a function field K/k such that $M_{gm}(K')(r)[r]$ is a retract of $M_{gm}(K)$ (see Corollary 4.2.2 below). Hence it suffices take H being the full subcategory of $\mathfrak{D}$ whose objects are $\prod_{l\in L} M_{gm}(K_l)$ (for bounded transcendence degrees of K_l/k).
3. The proposition implies that $\mathfrak{D}_s$ is exactly the Karoubization in $\mathfrak{D}$ of the triangulated category generated by comotives of all pro-schemes.
4. The author does not know whether one can describe weight decompositions for arbitrary objects of DM_{gm}^{eff} explicitly. Still, one can say something about these weight decompositions and weight complexes using their functoriality properties. In particular, knowing weight complexes for $X, Y \in ObjDM_{gm}^{eff}$ (or just $\in ObjDM^s$) one can describe the weight complex of $X \to Y$ up to a

homotopy equivalence as the corresponding cone (see Lemma 6.1.1 below). Besides, let $X \to Y \to Z$ be a distinguished triangle (in $\mathfrak{D}$). Then for any choice of $(X^{w\le 0}, X^{w\ge 1})$ and $(Z^{w\le 0}, Z^{w\ge 1})$ there exists a choice of $(Y^{w\le 0}, Y^{w\ge 1})$ such that there exist distinguished triangles $X^{w\le 0} \to Y^{w\le 0} \to Z^{w\le 0}$ and $X^{w\ge 1} \to Y^{w\ge 1} \to Z^{w\ge 1}$; see Lemma 1.5.4 of [6]. In particular, we obtain that j maps complexes (over $SmCor$) concentrated in degrees $\le j$ into $\mathfrak{D}_s^{w\le j}$ (we will prove a stronger statement in Remark 4.5.4(4) below). If $X \in Obj DM_{gm}^{eff}$ comes from a complex over $SmCor$ whose connecting morphisms satisfy certain codimension restrictions, these observations could be extended to an explicit description of a weight decomposition for it; cf. §7.4 of [6].

4.2 Direct summand results for comotives

Proposition 4.1.1 easily implies the following interesting result.

Theorem 4.2.1. *1. Let S be a primitive scheme; let S_0 be its dense sub-pro-scheme. Then $M_{gm}(S)$ is a direct summand of $M_{gm}(S_0)$.*
2. Suppose moreover that $S_0 = S \setminus T$ where T is a closed subscheme of S everywhere of codimension $r > 0$. Then we have $M_{gm}(S_0) \cong M_{gm}(S) \bigoplus M_{gm}(T)(r)[2r-1]$.

Proof. We can assume that S and S_0 are connected.
1. By Proposition 4.1.1(5), we have: $M_{gm}(S_0), M_{gm}(S) \in \mathfrak{D}_s^{w\le 0}$; $M_{gm}(\operatorname{Spec}(k(S)))$ could be assumed to be the zeroth term of their weight complexes for a choice of weight complexes compatible with some negative Postnikov weight towers for them; the embedding $S_0 \to S$ is compatible with $id_{M_{gm}(\operatorname{Spec}(k(S)))}$ (since we have a commutative triangle $\operatorname{Spec} k(S) \to S_0 \to S$ of pro-schemes). Hence Theorem 2.2.1(16) yields the result.
2. By Proposition 3.6.1 we have a distinguished triangle $M_{gm}(S_0) \to M_{gm}(S) \to M_{gm}(T)(r)[2r]$. By parts 4 and 5 of Proposition 4.1.1 we have $M_{gm}(S_0) \in \mathfrak{D}_s^{w\le 0}$, $M_{gm}(S) \in \mathfrak{D}_s^{w=0}$, $M_{gm}(T)(r)[2r] \in \mathfrak{D}_s^{w\le -r} \subset \mathfrak{D}_s^{w\le -1}$. Hence Theorem 2.2.1(8) yields the result.

□

Corollary 4.2.2. *1. Let S be a connected primitive scheme, let S_0 be its generic point. Then $M_{gm}(S)$ is a retract of $M_{gm}(S_0)$.*
2. Let K be a function field over k. Let K' be the residue fields for a geometric valuation v of K of rank r. Then $M_{gm}(K')(r)[r]$ is a retract of $M_{gm}(K)$.

Proof. 1. This is just a partial case of part 1 of the the theorem.
2. Obviously, it suffices to prove the statement in the case $r = 1$. Next, K is the function field of some normal projective variety over k. Hence there exists a $U \in SmVar$ such that: $k(U) = K$, v yields a non-empty closed subscheme of U (since the singular locus has codimension ≥ 2 in a normal variety). It easily follows that there exists a pro-scheme S (i.e. an inverse limit of smooth varieties) whose only points are the spectra of K and K_0. So, S is local, hence it is primitive.

By part 2 of the theorem, we have

$$M_{gm}(\operatorname{Spec} K) = M_{gm}(S) \bigoplus M_{gm}(\operatorname{Spec} K_0)(1)[1];$$

this concludes the proof.

□

Remark 4.2.3. 1. Note that we do not construct any explicit splitting morphisms in the decompositions above. Probably, one cannot choose any canonical splittings here (in the general case); so there is no (automatic) compatibility for any pair of related decompositions. Respectively, though comotives of (spectra of) function fields contain tons of direct summands, there seems to be no general way to decompose them into indecomposable summands.
2. Yet Proposition 3.6.1 easily yields that $M_{gm}(\operatorname{Spec} k(t)) \cong \mathbb{Z} \bigoplus \prod_E M_{gm}(E)(1)[1]$; here E runs through all closed points of $\mathbb{A}^1$ (considered as a scheme over k).

4.3 On cohomology of pro-schemes, and its direct summands

The results proved above immediately imply similar assertions for cohomology. We also construct a class of cohomology theories that respect homotopy limits.

Proposition 4.3.1. *Let $H : \mathfrak{D}_s \to \underline{A}$ be cohomological, S be a primitive scheme.*
1. Let S_0 be a dense sub-pro-scheme of S. Then $H(M_{gm}(S))$ is a direct summand of $H(M_{gm}(S_0))$.
2. Suppose moreover that $S_0 = S \setminus T$ where T is a closed subscheme of S of codimension $r > 0$. Then we have $H(M_{gm}(S_0)) \cong H(M_{gm}(S)) \bigoplus H(M_{gm}(T)(r)[2r-1])$.
3. Let S be connected, S_0 be the generic point of S. Then $H(M_{gm}(S))$ is a retract of $H(M_{gm}(S_0))$ in $\underline{A}$.
4. Let K be a function field over k. Let K' be the residue field for a geometric valuation v of K of rank r. Then $H(M_{gm}(K')(r)[r])$ is a retract of $H(M_{gm}(K))$ in $\underline{A}$.
5. Let $H' : DM_{gm}^{eff} \to \underline{A}$ be a cohomological functor, let $\underline{A}$ satisfy AB5. Then Proposition 1.2.1 allows to extend H' to a cohomological functor $H : \mathfrak{D} \to \underline{A}$ that converts inverse limits in $\mathfrak{D}'$ to the corresponding direct limits in $\underline{A}$.

Proof. 1. Immediate from Theorem 4.2.1(1).
2. Immediate from Theorem 4.2.1(2).
3. Immediate from Corollary 4.2.2(1).
4. Immediate from Corollary 4.2.2(2).
5. Immediate from Proposition 1.2.1; note that DM_{gm}^{eff} is skeletally small. Here in order to prove that H converts homotopy limits into direct limits we use part I2 of loc.cit. and Proposition 3.1.1(7).

□

Remark 4.3.2. 1. In the setting of assertion 5 we will call H an *extended* cohomology theory.
Note that for $H' = DM_{gm}^{eff}(-,Y)$, $Y \in ObjDM_{gm}^{eff}$, we have $H = \mathfrak{D}(-,Y)$; see (4).
2. Now recall that for any pro-scheme Z, any $i \geq 0$, $M_{gm}(Z)(i)$ (by definition) could be presented as a countable homotopy limit of geometric motives. Moreover, the same is true for all small countable products of $M_{gm}(Z_l)(i)$. Hence if H is extended, then the cohomology of $\prod M_{gm}(Z_l)(i)$ is the corresponding direct limit; this coincides with the definition given by (29) (cf. Remark 3.2.1). In particular, one can apply the results of Proposition 4.3.1 to the usual étale cohomology of pro-schemes mentioned (with values in Ab or in some category of Galois modules).
3. If H' is also a tensor functor (i.e. it converts tensor product in DM_{gm}^{eff} into tensor products in $D(\underline{A})$), then certainly the cohomology of $M_{gm}(K')(r)[r]$ is the corresponding tensor product of $H^*(M_{gm}(K'))$ with $H^*(\mathbb{Z}(r)[r])$. Note that the latter one is a retract of $H^*(G_m^r)$; we obtain the Tate twist for cohomology this way.

4.4 Coniveau spectral sequences for cohomology of (co)motives

Let $H : \mathfrak{D}_s^{op} \to \underline{A}$ be a cohomological functor, $X \in Obj\mathfrak{D}_s$.

Proposition 4.4.1. *1. Any choice of a weight spectral sequence $T(H,X)$ (see Theorem 2.4.2) corresponding to the Gersten weight structure w is canonical and $\mathfrak{D}_s$-functorial in X starting from E_2.*
2. $T(H,X)$ converges to $H(X)$.
3. Let H be an extended theory (see Remark 4.3.2), $X = M_{gm}(Z)$ for $Z \in SmVar$. Then any choice of $T(H,X)$ starting from E_2 is canonically isomorphic to the classical coniveau spectral sequence (converging to the H-cohomology of Z; see §1 of [8]).

Proof. 1. This is just a partial case of Theorem 2.4.2(I).
2. Immediate since w is bounded; see part I2 of loc.cit.
3. Recall that in the proof of Corollary 3.6.2 a certain Postnikov tower $Po(X)$ for X was obtained from certain 'geometric' Postnikov towers (in $j(C^b(SmCor))$) by passing to the homotopy limit. Now, the coniveau spectral sequence (for the H-cohomology of Z) in §1.2 of [8] was constructed by applying H to the same geometric towers and then passing to the inductive limit (in $\underline{A}$). Furthermore, Remark 4.3.2(2) yields that the latter limit is (naturally) isomorphic to the spectral sequence obtained via H from $Po(X)$. Next, since $Po(X)$ is a weight Postnikov tower for X (see Proposition 4.1.1(5)), we obtain that the latter spectral sequence is one of the possible choices for $T(H,X)$.
Lastly, assertion 1 yields that all other possible $T(H,X)$ (they depend on the choice of a weight Postnikov tower for X) starting from E_2 are also canonically isomorphic to the classical coniveau spectral sequence mentioned.

□

Remark 4.4.2. 1. Hence we proved (in particular) that classical coniveau spectral sequences (for cohomology theories that could be factorized through motives; this includes étale and singular cohomology of smooth varieties) are DM_{gm}^{eff}-functorial (starting from E_2); we also obtain such a functoriality for the coniveau filtration for cohomology! These facts are far from being obvious from the usual definition of the coniveau filtration and spectral sequences, and seem to be new (in the general case). So, we justified the title of the paper.
We also obtain certain coniveau spectral sequences for cohomology of singular varieties (for cohomology theories that could be factorized through DM_{gm}^{eff}; in the case $\operatorname{char} k > 0$ one also needs rational coefficients here).
2. Assertion 3 of the proposition yields a nice reason to call (any choice of) $T(H, X)$ a *coniveau spectral sequence* (for a general $H, \underline{A}$, and $X \in Obj\mathfrak{D}_s$); this will also distinguish (this version of) T from weight spectral sequences corresponding to other weight structures. We will give more justification for this term in Remark 4.5.4 below. So, the corresponding filtration could be called the (generalized) coniveau filtration.

4.5 An extension of results of Bloch and Ogus

Now we want to relate coniveau spectral sequences with the homotopy t-structure (in DM_{-}^{eff}). This would be a vast extension of the seminal results of §6 of [5] (i.e. of the calculation by Bloch and Ogus of the E_2-terms of coniveau spectral sequences) and of §6 of [11].
We should relate t (for DM_{-}^{eff}) and w; it turns out that they are orthogonal with respect to a certain quite natural nice duality.

Proposition 4.5.1. *For any $Y \in ObjDM_{-}^{eff}$ we extend $H' = DM_{-}^{eff}(-, Y)$ from DM_{gm}^{eff} to $\mathfrak{D} \supset \mathfrak{D}_s$ by the method of Proposition 1.2.1; we define $\Phi(X, Y) = H(X)$. Then the following statements are valid.*
1. Φ is a nice duality (see Definition 2.5.1).
2 w is left orthogonal to the homotopy t-structure t (on DM_{-}^{eff}) with respect to Φ.
3. $\Phi(-, Y)$ converts homotopy limits (in $\mathfrak{D}'$) into direct limits in Ab.

Proof. 1. By Proposition 2.5.6(1), the restriction of Φ to $DM_{gm}^{eff\,op} \times DM_{-}^{eff}$ is a nice duality. It remains to apply part 3 of loc.cit.
2. In the notation of Proposition 2.5.3, we take for D the set of all small products $\prod_{l \in L} M_{gm}(K_l)(n_l)[n_l] \in Obj\mathfrak{D}_s$; here $M_{gm}(K_l)$ denote comotives of (spectra of) some function fields over k, $n_l \geq 0$ and the transcendence degrees of K_l/k are bounded (cf. §4.1). Then D, Φ satisfy the assumptions of the proposition by Proposition 3.4.1(2) (see also Remark 4.3.2(2)).
3. Immediate from Proposition 4.3.1(3).

$\square$

Remark 4.5.2. 1. Suppose that we have an inductive family $Y_i \in ObjDM_{-}^{eff}$ connected by a compatible family of morphisms with some $Y \in DM_{-}^{eff}$ such

that: for any $Z \in ObjDM_{gm}^{eff}$ we have $DM_{-}^{eff}(Z,Y) \cong \varinjlim DM_{-}^{eff}(Z,Y_i)$ (via these morphisms $Y_i \to Y$). In such a situation it is reasonable to call Y a homotopy colimit of Y_i.
The definition of Φ in the proposition easily implies: for any $X \in Obj\mathfrak{D}$ we have $\Phi(X,Y) = \varinjlim \Phi(X,Y_i)$. So, one may say that all objects of $\mathfrak{D}$ are 'compact with respect to Φ', whereas part 3 of the proposition yields that all objects of DM_{-}^{eff} are 'cocompact with respect to Φ'. Note that no analogues of these nice properties can hold in the case of an adjacent weight and t-structure (defined on a single triangulated category).
2. Now, we could have replaced DM_{gm}^{eff} by DM_{gm} everywhere in the 'axiomatics' of $\mathfrak{D}$ (in Proposition 3.1.1). Then the corresponding category $\mathfrak{D}_{gm}$ could be used for our purposes (instead of $\mathfrak{D}$), since our arguments work for it also. Note that we can extend Φ to a nice duality $\mathfrak{D}_{gm}^{op} \times DM_{-}^{eff} \to Ab$; to this end it suffices for $Y \in ObjDM_{-}^{eff}$ to extend H' to DM_{gm} in the following way: $H'(X(-n)) = DM_{-}^{eff}(X, Y(n))$ for $X \in ObjDM_{gm}^{eff} \subset ObjDM_{gm}$, $n \geq 0$.
Moreover, the methods of §5.4.3 allow to define an invertible Tate twist functor on $\mathfrak{D}_{gm}$.

Corollary 4.5.3. *1. If H is represented by a $Y \in ObjDM_{-}^{eff}$ (via our Φ) then for a (co)motif X our coniveau spectral sequence $T(H,X)$ starting from E_2 could be naturally expressed in terms of the cohomology of X with coefficients in t-truncations of Y (as in Theorem 2.6.1).*
In particular, the coniveau filtration for $H^*(X)$ could be described as in part 2 of loc.cit.
2. For $U \in ObjDM_{gm}^{eff}$, $i \in \mathbb{Z}$, we have $U \in \mathfrak{D}_s^{w \leq i} \iff U \in DM_{-}^{eff\,t \leq i}$.

Proof. 1. Immediate from Proposition 4.5.1.
2. By Theorem 2.2.1(20), we should check whether $Z \perp U$ for any $Z = \prod_{l \in L} M_{gm}(K_l)(n_l)[n_l + r]$, where K_l are function fields over k, $n_l \geq 0$ and the transcendence degrees of K_l/k are bounded, $r > 0$ (see Proposition 4.1.1(2)). Moreover, since U is cocompact in $\mathfrak{D}$, it suffices to consider $Z = M_{gm}(K')(n)[n+r]$ (K'/k is a function field, $n \geq 0$). Lastly, Corollary 4.2.2(2) reduces the situation to the case $Z = M_{gm}(K)$ (K/k is a function field).
Hence (25) implies: $U \in \mathfrak{D}_s^{w \leq i}$ whenever for any $j > i$, any function field K/k, the stalk of $U^{t=j}$ at K is zero. Now, if $U \in DM_{-}^{eff\,t \leq i}$ then $U^{t=j} = 0$ for all $j > i$; hence all stalks of $U^{t=j}$ are zero. Conversely, if all stalks of $U^{t=j}$ at function fields are zero, then Corollary 4.19 of [26] yields $U^{t=j} = 0$ (see also Corollary 4.20 of loc.cit.); if $U^{t=j} = 0$ for all $j > i$ then $U \in DM_{-}^{eff\,t \leq i}$.

□

Remark 4.5.4. 1. Our comparison statement is true for Y-cohomology of an arbitrary $X \in ObjDM_{gm}^{eff}$; this extends to motives Theorem 6.4 of [11] (whereas the latter essentially extends the results of §6 of [5]). We obtain one more reason to call T (in this case) the coniveau spectral sequence for (cohomology of) motives.

2. If $Y \in ObjHI$, then $E_2(T)$ yields the Gersten resolution for Y (when X varies); this is why we called w the Gersten weight structure.
3. Now, let Y represent étale cohomology with coefficients in $\mathbb{Z}/l\mathbb{Z}$, l is prime to $\operatorname{char} k$ (Y is actually unbounded from above, yet this is not important). Then the t-truncations of Y represent $\mathbb{Z}/l\mathbb{Z}$-motivic cohomology by the (recently proved) Beilinson-Lichtenbaum conjecture (see [28]; this paper is not published at the moment). Hence Proposition 2.5.4(1) yields some new formulae for $\mathbb{Z}/l\mathbb{Z}$-motivic cohomology of X and for the 'difference' between étale and motivic cohomology. Note also that the virtual t-truncations (mentioned in loc.cit.) are exactly the D_2-terms of the alternative exact couple for $T(H,X)$ and for the version of the exact couple used in the current paper respectively (i.e. we consider exact couples coming from the two possible versions for a weight Postnikov tower for X, as described in Remark 2.1.3). See also §7.5 of [6] for more explicit results of this sort. It could also be interesting to study coniveau spectral sequences for singular cohomology; this could yield a certain theory of 'motives up to algebraic equivalence'; see Remark 7.5.3(3) of loc.cit. for more details.
5. Assertion 2 of the corollary yields that $\mathfrak{D}_s^{w\le 0} \cap ObjDM_{gm}^{eff}$ is large enough to recover w (in a certain sense); in particular, this assertion is similar to the definition of adjacent structures (see Remark 2.5.7). In contrast, $\mathfrak{D}_s^{w\ge 0} \cap ObjDM_{gm}^{eff}$ seems to be too small.

4.6 Base field change for coniveau spectral sequences; functoriality for an uncountable k

It can be easily seen (and well-known) that for any perfect field extension l/k there exist an extension of scalars functor $DM_{gm}^{eff}{}_k \to DM_{gm}^{eff}{}_l$ compatible with the extension of scalars for smooth varieties (and for $K^b(SmCor)$). In 5.4.2 below we will prove that this functor could be expanded to a functor $\mathrm{Ext}_{l/k} : \mathfrak{D}_k \to \mathfrak{D}_l$ that sends $M_{gm,k}(X)$ to $M_{gm,l}(X_l)$ for a pro-scheme X/k; this extension procedure is functorial with respect to embeddings of base fields. Moreover, $\mathrm{Ext}_{l/k}$ maps $\mathfrak{D}_{sk}$ into $\mathfrak{D}_{sl}$. Note the existence of base change for comotives does not follow from the properties of $\mathfrak{D}$ listed in Proposition 3.1.1; yet one can define base change for our model of comotives (described in §5 below) and (probably) for any other possible reasonable version of $\mathfrak{D}$.
Now we prove that base change for comotives yields base change for coniveau spectral sequences; it also allows to prove that these spectral sequences are motivically functorial for not necessary countable base fields.
In order to make the limit in Proposition 4.6.1(2) below well-defined, we assume that for any $X \in ObjDM_{gm}^{eff}$ there is a fixed representative Y, Z, p chosen, where: $Z, Y \in C^b(SmCor)$, $M_{gm}(Y) \cong M_{gm}(Z)$, $p \in C^b(SmCor)(Y,Z)$ yields a direct summand of $M_{gm}(Y)$ in DM_{gm}^{eff} that is isomorphic to X. We also assume that all the components of (X, Y, p) have fixed expressions in terms of algebraic equations over k; so one may speak about fields of definition for X.

Proposition 4.6.1. *Let l be a perfect field, $H : \mathfrak{D}_l \to \underline{A}$ be any cohomological functor (for an abelian $\underline{A}$). For any perfect $k \subset l$ we denote $H \circ \mathrm{Ext}_{l/k} : \mathfrak{D}_k \to \underline{A}$ by H_k.*

1. Let l be countable. Then for any $X \in Obj\mathfrak{D}_k$ the method of Proposition 2.7.3(II) yields some morphism $N_{l/k} : T_{w_k}(H_k, X) \to T_{w_l}(H, \mathrm{Ext}_{l/k}(X))$; this morphism is unique and $\mathfrak{D}_k$-functorial in X starting from E_2.

The correspondence $(l,k) \mapsto N_{l/k}$ is associative with respect to extensions of countable fields (starting from E_2); cf. part I3 of loc.cit.

2. Let l be a not (necessarily) countable perfect field, let $\underline{A}$ satisfy AB5.

For $X \in ObjDM_{gm}^{eff}{}_l$ we define $T_w(H,X) = \varinjlim_k T_{w_k}(H_k, X_k)$. Here we take the limit with respect to all perfect $k \subset l$ such that k is countable, X is defined over k; the connecting morphisms are given by the maps $N_{-/-}$ mentioned in assertion 1; we start our spectral sequences from E_2. Then $T_w(H,X)$ is a well-defined spectral sequence that is $DM_{gm}^{eff}{}_l$-functorial in X.

3. If $X = M_{gm,l}(Z)$, $Z \in SmVar$, H is as an extended theory, and $\underline{A}$ satisfies AB5, the spectral sequence given by the previous assertion is canonically isomorphic to the classical coniveau spectral sequence (for (H,Z); considered starting from E_2).

Proof. 1. By Proposition 2.7.3(II) it suffices to check that $\mathrm{Ext}_{l/k}$ is left weight-exact (with respect to weight structures in question). We take D being the class of all small products $\prod_{l \in L} M_{gm}(K_l)$, where $M_{gm}(K_l)$ denote comotives of (spectra of) function fields over k of bounded transcendence degree. Proposition 4.1.1 and Corollary 4.2.2(2) yield that any $X \in \mathfrak{D}_{s_k}^{w=0}$ is a retract of some element of D. It suffices to check that for any $X = \prod_{l \in L} M_{gm,k}(K_l)$ we have $\mathrm{Ext}_{l/k} X \in \mathfrak{D}_{s_l}^{w_l \le 0}$; here we recall that w_k is bounded and apply Lemma 2.7.5.

Now, X is the comotif of a certain pro-scheme, hence the same is true for $\mathrm{Ext}_{l/k} X$. It remains to apply Proposition 4.1.1(5).

2. By the associativity statement in the previous assertion, the limit is well-defined. Since $\underline{A}$ satisfies AB5, we obtain a spectral sequence indeed. Since we have k-motivic functoriality of coniveau spectral sequences over each k, we obtain l-motivic functoriality in the limit.

3. Again (as in the proof of Proposition 4.4.1(3)) we recall that the classical coniveau spectral sequence for this case is defined by applying H to 'geometric' Postnikov towers (coming from elements of L as in the proof of Corollary 3.6.2) and then passing to the limit (in $\underline{A}$) with respect to L. Our assertion follows easily, since each $l \in L$ is defined over some perfect countable $k \subset l$; the limit of the spectral sequences with respect to the subset of L defined over a fixed k is exactly $T_{w_k}(H_k, X_k)$ since H sends homotopy limits to inductive limits in $\underline{A}$ (being an extended theory).

Here we certainly use the functoriality of T starting from E_2.

□

Remark 4.6.2. 1. For a general $X \in ObjDM_{gm}^{eff}$ we only have a canonical choice of base change maps (for $T(H_{k_l}, X)$) starting from E_2; this is why we start our spectral sequence from the E_2-level.
2. Assertion 2 of the proposition is also valid for any comotif defined over a (perfect) countable subfield of l. Unfortunately, this does not seem to include comotives of function fields over l (of positive transcendence degrees, if l is not countable).

4.7 The Chow weight structure for $\mathfrak{D}$

Till the end of the section, we will either assume that $\operatorname{char} k = 0$, or that we deal with motives, comotives, and cohomology with rational coefficients (we will use the same notation for motives with integral and rational coefficients; cf. §6.3 below).
We prove that $\mathfrak{D}$ supports a weight structure that extends the Chow weight structure of DM_{gm}^{eff} (see §6.5 and Remark 6.6.1 of [6], and also [7]).
In this subsection we do not require k to be countable.

Proposition 4.7.1. *1. There exists a Chow weight structure on DM_{gm}^{eff} that is uniquely characterized by the condition that all $M_{gm}(P)$ for $P \in SmPrVar$ belong to its heart; it could be extended to a weight structure w_{Chow} on $\mathfrak{D}$.*
2. The heart of w_{Chow} is the category H_{Chow} of arbitrary small products of (effective) Chow motives.
3. We have $X \in \mathfrak{D}^{w_{Chow} \geq 0}$ if and only if $\mathfrak{D}(X, Y[i]) = \{0\}$ for any $Y \in ObjChow^{eff}$, $i > 0$.
4. There exists a t-structure t_{Chow} on $\mathfrak{D}$ that is right adjacent to w_{Chow} (see Remark 2.5.7). Its heart is the opposite category to $Chow^{eff*}$ (i.e. it is equivalent to $(\mathrm{AddFun}(Chow^{eff}, Ab))^{op}$).
5. w_{Chow} respects products i.e. $X_i \in \mathfrak{D}^{w_{Chow} \leq 0} \implies \prod X_i \in \mathfrak{D}^{w_{Chow} \leq 0}$ and $X_i \in \mathfrak{D}^{w_{Chow} \geq 0} \implies \prod X_i \in \mathfrak{D}^{w_{Chow} \geq 0}$.
6. For $\prod X_i$ there exists a weight decomposition: $\prod X_i \to \prod X_i^{w \leq 0} \to \prod X_i^{w \geq 1}$.
7. If $H : \mathfrak{D} \to \underline{A}$ is an extended theory, then the functor that sends X to the derived exact couple for $T_{w_{Chow}}(H, X)$ (see Theorem 2.4.2) converts all small products into direct sums.

Proof. 1. It was proved in (Proposition 6.5.3 and Remark 6.6.1 of) [6] that there exists a unique weight structure w'_{Chow} on DM_{gm}^{eff} such that $M_{gm}(P) \in \mathfrak{D}^{w'_{Chow}=0}$ for all $P \in SmPrVar$. Moreover, the heart of this structure is exactly $Chow^{eff} \subset DM_{gm}^{eff}$.
Now, DM_{gm}^{eff} is generated by $Chow^{eff}$. It easily follows that $\{M_{gm}(P),\ P \in SmPrVar\}$ weakly cogenerates $\mathfrak{D}$. Then the dual (see Theorem 2.2.1(1)) of Theorem 4.5.2(I2) of [6] yields that w'_{Chow} could be extended to a weight structure w_{Chow} for $\mathfrak{D}$. Moreover, the dual to part II1 of loc.cit. yields that for this extension we have: $\underline{Hw}_{Chow}$ is the idempotent completion of H_{Chow}.

2. It remains to prove that H_{Chow} is idempotent complete. This is obvious since $Chow^{eff}$ is.
3. This is just the dual of (27) in loc.cit.
4. The dual statement to part I2 of loc.cit. (cf. Remark 1.1.3(1)) yields the existence of t_{Chow}. Applying the dual of Theorem 4.5.2(II1) of [6] we obtain for the heart of t: $\underline{Ht}_{Chow} \cong (Chow_*^{eff})^{op}$.
5. Theorem 2.2.1(2) easily yields that $\mathfrak{D}^{w_{Chow}\le 0}$ is stable with respect to products. The stability of $\mathfrak{D}^{w_{Chow}\ge 0}$ with respect to products follows from assertion 3; here we recall that all objects of $Chow^{eff}$ are cocompact in $\mathfrak{D}$.
6. Immediate from the previous assertion; note that any small product of distinguished triangles is distinguished (see Remark 1.2.2 of [21]).
7. Since H is extended, it converts products in $\mathfrak{D}$ into direct sums in $\underline{A}$. Hence for any $X_i \in Obj\mathfrak{D}$ there exist a choice of exact couples for the corresponding weight spectral sequences for X_i and $\prod X_i$ that respects products i.e such that $D_1^{pq}T_{w_{Chow}}(H, \prod X_i) \cong \bigoplus_i D_1^{pq}T_{w_{Chow}}(H, X_i)$ and $E_1^{pq}T_{w_{Chow}}(H, \prod X_i) \cong \bigoplus_i E_1^{pq}T_{w_{Chow}}(H, X_i)$ (for all $p.q \in \mathbb{Z}$; this isomorphism is also compatible with the connecting morphisms of couples). Since $\underline{A}$ satisfies AB5, we obtain the isomorphism desired for D_2 and E_2-terms (note that those are uniquely determined by H and X).

□

Remark 4.7.2. 1. In Remark 2.4.3 of [6] it was shown that weight spectral sequences corresponding to the Chow weight structure are isomorphic to the classical (i.e. Deligne's) weight spectral sequences when the latter are defined (i.e. for singular or étale cohomology of varieties). Yet in order to specify the choice of a weight structure here we will call these spectral sequences *Chow-weight* ones.
2. All the assertions of the Proposition could be extended to arbitrary triangulated categories with negative families of cocompact weak cogenerators (sometimes one should also demand all products to exist; in assertion 7 we only need H to convert all products into direct sums).
3. Since (effective) Chow motives are cocompact in $\mathfrak{D}$, $\underline{Hw}_{Chow}$ is the category of 'formal products' of $Chow^{eff}$ i.e. $\mathfrak{D}(\prod_{l\in L} X_l, \prod_{i\in I} Y_i) = \prod_{i\in I}(\oplus_{l\in L} Chow^{eff}(X_l, Y_i))$ for $X_l, Y_l \in ObjChow^{eff} \subset Obj\mathfrak{D}$ (cf. Remark 4.5.3(2) of [6]).
4. Recall (see §7.1 of ibid.) that DM_-^{eff} supports (adjacent) Chow weight and t-structures (we will denote them by w'_{Chow} and t'_{Chow}, respectively). One could also check that these structures are right orthogonal to the corresponding Chow structures for $\mathfrak{D}$. Hence, applying Proposition 2.5.4(1) repeatedly one could relate the compositions of truncations (on $\mathfrak{D}_s \subset \mathfrak{D}$) via w and via t_{Chow} (resp. via w and via w_{Chow}) with truncations via t and via w'_{Chow} (resp. via t and via t'_{Chow}) on DM_-^{eff}; cf. §8.3 of [6]. One could also apply w_{Chow}-truncations and then w-truncations (i.e. compose truncations in the opposite order) when starting from an object of DM_{gm}^{eff}. Recall also that truncations via

t_{Chow} (and their compositions with t-truncations) are related with unramified cohomology; see Remark 7.6.2 of ibid.

4.8 Comparing Chow-weight and coniveau spectral sequences

Now we prove that Chow-weight and coniveau spectral sequences are naturally isomorphic for birational cohomology theories.

Proposition 4.8.1. *1. w_{Chow} for $\mathfrak{D}$ dominates w (for $\mathfrak{D}_s$) in the sense of §2.7.*
2. Let $H : DM_{gm}^{eff} \to \underline{A}$ be an extended cohomology theory in the sense of Remark 4.3.2; suppose that H is birational i.e. that $H(M_{gm}(P)(1)[i]) = 0$ for all $P \in SmPrVar$, $i \in \mathbb{Z}$. Then for any $X \in Obj\mathfrak{D}_s$ the Chow-weight spectral sequence $T_{w_{Chow}}(H, X)$ (corresponding to w_{Chow}) is naturally isomorphic starting from E_2 to (our) coniveau spectral sequence $T_w(H, X)$ via the comparison morphism M given by Proposition 2.7.3(I1).

Proof. 1. Let D be the class of all countable products $\prod_{l\in L} M_{gm}(K_l)$, where $M_{gm}(K_l)$ denote comotives of (spectra of) function fields over k of bounded transcendence degree. Proposition 4.1.1 and Corollary 4.2.2(2) yield that any $X \in \mathfrak{D}_s^{w=0}$ is a retract of some element of D. It suffices to check that any $X = \prod_{l\in L} M_{gm}(K_l)$ belongs to $\mathfrak{D}^{w_{Chow}\geq 0}$; here we recall that w is bounded and apply Lemma 2.7.5.
By Proposition 4.7.1(5), we can assume that L consists of a single element. In this case we have $\mathfrak{D}(M_{gm}(K_l), M_{gm}(P)[i]) = 0$ (this is a trivial case of Proposition 3.5.1); hence loc.cit. yields the result.
2. We take the same D and X as above.
Let $\operatorname{char} k = 0$. We choose $P_l \in SmPrVar$ such that K_l are their function fields. Since all $M_{gm}(P_l)$ are cocompact in $\mathfrak{D}$, we have a natural morphism $X \to \prod M_{gm}(P_l)$. By Proposition 2.7.3(I2), it suffices to check that $\operatorname{Cone}(X \to \prod M_{gm}(P_l)) \in \mathfrak{D}^{w_{Chow}\geq 0}$, $H(X) \cong H(\prod M_{gm}(P_l))$, and $E_2^{**}T_{w_{Chow}}(H, \operatorname{Cone}(X \to \prod M_{gm}(P_l))) = 0$.
By Proposition 4.7.1(7) we obtain: it suffices again to verify these statements in the case when L consists of a single element. Now, we have $\operatorname{Spec}(K_l) = \varprojlim M_{gm}(U)$ for $U \in SmVar$, $k(U) = K_l$. Therefore (27) yields: it suffices to verify assertions required for $Z = M_{gm}(U \to P)$ instead, where $U \in SmVar$, U is open in $P \in SmPrVar$.
The Gysin distinguished triangle for Voevodsky's motives (see Proposition 2.4.5 of [9]) easily yields by induction that $Z \in ObjDM_{gm}^{eff}(1)$.
Since $Chow^{eff}$ is $-\otimes \mathbb{Z}(1)[2]$-stable, we obtain that there exists a w_{Chow}-Postnikov tower for Z such that all of its terms are divisible by $\mathbb{Z}(1)$; this yields the vanishing of $E_2^{**}T_{w_{Chow}}(H, Z)$. Lastly, the fact that $Z \in DM_{gm}^{eff\, w'_{Chow}\geq 0}$ was (essentially) proved by easy induction (using the Gysin triangle) in the proof of Theorem 6.2.1 of [7].
In the case $\operatorname{char} k > 0$, de Jong's alterations allow to replace $M_{gm}(P_l)$ in the reasoning above by some Chow motives (with rational coefficients); see

Appendix B of [14]; we will not write down the details here.

□

Remark 4.8.2. Assertion 2 is not very actual for cohomology of smooth varieties since any $Z \in SmPrVar$ is birationally isomorphic to $P \in SmPrVar$ (at least for $\operatorname{char} k = 0$). Yet the statement becomes more interesting when applied for $X = M^c_{gm}(Z)$.

4.9 Birational motives; constructing the Gersten weight structure by gluing; other possible weight structures

An alternative way to prove Proposition 4.8.1(2) is to consider (following [15]) the category of *birational comotives*. It satisfies the following properties:
(i) All birational cohomology theories factorize through it.
(ii) Chow and Gersten weight structures induce the same weight structure on it (see Definition 2.7.1(4)).
(iii) More generally, for any $n \geq 0$ Chow and Gersten weight structures induce weight structures on the localizations $\mathfrak{D}(n)/\mathfrak{D}(n+1) \cong \mathfrak{D}_{bir}$ (we call these localizations *slices*) that differ only by a shift.
Moreover, one could 'almost recover' original Chow and Gersten weight structures starting from this single weight structure.
Now we describe the constructions and facts mentioned in more detail. We will be rather sketchy here, since we will not use the results of this subsection elsewhere in the paper. Possibly, the details will be written down in another paper.
As we will show in §5.4.3 below, the Tate twist functor could be extended (as an exact functor) from DM^{eff}_{gm} to $\mathfrak{D}$; this functor is compatible with (small) products.

Proposition 4.9.1. *I The functor $-\otimes\mathbb{Z}(1)[1]$ is weight-exact with respect to w on $\mathfrak{D}_s$; $-\otimes\mathbb{Z}(1)[2]$ is weight-exact with respect to w_{Chow} on $\mathfrak{D}$ (we will say that w is $-\otimes\mathbb{Z}(1)[1]$-stable, and w_{Chow} is $-\otimes\mathbb{Z}(1)[2]$-stable).*
II Let $\mathfrak{D}_{bir}$ denote the localization of $\mathfrak{D}$ by $\mathfrak{D}(1)$; B is the localization functor. We denote $B(\mathfrak{D}_s)$ by $\mathfrak{D}_{s,bir}$.
1. w_{Chow} induces a weight structure w'_{bir} on $\mathfrak{D}_{bir}$. Besides, w induces a weight structure w_{bir} on $\mathfrak{D}_{s,bir}$.
2. We have $\mathfrak{D}_{s,bir}^{w_{bir}\leq 0} \subset \mathfrak{D}_{bir}^{w'_{bir}\leq 0}$, $\mathfrak{D}_{s,bir}^{w_{bir}\geq 0} \subset \mathfrak{D}_{bir}^{w'_{bir}\geq 0}$ (i.e. the embedding $(\mathfrak{D}_{s,bir}, w_{bir}) \to (\mathfrak{D}_{bir}, w'_{bir})$ is weight-exact).
3. For any pro-scheme U we have $B(M_{gm}(U)) \in \mathfrak{D}_{s,bir}^{w_{bir}=0}$.

Proof. I This is easy, since the functors mentioned obviously map the corresponding hearts (of weight structures) into themselves.
II 1. By assertion I, w_{Chow} induces a weight structure on $\mathfrak{D}(1)$ (i.e. $\mathfrak{D}(1)$ is a triangulated category, $Obj\mathfrak{D}(1) \cap \mathfrak{D}^{w_{Chow}\leq 0}$ and $Obj\mathfrak{D}(1) \cap \mathfrak{D}^{w_{Chow}\geq 0}$ yield a weight structure on it). Hence by Proposition 8.1.1(1) of [6] we obtain existence

(and uniqueness) of w'_{bir}. The same argument also implies the existence of some w_{bir} on $\mathfrak{D}_{s,bir}$.
2. Now we compare w_{bir} with w'_{bir}. Since w is bounded, w_{bir} also is (see loc.cit.). Hence it suffices to check that $\underline{Hw}_{bir} \subset \underline{Hw'}_{bir}$ (see Theorem 2.2.1(19)).
Moreover, it suffices to check that for $X = \prod_{l\in L} M_{gm}(K_l)$ we have $B(X) \in \mathfrak{D}_{bir}^{w'_{bir}=0}$ (since $\mathfrak{D}_{bir}^{w'_{bir}=0}$ is Karoubi-closed in $\mathfrak{D}_{bir}$, here we also apply Proposition 4.7.1(2)). As in the proof of Proposition 4.8.1(2), we will consider the case $\operatorname{char} k = 0$; the case $\operatorname{char} k = p$ is treated similarly. Then we choose $P_l \in SmPrVar$ such that K_l are their function fields; we have a natural morphism $X \to \prod M_{gm}(P_l)$. It remains to check that $\operatorname{Cone}(X \to \prod M_{gm}(P_l)) \in \mathfrak{D}_s(1)$. Now, since $\mathfrak{D}_s(1)$ and the class of distinguished triangles are closed with respect to small products, it suffices to consider the case when L consists of a single element. In this case the statement is immediate from Corollary 3.6.2.
3. Immediate from Corollary 3.6.2.

□

Remark 4.9.2. 1. Assertion II easily implies Proposition 4.8.1(2).
Indeed, any extended birational H (as in loc.cit.) could be factorized as $G \circ B$ for a cohomological $G : \mathfrak{D}_{bir} \to \underline{A}$. Since B is weight-exact with respect to w_{Chow} (and its restriction to $\mathfrak{D}_s$ is weight-exact with respect to w), (the trivial case of) Proposition 2.7.3(I2) implies that for any $X \in Obj\mathfrak{D}$ (any choice) of $T_{w'_{bir}}(G, B(X))$ is naturally isomorphic starting from E_2 to any choice of $T_{w_{Chow}}(H, X)$; for any $X \in Obj\mathfrak{D}_s$ (any choice) of $T_{w_{bir}}(G, B(X))$ is naturally isomorphic starting from E_2 to any choice of $T_w(H, X)$.
It is also easily seen that the isomorphism $T_{w_{Chow}}(H, X) \to T_w(H, X)$ is compatible with the comparison morphism M (see loc.cit.).
2. The proof of existence of w_{bir} and of assertion 3 works with integral coefficients even if $\operatorname{char} k > 0$. Hence we obtain that that the category image $B(M_{gm}(U))$, $U \in SmVar$, is negative. We can apply this statement in $\underline{C}$ being the idempotent completion of $B(DM_{gm}^{eff})$ i.e. in the category of birational comotives. Hence Theorem 2.2.1(18) yields: there exists a weight structure for $\underline{C}$ whose heart is the category of birational Chow motives (defined as in §5 of [15]). Note also that one can pass to the inductive limit with respect to base change in this statement (cf. §4.6); hence one does not need to require k to be countable.

Now we explain that w and w_{Chow} could be 'almost recovered' from $(\mathfrak{D}_{bir}, w'_{bir})$. Exactly the same reasoning as above shows that for any $n > 0$ the localization of $\mathfrak{D}$ by $\mathfrak{D}(n)$ could be endowed with a weight structure w'_n compatible with w_{Chow}, whereas the localization of $\mathfrak{D}_s$ by $\mathfrak{D}_s(n)$ could be endowed with a weight structure w_n compatible with w.
Next, we have a short exact sequence of triangulated categories $\mathfrak{D}/\mathfrak{D}(n) \xrightarrow{i_*} \mathfrak{D}/\mathfrak{D}(n+1) \xrightarrow{j^*} \mathfrak{D}_{bir}$. Here the notation for functors comes from the 'classical' gluing data setting (cf. §8.2 of [6]); i_* could be given by $-\otimes \mathbb{Z}(1)[s]$ for any

$s \in \mathbb{Z}$, j^* is just the localization. Now, if we choose $s = 2$ then i_* is weight-exact with respect to w'_n and w'_{n+1}; if we choose $s = 1$ then the restriction of i_* to $\mathfrak{D}_s/\mathfrak{D}_s(n)$ is weight-exact with respect to w_n and w_{n+1}.

Next, an argument similar to the one used in §8.2 of [6] shows: for any short exact sequence $\underline{D} \xrightarrow{i_*} \underline{C} \xrightarrow{j^*} \underline{E}$ of triangulated categories, if $\underline{D}$ and $\underline{E}$ are endowed with weight structures, then there exist at most one weight structure on $\underline{C}$ such that both i_* and j^* are weight-exact (see also Lemma 4.6 of [3] for the proof of a similar statement for t-structures). Hence one can recover w_n and w'_n from (copies of) w'_{bir}; the main difference between them is that the first one is $-\otimes\mathbb{Z}(1)[1]$-stable, whereas the second one is $-\otimes\mathbb{Z}(1)[2]$-stable. It is quite amazing that weight structures corresponding to spectral sequences of quite distinct geometric origin differ just by [1] here! If one calls the filtration of $\mathfrak{D}$ by $\mathfrak{D}(n)$ the *slice filtration* (this term was already used by A. Huber, B. Kahn, M. Levine, V. Voevodsky, and other authors for other 'motivic' categories), then one may say that w_n and w'_n could be recovered from slices; the difference between them is 'how we shift the slices'.

Moreover, Theorem 8.2.3 of [6] shows: if both adjoints to i_* and j^* exist, then one can use this gluing data in order to glue (any pair) of weight structures for $\underline{D}$ and $\underline{E}$ into a weight structure for $\underline{C}$. So, suppose that we have a weight structure $w_{n,s}$ for $\mathfrak{D}/\mathfrak{D}(n)$ that is $-\otimes(1)[s]$-stable and compatible with w'_{bir} on all slices (in the sense described above; so $w'_n = w_{n,2}$, w_n is the restriction of $w_{n,1}$ to $\mathfrak{D}_s/\mathfrak{D}_s(n)$, and all $w_{1,s}$ coincide with w'_{bir}). General homological algebra (see Proposition 3.3 of [18]) yields that all the adjoints required do exist in our case. Hence one can construct $w_{n+1,s}$ for $\mathfrak{D}/\mathfrak{D}(n+1)$ that satisfies similar properties. So, $w_{n,s}$ exist for all $n > 0$ and all $s \in \mathbb{Z}$. Hence Gersten and Chow weight structures (for $\mathfrak{D}_s/\mathfrak{D}_s(n) \subset \mathfrak{D}/\mathfrak{D}(n)$) are members of a rather natural family of weight structures indexed by a single integral parameter. It could be interesting to study other members of it (for example, the one that is $-\otimes\mathbb{Z}(1)$-stable), though possibly w'_n is the only member of this family whose heart is cocompactly generated.

This approach could allow to construct w in the case of a not necessarily countable k. Note here that the system of $\mathfrak{D}_s/\mathfrak{D}_s(n)$ yields a fine approximation of $\mathfrak{D}_s$. Indeed, if $X \in SmPrVar$, $n \geq \dim X$, then Poincare duality yields: for any $Y \in ObjDM^{eff}_{gm}$ we have $DM^{eff}_{gm}(Y(n), M_{gm}(X)) \cong DM^{eff}_{gm}(Y \otimes X(n-\dim X)[-2\dim X], \mathbb{Z})$; this is zero if $n > \dim X$ since $\mathbb{Z}$ is a birational motif. Hence (by Yoneda's lemma) for any $n > 0$ the full subcategory of DM^{eff}_{gm} generated by motives of varieties of dimension less than n fully embeds into $DM^{eff}_{gm}/DM^{eff}_{gm}(n) \subset \mathfrak{D}/\mathfrak{D}(n)$.

It follows that the restrictions of $w_{n,s}$ to a certain series of (sufficiently small) subcategories of $\mathfrak{D}/\mathfrak{D}(n)$ are induced by a single $-\otimes(1)[s]$-stable weight structure w_s for the corresponding subcategory of $\mathfrak{D}$. Here for the corresponding subcategory of $\mathfrak{D}/\mathfrak{D}(n)$ (or $\mathfrak{D}$) one can take the union of the subcategories of $\mathfrak{D}/\mathfrak{D}(n)$ (resp. $\mathfrak{D}$) generated (in an appropriate sense) by comotives of (smooth) varieties of dimension $\leq r$ (with r running through all natural num-

bers). Note that this subcategory of $\mathfrak{D}$ contains DM_{gm}^{eff}.
We also relate briefly our results with the (conjectural) picture for t-structures described in [3]. There another (geometric) filtration for motives was considered; this filtration (roughly) differs from the filtration considered above by (a certain version of) Poincare duality. Now, conjecturally the gr_n of the category of birational motives with rational coefficients (cf. §4.2 of ibid.) should be (the homotopy category of complexes over) an abelian semisimple category. Hence it supports a t-structure which is simultaneously a weight structure. This structure should be the building block of all relevant weight and t-structures for comotives. Certainly, this picture is quite conjectural at the present moment.

Remark 4.9.3. The author also hopes to carry over (some of) the results of the current paper to relative motives (i.e. motives over a base scheme that is not a field), relative comotives, and their cohomology. One of the possible methods for this is the usage of gluing of weight structures (see §8.2 of [6], especially Remark 8.2.4(3) of loc.cit.). Possibly for this situation the 'version of $\mathfrak{D}$' that uses motives with compact support (see §6.4 below) could be more appropriate.

5 The construction of $\mathfrak{D}$ and $\mathfrak{D}'$; base change and Tate twists

Now we construct our categories $\mathfrak{D}'$ and $\mathfrak{D}$ using the differential graded categories formalism.
In §5.1 we recall the definitions of differential graded categories, modules over them, shifts and cones (of morphisms).
In §5.2 we recall main properties of the derived category of (modules over) a differential graded category.
In §5.3 we define $\mathfrak{D}'$ and $\mathfrak{D}$ as the categories opposite to the corresponding categories of modules; then we prove that they satisfy the properties required.
In §5.4 we use the differential graded modules formalism to define base change for motives (extension and restriction of scalars). This yields: our results on direct summands of comotives (and cohomology) of function fields (proved above) could be carried over to pro-schemes obtained from them via base change.
We also define tensoring of comotives by motives, as well as a certain 'cointernal Hom' (i.e. the corresponding left adjoint functor to $X \otimes -$ for $X \in ObjDM_{gm}^{eff}$). These results do not require k to be countable.

5.1 DG-categories and modules over them

We recall some basic definitions; cf. [16] and [12].
An additive category A is called graded if for any $P, Q \in ObjA$ there is a canonical decomposition $A(P,Q) \cong \oplus_i A^i(P,Q)$ defined; this decomposition satisfies $A^i(*,*) \circ A^j(*,*) \subset A^{i+j}(*,*)$. A differential graded category (cf. [12]) is a graded category endowed with an additive operator $\delta : A^i(P,Q) \to A^{i+1}(P,Q)$ for all $i \in \mathbb{Z}, P, Q \in ObjA$. δ should satisfy the equalities $\delta^2 = 0$ (so $A(P,Q)$ is a complex of abelian groups); $\delta(f \circ g) = \delta f \circ g + (-1)^i f \circ \delta g$ for any $P, Q, R \in ObjA$, $f \in A^i(P,Q)$, $g \in A(Q,R)$. In particular, $\delta(id_P) = 0$.

We denote δ restricted to morphisms of degree i by δ^i.

Now we give a simple example of a differential graded category.

For an additive category B we consider the category $B(B)$ whose objects are the same as for $C(B)$ whereas for $P=(P^i)$, $Q=(Q^i)$ we define $B(B)^i(P,Q)=\prod_{j\in\mathbb{Z}}B(P^j,Q^{i+j})$. Obviously $B(B)$ is a graded category. We will also consider a full subcategory $B^b(B)\subset B(B)$ whose objects are bounded complexes.

We set $\delta f=d_Q\circ f-(-1)^i f\circ d_P$, where $f\in B^i(P,Q)$, d_P and d_Q are the differentials in P and Q. Note that the kernel of $\delta^0(P,Q)$ coincides with $C(A)(P,Q)$ (the morphisms of complexes); the image of δ^{-1} are the morphisms homotopic to 0.

Note also that the opposite category to a differential graded category becomes differential graded also (with the same gradings and differentials) if we define $f^{op}\circ g^{op}=(-1)^{pq}(g\circ f)^{op}$ for g,f being composable homogeneous morphisms of degrees p and q, respectively.

For any differential graded A we define an additive category $H(A)$ (some authors denote it by $H^0(A)$); its objects are the same as for A; its morphisms are defined as

$$H(A)(P,Q)=\operatorname{Ker}\delta_A^0(P,Q)/\operatorname{Im}\delta_A^{-1}(P,Q).$$

In the case when $H(A)$ is triangulated (as a full subcategory of the category $\mathcal{K}(A)$ described below) we will say that A is a (differential graded) *enhancement* for $H(A)$.

We will also need $Z(A)$: $Obj Z(A)=Obj A$; $Z(A)(P,Q)=\operatorname{Ker}\delta_A^0(P,Q)$. We have an obvious functor $Z(A)\to H(A)$. Note that $Z(B(B))=C(B)$; $H(B(B))=K(B)$.

Now we define (left differential graded) modules over a small differential graded category A (cf. §3.1 of [16] or §14 of [12]): the objects DG-Mod(A) are those additive functors of the underlying additive categories $A\to B(Ab)$ that preserve gradings and differentials for morphisms. We define DG-Mod$(A)^i(F,G)$ as the set of transformations of additive functors of degree i; for $h\in$ DG-Mod$(A)^i(F,G)$ we define $\delta^i(h)=d_G\circ f-(-1)^i f\circ d_F$. We have a natural Yoneda embedding $Y: A^{op}\to$ DG-Mod(A) (one should apply Yoneda's lemma for the underlying additive categories); it is easily seen to be a full embedding of differential graded categories.

Now we define shifts and cones in DG-Mod(A) componentwisely. For $F\in Obj\,$DG-Mod(A) we set $F[1](X)=F(X)[1]$. For $h\in\operatorname{Ker}\delta^0_{\text{DG-Mod}(A)}(F,G)$ we define the object Cone(h): Cone$(h)(X)=$ Cone$(F(X)\to G(X))$ for all $X\in Obj A$.

Note that for $A=B(B)$ both of these definitions are compatible with the corresponding notions for complexes (with respect to the Yoneda embedding). We have a natural triangle of morphisms in $\delta^0_{\text{DG-Mod}(A)}$:

$$P\xrightarrow{f}P'\to\operatorname{Cone}(f)\to P[1]. \tag{30}$$

5.2 The derived category of a differential graded category

We define $\mathcal{K}(A) = H(\text{DG-Mod}(A))$. It was shown in §2.2 of [16] that $\mathcal{K}(A)$ is a triangulated category with respect to shifts and cones of morphisms that were defined above (i.e. a triangle is distinguished if it is isomorphic to those of the form (30)).
We will say that $f \in \operatorname{Ker} \delta^0_{\text{DG-Mod}(A)}(F, G)$ is a *quasi-isomorphism* if for any $X \in ObjA$ it yields an isomorphism $F(X) \to F(Y)$. We define $\mathcal{D}(A)$ as the localization of $\mathcal{K}(A)$ with respect to quasi-isomorphisms; so it is a triangulated category. Note that quasi-isomorphisms yield a localizing class of morphisms in $K(A)$. Moreover, the functor $X \to H^0(F(X)) : \mathcal{K}(A) \to Ab$ is corepresented by $\text{DG-Mod}(A)(X, -) \in Obj\mathcal{K}(A)$; hence for any $X \in ObjA$, $F \in Obj\mathcal{K}(A)$ we have

$$\mathcal{D}(A)(Y(X), F) \cong \mathcal{K}(A)(Y(X), F). \tag{31}$$

Hence we have an embedding $H(A)^{op} \to \mathcal{D}(A)$.
We define $\mathcal{C}(A)$ as $Z(\text{DG-Mod}(A))$. It is easily seen that $\mathcal{C}(A)$ is closed with respect to (small filtered) direct limits, and $\varinjlim F_l$ is given by $X \to \varinjlim F_l(X)$.
Now we recall (briefly) that differential graded modules admit certain 'resolutions' (i.e. any object is quasi-isomorphic to a *semi-free* one in the terms of §14 of [12]).

Proposition 5.2.1. *There exists a full triangulated $K' \subset \mathcal{K}(A)$ such that the projection $K(A) \to D(A)$ induces an equivalence $K' \approx \mathcal{D}(A)$. K' is closed with respect to all (small) coproducts.*

Proof. See §14.8 of [12] □

Remark 5.2.2. In fact, there exists a (Quillen) model structure for $\mathcal{C}(A)$ such that $\mathcal{D}(A)$ its homotopy category; see Theorem 3.2 of [16]. Moreover (for the first model structures mentioned in loc.cit) all objects of $\mathcal{C}(A)$ are fibrant, all objects coming from A are cofibrant. For this model structure two morphisms are homotopic whenever they become equal in $\mathcal{K}(A)$. So, one could take K' whose objects are the cofibrant objects of $\mathcal{C}(A)$.
Using these facts, one could verify most of Proposition 3.1.1 (for $\mathfrak{D}'$ and $\mathfrak{D}$ described below).

5.3 The construction of $\mathfrak{D}'$ and $\mathfrak{D}$; the proof of Proposition 3.1.1

It was proved in §2.3 of [4] (cf. also [19] or §8.3.1 of [7]) that DM^{eff}_{gm} could be described as $H(A)$, where A is a certain (small) differential graded category. Moreover, the functor $K^b(SmCor) \to DM^{eff}_{gm}$ could be presented as $H(f)$, where $f : B^b(SmCor) \to A$ is a differential graded functor. We will not describe the details for (any of) these constructions since we will not need them.
We define $\mathfrak{D}' = \mathcal{C}(A)^{op}$, $\mathfrak{D} = \mathcal{D}(A)^{op}$, p is the natural projection. We verify that these categories satisfy Proposition 3.1.1. Assertion 10 follows from the fact

that any localization of a triangulated category that possesses an enhancement is enhanceable also (see §§3.4–3.5 of [12]).
The embedding $H(A)^{op} \to \mathcal{D}(A)$ yields $DM_{gm}^{eff} \subset \mathfrak{D}'$. Since all objects coming from A are cocompact in $\mathcal{K}(A)^{op}$, Proposition 5.2.1 yields that the same is true in $\mathfrak{D}$. We obtain assertion 1.
$\mathfrak{D}'$ is closed with respect to inverse limits since $\mathcal{C}(A)$ is closed with respect to direct ones. Since the projection $\mathcal{C}(A) \to \mathcal{K}(A)$ respects coproducts (as well as all other (filtered) colimits), Proposition 5.2.1 yields that p respects products also. We obtain assertion 2.
The descriptions of $\mathcal{C}(A)$ and $\mathcal{D}(A)$ yields all the properties of shifts and cones required. This yields assertions 3, 4, and 6. Since $\mathcal{D}(A)$ is a localization of $\mathcal{K}(A)$, we also obtain assertion 5.
Next, since $\mathcal{D}(A)$ is a localization of $\mathcal{K}(A)$ with respect to quasi-isomorphisms, we obtain assertion 8.
Recall that filtered direct limits of exact sequences of abelian groups are exact. Hence for any $X \in Obj A \subset Obj \mathfrak{D}'$, $Y : L \to$ DG-Mod(A) we have

$$\mathcal{K}(A)(\text{DG-Mod}(A)(X,-), \varinjlim_l Y_l) = H^0((\varinjlim Y_l)(A))$$
$$= H^0(\varinjlim(Y_l(A))) = \varinjlim H^0(Y_l(A)) = \varinjlim_l \mathcal{K}(A)(\text{DG-Mod}(A)(X,-), Y_l).$$

Applying (31) we obtain assertion 7.
It remains to verify assertion 9 of loc.cit. Since the inverse limit with respect to a projective system is isomorphic to the inverse limit with respect to any its unbounded subsystem, and the same is true for $\varprojlim_1$ in the countable case, we can assume that I is the category of natural numbers, i.e. we have a sequence of F_i connected by morphisms.

In this case we have functorial morphisms $\varprojlim F_i \xrightarrow{f} \prod F_l \xrightarrow{g} \prod F_i$ as in (27). Hence it suffices to check that these morphisms yield a distinguished triangle in $\mathfrak{D}$. Note that $g \circ f = 0$; hence g could be factorized through a morphism $h : \operatorname{Cone} f \to \prod F_i$ in $\mathfrak{D}'$. Since for any $X \in Obj A$ the morphism $h^* : \prod_{\mathfrak{D}'} F_i(X) \to \operatorname{Cone} F(X)$ is a quasi-isomorphism, h becomes an isomorphism in $\mathfrak{D}$. This finishes the proof.

Remark 5.3.1. 1. Note that the only part of our argument when we needed k to be countable in the proof of assertion 9 of loc.cit.
2. The constructions of A (i.e. of the 'enhancement' for DM_{gm}^{eff} mentioned above) that were described in [4], [19], and in [7], are easily seen to be functorial with respect to base field change (see below). Still, the constructions mentioned are distinct and far from being the only ones possible; the author does not know whether all possible $\mathfrak{D}$ are isomorphic. Still, this makes no difference for cohomology (of pro-schemes); see Remark 4.3.2.
Moreover, note that assertion 10 of Proposition 3.1.1 was not very important for us (without if we would only have to consider a certain *weakly exact* weight complex functor in §6.1 below; see §3 of [6]). The author doubts that this condition follows from the other parts of Proposition 3.1.1.

5.4 Base change and Tate twists for comotives

Our differential graded formalism yields certain functoriality of comotives with respect to embeddings of base fields. We construct both extension and restriction of scalars (the latter one for the case of a finite extension of fields only). The construction of base change functors uses induction for differential graded modules. This method also allows to define certain tensor products and $Co-\underline{Hom}$ for comotives. In particular, we obtain a Tate twist functor which is compatible with (29) (and a left adjoint to it).
We note that the results of this subsection (probably) could not be deduced from the 'axioms' of $\mathfrak{D}$ listed in Proposition 3.1.1; yet they are quite natural.

5.4.1 Induction and restriction for differential graded modules: reminder

We recall certain results of §14 of [12] on functoriality of differential graded modules. These extend the corresponding (more or less standard) statements for modules over differential graded algebras (cf. §14.2 of ibid.).
If $f : A \to B$ is a functor of differential graded categories, we have an obvious *restriction* functor $f^* : \mathcal{C}(B) \to \mathcal{C}(A)$. It is easily seen that f^* also induces functors $\mathcal{K}(B) \to \mathcal{K}(A)$ and $\mathcal{D}(B) \to \mathcal{D}(A)$. Certainly, the latter functor respects homotopy colimits (i.e. the direct limits from $\mathcal{C}(B)$).
Now, it is not difficult to construct an *induction* functor $f_* : \text{DG-Mod}(A) \to \text{DG-Mod}(B)$ which is left adjoint to f^*; see §14.9 of ibid. By Example 14.10 of ibid, for any $X \in ObjA$ this functor sends $X^* = A(X,-)$ to $f(X)^*$.
f_* also induces functors $\mathcal{C}(A) \to \mathcal{C}(B)$ and $\mathcal{K}(A) \to \mathcal{K}(B)$. Restricting the latter one to the category of semi-free modules K' (see Proposition 5.2.1) one obtains a functor $Lf_* : \mathcal{D}(A) \to \mathcal{D}(B)$ which is also left adjoint to the corresponding f^*; see §14.12 of [12]. Since all functors of the type X^* are semi-free by definition, we have $Lf_*(X^*) = A(X,-) = Lf(X)^*$. It can also be shown that Lf_* respects direct limits of objects of A^{op} (considered as A-modules via the Yoneda embedding). In the case of countable limits this follows easily from the definition of semi-free modules and the expression of the homotopy colimit in $\mathcal{D}(A)$ as $\varinjlim X_i = \text{Cone}(\bigoplus X_i \to \bigoplus X_i)$ (this is just the dual to (27)). For uncountable limits, one could prove the fact using a 'resolution' of the direct limit similar to those described in §A3 of [21].

5.4.2 Extension and restriction of scalars for comotives

Now let l/k be an extension of perfect fields.
Recall that $\mathfrak{D}'$ and $\mathfrak{D}$ were described (in §5.3) in terms of modules over a certain differential graded category A. It was shown in [19] that the corresponding version of A is a tensor (differential graded) category; we also have an extension of scalars functor $A_k \to A_l$. It is most probable that both of these properties hold for the version of A described in [4] (note that they obviously hold for

$B^b(SmCor)$). Moreover, if l/k is finite, then we have the functor of restriction of scalars in inverse direction.
So, the induction for the corresponding differential graded modules yields an exact functor of extension of scalars $\mathrm{Ext}_{l/k} : \mathfrak{D}_k \to \mathfrak{D}_l$. The reasoning above shows that $\mathrm{Ext}_{l/k}$ is compatible with the 'usual' extension of scalars for smooth varieties (and complexes of smooth correspondences). Moreover, since $\mathrm{Ext}_{l/k}$ respects homotopy limits, this compatibility extends to comotives of pro-schemes and their products. It can also be easily shown that $\mathrm{Ext}_{l/k}$ respects Tate twists.
We immediately obtain the following result.

Proposition 5.4.1. *Let k be countable (and perfect), let $l \supset k$ be a perfect field.*
1. Let S be a connected primitive scheme over k, let S_0 be its generic point. Then $M_{gm}(S_l)$ is a retract of $M_{gm}(S_{0l})$ in $\mathfrak{D}_l$.
2. Let K be a function field over k. Let K' be the residue field for a geometric valuation v of K of rank r. Then $M_{gm}(K'_l(r)[r])$ is a retract of $M_{gm}(K_l)$ in $\mathfrak{D}_l$.

As in 4.3, this result immediately implies similar statements for any cohomology of pro-schemes mentioned (constructed from a cohomological $H : DM_{gm}^{eff}{}_l \to A$ via Proposition 1.2.1).
Next, if l/k is finite, induction for differential graded modules applied to the restriction of scalars for A's also yields a restriction of scalars functor $\mathrm{Res}_{l/k} : \mathfrak{D}_l \to \mathfrak{D}_k$. Similarly to $\mathrm{Ext}_{l/k}$, this functor is compatible with restriction of scalars for smooth varieties, pro-schemes, and complexes of smooth correspondences; it also respects Tate twists.
It follows: l/k is finite, then $\mathrm{Ext}_{l/k}$ maps $\mathfrak{D}_{sk}$ to $\mathfrak{D}_{sl}$; $\mathrm{Res}_{l/k}$ maps $\mathfrak{D}_{sl}$ to $\mathfrak{D}_{sk}$. Besides, if we also assume l to be countable, then both of these functors respect weight structures (i.e. they map $\mathfrak{D}_{s_k}^{w\le 0}$ to $\mathfrak{D}_{s_l}^{w\le 0}$, $\mathfrak{D}_{s_k}^{w\ge 0}$ to $\mathfrak{D}_{s_l}^{w\ge 0}$, and vice versa).

Remark 5.4.2. It seems that one can also define restriction of scalars via restriction of differential graded modules (applied to the extension of scalars for A's). To this end one needs to check the corresponding adjunction for DM_{gm}^{eff}; the corresponding (and related) statement for the motivic homotopy categories was proved by J. Ayoub. This would allow to define $\mathrm{Res}_{l/k}$ also in the case when l/k is infinite; this seems to be rather interesting if l is a function field over k. Note that $\mathrm{Res}_{l/k}$ (in this case) would (probably) also map $\mathfrak{D}_{s_l}^{w\le 0}$ to $\mathfrak{D}_{s_k}^{w\le 0}$ and $\mathfrak{D}_{s_l}^{w\ge 0}$ to $\mathfrak{D}_{s_k}^{w\ge 0}$ (if l is countable).

5.4.3 Tensor products and 'co-internal Hom' for comotives; Tate twists

Now, for $X \in ObjA$ we apply restriction and induction of differential graded modules for the functor $X \otimes - : A \to A$. Induction yields a certain functor

$X \otimes - : \mathfrak{D} \to \mathfrak{D}$, whereas restriction yields its left adjoint which we will denote by $Co - \underline{Hom}(X, -) : \mathfrak{D} \to \mathfrak{D}$. Both of them respect homotopy limits. Also, $X \otimes -$ is compatible with tensoring by X on DM^{eff}_{gm}. Besides, the isomorphisms classes of these functors only depend on the quasi-isomorphism class of X in DG-Mod(A). Indeed, it is easily seen that both $X \otimes Y$ and $Co - \underline{Hom}(X, Y)$ are exact with respect to X if we fix Y; since they are obviously zero for $X = 0$, it remains to note that quasi-isomorphic objects could be connected by a chain of quasi-isomorphisms.

Now suppose that X is a Tate motif i.e. $X = \mathbb{Z}(m)[n]$, $m > 0$, $n \in \mathbb{Z}$. Then we obtain that the formal Tate twists defined by (29) are the true Tate twists i.e. they are given by tensoring by X on $\mathfrak{D}$. Then recall the Cancellation Theorem for motives: (see Theorem 4.3.1 of [25], and [27])): $X \otimes -$ is a full embedding of DM^{eff}_{gm} into itself. Then one can deduce that $X \otimes -$ is fully faithful on $\mathfrak{D}$ also (since all objects of $\mathfrak{D}$ come from semi-free modules over A). Moreover, $Co - \underline{Hom}(X, -) \circ (X \otimes -)$ is easily seen to be isomorphic to the identity on $\mathfrak{D}$ (for such an X).

6 Supplements

We describe some more properties of comotives, as well as certain possible variations of our methods and results. We will be somewhat sketchy sometimes.

In §6.1 we define an additive category $\mathfrak{D}^{gen}$ of generic motives (a variation of those studied in [9]). We also prove that the exact conservative *weight complex* functor (that exists by the general theory of weight structures) could be modified to an exact conservative $WC : \mathfrak{D}_s \to K^b(\mathfrak{D}^{gen})$. Besides, we prove assertions on retracts of the pro-motif of a function field K/k, that are similar to (and follow from) those for its comotif.

In §6.2 we prove that HI has a nice description in terms of $\underline{Hw}$. This is a sort of Brown representability: a cofunctor $\underline{Hw} \to Ab$ is representable by a (homotopy invariant) sheaf with transfers whenever it converts all small products into direct sums. This result is similar to the corresponding results of §4 of [6] (on the connection between the hearts of adjacent structures).

In §6.3 we note that our methods could be used for motives (and comotives) with coefficients in an arbitrary commutative unital ring R; the most important cases are rational (co)motives and 'torsion' (co)motives.

In §6.4 we note that there exist natural motives of pro-schemes with compact support in DM^{eff}_{-}. It seems that one could construct alternative $\mathfrak{D}$ and $\mathfrak{D}'$ using this observation (yet this probably would not affect our main results significantly).

We conclude the section by studying which of our arguments could be extended to the case of an uncountable k.

6.1 The weight complex functor; relation with generic motives

We recall that the general formalism of weight structures yields a conservative exact weight complex functor $t : \mathfrak{D}_s \to K^b(\underline{Hw})$; it is compatible with Definition 2.1.2(9). Next we prove that one can compose it with a certain 'projection' functor without losing the conservativity.

Lemma 6.1.1. *There exists an exact conservative functor $t : \mathfrak{D}_s \to K^b(\underline{Hw})$ that sends $X \in Obj\mathfrak{D}_s$ to a choice of its weight complex (coming from any choice of a weight Postnikov tower for it).*

Proof. Immediate from Remark 6.2.2(2) and Theorem 3.3.1(V) of [6] (note that $\mathfrak{D}_s$ has a differential graded enhancement by Proposition 3.1.1(10)).

□

Now, since all objects of $\underline{Hw}$ are retracts of those that come via p from inverse limits of objects of $j(C^b(SmCor))$, we have a natural additive functor $\underline{Hw} \to \mathfrak{D}^{naive}$ (see §1.5). Its categorical image will be denoted by $\mathfrak{D}^{gen}$; this is a slight modification of Deglise's category of generic motives. We will denote the 'projection' $\underline{Hw} \to \mathfrak{D}^{gen}$ and $K^b(\underline{Hw}) \to K^b(\mathfrak{D}^{gen})$ by pr.

Theorem 6.1.2. *1. The functor $WC = pr \circ t : \mathfrak{D}_s \to K^b(\mathfrak{D}^{gen})$ is exact and conservative.*
2. Let S be a connected primitive scheme, let S_0 be its generic point. Then $pr(M_{gm}(S))$ is a retract of $pr(M_{gm}(S_0))$ in $\mathfrak{D}^{gen}$.
3. Let K be a function field over k. Let K' be the residue field for some geometric valuation v of K of rank r. Then $pr(M_{gm}(K')(r)[r])$ is a retract of $pr(M_{gm}(K))$ in $\mathfrak{D}^{gen}$.

Proof. 1. The exactness of WC is obvious (from Lemma 6.1.1). Now we check that WC is conservative.
By Proposition 3.1.1(8), it suffices to check: if $WC(X)$ is acyclic for some $X \in Obj\mathfrak{D}_s$, then $\mathfrak{D}(X, Y) = 0$ for all $Y \in ObjDM_{gm}^{eff}$. We denote the terms of $t(X)$ by X^i.
We consider the coniveau spectral sequence $T(H, X)$ for the functor $H = \mathfrak{D}(-, Y)$ (see Remark 4.4.2). Since $WC(X)$ is acyclic, we obtain that the complexes $\mathfrak{D}(X^{-i}, Y[j])$ are acyclic for all $j \in \mathbb{Z}$. Indeed, note that the restriction of a functor $\mathfrak{D}(X^{-i}, -)$ to DM_{gm}^{eff} could be expressed in terms of $pr(X^{-i})$; see Remark 3.2.1. Hence $E_2(T)$ vanishes. Since T converges (see Proposition 4.4.1(2)) we obtain the claim.
2. Immediate from Corollary 4.2.2(1).
3. Immediate from Corollary 4.2.2(2).

□

Remark 6.1.3. For $X = M_{gm}(Z)$, $Z \in SmVar$, it easily seen that $WC(X)$ could be described as a 'naive' limit of complexes of motives; cf. §1.5.
Now, the terms of $t(X)$ are just the factors of (some possible) weight Postnikov tower for X; so one can calculate them (at least, up to an isomorphism) for

$X = M_{gm}(Z)$. Unfortunately, it seems difficult to describe the boundary for $t(X)$ completely since $\underline{Hw}$ is finer than $\mathfrak{D}^{gen}$.

6.2 The relation of the heart of w with HI ('Brown representability')

In Theorem 4.4.2(4) of [6], for a pair of adjacent structures (w, t) for $\underline{C}$ (see Remark 2.5.7) it was proved that $\underline{Ht}$ is a full subcategory of $\underline{Hw}_*(=\mathrm{AddFun}(\underline{Hw}^{op}, Ab))$. This result cannot be extended to arbitrary orthogonal structures since our definition of a duality did not include any non-degenerateness conditions (in particular, Φ could be 0). Yet for our main example of orthogonal structures the statement is true; moreover, HI has a natural description in terms of $\underline{Hw}$. This statement is very similar to a certain Brown representability-type result (for adjacent structures) proved in Theorem 4.5.2(II.2) of ibid.
Note that $\underline{Hw}$ is closed with respect to arbitrary small products; see Proposition 4.1.1(2).

Proposition 6.2.1. *HI is naturally isomorphic to a full abelian subcategory $\underline{Hw}'_*$ of $\underline{Hw}_*$ that consists of functors that convert all products in $\underline{Hw}$ into direct sums (of the corresponding abelian groups).*

Proof. First, note that for any $G \in Obj DM_-^{eff}$ the functor $\mathfrak{D} \to Ab$ that sends $X \in Obj\mathfrak{D}$ to $\Phi(X, G)$ (Φ is the duality constructed in Proposition 4.5.1) is cohomological. Moreover, it converts homotopy limits into injective limits (of the corresponding abelian groups); hence its restriction to $\underline{Hw}$ belongs to $\underline{Hw}'_*$.
We obtain an additive functor $DM_{gm}^{eff} \to \underline{Hw}'_*$. In fact, it factorizes through HI (by (25)). For $G \in Obj HI$ we denote the functor $\underline{Hw} \to Ab$ obtained by G'.
Next, for any (additive) $F : \underline{Hw}^{op} \to Ab$ we define $F' : \mathfrak{D}_s \to Ab$ by:

$$F'(X) = (\mathrm{Ker}(F(X^0) \to F(X^{-1}))/\,\mathrm{Im}(F(X^1) \to F(X^0)); \qquad (32)$$

here X^i is a weight complex for X. It easily seen from Lemma 6.1.1 that F' is a well-defined cohomological functor. Moreover, Theorem 2.2.1(19) yields that F' vanishes on $\mathfrak{D}_s^{w \le -1}$ and on $\mathfrak{D}_s^{w \ge 1}$ (since it vanishes on $\mathfrak{D}_s^{w=i}$ for all $i \ne 0$). Hence F' defines an additive functor $F'' = F' \circ M_{gm} : SmCor^{op} \to Ab$ i.e. a presheaf with transfers. Since $M_{gm}(Z) \cong M_{gm}(Z \times \mathbb{A}^1)$ for any $Z \in SmVar$, F'' is homotopy invariant. We should check that F'' is actually a (Nisnevich) sheaf. By Proposition 5.5 of [26], it suffices to check that F'' is a Zariski sheaf. Now, the the Mayer-Vietoris triangle for motives (§2 of [25]) yields: to any Zariski covering $U \coprod V \to U \cup V$ there corresponds a long exact sequence

$$\dots \to F'(M_{gm}(U \cap V)[1]) \to F''(U \cup V) \to F''(U) \bigoplus F''(V) \to F''(U \cap V) \to \dots$$

Since $M_{gm}(U \cap V) \in \mathfrak{D}_s^{w \le 0}$ by part 5 of Proposition 4.1.1, we have $F'(M_{gm}(U \cap V)[1]) = \{0\}$; hence F'' is a sheaf indeed.

So, $F \mapsto F''$ yields an additive functor $\underline{Hw}_* \to HI$.

Now we check that the functor $G \mapsto G'$ (described above) and the restrictions of $F \mapsto F''$ to $\underline{Hw}'_* \subset \underline{Hw}_*$ yield mutually inverse equivalences of the categories in question.

(24) immediately yields that the functor $HI \to HI$ that sends $G \in Obj HI$ to $(G')''$ is isomorphic to id_{HI}.

Now for $F \in Obj\underline{Hw}'_*$ we should check: for any $P \in \mathfrak{D}_s^{w=0}$ we have a natural isomorphism $(F'')'(P) \cong F(P)$. Since $\underline{Hw}$ is the idempotent completion of H, it suffices to consider P being of the form $\prod_{l\in L} M_{gm}(K_l)(n_l)[n_l]$ (here K_l are function fields over k, $n_l \geq 0$; n_l and the transcendence degrees of K_l/k are bounded); see part 2 of Proposition 4.1.1. Moreover, since F converts products into direct sums, it suffices to consider $P = M_{gm}(K')(n)[n]$ (K'/k is a function field, $n \geq 0$). Lastly, part 2 of Corollary 4.2.2 reduces the situation to the case $P = M_{gm}(K)$ (K/k is a function field). Now, by the definition of the functor $G \mapsto G'$, we have $(F'')'(M_{gm}(K)) = \varinjlim_{l\in L} F''(M_{gm}(U_l))$, where $K = \varprojlim_{l\in L} U_l$, $U_l \in SmVar$. We have $F''(U_l) = \operatorname{Ker} F(M_{gm}(K)) \to F(\prod_{z\in U_l^1} M_{gm}(z)(1)[1])$; here U_l^1 is the set of points of U_l of codimension 1. Since $F(\prod_{z\in U_l^1} M_{gm}(z)(1)[1]) = \oplus_{z\in U_l^1} F(M_{gm}(z)(1)[1])$; we have $\varinjlim_{l\in L} F(\prod_{z\in U_l^1} M_{gm}(z)(1)[1]) = \{0\}$; this yields the result.

□

6.3 Motives and comotives with rational and torsion coefficients

Above we considered (co)motives with integral coefficients. Yet, as was shown in [20], one could do the theory of motives with coefficients in an arbitrary commutative associative ring with a unit R. One should start with the naturally defined category of R-correspondences: $Obj(SmCor_R) = SmVar$; for X, Y in $SmVar$ we set $SmCor_R(X,Y) = \bigoplus_U R$ for all integral closed $U \subset X \times Y$ that are finite over X and dominant over a connected component of X. Then one obtains a theory of motives that would satisfy all properties that are required in order to deduce the main results of this paper. So, we can define R-comotives and extend our results to them.

A well-known case of motives with coefficients are the motives with rational coefficients (note that $\mathbb{Q}$ is a flat $\mathbb{Z}$-algebra). Yet, one could also take $R = \mathbb{Z}/n\mathbb{Z}$ for any n prime to $\operatorname{char} k$.

So, the results of this paper are also valid for rational (co)motives and 'torsion' (co)motives.

Still, note that there could be idempotents for R-motives that do not come from integral ones. In particular, for the naturally defined rational motivic categories we have $DM_{gm}^{eff}\mathbb{Q} \neq DM_{gm}^{eff} \otimes \mathbb{Q}$; also $Chow^{eff}\mathbb{Q} \neq Chow^{eff} \otimes \mathbb{Q}$ (here $Chow^{eff}\mathbb{Q} \subset DM_{gm}^{eff}\mathbb{Q}$ denote the corresponding R-hulls). Certainly, this does not matter at all in the current paper.

6.4 Another possibility for $\mathfrak{D}$; motives with compact support of pro-schemes

In the case $\operatorname{char} k = 0$, Voevodsky developed a nice theory of motives with compact support that is compatible with Poincare duality; see Theorem 4.3.7 of [25]. Moreover, the explicit constructions of [25] yield that the functor of motif with compact support $M^c_{gm} : SmVar^{op} \to DM^{eff}_{gm}$ is compatible with a certain $j^c : SmVar^{op}_{fl} \to C^-(Shv(SmCor))$ (which sends X to the Suslin complex of $L^c(X)$, see §4.2 loc.cit.); this observation was kindly communicated to the author by Bruno Kahn). This allows to define $j^c(V)$ for a pro-scheme V as the corresponding direct limit (in $C(Shv(SmCor))$).

Starting from this observation, one could try to develop an analogue of our theory using the functor M^c_{gm}. One could consider $\mathfrak{D} = DM^{eff\,op}_-$; then it would contain $DM^{eff\,op}_{gm}$ as the full category of cocompact objects. It seems that our arguments could be carried over to this context. One can construct some $\mathfrak{D}'$ for this $\mathfrak{D}$ using certain differential graded categories.

Though motives with compact support are Poincare dual to ordinary motives of smooth varieties (up to a certain Tate twist), we do not have a covariant embedding $DM^{eff}_{gm} \to \mathfrak{D}$ (for this 'alternative' $\mathfrak{D}$), since (the whole) DM^{eff}_{gm} is not self-dual. Still, DM^{eff}_{gm} has a nice embedding into (Voevodsky's) self-dual category DM_{gm}; it contains an exhausting system of self-dual subcategories. Hence this alternative $\mathfrak{D}$ would yield a theory that is compatible with (though not 'isomorphic' to) the theory developed above.

Since the alternative version of $\mathfrak{D}$ is closely related with $DM^{eff\,op}_-$, it seems reasonable to call its objects comotives (as we did for the objects of 'our' $\mathfrak{D}$).

These observations show that one can dualize all the direct summands results of §4 to obtain their natural analogues for motives of pro-schemes with compact support. Indeed, to prove them we may apply the duals of our arguments in §4 without any problem; see part 2 of Remark 3.1.2. Note that we obtain certain direct summand statements for objects of DM^{eff}_- this way. This is an advantage of our 'axiomatic' approach in §3.1.

One could also take $\mathfrak{D}^{op} = \cup_{n\in\mathbb{Z}} DM^{eff}_{gm}(-n)$ (more precisely, this is the direct limit of copies of DM^{eff}_{gm} with connecting morphisms being $-\otimes\mathbb{Z}(1)$). Then we have a covariant embedding $DM^{eff}_{gm} \to DM_{gm} \to \mathfrak{D}$.

Note that both of these alternative versions of $\mathfrak{D}$ are not closed with respect to all (countable) products, and so not closed with respect to all (filtered countable) homotopy limits; yet they contain all products and homotopy limits that are required for our main arguments.

6.5 What happens if k is uncountable

We describe which of the arguments above could be applied in the case of an uncountable k (and for which of them the author has no idea how to achieve this). The author warns that he didn't check the details thoroughly here.

As we have already noted above, it is no problem to define $\mathfrak{D}$, $\mathfrak{D}'$, or even $\mathfrak{D}_s$ for any k. The main problem here that (if k is uncountable) the comotives of generic points of varieties (and of other pro-schemes) can usually be presented only as uncountable homotopy limits of motives of varieties. The general formalism of inverse limits (applied to the categories of modules over a differential graded category) allows to extend to this case all parts of Proposition 3.1.1 expect part 9. This actually means that instead of the short exact sequence (28) one obtains a spectral sequence whose E_1-terms are certain $\varprojlim^j$; here $\varprojlim^j$ is the j's derived functor of $\varprojlim_I$; cf. Appendix A of [21]. This does not seem to be catastrophic; yet the author has absolutely no idea how to control higher projective limits in the proof of Proposition 3.5.1; note that part 2 of loc.cit. is especially important for the construction of the Gersten weight structure.
Besides, the author does not know how to pass to an uncountable homotopy limit in the Gysin distinguished triangle. It seems that to this end one either needs to lift the functoriality of the (usual) motivic Gysin triangle to $\mathfrak{D}'$, or to find a way to describe the isomorphism class of an uncountable homotopy limit in $\mathfrak{D}$ in terms of $\mathfrak{D}$-only (i.e. without fixing any lifts to $\mathfrak{D}'$; this seems to be impossible in general). So, one could define the 'Gersten' weight tower for a comotif of a pro-scheme as as the homotopy limit of 'geometric towers' (as in the proof of Corollary 3.6.2); yet it seems to be rather difficult to calculate factors of such a tower. It seems that the problems mentioned do not become simpler for the alternative versions of $\mathfrak{D}$ described in §6.4. So, currently the author does not know how to prove the direct summand results of §4.2 if k is uncountable (they even could be wrong). The problem here that the splittings of §4.2 are not canonical (see Remark 4.2.3), so one cannot apply a limit argument (as in §4.6) here.
It seems that constructing the Gersten weight structure is easier for $\mathfrak{D}_s/\mathfrak{D}_s(n)$ (for some $n > 0$); see §4.9.
Lastly, one can avoid the problems with homotopy limits completely by restricting attention to the subcategory of Artin-Tate motives in DM_{gm}^{eff} (i.e. the triangulated category generated by Tate twists of motives of finite extensions of k, as considered in [30]). Note that coniveau spectral sequences for cohomology of such motives (could be chosen to be) very 'economic'.

References

[1] Balmer P., Schlichting M. Idempotent completion of triangulated categories// Journal of Algebra 236, no. 2 (2001), 819-834.

[2] Beilinson A., Bernstein J., Deligne P., Faisceaux pervers, Asterisque 100, 5–171.

[3] Beilinson A., Remarks on n-motives and correspondences at the generic point, in: Motives, polylogarithms and Hodge theory, part I, Irvine, CA, 1998, Int. Press Lect. Ser., 3, I, Int. Press, Somerville, MA, 2002, 35–46.

[4] Beilinson A., Vologodsky V. A guide to Voevodsky motives// Geom. Funct. Analysis, vol. 17, no. 6, 2008, 1709–1787.

[5] Bloch S., Ogus A. Gersten's conjecture and the homology of schemes// Ann. Sci. Éc. Norm. Sup. v.7 (1994), 181–202.

[6] Bondarko M.V., Weight structures vs. t-structures; weight filtrations, spectral sequences, and complexes (for motives and in general), to appear in J. of K-theory, http://arxiv.org/abs/0704.4003

[7] Bondarko M.V., Differential graded motives: weight complex, weight filtrations and spectral sequences for realizations; Voevodsky vs. Hanamura// J. of the Inst. of Math. of Jussieu, v.8 (2009), no. 1, 39–97, see also http://arxiv.org/abs/math.AG/0601713

[8] J.-L. Colliot-Thelène, R.T. Hoobler, B. Kahn, The Bloch-Ogus-Gabber Theorem; Algebraic K-Theory, in: Fields Inst. Commun., Vol. 16, Amer. Math. Soc., Providence, RI, 1997, 31–94.

[9] Deglise F. Motifs génériques, Rendiconti Sem. Mat. Univ. Padova, 119: 173–244, see also http://www.math.uiuc.edu/K-theory/0690/

[10] Deglise F. Around the Gysin triangle I, preprint,
http://arxiv.org/abs/0804.2415
http://www.math.uiuc.edu/K-theory/0764/

[11] Deglise F. Modules homotopiques (Homotopy modules), preprint, http://arxiv.org/abs/0904.4747

[12] Drinfeld V. DG quotients of DG categories// J. of Algebra 272 (2004), 643–691.

[13] Gelfand S., Manin Yu., Methods of homological algebra. 2nd ed. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. xx+372 pp.

[14] Huber A., Kahn B. The slice filtration and mixed Tate motives, Compos. Math. 142(4), 2006, 907–936.

[15] Kahn B., Sujatha R., Birational motives, I, preprint,
http://www.math.uiuc.edu/K-theory/0596/

[16] Keller B., On differential graded categories, International Congress of Mathematicians. Vol. II, 151–190, Eur. Math. Soc., Zürich, 2006.

[17] Krause H., Smashing subcategories and the telescope conjecture — an algebraic approach // Invent. math. 139, 2000, 99–133.

[18] Krause H., The stable derived category of a noetherian scheme// Comp. Math., 141:5 (2005), 1128–1162.

[19] Levine M. Mixed motives, Math. surveys and Monographs 57, AMS, Prov. 1998.

[20] Mazza C., Voevodsky V., Weibel Ch. Lecture notes on motivic cohomology, Clay Mathematics Monographs, vol. 2, see also `http://www.math.rutgers.edu/~weibel/MVWnotes/prova-hyperlink.pdf`

[21] Neeman A. Triangulated Categories. Annals of Mathematics Studies 148 (2001), Princeton University Press, viii+449 pp.

[22] Paranjape K., Some Spectral Sequences for Filtered Complexes and Applications// Journal of Algebra, v. 186, i. 3, 1996, 793–806.

[23] Pauksztello D., Compact cochain objects in triangulated categories and co-t-structures// Central European Journal of Mathematics, vol. 6, n. 1, 2008, 25–42.

[24] Rost M. Chow groups with coefficients// Doc. Math., 1 (16), 319–393, 1996.

[25] Voevodsky V. Triangulated category of motives, in: Voevodsky V., Suslin A., and Friedlander E., Cycles, transfers and motivic homology theories, Annals of Mathematical studies, vol. 143, Princeton University Press, 2000, 188–238.

[26] Voevodsky V. Cohomological theory of presheaves with transfers, same volume, 87–137.

[27] Vladimir Voevodsky Cancellation Theorem Doc. Math. Extra Volume: Andrei A. Suslin's Sixtieth Birthday (2010) 671–685.

[28] Voevodsky V. On motivic cohomology with $\mathbb{Z}/l$ coefficients, preprint, `http://arxiv.org/abs/0805.4430`

[29] Walker M., The primitive topology of a scheme// J. of Algebra 201 (1998), 655–685.

[30] Wildeshaus J., Notes on Artin-Tate motives, preprint, `http://www.math.uiuc.edu/K-theory/0918/`

Mikhail V. Bondarko
St. Petersburg State
University
Department of Mathematics
and Mechanics
Bibliotechnaya Pl. 2
198904 St. Petersburg
Russia
mbondarko@hotmail.com

On Equivariant Dedekind Zeta-Functions at $s = 1$

Dedicated to Professor Andrei Suslin

Manuel Breuning and David Burns

Received: November 19, 2009
Revised: April 26, 2010

Abstract. We study a refinement of an explicit conjecture of Tate concerning the values at $s = 1$ of Artin L-functions. We reinterpret this refinement in terms of Tamagawa number conjectures and then use this connection to obtain some important (and unconditional) evidence for our conjecture.

2010 Mathematics Subject Classification: 11R42, 11R33
Keywords and Phrases: Artin L-functions, equivariant zeta functions, leading terms

1. Introduction and statement of main results

This article studies a refinement of a conjecture of Tate concerning the values at $s = 1$ of Artin L-functions. We recall that Tate's conjecture was originally formulated in [26, Chap. I, Conj. 8.2] as an analogue of (Tate's reformulation of) the main conjecture of Stark on the leading terms at $s = 0$ of Artin L-functions and that the precise form of the 'regulators' and 'periods' that Tate introduced in this context were natural generalisations of earlier constructions of Serre in [24].

The refinement of Tate's conjecture that we study here was formulated by the present authors in [5, Conj. 3.3] and predicts an explicit formula for the leading term at $s = 1$ of the zeta-function of a finite Galois extension of number fields L/K in terms of the Euler characteristic of a certain perfect complex of $\mathrm{Gal}(L/K)$-modules (see (3) for a statement of this formula). In comparison to Tate's conjecture, this refinement predicts not only that the quotient by Tate's regulator of the leading term at $s = 1$ of the Artin L-function of a complex character χ of $\mathrm{Gal}(L/K)$ is an algebraic number but also that as χ varies these algebraic numbers should be related by certain types of integral congruence relations. We further recall that [5, Conj. 3.3] is also known to imply the '$\Omega(1)$-Conjecture' that was formulated by Chinburg in [13].

In the sequel we write $\mathbb{Q}(1)_L$ for the motive $h^0(\operatorname{Spec} L)(1)$, considered as defined over K and endowed with the natural action of the group ring $\mathbb{Q}[\mathrm{Gal}(L/K)]$. We recall that the 'equivariant Tamagawa number conjecture' applies in particular to pairs of the form $(\mathbb{Q}(1)_L, \mathbb{Z}[\mathrm{Gal}(L/K)])$ and was formulated by Flach and the second named author in [9] as a natural refinement of the seminal 'Tamagawa number conjecture' of Bloch and Kato [3]. The main technical result of the present article is then the following

Theorem 1.1. *Let L be a finite complex Galois extension of $\mathbb{Q}$. If Leopoldt's Conjecture is valid for L, then* [5, Conj. 3.3] *is equivalent to the equivariant Tamagawa number conjecture of* [9, Conj. 4] *for the pair* $(\mathbb{Q}(1)_L, \mathbb{Z}[\mathrm{Gal}(L/\mathbb{Q})])$.

Corollary 1.2. *If Leopoldt's Conjecture is valid for every number field, then for every Galois extension of number fields L/K the conjecture* [5, Conj. 3.3] *is equivalent to the conjecture* [9, Conj. 4] *for the pair* $(\mathbb{Q}(1)_L, \mathbb{Z}[\mathrm{Gal}(L/K)])$.

These results connect the explicit leading term formula of [5, Conj. 3.3] to a range of interesting results and conjectures. For example, [9, Conj. 4(iv)] is known to be a consequence of the 'main conjecture of non-commutative Iwasawa theory' that is formulated by Fukaya and Kato in [18, Conj. 2.3.2] and also of the 'main conjecture of non-commutative Iwasawa theory for Tate motives' that is formulated by Venjakob and the second named author in [12, Conj. 7.1]. Corollary 1.2 therefore allows one to regard the study of the explicit conjecture [5, Conj. 3.3] as an attempt to provide supporting evidence for these more general conjectures. Indeed, when taken in conjunction with the philosophy described by Huber and Kings in [19, §3.3] and by Fukaya and Kato in [18, §2.3.5], Corollary 1.2 suggests that, despite its comparatively elementary nature, [5, Conj. 3.3] may well play a particularly important role in the context of the very general conjecture of Fukaya and Kato.

In addition to the above consequences, our proof of Theorem 1.1 also answers an explicit question posed by Flach and the second named author in [7] (see Remark 5.1) and combines with previous work to give new evidence in support of the conjectures made in [5] including the following unconditional results.

Corollary 1.3. *If L is abelian over $\mathbb{Q}$, and K is any subfield of L, then both* [5, Conj. 3.3] *and* [5, Conj. 4.1] *are valid for the extension L/K.*

Corollary 1.4. *There exists a natural infinite family of quaternion extensions $L/\mathbb{Q}$ with the property that, if K is any subfield of L, then both* [5, Conj. 3.3] *and* [5, Conj. 4.1] *are valid for the extension L/K.*

We recall (from [5, Prop. 4.4(i)]) that [5, Conj. 4.1] is a natural refinement of the 'main conjecture of Stark at $s = 0$'. For details of connections between [5, Conj. 3.3 and Conj. 4.1] and other interesting conjectures of Chinburg, of Gruenberg, Ritter and Weiss and of Solomon see [5, Prop. 3.6 and Prop. 4.4] and the recent thesis of Jones [20].

The main contents of this article is as follows. In §2 we recall the explicit statement of [5, Conj. 3.3] and in §3 we review (and clarify) certain constructions in étale cohomology that are made in [8]. In §4 we make a detailed analysis of the p-adic completion of the perfect complex that occurs in [5, Conj. 3.3]. In §5 we prove Theorem 1.1 and in §6 we use Theorem 1.1 to prove Corollaries 1.2, 1.3 and 1.4.

Acknowledgements. It is a pleasure for us to thank Werner Bley and Matthias Flach and also the referee of [5] for some very useful comments about this paper. In addition, the first author would like to thank the Isaac Newton Institute where part of this paper was written.

2. The explicit leading term conjecture

In this section we quickly review [5, Conj. 3.3]. To do this it is necessary to summarise some background about K-theory and homological algebra.

2.1. K-theory. Let R be an integral domain of characteristic 0, E an extension of the field of fractions of R, and G a finite group. We denote the relative algebraic K-group associated to the ring homomorphism $R[G] \to E[G]$ by $K_0(R[G], E)$; a description of $K_0(R[G], E)$ in terms of generators and relations is given in [25, p. 215]. The group $K_0(R[G], E)$ is functorial in the pair (R, E) and also sits inside a long exact sequence of relative K-theory. In this paper we will use the homomorphisms $\partial^1_{R[G],E} : K_1(E[G]) \to K_0(R[G], E)$ and $\partial^0_{R[G],E} : K_0(R[G], E) \to K_0(R[G])$ from the latter sequence.

Let $\mathrm{Z}(E[G])^\times$ denote the multiplicative group of the centre of the finite dimensional semisimple E-algebra $E[G]$. The reduced norm induces a homomorphism $\mathrm{nr} : K_1(E[G]) \to \mathrm{Z}(E[G])^\times$ and we denote its image by $\mathrm{Z}(E[G])^{\times+}$. In this paper E will always be either $\mathbb{R}$ or $\mathbb{C}_p$ for some prime number p. In both cases the map nr is injective and hence we can use it to identify $K_1(E[G])$ and $\mathrm{Z}(E[G])^{\times+}$. In particular we will consider $\partial^1_{R[G],E}$ as a map $\mathrm{Z}(E[G])^{\times+} \to K_0(R[G], E)$. If $E = \mathbb{C}_p$ then $\mathrm{Z}(E[G])^{\times+} = \mathrm{Z}(E[G])^\times$.

For every prime p and embedding $j : \mathbb{R} \to \mathbb{C}_p$ there are induced homomorphisms $j_* : K_0(\mathbb{Z}[G], \mathbb{R}) \to K_0(\mathbb{Z}_p[G], \mathbb{C}_p)$ and $j_* : \mathrm{Z}(\mathbb{R}[G])^\times \to \mathrm{Z}(\mathbb{C}_p[G])^\times$. In [5, §2.1.2] it is shown that there exists a (unique) homomorphism $\hat{\partial}^1_G : \mathrm{Z}(\mathbb{R}[G])^\times \to K_0(\mathbb{Z}[G], \mathbb{R})$ which coincides with $\partial^1_{\mathbb{Z}[G],\mathbb{R}}$ on $\mathrm{Z}(\mathbb{R}[G])^{\times+}$ and is such that for every prime p and embedding $j : \mathbb{R} \to \mathbb{C}_p$ one has $j_* \circ \hat{\partial}^1_G = \partial^1_{\mathbb{Z}_p[G],\mathbb{C}_p} \circ j_* : \mathrm{Z}(\mathbb{R}[G])^\times \to K_0(\mathbb{Z}_p[G], \mathbb{C}_p)$.

2.2. Homological algebra. For our homological algebra constructions in this paper we use the same notations and sign conventions as in [5]. So in particular by a complex we mean a cochain complex of left R-modules for a ring R, we use the phrase 'distinguished triangle' in the sense specified in [5, §2.2.1] and by a perfect complex we mean a complex that in the derived category $\mathcal{D}(R)$ is isomorphic to a bounded complex of finitely generated projective left R-modules. The full triangulated subcategory of $\mathcal{D}(R)$ consisting of the perfect complexes will be denoted by $\mathcal{D}^{\mathrm{perf}}(R)$.

Now let R, E and G be as in §2.1. For any object C of $\mathcal{D}(R[G])$ we write $H^{\mathrm{ev}}(C)$ and $H^{\mathrm{od}}(C)$ for the direct sums $\oplus_{i\,\mathrm{even}} H^i(C)$ and $\oplus_{i\,\mathrm{odd}} H^i(C)$ where

i runs over all even and all odd integers respectively. A trivialisation t (over E) of a complex C in $\mathcal{D}^{\mathrm{perf}}(R[G])$ is an isomorphism of $E[G]$-modules of the form $t : H^{\mathrm{ev}}(C) \otimes_R E \xrightarrow{\cong} H^{\mathrm{od}}(C) \otimes_R E$. We write $\chi_{R[G],E}(C,t)$ for the Euler characteristic in $K_0(R[G],E)$ defined in [4, Definition 5.5]. To simplify notation in the sequel we write χ_G for $\chi_{\mathbb{Z}[G],\mathbb{R}}$.

We shall interpret certain complexes in the derived category in terms of Yoneda extension classes as in [8, p. 1353]. To be specific, for any complex E that is acyclic outside degrees 0 and $n \geq 1$ we associate the class in $\mathrm{Ext}_R^{n+1}(H^n(E), H^0(E))$ given by the truncated complex $E' := \tau^{\leq n}\tau^{\geq 0}E$ with the induced maps $H^0(E) \xrightarrow{\cong} H^0(E') \to (E')^0$ and $(E')^n \to H^n(E') \xrightarrow{\cong} H^n(E)$ considered as a Yoneda extension.

2.3. Notation for number fields. Let L be a number field. We write $\mathcal{O}_L$ for the ring of integers of L and $S(L)$ for the set of all places of L. For any place $w \in S(L)$ we denote the completion of L at w by L_w. For a non-archimedean place w we write $\mathcal{O}_w$ for the ring of integers of L_w, $\mathfrak{m}_w$ for the maximal ideal of $\mathcal{O}_w$ and $U_{L_w}^{(1)}$ for the group $1 + \mathfrak{m}_w$ of principal units in L_w.

If L is an extension of K and $v \in S(K)$ then $S_v(L)$ is the set of all places of L above v. We also use the notation $S_f(L)$ and $S_\infty(L)$ for the sets of all non-archimedean and archimedean places, $S_{\mathbb{R}}(L)$ for the set of real archimedean places and $S_{\mathbb{C}}(L)$ for the set of complex archimedean places.

From now on let L/K be a Galois extension of number fields with Galois group G. For w in $S(L)$ we let G_w denote the decomposition group of w. For any place v in $S(K)$ we set $L_v := \prod_{w \in S_v(L)} L_w$ and (if $v \in S_f(K)$) $\mathcal{O}_{L,v} := \prod_{w \in S_v(L)} \mathcal{O}_w$ and $\mathfrak{m}_{L,v} := \prod_{w \in S_v(L)} \mathfrak{m}_w$. Note that L_v, $\mathcal{O}_{L,v}$ and $\mathfrak{m}_{L,v}$ are G-modules in an obvious way.

Let S be a finite subset of $S(K)$. The G-stable set of places of L that lie above a place in S will also be denoted by S. This should not cause any confusion because places of K will be called v and places of L will be called w. For a finite subset S of $S(K)$ which contains all archimedean places we let $\mathcal{O}_{L,S}$ be the ring of S-integers in L. Note that $\mathcal{O}_{L,S}$ is a G-module and that if $S = S_\infty(K)$, then $\mathcal{O}_L = \mathcal{O}_{L,S}$.

2.4. The conjecture. Let L/K be a Galois extension of number fields with Galois group G. Let S be a finite subset of $S(K)$ which contains all archimedean places and all places which ramify in L/K and is such that $\mathrm{Pic}(\mathcal{O}_{L,S}) = 0$. In [5, Lemma 2.7(ii)] it is shown that the leading term $\zeta^*_{L/K,S}(1)$ at $s = 1$ of the S-truncated zeta-function of L/K belongs to $\mathrm{Z}(\mathbb{R}[G])^{\times +}$. In this subsection we recall the explicit conjectural description of $\hat{\partial}^1_G(\zeta^*_{L/K,S}(1))$ formulated in [5, Conj. 3.3].

For each $v \in S_\infty(K)$ we let $\exp : L_v \to L_v^\times$ denote the product of the (real or complex) exponential maps $L_w \to L_w^\times$ for $w \in S_v(L)$. If $v \in S_f(K)$, then for sufficiently large i the exponential map $\exp : \mathfrak{m}^i_{L,v} \to L_v^\times$ is the product of the p-adic exponential maps $\mathfrak{m}^i_w \to L_w^\times$ for $w \in S_v(L)$.

To state [5, Conj. 3.3] we need to choose certain lattices. For each $v \in S_f := S \cap S_f(K)$, with residue characteristic p, we choose a full projective $\mathbb{Z}_p[G]$-lattice $\mathcal{L}_v \subseteq \mathcal{O}_{L,v}$ which is contained in a sufficiently large power of $\mathfrak{m}_{L,v}$ to ensure that the exponential map is defined on $\mathcal{L}_v$. Let $\mathcal{L}$ be the full projective $\mathbb{Z}[G]$-sublattice of $\mathcal{O}_L$ which has p-adic completions

$$\mathcal{L} \otimes_{\mathbb{Z}} \mathbb{Z}_p = \Big(\prod_{v \in S_p(K) \setminus S} \mathcal{O}_{L,v} \Big) \times \Big(\prod_{v \in S_p(K) \cap S} \mathcal{L}_v \Big). \tag{1}$$

We set $L_S := \prod_{v \in S} L_v$ and $\mathcal{L}_S := \prod_{v \in S} \mathcal{L}_v$ (where $\mathcal{L}_v := L_v$ for each $v \in S_\infty(K)$) and we let $\exp_S$ denote the map $\mathcal{L}_S \to L_S^\times$ that is induced by the product of the respective exponential maps. We also write Δ_S for the natural diagonal embedding from $L^\times$ to $L_S^\times$.
Following the notation of [23, Chap. VIII] we write I_L for the group of idèles of L and regard $L^\times$ as embedded diagonally in I_L. The idèle class group is $C_L := I_L/L^\times$ and the S-idèle class group is $C_S(L) := I_L/(L^\times U_{L,S})$, where $U_{L,S} := \prod_{w \in S}\{1\} \times \prod_{w \notin S} \mathcal{O}_w^\times$. We remark that since $\mathrm{Pic}(\mathcal{O}_{L,S}) = 0$, the natural map $L_S^\times \to C_S(L)$ is surjective with kernel $\Delta_S(\mathcal{O}_{L,S}^\times)$. There is also a canonical invariant isomorphism $\mathrm{inv}_{L/K,S} : H^2(G, C_S(L)) \xrightarrow{\cong} \frac{1}{|G|}\mathbb{Z}/\mathbb{Z}$ and we write e_S^{glob} for the element of $\mathrm{Ext}^2_{\mathbb{Z}[G]}(\mathbb{Z}, C_S(L)) = H^2(G, C_S(L))$ that is sent by $\mathrm{inv}_{L/K,S}$ to $\frac{1}{|G|}$.
Let E_S be a complex in $\mathcal{D}(\mathbb{Z}[G])$ which corresponds (in the sense of the last paragraph of §2.2) to e_S^{glob}. Then by [5, Lemma 2.4] there is a unique morphism $\alpha_S : \mathcal{L}_S[0] \oplus \mathcal{L}[-1] \to E_S$ in $\mathcal{D}(\mathbb{Z}[G])$ for which $H^0(\alpha_S)$ is the composite $\mathcal{L}_S \xrightarrow{\exp_S} L_S^\times \to C_S(L)$ and $H^1(\alpha_S)$ is the restriction of the trace map $\mathrm{tr}_{L/\mathbb{Q}} : L \to \mathbb{Q}$ to $\mathcal{L}$. Let $E_S(\mathcal{L})$ be any complex which lies in a distinguished triangle in $\mathcal{D}(\mathbb{Z}[G])$ of the form

$$\mathcal{L}_S[0] \oplus \mathcal{L}[-1] \xrightarrow{\alpha_S} E_S \xrightarrow{\beta_S} E_S(\mathcal{L}) \xrightarrow{\gamma_S} . \tag{2}$$

To describe the cohomology of $E_S(\mathcal{L})$ we set $L_\infty := \prod_{w \in S_\infty(L)} L_w$ and write L_∞^0 for the kernel of the map $L_\infty \to \mathbb{R}$ defined by $(l_w)_{w \in S_\infty(L)} \mapsto \sum_{w \in S_\infty(L)} \mathrm{tr}_{L_w/\mathbb{R}}(l_w)$. We write $\exp_\infty$ for the product of the exponential maps $L_\infty \to L_\infty^\times$, Δ_∞ for the diagonal embedding $L^\times \to L_\infty^\times$ and $\log_\infty(\mathcal{O}_L^\times)$ for the full sublattice of L_∞^0 comprising elements x of L_∞ with $\exp_\infty(x) \in \Delta_\infty(\mathcal{O}_L^\times)$. In [5, Lemma 3.1] it is shown that $E_S(\mathcal{L})$ is a perfect complex of G-modules, that $E_S(\mathcal{L}) \otimes \mathbb{Q}$ is acyclic outside degrees -1 and 0, that $H^{-1}(\gamma_S)$ induces an identification of $H^{-1}(E_S(\mathcal{L}))$ with $\{x \in \mathcal{L}_S : \exp_S(x) \in \Delta_S(\mathcal{O}_L^\times)\}$ and that $H^0(\gamma_S)$ induces an identification of $H^0(E_S(\mathcal{L})) \otimes \mathbb{Q}$ with $\ker(\mathrm{tr}_{L/\mathbb{Q}})$. In addition, the projection $\mathcal{L}_S \to L_\infty$ induces an isomorphism of $\mathbb{Q}[G]$-modules from $\{x \in \mathcal{L}_S : \exp_S(x) \in \Delta_S(\mathcal{O}_L^\times)\} \otimes \mathbb{Q}$ to $\log_\infty(\mathcal{O}_L^\times) \otimes \mathbb{Q}$. With these identifications the isomorphism $\ker(\mathrm{tr}_{L/\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\cong} L_\infty^0 = \log_\infty(\mathcal{O}_L^\times) \otimes \mathbb{R}$ which is obtained by restricting the natural isomorphism $L \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\cong} L_\infty$ to $\ker(\mathrm{tr}_{L/\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{R}$ gives a trivialisation $\mu_L : H^0(E_S(\mathcal{L})) \otimes \mathbb{R} \xrightarrow{\cong} H^{-1}(E_S(\mathcal{L})) \otimes \mathbb{R}$ of $E_S(\mathcal{L})$. In [5, Conj.

3.3] it is conjectured that

$$\hat{\partial}^1_G(\zeta^*_{L/K,S}(1)) = -\chi_G(E_S(\mathcal{L}), \mu_L). \tag{3}$$

For a discussion of the basic properties of this conjecture see [5, §3]. In particular for a proof of the fact that this conjecture refines Tate's conjecture [26, Chap. I, Conj. 8.2] see [5, Prop. 3.6(i)].

3. Preliminaries concerning étale cohomology

To relate the conjectural equality (3) to [9, Conj. 4] we will use constructions in étale cohomology that are made in [8]. However, to do this certain aspects of the exposition in [8] require clarification and so in this section we review the relevant parts of these constructions.

We fix L/K and S as in §2.4 but for simplicity we also assume henceforth that S contains at least one non-archimedean place. For each $w \in S(L)$ we denote the algebraic closure of L in L_w by L_w^h. For $w \in S_f(L)$ we let $\mathcal{O}_w^h$ be the ring of integers in L_w^h; note that $\mathcal{O}_w^h$ is the henselization of (the localization of) $\mathcal{O}_L$ at w (compare [21, Chap. I, Exam. 4.10(a)]) and that L_w^h is the field of fractions of $\mathcal{O}_w^h$.

Similarly, for a place $v \in S(K)$ we define K_v^h as the algebraic closure of K in K_v. The inclusions $\mathcal{O}_{K,S} \subset K_v^h \subset K_v$ induce canonical maps $g_v^h : \operatorname{Spec} K_v^h \to \operatorname{Spec} \mathcal{O}_{K,S}$, $f_v : \operatorname{Spec} K_v \to \operatorname{Spec} K_v^h$ and $g_v = g_v^h \circ f_v : \operatorname{Spec} K_v \to \operatorname{Spec} \mathcal{O}_{K,S}$.

3.1. General conventions.

Let X be any scheme and $\mathcal{F}$ an étale sheaf on X, i.e. a sheaf on the étale site X_{et}. If Y is an étale X-scheme then we denote by $R\Gamma(Y, \mathcal{F})$ the complex in the derived category $\mathcal{D}(\mathbb{Z})$ which is obtained by applying the right derived functor of the section functor $\Gamma(Y, -)$ to the sheaf $\mathcal{F}$; thus $R\Gamma(Y, \mathcal{F})$ is defined up to canonical isomorphism in $\mathcal{D}(\mathbb{Z})$. If $Y = \operatorname{Spec} R$ for some commutative ring R, then we will write $R\Gamma(R, \mathcal{F})$ for $R\Gamma(\operatorname{Spec} R, \mathcal{F})$ and $H^i(R, \mathcal{F})$ for the cohomology groups $H^i(R\Gamma(R, \mathcal{F}))$.

Now let $v \in S(K)$, $w \in S_v(L)$ and let $\mathcal{F}$ be an étale sheaf on $\operatorname{Spec} K_v^h$. The G_w-action on $\operatorname{Spec} L_w^h$ induces a G_w-action on the sections $\Gamma(\operatorname{Spec} L_w^h, \mathcal{F})$ and hence the complex $R\Gamma(L_w^h, \mathcal{F})$ naturally lies in $\mathcal{D}(\mathbb{Z}[G_w])$. Similarly, if $\mathcal{F}$ is an étale sheaf on $\operatorname{Spec} \mathcal{O}_{K,S}$, then $R\Gamma(\mathcal{O}_{L,S}, \mathcal{F})$ belongs to $\mathcal{D}(\mathbb{Z}[G])$. Finally for $v \in S(K)$ and $\mathcal{F}$ an étale sheaf on $\operatorname{Spec} \mathcal{O}_{K,S}$ we can consider $\bigoplus_{w \in S_v(L)} R\Gamma(L_w^h, (g_v^h)^*\mathcal{F})$ as a complex in $\mathcal{D}(\mathbb{Z}[G])$. This is possible because there is a canonical isomorphism

$$\bigoplus_{w \in S_v(L)} R\Gamma(L_w^h, (g_v^h)^*\mathcal{F}) \cong R\Gamma\Big(\coprod_{w \in S_v(L)} \operatorname{Spec} L_w^h, (g_v^h)^*\mathcal{F}\Big),$$

and $\coprod_{w \in S_v(L)} \operatorname{Spec} L_w^h$ is a Galois covering of $\operatorname{Spec} K_v^h$ with group G. Of course the same is true with L_w^h and g_v^h replaced by L_w and g_v respectively.

3.2. Local cohomology. Let v be a place of K and $w \in S_v(L)$. Recall that $f_v : \operatorname{Spec} K_v \to \operatorname{Spec} K_v^h$ corresponds to the inclusion $K_v^h \to K_v$. For any étale sheaf $\mathcal{F}$ on $\operatorname{Spec} K_v^h$ the canonical map $R\Gamma(L_w^h, \mathcal{F}) \longrightarrow R\Gamma(L_w, f_v^* \mathcal{F})$ is an isomorphism in $\mathcal{D}(\mathbb{Z}[G_w])$. Indeed, if $\overline{L_w}$ is an algebraic closure of L_w and $\overline{L_w^h}$ is the algebraic closure of L_w^h in $\overline{L_w}$, then the restriction map gives an isomorphism $\operatorname{Gal}(\overline{L_w}/K_v) \xrightarrow{\cong} \operatorname{Gal}(\overline{L_w^h}/K_v^h)$. Thus, upon identifying étale cohomology and Galois cohomology the claimed isomorphism follows.
If $\mathcal{F} = \mathbb{G}_m$ on $(\operatorname{Spec} K_v^h)_{\mathrm{et}}$, then $f_v^* \mathbb{G}_m$ is not isomorphic to the sheaf $\mathbb{G}_m$ on $(\operatorname{Spec} K_v)_{\mathrm{et}}$. However the complexes $R\Gamma(L_w^h, \mathbb{G}_m) \cong R\Gamma(L_w, f_v^* \mathbb{G}_m)$ and $R\Gamma(L_w, \mathbb{G}_m)$ are related as follows.

Lemma 3.1. *There is a distinguished triangle in $\mathcal{D}(\mathbb{Z}[G_w])$*

$$R\Gamma(L_w^h, \mathbb{G}_m) \longrightarrow R\Gamma(L_w, \mathbb{G}_m) \longrightarrow (L_w^\times/(L_w^h)^\times)[0] \longrightarrow,$$

whose cohomology sequence in degree 0 identifies with the canonical short exact sequence $0 \longrightarrow (L_w^h)^\times \longrightarrow L_w^\times \longrightarrow L_w^\times/(L_w^h)^\times \longrightarrow 0$. The G_w-module $L_w^\times/(L_w^h)^\times$ is uniquely divisible and hence cohomologically trivial.

Proof. There is a canonical injection $f_v^* \mathbb{G}_m \to \mathbb{G}_m$ of sheaves on $(\operatorname{Spec} K_v)_{\mathrm{et}}$ such that the sequence

$$0 \longrightarrow f_v^* \mathbb{G}_m \longrightarrow \mathbb{G}_m \longrightarrow \mathbb{G}_m/f_v^* \mathbb{G}_m \longrightarrow 0$$

corresponds to the exact sequence $0 \to \overline{L_w^h}^\times \to \overline{L_w}^\times \to \overline{L_w}^\times/\overline{L_w^h}^\times \to 0$ of $\operatorname{Gal}(\overline{L_w}/K_v)$-modules. Now $\overline{L_w}^\times/\overline{L_w^h}^\times$ is uniquely divisible. Also, the isomorphism $\operatorname{Gal}(\overline{L_w}/K_v) \cong \operatorname{Gal}(\overline{L_w^h}/K_v^h)$ combines with Hilbert's Theorem 90 to imply $H^0(\operatorname{Gal}(\overline{L_w}/L_w), \overline{L_w}^\times/\overline{L_w^h}^\times) = L_w^\times/(L_w^h)^\times$ as G_w-modules. It follows that $L_w^\times/(L_w^h)^\times$ is uniquely divisible and hence cohomologically trivial (as a G_w-module). In addition, by applying $R\Gamma(L_w, -)$ to the displayed exact sequence we obtain the claimed distinguished triangle. $\square$

Lemma 3.2. *There are canonical isomorphisms of G_w-modules*

$$H^i(L_w, \mathbb{G}_m) \cong \begin{cases} L_w^\times & \text{if } i = 0, \\ 0 & \text{if } i = 1, \\ \operatorname{Br}(L_w) & \text{if } i = 2. \end{cases}$$

If w is non-archimedean then $H^i(L_w, \mathbb{G}_m) = 0$ for $i \geq 3$ and the local invariant isomorphism gives a canonical identification $\operatorname{Br}(L_w) \cong \mathbb{Q}/\mathbb{Z}$. With respect to this identification the class of $R\Gamma(L_w, \mathbb{G}_m)$ in $\operatorname{Ext}^3_{\mathbb{Z}[G_w]}(\mathbb{Q}/\mathbb{Z}, L_w^\times) \cong H^2(G_w, L_w^\times)$ is the local canonical class.

Proof. This is [8, Prop. 3.5.(a)]. $\square$

3.3. Cohomology with compact support. For any étale sheaf $\mathcal{F}$ on $\operatorname{Spec} \mathcal{O}_{K,S}$ we define the complex $R\Gamma_c(\mathcal{O}_{L,S}, \mathcal{F})$ in $\mathcal{D}(\mathbb{Z}[G])$ by

$$R\Gamma_c(\mathcal{O}_{L,S}, \mathcal{F}) := \operatorname{cone}\left(R\Gamma(\mathcal{O}_{L,S}, \mathcal{F}) \to \bigoplus_{w \in S} R\Gamma(L_w^h, (g_{v(w)}^h)^* \mathcal{F})\right)[-1], \tag{4}$$

where, for every $w \in S$, $v(w)$ denotes the place of K below w. Thus this complex lies in a distinguished triangle

$$(5) \qquad R\Gamma_c(\mathcal{O}_{L,S},\mathcal{F}) \longrightarrow R\Gamma(\mathcal{O}_{L,S},\mathcal{F}) \longrightarrow \bigoplus_{w\in S} R\Gamma(L_w^h,(g_{v(w)}^h)^*\mathcal{F}) \longrightarrow .$$

In [8, (3)] a complex $R\Gamma_c(\mathcal{O}_{L,S},\mathcal{F})$ is defined just as in (4) but with L_w^h and $g_{v(w)}^h$ replaced by L_w and $g_{v(w)}$ respectively. However, the observation made at the beginning of §3.2 ensures that this definition coincides with that given above.

3.3.1. *The complex* $R\Gamma_c(\mathcal{O}_{L,S},\mathbb{G}_m)$. We define a G-module $C_S^h(L)$ in the same way as $C_S(L)$ is defined in §2.4 but with L_w replaced by L_w^h for each $w \in S(L)$ and $\mathcal{O}_w$ replaced by $\mathcal{O}_w^h$ for each $w \in S_f(L)$. Then, since we assume $\mathrm{Pic}(\mathcal{O}_{L,S}) = 0$, the natural map $\prod_{w\in S}(L_w^h)^\times \to C_S^h(L)$ is surjective with kernel $\mathcal{O}_{L,S}^\times$.

Lemma 3.3. *There are canonical isomorphisms of G-modules*

$$H^i(R\Gamma_c(\mathcal{O}_{L,S},\mathbb{G}_m)) \cong \begin{cases} C_S^h(L) & \textit{if } i=1,\\ \mathbb{Q}/\mathbb{Z} & \textit{if } i=3,\\ 0 & \textit{otherwise.}\end{cases}$$

Proof. We first note that there are canonical isomorphisms of G-modules

$$H^i(\mathcal{O}_{L,S},\mathbb{G}_m) \cong \begin{cases} \mathcal{O}_{L,S}^\times & \text{if } i=0,\\ 0 & \text{if } i=1,\\ \ker\left(\mathrm{Br}(L)\to\bigoplus_{w\notin S}\mathrm{Br}(L_w)\right) & \text{if } i=2,\\ \bigoplus_{w\in S_{\mathbb{R}}(L)} H^i(L_w,\mathbb{G}_m) & \text{if } i\geq 3,\end{cases}$$

(cf. [22, Chap. II, Prop. 2.1, Rem. 2.2] and recall that $\mathrm{Pic}(\mathcal{O}_{L,S}) = 0$ and $S_f \neq \emptyset$). Now, for every $w \in S$ one has $(g_{v(w)}^h)^*\mathbb{G}_m = \mathbb{G}_m$ on $(\mathrm{Spec}\, K_{v(w)}^h)_{\mathrm{et}}$ because $K_{v(w)}^h$ is an algebraic extension of K. The cohomology sequence of the distinguished triangle (5) with $\mathcal{F} = \mathbb{G}_m$ thus combines with Lemmas 3.1 and 3.2 and the above displayed isomorphisms to give exact sequences

$$0 \to H^0(R\Gamma_c(\mathcal{O}_{L,S},\mathbb{G}_m)) \to \mathcal{O}_{L,S}^\times \to \bigoplus\nolimits_{w\in S}(L_w^h)^\times \to H^1(R\Gamma_c(\mathcal{O}_{L,S},\mathbb{G}_m)) \to 0$$

and

$$0 \to H^2(R\Gamma_c(\mathcal{O}_{L,S},\mathbb{G}_m)) \to \ker\left(\mathrm{Br}(L)\to\bigoplus\nolimits_{w\notin S}\mathrm{Br}(L_w)\right) \to \bigoplus\nolimits_{w\in S}\mathrm{Br}(L_w) \to H^3(R\Gamma_c(\mathcal{O}_{L,S},\mathbb{G}_m)) \to 0$$

and an equality $H^i(R\Gamma_c(\mathcal{O}_{L,S},\mathbb{G}_m)) = 0$ for each $i \geq 4$. All maps here are the canonical ones, thus for $i = 0$ and $i = 1$ the claimed description follows immediately and for $i = 2$ and $i = 3$ it follows by using the canonical exact sequence $0 \to \mathrm{Br}(L) \to \bigoplus_{w\in S(L)}\mathrm{Br}(L_w) \to \mathbb{Q}/\mathbb{Z} \to 0$. □

3.3.2. *The complex* $\widehat{R\Gamma}_c(\mathcal{O}_{L,S}, \mathbb{G}_m)$. Recall that for every $w \in S$ there is a canonical map $g_{v(w)} : \operatorname{Spec} K_{v(w)} \to \operatorname{Spec} \mathcal{O}_{K,S}$ of schemes and an inclusion $g^*_{v(w)}\mathbb{G}_m \to \mathbb{G}_m$ of étale sheaves on $\operatorname{Spec} K_{v(w)}$. Thus we can consider the composite morphism

$$R\Gamma(\mathcal{O}_{L,S}, \mathbb{G}_m) \longrightarrow \bigoplus_{w\in S} R\Gamma(L_w, g^*_{v(w)}\mathbb{G}_m) \longrightarrow \bigoplus_{w\in S} R\Gamma(L_w, \mathbb{G}_m)$$

in $\mathcal{D}(\mathbb{Z}[G])$. We then define the complex $\widehat{R\Gamma}_c(\mathcal{O}_{L,S}, \mathbb{G}_m)$ by setting

$$\widehat{R\Gamma}_c(\mathcal{O}_{L,S}, \mathbb{G}_m) := \operatorname{cone}\left(R\Gamma(\mathcal{O}_{L,S}, \mathbb{G}_m) \longrightarrow \bigoplus_{w\in S} R\Gamma(L_w, \mathbb{G}_m)\right)[-1].$$

Lemma 3.4. *There are canonical isomorphisms of G-modules*

$$H^i(\widehat{R\Gamma}_c(\mathcal{O}_{L,S}, \mathbb{G}_m)) \cong \begin{cases} C_S(L) & \text{if } i = 1, \\ \mathbb{Q}/\mathbb{Z} & \text{if } i = 3, \\ 0 & \text{otherwise.} \end{cases}$$

The class of $\widehat{R\Gamma}_c(\mathcal{O}_{L,S}, \mathbb{G}_m)[1]$ *in* $\operatorname{Ext}^3_{\mathbb{Z}[G]}(\mathbb{Q}/\mathbb{Z}, C_S(L)) \cong H^2(G, C_S(L))$ *is the global canonical class.*

Proof. The computation of the cohomology is similar to the proof of Lemma 3.3, except that the role of (5) is now played by the distinguished triangle

$$\widehat{R\Gamma}_c(\mathcal{O}_{L,S}, \mathbb{G}_m) \longrightarrow R\Gamma(\mathcal{O}_{L,S}, \mathbb{G}_m) \longrightarrow \bigoplus_{w\in S} R\Gamma(L_w, \mathbb{G}_m) \longrightarrow \tag{6}$$

that is induced by the definition of $\widehat{R\Gamma}_c(\mathcal{O}_{L,S}, \mathbb{G}_m)$. In degree 1 we also use the fact that, since $\operatorname{Pic}(\mathcal{O}_{L,S}) = 0$, $C_S(L)$ is canonically isomorphic to the cokernel of the diagonal embedding $\mathcal{O}^\times_{L,S} \to \prod_{w\in S} L^\times_w$. For the extension class see [8, Prop. 3.5(b)] (but note that the result and proof in [8] apply to $\widehat{R\Gamma}_c(\mathcal{O}_{L,S}, \mathbb{G}_m)$ rather than to $R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{G}_m)$ as incorrectly stated in loc. cit.). □

Lemma 3.5. *There is a distinguished triangle in* $\mathcal{D}(\mathbb{Z}[G])$

$$R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{G}_m) \longrightarrow \widehat{R\Gamma}_c(\mathcal{O}_{L,S}, \mathbb{G}_m) \longrightarrow \bigoplus_{w\in S} (L^\times_w/(L^h_w)^\times)[-1] \longrightarrow$$

which on cohomology in degree 1 *induces the canonical exact sequence*

$$0 \to C^h_S(L) \to C_S(L) \to \prod_{w\in S} L^\times_w/(L^h_w)^\times \to 0$$

and on cohomology in degree 3 *induces the identity map* $\mathbb{Q}/\mathbb{Z} \xrightarrow{=} \mathbb{Q}/\mathbb{Z}$.

Proof. This follows upon combining the distinguished triangle in Lemma 3.1 for each $w \in S$ with the distinguished triangle (5) with $\mathcal{F} = \mathbb{G}_m$ and the distinguished triangle (6). □

4. Pro-p-completion

Let L/K be a Galois extension of number fields, $G = \mathrm{Gal}(L/K)$, and S a set of places of K as in §2.4. We will assume throughout this section that L is totally complex. We fix a prime number p and also assume henceforth that S contains all places of residue characteristic p. As in §2.4 we choose lattices $\mathcal{L}_v$ for $v \in S_f$ and define $\mathcal{L}$ by (1). We fix an algebraic closure $\overline{K}$ of K containing L and write K_S for the maximal extension of K inside $\overline{K}$ which is unramified outside S. For each natural number n we write μ_{p^n} for the group of p^n-th roots of unity in $\overline{K}$ and let $\mathbb{Z}_p(1)$ denote the continuous $\mathrm{Gal}(K_S/K)$-module $\varprojlim_n \mu_{p^n}$ where the limit is taken with respect to p-th power maps. In this section we relate $E_S(\mathcal{L}) \otimes \mathbb{Z}_p$ to the explicit complex $R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(1))$ that is defined in [9, p. 522]. For convenience we often abbreviate $R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(1))$ to $R\Gamma_c(\mathbb{Z}_p(1))$.
For any abelian group A and natural number m we write $A_{[m]}$ for the kernel of the endomorphism given by multiplication by m. For each natural number n we consider the $\mathbb{Z}/p^n[G]$-module $\prod_{w \in S_\infty(L)} (L_w^\times)_{[p^n]} \subset L_\infty^\times$. We then define a $\mathbb{Z}_p[G]$-module by setting $L(1)_p := \varprojlim_n \left(\prod_{w \in S_\infty(L)} (L_w^\times)_{[p^n]}\right)$ where the transition morphisms are the p-th power maps. We set $L_p := \prod_{w \in S_p(L)} L_w$ and note that $\mathcal{L}_p := \prod_{v \in S_p(K)} \mathcal{L}_v$ is a full projective $\mathbb{Z}_p[G]$-sublattice of L_p. We write λ_p for the natural localization map $\mathcal{O}_L^\times \otimes \mathbb{Z}_p \to \prod_{w \in S_p(L)} U_{L_w}^{(1)}$. Recall that Leopoldt's Conjecture for the field L and prime number p is the statement that λ_p is injective. With these notations we can now describe the cohomology of the complex $R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Lemma 4.1. *If λ_p is injective (as predicted by Leopoldt's Conjecture for the field L and prime p), then there are canonical isomorphisms*

$$H^i(R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(1))) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong \begin{cases} L(1)_p \otimes_{\mathbb{Z}_p} \mathbb{Q}_p & \text{if } i = 1, \\ \mathrm{cok}(\lambda_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p & \text{if } i = 2, \\ \mathbb{Q}_p & \text{if } i = 3, \\ 0 & \text{otherwise.} \end{cases}$$

Before proving Lemma 4.1 we first state the main result of this section and introduce some further notation.

Proposition 4.2. *There is a distinguished triangle in $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}_p[G])$ of the form*

$$\mathcal{L}_p[0] \oplus \mathcal{L}_p[-1] \longrightarrow R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(1))[2] \longrightarrow E_S(\mathcal{L}) \otimes \mathbb{Z}_p \longrightarrow . \tag{7}$$

Now assume that λ_p is injective (as predicted by Leopoldt's Conjecture for the field L and prime p). With respect to the isomorphisms in Lemma 4.1 and the description of the cohomology groups $H^i(E_S(\mathcal{L})) \otimes \mathbb{Q}$ given in §2.4, the image *under* $-\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ *of the cohomology sequence of (7) is equal to*

$$\begin{array}{c} 0 \longrightarrow L(1)_p\otimes_{\mathbb{Z}_p}\mathbb{Q}_p \xrightarrow{\theta_1} H^{-1}(E_S(\mathcal{L}))\otimes\mathbb{Q}_p \\ \xrightarrow{\theta_2} L_p \xrightarrow{\exp_p} \mathrm{cok}(\lambda_p)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p \xrightarrow{0} H^0(E_S(\mathcal{L}))\otimes\mathbb{Q}_p \\ \xrightarrow{\subset} L_p \xrightarrow{\mathrm{tr}_{L_p/\mathbb{Q}_p}} \mathbb{Q}_p \longrightarrow 0 \end{array} \tag{8}$$

where θ_1 *sends an element* $(r_w\cdot\{\exp(2\pi\sqrt{-1}/p^n)\}_{n\geq 0})_{w\in S_\infty(L)}$ *of* $L(1)_p\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ *to the element* $(r_w\cdot 2\pi\sqrt{-1})_{w\in S_\infty(L)}$ *of* $\ker(\exp_\infty)\otimes\mathbb{Q}_p\subset H^{-1}(E_S(\mathcal{L}))\otimes\mathbb{Q}_p$ *and* θ_2 *is induced by the projection* $L_S\to L_p$.

In the proofs of Lemma 4.1 and Proposition 4.2 we will need the complex $R\Gamma_c(\mu_{p^n}) := R\Gamma_c(\mathcal{O}_{L,S},\mu_{p^n})$ for each natural number n. This complex can be considered in two different ways. On the one hand, since μ_{p^n} is a continuous $\mathrm{Gal}(K_S/K)$-module, we can consider $R\Gamma_c(\mu_{p^n})$ as the concrete complex of $\mathbb{Z}/p^n[G]$-modules that is constructed using continuous cochains in [9, p. 522]. On the other hand, there is a natural étale sheaf μ_{p^n} on $\mathrm{Spec}\,\mathcal{O}_{K,S}$ and we can consider the cohomology with compact support as defined in §3.3. However this will not cause any confusion because it can be shown that these two possible definitions of $R\Gamma_c(\mu_{p^n})$ agree (up to canonical isomorphism), and whenever it is necessary to distinguish between these two constructions of $R\Gamma_c(\mu_{p^n})$ we will emphasize which one we are using.

Proof of Lemma 4.1. Recall that the complex $R\Gamma_c(\mathbb{Z}_p(1))$ defined in [9, p. 522] is equal to $\varprojlim_n R\Gamma_c(\mu_{p^n})$, where $R\Gamma_c(\mu_{p^n})$ denotes the complex constructed using continuous cochains and the transition morphisms are induced by the p-th power map $\mu_{p^{n+1}}\to\mu_{p^n}$. From the exact sequence $0\to\mu_{p^n}\to\mathbb{G}_m\xrightarrow{p^n}\mathbb{G}_m\to 0$ of étale sheaves on $\mathrm{Spec}\,\mathcal{O}_{K,S}$ we obtain the distinguished triangle

$$R\Gamma_c(\mu_{p^n})\xrightarrow{\theta} R\Gamma_c(\mathcal{O}_{L,S},\mathbb{G}_m)\xrightarrow{p^n} R\Gamma_c(\mathcal{O}_{L,S},\mathbb{G}_m)\longrightarrow \tag{9}$$

in $\mathcal{D}(\mathbb{Z}[G])$. To compute the modules $H^i(R\Gamma_c(\mathbb{Z}_p(1)))$ explicitly we combine the cohomology sequence of (9) with the identifications of Lemma 3.3 and then pass to the inverse limit over n. In particular, since each module $L_w^\times/(L_w^h)^\times$ is uniquely divisible (by Lemma 3.1), one obtains in this way canonical isomorphisms

$$H^i(R\Gamma_c(\mathbb{Z}_p(1)))\cong\begin{cases}\varprojlim_n C_S(L)_{[p^n]} & \text{if } i=1,\\ \varprojlim_n C_S(L)/p^n & \text{if } i=2,\\ \mathbb{Z}_p & \text{if } i=3,\\ 0 & \text{otherwise.}\end{cases} \tag{10}$$

To describe this cohomology more explicitly we use the natural exact sequence of finite G-modules

$$(11)\quad 0 \to (\mathcal{O}_{L,S}^\times)_{[p^n]} \xrightarrow{\Delta_S} \prod_{w\in S}(L_w^\times)_{[p^n]} \to C_S(L)_{[p^n]} \\ \to \mathcal{O}_{L,S}^\times/p^n \xrightarrow{\Delta_S/p^n} \prod_{w\in S} L_w^\times/p^n \to C_S(L)/p^n \to 0.$$

For each place (resp. finite place) w of L we write $L_w^\times\hat{\otimes}\mathbb{Z}_p$ (resp. $\mathcal{O}_{L_w}^\times\hat{\otimes}\mathbb{Z}_p$) for the pro-$p$-completion of $L_w^\times$ (resp. $\mathcal{O}_{L_w}^\times$). Note that $\mathcal{O}_{L_w}^\times\hat{\otimes}\mathbb{Z}_p \cong U_{L_w}^{(1)}$ if $w \in S_p(L)$, and that $\mathcal{O}_{L_w}^\times\hat{\otimes}\mathbb{Z}_p$ is finite if $w \in S_f(L)\setminus S_p(L)$. Hence from the commutative diagram

$$\begin{tikzcd} \mathcal{O}_L^\times \otimes \mathbb{Z}_p \arrow[r] \arrow[d, "\subset"] & \prod_{w\in S_f}\mathcal{O}_{L_w}^\times\hat{\otimes}\mathbb{Z}_p \arrow[d, "\subset"] \\ \mathcal{O}_{L,S}^\times \otimes \mathbb{Z}_p \arrow[r, "\varprojlim_n \Delta_S/p^n"] & \prod_{w\in S} L_w^\times\hat{\otimes}\mathbb{Z}_p \end{tikzcd}$$

we can deduce that the map $\varprojlim_n \Delta_S/p^n$ is injective (since $\lambda_p : \mathcal{O}_L^\times \otimes \mathbb{Z}_p \to \prod_{w\in S_p(L)} U_{L_w}^{(1)}$ is injective by assumption), and that $\mathrm{cok}\big(\varprojlim_n \Delta_S/p^n\big)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p = \mathrm{cok}(\lambda_p)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$.

Now the limit $\varprojlim_n(\mathcal{O}_{L,S}^\times)_{[p^n]}$ vanishes and one has $\varprojlim_n \prod_{w\in S}(L_w^\times)_{[p^n]} = \varprojlim_n \prod_{w\in S_\infty(L)}(L_w^\times)_{[p^n]} = L(1)_p$. By passing to the inverse limit over n the sequence (11) thus induces identifications $\varprojlim_n C_S(L)_{[p^n]} = L(1)_p$ and $\varprojlim_n C_S(L)/p^n = \mathrm{cok}\big(\varprojlim_n \Delta_S/p^n\big)$. The explicit description of $H^i(R\Gamma_c(\mathbb{Z}_p(1)))\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ given in Lemma 4.1 therefore follows from (10) and the identification $\mathrm{cok}\big(\varprojlim_n \Delta_S/p^n\big)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p = \mathrm{cok}(\lambda_p)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ described above. □

The proof of Proposition 4.2 will occupy the rest of this section. As the first step in this proof we introduce a useful auxiliary complex.

Lemma 4.3. *There exists a complex Q in $\mathcal{D}(\mathbb{Z}[G])$ which corresponds (in the sense of the third paragraph of §2.2) to the extension class e_S^{glob} and also possesses all of the following properties.*

(i) *Q is a complex of $\mathbb{Z}$-torsion-free G-modules of the form $Q^{-1} \to Q^0 \to Q^1$ (where the first term is placed in degree -1).*

(ii) *The morphism α_S used in the distinguished triangle (2) is represented by a morphism of complexes of G-modules $\alpha : \mathcal{L}_S[0]\oplus\mathcal{L}[-1] \to Q$.*

(iii) *For each natural number n the complex Q/p^n consists of finite projective $\mathbb{Z}/p^n[G]$-modules.*

Proof. At the outset we fix a representative of e_S^{glob} of the form $A \xrightarrow{\delta} B$ as in [5, Rem. 3.2] with B a finitely generated projective $\mathbb{Z}[G]$-module. We write d^{-1} for the composite of $\exp_S : \mathcal{L}_S \to C_S(L)$ and the inclusion $C_S(L) \subset A$. Since $\mathrm{cok}(\exp_S)$ is finite we may choose a finitely generated free $\mathbb{Z}[G]$-module F and

a homomorphism $\pi : F \to A$ such that the morphism $(d^{-1}, \pi) : \mathcal{L}_S \oplus F \to A$ is surjective. We take Q to be the complex $\ker((d^{-1}, \pi)) \xrightarrow{\subseteq} \mathcal{L}_S \oplus F \xrightarrow{\delta\circ(d^{-1},\pi)} B$ where the first term is placed in degree -1. Then (d^{-1}, π) restricts to give a surjection $\ker(\delta \circ (d^{-1}, \pi)) \to C_S(L)$ which induces an identification of $H^0(Q)$ with $C_S(L)$. Via this identification, the morphism from Q to $A \to B$ that is equal to (d^{-1}, π) in degree 0 and to the identity map in degree 1 induces the identity map on cohomology in each degree and so Q represents e_S^{glob}. Further, we obtain a morphism α as in claim (ii) by defining α^0 to be the inclusion $\mathcal{L}_S \subset \mathcal{L}_S \oplus F$ and α^1 to be any lift $\mathcal{L} \xrightarrow{\mathrm{tr}'} B$ of $\mathcal{L} \xrightarrow{\mathrm{tr}} \mathbb{Z}$ through the given surjection $B \to \mathbb{Z}$.
It is easy to see that $(\mathcal{L}_S \oplus F)/p^n$ and B/p^n are finite and projective as $\mathbb{Z}/p^n[G]$-modules. So to prove claim (iii) it remains to show that $\ker((d^{-1}, \pi))/p^n$ is a finite projective $\mathbb{Z}/p^n[G]$-module. The proof of [5, Lemma 3.1] shows that $\ker(\mathcal{L}_S \to C_S(L))$ is finitely generated, from which we can deduce that $\ker((d^{-1}, \pi))$ is finitely generated. Since furthermore $\ker((d^{-1}, \pi))$ is $\mathbb{Z}$-torsion-free, it follows that $\ker((d^{-1}, \pi))$ is in fact $\mathbb{Z}$-free. But the exact sequence $0 \to \ker((d^{-1}, \pi)) \to \mathcal{L}_S \oplus F \to A \to 0$ implies that the G-module $\ker((d^{-1}, \pi))$ is cohomologically trivial, and any cohomologically trivial $\mathbb{Z}$-free $\mathbb{Z}[G]$-module is a projective $\mathbb{Z}[G]$-module. From this it immediately follows that $\ker((d^{-1}, \pi))/p^n$ is finite and projective as $\mathbb{Z}/p^n[G]$-module, as required. $\square$

We now fix a complex Q as in Lemma 4.3, and set $Q_{\lim} := \varprojlim_n Q/p^n$ where the inverse limit is taken with respect to the natural transition morphisms. To compute the cohomology $H^i(Q_{\lim}) = \varprojlim_n H^i(Q/p^n)$ we use the short exact sequence $0 \to Q \xrightarrow{p^n} Q \to Q/p^n \to 0$ together with the identifications $H^0(Q) = C_S(L)$ and $H^1(Q) = \mathbb{Z}$ to compute the cohomology of Q/p^n and then pass to the inverse limit over n. We find that (similar to the proof of Lemma 4.1) $H^{-1}(Q_{\lim}) = \varprojlim_n C_S(L)_{[p^n]}$, $H^0(Q_{\lim}) = \varprojlim_n C_S(L)/p^n$, $H^1(Q_{\lim}) = \mathbb{Z}_p$, and $H^i(Q_{\lim}) = 0$ otherwise. Hence, if we assume that Leopoldt's Conjecture is valid for L at the prime p and use the same identifications as in the proof of Lemma 4.1, then we obtain isomorphisms

$$H^i(Q_{\lim}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong \begin{cases} L(1)_p \otimes_{\mathbb{Z}_p} \mathbb{Q}_p & \text{if } i = -1, \\ \mathrm{cok}(\lambda_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p & \text{if } i = 0, \\ \mathbb{Q}_p & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

Lemma 4.4. *There exists an isomorphism $Q_{\lim} \cong R\Gamma_c(\mathbb{Z}_p(1))[2]$ in $\mathcal{D}(\mathbb{Z}_p[G])$. Further, if Leopoldt's Conjecture is valid for L at the prime p and we use the isomorphisms in Lemma 4.1 and (12) to identify the cohomology groups of $R\Gamma_c(\mathbb{Z}_p(1))[2] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $Q_{\lim} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ respectively, then this isomorphism induces the identity map in each degree of cohomology after tensoring with $\mathbb{Q}_p$.*

Proof. Applying $R\Gamma_c$ to the short exact sequence $0 \to \mu_{p^n} \to \mathbb{G}_m \xrightarrow{p^n} \mathbb{G}_m \to 0$ and combining the resulting distinguished triangle with the triangle of Lemma 3.5 and the fact that each module $L_w^\times/(L_w^h)^\times$ is uniquely divisible (by Lemma 3.1) one obtains the following commutative diagram of distinguished triangles

$$\begin{tikzcd} R\Gamma_c(\mu_{p^n}) \arrow[r] \arrow[d, equal] & R\Gamma_c(\mathcal{O}_{L,S},\mathbb{G}_m) \arrow[r, "p^n"] \arrow[d] & R\Gamma_c(\mathcal{O}_{L,S},\mathbb{G}_m) \arrow[r] \arrow[d] & {} \\ R\Gamma_c(\mu_{p^n}) \arrow[r] & \widehat{R\Gamma}_c(\mathcal{O}_{L,S},\mathbb{G}_m) \arrow[r, "p^n"] & \widehat{R\Gamma}_c(\mathcal{O}_{L,S},\mathbb{G}_m) \arrow[r] & {} \end{tikzcd} \tag{13}$$

Rotating the lower row of (13) (without changing the signs of the maps) gives the distinguished triangle

$$\widehat{R\Gamma}_c(\mathcal{O}_{L,S},\mathbb{G}_m)[1] \xrightarrow{p^n} \widehat{R\Gamma}_c(\mathcal{O}_{L,S},\mathbb{G}_m)[1] \xrightarrow{\varrho_n'} R\Gamma_c(\mu_{p^n})[2] \to .$$

It is not difficult to see that one obtains the same identifications for $H^i(R\Gamma_c(\mu_{p^n}))$ (and hence also for $H^i(R\Gamma_c(\mathbb{Z}_p(1))) = \varprojlim_n H^i(R\Gamma_c(\mu_{p^n})))$ if one computes the cohomology using this distinguished triangle instead of the first row of (13).

Let $\widehat{Q}$ denote the complex

$$Q^{-1} \to Q^0 \to Q^1 \to \mathbb{Q}$$

where Q^{-1} is placed in degree -1, the first two arrows are the differentials of Q and the third is the natural map $Q^1 \to H^1(Q) = \mathbb{Z} \subset \mathbb{Q}$. Associated to the natural short exact sequence $0 \to \widehat{Q} \xrightarrow{p^n} \widehat{Q} \to Q/p^n \to 0$ is a distinguished triangle

$$\widehat{Q} \xrightarrow{p^n} \widehat{Q} \xrightarrow{\varrho_n} Q/p^n \to .$$

It is easy to see that one obtains the same identifications for $H^i(Q/p^n)$ (and hence also for $H^i(Q_{\lim}) = \varprojlim_n H^i(Q/p^n))$ if one computes the cohomology using this distinguished triangle instead of the short exact sequence $0 \to Q \xrightarrow{p^n} Q \to Q/p^n \to 0$.

The second assertion of Lemma 3.4 combines with the fact that Q corresponds to e_S^{glob} to imply the existence of an isomorphism $\xi : \widehat{Q} \cong \widehat{R\Gamma}_c(\mathcal{O}_{L,S},\mathbb{G}_m)[1]$ in $\mathcal{D}(\mathbb{Z}[G])$ which induces the identity map on each degree of cohomology.

We now consider the following diagram in $\mathcal{D}(\mathbb{Z}[G])$

$$\begin{tikzcd} \widehat{Q} \arrow[r, "p^n"] \arrow[d, "\xi"] & \widehat{Q} \arrow[r, "\varrho_n"] \arrow[d, "\xi"] & Q/p^n \arrow[r] & {} \\ \widehat{R\Gamma}_c(\mathcal{O}_{L,S},\mathbb{G}_m)[1] \arrow[r, "p^n"] & \widehat{R\Gamma}_c(\mathcal{O}_{L,S},\mathbb{G}_m)[1] \arrow[r, "\varrho_n'"] & R\Gamma_c(\mu_{p^n})[2] \arrow[r] & {} \end{tikzcd} \tag{14}$$

Since the left hand square of (14) commutes there exists an isomorphism

$$\xi_n : Q/p^n \to R\Gamma_c(\mu_{p^n})[2]$$

in $\mathcal{D}(\mathbb{Z}[G])$ that makes the diagram into an isomorphism of distinguished triangles. In fact the isomorphisms ξ_n can be chosen to be compatible with the inverse systems over n, i.e. such that for every n the square

$$\begin{tikzcd} Q/p^n \arrow[r, "\xi_n"] \arrow[d] & R\Gamma_c(\mu_{p^n})[2] \arrow[d] \\ Q/p^{n-1} \arrow[r, "\xi_{n-1}"] & R\Gamma_c(\mu_{p^{n-1}})[2] \end{tikzcd}$$

commutes in $\mathcal{D}(\mathbb{Z}[G])$. This can be seen for example as follows: if we compute $\widehat{R\Gamma}_c(\mathcal{O}_{L,S}, \mathbb{G}_m)$ and $R\Gamma_c(\mu_{p^n})$ using the concrete realisation of all chain complexes given by the Godement resolution of the sheaves (as described, for example, in [21, Chap. III, Rem. 1.20(c)]), then we obtain a short exact sequence

$$0 \to R\Gamma_c(\mu_{p^n}) \to \widehat{R\Gamma}_c(\mathcal{O}_{L,S}, \mathbb{G}_m) \xrightarrow{p^n} \widehat{R\Gamma}_c(\mathcal{O}_{L,S}, \mathbb{G}_m) \to 0.$$

Then both the top and the bottom row of (14) are canonically isomorphic to the distinguished triangles coming from short exact sequences (i.e. the distinguished triangles which are constructed using mapping cones), and for such distinguished triangles the statement is easy to see.
To be able to pass to the inverse limit we must replace the maps ξ_n in $\mathcal{D}(\mathbb{Z}[G])$ by actual maps of complexes. Since both Q/p^n and $R\Gamma_c(\mu_{p^n})[2]$ are cohomologically bounded complexes of $\mathbb{Z}/p^n[G]$-modules, the natural restriction of scalars homomorphism

$$\mathrm{Hom}_{\mathcal{D}(\mathbb{Z}_p[G])}(Q/p^n, R\Gamma_c(\mu_{p^n})[2]) \to \mathrm{Hom}_{\mathcal{D}(\mathbb{Z}[G])}(Q/p^n, R\Gamma_c(\mu_{p^n})[2]) \tag{15}$$

is bijective (cf. [8, Lemma 17]). Thus for each n the map $\xi_n : Q/p^n \to R\Gamma_c(\mu_{p^n})[2]$ can be represented as $Q/p^n \xleftarrow{\sim} T_n \xrightarrow{\sim} R\Gamma_c(\mu_{p^n})[2]$ where T_n is a complex of $\mathbb{Z}_p[G]$-modules and $Q/p^n \xleftarrow{\sim} T_n$ and $T_n \xrightarrow{\sim} R\Gamma_c(\mu_{p^n})[2]$ are quasi-isomorphisms of complexes of $\mathbb{Z}_p[G]$-modules. By choosing a projective resolution we can assume that T_n is a bounded above complex of projective $\mathbb{Z}_p[G]$-modules. There exists a morphism $T_n \to T_{n-1}$ in $\mathcal{D}(\mathbb{Z}_p[G])$ such that the diagram

$$\begin{tikzcd} Q/p^n \arrow[d] & T_n \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, dashed] & R\Gamma_c(\mu_{p^n})[2] \arrow[d] \\ Q/p^{n-1} & T_{n-1} \arrow[l, "\sim"'] \arrow[r, "\sim"] & R\Gamma_c(\mu_{p^{n-1}})[2] \end{tikzcd}$$

commutes in $\mathcal{D}(\mathbb{Z}_p[G])$. Since T_n is a bounded above complex of projective $\mathbb{Z}_p[G]$-modules, the morphism $T_n \to T_{n-1}$ in $\mathcal{D}(\mathbb{Z}_p[G])$ can be realised by an actual map of complexes, and the above diagram will commute up to homotopy. The same argument as in [8, p. 1367] shows that after modifying the horizontal maps in this diagram by homotopies we can assume that the diagram is commutative. Finally, we can add suitable acyclic complexes to the T_n to guarantee that the maps $T_n \to T_{n-1}$ are surjective.

To summarise, we have constructed morphisms of inverse systems of complexes of $\mathbb{Z}_p[G]$-modules $(Q/p^n) \leftarrow (T_n) \rightarrow (R\Gamma_c(\mu_{p^n})[2])$ such that for each n the composite $Q/p^n \xleftarrow{\sim} T_n \xrightarrow{\sim} R\Gamma_c(\mu_{p^n})[2]$ considered as a map in $\mathcal{D}(\mathbb{Z}[G])$ is equal to ξ_n. Furthermore the transition maps in each inverse system are surjective. Passing to the inverse limit gives morphisms of complexes of $\mathbb{Z}_p[G]$-modules

$$Q_{\lim} = \varprojlim_n Q/p^n \longleftarrow \varprojlim_n T_n \longrightarrow \varprojlim_n R\Gamma_c(\mu_{p^n})[2] = R\Gamma_c(\mathbb{Z}_p(1))[2].$$

Now [8, Lemma 9] implies that these morphisms are quasi-isomorphisms and that the resulting map $Q_{\lim} \to R\Gamma_c(\mathbb{Z}_p(1))[2]$ in $\mathcal{D}(\mathbb{Z}_p[G])$ has the required properties. $\square$

We now fix a morphism α as in Lemma 4.3(ii). Then, for each natural number n one has a commutative diagram of morphisms of complexes of G-modules

$$(16) \qquad \begin{tikzcd} \mathcal{L}_S[0] \oplus \mathcal{L}[-1] \arrow[r, "\alpha"] \arrow[d, hook, "p^n"] & Q \arrow[r, "\beta"] \arrow[d, hook, "p^n"] & \mathrm{cone}(\alpha) \arrow[r, "\gamma"] \arrow[d, hook, "p^n"] & {} \\ \mathcal{L}_S[0] \oplus \mathcal{L}[-1] \arrow[r, "\alpha"] \arrow[d, two heads] & Q \arrow[r, "\beta"] \arrow[d, two heads] & \mathrm{cone}(\alpha) \arrow[r, "\gamma"] \arrow[d, two heads] & {} \\ \mathcal{L}_S/p^n[0] \oplus \mathcal{L}/p^n[-1] \arrow[r, "\alpha/p^n"] & Q/p^n \arrow[r, "\beta/p^n"] & \mathrm{cone}(\alpha/p^n) \arrow[r, "\gamma/p^n"] & {} \end{tikzcd}$$

In this diagram the maps β and γ come from the definition of cone(α) and so the first (and second) row is an explicit representative of the triangle (2). Also, the columns are the short exact sequences which result from the fact that $\mathcal{L}_S$, $\mathcal{L}$ and all terms of Q (and hence also of cone(α)) are $\mathbb{Z}$-torsion-free. Now $\mathcal{L}_p$ is canonically isomorphic to both $\varprojlim_n \mathcal{L}_S/p^n$ and $\varprojlim_n \mathcal{L}/p^n$. Furthermore, as cone$(\alpha)$ is a perfect complex of $\mathbb{Z}$-torsion-free modules, there is a natural isomorphism $\mathrm{cone}(\alpha) \otimes \mathbb{Z}_p \cong \varprojlim_n \mathrm{cone}(\alpha)/p^n$ in $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}_p[G])$, and clearly $\varprojlim_n \mathrm{cone}(\alpha)/p^n \cong \varprojlim_n \mathrm{cone}(\alpha/p^n) \cong \mathrm{cone}(\varprojlim_n \alpha/p^n)$ (where in all cases the limits are taken with respect to the natural transition morphisms). Hence, upon passing to the inverse limit of the lower row of (16), we obtain a distinguished triangle in $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}_p[G])$ of the form

$$(17) \qquad \mathcal{L}_p[0] \oplus \mathcal{L}_p[-1] \xrightarrow{\varprojlim_n \alpha/p^n} Q_{\lim} \xrightarrow{\varprojlim_n \beta/p^n} \mathrm{cone}(\alpha) \otimes \mathbb{Z}_p \xrightarrow{\varprojlim_n \gamma/p^n} .$$

The distinguished triangle (17) together with the isomorphism $Q_{\lim} \cong R\Gamma_c(\mathbb{Z}_p(1))[2]$ from Lemma 4.4 show the existence of a triangle of the form (7).

It remains to show that if Leopoldt's Conjecture is valid for L at the prime p and we use the identifications of the cohomology of $Q_{\lim}$ given in (12), then after tensoring with $\mathbb{Q}_p$ the long exact sequence of cohomology of the triangle (17) is equal to (8). Now the identifications of the cohomology of the three terms in (17) come from the columns in (16). In particular we have natural isomorphisms $H^i(\mathcal{L}_p[0] \oplus \mathcal{L}_p[-1]) \cong \varprojlim_n H^i(\mathcal{L}_S[0] \oplus \mathcal{L}_S[-1])/p^n$ and $H^i(\mathrm{cone}(\alpha) \otimes \mathbb{Z}_p) \cong$

$\varprojlim_n H^i(\mathrm{cone}(\alpha))/p^n$ for all i, and $H^i(Q_{\lim}) \cong \varprojlim_n H^i(Q)/p^n$ for $i = 0$ and $i = 1$. Therefore by considering the cohomology sequences of the second and third rows in (16), we can easily deduce the explicit description of all maps in (8) except for the map $L(1)_p \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = H^{-1}(Q_{\lim}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to H^{-1}(E_S(\mathcal{L})) \otimes \mathbb{Q}_p = \log_\infty(\mathcal{O}_L^\times) \otimes \mathbb{Q}_p$.
To compute this map we consider the following diagram.

$$\begin{tikzcd}
 & & H^{-1}(Q/p^n) \arrow[r, "H^{-1}(\beta/p^n)"] \arrow[d, hook] & H^{-1}(\mathrm{cone}(\alpha)/p^n) \\
 & H^0(\mathcal{L}_S[0] \oplus \mathcal{L}[-1]) \arrow[r, "H^0(\alpha)"] \arrow[d, "p^n"] & H^0(Q) \arrow[d, "p^n"] & \\
H^{-1}(\mathrm{cone}(\alpha)) \arrow[r, hook, "H^{-1}(\gamma)"] \arrow[d] & H^0(\mathcal{L}_S[0] \oplus \mathcal{L}[-1]) \arrow[r, "H^0(\alpha)"] & H^0(Q) & \\
H^{-1}(\mathrm{cone}(\alpha)/p^n) & & &
\end{tikzcd}$$

By an easy computation with cochains one shows that if an element of $H^0(\mathcal{L}_S[0] \oplus \mathcal{L}[-1])$ lies in the kernel of $H^0(\mathcal{L}_S[0] \oplus \mathcal{L}[-1]) \xrightarrow{p^n \cdot H^0(\alpha)} H^0(Q)$, then its images under the two maps

$$H^0(\mathcal{L}_S[0] \oplus \mathcal{L}[-1]) \xrightarrow{H^0(\alpha)} H^0(Q) \leftarrow H^{-1}(Q/p^n) \xrightarrow{H^{-1}(\beta/p^n)} H^{-1}(\mathrm{cone}(\alpha)/p^n)$$

and

$$H^0(\mathcal{L}_S[0] \oplus \mathcal{L}[-1]) \xrightarrow{p^n} H^0(\mathcal{L}_S[0] \oplus \mathcal{L}[-1]) \xleftarrow{H^{-1}(\gamma)} H^{-1}(\mathrm{cone}(\alpha)) \\ \to H^{-1}(\mathrm{cone}(\alpha)/p^n)$$

coincide (note that the inverse arrows make sense in this context). By considering the elements $(r_w \cdot 2\pi\sqrt{-1}/p^n)_{w \in S_\infty} \in L_\infty \subseteq \mathcal{L}_S = H^0(\mathcal{L}_S[0] \oplus \mathcal{L}[-1])$ for $r_w \in \mathbb{Z}$ we see that the map $H^{-1}(Q/p^n) \to H^{-1}(\mathrm{cone}(\alpha)/p^n)$ sends the image of $(r_w \cdot \exp(2\pi\sqrt{-1}/p^n))_{w \in S_\infty(L)} \in (L_S^\times)_{[p^n]} \subset L_S^\times$ in $C_S(L)_{[p^n]} = H^{-1}(Q/p^n)$ to the image of the element $(r_w \cdot 2\pi\sqrt{-1})_{w \in S_\infty(L)} \in \ker(\exp_\infty) \subseteq H^{-1}(\mathrm{cone}(\alpha))$ in $H^{-1}(\mathrm{cone}(\alpha)/p^n)$. Passing to the inverse limit gives the desired description of θ_1. This completes the proof of Proposition 4.2.
q

5. The proof of Theorem 1.1

In this section we prove Theorem 1.1. Let L/K be a Galois extension of number fields with Galois group G. We define an element of $K_0(\mathbb{Z}[G], \mathbb{R})$ by setting

$$T\Omega(L/K, 1) := \hat{\partial}_G^1(\zeta_{L/K,S}^*(1)) + \chi_G(E_S(\mathcal{L}), \mu_L)$$

where the terms on the right hand side are as in §2.4. The element $T\Omega(L/K, 1)$ depends only upon L/K (see [5, Prop. 3.4]), and the conjectural equality (3) asserts that $T\Omega(L/K,1)$ vanishes. We also recall that [9, Conj. 4(iv)] for the pair $(\mathbb{Q}(1)_L, \mathbb{Z}[G])$ asserts the vanishing of an element $T\Omega(\mathbb{Q}(1)_L, \mathbb{Z}[G])$ of $K_0(\mathbb{Z}[G], \mathbb{R})$ that is defined (unconditionally) in [9, Conj. 4(iii)]. To prove Theorem 1.1 it is therefore enough to prove the following result.

Proposition 5.1. *Let L be a complex Galois extension of $\mathbb{Q}$ and $G = \mathrm{Gal}(L/\mathbb{Q})$. If Leopoldt's Conjecture is valid for L and all prime numbers p, then $T\Omega(L/\mathbb{Q},1) = T\Omega(\mathbb{Q}(1)_L, \mathbb{Z}[G])$.*

Remark 5.1. Recall that we write $\partial^0_{\mathbb{Z}[G],\mathbb{R}}$ for the natural homomorphism of K-groups $K_0(\mathbb{Z}[G], \mathbb{R}) \to K_0(\mathbb{Z}[G])$. The argument of [5, Prop. 3.6(ii)] combines with the equality of Proposition 5.1 to imply that if Leopoldt's Conjecture is valid, then $\partial^0_{\mathbb{Z}[G],\mathbb{R}}(T\Omega(\mathbb{Q}(1)_L, \mathbb{Z}[G]))$ is equal to the element $\Omega(L/K,1)$ of $K_0(\mathbb{Z}[G])$ defined by Chinburg in [13]. Proposition 5.1 therefore answers the question raised in [7, Question 1.54].

5.1. Preliminaries. From now on let $L/\mathbb{Q}$ be a complex Galois extension with Galois group G. For each p and each embedding $j : \mathbb{R} \to \mathbb{C}_p$ there is an induced homomorphism $j_* : K_0(\mathbb{Z}[G], \mathbb{R}) \to K_0(\mathbb{Z}_p[G], \mathbb{C}_p)$ and it is known that $\bigcap_{p,j} \ker(j_*) = \{0\}$ where p runs over all primes and j over all embeddings $\mathbb{R} \to \mathbb{C}_p$ (cf. [5, Lemma 2.1]). To prove Proposition 5.1 it is thus enough to prove that for all p and j one has

$$j_*(T\Omega(L/\mathbb{Q},1)) = j_*(T\Omega(\mathbb{Q}(1)_L, \mathbb{Z}[G])). \tag{18}$$

The proof of this equality will occupy the rest of this section.

We fix a prime p and in the sequel assume that Leopoldt's Conjecture is valid for L and p. We also fix an embedding $j : \mathbb{R} \to \mathbb{C}_p$ and often suppress it from our notation; so in particular in a tensor product of the form $- \otimes_{\mathbb{R}} \mathbb{C}_p$ we consider $\mathbb{C}_p$ as an $\mathbb{R}$-module via j. Just as in §4 we will always assume that S contains all places of residue characteristic p.

In the following we will need to use the language of virtual objects. To this end we consider the Picard categories $\mathcal{V}(\mathbb{Z}_p[G])$, $\mathcal{V}(\mathbb{C}_p[G])$ and $\mathcal{V}(\mathbb{Z}_p[G], \mathbb{C}_p[G])$ discussed in [4, §5]. We fix a unit object $\mathbf{1}_{\mathcal{V}(\mathbb{C}_p[G])}$ of $\mathcal{V}(\mathbb{C}_p[G])$ and for each object X of $\mathcal{V}(\mathbb{C}_p[G])$ we fix an inverse, i.e. an object X^{-1} of $\mathcal{V}(\mathbb{C}_p[G])$ together with an isomorphism $X \otimes X^{-1} \cong \mathbf{1}_{\mathcal{V}(\mathbb{C}_p[G])}$ in $\mathcal{V}(\mathbb{C}_p[G])$. We also write $\iota : \pi_0\mathcal{V}(\mathbb{Z}_p[G], \mathbb{C}_p[G]) \cong K_0(\mathbb{Z}_p[G], \mathbb{C}_p)$ for the group isomorphism described in [4, Lemma 5.1].

We need to slightly generalise the definition of a trivialised complex and its Euler characteristic. If P is a perfect complex of $\mathbb{Z}_p[G]$-modules and $\tau : [H^{\mathrm{ev}}(P \otimes_{\mathbb{Z}_p} \mathbb{C}_p)] \to [H^{\mathrm{od}}(P \otimes_{\mathbb{Z}_p} \mathbb{C}_p)]$ an isomorphism in $\mathcal{V}(\mathbb{C}_p[G])$, then we will sometimes call the pair (P, τ) a trivialised complex. Its Euler characteristic $\chi_{\mathbb{Z}_p[G],\mathbb{C}_p}(P, \tau)$ is defined as in [4, Definition 5.5] except that $[t]$ is replaced by τ. Clearly any trivialised complex (P, t) as in §2.2 gives rise to the trivialised complex $(P, [t])$ in the new sense, but in general not every trivialisation $\tau : [H^{\mathrm{ev}}(P \otimes_{\mathbb{Z}_p} \mathbb{C}_p)] \to [H^{\mathrm{od}}(P \otimes_{\mathbb{Z}_p} \mathbb{C}_p)]$ of P will be of the form $[t]$ for some isomorphism $t : H^{\mathrm{ev}}(P \otimes_{\mathbb{Z}_p} \mathbb{C}_p) \to H^{\mathrm{od}}(P \otimes_{\mathbb{Z}_p} \mathbb{C}_p)$.

5.2. The element $j_*(T\Omega(L/\mathbb{Q}, 1))$. We set $R\Gamma_c(\mathbb{Z}_p(1)) := R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(1))$ and also $H_c^i(\mathbb{C}_p(1)) := H^i(R\Gamma_c(\mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} \mathbb{C}_p)$. Furthermore we write $H_c^{\mathrm{ev}}(\mathbb{C}_p(1))$ and $H_c^{\mathrm{od}}(\mathbb{C}_p(1))$ for the direct sums $\oplus_{i \text{ even}} H_c^i(\mathbb{C}_p(1))$ and $\oplus_{i \text{ odd}} H_c^i(\mathbb{C}_p(1))$ respectively.

We start by defining an isomorphism

$$\psi : [H_c^{\mathrm{ev}}(\mathbb{C}_p(1))] \otimes [\mathrm{im}(\theta_2) \otimes_{\mathbb{Q}_p} \mathbb{C}_p] \xrightarrow{\cong} [H_c^{\mathrm{od}}(\mathbb{C}_p(1))] \otimes [\mathrm{im}(\theta_2) \otimes_{\mathbb{Q}_p} \mathbb{C}_p]$$

in $\mathcal{V}(\mathbb{C}_p[G])$ which is induced by the identifications from Lemma 4.1, the exact sequence (8) in Proposition 4.2, and μ_L. More precisely, we let ψ be the following composite map.

$$\begin{aligned}
&[H_c^2(\mathbb{C}_p(1))] \otimes [\mathrm{im}(\theta_2) \otimes_{\mathbb{Q}_p} \mathbb{C}_p] \\
&\qquad\xrightarrow{\alpha_1} [L_p \otimes_{\mathbb{Q}_p} \mathbb{C}_p] \\
&\qquad\xrightarrow{\alpha_2} [H^0(E_S(\mathcal{L})) \otimes \mathbb{C}_p] \otimes [\mathbb{C}_p] \\
&\qquad\xrightarrow{\alpha_3} [H^{-1}(E_S(\mathcal{L})) \otimes \mathbb{C}_p] \otimes [\mathbb{C}_p] \\
&\qquad\xrightarrow{\alpha_4} [L(1)_p \otimes_{\mathbb{Z}_p} \mathbb{C}_p] \otimes [\mathrm{im}(\theta_2) \otimes_{\mathbb{Q}_p} \mathbb{C}_p] \otimes [\mathbb{C}_p] \\
&\qquad\xrightarrow{\alpha_5} [H_c^1(\mathbb{C}_p(1)) \oplus H_c^3(\mathbb{C}_p(1))] \otimes [\mathrm{im}(\theta_2) \otimes_{\mathbb{Q}_p} \mathbb{C}_p].
\end{aligned}$$

Here α_1 is induced by the isomorphism $H_c^2(\mathbb{C}_p(1)) \cong \mathrm{cok}(\lambda_p) \otimes_{\mathbb{Z}_p} \mathbb{C}_p$ and the short exact sequence

$$\mathrm{im}(\theta_2) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \overset{\subset}{\hookrightarrow} L_p \otimes_{\mathbb{Q}_p} \mathbb{C}_p \xrightarrow{\exp_p} \mathrm{cok}(\lambda_p) \otimes_{\mathbb{Z}_p} \mathbb{C}_p, \tag{19}$$

α_2 and α_4 are induced by the short exact sequences

$$H^0(E_S(\mathcal{L})) \otimes \mathbb{C}_p \hookrightarrow L_p \otimes_{\mathbb{Q}_p} \mathbb{C}_p \xrightarrow{\mathrm{tr}} \mathbb{C}_p \tag{20}$$

and

$$L(1)_p \otimes_{\mathbb{Z}_p} \mathbb{C}_p \xrightarrow{\theta_1 \otimes_{\mathbb{Q}_p} \mathbb{C}_p} H^{-1}(E_S(\mathcal{L})) \otimes \mathbb{C}_p \xrightarrow{\theta_2 \otimes_{\mathbb{Q}_p} \mathbb{C}_p} \mathrm{im}(\theta_2) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \tag{21}$$

respectively, $\alpha_3 = [\mu_L \otimes_{\mathbb{R}} \mathbb{C}_p] \otimes \mathrm{id}$, and α_5 is induced by the isomorphisms $H_c^1(\mathbb{C}_p(1)) \cong L(1)_p \otimes_{\mathbb{Z}_p} \mathbb{C}_p$ and $H_c^3(\mathbb{C}_p(1)) \cong \mathbb{C}_p$.

Now by the properties of a Picard category there exists a unique isomorphism

$$\nu : [H_c^{\mathrm{ev}}(\mathbb{C}_p(1))] \xrightarrow{\cong} [H_c^{\mathrm{od}}(\mathbb{C}_p(1))]$$

in $\mathcal{V}(\mathbb{C}_p[G])$ such that $\psi = \nu \otimes \mathrm{id}$. We will consider this isomorphism as a trivialisation of the complex $R\Gamma_c(\mathbb{Z}_p(1))$.

Lemma 5.2. *In $K_0(\mathbb{Z}_p[G], \mathbb{C}_p)$ one has*

$$j_*(T\Omega(L/\mathbb{Q}, 1)) = \partial^1_{\mathbb{Z}_p[G],\mathbb{C}_p}\big(j_*(\zeta^*_{L/\mathbb{Q},S}(1))\big) + \chi_{\mathbb{Z}_p[G],\mathbb{C}_p}(R\Gamma_c(\mathbb{Z}_p(1)), \nu).$$

Proof. To simplify the notation we will abbreviate '$\chi_{\mathbb{Z}_p[G],\mathbb{C}_p}$' to '$\chi_p$'.

It is clear that $j_*(\hat{\partial}^1_G(\zeta^*_{L/\mathbb{Q},S}(1))) = \partial^1_{\mathbb{Z}_p[G],\mathbb{C}_p}(j_*(\zeta^*_{L/\mathbb{Q},S}(1)))$ (compare §2.1) and also $j_*(\chi_{\mathbb{Z}[G],\mathbb{R}}(E_S(\mathcal{L}), \mu_L)) = \chi_p(E_S(\mathcal{L}) \otimes \mathbb{Z}_p, \mu_L \otimes_{\mathbb{R}} \mathbb{C}_p)$. Moreover it follows from [4, Prop. 5.6.3] that $\chi_p(R\Gamma_c(\mathbb{Z}_p(1)),\nu) = \chi_p(R\Gamma_c(\mathbb{Z}_p(1))[2],\nu)$. It is thus enough to prove that in $K_0(\mathbb{Z}_p[G],\mathbb{C}_p)$ one has

$$\chi_p(E_S(\mathcal{L})\otimes\mathbb{Z}_p, \mu_L\otimes_{\mathbb{R}}\mathbb{C}_p) = \chi_p(R\Gamma_c(\mathbb{Z}_p(1))[2],\nu). \tag{22}$$

To do this we will apply the additivity criterion of [4, Theorem 5.7] to the exact triangle (7) in Proposition 4.2. On the complex $\mathcal{L}_p[0]\oplus\mathcal{L}_p[-1]$ we consider the trivialisation given by the identity map $\mathrm{id}:\mathcal{L}_p\otimes_{\mathbb{Z}_p}\mathbb{C}_p\to\mathcal{L}_p\otimes_{\mathbb{Z}_p}\mathbb{C}_p$, on $R\Gamma_c(\mathbb{Z}_p(1))[2]$ we consider the trivialisation ν, and on $E_S(\mathcal{L})\otimes\mathbb{Z}_p$ we consider the trivialisation $\mu_L\otimes_{\mathbb{R}}\mathbb{C}_p$. Note that the additivity criterion in [4] is only stated for trivialisations as defined in §2.2, however it is easy to check that it remains valid for generalised trivialisations as defined in §5.1.

In our context, the map a in [4, Theorem 5.7] is the map $\mathcal{L}_p[0]\oplus\mathcal{L}_p[-1]\to R\Gamma_c(\mathbb{Z}_p(1))[2]$ in the distinguished triangle (7), and $\Sigma=\mathbb{C}_p[G]$. Therefore $\ker(H^{\mathrm{ev}}a_\Sigma) = \mathrm{im}(\theta_2)\otimes_{\mathbb{Q}_p}\mathbb{C}_p$ and $\ker(H^{\mathrm{od}}a_\Sigma) = L_p^0\otimes_{\mathbb{Q}_p}\mathbb{C}_p$ where $L_p^0 = \ker(\mathrm{tr}_{L_p/\mathbb{Q}_p}: L_p\to\mathbb{Q}_p)$. To apply the additivity criterion we must show that the following diagram commutes in $\mathcal{V}(\mathbb{C}_p[G])$.

$$\begin{tikzcd}
\begin{array}{c}[\mathrm{cok}(\lambda_p)\otimes_{\mathbb{Z}_p}\mathbb{C}_p]\\ \otimes[\mathrm{im}(\theta_2)\otimes_{\mathbb{Q}_p}\mathbb{C}_p]\otimes[L_p^0\otimes_{\mathbb{Q}_p}\mathbb{C}_p]\end{array} \arrow[r, "s^{\mathrm{ev}}"] \arrow[d, "\nu\otimes\mathrm{id}\otimes[-\mathrm{id}]"] & {[L_p\otimes_{\mathbb{Q}_p}\mathbb{C}_p]\otimes[H^0(E_S(\mathcal{L}))\otimes\mathbb{C}_p]} \arrow[d, "\mathrm{id}\otimes[\mu_L\otimes_{\mathbb{R}}\mathbb{C}_p]"] \\
\begin{array}{c}[L(1)_p\otimes_{\mathbb{Z}_p}\mathbb{C}_p\oplus\mathbb{C}_p]\\ \otimes[\mathrm{im}(\theta_2)\otimes_{\mathbb{Q}_p}\mathbb{C}_p]\otimes[L_p^0\otimes_{\mathbb{Q}_p}\mathbb{C}_p]\end{array} \arrow[r, "s^{\mathrm{od}}"] & {[L_p\otimes_{\mathbb{Q}_p}\mathbb{C}_p]\otimes[H^{-1}(E_S(\mathcal{L}))\otimes\mathbb{C}_p]}
\end{tikzcd}$$

Here the horizontal maps are induced by the even respectively odd part of the cohomology sequence (8) after tensoring with $\mathbb{C}_p$, i.e. the top horizontal map s^{ev} is induced by the short exact sequence (19) and the isomorphism

$$H^0(E_S(\mathcal{L}))\otimes\mathbb{C}_p\cong L_p^0\otimes_{\mathbb{Q}_p}\mathbb{C}_p, \tag{23}$$

and the bottom horizontal map s^{od} is induced by (21) and

$$L_p^0\otimes_{\mathbb{Q}_p}\mathbb{C}_p \xhookrightarrow{\ \subset\ } L_p\otimes_{\mathbb{Q}_p}\mathbb{C}_p \xrightarrow{\mathrm{tr}} \mathbb{C}_p. \tag{24}$$

To see the commutativity of the above diagram we will show that the automorphism

$$\kappa := (\mathrm{id}\otimes[\mu_L\otimes_{\mathbb{R}}\mathbb{C}_p])^{-1}\circ(s^{\mathrm{od}})\circ(\nu\otimes\mathrm{id}\otimes[-\mathrm{id}])\circ(s^{\mathrm{ev}})^{-1}$$

of $[L_p\otimes_{\mathbb{Q}_p}\mathbb{C}_p]\otimes[H^0(E_S(\mathcal{L}))\otimes\mathbb{C}_p]$ is the identity map. For this we use the isomorphism

$$[L_p\otimes_{\mathbb{Q}_p}\mathbb{C}_p]\otimes[H^0(E_S(\mathcal{L}))\otimes\mathbb{C}_p]\cong[L_p^0\otimes_{\mathbb{Q}_p}\mathbb{C}_p]\otimes[\mathbb{C}_p]\otimes[L_p^0\otimes_{\mathbb{Q}_p}\mathbb{C}_p]$$

which is induced by the short exact sequence (24) and the isomorphism (23). Using $\nu\otimes\mathrm{id}\otimes[-\mathrm{id}]=\psi\otimes[-\mathrm{id}]$ and the definition of ψ, it is easy to see that then κ becomes the automorphism of $[L_p^0\otimes_{\mathbb{Q}_p}\mathbb{C}_p]\otimes[\mathbb{C}_p]\otimes[L_p^0\otimes_{\mathbb{Q}_p}\mathbb{C}_p]$ which is given by (using the obvious abuse of notation)

$$a\otimes b\otimes c\mapsto[-\mathrm{id}](c)\otimes b\otimes a,$$

i.e. the morphism in $\mathcal{V}(\mathbb{C}_p[G])$ which swaps the two copies of $[L_p^0 \otimes_{\mathbb{Q}_p} \mathbb{C}_p]$ composed with the map $[-\mathrm{id}]$ on one of the two copies. It now follows from the general properties of a determinant functor (see e.g. [15, §4.9]), that this automorphism (and hence also κ) is the identity morphism as required.
The additivity criterion [4, Theorem 5.7] now implies that

$$\chi_p(R\Gamma_c(\mathbb{Z}_p(1))[2], \nu) = \chi_p(\mathcal{L}_p[0] \oplus \mathcal{L}_p[-1], \mathrm{id}) + \chi_p(E_S(\mathcal{L}) \otimes \mathbb{Z}_p, \mu_L \otimes_{\mathbb{R}} \mathbb{C}_p).$$

Since clearly $\chi_p(\mathcal{L}_p[0] \oplus \mathcal{L}_p[-1], \mathrm{id}) = 0$ this completes the proof of (22) and hence of Lemma 5.2. □

5.3. The element $j_*(T\Omega(\mathbb{Q}(1)_L, \mathbb{Z}[G]))$. The motive $\mathbb{Q}(1)_L$ is pure of weight -2. The argument of [10, §2] therefore shows that

$$j_*(T\Omega(\mathbb{Q}(1)_L, \mathbb{Z}[G])) = \partial^1_{\mathbb{Z}_p[G],\mathbb{C}_p}\big(j_*(\zeta^*_{L/\mathbb{Q},S}(1))\big) + \iota\big(([R\Gamma_c(\mathbb{Z}_p(1))], \omega)\big) \tag{25}$$

with ω the composite morphism

$$[R\Gamma_c(\mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} \mathbb{C}_p] \xrightarrow{\tilde\vartheta_p \otimes_{\mathbb{Q}_p} \mathbb{C}_p} [\Xi(\mathbb{Q}(1)_L) \otimes_{\mathbb{Q}} \mathbb{C}_p] \xrightarrow{\vartheta_\infty \otimes_{\mathbb{R}} \mathbb{C}_p} \mathbf{1}_{\mathcal{V}(\mathbb{C}_p[G])}$$

where $\tilde\vartheta_p$ and ϑ_∞ are as defined in [10, p. 479, resp. p. 477]. Indeed, whilst the argument of [10, §2] is phrased solely in terms of abelian groups G it extends immediately to the general case upon replacing graded determinants by virtual objects and then (25) is the non-abelian generalisation of the equality [10, (11)]. Given the observations of [7, §1.1, §1.3] it is also a straightforward exercise to explicate the space $\Xi(\mathbb{Q}(1)_L)$ and the morphisms $\tilde\vartheta_p$ and ϑ_∞. To describe the result we introduce further notation. We write $\Sigma(L)$ for the set of all complex embeddings $L \to \mathbb{C}$ and consider $\bigoplus_{\Sigma(L)} \mathbb{C}$ as a $G \times \mathrm{Gal}(\mathbb{C}/\mathbb{R})$-module where G acts via L and $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts diagonally. We write H_B for the $G \times \mathrm{Gal}(\mathbb{C}/\mathbb{R})$-submodule $\bigoplus_{\Sigma(L)} 2\pi\sqrt{-1} \cdot \mathbb{Z}$ of $\bigoplus_{\Sigma(L)} \mathbb{C}$ and let H_B^+ and $\big(\bigoplus_{\Sigma(L)} \mathbb{C}\big)^+$ denote the G-submodules comprising elements invariant under the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. We also set $H_f^1 := \mathrm{im}(\lambda_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Then ω is equal to the composite

$$\begin{aligned}
[R\Gamma_c(\mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} \mathbb{C}_p] &\cong [H_c^1(\mathbb{C}_p(1))]^{-1} \otimes [H_c^2(\mathbb{C}_p(1))] \otimes [H_c^3(\mathbb{C}_p(1))]^{-1} \\
&\cong [H_c^1(\mathbb{C}_p(1))]^{-1} \otimes \big([H_f^1 \otimes_{\mathbb{Q}_p} \mathbb{C}_p] \otimes [H_c^2(\mathbb{C}_p(1))]\big) \\
&\qquad \otimes [H_f^1 \otimes_{\mathbb{Q}_p} \mathbb{C}_p]^{-1} \otimes [H_c^3(\mathbb{C}_p(1))]^{-1} \\
&\cong \big([H_B^+ \otimes \mathbb{C}_p]^{-1} \otimes [L \otimes_{\mathbb{Q}} \mathbb{C}_p]\big) \\
&\qquad \otimes \big([\mathcal{O}_L^\times \otimes \mathbb{C}_p]^{-1} \otimes [\mathbb{C}_p]^{-1}\big) \\
&\cong \Big[\prod\nolimits_{S_\infty(L)} \mathbb{C}_p\Big] \otimes \Big[\prod\nolimits_{S_\infty(L)} \mathbb{C}_p\Big]^{-1} \\
&\cong \mathbf{1}_{\mathcal{V}(\mathbb{C}_p[G])}
\end{aligned} \tag{26}$$

where the maps are defined as follows. The first, second and fifth maps are clear. The third map is induced by the exact sequence

$$0 \to L(1)_p \otimes_{\mathbb{Z}_p} \mathbb{C}_p \xrightarrow{\cong} H^1_c(\mathbb{C}_p(1)) \xrightarrow{0} H^1_f \otimes_{\mathbb{Q}_p} \mathbb{C}_p \xrightarrow{\subseteq} \prod_{w \in S_p(L)} U^{(1)}_{L_w} \otimes_{\mathbb{Z}_p} \mathbb{C}_p$$
$$\xrightarrow{\pi} H^2_c(\mathbb{C}_p(1)) \to 0 \to 0 \to H^3_c(\mathbb{C}_p(1)) \xrightarrow{\cong} \mathbb{C}_p \to 0,$$

where π is induced by the identification $H^2_c(\mathbb{C}_p(1)) \cong \mathrm{cok}(\lambda_p) \otimes_{\mathbb{Z}_p} \mathbb{C}_p$ from Lemma 4.1 (this sequence is the cohomology sequence of the distinguished triangle of [10, (3)] with $M = \mathbb{Q}(1)_L$ and $A = \mathbb{Q}[G]$), together with the isomorphism $L(1)_p \cong H^+_B \otimes \mathbb{Z}_p$ that sends an element $\big(n_w \cdot \{\exp(2\pi\sqrt{-1}/p^n)\}_{n\geq 0}\big)_{w \in S_\infty(L)}$ in $L(1)_p$ to the element $\big(n_{w_\sigma} \cdot \hat{\sigma}(2\pi\sqrt{-1})\big)_{\sigma \in \Sigma(L)}$ in $H^+_B \otimes \mathbb{Z}_p$ (where w_σ denotes the place of L corresponding to σ, and $\hat{\sigma} : L_{w_\sigma} \to \mathbb{C}$ is the unique continuous extension of σ), the isomorphism

$$(27) \qquad \prod_{w \in S_p(L)} U^{(1)}_{L_w} \otimes_{\mathbb{Z}_p} \mathbb{C}_p \cong \Big(\prod_{w \in S_p(L)} L_w\Big) \otimes_{\mathbb{Q}_p} \mathbb{C}_p = L \otimes_{\mathbb{Q}} \mathbb{C}_p$$

induced by the p-adic logarithm maps $U^{(1)}_{L_w} \to L_w$, and the isomorphism $\lambda_p \otimes_{\mathbb{Z}_p} \mathbb{C}_p : \mathcal{O}^\times_L \otimes \mathbb{C}_p \cong H^1_f \otimes_{\mathbb{Q}_p} \mathbb{C}_p$. The fourth map is induced by (the image under $- \otimes_{\mathbb{R}} \mathbb{C}_p$ of) the short exact sequence

$$(28) \qquad \mathcal{O}^\times_L \otimes \mathbb{R} \xhookrightarrow{\mathrm{Reg}} \prod\nolimits_{S_\infty(L)} \mathbb{R} \twoheadrightarrow \mathbb{R}$$

where $\mathrm{Reg} : \mathcal{O}^\times_L \otimes \mathbb{R} \to \prod_{S_\infty(L)} \mathbb{R}$ denotes the usual regulator map $u \otimes r \mapsto r \cdot (2\log|\sigma_w(u)|)_{w \in S_\infty(L)}$ (here σ_w is a complex embedding of L corresponding to the place w), the natural isomorphism

$$(29) \qquad \Big(\bigoplus\nolimits_{\Sigma(L)} \mathbb{C}\Big)^+ \cong L \otimes_{\mathbb{Q}} \mathbb{R}$$

and (the image under $- \otimes_{\mathbb{R}} \mathbb{C}_p$ of) the short exact sequence

$$(30) \qquad H^+_B \otimes \mathbb{R} \xhookrightarrow{\subset} \Big(\bigoplus\nolimits_{\Sigma(L)} \mathbb{C}\Big)^+ \twoheadrightarrow \prod\nolimits_{S_\infty(L)} \mathbb{R}$$

in which the second arrow sends each element $(z_\sigma)_{\sigma \in \Sigma(L)}$ of $\big(\bigoplus_{\Sigma(L)} \mathbb{C}\big)^+$ to $(z_{\sigma_w} + z_{\overline{\sigma_w}})_{w \in S_\infty(L)}$ in $\prod_{S_\infty(L)} \mathbb{R}$ (where σ_w and $\overline{\sigma_w}$ denote the two complex embeddings of L corresponding to the place w).

5.4. Completion of the proof. Let

$$\psi' : [H^{\mathrm{ev}}_c(\mathbb{C}_p(1))] \otimes [H^1_f \otimes_{\mathbb{Q}_p} \mathbb{C}_p] \xrightarrow{\cong} [H^{\mathrm{od}}_c(\mathbb{C}_p(1))] \otimes [H^1_f \otimes_{\mathbb{Q}_p} \mathbb{C}_p]$$

denote the composite isomorphism

$$\begin{aligned}
[H^2_c(\mathbb{C}_p(1))] \otimes [H^1_f \otimes_{\mathbb{Q}_p} \mathbb{C}_p] &\xrightarrow{\alpha'_1} [L \otimes_{\mathbb{Q}} \mathbb{C}_p] \\
&\xrightarrow{\alpha'_2} \Big[\Big(\bigoplus\nolimits_{\Sigma(L)} \mathbb{C}\Big)^+ \otimes_{\mathbb{R}} \mathbb{C}_p\Big] \\
&\xrightarrow{\alpha'_3} [H^+_B \otimes \mathbb{C}_p] \otimes \Big[\prod\nolimits_{S_\infty(L)} \mathbb{C}_p\Big] \\
&\xrightarrow{\alpha'_4} [L(1)_p \otimes_{\mathbb{Z}_p} \mathbb{C}_p] \otimes [\mathcal{O}^\times_L \otimes \mathbb{C}_p] \otimes [\mathbb{C}_p] \\
&\xrightarrow{\alpha'_5} [H^1_c(\mathbb{C}_p(1)) \oplus H^3_c(\mathbb{C}_p(1))] \otimes [H^1_f \otimes_{\mathbb{Q}_p} \mathbb{C}_p]
\end{aligned}$$

where α'_1 is induced by the short exact sequence

$$H^1_f \otimes_{\mathbb{Q}_p} \mathbb{C}_p \hookrightarrow \prod\nolimits_{w \in S_p(L)} U^{(1)}_{L_w} \otimes_{\mathbb{Z}_p} \mathbb{C}_p \twoheadrightarrow H^2_c(\mathbb{C}_p(1))$$

and the isomorphism (27), the map α'_2 is induced by the isomorphism (29), the map α'_3 is induced by (the image under $- \otimes_{\mathbb{R}} \mathbb{C}_p$ of) the short exact sequence (30), α'_4 is induced by (the image under $- \otimes_{\mathbb{R}} \mathbb{C}_p$ of) the short exact sequence (28) and the isomorphism $H^+_B \otimes \mathbb{C}_p \cong L(1)_p \otimes_{\mathbb{Z}_p} \mathbb{C}_p$, and α'_5 is induced by the isomorphisms $H^1_c(\mathbb{C}_p(1)) \cong L(1)_p \otimes_{\mathbb{Z}_p} \mathbb{C}_p$, $H^3_c(\mathbb{C}_p(1)) \cong \mathbb{C}_p$ and $\mathcal{O}^\times_L \otimes \mathbb{C}_p \cong H^1_f \otimes_{\mathbb{Q}_p} \mathbb{C}_p$.

Let $\nu' : [H^{\mathrm{ev}}_c(\mathbb{C}_p(1))] \xrightarrow{\cong} [H^{\mathrm{od}}_c(\mathbb{C}_p(1))]$ be the unique isomorphism in $\mathcal{V}(\mathbb{C}_p[G])$ such that $\nu' \otimes \mathrm{id} = \psi'$. We recall that the Euler characteristic $\chi_{\mathbb{Z}_p[G],\mathbb{C}_p}(R\Gamma_c(\mathbb{Z}_p(1)), \nu')$ is defined to be $\iota\big(([R\Gamma_c(\mathbb{Z}_p(1))], \lambda)\big)$, where λ is the composite isomorphism

$$\begin{aligned}
[R\Gamma_c(\mathbb{C}_p(1))] &\cong [H^{\mathrm{ev}}_c(\mathbb{C}_p(1))] \otimes [H^{\mathrm{od}}_c(\mathbb{C}_p(1))]^{-1} \\
&\xrightarrow{\nu' \otimes \mathrm{id}} [H^{\mathrm{od}}_c(\mathbb{C}_p(1))] \otimes [H^{\mathrm{od}}_c(\mathbb{C}_p(1))]^{-1} \cong 1_{\mathcal{V}(\mathbb{C}_p[G])}
\end{aligned}$$

in $\mathcal{V}(\mathbb{C}_p[G])$ (compare [4, Definition 5.5]). Now by comparing ω and λ one can show that

$$\iota\big(([R\Gamma_c(\mathbb{Z}_p(1))], \omega)\big) = \chi_{\mathbb{Z}_p[G],\mathbb{C}_p}(R\Gamma_c(\mathbb{Z}_p(1)), \nu'). \tag{31}$$

The isomorphism (27) restricts to an isomorphism

$$\varphi : H^1_f \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong \mathrm{im}(\theta_2) \otimes_{\mathbb{Q}_p} \mathbb{C}_p$$

of $\mathbb{C}_p[G]$-modules and we will show below that the following diagram in $\mathcal{V}(\mathbb{C}_p[G])$ is commutative.

$$\begin{tikzcd}
{[H^{\mathrm{ev}}_c(\mathbb{C}_p(1))] \otimes [H^1_f \otimes_{\mathbb{Q}_p} \mathbb{C}_p]} \arrow[r, "\mathrm{id} \otimes [\varphi]"] \arrow[d, "\nu' \otimes \mathrm{id}"] & {[H^{\mathrm{ev}}_c(\mathbb{C}_p(1))] \otimes [\mathrm{im}(\theta_2) \otimes_{\mathbb{Q}_p} \mathbb{C}_p]} \arrow[d, "\nu \otimes \mathrm{id}"] \\
{[H^{\mathrm{od}}_c(\mathbb{C}_p(1))] \otimes [H^1_f \otimes_{\mathbb{Q}_p} \mathbb{C}_p]} \arrow[r, "\mathrm{id} \otimes [\varphi]"] & {[H^{\mathrm{od}}_c(\mathbb{C}_p(1))] \otimes [\mathrm{im}(\theta_2) \otimes_{\mathbb{Q}_p} \mathbb{C}_p]}
\end{tikzcd}$$

From this diagram it follows that $\nu = \nu'$. In view of Lemma 5.2 and equations (25) and (31) this implies the required equality (18) and hence Proposition 5.1. It now only remains to show that the above diagram in $\mathcal{V}(\mathbb{C}_p[G])$ is commutative. For this we consider the following diagram.

$$\begin{tikzcd}
{[H_c^2(\mathbb{C}_p(1))] \otimes [H_f^1 \otimes_{\mathbb{Q}_p} \mathbb{C}_p]} \arrow[r, "\mathrm{id}\otimes[\varphi]"] \arrow[d, "\alpha_1'"] & {[H_c^2(\mathbb{C}_p(1))] \otimes [\mathrm{im}(\theta_2) \otimes_{\mathbb{Q}_p} \mathbb{C}_p]} \arrow[d, "\alpha_1"] \\
{[L \otimes_{\mathbb{Q}} \mathbb{C}_p]} \arrow[r, equal] \arrow[d, "\alpha_3'\circ\alpha_2'"] & {[L \otimes_{\mathbb{Q}} \mathbb{C}_p]} \arrow[d, "\alpha_3\circ\alpha_2"] \\
{[H_B^+ \otimes \mathbb{C}_p] \otimes \left[\left(\prod_{S_\infty(L)} \mathbb{R}\right) \otimes_{\mathbb{R}} \mathbb{C}_p\right]} \arrow[d, "\beta_1"] & {[H^{-1}(E_S(\mathcal{L})) \otimes \mathbb{C}_p] \otimes [\mathbb{C}_p]} \arrow[d, "\alpha_4"] \\
{\begin{array}{c}[L(1)_p \otimes_{\mathbb{Z}_p} \mathbb{C}_p] \\ \otimes[\mathrm{im}(\theta_2) \otimes_{\mathbb{Q}_p} \mathbb{C}_p] \otimes [\mathbb{C}_p]\end{array}} \arrow[r, equal] \arrow[d, "\beta_2"] & {\begin{array}{c}[L(1)_p \otimes_{\mathbb{Z}_p} \mathbb{C}_p] \\ \otimes[\mathrm{im}(\theta_2) \otimes_{\mathbb{Q}_p} \mathbb{C}_p] \otimes [\mathbb{C}_p]\end{array}} \arrow[d, "\alpha_5"] \\
{\begin{array}{c}[H_c^1(\mathbb{C}_p(1)) \oplus H_c^3(\mathbb{C}_p(1))] \\ \otimes[H_f^1 \otimes_{\mathbb{Q}_p} \mathbb{C}_p]\end{array}} \arrow[r, "\mathrm{id}\otimes[\varphi]"] & {\begin{array}{c}[H_c^1(\mathbb{C}_p(1)) \oplus H_c^3(\mathbb{C}_p(1))] \\ \otimes[\mathrm{im}(\theta_2) \otimes_{\mathbb{Q}_p} \mathbb{C}_p]\end{array}}
\end{tikzcd}$$

Here the maps α_i and α_i' are as above. The map β_1 is induced by the isomorphism $L(1)_p \otimes_{\mathbb{Z}_p} \mathbb{C}_p \cong H_B^+ \otimes \mathbb{C}_p$ and the short exact sequence

$$\mathrm{im}(\theta_2) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \hookrightarrow \left(\prod_{S_\infty(L)} \mathbb{R}\right) \otimes_{\mathbb{R}} \mathbb{C}_p \twoheadrightarrow \mathbb{C}_p \tag{32}$$

which is obtained by applying $- \otimes_{\mathbb{R}} \mathbb{C}_p$ to the short exact sequence (28) and using the identification $\mathcal{O}_L^\times \otimes \mathbb{C}_p \cong \mathrm{im}(\theta_2) \otimes_{\mathbb{Q}_p} \mathbb{C}_p$, and the map β_2 is induced by the isomorphisms $H_c^1(\mathbb{C}_p(1)) \cong L(1)_p \otimes_{\mathbb{Z}_p} \mathbb{C}_p$, $H_c^3(\mathbb{C}_p(1)) \cong \mathbb{C}_p$ and φ.
By definition the composite of the right vertical maps is $\psi = \nu \otimes \mathrm{id}$. Furthermore it is not difficult to see that $\beta_2 \circ \beta_1 = \alpha_5' \circ \alpha_4'$, hence the composite of the left vertical maps is $\psi' = \nu' \otimes \mathrm{id}$.
Clearly the bottom square is commutative. The isomorphism of short exact sequences

$$\begin{tikzcd}
H_f^1 \otimes_{\mathbb{Q}_p} \mathbb{C}_p \arrow[r, hook] \arrow[d, "\varphi"] & \prod_{w \in S_p(L)} U_{L_w}^{(1)} \otimes_{\mathbb{Z}_p} \mathbb{C}_p \arrow[r, two heads] \arrow[d, "\cong"] & H_c^2(\mathbb{C}_p(1)) \arrow[d, "\cong"] \\
\mathrm{im}(\theta_2) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \arrow[r, hook] & L \otimes_{\mathbb{Q}} \mathbb{C}_p \arrow[r, two heads] & \mathrm{cok}(\lambda_p) \otimes_{\mathbb{Z}_p} \mathbb{C}_p
\end{tikzcd}$$

implies that the top square is commutative. The commutativity of the middle rectangle follows from the properties of a determinant functor applied to the following commutative diagram of short exact sequences.

$$\begin{tikzcd}
L(1)_p \otimes_{\mathbb{Z}_p} \mathbb{C}_p \arrow[r, hook, "\theta_1 \otimes_{\mathbb{Q}_p} \mathbb{C}_p"] \arrow[d, "\cong"] & H^{-1}(E_S(\mathcal{L})) \otimes \mathbb{C}_p \arrow[r, two heads, "\theta_2 \otimes_{\mathbb{Q}_p} \mathbb{C}_p"] \arrow[d, hook] & \mathrm{im}(\theta_2) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \arrow[d, hook] \\
H_B^+ \otimes \mathbb{C}_p \arrow[r, hook] & L \otimes_{\mathbb{Q}} \mathbb{C}_p \arrow[r, two heads] \arrow[d, two heads, "\mathrm{tr}"] & \left(\prod_{S_\infty(L)} \mathbb{R}\right) \otimes_{\mathbb{R}} \mathbb{C}_p \arrow[d, two heads] \\
 & \mathbb{C}_p \arrow[r, equal] & \mathbb{C}_p
\end{tikzcd}$$

Here the top horizontal and right vertical short exact sequences are (21) and (32) respectively. The middle horizontal short exact sequence comes from combining (30) with the isomorphism (29), and the middle vertical short exact sequence comes from combining (20) with the isomorphism $\mu_L \otimes_{\mathbb{R}} \mathbb{C}_p$. The commutativity of this diagram is easily checked.

6. The proofs of Corollaries 1.2, 1.3 and 1.4

In this section we use Theorem 1.1 to prove Corollaries 1.2, 1.3 and 1.4.

6.1. The proof of Corollary 1.2. Let F/E be a Galois extension of number fields and set $\Gamma := \mathrm{Gal}(F/E)$. Let L be a totally complex finite Galois extension of $\mathbb{Q}$ containing F and set $G := \mathrm{Gal}(L/\mathbb{Q})$. We write π for the natural composite homomorphism $K_0(\mathbb{Z}[G], \mathbb{R}) \to K_0(\mathbb{Z}[\mathrm{Gal}(L/E)], \mathbb{R}) \to K_0(\mathbb{Z}[\Gamma], \mathbb{R})$ where the first arrow is restriction and the second projection. Then it is known that $\pi(T\Omega(L/\mathbb{Q}, 1)) = T\Omega(F/E, 1)$ and $\pi(T\Omega(\mathbb{Q}(1)_L, \mathbb{Z}[G])) = T\Omega(\mathbb{Q}(1)_F, \mathbb{Z}[\Gamma])$ (see [5, Prop. 3.5] and [9, Prop. 4.1]). In particular, to prove that $T\Omega(F/E, 1) = T\Omega(\mathbb{Q}(1)_F, \mathbb{Z}[\Gamma])$ it is enough to prove that $T\Omega(L/\mathbb{Q}, 1) = T\Omega(\mathbb{Q}(1)_L, \mathbb{Z}[G])$. Given this observation, Corollary 1.2 is an immediate consequence of Theorem 1.1.

6.2. The proof of Corollary 1.3. By the functorial properties of the conjectures (see [5, Prop. 3.5 and Rem. 4.2]) it suffices to consider the case $K = \mathbb{Q}$ and L totally complex. Since L is abelian over $\mathbb{Q}$, Leopoldt's Conjecture is known to be valid for L and all primes p [6]. In addition, the validity of [9, Conj. 4(iv)] for the pair $(\mathbb{Q}(1)_L, \mathbb{Z}[\mathrm{Gal}(L/\mathbb{Q})])$ has been proved by Flach and the second named author in [11, Cor. 1.2]. (The proof of [11, Cor. 1.2] relies on certain 2-adic results of Flach in [16] and unfortunately the relevant results in [16] are now known to contain errors. However, in [17] Flach has recently provided the necessary corrections so that, in particular, the result of [11, Cor. 1.2] is valid as stated.) Given the validity of [9, Conj. 4(iv)] for $(\mathbb{Q}(1)_L, \mathbb{Z}[\mathrm{Gal}(L/\mathbb{Q})])$, the first assertion of Corollary 1.3 follows immediately from Theorem 1.1.

We now assume that [5, Conj. 3.3] is valid for $L/\mathbb{Q}$. Then [5, Theorem 5.2] implies that [5, Conj. 4.1] is valid for $L/\mathbb{Q}$ if and only if [5, Conj. 5.3] is valid for $L/\mathbb{Q}$. Also, in [5, Rem. 5.4] it is shown that [5, Conj. 5.3] is equivalent to

the earlier conjecture [2, Conj. 4.1]. To prove the second assertion of Corollary 1.3 we therefore need only note that [2, Conj. 4.1] is proved for abelian extensions $L/\mathbb{Q}$ of odd conductor in [2, Cor. 6.2] and for abelian extensions $L/\mathbb{Q}$ of arbitrary conductor in [11, Theorem 1.1] (see in particular the discussion at the end of [11, §3.1]).
This completes the proof of Corollary 1.3.

Remark 6.1. By using the main result of Bley in [1] one can prove an analogue of Corollary 1.3 for certain classes of abelian extensions of imaginary quadratic fields.

6.3. The proof of Corollary 1.4. Let p, q and r be distinct (odd) rational primes which satisfy $p \equiv r \equiv -q \equiv 3 \pmod 4$ and are such that the Legendre symbols $(\frac{p}{q})$ and $(\frac{r}{q})$ are both equal to -1. Then if ℓ is any odd prime such that $(\frac{\ell}{pr}) = -(\frac{\ell}{q}) = 1$ Chinburg has shown that there exists a unique totally complex field $L_{p,q,r,\ell}$ which contains $\mathbb{Q}(\sqrt{pr}, \sqrt{q})$, is Galois over $\mathbb{Q}$ with group isomorphic to the quaternion group of order 8 and is such that $L_{p,q,r,\ell}/\mathbb{Q}$ is ramified precisely at p, q, r, ℓ and infinity (cf. [14, Prop. 4.1.3]). We observe that the primes $p = 3, q = 5$ and $r = 7$ satisfy the congruence conditions described above and will now prove that the conjectures [5, Conj. 3.3] and [5, Conj. 4.1] are both valid for any extension of the form $L_{3,5,7,\ell}/K$. To do this we set $L_\ell := L_{3,5,7,\ell}$ and $G_\ell := \mathrm{Gal}(L_{3,5,7,\ell}/\mathbb{Q})$.
We note first that L_ℓ/K is tamely ramified and we recall that for any tamely ramified extension of number fields F/E the element $T\Omega^{\mathrm{loc}}(F/E, 1)$ that is defined in [5, §5.1.1] vanishes (by [5, Prop. 5.7(i)]) and hence that the conjectures [5, Conj. 3.3] and [5, Conj. 4.1] are equivalent for F/E (by [5, Theorem 5.2]). It therefore suffices for us to prove that [5, Conj. 3.3] is valid for all extensions L_ℓ/K. We recall that this is equivalent to asserting that the element $T\Omega(L_\ell/K, 1)$ of $K_0(\mathbb{Z}[\mathrm{Gal}(L_\ell/K)], \mathbb{R})$ that is defined in [5, §3.2] vanishes. Taking account of the functorial behaviour described in [5, Prop. 3.5(i)] it is therefore enough to prove that each element $T\Omega(L_\ell/\mathbb{Q}, 1)$ vanishes.
We claim next that $T\Omega(L_\ell/\mathbb{Q}, 1)$ belongs to the subgroup $K_0(\mathbb{Z}[G_\ell], \mathbb{Q})_{\mathrm{tor}}$ of $K_0(\mathbb{Z}[G_\ell], \mathbb{R})$. Indeed, since $T\Omega^{\mathrm{loc}}(L_\ell/\mathbb{Q}, 1)$ vanishes the equality of [5, Theorem 5.2] implies $T\Omega(L_\ell/\mathbb{Q}, 1) = \psi^*_{G_\ell}(T\Omega(L_\ell/\mathbb{Q}, 0))$ where $\psi^*_{G_\ell}$ is the involution of $K_0(\mathbb{Z}[G_\ell], \mathbb{R})$ defined in [5, §2.1.4] and $T\Omega(L_\ell/\mathbb{Q}, 0)$ the element of $K_0(\mathbb{Z}[G_\ell], \mathbb{R})$ defined in [5, §4]. Now $\psi^*_{G_\ell}$ preserves the subgroup $K_0(\mathbb{Z}[G_\ell], \mathbb{Q})_{\mathrm{tor}}$ and from [5, Prop. 4.4(ii)] one knows that $T\Omega(L_\ell/\mathbb{Q}, 0)$ belongs to $K_0(\mathbb{Z}[G_\ell], \mathbb{Q})_{\mathrm{tor}}$ if the 'strong Stark conjecture' of Chinburg is valid for $L_\ell/\mathbb{Q}$. It thus suffices to recall that, since every complex character of G_ℓ is rational valued, the strong Stark conjecture for $L_\ell/\mathbb{Q}$ has been proved by Tate in [26, Chap. II].
We write F_ℓ for the maximal abelian extension of $\mathbb{Q}$ in L_ℓ (and note that $F_\ell/\mathbb{Q}$ is biquadratic). Then, since the element $T\Omega(L_\ell/\mathbb{Q}, 1)$ belongs to $K_0(\mathbb{Z}[G_\ell], \mathbb{Q})_{\mathrm{tor}}$, the result of [10, Lemma 4] implies $T\Omega(L_\ell/\mathbb{Q}, 1)$ vanishes if it belongs to the kernels of both the natural projection homomorphism

$q : K_0(\mathbb{Z}[G_\ell], \mathbb{R}) \to K_0(\mathbb{Z}[\mathrm{Gal}(F_\ell/\mathbb{Q})], \mathbb{R})$ and the connecting homomorphism $\partial^0_{\mathbb{Z}[G_\ell],\mathbb{R}} : K_0(\mathbb{Z}[G_\ell], \mathbb{R}) \to K_0(\mathbb{Z}[G_\ell])$.

Now from [5, Prop. 3.6(ii)] one knows that $T\Omega(L_\ell/\mathbb{Q}, 1)$ belongs to $\ker(\partial^0_{\mathbb{Z}[G_\ell],\mathbb{R}})$ if Chinburg's 'Ω_1-Conjecture' [13, Question 3.2] is valid for $L_\ell/\mathbb{Q}$. In addition, the equality of [13, (3.2)] shows that the Ω_1-Conjecture is valid for $L_\ell/\mathbb{Q}$ if the 'Ω_3-Conjecture' [13, Conj. 3.1] and 'Ω_2-Conjecture' [13, Question 3.1] are both valid for $L_\ell/\mathbb{Q}$. But Chinburg proves the Ω_3-Conjecture for $L_\ell/\mathbb{Q}$ in [14] and, since $L_\ell/\mathbb{Q}$ is tamely ramified, the validity of the Ω_2-Conjecture for $L_\ell/\mathbb{Q}$ follows directly from [13, Theorems 3.2 and 3.3].

At this stage it suffices to prove that $T\Omega(L_\ell/\mathbb{Q}, 1)$ belongs to $\ker(q)$. But, by [5, Prop. 3.5(ii)], this is equivalent to asserting that [5, Conj. 3.3] is valid for the extension $F_\ell/\mathbb{Q}$ and since $F_\ell/\mathbb{Q}$ is abelian this follows from Corollary 1.3. This completes the proof of Corollary 1.4.

References

[1] W. Bley, On the equivariant Tamagawa number conjecture for abelian extensions of a quadratic imaginary field, Doc. Math. 11 (2006), 73–118.

[2] W. Bley, D. Burns, Equivariant epsilon constants, discriminants and étale cohomology, Proc. London Math. Soc. 87 (2003), 545–590.

[3] S. Bloch, K. Kato, L-functions and Tamagawa numbers of motives, in: The Grothendieck Festschrift, vol. 1, pp. 333–400, Progress in Math. 86 (Birkhäuser, Boston, 1990).

[4] M. Breuning, D. Burns, Additivity of Euler characteristics in relative algebraic K-groups, Homology, Homotopy Appl. 7 (2005), No. 3, 11–36.

[5] M. Breuning, D. Burns, Leading terms of Artin L-functions at $s = 0$ and $s = 1$, Compos. Math. 143 (2007), 1427–1464.

[6] A. Brumer, On the units of algebraic number fields, Mathematika 14 (1967), 121–124.

[7] D. Burns, M. Flach, Motivic L-functions and Galois module structures, Math. Ann. 305 (1996), 65–102.

[8] D. Burns, M. Flach, On Galois structure invariants associated to Tate motives, Amer. J. Math. 120 (1998), 1343–1397.

[9] D. Burns, M. Flach, Tamagawa numbers for motives with (non-commutative) coefficients, Doc. Math. 6 (2001), 501–570.

[10] D. Burns, M. Flach, Tamagawa numbers for motives with (noncommutative) coefficients, II, Amer. J. Math. 125 (2003), 475–512.

[11] D. Burns, M. Flach, On the equivariant Tamagawa number conjecture for Tate motives, II, Doc. Math., Extra volume: John H. Coates' Sixtieth Birthday (2006), 133–163.

[12] D. Burns, O. Venjakob, On descent theory and main conjectures in non-commutative Iwasawa theory, to appear in J. Inst. Math. Jussieu (2010).

[13] T. Chinburg, Exact sequences and Galois module structure, Ann. of Math. 121 (1985), 351–376.

[14] T. Chinburg, The analytic theory of multiplicative Galois structure, Memoirs of the Amer. Math. Soc. 77 (1989).
[15] P. Deligne, Le déterminant de la cohomologie, in: Current trends in arithmetical algebraic geometry (Arcata, CA, 1985), pp. 93–177, Contemp. Math. 67 (Amer. Math. Soc., Providence, 1987).
[16] M. Flach, The equivariant Tamagawa number conjecture: a survey (with an appendix by C. Greither), Contemp. Math. 358 (2004), 79–126.
[17] M. Flach, On the cyclotomic main conjecture for the prime 2, preprint 2009.
[18] T. Fukaya, K. Kato, A formulation of conjectures on p-adic zeta functions in non-commutative Iwasawa theory, Proc. St. Petersburg Math. Soc. Vol. XII, 1–85, Amer. Math. Soc. Transl. Ser. 2, 219 (Amer. Math. Soc., Providence, 2006).
[19] A. Huber, G. Kings, Equivariant Bloch-Kato conjecture and non-abelian Iwasawa main conjecture, Proc. of the International Congress of Mathematicians, vol. II (Beijing, 2002), 149–162.
[20] A. Jones, Dirichlet L-functions at $s = 1$, PhD Thesis, King's College London, 2007.
[21] J. S. Milne, Étale cohomology (Princeton University Press, Princeton, 1980).
[22] J. S. Milne, Arithmetic duality theorems (Academic Press, Boston, 1986).
[23] J. Neukirch, A. Schmidt, K. Wingberg, Cohomology of number fields (Springer-Verlag, Berlin, 2000).
[24] J.-P. Serre, Sur le résidue de la fonction zêta p-adique d'un corps de nombres, C.R. Acad. Sci. Paris 278 (1978), 183–188.
[25] R. G. Swan, Algebraic K-theory, Lecture Notes in Math. 76 (Springer-Verlag, Berlin, 1968).
[26] J. Tate, Les conjectures de Stark sur les fonctions L d'Artin en $s = 0$ (Birkhäuser, Boston, 1984).

Manuel Breuning
King's College London
Department of Mathematics
London WC2R 2LS
England
manuel.breuning@kcl.ac.uk

David Burns
King's College London
Department of Mathematics
London WC2R 2LS
England
david.burns@kcl.ac.uk

Variations on a Theme of Groups Splitting by a Quadratic Extension and Grothendieck-Serre Conjecture for Group Schemes F_4 with Trivial g_3 Invariant

V. Chernousov[1]

Received: August 17, 2009
Revised: February 1, 2010

Abstract. We study structure properties of reductive group schemes defined over a local ring and splitting over its étale quadratic extension. As an application we prove Serre–Grothendieck conjecture on rationally trivial torsors over a local regular ring containing a field of characteristic 0 for group schemes of type F_4 with trivial g_3 invariant.

2010 Mathematics Subject Classification: 20G07, 20G10, 20G15, 20G41
Keywords and Phrases: Linear algebraic groups, exceptional groups, torsors, non-abelian cohomology, local regular rings, Grothendieck–Serre conjecture

To A. Suslin on his 60th birthday

1 Introduction

In the present paper we prove the Grothendieck-Serre conjecture on rationally trivial torsors for group schemes of type F_4 whose generic fiber has trivial g_3 invariant. The Grothendieck-Serre conjecture [Gr58], [Gr68], [S58] asserts that if R is a regular local ring and if G is a reductive group scheme defined over R then a G-torsor over R is trivial if and only if its fiber at the generic point of Spec (R) is trivial. In other words the kernel of a natural map $H^1_{ét}(R,G) \to H^1_{ét}(K,G)$ where K is a quotient field of R is trivial.

[1]Partially supported by the Canada Research Chairs Program and an NSERC research grant

Many people contributed to this conjecture by considering various particular cases. If R is a discrete valuation ring the conjecture was proved by Y. Nisnevich [N]. If R contains a field k and G is defined over k this is due to J.-L. Colliot-Thélène, M. Ojanguren [CTO] when k is infinite perfect and it is due to M. S. Raghunathan [R94], [R95] when k is infinite. The case of tori was done by J.-L. Colliot-Thélène and J.-L. Sansuc [CTS]. For certain simple simply connected group of classical type the conjecture was proved by Ojanguren, Panin, Suslin and Zainoulline [PS], [OP], [Z], [OPZ]. For a recent progress on isotropic group schemes we refer to preprints [PSV], [Pa09], [PPS].[2]

In the paper we deal with a still open case related to group schemes of type F_4. Recall that if G is a group of type F_4 defined over a field k of characteristic $\neq 2,3$ one can associate (cf. [S93], [GMS03], [PetRac], [Ro]) cohomological invariants $f_3(G)$, $f_5(G)$ and $g_3(G)$ of G in $H^3(k,\mu_2)$, $H^5(k,\mu_2)$ and $H^3(k,\mathbb{Z}/3\mathbb{Z})$ respectively. The group G can be viewed as the automorphism group of a corresponding 27-dimensional Jordan algebra J. The invariant $g_3(G)$ vanishes if and only if J is reduced, i.e. it has zero divisors. The main result of the paper is the following.

Theorem 1. *Let R be a regular local ring containing a field of characteristic* 0. *Let G be a group scheme of type F_4 over R such that its fiber at the generic point of* $\operatorname{Spec}(R)$ *has trivial g_3 invariant. Then the canonical mapping $H^1_{ét}(R,G) \to H^1_{ét}(K,G)$ where K is a quotient field of R has trivial kernel.*

We remark that for a group scheme G of type F_4 we have $\operatorname{Aut}(G) \simeq G$, so that by the twisting argument the above theorem is equivalent to the following:

Theorem 2. *Let R be as above and let G_0 be a split group scheme of type F_4 over R. Let $H^1_{ét}(R,G_0)_{\{g_3=0\}} \subset H^1_{ét}(R,G_0)$ be the subset consisting of isomorphism classes $[\mathcal{T}]$ of G_0-torsors such that the corresponding twisted group $({}^{\mathcal{T}}G_0)_K$ has trivial g_3 invariant. Then a canonical mapping*

$$H^1_{ét}(R,G_0)_{\{g_3=0\}} \to H^1_{ét}(K,G_0)$$

is injective, i.e. two G_0-torsors in $H^1_{ét}(R,G_0)_{\{g_3=0\}}$ are isomorphic over R if and only if they are isomorphic over K.

The characteristic restriction in the theorem is due to the fact that the purity result [ChP] is used in the proof and the latter is based on the use of the main result in [P09] on rationally isotropic quadratic spaces which was proven in characteristic zero only (the resolution of singularities is involved in that proof). We remark that if the Panin's result is true in full generality (except probably characteristic 2 case) then our arguments can be easily modify in such way that the theorem holds for all regular local rings where 2 is invertible.[3]

[2] We also remark that experts know the proof of the conjecture for group schemes of type G_2 but it seems to us that a proof is not available in the literature.

[3] I. Panin has informed the author that his main theorem in [P09] holds for quadratic spaces defined over a regular local ring containing an infinite perfect field.

The proof of the theorem heavily depends on the fact that group schemes of type F_4 with trivial g_3 invariant are split by an étale quadratic extension of the ground ring R. This is why the main body of the paper consists of studying structure properties of simple group schemes of an arbitrary type over R (resp. K) splitting by an étale quadratic extension S/R (resp. L/K) which is of independent interest.

We show that the structure of such group schemes is completely determined by a finite family of units in R which we call structure constants of G. These constants depend on a chosen maximal torus $T \subset G$ defined over R and splitting over S. Such a torus is not unique in G. Giving two tori T and T' we find formulas which express structure constants of G related to T in terms of that of related to T' and this leads us quickly to the proof of the main theorem.

Of course we are using a group point view. It seems plausible that our proof can be carried over in terms of Jordan algebras and their trace quadratic forms, but we do not try to do it here.

The paper is divided into four parts. We begin by introducing notation, terminology that are used throughout the paper as well as by reminding properties of algebraic groups defined over a field and splitting by a quadratic field extension. This is followed by two sections on explicit formulas for cohomological invariants f_3 and f_5 in terms of structure constants for groups of type F_4 and their classification. In the third part of the paper we study structure properties of group schemes splitting by an étale quadratic extension of the ground ring. The proof of the main theorem is the content of the last section.

Notation. Let R be a (commutative) ring. We let G_0 denote a split reductive group scheme over R and we let $T_0 \subset G_0$ denote a maximal split torus over R. We denote by $\Sigma(G_0, T_0)$ the root system of G_0 with respect to T_0. We use standard terminology related to algebraic groups over rings. For the definition of reductive group schemes (and in particular split reductive group schemes), maximal tori, root systems of split group schemes and their properties we refer to [SGA3].

We number the simple roots as in [Bourb68].

Acknowledgments. We thank the referee for useful comments and remarks which helped to improve the exposition.

2 Lemma on representability of units by quadratic forms

Throughout the paper R denotes a (commutative) ring where 2 is invertible and $R^\times$ denotes the group of invertible elements of R. Also, all fields considered in the paper have characteristic $\neq 2$.

If R is a local ring with the maximal ideal M we let $k = \overline{R} = R/M$. Similarly, if V is a free module on rank n over R we let $\overline{V} = V \otimes_R \overline{R} = V \otimes_R k$ and for a vector $v \in V$ we set $\overline{v} = v \otimes 1$. If R is a regular local ring it is a unique factorization domain ([Ma, Theorem 48, page 142]). Throughout the paper a quotient field of R will be denoted by K.

Let $f = \sum_{i=1}^n a_i x_i^2$ be a quadratic form over R where $a_1, \dots, a_n \in R^\times$ given on a free R-module V. If $I \subset \{1, \dots, n\}$ is a non-empty subset we denote by $f_I = \sum_{i\in I} a_i x_i^2$ the corresponding subform of f. If $v = (v_1, \dots, v_n) \in V$ we set $f_I(v) = \sum_{i\in I} a_i v_i^2$. Finally, let $g = \prod_I f_i$ where the product is taken over all non-empty subsets of $\{1, \dots, n\}$. For a vector v we set $g(v) = \prod_I f_I(v)$.

Lemma 3. *Let f and g be as above. Assume that (the residue field) k is infinite. Let $a \in R^\times$ be a unit such that $f(v) = a$ for some vector $v \in V$. Then there exists a vector $u \in V$ such that $f(u) = a$ and $g(u)$ is a unit.*

Proof. If $n = 1$, v has the required properties. Hence me may assume $n \geq 2$. If $w \in V$ is a vector whose length $f(w)$ with respect to f is a unit we denote by τ_w an orthogonal reflection with respect to w given by

$$\tau_w(x) = x - 2f(x, w)f(w)^{-1}w$$

for all vectors x in V. Since orthogonal reflections preserve length of vectors it suffices to find vectors $w_1, \dots, w_s \in V$ such that $g(\tau_{w_1} \cdots \tau_{w_s}(v))$ is a unit. For that, in turn, it suffices to find $\overline{w}_1, \dots, \overline{w}_s \in \overline{V}$ such that $\overline{g}(\tau_{\overline{w}_1} \cdots \tau_{\overline{w}_s}(\overline{v})) \neq 0$. It follows that we can pass to a vector space $\overline{V}$ over k. Consider a quadric

$$Q_{\overline{a}} = \{\, x \in \overline{V} \mid \overline{f}(x) = \overline{a} \,\}$$

defined over k. We have $\overline{v} \in Q_{\overline{a}}(k)$, hence $Q_{\overline{a}}(k) \neq \emptyset$ implying $Q_{\overline{a}}$ is a rational variety over k.
Let $U \subset \overline{V}$ be an open subset given by $\overline{g}(x) \neq 0$. It is easy to see that $Q_{\overline{a}} \cap U \neq \emptyset$ (indeed, if we pass to an algebraic closure $\bar{k}$ of k then obviously we have $U(\bar{k}) \cap Q_{\overline{a}}(\bar{k}) \neq \emptyset$). Since k is infinite, k-points of $Q_{\overline{a}}$ are dense in $Q_{\overline{a}}$. Hence $Q_{\overline{a}}(k) \cap U$ is nonempty. Take a vector $\overline{w} \in Q_{\overline{a}}(k) \cap U$. Since the orthogonal group $\mathrm{O}(\overline{f})$ acts transitively on vectors of $Q_{\overline{a}}$ there exists $\overline{s} \in \mathrm{O}(\overline{f})$ such that $\overline{w} = \overline{s}(\overline{v})$. It remains to note that orthogonal reflections generate $O(\overline{f})$. □

3 Algebraic groups splitting by quadratic field extensions

The aim of this section is to remind structure properties of a simple simply connected algebraic group G defined over a field K and splitting over its quadratic extension L/K. There is nothing special in type F_4 and we will assume in this section that G is of an arbitrary type of rank n. The only technical restriction which we need later on to simplify the exposition of the material on the structure of such groups relates to the Weyl group W of G. Namely, we will assume that W contains -1, i.e. an element which takes an arbitrary root α into $-\alpha$.[4]
Let τ be the nontrivial automorphism of L/K. If $B_L \subset G_L$ is a Borel subgroup over L in G_L in generic position then $B_L \cap \tau(B_L) = T$ is a maximal torus in

[4]For groups G splitting over a quadratic extension of the ground field and whose whose Weyl group doesn't contain -1 the Galois descent data looks more complicated; for instance, Lemma 4 doesn't hold for them.

G_L. Clearly, it is defined over K and splitting over L (because it is contained in B_L and all tori in B_L are L-split).

Lemma 4. *T is anisotropic over K.*

Proof. The Galois group of L/K acts in a natural way on characters of T and hence on the root system $\Sigma = \Sigma(G_K, T)$ of G_K with respect to T_K. Thus we have a natural embedding $\mathrm{Gal}\,(L/F) \hookrightarrow W$ which allows us to view τ as an element of W. Since the intersection of two Borel subgroups B_L and $\tau(B_L)$ is a maximal torus in G_L, one of them, say $\tau(B_L)$, is the opposite Borel subgroup to the second one B_L with respect to the ordering on Σ determined by the pair (T_L, B_L). One knows that W contains a unique element which takes B_L to $\tau(B_L) = B_L^-$. Since $-1 \in W$ such an element is necessary -1. Of course this implies $\tau = -1$, hence τ acts on characters of T as -1. In particular T is K-anisotropic. $\square$

Our Borel subgroup B_L determines an ordering of the root system Σ of G_L, hence the system of simple roots $\Pi = \{\alpha_1, \dots, \alpha_n\}$. Let Σ^+ (resp. Σ^-) be the set of positive (resp. negative) roots. Let us choose a Chevalley basis [St]

$$\{H_{\alpha_1}, \dots H_{\alpha_n},\ X_\alpha,\ \alpha \in \Sigma\} \tag{5}$$

in the Lie algebra $\mathfrak{g}_L = \mathcal{L}(G_L)$ of G_L corresponding to the pair (T_L, B_L). Recall that elements from (5) are eigenvectors of T_L with respect to the adjoint representation $ad : G \to \mathrm{End}\,(\mathfrak{g}_L)$ satisfying some additional relations; in particular for each $t \in T_L$ we have

$$tX_\alpha t^{-1} = \alpha(t) X_\alpha \tag{6}$$

where $\alpha \in \Sigma$ and $tH_{\alpha_i}t^{-1} = H_{\alpha_i}$. A Chevalley basis is unique up to signs and automorphisms of $\mathfrak{g}_L$ which preserve B_L and T_L (see [St], §1, Remark 1).
Since G_L is a Chevalley group over L, the structure of $G(L)$ as an abstract group, i.e. its generators and relations, is well known. For more details and proofs of all standard facts about $G(L)$ used in this paper we refer to [St]. Recall that $G(L)$ is generated by the so-called root subgroups $U_\alpha = \langle x_\alpha(u) \mid u \in L\rangle$, where $\alpha \in \Sigma$ and T is generated by the one-parameter subgroups

$$T_\alpha = T \cap G_\alpha = \mathrm{Im}\, h_\alpha$$

Here G_α is the subgroup generated by $U_{\pm\alpha}$ and $h_\alpha : G_{m,L} \to T_L$ is the corresponding cocharacter (coroot) of T. Furthermore, since G_L is a simply connected group, the following relations hold in G_L (cf. [St], Lemma 28 b), Lemma 20 c)):

(i) $T \simeq T_{\alpha_1} \times \cdots \times T_{\alpha_n}$;

(ii) for any two roots $\alpha,\ \beta \in \Sigma$ and $t, u \in L$ we have

$$h_\alpha(t)\, x_\beta(u)\, h_\alpha(t)^{-1} = x_\beta(t^{\langle \beta, \alpha\rangle} u)$$

where $\langle \beta, \alpha \rangle = 2\,(\beta,\alpha)/(\alpha,\alpha)$ and

$$h_\alpha(t) X_\beta h_\alpha(t)^{-1} = t^{\langle \beta,\alpha\rangle} X_\beta \tag{7}$$

If $\Delta \subset \Sigma^+$ is a subset, we let G_Δ denote the subgroup generated by $U_{\pm\alpha}$, $\alpha \in \Delta$. We shall now describe explicitly the K-structure of G, i.e. the action of τ on the generators $\{x_\alpha(u),\, \alpha \in \Sigma\}$ of G_L. As we already know $\tau(\alpha) = -\alpha$ for any $\alpha \in \Sigma$ and this implies $T_\alpha \simeq R^{(1)}_{L/K}(G_{m,L})$ (see [V, 4.9, Example 6]).
Let $\alpha \in \Sigma$. Since $\tau(\alpha) = -\alpha$ there exists a constant $c_\alpha \in L^\times$ such that $\tau(X_\alpha) = c_\alpha X_{-\alpha}$. It follows that the action of τ on $G(L)$ is determined completely by the family $\{c_\alpha,\, \alpha \in \Sigma\}$. We call these constants by *structure constants* of G with respect to T and Chevalley basis (5). Of course, they depend on the choice of T and a Chevalley basis. We summarize their properties in the following two lemmas (for their proofs we refer to [Ch, Lemmas 4.4, 4.5, 4.11]).

Lemma 8. *Let $\alpha \in \Sigma$. Then we have*
(i) $c_{-\alpha} = c_\alpha^{-1}$;
(ii) $c_\alpha \in K^\times$;
(iii) *if $\beta \in \Sigma$ is a root such that $\alpha + \beta \in \Sigma$, then $c_{\alpha+\beta} = -c_\alpha\, c_\beta$; in particular, the family $\{c_\alpha,\, \alpha \in \Sigma\}$ is determined completely by its subfamily $\{c_{\alpha_1}, \ldots, c_{\alpha_n}\}$.*

Lemma 9. (i) *$\tau[\,x_\alpha(u)\,] = x_{-\alpha}(c_\alpha \tau(u))$ for every $u \in L$ and every $\alpha \in \Sigma$.*
(ii) *Let $L = K(\sqrt{d})$. Then the subgroup G_α of G is isomorphic to* $\mathrm{SL}\,(1, D)$ *where D is a quaternion algebra over K of the form $D = (d, c_\alpha)$.*

4 Moving tori

We follow the notation of the previous section. The family $\{c_\alpha,\, \alpha \in \Sigma\}$ determining the action of τ on $G(L)$ depends on a chosen Borel subgroup B_L and the corresponding Chevalley basis. Given another Borel subgroup and Chevalley basis we get another family of constants and we now are going to describe the relation between the old ones and the new ones.
Let $B'_L \subset G_L$ be a Borel subgroup over L such that the intersection $T' = B'_L \cap \tau(B'_L)$ is a maximal K-anisotropic torus. Clearly both tori T and T' are isomorphic over K (because both of them are isomorphic to the direct product of n copies of $R^{(1)}_{L/K}(G_{m,L})$). Furthermore, there exists a K-isomorphism $\lambda : T \to T'$ preserving positive roots, i.e. which takes $(\Sigma')^+ = \Sigma(G,T')^+$ into $\Sigma^+ = \Sigma(G,T)^+$. Any such isomorphism can be extended to an inner automorphism

$$i_g : G \longrightarrow G, \quad x \to g\,x\,g^{-1}$$

for some $g \in G(K_s)$, where K_s is a separable closure of K, which takes B_L into B'_L (see [Hum], Theorem 32.1). Note that g is not unique since for any $t \in T(K_s)$ the inner conjugation by gt also extends λ and it takes B_L into B'_L.

Lemma 10. *The element g can be chosen in $G(L)$.*

Proof. Take an arbitrary $g' \in G(K_s)$ such that $i_{g'}$ extends λ and $i_{g'}(B_L) = B'_L$. Since the restriction $i_{g'}\,|_T$ is a K-defined isomorphism, we have

$$t_\sigma = (g')^{-1+\sigma} \in T(K_s)$$

for any $\sigma \in \mathrm{Gal}\,(K_s/K)$. The family $\{t_\sigma,\, \sigma \in \mathrm{Gal}\,(K_s/F)\}$ determines a cocycle $\xi = (t_\sigma) \in Z^1(K,T)$. Since T splits over L, $res_L(\xi)$ viewed as a cocycle in T is trivial, by Hilbert's Theorem 90. It follows there is $z \in T(K_s)$ such that $t_\sigma = z^{1-\sigma}$, $\sigma \in \mathrm{Gal}\,(K_s/L)$. Then $g = g'z$ is stable under $\mathrm{Gal}\,(K_s/L)$. This implies $g \in G(L)$ and clearly we have $gB_Lg^{-1} = B'_L$. □

Let g be an element from Lemma 10 and let $t = g^{-1+\tau}$. Since $t \in T(L)$, it can be written uniquely as a product $t = h_{\alpha_1}(t_1)\cdots h_{\alpha_n}(t_n)$, where $t_1,\dots,t_n \in L^\times$ are some parameters.

Lemma 11. *We have $t_1,\dots,t_n \in K^\times$.*

Proof. We first note that, by the construction of t, we have $t\,\tau(t) = 1$. Since τ acts on characters of T as multiplication by -1 we have $\tau(h_{\alpha_i}(t_i)) = h_{\alpha_i}(1/\tau(t_i))$ for every $i = 1,\dots,n$. Also, the equality $t\,\tau(t) = 1$ implies $h_{\alpha_i}(t_i)h_{\alpha_i}(1/\tau(t_i)) = 1$, hence $t_i = \tau(t_i)$. □

The set

$$\{H'_{\alpha_1} = gH_{\alpha_1}g^{-1},\dots,H'_{\alpha_n} = gH_{\alpha_n}g^{-1},\, X'_\alpha = gX_\alpha g^{-1},\, \alpha \in \Sigma\} \tag{12}$$

is a Chevalley basis related to the pair (T',B'_L). Let $\{c'_\alpha,\, \alpha \in \Sigma\}$ be the corresponding structure constants of G with respect to T' and Chevalley basis (12).

Lemma 13. *For every root $\alpha \in \Sigma'$ one has $c'_\alpha = t_1^{-\langle\alpha,\alpha_1\rangle}\cdots t_n^{-\langle\alpha,\alpha_n\rangle}\cdot c_\alpha$.*

Proof. Apply τ to the equality $X'_\alpha = gX_\alpha g^{-1}$ and use relation (7). □

Our element g constructed in Lemma 10 has the property $g^{-1+\tau} \in T(L)$. Conversely, it is easy to see that an arbitrary $g \in G(L)$ with this property gives rise to a new pair (B'_L,T') and hence to the new structure constants $\{c'_\alpha\}$ which are given by the formulas in Lemma 13. Thus we have

Lemma 14. *Let $g \in G(L)$ be an element such that $t = g^{-1+\tau} \in T(L)$. Then $T' = gTg^{-1}$ is a K-defined maximal torus splitting over L and the restriction of the inner automorphism i_g to T is a K-defined isomorphism. The structure constants $\{c'_\alpha\}$ related to T' are given by the formulas in Lemma 13.*

Example 15. Let G,T be as above and let $\Sigma = \Sigma(G,T)$. Take an element

$$g = x_{-\alpha}(-c_\alpha v)x_\alpha\left(\frac{-\tau(v)}{1-c_\alpha v\tau(v)}\right) \tag{16}$$

where $\alpha \in \Sigma$ is an arbitrary root and $v \in L^\times$ is such that $1 - c_\alpha v\tau(v) \neq 0$. One easily checks that

$$g^{-1+\tau} = h_\alpha \left(\frac{1}{1 - c_\alpha v\tau(v)} \right)$$

and hence g gives rise to a new torus $T' = gTg^{-1}$ and to a new structure constants.

Definition 17. *We say that we apply an elementary transformation of T with respect to a root α and a parameter $v \in L^\times$ when we move from T to $T' = gTg^{-1}$ where g is given by* (16) *and* $1 - c_\alpha v\tau(v) \neq 0$.

Remark 18. The main property of an elementary transformation with respect to a root α is that the new structure constant c'_β with respect to T' doesn't change (up to squares) if β is orthogonal to α or $\langle \beta, \alpha \rangle = \pm 2$ and it is equal to $(1 - c_\alpha v\tau(v))c_\beta$ (up to squares) if $\langle \beta, \alpha \rangle = \pm 1$. Thus in the context of algebraic groups this an analogue of an elementary chain equivalence of quadratic forms.

Remark 19. An arbitrary reduced norm in the quaternion algebra $D = (d, c_\alpha)$ can be written as a product of two elements of the form $1 - c_\alpha v\tau(v)$, hence in the case $\langle \beta, \alpha \rangle = \pm 1$ we can change c_β by any reduced norm in D.

5 Cohomological interpretation

While considering cohomological invariants of G of type F_4 sometimes it is convenient to consider G as a twisting group. Let G^{ad} be the corresponding adjoint group. Note that groups of type F_4 are simply connected and adjoint so that for them we have $G = G^{ad}$. Let G_0 (resp. G_0^{ad}) be a K-split simple simply connected (resp. adjoint) group of the same type as G^{ad} and let $T_0 \subset G_0$ (resp. $T_0^{ad} \subset G_0^{ad}$) be a maximal K-split torus. We denote by $c \in \mathrm{Aut}(G_0)$ an element such that $c^2 = 1$ and $c(t) = t^{-1}$ for every $t \in T_0$ (it is known that such an automorphism exists, see e.g. [DG], Exp. XXIV, Prop. 3.16.2, p. 355). We assume additionally that $c \in N_{G_0^{ad}}(T_0^{ad})$.

Remark 20. In general case c can not be lifted to $N_{G_0}(T_0)$. However it is known that if G_0 has type D_4 or F_4 such an element can be chosen inside the normalizer $N_{G_0}(T_0)$ of T_0. So when we deal with such groups we will assume that $c \in N_{G_0}(T_0)$.

Lemma 21. *Let $t \in T_0^{ad}(K)$ and let $a_\tau = ct$. Then $\xi = (a_\tau)$ is a cocycle in $Z^1(L/K, G_0^{ad}(L))$.*

Proof. We need to check that $a_\tau \tau(a_\tau) = 1$. Indeed,

$$a_\tau \tau(a_\tau) = ct\,\tau(ct) = ctct = t^{-1}t = 1$$

as required. □

For further reference we note that every cocycle $\eta \in Z^1(K, G_0^{ad})$ acts by inner conjugation on both G_0 and G_0^{ad} and hence we can twist ${}^{\eta}G_0$, ${}^{\eta}G_0^{ad}$ both groups. Since G_0^{ad} is adjoint the character group of T_0^{ad} is generated by simple roots $\{\alpha_1, \dots, \alpha_n\}$ of the root system $\Sigma = \Sigma(G_0^{ad}, T_0^{ad})$ of G_0^{ad} with respect to T_0^{ad}. Choose a decomposition $T_0^{ad} = G_m \times \cdots \times G_m$ such that the canonical embeddings $\pi_i : G_m \to T_0^{ad}$ onto the ith factor, $i = 1, \dots, n$, are the cocharacters dual to $\alpha_1, \dots, \alpha_n$.

Proposition 22. *Let G be as above with structure constants $c_{\alpha_1}, \dots, c_{\alpha_n}$. Let $\xi = (a_\tau)$ where $a_\tau = ct$ and $t = \prod_i \pi_i(c_{\alpha_i})$. Then the twisted group ${}^{\xi}G_0$ is isomorphic to G over K.*

Proof. It is known that $cX_\alpha c^{-1} = X_{-\alpha}$ and according to (6) we have $tX_\alpha t^{-1} = \alpha(t)X_\alpha$ for every root $\alpha \in \Sigma$. Since the cocharacters $\pi_1, \dots, \pi_n$ are dual to the roots $\alpha_1, \dots, \alpha_n$, we have $\langle \pi_i, \alpha_j \rangle = \delta_{ij}$, hence

$$\pi_i(c_{\alpha_i}) X_{\alpha_i} \pi_i(c_{\alpha_i})^{-1} = c_{\alpha_i} X_{\alpha_i}$$

and

$$\pi_i(c_{\alpha_i}) X_{\alpha_j} \pi_i(c_{\alpha_i})^{-1} = X_{\alpha_j}$$

if $i \neq j$. Thus for the twisted group ${}^{\xi}G_0$ the structure constant for the simple root α_i, $i = 1, \dots, n$, is c_{α_i} because

$$X_{\alpha_i} \to a_\tau X_{\alpha_i} a_\tau^{-1} = (c \prod_i \pi_i(c_{\alpha_i})) X_{\alpha_i} (c \prod_i \pi_i(c_{\alpha_i}))^{-1} = c_{\alpha_i} X_{-\alpha_i}.$$

If $\alpha \in \Sigma$ is an arbitrary root, then by Lemma 8 the structure constant c_α of ${}^{\xi}G_0$ can be expressed uniquely in terms of the constants $c_{\alpha_1}, \dots, c_{\alpha_n}$, so that the twisted group ${}^{\xi}G_0$ has the same structure constants as G. It follows that the Lie algebras $\mathcal{L}(G)$ and $\mathcal{L}({}^{\xi}G_0)$ of G and ${}^{\xi}G_0$ have the same Galois descent data. This yields $\mathcal{L}(G) \simeq \mathcal{L}({}^{\xi}G_0)$ and as a consequence we obtain that their automorphism groups (and in particular their connected components) are isomorphic over K as well. □

Remark 23. Assume that R is a domain where 2 is invertible with a field of fractions K and G_0 is a split group scheme over R. Let $S = R(\sqrt{d})$ be an étale quadratic extension of R where d is a unit in R. Let τ be the generator of $\mathrm{Gal}\,(S/R)$. Assume that $c_{\alpha_1}, \dots, c_{\alpha_n} \in R^\times$. Then we may view $\xi = (a_\tau)$ where $a_\tau = c \prod_i \pi_i(c_{\alpha_i})$ as a cocycle in $Z^1(S/R, G_0^{ad}(S))$ and hence the twisted group ${}^{\xi}G_0$ is a group scheme over R whose fiber at the generic point of $\mathrm{Spec}\,(R)$ is isomorphic to G_K.

As an application of the above proposition we get

Lemma 24. *Let G and G' be groups over K and splitting over L with structure constants $\{c_{\alpha_1}, \dots, c_{\alpha_n}\}$ and $\{c_{\alpha_1}u_1, \dots, c_{\alpha_n}u_n\}$ where $u_1, \dots, u_n$ are in the image of the norm map $N_{L/K} : L^\times \to K^\times$. Then G and G' are isomorphic over K.*

Proof. Let $u_i = N_{L/K}(v_i)$. By Proposition 22, we have G and G' are twisted forms of G_0 by means of cocycles $\xi = (a_\tau)$ and $\xi' = (a'_\tau)$ with coefficients in $G_0^{ad}(S)$ where $a_\tau = c\prod_i \pi_i(c_{\alpha_i})$ and $a'_\tau = c\prod_i \pi_i(c_{\alpha_i}u_i)$. Since T_0^{ad} is a K-split torus and since π_i is a K-defined morphism we have $\tau(\pi_i(v_i)) = \pi_i(\tau(v_i))$. Also, we have $c^2 = 1$ and $c\pi_i(v_i)c^{-1} = \pi_i(v_i^{-1})$. Then it easily follows

$$a_\tau = \left(\prod_i \pi_i(v_i)\right) a'_\tau \left(\prod_i \pi_i(v_i)\right)^{-\tau}$$

and this implies ξ is equivalent to ξ'. □

The statement of the lemma can be equivalently reformulated as follows.

Corollary 25. *Let $T \subset G$ be a maximal torus with the structure constants $\{c_{\alpha_1}, \ldots, c_{\alpha_n}\}$ and let $u_1, \ldots, u_n \in N_{L/K}(L^\times)$. Then G contains a maximal torus T' whose structure constants are $\{c_{\alpha_1}u_1, \ldots, c_{\alpha_n}u_n\}$.*

6 Strongly inner forms of type D_4

For later use we need some classification results on strongly inner forms of type ${}^1D_4$; in other words we need an explicit description of the image of $H^1(K, G_0) \to H^1(K, \mathrm{Aut}(G_0))$ where G_0 is a simple simply connected group over a field K of type D_4.

For an arbitrary cocycle $\xi \in Z^1(K, G_0)$ the twisted group $G = {}^\xi G_0$ is isomorphic to $\mathrm{Spin}(f)$ where f is an 8-dimensional quadratic form having trivial discriminant and trivial Hasse-Witt invariant. By Merkurjev's theorem [M], f belongs to I^3 where I is the fundamental ideal of even dimensional quadratic forms in the Witt group $W(K)$. We may assume that f represents 1 (because $\mathrm{Spin}(f) \simeq \mathrm{Spin}(af)$ for $a \in K^\times$). Since $\dim f = 8$, by the Arason-Pfister Hauptsatz, f is a 3-fold Pfister form over K and as a consequence we obtain G is splitting over a quadratic extension L/K of K, say $L = K(\sqrt{d})$.

Lemma 26. *There exist parameters $u_1, \ldots, u_4 \in K^\times$ such that $G \simeq {}^\eta G_0$ where η is of the form $\eta = (a_\tau)$ and $a_\tau = c\prod_i h_{\alpha_i}(u_i)$.*

Proof. By Remark 20 we may assume that $c \in N_{G_0}(T_0)$. Let ξ' be the image of ξ in $H^1(K, G_0^{ad})$ and let c' be the image of c in G_0^{ad}. By Proposition 22, we may assume that ξ' is of the form $\xi' = (a'_\tau)$ where $a'_\tau = c'\prod_i \pi_i(c_{\alpha_i})$ and c_{α_i} are structure constants of $G^{ad} = {}^{\xi'}G_0^{ad}$ with respect to some maximal torus in G^{ad} defined over K and splitting over L.

The element c gives rise to a cocycle $\lambda = (b_\tau) \in Z^1(L/K, G_0(L))$ where $b_\tau = c$. Twisting G_0 by λ yields a commutative diagram

$$\begin{array}{ccc} H^1(K, G_0) & \xrightarrow{f_1} & H^1(K, {}^\lambda G_0) \\ \downarrow & & \downarrow \\ H^1(K, G_0^{ad}) & \xrightarrow{f_2} & H^1(K, {}^\lambda G_0) \end{array}$$

where f_1 and f_2 are the canonical bijections. Let $f_2(\xi') = \xi''$. It is of the form $\xi'' = (a''_\tau)$ where $a''_\tau = \prod_i \pi_i(c_{\alpha_i})$; hence $f_2(\xi')$ takes values in a maximal torus $T^{ad} = {}^\lambda T_0^{ad}$ of ${}^\lambda(G_0^{ad})$ defined over K and splitting over L.
Let Z be the center of G_0. We have an exact sequence

$$0 \to Z \to {}^\lambda T_0 \to T^{ad} \to 1$$

It induces a morphism $f_3 : H^1(K, T^{ad}) \to H^2(K, Z)$. Since c' and ξ' can be lifted to G_0, we have $f_3(\xi'') = 0$. Hence ξ'' has a lifting into the torus ${}^\lambda T_0$, say $\tilde\eta \in H^1(L/K, {}^\lambda T_0)$. Going back to $H^1(K, T_0)$ we see that $\eta = f_1^{-1}(\tilde\eta)$ has the required property. □

Since we are interesting in the description of $G = {}^\xi G_0$ we may assume without loss of generality that $\xi = \eta$. It is known that $Z \simeq \mu_2 \times \mu_2$ (see [PR94, §6.5]), hence Z contains three elements of order 2. They give rise to three homomorphisms $\phi_i : G_0 \to \mathrm{SO}(f_0)$ where $i = 1, 2, 3$ and f_0 is a split 8-dimensional quadratic form. The images $\phi_i(\xi)$, $i = 1, 2, 3$, of ξ in $Z^1(K, \mathrm{SO}(f_0))$ correspond to three quadratic form f_1, f_2, f_3 and we are going to give an explicit description of f_i in terms of the parameters u_1, u_2, u_3, u_4 and d.

LEMMA 27. *Up to numbering we have $f_1 = u_3 f$, $f_2 = u_4 f$ and $f_3 = u_3 u_4 f$ where $f = \langle\langle d, v_1, v_2 \rangle\rangle$ and $v_1 = u_1 u_3^{-1} u_4^{-1}$, $v_2 = u_2$. In particular G is split over a field extension E/K if and only if so is f_E.*

Proof. One easily checks that Z is generated by

$$h_{\alpha_1}(-1) h_{\alpha_3}(-1) \ \text{ and } \ h_{\alpha_1}(-1) h_{\alpha_4}(-1).$$

We now rewrite the cocycle $\xi = (a_\tau)$ in the form

$$a_\tau = c h_{\alpha_1}(v_1) h_{\alpha_2}(v_2) z_1 z_2$$

where $v_1 = u_1 u_3^{-1} u_4^{-1}$, $v_2 = u_2$ and

$$z_1 = h_{\alpha_1}(u_3) h_{\alpha_3}(u_3), \ z_2 = h_{\alpha_1}(u_4) h_{\alpha_4}(u_4).$$

Using relation (7) we find that the structure constants of G with respect to the twisted torus $T = {}^\xi T_0$ up to squares are $c_{\alpha_2} = v_1$ and $c_{\alpha_1} = c_{\alpha_3} = c_{\alpha_4} = v_2$. Also, applying the same twisting argument as in [ChS, 4.1] we find that up to numbering we have $f_1 = u_3 f$, $f_2 = u_4 f$ and $f_3 = u_3 u_4 f$ where

$$f = \langle\langle d, v_1, v_2 \rangle\rangle = \langle\langle d, c_{\alpha_1}, c_{\alpha_2} \rangle\rangle .$$

□

We are now going to show that we don't change the equivalence class $[\xi]$ if we multiply the parameters u_3, u_4 in the expression for ξ by elements in $K^\times$ represented by f. Let V, V_1, V_2, V_3 be 8-dimensional vector space over K equipped with the quadratic forms f, f_1, f_2, f_3.

Proposition 28. *Let $w_1, w_2 \in V$ be two anisotropic vectors and let $a = f(w_1)$, $b = f(w_2)$. Let $\xi' = (a'_\tau)$ where $a'_\tau = ch_{\alpha_1}(v_1)h_{\alpha_2}(v_2)z'_1z'_2$ and*

$$z'_1 = h_{\alpha_1}(au_3)h_{\alpha_3}(au_3), \quad z'_2 = h_{\alpha_1}(bu_4)h_{\alpha_4}(bu_4).$$

Then ξ' is equivalent to ξ.

Proof. Consider two embeddings $\psi_1\, \psi_2 : \mu_2 \to G_0$ given by

$$-1 \to h_{\alpha_1}(-1)h_{\alpha_3}(-1)$$

and

$$-1 \to h_{\alpha_1}(-1)h_{\alpha_4}(-1).$$

Up to numbering we may assume that

$${}^{\xi}G_0/\psi_1(\mu_2) \simeq \mathrm{SO}(f_1) \quad \text{and} \quad {}^{\xi}G_0/\psi_2(\mu_2) \simeq \mathrm{SO}(f_2).$$

We also have a canonical bijection $H^1(K, G_0) \to H^1(K, {}^{\xi}G_0)$ (translation by ξ) under which ξ' goes to $\eta = (h_{\alpha_1}(a)h_{\alpha_3}(a)h_{\alpha_1}(b)h_{\alpha_4}(b))$ and we need to show that η is trivial in $H^1(K, {}^{\xi}G_0)$.
We now note that η is the product of two cocycles $\eta_1 = (h_{\alpha_1}(a)h_{\alpha_3}(a))$ and $\eta_2 = (h_{\alpha_1}(b)h_{\alpha_4}(b))$ first of which being in the image of $\psi_1^* : H^1(K, \mu_2) \to H^1(K, {}^{\xi}G_0)$ induced by ψ_1 and the second one being in the image of $\psi_2^* : H^1(K, \mu_2) \to H^1(K, {}^{\xi}G_0)$ induced by ψ_2. We may identify $H^1(K, \mu_2) = K^\times/(K^\times)^2$. It is known that $\operatorname{Ker} \psi_1^*$ (resp. $\operatorname{Ker} \psi_2^*$) consists of spinor norms of f_1 (resp. f_2). Thus the statement of the proposition is amount to saying that a, b are spinor norms for the twisted group $G = {}^{\xi}G_0$ with respect to the quadratic forms f_1 and f_2 respectively. Since spinor norms of f_i are generated by $f_i(s_1)f_i(s_2)$ where $s_1, s_2 \in V_i$ are anisotropic vectors and since f_i is proportional to f we are done. □

Remark 29. Assume that R and S are as in Remark 23. Take a cocycle $\xi = (a_\tau)$ in $Z^1(S/R, G_0(S))$ given by $a_\tau = ch_{\alpha_1}(u_1) \cdots h_{\alpha_4}(u_4)$ where $u_1, \dots, u_4 \in R^\times$. Then arguing literally verbatim we find that the twisted group $G = {}^{\xi}G_0$ is isomorphic to $\mathrm{Spin}(f)$ where f is a 3-fold Pfister form given by $f = \langle\langle\, d, u_2, u_1u_3u_4 \,\rangle\rangle$ and that for all units $a, b \in R^\times$ represented by f the cocycle ξ' from Proposition 28 is equivalent to ξ.

Proposition 30. *Let G be as above and let $f = \langle\langle\, d, v_1, v_2 \,\rangle\rangle$ be the corresponding 3-fold Pfister form. Assume that f has another presentation $f = \langle\langle\, d, a, b \,\rangle\rangle$ over K. Then there exists a maximal torus $T' \subset G$ defined over K and splitting over L such that structure constants of G with respect to T' (up to squares) are $c'_{\alpha_1} = a$ and $c'_{\alpha_2} = b$.*

Proof. We proved in Lemma 27 that the structure constants of G with respect to the torus $T = {}^{\xi}T_0$ are $c_{\alpha_1} = v_2$ and $c_{\alpha_1} = v_1$. We now construct a sequence of elementary transformations of T with respect to the roots α_1 and α_2 such that

at the end we arrive to a torus with the required structure constants. Recall that, by Remarks 18 and 19, an application of an elementary transformation of T with respect to α_1 (resp. α_2) does not change c_{α_1} (resp. c_{α_2}) modulo squares and multiplies c_{α_2} (resp. c_{α_1}) by a reduced norm from the quaternion algebra (d, c_{α_1}) (resp. (d, c_{α_2})).
By Witt cancellation we may write a in the form $a = w_1 c_{\alpha_1} + w_2 c_{\alpha_2} - w_3 c_{\alpha_1} c_{\alpha_2}$ where $w_1, w_2, w_3 \in N_{L/K}(L^\times)$. By Corollary 25, passing to another maximal torus and Chevalley basis (if necessary) we may assume without loss of generality that $w_1 = w_2 = 1$ and hence we may assume that a is of the form $a = c_{\alpha_1}(1 - w_3 c_{\alpha_2}) + c_{\alpha_2}$ where w_3 is still in $N_{L/K}(L^\times)$.
If $1 - w_3 c_{\alpha_2} = 0$ then $a = c_{\alpha_2}$ and we pass to the last paragraph of the proof. Otherwise applying a proper elementary transformation with respect to α_2 we pass to a new torus with structure constants $c'_{\alpha_1} = c_{\alpha_1}(1 - w_3 c_{\alpha_2})$ and $c'_{\alpha_2} = c_{\alpha_2}$. Thus abusing notation without loss of generality we may assume

$$a = c_{\alpha_1} + c_{\alpha_2} = c_{\alpha_1}(1 - (-c_{\alpha_1})^{-1} c_{\alpha_2}).$$

Applying again a proper elementary transformation with respect to α_1 we can pass to a torus whose second structure constant is $(-c_{\alpha_1})^{-1} c_{\alpha_2}$, so that we may assume $a = c_{\alpha_1}(1 - c_{\alpha_2})$. Lastly, applying an elementary transformation with respect to α_2 we pass to a torus such that $a = c_{\alpha_1}$.
We finally observe that from

$$\langle\langle\, d, c_{\alpha_1}, c_{\alpha_2} \,\rangle\rangle = \langle\langle\, d, a, b \,\rangle\rangle = \langle\langle\, d, c_{\alpha_1}, b \,\rangle\rangle$$

it follows that b is of the form $b = w c_{\alpha_2}$ where $w \in \mathrm{Nrd}\,(d, c_{\alpha_1})$. So a proper elementary transformation with respect to α_1 completes the proof. □

7 Alternative formulas for f_3 and f_5 invariants

We are going to apply the previous technique to produce explicit formulas for the f_3 and f_5 invariants of a group G of type F_4 over a field K of characteristic $\neq 2$ with trivial g_3 invariant. Recall (cf. [S93], [GMS03], [PetRac]) that given such G one can associate the cohomological invariants $f_3(G) \in H^3(K, \mu_2)$ and $f_5(G) \in H^5(K, \mu_2)$ with the following properties (cf. [Sp], [Ra]):

(a) *The group G is split over a field extension E/K if and only if $f_3(G)$ is trivial over E;*

(b) *The group G is isotropic over a field extension E/K if and only if $f_5(G)$ is trivial over E.*

These two invariants f_3, f_5 are symbols given in terms of the trace quadratic form of the Jordan algebra J corresponding to G and hence we may associate to them 3-fold and 5-fold Pfister forms. Abusing notation we denote them by the same symbols $f_3(G)$ and $f_5(G)$. It is well known that $f_3(G)$ and $f_5(G)$ completely classify groups of type F_4 with trivial g_3 invariant (see [Sp], [S93]) and we would like to produce explicit formulas of $f_3(G)$ and $f_5(G)$ in group terms only in order to generalize them later on to the case of local rings.

It follows from (a) that our group G is splitting by a quadratic extension. Indeed, if $f_3(G) = (d) \cup (a) \cup (b)$ then passing to $L = K(\sqrt{d})$ we get G_L has trivial f_3 invariant and as a consequence G is L-split by property (a).

We next construct a subgroup H in G of type D_4 and compute structure constants of G and H. By Proposition 22 we may view G as a twisted group ${}^{\xi}G_0$ where $\xi = (a_\tau)$, $a_\tau = c\prod_{i=1}^{4} h_{\alpha_i}(u_i)$ and $u_1, \ldots, u_4 \in K^\times$ where G_0 is a split group of type F_4. Looking at the tables in [Bourb68] we find that the subroot system Σ' in $\Sigma(G_0, T_0)$ generated by the long roots has type D_4. One checks that

$$\beta_1 = -\epsilon_1 - \epsilon_2, \quad \beta_2 = \alpha_1, \quad \beta_3 = \alpha_2, \quad \beta_4 = \epsilon_3 + \epsilon_4$$

is its basis. Since $\epsilon_3 + \epsilon_4 = \alpha_2 + 2\alpha_3$ and $\epsilon_1 + \epsilon_2 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$, it follows that the cocharacters $h_{\epsilon_3+\epsilon_4}$ and $h_{\epsilon_1+\epsilon_2}$ are equal to

$$h_{\epsilon_3+\epsilon_4} = h_{\alpha_2} + h_{\alpha_3} \quad \text{and} \quad h_{\epsilon_1+\epsilon_2} = 2h_{\alpha_1} + 3h_{\alpha_2} + 2h_{\alpha_3} + h_{\alpha_4}$$

so that

$$h_{\epsilon_3+\epsilon_4}(u) = h_{\alpha_2}(u)h_{\alpha_3}(u) \tag{31}$$

and

$$h_{\epsilon_1+\epsilon_2}(u) = h_{\alpha_1}(u^2)h_{\alpha_2}(u^3)h_{\alpha_3}(u^2)h_{\alpha_4}(u) \tag{32}$$

for all parameters $u \in L^\times$.

These relations shows that a_τ can be rewritten in the form

$$a_\tau = ch_{\alpha_1}(v_1)h_{\alpha_2}(v_2)\,[h_{\epsilon_1+\epsilon_2}(v_3)h_{\alpha_2}(v_3)]\,[h_{\epsilon_3+\epsilon_4}(v_4)h_{\alpha_2}(v_4)] \tag{33}$$

where $v_1, v_2, v_3, v_4 \in K^\times$.

Let H_0 be the subgroup in G_0 generated by Σ'. It is stable with respect to the conjugation by a_τ, hence G contains the subgroup $H = {}^{\xi}H_0$ of type D_4. Using (7) we easily find that modulo squares in $K^\times$ one has $c_{\alpha_3} = v_2v_3$ and $c_{\alpha_4} = v_4$ and $c_{\alpha_1} = v_2$, $c_{\alpha_2} = v_1$; in particular c_{α_1}, c_{α_2} don't depend on v_3, v_4 modulo squares.

Recall that two n-fold Pfister forms, say g_1 and g_2, are isomorphic over the ground field K if and only if g_1 is hyperbolic over the function field of $K(g_2)$ of g_2.

Theorem 34. *One has $f_3(G) = (d) \cup (c_{\alpha_1}) \cup (c_{\alpha_2})$.*

Proof. Let $f = \langle\langle\, d, c_{\alpha_1}, c_{\alpha_2}\,\rangle\rangle$ and let E be the function field of f. According to property (a), it suffices to show that G is split over E or H is split over E. But in Lemma 27 we showed that $H \simeq \mathrm{Spin}(f)$ and so we are done. $\square$

The following proposition shows that the structure constants c_{α_3} are c_{α_4} of G are well defined modulo values of $f = f_3(G) = \langle\langle\, d, c_{\alpha_1}, c_{\alpha_2}\,\rangle\rangle$.

Proposition 35. *Let $a, b \in K^\times$ be represented by f over K. Then there exists a maximal torus $T' \subset G$ defined over K and splitting over L such that modulo squares G has structure constants $c_{\alpha_1}, c_{\alpha_2}, ac_{\alpha_3}, bc_{\alpha_4}$ with respect to T'.*

Proof. According to Proposition 28, if multiply the parameters v_3, v_4 in the expression (33) by a, b respectively we obtain a cocycle equivalent to ξ, so the result follows. $\square$

Theorem 36. *One has $f_5(G) = (d) \cup (c_{\alpha_1}) \cup (c_{\alpha_2}) \cup (c_{\alpha_3}) \cup (c_{\alpha_4})$.*

Proof. Let $g = \langle\langle\, d, c_{\alpha_1}, c_{\alpha_2}, c_{\alpha_3}, c_{\alpha_4}\,\rangle\rangle$. Arguing as in Theorem 34 and using property (b) we may assume that g is split and we have to prove that G is isotropic. Since g is split we may write c_{α_4} in the form

$$c_{\alpha_4} = a^{-1}(1 - bc_{\alpha_3}) \tag{37}$$

where a, b are represented by $f = \langle\langle\, d, c_{\alpha_1}, c_{\alpha_2}\,\rangle\rangle$. Our aim is to pass (with the use of elementary transformations) to a new torus $T' \subset G$ defined over K and splitting over L such that the new structure constant c'_{α_4} related to T' is equal to 1 modulo squares. The last would imply that the corresponding subgroup G_{α_4} of G is isomorphic to SL_2 by Lemma 9 (ii) and this would show that G is isotropic as required.
By Proposition 35 there exists a maximal torus T' in G such that two last structure constants related to T' are $c'_{\alpha_3} = bc_{\alpha_3}$ and $c'_{\alpha_4} = ac_{\alpha_4}$. Then by (37) we have $c'_{\alpha_4} = 1 - c'_{\alpha_3}$. Applying a proper elementary transformation with respect to α_3 we pass to the third torus T'' for which $c''_{\alpha_4} = 1$ modulo squares and we are done. $\square$

8 Classification of groups of type F_4 with trivial g_3 invariant

The theorem below is due to T. Springer [Sp]. In this section we produce an alternative short proof which can be easily adjusted to the case of local rings.

Theorem 38. *Let G_0 be a split group of type F_4 over a field K. A mapping*

$$H^1_{\acute{e}t}(K, G_0)_{\{g_3=0\}} \to H^3(K, \mu_2) \times H^5(K, \mu_2)$$

given by $G \to (f_3(G), f_5(G))$ is injective.

We need the following preliminary result.

Proposition 39. *Let G be a group of type F_4 defined over K and splitting over L with structure constants $c_{\alpha_1}, \ldots, c_{\alpha_4}$ with respect to a torus T. Let $a \in K^\times$ be represented by $g = \langle\langle\, d, c_{\alpha_1}, c_{\alpha_2}, c_{\alpha_3}\,\rangle\rangle$ over K. Then there is a maximal torus $T' \subset G$ such that the corresponding structure constants are $c_{\alpha_1}, c_{\alpha_2}, c_{\alpha_3}, ac_{\alpha_4}$ modulo squares.*

Proof. Write a in the form $a = a_1(1 - a_2 c_{\alpha_3})$ where a_1, a_2 are represented by $f = \langle\langle\, d, c_{\alpha_1}, c_{\alpha_2}\,\rangle\rangle$. By Proposition 35 the structure constants c_{α_3} and c_{α_4} are well defined modulo values of f. Hence passing to another maximal torus in G we may assume without loss of generality that $a_1 = a_2 = 1$ so that $a = 1 - c_{\alpha_3}$. Since $1 - c_{\alpha_3}$ is a reduced norm in the quaternion algebra (d, c_{α_3}) a proper elementary transformation with respect to α_3 lead us to a torus whose first three structure constants are the same modulo squares and the last one is $(1 - c_{\alpha_3})c_{\alpha_4}$. □

Proof of Theorem 38. Let G, G' be two groups of type F_4 over K such that $f_3(G) = f_3(G')$ and $f_5(G) = f_5(G')$. Choose a quadratic extension L/K splitting $f_3(G)$. It splits both G and G'. Our strategy is to show that G, G' contain maximal tori defined over K and splitting over L with the same structure constants.

Choose arbitrary maximal tori $T \subset G$, $T' \subset G'$ defined over K and splitting over L. Let $c_{\alpha_1}, \dots, c_{\alpha_4}$ and $c'_{\alpha_1}, \dots, c'_{\alpha_4}$ be the corresponding structure constants. As we know, G, G' contain subgroups H, H' of type D_4 over K generated by the long roots. By Theorem 34 we have $f_3(G) = (d) \cup (c_{\alpha_1}) \cup (c_{\alpha_2})$ and $f_3(G') = (d) \cup (c'_{\alpha_1}) \cup (c'_{\alpha_2})$, hence

$$\langle\langle\, d, c_{\alpha_1}, c_{\alpha_2}\,\rangle\rangle = \langle\langle\, d, c'_{\alpha_1}, c'_{\alpha_2}\,\rangle\rangle.$$

Then according to Proposition 30 applied to H' and $f = \langle\langle\, d, c_{\alpha_1}, c_{\alpha_2}\,\rangle\rangle$ we may assume without loss of generality that $c_{\alpha_1} = c'_{\alpha_1}$ and $c_{\alpha_2} = c'_{\alpha_2}$.

We next show that up to choice of maximal tori in G and G' we also may assume that $c_{\alpha_3} = c'_{\alpha_3}$. Since $f_5(G) = f_5(G')$ we get

$$\langle\langle\, d, c_{\alpha_1}, c_{\alpha_2}, c_{\alpha_3}, c_{\alpha_4}\,\rangle\rangle = \langle\langle\, d, c_{\alpha_1}, c_{\alpha_2}, c'_{\alpha_3}, c'_{\alpha_4}\,\rangle\rangle. \tag{40}$$

By Witt cancellation we can write c'_{α_3} in the form $c'_{\alpha_3} = a_1 c_{\alpha_3} + a_2 c_{\alpha_4} - a_3 c_{\alpha_3} c_{\alpha_4}$ where a_1, a_2, a_3 are values of f. By Proposition 35 we may assume without loss of generality that $a_1 = a_2 = 1$. Arguing as in Proposition 30 we may pass to another maximal torus in G' such that the corresponding structure constants are

$$c'_{\alpha_1} = c_{\alpha_1},\ c'_{\alpha_2} = c_{\alpha_2},\ c'_{\alpha_3} = c_{\alpha_3}.$$

Finally, from (40) it follows that $c'_{\alpha_4} = a c_{\alpha_4}$ for some $a \in K^\times$ represented by $g = \langle\langle\, d, c_{\alpha_1}, c_{\alpha_2}, c_{\alpha_3}\,\rangle\rangle$. Application of Proposition 39 completes the proof. □

9 Group schemes splitting by étale quadratic extensions

We now pass to a simple simply connected group scheme G of an arbitrary type of rank n defined over a local ring R where 2 is invertible and splitting by an étale quadratic extension $S = R(\sqrt{u}) \simeq R[t]/(t^2 - u)$ of R where $u \in R^\times$. We assume that R is a domain with a quotient field K and with a residue field k and we assume u is not square in $K^\times$. We also denote $L = S \otimes_R K$ and

$l = S \otimes_R k$. Abusing notation we denote the nontrivial automorphisms of S/R, L/K and l/k by the same letter τ.
Let $\mathfrak{g}$ be the Lie algebra of G. As usual we set

$$\mathfrak{g}_S = \mathfrak{g} \otimes_R S, \quad \mathfrak{g}_K = \mathfrak{g} \otimes_R K, \quad \mathfrak{g}_L = \mathfrak{g} \otimes_R L$$

and

$$\overline{\mathfrak{g}} = \mathfrak{g}_k = \mathfrak{g} \otimes_R k, \quad \overline{\mathfrak{g}}_S = \mathfrak{g}_l = \mathfrak{g}_S \otimes_S l.$$

Let $\mathfrak{b}_S$ be a Borel subalgebra in $\mathfrak{g}_S$. We say that it is in *a generic position* if $\overline{\mathfrak{b}}_S \cap \tau(\overline{\mathfrak{b}}_S)$ is a Cartan subalgebra in $\overline{\mathfrak{g}}_l$. This amounts to saying that $\overline{\mathfrak{b}}_S \cap \tau(\overline{\mathfrak{b}}_S)$ has dimension n over l.
We will systematically use below the fact that in a split simple Lie algebra defined over a field the intersection of two Borel subalgebras contains a split Cartan subalgebra; in particular this intersection has dimension at least n.

Lemma 41. *The Lie algebra $\mathfrak{g}_S$ contains Borel subalgebras in generic position.*

Proof. Let $\mathcal{B}$ and $\overline{\mathcal{B}}$ be the varieties of Borel subalgebras in the split Lie algebras $\mathfrak{g}_S$ and $\mathfrak{g}_l$ respectively. Passing to residues we have a canonical mapping $\mathcal{B} \to \overline{\mathcal{B}}$ whose image is dense (because $\mathfrak{g}_S$ is split). Let $U \subset \overline{\mathcal{B}}$ be an open subset in Zariski topology consisting of Borel subalgebras $\mathfrak{b}_l$ such that $\mathfrak{b}_l \cap \tau(\mathfrak{b}_l)$ has dimension n. Since $\mathcal{B}(S)$ is dense in $\mathcal{B}$ there exists a Borel subalgebra $\mathfrak{b}_S$ in $\mathfrak{g}_S$ over S whose image in $\overline{\mathcal{B}}$ is contained in U. □

Lemma 42. *Let $\mathfrak{b}_S \subset \mathfrak{g}_S$ be a Borel subalgebra in generic position. Then a submodule $\mathfrak{t}_S = \mathfrak{b}_S \cap \tau(\mathfrak{b}_S)$ of $\mathfrak{b}_S$ has rank n.*

Proof. Let $M_S \subset S$ be a maximal ideal. Our subalgebra $\mathfrak{t}_S$ is given as an intersection of two free submodules in $\mathfrak{g}_S$ of codimensions m, where m is the number of positive roots in $\mathfrak{g}_S$, each of them being a direct summand in $\mathfrak{g}_S$. So $\mathfrak{t}_S$ consists of all solutions of a linear system of m equations in $m+n$ variables. The space of solutions of this system modulo M coincides with the intersection $\overline{\mathfrak{b}}_S \cap \tau(\overline{\mathfrak{b}}_S)$ and hence it has dimension n. This implies that the linear system has a minor of size $m \times m$ whose determinant is a unit in S and we are done. □

Our next aim is to show that the Galois descent data for the generic fiber G_K of G described in previous sections can be pushed down at the level of R. As usual we will assume that the Weyl group of G contains -1.

Proposition 43. *Let $\mathfrak{b}_S \subset \mathfrak{g}_S$ be a Borel subalgebra in generic position and let $\mathfrak{t}_S = \mathfrak{b}_S \cap \tau(\mathfrak{b}_S)$. Then $\mathfrak{t}_S$ is a split Cartan subalgebra of $\mathfrak{g}_S$ contained in $\mathfrak{b}_S$.*

Proof. Let $\mathfrak{u}_S$ be the ideal in $\mathfrak{b}_S$ consisting of nilpotent elements. It is complimented in $\mathfrak{b}_S$ by a split Cartan algebra and hence $\mathfrak{b}_S/\mathfrak{u}_S$ is isomorphic to a split Cartan subalgebra in $\mathfrak{b}_S$. We want to show that a canonical projection $p : \mathfrak{b}_S \to \mathfrak{b}_S/\mathfrak{u}_S$ restricted at $\mathfrak{t}_S$ is an isomorphism.

Let $\mathfrak{b}_L = \mathfrak{b}_S \otimes_S L$ be a generic fiber of $\mathfrak{b}_S$. We already know that $\mathfrak{t}_L = \mathfrak{b}_L \cap \tau(\mathfrak{b}_L)$ has dimension n over L, so it is a split Cartan algebra in $\mathfrak{g}_L$. Since $\mathfrak{t}_S$ embeds into $\mathfrak{t}_L$, it is a commutative Lie subalgebra contained in $\mathfrak{b}_S$ and consisting of diagonalizable semisimple elements. So injectivity of p follows.
As for surjectivity, it suffices to prove it modulo maximal ideal $M_R \subset R$. In the course of proving of Lemma 42 we saw that $\mathfrak{t}_S$ is the space of solutions of the linear system of m equations in $m+n$ variables whose matrix modulo M has rank m. It follows $\mathfrak{t}_S$ modulo M has dimension n and we are done. $\square$

Let now $\mathfrak{t}_S$ be as in Proposition 43 and let $\mathfrak{t} = \mathfrak{t}_S^{\langle\tau\rangle}$ be the invariant subspace. By descent we have $\mathfrak{t} \otimes_R S = \mathfrak{t}_S$, hence $\mathfrak{t}$ is an R-defined Cartan subalgebra splitting over S. Let B_S be a Borel subgroup in G_S corresponding to $\mathfrak{b}_S$. The connected component of the automorphism group of a pair $(\mathfrak{b}_S, \mathfrak{t}_S)$ gives rise to a maximal torus T_S in B_S. It is R-defined and S-split because so is $\mathfrak{t}$. Let us choose a Chevalley basis

$$\{H_{\alpha_1}, \dots H_{\alpha_n},\ X_\alpha,\ \alpha \in \Sigma\}$$

in $\mathfrak{g}_S$ corresponding to (T_S, B_S). Since W contains -1, we know that τ acts on the root system $\Sigma = \Sigma(G_S, T_S)$ as -1. Now repeating verbatim the arguments in [Ch] we easily find that for every root $\alpha \in \Sigma$ there exists a constants $c_\alpha \in R$ such that $\tau(X_\alpha) = c_\alpha X_{-\alpha}$ and hence the action of τ on $G(S)$ is determined completely by the family $\{c_\alpha,\ \alpha \in \Sigma\}$. We call these constants by *structure constants* of G with respect to T.
As in [Ch] one checks that the structure constants satisfy the relations given in Lemmas 8, 9. Also, as in Example 15 we may obviously define the notion of an elementary transformation with respect to a root $\alpha \in \Sigma$ (because root subgroups U_α are defined over S).

Remark 44. We note that the structure constants $\{\, c_\alpha \mid \alpha \in \Sigma \,\}$ are units in R. Indeed, by our construction we have surjections $\mathfrak{b}_S \to \overline{\mathfrak{b}}_S$ and $\mathfrak{b}_S \cap \tau(\mathfrak{b}_S) \to \overline{\mathfrak{b}}_S \cap \tau(\overline{\mathfrak{b}}_S)$. Hence the residues of c_α are structure constants of $\overline{G} = G \otimes_R k$ in the corresponding basis.

10 Proof of Theorem 2

Let R be a ring satisfying all hypothesis in Theorem 2. As usual we denote its quotient field by K. Let G_0 be a split group of type F_4 over R and let $[\xi] \in H^1(R, G_0)_{\{g_3=0\}}$. We first claim that the twisted group $G = {}^\xi G_0$ is split by an étale quadratic extension of R. The proof is based on the following.

Lemma 45. *There exist $u, v, w \in R^\times$ such that $f_3(G_K) = (u) \cup (v) \cup (w)$.*

Proof. Let $f_3(G_K) = (a) \cup (b) \cup (c)$ where $a, b, c \in K^\times$. By [ChP] the functor of 3-fold Pfister forms satisfies purity, hence it suffices to show that $f_3(G)$ is unramified at prime ideals of R of height 1.

Let $\mathfrak{p} \subset R$ be a prime ideal of height 1 and let $v = v_{\mathfrak{p}}$ be the corresponding discrete valuation on K with the residue field $k(v) = R/\mathfrak{p}$. We need to show that the image of $f_3(G_K)$ under the boundary map $\partial_{v,K} : H^3(K, \mathbb{Z}/2) \to H^2(k(v), \mathbb{Z}/2)$ is trivial. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
H^1(R,G_0) & \xrightarrow{\phi_1} & H^1(K,G_0) & \xrightarrow{\mathcal{R}_{G_0,K}} & H^3(K,\mu_6^{\otimes 2}) & \xrightarrow{\partial_{v,K}} & H^2(k(v),\mu_6) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow{=} \\
H^1(R_v,G_0) & \xrightarrow{\phi_2} & H^1(K_v,G_0) & \xrightarrow{\mathcal{R}_{G_0,K_v}} & H^3(K_v,\mu_6^{\otimes 2}) & \xrightarrow{\partial_{v,K_v}} & H^2(k(v),\mu_6)
\end{array}$$

Here $\mathcal{R}_{G_0}$ is the Rost invariant for G_0 (see [GMS03]). Since $g_3(G_K) = 0$ and since $G_K = ({}^{\xi}G_0)_K$ we have $f_3(G_K) = \mathcal{R}_{G_0,K}(\phi_1(\xi))$. By [G00, Theorem 2], we also have $\partial_{v,K_v} \circ \mathcal{R}_{G_0,K_v} \circ \phi_2 = 0$. This yields immediately $(\partial_{v,K} \circ \mathcal{R}_{G_0,K} \circ \phi_1)(\xi) = 0$ as required. ☐

Proposition 46. *G is split by an étale quadratic extension of R.*

Proof. By Lemma 45 we have $f_3(G_K) = (u) \cup (v) \cup (w)$ where $u, v, w \in R^\times$. Take $S = R(\sqrt{u})$ and we claim G_S is split. One of the following two cases occurs.
If $u \in (K^\times)^2$ then we have $f_3(G_K) = 0$. It follows $\mathcal{R}_{G_0}([\xi_K]) = f_3(G_K) = 0$. Since the kernel of the Rost invariant for split groups of type F_4 defined over K is trivial we have $[\xi_K] = 0$. Since by [CTO], [R94], [R95] Grothendieck–Serre conjecture holds for G_0 we conclude $\xi = 0$, i.e. G is already split over R.
Assume now that $u \notin (K^\times)^2$. Let L be a quotient field of S. Arguing along the same lines we first get $\mathcal{R}_{G_0}([\xi_L]) = 0$ and then G_S is split. ☐

The following lemma is an R-analogue of Corollary 25.

Lemma 47. *Let $T \subset G$ be a maximal torus with the structure constants $\{c_{\alpha_1}, \dots, c_{\alpha_4}\}$ and let $u_1, \dots, u_4 \in N_{S/R}(S^\times)$. Then G contains a maximal torus T' whose structure constants are $\{c_{\alpha_1}u_1, \dots, c_{\alpha_n}u_4\}$.*

Proof. Apply the same argument as in Lemma 24 with the use of Remark 23. ☐

Proof of Theorem 2. Let $[\xi], [\xi'] \in H^1(R, G_0)_{\{g_3=0\}}$ be two classes and let G, G' be the corresponding twisted group schemes over R. Assume that the generic fibers G_K, G'_K of G and G' are isomorphic over K. If G_K is K-split, there is nothing to prove, because Grothendieck-Serre conjecture is already proven for G_0, and so we may assume that G_K, G'_K are not split over K (and hence G, G' are not split over R) which amounts to saying that $f_3(G_K) \neq 0$ and $f_3(G'_K) \neq 0$.
By Proposition 46 there exists an étale quadratic extension $S = R(\sqrt{d})$, where $d \in R^\times$, splitting G. Of course, it is split G' as well. It now suffices to show

that G, G' contain maximal tori T, T' defined over R and splitting over S and such that the corresponding structure constants for G and G' are the same.
Let T, T' be arbitrary R-defined and S-splitting maximal tori in G, G'. Let $c_{\alpha_1}, \ldots, c_{\alpha_4}$ and $c'_{\alpha_1}, \ldots, c'_{\alpha_4}$ be structure constants of G, G' with respect to T and T'. By Theorem 34 we have $f_3(G_K) = (d) \cup (c_{\alpha_1}) \cup (c_{\alpha_2})$ and $f_3(G'_K) = (d) \cup (c'_{\alpha_1}) \cup (c'_{\alpha_2})$. Since $f_3(G_K) = f_3(G'_K)$ we get

$$\langle\langle\, d, c_{\alpha_1}, c_{\alpha_2} \,\rangle\rangle_K \overset{K}{\simeq} \langle\langle\, d, c'_{\alpha_1}, c'_{\alpha_2} \,\rangle\rangle_K$$

and hence

$$\langle\langle\, d, c_{\alpha_1}, c_{\alpha_2} \,\rangle\rangle \overset{R}{\simeq} \langle\langle\, d, c'_{\alpha_1}, c'_{\alpha_2} \,\rangle\rangle.$$

We first claim that up to choice of T and T' we may assume that $c_{\alpha_1} = c'_{\alpha_1}$ and $c_{\alpha_2} = c'_{\alpha_2}$. The proof of the claim is completely similar to that of Proposition 30. Namely, by Witt cancellation and by Lemma 3 we may write c'_{α_1} in the form $c'_{\alpha_1} = w_1 c_{\alpha_1} + w_2 c_{\alpha_2} - w_3 c_{\alpha_1} c_{\alpha_2}$ where $w_1, w_2, w_3 \in N_{S/R}(S^\times)$ and $w_1 c_{\alpha_1} - w_3 c_{\alpha_1} c_{\alpha_2}$ is a unit in R. By Lemma 47, passing to another maximal torus in G (if necessary) we may assume that $w_1 = w_2 = 1$ and then $c'_{\alpha_1} = c_{\alpha_1}(1 - w_3 c_{\alpha_2}) + c_{\alpha_2}$ where w_3 is still in $N_{S/R}(S^\times)$ and $1 - w_3 c_{\alpha_2}$ is a unit in R. The rest of the proof is the same as in Proposition 30.
We next claim that up to choice of T and T' we may additionally assume that $c_{\alpha_3} = c'_{\alpha_3}$. To prove it we are just copying the related part of the proof of Theorem 38. Arguing as in Proposition 22 we conclude that up to equivalence ξ and ξ' are of the form $\xi = (a_\tau)$ and $\xi' = (a'_\tau)$ where $a_\tau = c \prod_{i=1}^n h_{\alpha_i}(u_i)$ and $a'_\tau = c \prod_{i=1}^n h_{\alpha_i}(u'_i)$, so that, by Remark 29, G and G' contain simple simply connected subgroups H and H' generated by long roots such that $H \simeq H' \simeq \mathrm{Spin}\,(f)$ where $f = \langle\langle\, d, c_{\alpha_1}, c_{\alpha_2} \,\rangle\rangle$. Furthermore arguing as in Proposition 35 with the use of the second part of Remark 29 we see that the structure constants c_{α_3}, c_{α_4}, c'_{α_3}, c'_{α_4} are well defined modulo units in R represented by f.
Since $f_5(G_K) = f_5(G'_K)$ we get

$$\langle\langle\, d, c_{\alpha_1}, c_{\alpha_2}, c_{\alpha_3}, c_{\alpha_4} \,\rangle\rangle \overset{K}{\simeq} \langle\langle\, d, c_{\alpha_1}, c_{\alpha_2}, c'_{\alpha_3}, c'_{\alpha_4} \,\rangle\rangle$$

and hence

$$\langle\langle\, d, c_{\alpha_1}, c_{\alpha_2}, c_{\alpha_3}, c_{\alpha_4} \,\rangle\rangle \overset{R}{\simeq} \langle\langle\, d, c_{\alpha_1}, c_{\alpha_2}, c'_{\alpha_3}, c'_{\alpha_4} \,\rangle\rangle. \tag{48}$$

By Witt cancellation we can write c'_{α_3} in the form $c'_{\alpha_3} = a_1 c_{\alpha_3} + a_2 c_{\alpha_4} - a_3 c_{\alpha_3} c_{\alpha_4}$ where a_1, a_2, a_3 are units in R represented by f and $a_1 c_{\alpha_3} - a_3 c_{\alpha_3} c_{\alpha_4}$ is also a unit in R. Since c_{α_3}, c_{α_4} are defined modulo values of f passing to another maximal torus in G we may assume without loss of generality that $a_1 = a_2 = 1$. The rest of the proof is the same as in Proposition 30.
Finally we claim that we may assume that $c_{\alpha_4} = c'_{\alpha_4}$. Indeed, from (48) and Witt cancellation we conclude that c'_{α_4} is of the form $c'_{\alpha_4} = a c_{\alpha_4}$ where a is a unit in R represented by $\langle\langle\, d, c_{\alpha_1}, c_{\alpha_2}, c_{\alpha_3} \,\rangle\rangle$. Copying the proof of Proposition 39 we easily complete the proof of the claim. Thus Theorem 2 is proven. $\square$

References

[Bourb68] N. Bourbaki, Groupes et Algèbres de Lie, Chapitres 4,5 et 6, Hermann, Paris 1968.

[Ch] V. Chernousov, The kernel of the Rost invariant, Serre's Conjecture II and the Hasse principle for quasi-split groups ${}^{3,6}D_4$, E_6, E_7, Math. Annalen 326 (2003), 297– 330.

[ChS] V. Chernousov, J.-P. Serre, Lower bounds for essential dimensions via orthogonal representations, Journal of Algebra, 305 (2006), 1055 – 1070.

[ChP] V. Chernousov, I. Panin, Purity of G_2-torsors, C. R. Math. Acad. Sci. Paris, 345 (2007), 307–312.

[CTO] J.-L. Colliot-Thélène, M. Ojanguren, Espaces principaux homogènes génériquement triviaux, Publ. Math. IHES 75 (1992), 97–122.

[CTS] J.-L. Colliot-Thélène, J.-J. Sansuc, Principal homogeneous spaces under flasque tori: Applications, Journal of Algebra 106 (1987), 148–205.

[DG] M. Demazure, A. Grothendieck, Structure des Schémas en Groupes Réductifs, SGA 3 III, LN 153, Springer-Verlag, 1970.

[GMS03] R. S. Garibaldi, A. Merkurjev, J.-P. Serre, Cohomological invariants in Galois cohomology, University Lecture Series, vol. 28, Amer. Math. Soc., 2003.

[G00] P. Gille, Invariants cohomologiques de Rost en caractéristique positive, K-Theory 21 (2000), 57–100.

[Gr58] A. Grothendieck, Torsion homologique et sections rationnelles, *Anneaux de Chow et applications*, Séminaire Chevalley, 2^e année, Secrétariat mathématique, Paris, 1958.

[Gr68] A. Grothendieck, Le groupe de Brauer II, in *Dix exposés sur la cohomologie des schémas*, Amsterdam, North-Holland, 1968.

[Hum] J. Humphreys, Linear Algebraic Groups, Springer-Verlag 1975.

[Ma] H. Matsumura, Commutative algebra, W.A. Benjamin Co., New York, 1970.

[M] A. Merkurjev, On the norm residue symbol of degree 2, Dokl. Acad. Nauk SSSR, 261 (1981), 542–547.

[N] Y. Nisnevich, Rationally trivial principal homogeneous spaces and arithmetic of reductive group schemes over Dedekind rings, C. R. Math. Acad. Sci. Paris, Série I, 299 (1984), 5–8.

[OP] M. Ojangurem, I. Panin, Rationally trivial hermitian spaces are locally trivial, Math. Z. 237 (2001), 181–198.

[OPZ] M. Ojanguren, I. Panin, K. Zainoulline, On the norm principle for quadratic forms, J. Ramanujan Math. Soc. 19 (2004), 1–12.

[P09] I. Panin, Rationally isotropic quadratic spaces are locally isotropic, Invent. Math., 176 (2009), 397-403.

[Pa09] I. Panin, On Grothendieck-Serre conjecture concerning principal G-bundles over reductive group schemes: II, Preprint 2009.

[PSV] I. Panin, A. Stavrova, N. Vavilov, On Grothendieck-Serre conjecture concerning principal G-bundles over reductive group schemes: I, Preprint 2009.

[PPS] I. Panin, V. Petrov, A. Stavrova, On Grothendieck-Serre conjecture for simple adjoint group schemes of types E_6, E_7, Preprint 2009.

[PetRac] H. Petersson, M. Racine, On the invariants mod 2 of Albert Algebras, Journal of Algebra 174 (1995), 1049–1072.

[PS] I. Panin, A. Suslin, On a conjecture of Grothendieck concerning Azumaya algebras, St. Petersburg Math. J. 9 (1998), 851–858.

[PR94] V. Platonov, A. Rapinchuk, Algebraic Groups and Number Theory, Academic Press 1994.

[Ra] M. Racine, A note on Quadratic Jordan Algebras of degree 3, Trans. Amer. Math. Soc. 164 (1972), 93–103.

[R94] M. S. Raghunathan, Principal bundles admitting a rational section, Invent. Math. 116 (1994), 409– 423.

[R95] M. S. Raghunathan, Erratum: Principal bundles admitting a rational section, Invent. Math. 121 (1995), 223.

[Ro] M. Rost, A (mod 3) invariant for exceptional Jordan algebras, C. R. Acad. Sci. Paris, Série I 315 (1991), 823–827.

[S58] J.-P. Serre, Espaces fibrés algébriques, in *Anneaux de Chow et applications*, Séminaire Chevalley, 2^e année, Secrétariat mathématique, Paris, 1958.

[S93] J.–P. Serre, Cohomologie galoisienne: progrès et problèmes, Sém. Bourbaki 1993–1994, exposé 783.

[Sp] T. A. Springer, The classification of reduced exceptional simple Jordan algebras, Indag. Math. 22 (1960), 414–422.

[SGA3] Séminaire de Géométrie algébrique de l'I.H.E.S., 1963–1964, Schémas en groupes, dirigé par M. Demazure et A. Grothendieck, Lecture Notes in Math. 151–153, Springer (1970).

[St] R. Steinberg, Lectures on Chevalley Groups, Yale University 1967.

[V] V. Voskresenskii, Algebraic Groups and Their Birational Invariants, Translation of Mathematical Monographs, Volume 179, AMS.

[Z] K. Zainoulline, On Grothendieck conjecture about principal homogeneous spaces for some classical algebraic groups, St. Petersburg Math. J. 12 (2001), 117–143.

V. Chernousov
Department of Mathematics
University of Alberta
Edmonton, Alberta
T6G 2G1, Canada
chernous@math.ualberta.ca

Power Reductivity over an Arbitrary Base

À André Sousline pour son soixantième anniversaire

Voor Andree Soeslin op zijn zestigste verjaardag

Vincent Franjou[1] and Wilberd van der Kallen[2]

Received: November 27, 2008
Revised: September 29, 2009

Abstract. Our starting point is Mumford's conjecture, on representations of Chevalley groups over fields, as it is phrased in the preface of *Geometric Invariant Theory*. After extending the conjecture appropriately, we show that it holds over an arbitrary commutative base ring. We thus obtain the first fundamental theorem of invariant theory (often referred to as Hilbert's fourteenth problem) over an arbitrary Noetherian ring. We also prove results on the Grosshans graded deformation of an algebra in the same generality. We end with tentative finiteness results for rational cohomology over the integers.

2010 Mathematics Subject Classification: 20G35; 20G05; 20G10
Keywords and Phrases: Chevalley group; Hilbert's 14th; cohomology; Geometric reductivity.

1 Introduction

The following statement may seem quite well known:

Theorem 1. *Let* $\mathbf{k}$ *be a Dedekind ring and let* G *be a Chevalley group scheme over* $\mathbf{k}$. *Let* A *be a finitely generated commutative* $\mathbf{k}$*-algebra on which* G *acts rationally through algebra automorphisms. The subring of invariants* A^G *is then a finitely generated* $\mathbf{k}$*-algebra.*

[1]LMJL - Laboratoire de Mathématiques Jean Leray, CNRS/Université de Nantes. The author acknowledges the hospitality and support of CRM Barcelona during the tuning of the paper.
[2]The project started in Nantes, the author being the first visitor in the MATPYL program of CNRS' FR 2962 "Mathématiques des Pays de Loire".

Indeed, R. Thomason proved [21, Theorem 3.8] the statement for any Noetherian Nagata ring $\mathbf{k}$. Thomason's paper deals with quite a different theme, that is the existence of equivariant resolutions by free modules. Thomason proves that equivariant sheaves can be resolved by equivariant vector bundles. He thus solves a conjecture of Seshadri [19, question 2 p.268]. The affirmative answer to Seshadri's question is explained to yield Theorem 1 in the same paper [19, Theorem 2 p.263]. The finesse only illustrates that the definition of geometric reductivity in [19] does not suit well an arbitrary base. Indeed, Seshadri does not follow the formulation in Mumford's book's introduction [GIT, Preface], and uses polynomials instead [19, Theorem 1 p.244]. This use of a dual in the formulation seems to be why one requires Thomason's result [21, Corollary 3.7]. One can rather go back to the original formulation in terms of symmetric powers as illustrated by the following:

Definition 2. Let $\mathbf{k}$ be a ring and let G be an algebraic group over $\mathbf{k}$. The group G is *power-reductive* over $\mathbf{k}$ if the following holds.

Property (Power reductivity). *Let L be a cyclic $\mathbf{k}$-module with trivial G-action. Let M be a rational G-module, and let φ be a G-module map from M onto L. Then there is a positive integer d such that the d-th symmetric power of φ induces a surjection:*

$$(S^dM)^G \to S^dL.$$

We show in Section 3 that power-reductivity holds for Chevalley group schemes G, without assumption on the commutative ring $\mathbf{k}$. Note that this version of reductivity is exactly what is needed in Nagata's treatment of finite generation of invariants. We thus obtain:

Theorem 3. *Let $\mathbf{k}$ be a Noetherian ring and let G be a Chevalley group scheme over $\mathbf{k}$. Let A be a finitely generated commutative $\mathbf{k}$-algebra on which G acts rationally through algebra automorphisms. The subring of invariants A^G is then a finitely generated $\mathbf{k}$-algebra.*

There is a long history of cohomological finite generation statements as well, where the algebra of invariants $A^G = \mathrm{H}^0(G, A)$ is replaced by the whole algebra $\mathrm{H}^*(G, A)$ of the derived functors of invariants. Over fields, Friedlander and Suslin's solution in the case of finite group schemes [8] lead to the conjecture in [13], now a theorem of Touzé [22]. In Section 5, we generalize to an arbitrary (Noetherian) base Grosshans' results on his filtration [9]. These are basic tools for obtaining finite generation statements on cohomology. In Section 6, we apply our results in an exploration of the case when the base ring is $\mathbb{Z}$. Section 4 presents results of use in Section 5 and Section 3. Our results support the hope that Touzé's theorem extends to an arbitrary base.

2 Power reductivity and Hilbert's 14th

2.1 Power surjectivity

To deal with the strong form of integrality we encounter, we find it convenient to make the following definition.

Definition 4. A morphism of $\mathbf{k}$-algebras: $\phi : S \to R$ is *power-surjective* if every element of R has a power in the image of ϕ. It is *universally power-surjective* if for every $\mathbf{k}$-algebra A, the morphism of $\mathbf{k}$-algebras $A \otimes \phi$ is power-surjective, that is: for every $\mathbf{k}$-algebra A, for every x in $A \otimes R$, there is a positive integer n so that x^n lies in the image of $A \otimes \phi$.

If $\mathbf{k}$ contains a field, one does not need arbitrary positive exponents n, but only powers of the characteristic exponent of $\mathbf{k}$ (compare [20, Lemma 2.1.4, Exercise 2.1.5] or Proposition 41 below). Thus if $\mathbf{k}$ is a field of characteristic zero, any universally power-surjective morphism of $\mathbf{k}$-algebras is surjective.

2.2 Consequences

We start by listing consequences of power reductivity, as defined in the introduction (Definition 2).

Convention 5. An algebraic group over our commutative ring $\mathbf{k}$ is always assumed to be a flat affine group scheme over $\mathbf{k}$. Flatness is essential, as we tacitly use throughout that the functor of taking invariants is left exact.

Proposition 6 (Lifting of invariants). *Let $\mathbf{k}$ be a ring and let G be a power-reductive algebraic group over $\mathbf{k}$. Let A be a finitely generated commutative $\mathbf{k}$-algebra on which G acts rationally through algebra automorphisms. If J is an invariant ideal in A, the map induced by reducing mod J:*

$$A^G \to (A/J)^G$$

is power-surjective.

For an example over $\mathbb{Z}$, see 2.3.2.

Remark 7. Let G be power reductive and let $\phi : A \to B$ be a power-surjective G-map of $\mathbf{k}$-algebras. One easily shows that $A^G \to B^G$ is power-surjective.

Theorem 8 (Hilbert's fourteenth problem). *Let $\mathbf{k}$ be a Noetherian ring and let G be an algebraic group over $\mathbf{k}$. Let A be a finitely generated commutative $\mathbf{k}$-algebra on which G acts rationally through algebra automorphisms. If G is power-reductive, then the subring of invariants A^G is a finitely generated $\mathbf{k}$-algebra.*

Proof. We apply [20, p. 23–26]. It shows that Theorem 8 relies entirely on the conclusion of Proposition 6, which is equivalent to the statement [20, Lemma 2.4.7 p. 23] that, for a surjective G-map $\phi : A \to B$ of $\mathbf{k}$-algebras, the induced map on invariants $A^G \to B^G$ is power-surjective. To prove that power reductivity implies this, consider an invariant b in B, take for L the cyclic module $\mathbf{k}.b$ and for M any submodule of A such that $\phi(M) = L$. We conclude with a commuting diagram:

$$\begin{tikzcd}
(S^dM)^G \arrow[r] \arrow[d, "S^d\phi"] & (S^dA)^G \arrow[r] \arrow[d] & A^G \arrow[d, "\phi^G"] \\
S^dL \arrow[r] & (S^dB)^G \arrow[r] & B^G.
\end{tikzcd}$$

□

Theorem 9 (Hilbert's fourteenth for modules). *Let* $\mathbf{k}$ *be a Noetherian ring and let* G *be a power-reductive algebraic group over* $\mathbf{k}$. *Let* A *be a finitely generated commutative* $\mathbf{k}$*-algebra on which* G *acts rationally, and let* M *be a Noetherian* A*-module, with an equivariant structure map* $A \otimes M \to M$. *If* G *is power-reductive, then the module of invariants* M^G *is Noetherian over* A^G.

Proof. As in [14, 2.2], consider either the symmetric algebra of M on A, or the 'semi-direct product ring' $A \ltimes M$ as in Proposition 57, whose underlying G-module is $A \oplus M$, with product given by $(a_1, m_1)(a_2, m_2) = (a_1a_2, a_1m_2 + a_2m_1)$. □

2.3 Examples

2.3.1

Let $\mathbf{k} = \mathbb{Z}$. Consider the group SL_2 acting on 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ by conjugation. Let L be the line of homotheties in $M := M_2(\mathbb{Z})$. Write $V^\#$ to indicate the dual module $\mathrm{Hom}_{\mathbb{Z}}(V, \mathbb{Z})$ of a $\mathbb{Z}$-module V. The restriction: $M^\# \to L^\#$ extends to

$$\mathbb{Z}[M] = \mathbb{Z}[a, b, c, d] \to \mathbb{Z}[\lambda] = \mathbb{Z}[L].$$

Taking SL_2-invariants:

$$\mathbb{Z}[a, b, c, d]^{\mathrm{SL}_2} = \mathbb{Z}[t, D] \to \mathbb{Z}[\lambda],$$

the trace $t = a + d$ is sent to 2λ, so λ does not lift to an invariant in $M^\#$. The determinant $D = ad - bc$ is sent to λ^2 however, illustrating power reductivity of SL_2.

2.3.2

Similarly, the adjoint action of SL_2 on $\mathfrak{sl}_2$ is such that $u(a) := \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ sends $X, H, Y \in \mathfrak{sl}_2$ respectively to $X + aH - a^2Y, H - 2aY, Y$. This action extends to the symmetric algebra $S^*(\mathfrak{sl}_2)$, which is a polynomial ring in variables X, H, Y. Take $\mathbf{k} = \mathbb{Z}$ again. The mod 2 invariant H does not lift to an integral invariant, but $H^2 + 4XY$ is an integral invariant, and it reduces mod 2 to the square H^2 in $\mathbb{F}_2[X, H, Y]$. This illustrates power reductivity with modules that are not flat, and the strong link between integral and modular invariants.

2.3.3

Consider the group U of 2×2 upper triangular matrices with diagonal 1: this is just an additive group. Let it act on M with basis $\{x, y\}$ by linear substitutions: $u(a)$ sends x, y respectively to $x, ax + y$. Sending x to 0 defines $M \to L$, and since $(S^*M)^U = \mathbf{k}[x]$, power reductivity fails.

2.4 Equivalence of power reductivity with property (Int)

Following [14], we say that a group G satisfies (Int) if $(A/J)^G$ is integral over the image of A^G for every A and J with G action. Note that if $(A/J)^G$ is a Noetherian A^G-module (compare Theorem 9), it must be integral over the image of A^G. As explained in [14, Theorem 2.8], when $\mathbf{k}$ is a field, the property (Int) is equivalent to geometric reductivity, which is equivalent to power-reductivity by [20, Lemma 2.4.7 p. 23]. In general, property (Int) is still equivalent to power-reductivity. But geometric reductivity in the sense of [19] looks too weak.

Proposition 10. *An algebraic group G has property (Int) if, and only if, it is power-reductive.*

Proof. By Proposition 6, power reductivity implies property (Int). We prove the converse. Let $\phi : M \to L$ be as in the formulation of power reductivity in Definition 2. Choose a generator b of L. Property (Int) gives a polynomial $t^n + a_1 t^{n-1} + \cdots + a_n$ with b as root, and with a_i in the image of $S^*(\varphi) : (S^*M)^G \to S^*L$. As b is homogeneous of degree one, we may assume $a_i \in S^i\varphi((S^iM)^G)$. Write a_i as $r_i b^i$ with $r_i \in \mathbf{k}$. Put $r = 1 + r_1 + \cdots r_n$. Then $rb^n = 0$, and $r^{(n-1)!}$ annihilates $b^{n!}$. Since $a_i^{n!/i}$ lies in the image of $S^{n!}\varphi : (S^{n!}M)^G \to S^{n!}L$, the cokernel of this map is annihilated by $r_i^{n!/i}$. Together $r^{(n-1)!}$ and the $r_i^{n!/i}$ generate the unit ideal. So the cokernel vanishes. □

Example 11. Let G be a finite group, viewed as an algebraic group over $\mathbf{k}$. Then A is integral over A^G, because a is a root of $\prod_{g \in G}(x - g(a))$. (This goes back to Emmy Noether [18].) Property (Int) follows easily. Hence G is power reductive.

3 Mumford's conjecture over an arbitrary base

This section deals with the following generalization of the Mumford conjecture.

Theorem 12 (Mumford conjecture). *A Chevalley group scheme is power-reductive for every base.*

By a Chevalley group scheme over $\mathbb{Z}$, we mean a connected split reductive algebraic $\mathbb{Z}$-group $G_{\mathbb{Z}}$, and, by a Chevalley group scheme over a ring $\mathbf{k}$, we mean an algebraic $\mathbf{k}$-group $G = G_{\mathbf{k}}$ obtained by base change from such a $G_{\mathbb{Z}}$. We want to establish the following:

Property. *Let $\mathbf{k}$ be a commutative ring. Let L be a cyclic $\mathbf{k}$-module with trivial G-action. Let M be a rational G-module, and let φ be a G-module map from M onto L. Then there is a positive integer d such that the d-th symmetric power of φ induces a surjection:*

$$(S^d M)^G \to S^d L.$$

3.1 Reduction to local rings

We first reduce to the case of a local ring. For each positive integer d, consider the ideal in $\mathbf{k}$ formed by those scalars which are hit by an invariant in $(S^d M)^G$, and let:

$$\mathfrak{I}_d(\mathbf{k}) := \{x \in \mathbf{k} \mid \exists m \in \mathbb{N},\ x^m.S^d L \subset S^d\varphi((S^d M)^G)\}$$

be its radical. Note that these ideals form a monotone family: if d divides d', then $\mathfrak{I}_d$ is contained in $\mathfrak{I}_{d'}$. We want to show that $\mathfrak{I}_d(\mathbf{k})$ equals $\mathbf{k}$ for some d. To that purpose, it is enough to prove that for each maximal ideal $\mathfrak{M}$ in $\mathbf{k}$, the localized $\mathfrak{I}_d(\mathbf{k})_{(\mathfrak{M})}$ equals the local ring $\mathbf{k}_{(\mathfrak{M})}$ for some d. Notice that taking invariants commutes with localization. Indeed the whole Hochschild complex does and localization is exact. As a result, the localized $\mathfrak{I}_d(\mathbf{k})_{(\mathfrak{M})}$ is equal to the ideal $\mathfrak{I}_d(\mathbf{k}_{(\mathfrak{M})})$. This shows that it is enough to prove the property for a local ring $\mathbf{k}$.

For the rest of this proof, $\mathbf{k}$ denotes a local ring with residual characteristic p.

3.2 Reduction to cohomology

As explained in Section 3.5, we may assume that G is semisimple simply connected. Replacing M if necessary by a submodule that still maps onto L, we may assume that M is finitely generated.

We then reduce the desired property to cohomological algebra. To that effect, if X is a G-module, consider the evaluation map on the identity id_X: $\mathrm{Hom}_{\mathbf{k}}(X,X)^\# \to \mathbf{k}$ (we use $V^\#$ to indicate the dual module $\mathrm{Hom}_{\mathbf{k}}(V,\mathbf{k})$ of a module V). If X is $\mathbf{k}$-free of finite rank d, then $S^d(\mathrm{Hom}_{\mathbf{k}}(X,X)^\#)$ contains the determinant. The determinant is G-invariant, and its evaluation at id_X is equal to 1. Let b a $\mathbf{k}$-generator of L and consider the composite:

$$\psi : \mathrm{Hom}_{\mathbf{k}}(X,X)^\# \to \mathbf{k} \to \mathbf{k}.b = L.$$

Its d-th power $S^d\psi$ sends the determinant to b^d. Suppose further that ψ lifts to M by a G-equivariant map. Then, choosing d to be the $\mathbf{k}$-rank of X, the d-th power of the resulting map $S^d(\mathrm{Hom}_{\mathbf{k}}(X,X)^{\#}) \to S^dM$ sends the determinant to a G-invariant in S^dM, which is sent to b^d through $S^d\varphi$. This would establish the property.

$$\begin{tikzcd} & \mathrm{Hom}_{\mathbf{k}}(X,X)^{\#} \arrow[dl, dashed] \arrow[d, "\psi"] \\ M \arrow[r, "\varphi"'] & L \end{tikzcd}$$

The existence of a lifting would follow from the vanishing of the extension group:

$$\mathrm{Ext}^1_G((\mathrm{Hom}_{\mathbf{k}}(X,X)^{\#}, \mathrm{Ker}\varphi),$$

or, better, from acyclicity, *i.e.* the vanishing of all positive degree Ext-groups. Inspired by the proof of the Mumford conjecture in [6, (3.6)], we choose X to be an adequate Steinberg module. To make this choice precise, we need notations, essentially borrowed from [6, 2].

3.3 Notations

We decide as in [11], and contrary to [12] and [6], that the roots of the standard Borel subgroup B are negative. The opposite Borel group B^+ of B will thus have positive roots. We also fix a Weyl group invariant inner product on the weight lattice $X(T)$. Thus we can speak of the length of a weight.

For a weight λ in the weight lattice, we denote by λ as well the corresponding one-dimensional rational B-module (or sometimes B^+-module), and by ∇_λ the costandard module (Schur module) $\mathrm{ind}_B^G\lambda$ induced from it. Dually, we denote by Δ_λ the standard module (Weyl module) of highest weight λ. So $\Delta_\lambda = \mathrm{ind}_{B^+}^G(-\lambda)^{\#}$. We shall use that, over $\mathbb{Z}$, these modules are $\mathbb{Z}$-free [11, II Ch. 8].

We let ρ be half the sum of the positive roots of G. It is also the sum of the fundamental weights. As G is simply connected, the fundamental weights are weights of B.

Let p be the characteristic of the residue field of the local ring $\mathbf{k}$. When p is positive, for each positive integer r, we let the weight σ_r be $(p^r-1)\rho$. When p is 0, we let σ_r be $r\rho$. Let St_r be the G-module $\nabla_{\sigma_r} = \mathrm{ind}_B^G\sigma_r$; it is a usual Steinberg module when $\mathbf{k}$ is a field of positive characteristic.

3.4

We shall use the following combinatorial lemma:

Lemma 13. *Let R be a positive real number. If r is a large enough integer, for all weights μ of length less than R, $\sigma_r+\mu$ is dominant.*

So, if r is a large enough integer to satisfy the condition in Lemma 13, for all G-modules M with weights that have length less than R, all the weights in $\sigma_r \otimes M$ are dominant. Note that in the preceding discussion, the G-module M is finitely generated. Thus the weights of M, and hence of $\mathrm{Ker}\varphi$, are bounded. Thus, Theorem 12 is implied by the following proposition.

Proposition 14. *Let R be a positive real number, and let r be as in Lemma 13 . For all local rings $\mathbf{k}$ with characteristic p residue field, for all G-module N with weights of length less than R, and for all positive integers n:*

$$\mathrm{Ext}^n_G((\mathrm{Hom}_{\mathbf{k}}(St_r, St_r)^{\#}, N) = 0 \ .$$

Proof. First, the result is true when $\mathbf{k}$ is a field. Indeed, we have chosen St_r to be a self-dual Steinberg module, so, for each positive integer n:

$$\mathrm{Ext}^n_G((\mathrm{Hom}_{\mathbf{k}}(St_r, St_r)^{\#}, N) = \mathrm{H}^n(G, St_r \otimes St_r \otimes N) = \mathrm{H}^n(B, St_r \otimes \sigma_r \otimes N).$$

Vanishing follows by [6, Corollary (3.3')] or the proof of [6, Corollary (3.7)].
Suppose now that N is defined over $\mathbb{Z}$ by a free $\mathbb{Z}$-module, in the following sense: $N = N_{\mathbb{Z}} \otimes_{\mathbb{Z}} V$ for a $\mathbb{Z}$-free $G_{\mathbb{Z}}$-module $N_{\mathbb{Z}}$ and a $\mathbf{k}$-module V with trivial G action. We then use the universal coefficient theorem [4, A.X.4.7] (see also [11, I.4.18]) to prove acyclicity in this case.
Specifically, let us note $Y_{\mathbb{Z}} := \mathrm{Hom}_{\mathbb{Z}}((St_r)_{\mathbb{Z}}, (St_r)_{\mathbb{Z}}) \otimes N_{\mathbb{Z}}$, so that, using base change (Proposition 16 for $\lambda = \sigma_r$):

$$\mathrm{Ext}^n_G((\mathrm{Hom}_{\mathbf{k}}(St_r, St_r)^{\#}, N) = \mathrm{H}^n(G, Y_{\mathbb{Z}} \otimes V).$$

This cohomology is computed [7, II.3] (see also [11, I.4.16]) by taking the homology of the Hochschild complex $C(G, Y_{\mathbb{Z}} \otimes V)$. This complex is isomorphic to the complex obtained by tensoring with V the integral Hochschild complex $C(G_{\mathbb{Z}}, Y_{\mathbb{Z}})$. Since the latter is a complex of torsion-free abelian groups, we deduce, by the universal coefficient theorem applied to tensoring with a characteristic p field k, and the vanishing for the case of such a field, that: $\mathrm{H}^n(G_{\mathbb{Z}}, Y_{\mathbb{Z}}) \otimes k = 0$, for all positive n. We apply this when k is the residue field of $\mathbb{Z}_{(p)}$; note that if $p = 0$ the residue field k is just $\mathbb{Q}$. Since the cohomology $\mathrm{H}^n(G_{\mathbb{Z}}, Y_{\mathbb{Z}})$ is finitely generated [11, B.6], the Nakayama lemma implies that: $\mathrm{H}^n(G_{\mathbb{Z}}, Y_{\mathbb{Z}}) \otimes \mathbb{Z}_{(p)} = 0$, for all positive n. And $\mathrm{H}^n(G_{\mathbb{Z}}, Y_{\mathbb{Z}}) \otimes \mathbb{Z}_{(p)} = \mathrm{H}^n(G_{\mathbb{Z}}, Y_{\mathbb{Z}} \otimes \mathbb{Z}_{(p)})$ because localization is exact. The complex $C(G_{\mathbb{Z}}, Y_{\mathbb{Z}} \otimes \mathbb{Z}_{(p)})$ is a complex of torsion-free $\mathbb{Z}_{(p)}$-modules, we thus can apply the universal coefficient theorem to tensoring with V. The vanishing of $\mathrm{H}^n(G, Y_{\mathbb{Z}} \otimes \mathbb{Z}_{(p)} \otimes V) = \mathrm{H}^n(G, Y_{\mathbb{Z}} \otimes V)$ follows.
For the general case, we proceed by descending induction on the highest weight of N. To perform the induction, we first choose a total order on weights of length less than R, that refines the usual dominance order of [11, II 1.5]. Initiate the induction with $N = 0$. For the induction step, consider the highest weight μ in N and let N_μ be its weight space. By the preceding case, we obtain vanishing for $\Delta_{\mu_{\mathbb{Z}}} \otimes_{\mathbb{Z}} N_\mu$. Now, by Proposition 21, $\Delta_{\mu_{\mathbb{Z}}} \otimes_{\mathbb{Z}} N_\mu$ maps to

N, and the kernel and the cokernel of this map have lower highest weight. By induction, they give vanishing cohomology. Thus $\mathrm{Hom}_{\mathbf{k}}(St_r, St_r) \otimes N$ is in an exact sequence where three out of four terms are acyclic, hence it is acyclic. □

This concludes the proof of Theorem 12.

3.5 Reduction to simply connected group schemes

Let $Z_{\mathbb{Z}}$ be the center of $G_{\mathbb{Z}}$ and let Z be the corresponding subgroup of G. It is a diagonalisable group scheme, so $M^Z \to L$ is also surjective. We may replace M with M^Z and G with G/Z, in view of the general formula $M^G = (M^Z)^{G/Z}$, see [11, I 6.8(3)]. So now G has become semisimple, but of adjoint type rather than simply connected type. So choose a simply connected Chevalley group scheme $\tilde{G}_{\mathbb{Z}}$ with center $\tilde{Z}_{\mathbb{Z}}$ so that $\tilde{G}_{\mathbb{Z}}/\tilde{Z}_{\mathbb{Z}} = G_{\mathbb{Z}}$. We may now replace G with $\tilde{G}$.

Remark 15. Other reductions are possible, to enlarge the supply of power reductive algebraic groups. For instance, if G has a normal subgroup N so that both N and G/N are power reductive, then so is G (for a proof, use Remark 7). And if $\mathbf{k} \to R$ is a faithfully flat extension so that G_R is power reductive, then G is already power reductive. So twisted forms are allowed, compare the discussion in [19, p. 239].

4 Generalities

This section collects known results over an arbitrary base, their proof, and correct proofs of known results over fields, for use in the other sections. The part up to subsection 4.3 is used, and referred to, in the previous section.

4.1 Notations

Throughout this paper, we let G be a semisimple Chevalley group scheme over the commutative ring $\mathbf{k}$. We keep the notations of Section 3.3. In particular, the standard parabolic B has negative roots. Its standard torus is T, its unipotent radical is U. The opposite Borel B^+ has positive roots and its unipotent radical is U^+. For a standard parabolic subgroup P its unipotent radical is $R_u(P)$. For a weight λ in $X(T)$, $\nabla_\lambda = \mathrm{ind}_B^G \lambda$ and $\Delta_\lambda = \mathrm{ind}_{B^+}^G(-\lambda)^\#$.

4.2

We first recall base change for costandard modules.

Proposition 16. *Let λ be a weight, and denote also by $\lambda = \lambda_{\mathbb{Z}} \otimes \mathbf{k}$ the B-module $\mathbf{k}$ with action by λ. For any ring $\mathbf{k}$, there is a natural isomorphism:*

$$\mathrm{ind}_{B_{\mathbb{Z}}}^{G_{\mathbb{Z}}} \lambda_{\mathbb{Z}} \otimes \mathbf{k} \cong \mathrm{ind}_B^G \lambda$$

In particular, $\mathrm{ind}_B^G \lambda$ is nonzero if and only if λ is dominant.

Proof. First consider the case when λ is not dominant. Then $\mathrm{ind}_B^G\lambda$ vanishes when $\mathbf{k}$ is a field [11, II.2.6], so both $\mathrm{ind}_{B_\mathbb{Z}}^{G_\mathbb{Z}}\lambda_\mathbb{Z}$ and the torsion in $R^1\mathrm{ind}_{B_\mathbb{Z}}^{G_\mathbb{Z}}\lambda_\mathbb{Z}$ must vanish. Then $\mathrm{ind}_B^G\lambda$ vanishes as well for a general $\mathbf{k}$ by the universal coefficient theorem.
In the case when λ is dominant, $R^1\mathrm{ind}_{B_\mathbb{Z}}^{G_\mathbb{Z}}\lambda_\mathbb{Z}$ vanishes by Kempf's theorem [11, II 8.8(2)]. Thus, by [11, I.4.18b)]: $\mathrm{ind}_{B_\mathbb{Z}}^{G_\mathbb{Z}}\lambda_\mathbb{Z} \otimes \mathbf{k} \cong \mathrm{ind}_B^G\lambda$. □

Proposition 17 (Tensor identity for weights). *Let λ be a weight, and denote again by λ the B-module $\mathbf{k}$ with action by λ. Let N be a G-module. There is a natural isomorphism:*

$$\mathrm{ind}_B^G(\lambda \otimes N) \cong (\mathrm{ind}_B^G\lambda) \otimes N.$$

Remark 18. The case when N is $\mathbf{k}$-flat is covered by [11, I.4.8]. We warn the reader against Proposition I.3.6 in the 1987 first edition of the book. Indeed, suppose we always had $\mathrm{ind}_B^G(M \otimes N) \cong (\mathrm{ind}_B^G M) \otimes N$. Take $\mathbf{k} = \mathbb{Z}$ and $N = \mathbb{Z}/p\mathbb{Z}$. The universal coefficient theorem would then imply that $R^1\mathrm{ind}_B^G M$ never has torsion. Thus $R^i\mathrm{ind}_B^G M$ would never have torsion for positive i. It would make [1, Cor. 2.7] contradict the Borel–Weil–Bott theorem.

Proof. Recall that for a B-module M one may define $\mathrm{ind}_B^G(M)$ as $(\mathbf{k}[G]\otimes M)^B$, where $\mathbf{k}[G]\otimes M$ is viewed as a $G\times B$-module with G acting by left translation on $\mathbf{k}[G]$, B acting by right translation on $\mathbf{k}[G]$, and B acting the given way on M. Let N_{triv} denote N with trivial B action. There is a B-module isomorphism $\psi : \mathbf{k}[G] \otimes \lambda \otimes N \to \mathbf{k}[G] \otimes \lambda \otimes N_{\mathrm{triv}}$, given in non-functorial notation by:

$$\psi(f \otimes 1 \otimes n) : x \mapsto f(x) \otimes 1 \otimes xn.$$

So ψ is obtained by first applying the comultiplication $N \to \mathbf{k}[G] \otimes N$, then the multiplication $\mathbf{k}[G]\otimes\mathbf{k}[G] \to \mathbf{k}[G]$. It sends $(\mathbf{k}[G]\otimes\lambda\otimes N)^B$ to $(\mathbf{k}[G]\otimes\lambda\otimes N_{\mathrm{triv}})^B = (\mathbb{Z}[G_\mathbb{Z}]\otimes_\mathbb{Z}\lambda_\mathbb{Z}\otimes_\mathbb{Z}N_{\mathrm{triv}})^B$. Now recall from the proof of Proposition 16 that the torsion in $R^1\mathrm{ind}_{B_\mathbb{Z}}^{G_\mathbb{Z}}\lambda_\mathbb{Z}$ vanishes. By the universal coefficient theorem we get that $(\mathbb{Z}[G_\mathbb{Z}]\otimes_\mathbb{Z}\lambda_\mathbb{Z}\otimes_\mathbb{Z}N_{\mathrm{triv}})^B$ equals $(\mathbf{k}[G]\otimes\lambda)^B\otimes N_{\mathrm{triv}}$. To make these maps into G-module maps, one must use the given G-action on N as the action on N_{triv}. So B acts on N, but not N_{triv}, and for G it is the other way around. One sees that $(\mathbf{k}[G]\otimes\lambda)^B\otimes N_{\mathrm{triv}}$ is just $(\mathrm{ind}_B^G\lambda)\otimes N$. □

Proposition 19. *For a G-module M, there are only dominant weights in M^{U^+}.*

Proof. Let λ be a nondominant weight. Instead we show that $-\lambda$ is no weight of M^U, or that $\mathrm{Hom}_B(-\lambda, M)$ vanishes. By the tensor identity of Proposition 17: $\mathrm{Hom}_B(-\lambda, M) = \mathrm{Hom}_B(k, \lambda\otimes M) = \mathrm{Hom}_G(k, \mathrm{ind}_B^G(\lambda\otimes M)) = \mathrm{Hom}_G(k, \mathrm{ind}_B^G\lambda\otimes M)$ which vanishes by Proposition 16. □

Proposition 20. *Let λ be a dominant weight. The restriction (or evaluation) map $\mathrm{ind}_B^G\lambda \to \lambda$ to the weight space of weight λ is a T-module isomorphism.*

Proof. Over fields of positive characteristic this is a result of Ramanathan [12, A.2.6]. It then follows over $\mathbb{Z}$ by the universal coefficient theorem applied to the complex $\mathrm{ind}_{B_{\mathbb{Z}}}^{G_{\mathbb{Z}}}\lambda_{\mathbb{Z}} \to \lambda_{\mathbb{Z}} \to 0$. For a general $\mathbf{k}$, apply proposition 16. □

Proposition 21 (Universal property of Weyl modules). *Let λ be a dominant weight. For any G-module M, there is a natural isomorphism*

$$\mathrm{Hom}_G(\Delta_\lambda, M) \cong \mathrm{Hom}_{B^+}(\lambda, M).$$

In particular, if M has highest weight λ, then there is a natural map from $\Delta_{\lambda_{\mathbb{Z}}} \otimes_{\mathbb{Z}} M_\lambda$ to M, its kernel has lower weights, and λ is not a weight of its cokernel.

Proof. By the tensor identity Proposition 17: $\mathrm{ind}_{B^+}^G(-\lambda \otimes M) \cong \mathrm{ind}_{B^+}^G(-\lambda) \otimes M$. Thus $\mathrm{Hom}_G(\Delta_\lambda, M) = \mathrm{Hom}_G(\mathbf{k}, \mathrm{ind}_{B^+}^G(-\lambda) \otimes M) = \mathrm{Hom}_{B^+}(\mathbf{k}, -\lambda \otimes M) = \mathrm{Hom}_{B^+}(\lambda, M)$. If M has highest weight λ, then $M_\lambda = \mathrm{Hom}_{B^+}(\lambda, M)$. Tracing the maps, the second part follows from Proposition 20. □

4.3 Notations

We now recall the notations from [13, §2.2]. Let the *Grosshans height function* $\mathrm{ht} : X(T) \to \mathbb{Z}$ be defined by:

$$\mathrm{ht}\,\gamma = \sum_{\alpha>0} \langle \gamma, \alpha^\vee \rangle.$$

For a G-module M, let $M_{\leq i}$ denote the largest G-submodule with weights λ that all satisfy: $\mathrm{ht}\,\lambda \leq i$. Similarly define $M_{<i} = M_{\leq i-1}$. For instance, $M_{\leq 0} = M^G$. We call the filtration

$$0 \subseteq M_{\leq 0} \subseteq M_{\leq 1} \cdots$$

the *Grosshans filtration*, and we call its associated graded the *Grosshans graded* $\mathrm{gr}\,M$ of M. We put: $\mathrm{hull}_\nabla(\mathrm{gr}\,M) = \mathrm{ind}_B^G M^{U^+}$.
Let A be a commutative $\mathbf{k}$-algebra on which G acts rationally through $\mathbf{k}$-algebra automorphisms. The Grosshans graded algebra $\mathrm{gr}\,A$ is given in degree i by:

$$\mathrm{gr}_i\,A = A_{\leq i}/A_{<i}.$$

4.4 Erratum

When $\mathbf{k}$ is a field, one knows that $\mathrm{gr}\,A$ embeds in a good filtration hull, which Grosshans calls R in [10], and which we call $\mathrm{hull}_\nabla(\mathrm{gr}\,A)$:

$$\mathrm{hull}_\nabla(\mathrm{gr}\,A) = \mathrm{ind}_B^G A^{U^+}.$$

When $\mathbf{k}$ is a field of positive characteristic p, it was shown by Mathieu [16, Key Lemma 3.4] that this inclusion is power-surjective: for every $b \in \mathrm{hull}_\nabla(\mathrm{gr}\,A)$, there is an r so that b^{p^r} lies in the subalgebra $\mathrm{gr}\,A$.

This result's exposition in [13, Lemma 2.3] relies on [12, Sublemma A.5.1]. Frank Grosshans has pointed out that the proof of this sublemma is not convincing beyond the reduction to the affine case. Later A. J. de Jong actually gave a counterexample to the reasoning. The result itself is correct and has been used by others. As power surjectivity is a main theme in this paper, we take the opportunity to give a corrected treatment. Mathieu's result will be generalized to an arbitrary base $\mathbf{k}$ in Section 5.

Proposition 22. *Let $\mathbf{k}$ be an algebraically closed field of characteristic $p > 0$. Let both A and B be commutative $\mathbf{k}$-algebras of finite type over $\mathbf{k}$, with B finite over A. Put $Y = \operatorname{Spec}(A)$, $X = \operatorname{Spec}(B)$. Assume $X \to Y$ gives a bijection between $\mathbf{k}$ valued points. Then for all $b \in B$ there is an m with $b^{p^m} \in A$.*

Proof. The result follows easily from [15, Lemma 13]. We shall argue instead by induction on the Krull dimension of A.
Say B as an A-module is generated by d elements $b_1, \dots, b_d$. Let $\mathfrak{p}_1, \dots \mathfrak{p}_s$ be the minimal prime ideals of A.
Suppose we can show that for every i, j we have $m_{i,j}$ so that $b_j^{p^{m_{i,j}}} \in A + \mathfrak{p}_i B$. Then for every i we have m_i so that $b^{p^{m_i}} \in A + \mathfrak{p}_i B$ for every $b \in B$. Then $b^{p^{m_1 + \cdots m_s}} \in A + \mathfrak{p}_1 \cdots \mathfrak{p}_s B$ for every $b \in B$. As $\mathfrak{p}_1 \cdots \mathfrak{p}_s$ is nilpotent, one finds m with $b^{p^m} \in A$ for all $b \in B$. The upshot is that it suffices to prove the sublemma for the inclusion $A/\mathfrak{p}_i \subset B/\mathfrak{p}_i B$. [It is an inclusion because there is a prime ideal $\mathfrak{q}_i$ in B with $A \cap \mathfrak{q}_i = \mathfrak{p}_i$.] Therefore we further assume that A is a domain.
Let $\mathfrak{r}$ denote the nilradical of B. If we can show that for all $b \in B$ there is m with $b^{p^m} \in A + \mathfrak{r}$, then clearly we can also find a u with $b^{p^u} \in A$. So we may as well replace $A \subset B$ with $A \subset B/\mathfrak{r}$ and assume that B is reduced. But then at least one component of $\operatorname{Spec}(B)$ must map onto $\operatorname{Spec}(A)$, so bijectivity implies there is only one component. In other words, B is also a domain.
Choose t so that the field extension $\operatorname{Frac}(A) \subset \operatorname{Frac}(AB^{p^t})$ is separable. (So it is the separable closure of $\operatorname{Frac}(A)$ in $\operatorname{Frac}(B)$.) As $X \to \operatorname{Spec}(AB^{p^t})$ is also bijective, we have that $\operatorname{Spec}(AB^{p^t}) \to \operatorname{Spec}(A)$ is bijective. It clearly suffices to prove the proposition for $A \subset AB^{p^t}$. So we replace B with AB^{p^t} and further assume that $\operatorname{Frac}(B)$ is separable over $\operatorname{Frac}(A)$.
Now $X \to Y$ has a degree which is the degree of the separable field extension. There is a dense subset U of Y so that this degree is the number of elements in the inverse image of a point of U. [Take a primitive element of the field extension, localize to make its minimum polynomial monic over A, invert the discriminant.] Thus the degree must be one because of bijectivity. So we must now have that $\operatorname{Frac}(B) = \operatorname{Frac}(A)$.
Let $\mathfrak{c} = \{ b \in B \mid bB \subset A \}$ be the conductor of $A \subset B$. We know it is nonzero. If it is the unit ideal then we are done. Suppose it is not. By induction applied to $A/\mathfrak{c} \subset B/\mathfrak{c}$ (we need the induction hypothesis for the original problem without any of the intermediate simplifications) we have that for each $b \in B$ there is an m so that $b^{p^m} \in A + \mathfrak{c} = A$. ∎

4.5

This subsection prepares the ground for the proof of the theorems in Section 5. We start with the ring of invariants $\mathbf{k}[G/U]$ of the action of U by right translation on $\mathbf{k}[G]$.

Lemma 23. *The* $\mathbf{k}$*-algebra* $\mathbf{k}[G/U]$ *is finitely generated.*

Proof. We have:

$$k[G/U] = \bigoplus_{\lambda\in X(T)} k[G/U]_{-\lambda} = \bigoplus_{\lambda\in X(T)} (k[G]\otimes\lambda)^B = \bigoplus_{\lambda\in X(T)} \nabla_\lambda.$$

By Proposition 16, this equals the sum $\oplus_\lambda \nabla_\lambda$ over dominant weights λ only. When G is simply connected, every fundamental weight is a weight, and the monoid of dominant λ is finitely generated. In general, some multiple of a fundamental weight is in $X(T)$ and there are only finitely many dominant weights modulo these multiples. So the monoid is still finitely generated by Dickson's Lemma [5, Ch. 2 Thm. 7]. The maps $\nabla_\lambda \otimes \nabla_\mu \to \nabla_{\lambda+\mu}$ are surjective for dominant λ, μ, because this is so over $\mathbb{Z}$, by base change and surjectivity for fields [11, II, Proposition 14.20]. This implies the result. □

In the same manner one shows:

Lemma 24. *If the* $\mathbf{k}$*-algebra* A^U *is finitely generated, so is* $\operatorname{hull}_\nabla \operatorname{gr} A = \operatorname{ind}_B^G A^{U^+}$.

Proof. Use that A^{U^+} is isomorphic to A^U as $\mathbf{k}$-algebra. □

Lemma 25. *Suppose* $\mathbf{k}$ *is Noetherian. If the* $\mathbf{k}$*-algebra* A *with* G *action is finitely generated, then so is* A^U.

Proof. By the transfer principle [9, Ch. Two]:

$$A^U = \operatorname{Hom}_U(k, A) = \operatorname{Hom}_G(k, \operatorname{ind}_U^G A) = (A \otimes k[G/U])^G.$$

Now apply Lemma 23 and Theorem 3. □

Lemma 26. *If* M *is a* G*-module, there is a natural injective map*

$$\operatorname{gr} M \hookrightarrow \operatorname{hull}_\nabla(\operatorname{gr} M) = \operatorname{ind}_B^G M^{U^+}.$$

Proof. By Lemma 19, the weights of M^{U^+} are dominant. If one of them, say λ, has Grosshans height i, the universal property of Weyl modules (Proposition 21) shows that $(M^{U^+})_\lambda$ is contained in a G-submodule with weights that do not have a larger Grosshans height. So the weight space $(M^{U^+})_\lambda$ is contained in $M_{\leq i}$, but not $M_{<i}$. We conclude that the T-module $\oplus_i (\operatorname{gr}_i M)^{U^+}$ may be identified with the T-module M^{U^+}. It remains to embed $\operatorname{gr}_i M$ into $\operatorname{ind}_B^G (\operatorname{gr}_i M)^{U^+}$.

The T-module projection of $\mathrm{gr}_i M$ onto $(\mathrm{gr}_i M)^{U^+}$ may be viewed as a B-module map, and then, it induces a G-module map $\mathrm{gr}_i M \to \mathrm{ind}_B^G((\mathrm{gr}_i M)^{U^+})$, which restricts to an isomorphism on $(\mathrm{gr}_i M)^{U^+}$ by Proposition 20. So its kernel has weights with lower Grosshans height, and must therefore be zero. □

In the light of Lemma 26, one may write:

Definition 27. A G-module M has *good Grosshans filtration* if the embedding of $\mathrm{gr}\, M$ into $\mathrm{hull}_\nabla(\mathrm{gr}\, M)$ is an isomorphism.

It is worth recording the following characterization, just like in the case where $\mathbf{k}$ is a field.

Proposition 28 (Cohomological criterion). *For a G-module M, the following are equivalent.*

i. M has good Grosshans filtration.

ii. $\mathrm{H}^1(G, M \otimes \mathbf{k}[G/U])$ *vanishes.*

iii. $\mathrm{H}^n(G, M \otimes \mathbf{k}[G/U])$ *vanishes for all positive n.*

Proof. Let M have good Grosshans filtration. We have to show that $M \otimes \mathbf{k}[G/U]$ is acyclic. First, for each integer i, $\mathrm{gr}_i M \otimes \mathbf{k}[G/U]$ is a direct sum of modules of the form $\mathrm{ind}_B^G\lambda \otimes \mathrm{ind}_B^G\mu \otimes N$, where G acts trivially on N. Such modules are acyclic by [11, B.4] and the universal coefficient theorem. As each $\mathrm{gr}_i M \otimes \mathbf{k}[G/U]$ is acyclic, so is each $M_{\leq i} \otimes \mathbf{k}[G/U]$, and thus $M \otimes \mathbf{k}[G/U]$ is acyclic.
Conversely, suppose that M does not have good Grosshans filtration. Choose i so that $M_{<i}$ has good Grosshans filtration, but $M_{\leq i}$ does not. Choose λ so that $\mathrm{Hom}(\Delta_\lambda, \mathrm{hull}(\mathrm{gr}_i M)/\mathrm{gr}_i M)$ is nonzero. Note that λ has Grosshans height below i. As $\mathrm{Hom}(\Delta_\lambda, \mathrm{hull}(\mathrm{gr}_i M))$ vanishes, $\mathrm{Ext}^1_G(\Delta_\lambda, \mathrm{gr}_i M) = \mathrm{H}^1(G, \mathrm{gr}_i M \otimes \nabla_\lambda)$ does not. Since $M_{<i} \otimes \mathbf{k}[G/U] = \oplus_{\mu \text{ dominant}} M_{<i} \otimes \nabla_\mu$ is acyclic, $\mathrm{H}^1(G, M_{\leq i} \otimes \nabla_\lambda)$ is nonzero as well. Now use that $\mathrm{Hom}(\Delta_\lambda, M/M_{\leq i})$ vanishes, and conclude that $\mathrm{H}^1(G, M \otimes \mathbf{k}[G/U])$ does not vanish. □

5 Grosshans graded, Grosshans hull and powers

5.1

When G is a semisimple group over a field $\mathbf{k}$, Grosshans has introduced a filtration on G-modules. As recalled in Section 4.3, it is the filtration associated to the function defined on $X(T)$ by: $\mathrm{ht}\,\gamma = \sum_{\alpha>0}\langle\gamma, \alpha^\vee\rangle$. Grosshans has proved some interesting results about its associated graded, when the G-module is a $\mathbf{k}$-algebra A with rational G action. We now show how these results generalize to an arbitrary Noetherian base $\mathbf{k}$, and we draw some conclusions about $\mathrm{H}^*(G, A)$. All this suggests that the finite generation conjecture of [13] (see also [14]) deserves to be investigated in the following generality.

Problem. *Let* $\mathbf{k}$ *be a Noetherian ring and let* G *be a Chevalley group scheme over* $\mathbf{k}$*. Let* A *be a finitely generated commutative* $\mathbf{k}$*-algebra on which* G *acts rationally through algebra automorphisms. Is the cohomology ring* $\mathrm{H}^*(G, A)$ *a finitely generated* $\mathbf{k}$*-algebra?*

Let $\mathbf{k}$ be an arbitrary commutative ring.

Theorem 29 (Grosshans hull and powers). *The natural embedding of* $\operatorname{gr} A$ *in* $\operatorname{hull}_\nabla(\operatorname{gr} A)$ *is power-surjective.*

This will then be used to prove:

Theorem 30 (Grosshans hull and finite generation). *If the ring* $\mathbf{k}$ *is Noetherian, then the following are equivalent.*

i. The $\mathbf{k}$*-algebra* A *is finitely generated;*

ii. For every standard parabolic P*, the* $\mathbf{k}$*-algebra of invariants* $A^{R_u(P)}$ *is finitely generated;*

iii. The $\mathbf{k}$*-algebra* $\operatorname{gr} A$ *is finitely generated;*

iv. The $\mathbf{k}$*-algebra* $\operatorname{hull}_\nabla(\operatorname{gr} A)$ *is finitely generated.*

Remark 31. Consider a reductive Chevalley group scheme G. As the Grosshans height is formulated with the help of coroots $\alpha^\vee$, only the semisimple part of G is relevant for it. But of course everything is compatible with the action of the center of G also. We leave it to the reader to reformulate our results for reductive G. We return to the assumption that G is semisimple.

Theorem 32. *Let* A *be a finitely generated commutative* $\mathbf{k}$*-algebra. If* $\mathbf{k}$ *is Noetherian, there is a positive integer* n *so that:*

$$n \operatorname{hull}_\nabla(\operatorname{gr} A) \subseteq \operatorname{gr} A.$$

In particular $\mathrm{H}^i(G, \operatorname{gr} A)$ *is annihilated by* n *for positive* i*.*

This is stronger than the next result.

Theorem 33 (generic good Grosshans filtration). *Let* A *be a finitely generated commutative* $\mathbf{k}$*-algebra. If* $\mathbf{k}$ *is Noetherian, there is a positive integer* n *so that* $A[1/n]$ *has good Grosshans filtration. In particular* $\mathrm{H}^i(G, A) \otimes \mathbb{Z}[1/n] = \mathrm{H}^i(G, A[1/n])$ *vanishes for positive* i*.*

Remark 34. Of course $A[1/n]$ may vanish altogether, as we are allowed to take the characteristic for n, when that is positive.

Theorem 35. *Let* A *be a finitely generated commutative* $\mathbf{k}$*-algebra. If* $\mathbf{k}$ *is Noetherian, for each prime number* p*, the algebra map* $\operatorname{gr} A \to \operatorname{gr}(A/pA)$ *is power-surjective.*

5.2

We start with a crucial special case. Let $\mathbf{k} = \mathbb{Z}$. Let $\lambda \in X(T)$ be dominant. Let S' be the graded algebra with degree n part:

$$S'_n = \nabla_{n\lambda} = \Gamma(G/B, \mathcal{L}(n\lambda)).$$

Let us view Δ_λ as a submodule of ∇_λ with common λ weight space (the 'minimal admissible lattice' embedded in the 'maximal admissible lattice'). Let S be the graded subalgebra generated by Δ_λ in the graded algebra S'. If we wish to emphasize the dependence on λ, we write $S'(\lambda)$ for S', $S(\lambda)$ for S. Consider the map

$$G/B \to \mathbb{P}_{\mathbb{Z}}(\Gamma(G/B, \mathcal{L}(\lambda))^{\#})$$

given by the 'linear system' ∇_λ on G/B. The projective scheme $\mathrm{Proj}(S')$ corresponds with its image, which, by direct inspection, is isomorphic to G/P, where P is the stabilizer of the weight space with weight $-\lambda$ of $\nabla_\lambda^{\#}$. Indeed that weight space is the image of B/B, compare Proposition 20 and [11, II.8.5]. The inclusion $\phi : S \hookrightarrow S'$ induces a morphism from an open subset of $\mathrm{Proj}(S')$ to $\mathrm{Proj}(S)$. This open subset is called $G(\phi)$ in [EGA II, 2.8.1].

Lemma 36. *The morphism* $\mathrm{Proj}(S') \to \mathrm{Proj}(S)$ *is defined on all of* $G/P = \mathrm{Proj}(S')$.

Proof. As explained in [EGA II, 2.8.1], the domain $G(\phi)$ contains the principal open subset $D_+(s)$ of $\mathrm{Proj}(S')$ for any $s \in S_1$. Consider in particular a generator s of the λ weight space of ∇_λ. It is an element in S_1, and, by Lemma 20, it generates the free $\mathbf{k}$-module $\Gamma(P/P, \mathcal{L}(\lambda))$. Thus, the minimal Schubert variety P/P is contained in $D_+(s)$. We then conclude by homogeneity: s is also U^+ invariant, so in fact the big cell $\Omega = U^+P/P$ is contained in $D_+(s)$, and the domain $G(\phi)$ contains the big cell Ω. Then it also contains the Weyl group translates $w\Omega$, and thus it contains all of G/P. ☐

Lemma 37. *The graded algebra S' is integral over its subalgebra S.*

Proof. We also put a grading on the polynomial ring $S'[z]$, by assigning degree one to the variable z. One calls $\mathrm{Proj}(S'[z])$ the projective cone of $\mathrm{Proj}(S')$ [EGA II, 8.3]. By [EGA II, 8.5.4], we get from Lemma 36 that $\hat{\Phi} : \mathrm{Proj}(S'[z]) \to \mathrm{Proj}(S[z])$ is everywhere defined. Now by [EGA II, Th (5.5.3)], and its proof, the maps $\mathrm{Proj}(S'[z]) \to \mathrm{Spec}\,\mathbb{Z}$ and $\mathrm{Proj}(S[z]) \to \mathrm{Spec}\,\mathbb{Z}$ are proper and separated, so $\hat{\Phi}$ is proper by [EGA II, Cor (5.4.3)]. But now the principal open subset $D_+(z)$ associated to z in $\mathrm{Proj}(S'[z])$ is just $\mathrm{Spec}(S')$, and its inverse image is the principal open subset associated to z in $\mathrm{Proj}(S[z])$, which is $\mathrm{Spec}(S)$ (compare [EGA II, 8.5.5]). So $\mathrm{Spec}(S) \to \mathrm{Spec}(S')$ is proper, and S' is a finitely generated S-module by [EGA III, Prop (4.4.2)]. ☐

Lemma 38. *There is a positive integer t so that tS' is contained in S.*

Proof. Clearly $S' \otimes \mathbb{Q} = S \otimes \mathbb{Q}$, so the result follows from Lemma 37. ☐

Let p be a prime number. Recall from 4.4 the result of Mathieu [16, Key Lemma 3.4] that, for every element b of S'/pS', there is a positive r so that $b^{p^r} \in (S+pS')/(pS') \subseteq S'/pS'$.
By Lemma 38 and Proposition 41 below this implies

Lemma 39. *The inclusion $S \to S'$ is universally power-surjective.*

5.3

We briefly return to power surjectivity for a general commutative ring $\mathbf{k}$.

Definition 40. Let t be a positive integer and let $f : Q \to R$ a $\mathbf{k}$-algebra homomorphism. We say that f is t-power-surjective if for every $x \in R$ there is a power t^n with $x^{t^n} \in f(Q)$.

Proposition 41. *Let $f : Q \to R$ be a $\mathbf{k}$-algebra homomorphism and Y a variable.*

- *If $f \otimes \mathbf{k}[Y] : Q[Y] \to R[Y]$ is power-surjective, then $Q \to R/pR$ is p-power-surjective for every prime p;*
- *Assume t is a positive integer such that $tR \subset f(Q)$. If $Q \to R/pR$ is p-power-surjective for every prime p dividing t, then f is universally power-surjective.*

Proof. First suppose $f \otimes \mathbf{k}[Y] : Q[Y] \to R[Y]$ is power-surjective. Let $x \in R/pR$. We have to show that x^{p^n} lifts to Q for some n. As $R[Y] \to (R/pR)[Y]$ is surjective, the composite $Q[Y] \to (R/pR)[Y]$ is also power-surjective. Choose n prime to p and m so that $(x+Y)^{np^m}$ lifts to $Q[Y]$. Rewrite $(x+Y)^{np^m}$ as $(x^{p^m}+Y^{p^m})^n$ and note that the coefficient nx^{p^m} of $Y^{(n-1)p^m}$ must lift to Q. Now use that n is invertible in $\mathbf{k}/p\mathbf{k}$.
Next suppose $tR \subset f(Q)$ and $Q \to R/pR$ is p-power-surjective for every prime p dividing t. Let C be a $\mathbf{k}$-algebra. We have to show that $f \otimes C : Q \otimes C \to R \otimes C$ is power-surjective. Since $f \otimes C : Q \otimes C \to R \otimes C$ satisfies all the conditions that $f : Q \to R$ does, we may as well simplify notation and suppress C. For $x \in R$ we have to show that some power lifts to Q. By taking repeated powers we can get x in $f(Q) + pR$ for every prime p dividing t. So if $p_1,\dots,p_m$ are the primes dividing t, we can arrange that x lies in the intersection of the $f(Q) + p_iR$, which is $f(Q) + p_1 \cdots p_m R$. Now by taking repeated $p_1 \cdots p_m$-th powers, one pushes it into $f(Q) + (p_1 \cdots p_m)^n R$ for any positive n, eventually into $f(Q) + tR \subseteq f(Q)$. □

5.4

We come back to the $\mathbf{k}$-algebra A, and consider the inclusion $\operatorname{gr} A \hookrightarrow \operatorname{hull}_\nabla(\operatorname{gr} A)$, as in Theorem 29.

Notations 42. Let λ be a dominant weight and let $b \in A^{U^+}$ be a weight vector of weight λ. Then we define $\psi_b : S'(\lambda) \otimes \mathbf{k} \to \mathrm{hull}_\nabla(\mathrm{gr}\, A)$ as the algebra map induced by the B-algebra map $S'(\lambda) \otimes \mathbf{k} \to A^{U^+}$ which sends the generator (choose one) of the λ weight space of ∇_λ to b.

Lemma 43. *For each c in the image of ψ_b, there is a positive integer s so that $c^s \in \mathrm{gr}\, A$.*

Proof. The composite of $S \otimes \mathbf{k} \to S' \otimes \mathbf{k}$ with ψ_b factors through $\mathrm{gr}\, A$, so this follows from Lemma 39. □

Proof of Theorem 29. For every $b \in \mathrm{hull}_\nabla(\mathrm{gr}\, A)$, there are $b_1,\ldots,\, b_s$ of respective weights $\lambda_1,\ldots,\, \lambda_s$ so that b lies in the image of $\psi_{b_1} \otimes \cdots \otimes \psi_{b_s}$. As $\bigotimes_{i=1}^s S(\lambda_i) \to \bigotimes_{i=1}^s S'(\lambda_i)$ is universally power-surjective by lemma 39, lemma 43 easily extends to tensor products. □

Lemma 44. *Suppose* $\mathbf{k}$ *is Noetherian. If* $\mathrm{hull}_\nabla(\mathrm{gr}\, A)$ *is a finitely generated* $\mathbf{k}$*-algebra, so is* $\mathrm{gr}\, A$.

Proof. Indeed, $\mathrm{hull}_\nabla(\mathrm{gr}\, A)$ is integral over $\mathrm{gr}\, A$ by Theorem 29. Then it is integral over a finitely generated subalgebra of $\mathrm{gr}\, A$, and it is a Noetherian module over that subalgebra. □

Lemma 45. *If* $\mathrm{gr}\, A$ *is finitely generated as a* $\mathbf{k}$*-algebra, then so is* A.

Proof. Say $j_1, \ldots, j_n$ are nonnegative integers and $a_i \in A_{\leq j_i}$ are such that the classes $a_i + A_{<j_i} \in \mathrm{gr}_{j_i} A$ generate $\mathrm{gr}\, A$. Then the a_i generate A. □

Lemma 46. *Suppose* $\mathbf{k}$ *is Noetherian. If* A^U *is a finitely generated* $\mathbf{k}$*-algebra, so is* A.

Proof. Combine Lemmas 24, 44, 45. □

Lemma 47. *Let* P *be a standard parabolic subgroup. Suppose* $\mathbf{k}$ *is Noetherian. Then* A *is a finitely generated* $\mathbf{k}$*-algebra if and only if* $A^{R_u(P)}$ *is one.*

Proof. Let V be the intersection of U with the semisimple part of the standard Levi subgroup of P. Then $U = VR_u(P)$ and $A^U = (A^{R_u(P)})^V$. Suppose that A is a finitely generated $\mathbf{k}$-algebra. Then $A^U = (A^{R_u(P)})^V$ is one also by Lemma 25, and so is $A^{R_u(P)}$ by Lemma 46 (applied with a different group and a different algebra).
Conversely, if $A^{R_u(P)}$ is a finitely generated $\mathbf{k}$-algebra, Lemma 25 (with that same group and algebra) implies that $A^U = (A^{R_u(P)})^V$ is finitely generated, and thus A is as well, by Lemma 46. □

Proof of Theorem 30. Combine Lemmas 47, 25, 24, 44, 45. □

Proof of Theorem 32. Let $\mathbf{k}$ be Noetherian and let A be a finitely generated $\mathbf{k}$-algebra. By Theorem 30, the $\mathbf{k}$-algebra $\mathrm{hull}_\nabla(\mathrm{gr}\, A)$ is finitely generated. So we may choose $b_1,\dots,b_s$, so that $\psi_{b_1} \otimes \cdots \otimes \psi_{b_s}$ has image $\mathrm{hull}_\nabla(\mathrm{gr}\, A)$. By extending Lemma 38 to tensor products, we can argue as in the proof of Lemma 43 and Theorem 29, and see that there is a positive integer n so that $n\, \mathrm{hull}_\nabla(\mathrm{gr}\, A) \subseteq \mathrm{gr}\, A$. Now, $\mathrm{hull}_\nabla(\mathrm{gr}\, A) \otimes \mathbf{k}[G/U]$ is acyclic by Proposition 28, and thus its summand $\mathrm{hull}_\nabla(\mathrm{gr}\, A)$ is acyclic as well. It follows that $\mathrm{H}^i(G, \mathrm{gr}\, A)$ is a quotient of $\mathrm{H}^{i-1}(G, \mathrm{hull}_\nabla(\mathrm{gr}\, A)/\, \mathrm{gr}\, A)$, which is annihilated by n. □

Proof of Theorem 33. Take n as in Theorem 32, and use that localization is exact. □

Proof of Theorem 35. It suffices to show that the composite:

$$\mathrm{gr}\, A \to \mathrm{gr}(A/pA) \to \mathrm{hull}_\nabla(\mathrm{gr}(A/pA))$$

is power-surjective. It coincides with the composite

$$\mathrm{gr}\, A \to \mathrm{hull}_\nabla(\mathrm{gr}(A)) \to \mathrm{hull}_\nabla(\mathrm{gr}(A/pA)).$$

Now $A^{U^+} \to (A/pA)^{U^+}$ is p-power-surjective by a combination of Theorem 12, Proposition 6, Proposition 41 and the transfer principle [9, Ch. Two] as used in 25. After inducing up, $\mathrm{hull}_\nabla(\mathrm{gr}(A)) \to \mathrm{hull}_\nabla(\mathrm{gr}(A/pA))$ is still p-power-surjective, indeed the same p-power is sufficient. And $\mathrm{gr}\, A \to \mathrm{hull}_\nabla(\mathrm{gr}(A))$ is power-surjective by Theorem 29. □

6 Finiteness properties of cohomology algebras

In this section we study finiteness properties of $\mathrm{H}^*(G, A)$, primarily when the base ring $\mathbf{k}$ is $\mathbb{Z}$. We shall always assume that the commutative algebra A is finitely generated over the ring $\mathbf{k}$, with rational action of a Chevalley group scheme G. Further, M will be a noetherian A-module with compatible G-action. Torsion will refer to torsion as an abelian group, not as an A-module. We say that V has bounded torsion if there is a positive integer that annihilates the torsion subgroup V_{tors}.

Lemma 48. *A noetherian module over a graded commutative ring has bounded torsion.* □

Recall that we call a homomorphism $f : R \to S$ of graded commutative algebras *noetherian* if f makes S into a noetherian left R-module. Recall that CFG refers to cohomological finite generation. The main result of this section is the following.

Theorem 49 (Provisional CFG). *Suppose* $\mathbf{k} = \mathbb{Z}$.

- *Every* $\mathrm{H}^m(G, M)$ *is a noetherian* A^G*-module.*

- *If* $\mathrm{H}^*(G,A)$ *is a finitely generated algebra, then* $\mathrm{H}^*(G,M)$ *is a noetherian* $\mathrm{H}^*(G,A)$*-module.*
- $\mathrm{H}^*(G,\mathrm{gr}\,A)$ *is a finitely generated algebra.*
- *If* $\mathrm{H}^*(G,A)$ *has bounded torsion, then the reduction* $\mathrm{H}^{\mathrm{even}}(G,A) \to \mathrm{H}^{\mathrm{even}}(G,A/pA)$ *is power-surjective for every prime number* p*.*
- *If* $\mathrm{H}^{\mathrm{even}}(G,A) \to \mathrm{H}^{\mathrm{even}}(G,A/pA)$ *is noetherian for every prime number* p*, then* $\mathrm{H}^*(G,A)$ *is a finitely generated algebra.*

Remark 50. Note that the first statement would fail badly, by [11, I 4.12], if one replaced G with the additive group scheme $\mathbb{G}_a$. This may explain why our proof is far from elementary.

We hope to show in the future that $\mathrm{H}^{\mathrm{even}}(G,A) \to \mathrm{H}^{\mathrm{even}}(G,A/pA)$ is noetherian for every prime number p. The theorem suggests to ask:

Problem. *Let* $\mathbf{k}$ *be a Noetherian ring and let* G *be a Chevalley group scheme over* $\mathbf{k}$*. Let* A*,* Q *be finitely generated commutative* $\mathbf{k}$*-algebras on which* G *acts rationally through algebra automorphisms. Let* $f : A \to Q$ *be a power-surjective equivariant homomorphism. Is* $\mathrm{H}^*(G,A) \to \mathrm{H}^*(G,Q)$ *power-surjective?*

We will need the recent theorem of Touzé [22, Thm 1.1], see also [22, Thm 1.5],

Theorem 51 (CFG over a field). *If* $\mathbf{k}$ *is a field, then* $\mathrm{H}^*(G,A)$ *is a finitely generated* $\mathbf{k}$*-algebra and* $\mathrm{H}^*(G,M)$ *is a noetherian* $\mathrm{H}^*(G,A)$*-module.*

Remark 52. If $\mathbf{k}$ is a commutative ring and V is a $G_{\mathbf{k}}$-module, then the comultiplication $V \to V \otimes_{\mathbf{k}} \mathbf{k}[G]$ gives rise to a comultiplication $V \to V \otimes_{\mathbb{Z}} \mathbb{Z}[G]$ through the identification $V \otimes_{\mathbf{k}} \mathbf{k}[G] = V \otimes_{\mathbb{Z}} \mathbb{Z}[G]$. So one may view V as a $G_{\mathbb{Z}}$-module. Further $\mathrm{H}^*(G_{\mathbf{k}},V)$ is the same as $\mathrm{H}^*(G_{\mathbb{Z}},V)$, because the Hochschild complexes are the same. So if $\mathbf{k}$ is finitely generated over a field F, then the conclusions of the (CFG) theorem still hold, because $\mathrm{H}^*(G,A) = \mathrm{H}^*(G_F,A)$. We leave it to the reader to try a limit argument to deal with the case where $\mathbf{k}$ is essentially of finite type over a field.

First let the ring $\mathbf{k}$ be noetherian. We are going to imitate arguments of Benson–Habegger [3]. We thank Dave Benson for the reference.

Lemma 53. *Let* $m > 1$*,* $n > 1$*. The reduction* $\mathrm{H}^{\mathrm{even}}(G,A/mnA) \to \mathrm{H}^{\mathrm{even}}(G,A/nA)$ *is power-surjective.*

Proof. We may assume m is prime. By the Chinese Remainder Theorem we may then also assume that n is a power of that same prime. (If n is prime to m the Lemma is clear.) Let $x \in \mathrm{H}^{\mathrm{even}}(G,A/nA)$. We show that some power x^{m^r} of x lifts. Arguing as in the proof of Proposition 41 we may assume x is homogeneous. Let I be the kernel of $A/mnA \to A/nA$. Note that m annihilates I, hence also $\mathrm{H}^*(G,I)$. Further I is an A/nA-module and the connecting homomorphism $\partial : \mathrm{H}^i(G,A/nA) \to \mathrm{H}^{i+1}(G,I)$ satisfies the Leibniz rule. So $\partial(x^m) = mx^{m-1}\partial(x) = 0$ and x^m lifts. $\square$

Proposition 54. *If* $\mathrm{H}^*(G,A)$ *has bounded torsion, then* $\mathrm{H}^{\mathrm{even}}(G,A) \to \mathrm{H}^{\mathrm{even}}(G,A/pA)$ *is power-surjective for every prime number* p.

Proof. Assume $\mathrm{H}^*(G,A)$ has bounded torsion. Write $\mathrm{H}^{\mathrm{pos}}$ for $\bigoplus_{i>0}\mathrm{H}^i$. Let p be a prime number. Choose a positive multiple n of p so that $n\mathrm{H}^{\mathrm{pos}}(G,A)=0$ and $nA_{\mathrm{tors}}=0$. We have an exact sequence

$$\cdots \to \mathrm{H}^i(G,A_{\mathrm{tors}}) \to \mathrm{H}^i(G,A) \to \mathrm{H}^i(G,A/A_{\mathrm{tors}}) \to \cdots .$$

Multiplication by n^2 is zero on $\mathrm{H}^{\mathrm{pos}}(G,A/A_{\mathrm{tors}})$, so $\mathrm{H}^{\mathrm{pos}}(G,A/A_{\mathrm{tors}}) \hookrightarrow \mathrm{H}^{\mathrm{pos}}(G,A/n^2A+A_{\mathrm{tors}})$. We have exact sequences

$$0 \to A/A_{\mathrm{tors}} \xrightarrow{\times n^2} A \to A/n^2A \to 0$$

and

$$0 \to A/n^2A + A_{\mathrm{tors}} \xrightarrow{\times n^2} A/n^4A \to A/n^2A \to 0.$$

Consider the diagram

$$\begin{tikzcd}
\mathrm{H}^{2i}(G,A) \arrow[r] \arrow[d] & \mathrm{H}^{2i}(G,A/n^2A) \arrow[r, "\partial_1"] \arrow[d, equal] & \mathrm{H}^{2i+1}(G,A/A_{\mathrm{tors}}) \arrow[d, hook] \\
\mathrm{H}^{2i}(G,A/n^4A) \arrow[r] & \mathrm{H}^{2i}(G,A/n^2A) \arrow[r, "\partial_2"] & \mathrm{H}^{2i+1}(G,A/n^2A+A_{\mathrm{tors}})
\end{tikzcd}$$

If $x \in \mathrm{H}^{2j}(G,A/n^2A)$, put $i = jn^2$. The image $n^2x^{n^2-1}\partial_2(x)$ in $\mathrm{H}^{2i+1}(G,A/n^2A + A_{\mathrm{tors}})$ of x^{n^2} vanishes, hence $\partial_1(x^{n^2})$ vanishes in $\mathrm{H}^{2i+1}(G,A/A_{\mathrm{tors}})$, and x^{n^2} lifts to $\mathrm{H}^{2i}(G,A)$. As $\mathrm{H}^{\mathrm{even}}(G,A/n^2A) \to \mathrm{H}^{\mathrm{even}}(G,A/pA)$ is power-surjective by Lemma 53, we conclude that for every homogeneous $y \in \mathrm{H}^{\mathrm{even}}(G,A/pA)$ some power lifts all the way to $\mathrm{H}^{\mathrm{even}}(G,A)$. We want to show more, namely that $\mathrm{H}^{\mathrm{even}}(G,A) \to \mathrm{H}^{\mathrm{even}}(G,A/pA)$ is universally power-surjective. By Proposition 41 we need to show that the power of y may be taken of the form y^{p^r}. Localize with respect to the multiplicative system $S = (1+p\mathbb{Z})$ in $\mathbb{Z}$. The p-primary torsion is not affected and all the other torsion disappears, so n may be taken a power of p. The proofs then produce that some y^{p^r} lifts to $\mathrm{H}^{\mathrm{even}}(G,S^{-1}A)$. Now just remove the denominator, which acts trivially on y. $\square$

We now restrict to the case $\mathbf{k} = \mathbb{Z}$. (More generally, one could take for $\mathbf{k}$ a noetherian ring so that for every prime number p the ring $\mathbf{k}/p\mathbf{k}$ is essentially of finite type over a field.)

Proposition 55. *Suppose* $\mathbf{k} = \mathbb{Z}$. *If* $\mathrm{H}^*(G,A)$ *has bounded torsion, then* $\mathrm{H}^*(G,A)$ *is a finitely generated algebra.*

Proof. By Theorem 33 we may choose a prime number p and concentrate on the p-primary part. Say by tensoring $\mathbb{Z}$ and A with $\mathbb{Z}_{(p)}$. So now $\mathrm{H}^{\mathrm{pos}}(G,A)$

is p-primary torsion and A is a $\mathbb{Z}_{(p)}$-algebra. We know that $\mathrm{H}^{\mathrm{even}}(G,A) \to \mathrm{H}^{\mathrm{even}}(G,A/pA)$ is power-surjective. By power surjectivity and the (CFG) Theorem 51, we choose an A^G-subalgebra C of $\mathrm{H}^*(G,A)$, generated by finitely many homogeneous elements, so that $C \to \mathrm{H}^*(G,A/pA)$ is noetherian. Again by the (CFG) Theorem 51 it follows that $\mathrm{H}^*(G,A/pA) \to \mathrm{H}^*(G,A/pA+A_{\mathrm{tors}})$ is noetherian, so that $C \to \mathrm{H}^*(G,A/pA+A_{\mathrm{tors}})$ is also noetherian.
Let N be the image of $\mathrm{H}^{\mathrm{pos}}(G,A/A_{\mathrm{tors}})$ in $\mathrm{H}^{\mathrm{pos}}(G,A/pA+A_{\mathrm{tors}}))$. As a C-module, it is isomorphic to $\mathrm{H}^{\mathrm{pos}}(G,A/A_{\mathrm{tors}})/p\mathrm{H}^{\mathrm{pos}}(G,A/A_{\mathrm{tors}})$. Choose homogeneous $v_i \in \mathrm{H}^{\mathrm{pos}}(G,A/A_{\mathrm{tors}})$ so that their images generate N. Say V is the C-span of the v_i. We have $\mathrm{H}^{\mathrm{pos}}(G,A/A_{\mathrm{tors}})+V \subseteq p\mathrm{H}^{\mathrm{pos}}(G,A/A_{\mathrm{tors}})+V$. Iterating this we get $\mathrm{H}^{\mathrm{pos}}(G,A/A_{\mathrm{tors}})+V \subseteq p^r\mathrm{H}^{\mathrm{pos}}(G,A/A_{\mathrm{tors}})+V$ for any $r>0$. But $\mathrm{H}^*(G,A)$ and $\mathrm{H}^*(G,A_{\mathrm{tors}})$ have bounded torsion, so $\mathrm{H}^*(G,A/A_{\mathrm{tors}})$ also has bounded torsion. It follows that $\mathrm{H}^{\mathrm{pos}}(G,A/A_{\mathrm{tors}})=V$. We conclude that $\mathrm{H}^{\mathrm{pos}}(G,A/A_{\mathrm{tors}})$ is a noetherian C-module.
Now let us look at $\mathrm{H}^{\mathrm{pos}}(G,A_{\mathrm{tors}})$. Filter $A_{\mathrm{tors}} \supseteq pA_{\mathrm{tors}} \supseteq p^2A_{\mathrm{tors}} \supseteq \cdots \supseteq 0$. By the (CFG) theorem $\mathrm{H}^{\mathrm{pos}}(G,p^iA_{\mathrm{tors}}/p^{i+1}A_{\mathrm{tors}})$ is a noetherian $\mathrm{H}^*(G,A/pA)$-module, hence a noetherian C-module. So $\mathrm{H}^{\mathrm{pos}}(G,A_{\mathrm{tors}})$ is also a noetherian C-module and thus $\mathrm{H}^*(G,A)$ is one. It follows that $\mathrm{H}^*(G,A)$ is a finitely generated A^G-algebra. And A^G itself is finitely generated by Theorem 3. □

Proposition 56. *Let* $\mathbf{k}=\mathbb{Z}$. *Then* $\mathrm{H}^*(G,\operatorname{gr} A)$ *is a finitely generated algebra.*

Proof. By Theorem 32 the algebra $\mathrm{H}^*(G,\operatorname{gr} A)$ has bounded torsion, so Proposition 55 applies. □

Proposition 57. *Let* $\mathbf{k}=\mathbb{Z}$. *Then* $\mathrm{H}^m(G,M)$ *is a noetherian* A^G*-module.*

Proof. Form the 'semi-direct product ring' $A \ltimes M$ whose underlying G-module is $A \oplus M$, with product given by $(a_1,m_1)(a_2,m_2)=(a_1a_2,a_1m_2+a_2m_1)$. It suffices to show that $\mathrm{H}^m(G,A\ltimes M)$ is a noetherian $\mathrm{H}^0(G,A\ltimes M)$-module. In other words, we may forget M and just ask if $\mathrm{H}^m(G,A)$ is a noetherian A^G-module. Now $\mathrm{H}^*(G,\operatorname{gr} A)$ is a finitely generated algebra and $\mathrm{H}^0(G,\operatorname{gr} A)=\operatorname{gr}_0 A$, so in the spectral sequence

$$E(A): E_1^{ij}=\mathrm{H}^{i+j}(G,\operatorname{gr}_{-i} A) \Rightarrow \mathrm{H}^{i+j}(G,A)$$

the $\bigoplus_{i+j=t} E_1^{ij}$ are noetherian A^G-modules for each t. So for fixed t there are only finitely many nonzero $E_1^{i,t-i}$ and the result follows. □

Proposition 58. *Let* $\mathbf{k}=\mathbb{Z}$. *If* $\mathrm{H}^{\mathrm{even}}(G,A)\to\mathrm{H}^{\mathrm{even}}(G,A/pA)$ *is noetherian for every prime number* p*, then* $\mathrm{H}^*(G,A)$ *is a finitely generated algebra.*

Proof. We argue as in the proof of Proposition 55. We may no longer know that $\mathrm{H}^*(G,A)$ has bounded torsion, but for every $m>0$ we know that $\mathrm{H}^m(G,A/A_{\mathrm{tors}})$ is a noetherian A^G-module, hence has bounded torsion. Instead of $\mathrm{H}^{\mathrm{pos}}(G,A/A_{\mathrm{tors}})+V \subseteq p\mathrm{H}^{\mathrm{pos}}(G,A/A_{\mathrm{tors}})+V$, we use $\mathrm{H}^m(G,A/A_{\mathrm{tors}})+V \subseteq p\mathrm{H}^m(G,A/A_{\mathrm{tors}})+V$. We find that $\mathrm{H}^m(G,A/A_{\mathrm{tors}})\subseteq V$ for all $m>0$ and thus $\mathrm{H}^{\mathrm{pos}}(G,A/A_{\mathrm{tors}})=V$ again. Finish as before. □

Corollary 59. *Let* $\mathbf{k} = \mathbb{Z}$. *If* $\mathrm{H}^{\mathrm{even}}(G, A) \to \mathrm{H}^{\mathrm{even}}(G, A/pA)$ *is power-surjective for every prime number* p*, then* $\mathrm{H}^*(G, A)$ *is a finitely generated algebra.*

Proposition 60. *Let* $\mathbf{k} = \mathbb{Z}$. *If* $\mathrm{H}^*(G, A)$ *is a finitely generated algebra, then* $\mathrm{H}^*(G, M)$ *is a noetherian* $\mathrm{H}^*(G, A)$*-module.*

Proof. Let $\mathrm{H}^*(G, A)$ be a finitely generated algebra. So it has bounded torsion and $\mathrm{H}^{\mathrm{even}}(G, A) \to \mathrm{H}^{\mathrm{even}}(G, A/pA)$ is power-surjective for every prime number p. We argue again as in the proof of Proposition 55.
By Theorem 33, applied to $A \ltimes M$, we may choose a prime number p and concentrate on the p-primary part, so $\mathrm{H}^{\mathrm{pos}}(G, M)$ is p-primary torsion and A is a $\mathbb{Z}_{(p)}$-algebra. Write $C = \mathrm{H}^*(G, A)$. By power surjectivity and the (CFG) Theorem 51, $C \to \mathrm{H}^*(G, A/pA)$ is noetherian. Again by the (CFG) Theorem 51 it follows that $\mathrm{H}^*(G, M/pM + M_{\mathrm{tors}})$ is a noetherian $\mathrm{H}^*(G, A/pA)$-module, hence a noetherian C-module.
Let N be the image of $\mathrm{H}^{\mathrm{pos}}(G, M/M_{\mathrm{tors}})$ in $\mathrm{H}^{\mathrm{pos}}(G, M/pM + M_{\mathrm{tors}})$). As a C-module, it is isomorphic to $\mathrm{H}^{\mathrm{pos}}(G, M/M_{\mathrm{tors}})/p\mathrm{H}^{\mathrm{pos}}(G, M/M_{\mathrm{tors}})$. Choose homogeneous $v_i \in \mathrm{H}^{\mathrm{pos}}(G, M/M_{\mathrm{tors}})$ so that their images generate N. Say V is the C-span of the v_i. We have $\mathrm{H}^m(G, M/M_{\mathrm{tors}}) + V \subseteq p\mathrm{H}^m(G, M/M_{\mathrm{tors}}) + V$ for $m > 0$. Iterating this we get $\mathrm{H}^m(G, M/M_{\mathrm{tors}}) + V \subseteq p^r\mathrm{H}^m(G, M/M_{\mathrm{tors}}) + V$ for any $r > 0$, $m > 0$. But $\mathrm{H}^m(G, M/M_{\mathrm{tors}})$ is a noetherian A^G-module, hence has bounded torsion. It follows that $\mathrm{H}^m(G, M/M_{\mathrm{tors}}) \subseteq V$ for all $m > 0$, and $\mathrm{H}^{\mathrm{pos}}(G, M/M_{\mathrm{tors}}) = V$. So $\mathrm{H}^{\mathrm{pos}}(G, M/M_{\mathrm{tors}})$ is a noetherian C-module.
Now let us look at $\mathrm{H}^{\mathrm{pos}}(G, M_{\mathrm{tors}})$. Filter $M_{\mathrm{tors}} \supseteq pM_{\mathrm{tors}} \supseteq p^2M_{\mathrm{tors}} \supseteq \cdots \supseteq 0$. By the (CFG) theorem $\mathrm{H}^{\mathrm{pos}}(G, p^iM_{\mathrm{tors}}/p^{i+1}M_{\mathrm{tors}})$ is a noetherian $\mathrm{H}^*(G, A/pA)$-module, hence a noetherian C-module. So $\mathrm{H}^{\mathrm{pos}}(G, M_{\mathrm{tors}})$ is also a noetherian C-module and thus $\mathrm{H}^*(G, M)$ is one. □

Theorem 49 has been proven.

References

[1] Henning Haahr Andersen, *Cohomology of line bundles on G/B.* Annales scientifiques de l'École Normale Supérieure Sér. 4, 12, no. 1 (1979), 85–100.

[2] Henning Haahr Andersen and Upendra Kulkarni, *Sum formulas for reductive algebraic groups.* Adv. Math. 217, no. 1 (2008), 419–447.

[3] David J. Benson and Nathan Habegger, *Varieties for modules and a problem of Steenrod.* Journal of Pure and Applied Algebra 44 (1987), 13–34.

[4] Nicolas Bourbaki, Éléments de mathématique. Algèbre. Chapitre 10. Algèbre homologique. Masson, Paris, 1980. vii+216 pp.

[5] David Cox, John Little, and Donal O'Shea, Ideals, Varieties and Algorithms. Undergraduate texts in Mathematics, Springer-Verlag, New York, 1992

[6] Edward Cline, Brian Parshall, Leonard Scott, and Wilberd van der Kallen, *Rational and generic cohomology.* Invent. Math. 39 (1977), 143–163.

[7] Michel Demazure and Pierre Gabriel, Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs. Masson & Cie, Éditeur, Paris; North-Holland Publishing Co., Amsterdam, 1970. xxvi+700 pp.

[8] Eric Friedlander and Andrei Suslin, *Cohomology of finite group scheme over a field.* Invent. Math 127 (1997) 209–270.

[9] Frank D. Grosshans, Algebraic Homogeneous Spaces and Invariant Theory. Lecture Notes in Mathematics, 1673. Springer-Verlag, Berlin, 1997.

[10] Frank D. Grosshans, *Contractions of the actions of reductive algebraic groups in arbitrary characteristic.* Invent. Math. 107 (1992), 127–133.

[EGA II] Alexandre Grothendieck, Éléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné) : II. Étude globale élémentaire de quelques classes de morphismes. Publications Mathématiques de l'IHÉS 8 (1961), 5–222.

[EGA III] Alexandre Grothendieck, Éléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné) : III. Étude cohomologique des faisceaux cohérents, Première partie. Publications Mathématiques de l'IHÉS 11 (1961), 5–167.

[11] Jens Carsten Jantzen, Representations of Algebraic Groups. Second edition. Mathematical Surveys and Monographs, 107. American Mathematical Society, Providence, RI, 2003.

[12] Wilberd van der Kallen, Lectures on Frobenius splittings and B-modules. Notes by S. P. Inamdar. Tata Institute of Fundamental Research, Bombay, and Springer-Verlag, Berlin, 1993.

[13] Wilberd van der Kallen, *Cohomology with Grosshans graded coefficients.* In: Invariant Theory in All Characteristics. Edited by H. E. A. Eddy Campbell and David L. Wehlau. CRM Proceedings and Lecture Notes, Volume 35 (2004), 127–138. Amer. Math. Soc., Providence, RI.

[14] Wilberd van der Kallen, *Reductive group with finitely generated cohomology algebras.* In: Algebraic Groups and Homogeneous Spaces, Mumbai 2004. Edited by Vikram B. Mehta. Tata Inst. Fund. Res. Stud. Math. (2007), 301–314. Tata Inst. Fund. Res., Mumbai.

[15] Olivier Mathieu, *Construction d'un groupe de Kac–Moody et applications.* Compos. Math. 69 (1989), 37–60.

[16] Olivier Mathieu, *Filtrations of G-modules.* Ann. Sci. École Norm. Sup. 23 (1990), 625–644.

[GIT] David Mumford, Geometric Invariant Theory. Ergebnisse der Mathematik und ihrer Grenzgebiete, Neue Folge, Band 34. Springer-Verlag, Berlin-New York 1965 vi+145 pp.

[17] Nagata Masayoshi, *On the 14th problem of Hilbert.* Am. J. Math. 81 (1959), 766–772.

[18] Emmy Noether, *Der Endlichkeitssatz der Invarianten endlicher linearer Gruppen der Charakteristik p.* Nachr. Ges. Wiss. Göttingen (1926) 28–35

[19] Conjeevaram S. Seshadri, *Geometric reductivity over arbitrary base.* Advances in Math. 26, no. 3, (1977) 225–274

[20] Tonny Albert Springer, Invariant Theory. Lecture Notes in Mathematics, Vol. 585. Springer-Verlag, Berlin-New York, 1977

[21] Robert W. Thomason, *Equivariant resolution, linearization, and Hilbert's fourteenth problem over arbitrary base schemes.* Adv. in Math. 65, no. 1 (1987), 16–34.

[22] Antoine Touzé and Wilberd van der Kallen, *Bifunctor cohomology and Cohomological finite generation for reductive groups*, Duke Math. J. 151, no. 2 (2010), 251–278.

Vincent Franjou
Mathématiques
UFR Sciences & Techniques
BP 92208
F-44322 Nantes Cedex 3
Vincent.Lastname@univ-nantes.fr

Wilberd van der Kallen
Mathematisch Instituut
Universiteit Utrecht
P.O. Box 80.010
NL-3508 TA Utrecht
Initial.vanderKallen@uu.nl

Generalized Support Varieties for Finite Group Schemes

to Andrei Suslin, with great admiration

Eric M. Friedlander[1] and Julia Pevtsova[2]

Received: Sptember 21, 2009
Revised: April 27, 2010

Abstract. We construct two families of refinements of the (projectivized) support variety of a finite dimensional module M for a finite group scheme G. For an arbitrary finite group scheme, we associate a family of *non-maximal rank varieties* $\Gamma^j(G)_M$, $1 \leq j \leq p-1$, to a kG-module M. For G infinitesimal, we construct a finer family of locally closed subvarieties $V^{\underline{a}}(G)_M$ of the variety of one parameter subgroups of G for any partition $\underline{a}$ of $\dim M$. For an arbitrary finite group scheme G, a kG-module M of constant rank, and a cohomology class ζ in $\mathrm{H}^1(G,M)$ we introduce the *zero locus* $Z(\zeta) \subset \Pi(G)$. We show that $Z(\zeta)$ is a closed subvariety, and relate it to the non-maximal rank varieties. We also extend the construction of $Z(\zeta)$ to an arbitrary extension class $\zeta \in \mathrm{Ext}^n_G(M,N)$ whenever M and N are kG-modules of constant Jordan type.

2010 Mathematics Subject Classification: 16G10, 20C20, 20G10

0. Introduction

In the remarkable papers [21], D. Quillen identified the spectrum of the (even dimensional) cohomology of a finite group $\mathrm{Spec}\,\mathrm{H}^\bullet(G,k)$ where k is some field of characteristic p dividing the order of the group. The variety $\mathrm{Spec}\,\mathrm{H}^\bullet(G,k)$ is the "control space" for certain geometric invariants of finite dimensional kG-modules. These invariants, *cohomological support varieties* and *rank varieties*, were initially introduced and studied in [1] and [6]. Over the last twenty five years, many authors have been investigating these varieties inside $\mathrm{Spec}\,\mathrm{H}^\bullet(G,k)$ in order to provide insights into the structure, behavior, and properties of kG-modules. The initial theory for finite groups has been extended to a much more general family of finite group schemes, starting with the work of [13] for p-restricted Lie algebras. The resulting theory of support varieties for modules for finite group schemes satisfies all of the axioms of a "support data" of tensor

[1]partially supported by the NSF # DMS 0909314
[2]partially supported by the NSF # DMS 0800950

triangulated categories as defined in [2]. Thus, for example, this theory provides a classification of tensor–ideal, thick subcategories of the stable module category of a finite group scheme G.

In this present paper, we embark on a different perspective of geometric invariants for kG-modules for a finite group scheme G. We introduce a new family of invariants, "generalized support varieties", which stratify the support variety of a finite dimensional kG-module M. The construction comes from considering ranks of nilpotent operators on M which leads to an alternative name *non-maximal rank varieties.* As finer invariants, they capture more structure of a module M and can distinguish between modules with the same support varieties. In particular, the generalized support varieties are always proper subvarieties of the control space $\operatorname{Spec} \mathrm{H}^\bullet(G, k)$ whereas the support variety often coincides with the entire control space. On the other hand, they necessarily lack certain good behavior with respect to tensor products and distinguished triangles in the stable module category of kG. However, as we shall try to convince the reader, these varieties provide interesting and useful tools in the further study of the representation theory of finite groups and their generalizations.

Since the module category of a finite group scheme G is wild except for very special G, our goals are necessarily more modest than the classification of all (finite dimensional) kG-modules. Two general themes that we follow when introducing our new varieties associated to representations are the formulation of invariants which distinguish various known classes of modules and the construction of modules with specified invariants.

In Section 1, we summarize some of our earlier work, and that of others, concerning support varieties of kG-modules. We emphasize the formulation of support varieties in terms of π-points, since the fundamental structure underlying our new invariants is the scheme $\Pi(G)$ of equivalence classes of π-points. Also in this section, we recall maximal Jordan types of kG-modules and the non-maximal subvariety $\Gamma(G)_M \subset M$ refining the support variety $\Pi(G)_M$ for a finite dimensional kG-module M.

If G is an infinitesimal group scheme, one formulation of support varieties is in terms of the affine scheme $V(G)$ of infinitesimal subgroups of G. For any Jordan type $\underline{a} = \sum_{i=1}^p a_i[i]$ and any finite dimensional kG-module M (with G infinitesimal), we associate in Section 2 subvarieties $V^{\leq \underline{a}}(G)_M$ and $V^{\underline{a}}(G)_M$ of $V(G)$. Determination of these refined support varieties is enabled by earlier computations of the global p-nilpotent operator $\Theta_G : M \otimes k[V(G)] \to M \otimes k[V(G)]$ which was introduced and studied in [17].

We require a refinement of one of the main theorems of [18] recalled as Theorem 1.5. Section 3 outlines the original proof due to A. Suslin and the authors, and points out the minor modifications required to establish the fact that whether or not a kG-module has maximal j-rank at a π-point depends only upon the equivalence class of that π-point (Theorem 3.6). This is the key result needed to establish that the non-maximal rank varieties are well–defined for all finite group schemes.

In Section 4, we consider closed subvarieties $\Gamma^j(G)_M \subset \Pi(G)$ for any finite group scheme, finite dimensional kG-module M, and integer $j, 1 \leq j < p$, *the non-maximal rank varieties*. We establish some properties of these varieties and work out a few examples to suggest how these invariants can distinguish certain non-isomorphic kG-modules.

In the concluding Section 5, we employ π-points to associate a closed subvariety $Z(\zeta) \subset \Pi(G)$ to a cohomology class $\zeta \in \mathrm{H}^1(G, M)$ provided that M is a kG-module of constant rank. One of the key properties of $Z(\zeta)$ is that $Z(\zeta) = \emptyset$ if and only if the extension $0 \to M \to E_\zeta \to k \to 0$ satisfies the condition that E_ζ is also a kG-module of constant rank. We show that $Z(\zeta)$ is often homeomorphic to $\Gamma^1(G)_{E_\zeta}$ which allows us to conclude that $Z(\zeta)$ is closed. Taking M to be an odd degree Heller shift of the trivial module k, we recover the familiar zero locus of a class in $\mathrm{H}^{2n}(G, k)$ in the special case $M = k$. Finally, we generalize this construction to extension classes $\xi \in \mathrm{Ext}^n_G(M, N)$ for kG-modules M and N of constant Jordan type and any $n \geq 0$.

We abuse terminology in this paper by referring to a (Zariski) closed subset of an affine or projective variety as a subvariety. Should one wish, one could always impose the reduced scheme structure on such "subvarieties".

We would like to thank Jon Carlson for pointing out to us that maximal ranks do not behave well under tensor product, Rolf Farnsteiner for his insights into components of the Auslander-Reiten quiver, and the referee for several useful comments. The second author gratefully acknowledges the support of MSRI during her postdoctoral appointment there.

1. Recollection of Π-point schemes and support varieties

Throughout, k will denote an arbitrary field of characteristic $p > 0$. Unless explicit mention is made to the contrary, G will denote a finite group scheme over k with finite dimensional coordinate algebra $k[G]$. We denote by kG the Hopf algebra dual to $k[G]$, and refer to kG as the group algebra of G. Thus, (left) kG-modules are naturally equivalent to (left) $k[G]$-comodules, which are equivalent to (left) rational G-modules (see [20, ch.1]). If M is a kG-module and K/k is a field extension, then we denote by M_K the KG-module obtained by base change.

We shall identify $\mathrm{H}^*(G, k)$ with $\mathrm{H}^*(kG, k)$.

Definition 1.1. ([16]) The Π-*point scheme* of a finite group scheme G is the k-scheme of finite type whose points are equivalence classes of π-points of G and whose scheme structure is defined in terms of the category of kG-modules. In more detail,

(1) A π-point of G is a (left) flat map of K-algebras $\alpha_K : K[t]/t^p \to KG$ for some field extension K/k with the property that there exists a unipotent abelian subgroup scheme $i : C_K \subset G_K$ defined over K such that α_K factors through $i_* : KC_K \to KG_K = KG$.

(2) If $\alpha_K : K[t]/t^p \to KG$, $\beta_L : L[t]/t^p \to LG$ are two π-points of G, then α_K is said to be a *specialization* of β_L, provided that for any finite dimensional kG-module M, $\alpha_K^*(M_K)$ being free as $K[t]/t^p$-module implies that $\beta_L^*(M_L)$ is free as $L[t]/t^p$-module.

(3) Two π-points $\alpha_K : K[t]/t^p \to KG$, $\beta_L : L[t]/t^p \to LG$ are said to be *equivalent*, written $\alpha_K \sim \beta_L$, if they satisfy the following condition for all finite dimensional kG-modules M: $\alpha_K^*(M_K)$ is free as $K[t]/t^p$-module if and only if $\beta_L^*(M_L)$ is free as $L[t]/t^p$-module.

(4) A subset $V \subset \Pi(G)$ is closed if and only if there exists a finite dimensional kG-module M such that V equals

$$\Pi(G)_M = \{[\alpha_K] \mid \alpha_K^*(M_K) \text{ is not free as } K[t]/t^p - \text{module}\}$$

The closed subset $\Pi(G)_M \subset \Pi(G)$ is called the Π-*support of* M.

(5) The topological space $\Pi(G)$ of equivalence classes of π-points can be endowed with a scheme structure based on representation theoretic properties of G (see [16, §7]).

We denote by

$$\mathrm{H}^\bullet(G,k) = \begin{cases} \mathrm{H}^*(G,k), & \text{if } p = 2, \\ \mathrm{H}^{\mathrm{ev}}(G,k) & \text{if } p > 2. \end{cases}$$

The *cohomological support variety* $|G|_M$ of a kG-module M is the closed subspace of $\operatorname{Spec} \mathrm{H}^\bullet(G,k)$ defined as the variety of the ideal $\operatorname{Ann}_{\mathrm{H}^\bullet(G,k)} \operatorname{Ext}_G^*(M,M) \subset \mathrm{H}^\bullet(G,k)$.

Theorem 1.2. [16, 7.5] *Let G be a finite group scheme, and M be a finite dimensional kG-module. Denote by* $\operatorname{Proj} \mathrm{H}^\bullet(G,k)$ *the projective k-scheme associated to the commutative, graded k-algebra* $\mathrm{H}^\bullet(G,k)$. *Then there is an isomorphism of k-schemes*

$$\Phi_G : \operatorname{Proj} \mathrm{H}^\bullet(G,k) \simeq \Pi(G)$$

which restricts to a homeomorphism of closed subspaces

$$\operatorname{Proj}(|G|_M) \simeq \Pi(G)_M$$

for all finite dimensional kG-modules M.

We (implicitly) identify $\operatorname{Proj} \mathrm{H}^\bullet(G,k)$ with $\Pi(G)$ via this isomorphism.

We consider the *stable module category* $\operatorname{stmod} kG$. Recall that the Heller shift $\Omega(M)$ of M is the kernel of the minimal projective cover $P(M) \twoheadrightarrow M$, and the inverse Heller shift $\Omega^{-1}(M)$ is the cokernel of the embedding of M into its injective hull, $M \hookrightarrow I(M)$.

The objects of $\operatorname{stmod} kG$ are finite dimensional kG-modules. The morphisms are equivalence classes where two morphisms are equivalent if they differ by a morphism which factors through a projective module,

$$\operatorname{Hom}_{\operatorname{stmod} kG}(M,N) = \operatorname{Hom}_{kG}(M,N)/\operatorname{PHom}_{kG}(M,N).$$

The stable module category has a tensor triangulated structure: the triangles are induced by exact sequences, the shift operator is given by the inverse Heller operator Ω^{-1}, and the tensor product is the standard tensor product in the category of kG-modules. Two kG-modules M, N are stably isomorphic if and only if they are isomorphic as kG-modules up to a projective direct summand. The association $M \mapsto \Pi(G)_M$ fits the abstractly defined "theory of supports" for the stable module category of G (as defined in [2]). Some of the basic properties of this theory are summarized in the next theorem (see [16]).

Theorem 1.3. *Let G be a finite group scheme and let M, N be finite dimensional kG-modules.*

(1) $\Pi(G)_M = \emptyset$ *if and only if M is projective as a kG-module.*
(2) $\Pi(G)_{M\oplus N} = \Pi(G)_M \cup \Pi(G)_N$.
(3) $\Pi(G)_{M\otimes N} = \Pi(G)_M \cap \Pi(G)_N$.
(4) $\Pi(G)_M = \Pi(G)_{\Omega M}$.
(5) *If $M \to N \to Q \to \Omega^{-1}M$ is an exact triangle in the stable module category* $\operatorname{stmod}(kG)$ *then* $\Pi(G)_N \subset \Pi(G)_M \cup \Pi(G)_Q$.
(6) *If p does not divide the dimension of M, then* $\Pi(G)_M = \Pi(G)$.

The last property of Theorem 1.3 indicates that $M \mapsto \Pi(G)_M$ is a somewhat crude invariant.

We next recall the use of Jordan types in order to refine this theory. The isomorphism type of a finite dimensional $k[t]/t^p$-module M is said to be the Jordan type of M. We denote the Jordan type of M by $\mathrm{JType}(M)$, and write $\mathrm{JType}(M) = \sum_{i=1}^p a_i[i]$; in other words, as a $k[t]/t^p$-module $M \simeq \bigoplus_{i=1}^p([i])^{\oplus a_i}$ where $[i] = k[t]/t^i$. Thus, we may (and will) view a Jordan type $\mathrm{JType}(M)$ as a partition of $m = \dim M$ into subsets each of which has cardinality $\le p$.

We shall compare Jordan types using the *dominance order*. Let $\underline{n} = [n_k \ge \ldots \ge n_2 \ge n_1]$, $\underline{m} = [m_k \ge \ldots \ge m_2 \ge m_1]$ be two partitions of N. Then $\underline{n}$ *dominates* $\underline{m}$, written $\underline{n} \ge \underline{m}$, iff

$$\sum_{i=j}^{k} n_i \ \ge\ \sum_{i=j}^{k} m_i. \tag{1.3.1}$$

for all j, $1 \le j \le k$. For $k[t]/t^p$-modules M, N, we say that $\mathrm{JType}(M) \ge \mathrm{JType}(N)$ if the partition corresponding to $\mathrm{JType}(M)$ dominates the partition corresponding to $\mathrm{JType}(N)$. The dominance order on Jordan types can be reformulated in the following way.

Lemma 1.4. *Let M, N be $k[t]/t^p$-modules of dimension m. Then* $\mathrm{JType}(M) \ge \mathrm{JType}(N)$ *if and only if*

$$\mathrm{rk}(t^j, M) \ge \mathrm{rk}(t^j, N)$$

for all j, $1 \le j < p$, where $\mathrm{rk}(t^j, M)$ denotes the rank of the operator t^j on M.

Proof. If $\mathrm{JType}(M) = \sum\limits_{i=1}^{p} a_i[i]$, then

$$\mathrm{rk}(t^j, M) = \sum_{i=j+1}^{p} a_i(i-j). \tag{1.4.1}$$

The statement now follows from [10, 6.2.2]. □

The following theorem plays a key role in our formulation of geometric invariants for a kG-module M that are finer than the Π-support $\Pi(G)_M$. In Section 3, we outline the proof of this theorem in order to prove the related, but sharper, Theorem 3.6. We say that a π-point α_K has maximal Jordan type for a kG-module M if there does not exist a π-point β_L such that $\mathrm{JType}(\alpha_K^*(M_K)) < \mathrm{JType}(\beta_L^*(M_L))$.

Theorem 1.5. [18, 4.10] *Let G be a finite group scheme over k and M a finite dimensional kG-module. Let $\alpha_K : K[t]/t^p \to KG$ be a π-point of G which has maximal Jordan type for M. Then for any π-point $\beta_L : L[t]/t^p \to LG$ which specializes to α_K, the Jordan type of $\alpha_K^*(M_K)$ equals the Jordan type of $\beta_L^*(M_L)$; in particular, if $\alpha_K \sim \beta_L$, then the Jordan type of $\alpha_K^*(M_K)$ equals the Jordan type of $\beta_L^*(M_L)$.*

The following class of kG-modules was introduced in [8] and further studied in [7], [9], [4], [5].

Definition 1.6. A finite dimensional kG-module M is said to be of *constant Jordan type* if the Jordan type of $\alpha_K^*(M_K)$ is the same for every π-point α_K of G. By Theorem 1.5, M has constant Jordan type $\underline{a}$ if and only if for each point of $\Pi(G)$ there is some representative α_K of that point with $\mathrm{JType}(\alpha_K^*(M)) = \underline{a}$.

Theorem 1.5 justifies the following definition (see [18, 5.1]).

Definition 1.7. ([18, 5.1]) Let M be a finite dimensional representation of a finite group scheme G. We define $\Gamma(G)_M \subset \Pi(G)$ to be the subset of equivalence classes of π-points $\alpha_K : K[t]/t^p \to KG$ such that $\mathrm{JType}(\alpha_K^*(M_K))$ is not maximal among Jordan types $\mathrm{JType}(\beta_L^*(M_L))$ where β_L runs over all π-points of G.

To conclude this summary, we recall certain properties of the association $M \mapsto \Gamma(G)_M$.

Proposition 1.8. *Let G be a finite group scheme and let M, N be finite dimensional kG-modules. Then $\Gamma(G)_M \subset \Pi(G)$ is a closed subvariety satisfying the following properties:*

1. *If M and N are stably isomorphic, then $\Gamma(G)_M = \Gamma(G)_N$.*
2. *$\Gamma(G)_M \subset \Pi(G)_M$ with equality if and only if $\Pi(G)_M \neq \Pi(G)$.*
3. *$\Gamma(G)_M$ is empty if and only if M has constant Jordan type.*
4. *If M has constant Jordan type, then $\Gamma(G)_{M\oplus N} = \Gamma(G)_N$.*
5. *If $\Pi(G)$ is irreducible, then N has constant non-projective Jordan type if and only if $\Gamma(G)_{M\otimes N} = \Gamma(G)_M$ for any kG-module M.*

(6) *If* $\Pi(G)$ *is irreducible, then*

$$\Gamma(G)_{M\otimes N} \ = \ (\Gamma(G)_M \cup \Gamma(G)_N) \cap (\Pi(G)_M \cap \Pi(G)_N).$$

Proof. If M and N are stably isomorphic then $M = N \oplus P$ or $N = M \oplus P$ with P projective. Since projective modules have constant Jordan type, (1) becomes a special case of (4). The fact that $\Gamma(G)_M \subset \Pi(G)$ is closed is proved in [18, 5.2]. Properties (2) and (3) follow essentially from definitions. Property (4) follows from the additivity of the dominance order. Properties (5) and (6) are the statements of [8, 4.9] and [8, 4.7] respectively. □

2. Generalized support varieties for infinitesimal group schemes

Before considering refinements of $\Gamma(G)_M \subset \Pi(G)$ in Section 3 for a general finite group scheme G, we specialize in this section to infinitesimal group schemes and work with the affine variety $V(G)$. First, we recall some definitions and several fundamental results from [23], [24].

A finite group scheme is called *infinitesimal* if its coordinate algebra $k[G]$ is local. Important examples of infinitesimal group schemes are Frobenius kernels of algebraic groups (see [20]). An infinitesimal group scheme is said to have height less or equal to r if for any x in $\mathrm{Rad}(k[G])$, $x^{p^r} = 0$.

Let $\mathbb{G}_a$ be the additive group, and $\mathbb{G}_{a(r)}$ be the r-th Frobenius kernel of $\mathbb{G}_a$. A *one-parameter subgroup* of height r of G over a commutative k-algebra A is a map of group schemes over A of the form $\mu : \mathbb{G}_{a(r),A} \to G_A$. Here, $\mathbb{G}_{a(r),A}$, G_A are group schemes over A defined as the base changes from k to A of $\mathbb{G}_{a(r)}$, G. Let $k[\mathbb{G}_{a(r)}] = k[T]/T^{p^r}$, and $k\mathbb{G}_{a(r)} = k[u_0, \dots, u_{r-1}]/(u_0^p, \dots, u_{r-1}^p)$, indexed so that the Frobenius map $F : \mathbb{G}_{a(r)} \to \mathbb{G}_{a(r)}$ satisfies $F_*(u_i) = u_{i-1}, i > 0; F_*(u_0) = 0$. We define

$$\epsilon : k[u]/u^p \ \to \ k\mathbb{G}_{a(r)} = k[u_0, \dots, u_{r-1}]/(u_0^p, \dots, u_{r-1}^p) \tag{2.0.1}$$

to be the map sending u to u_{r-1}. Thus, ϵ is a map of group algebras but not of Hopf algebras in general. In fact, the map ϵ is induced by a group scheme homomorphism if and only if $r = 1$ in which case ϵ is an isomorphism.

Theorem 2.1. [23] *Let G be an infinitesimal group scheme of height $\leq r$. Then there is an affine group scheme $V(G)$ which represents the functor sending a commutative k-algebra A to the set* $\mathrm{Hom}_{\mathrm{gr.sch}/A}(\mathbb{G}_{a(r),A}, G_A)$.

Thus, a point $v \in V(G)$ naturally corresponds to a 1-parameter subgroup

$$\mu_v : \mathbb{G}_{a(r),k(v)} \longrightarrow G_{k(v)}$$

where $k(v)$ is the residue field of v.

Theorem 2.2. [24] *(1). The closed subspaces of $V(G)$ are the subsets of the form*

$$V(G)_M = \{v \in V(G) \,|\, \epsilon^* \mu_v^*(M_{k(v)}) \text{ is not free as a module over } k(v)[u]/u^p\}$$

for some finite dimensional kG-module M.

(2). There is a natural p-isogeny $V(G) \longrightarrow \operatorname{Spec} \mathrm{H}^\bullet(G,k)$ *which restricts to a homeomorphism* $V(G)_M \simeq |G|_M$ *for any finite dimensional kG-module M.*

Theorem 1.2 implies that the spaces $\Pi(G)$ and $\operatorname{Proj} k[V(G)]$ are also homeomorphic (see [16] for a natural direct relationship between $\Pi(G)$ and $V(G)$ for an infinitesimal group scheme).
Let $\mu_{v*} : k(v)\mathbb{G}_{a(r)} \to k(v)G$ be the map on group algebras induced by the one-parameter subgroup $\mu_v : \mathbb{G}_{a(r)} \to G$. We denote by θ_v the nilpotent element of $k(v)G$ which is the image u under the composition

$$k(v)[u]/u^p \xrightarrow{\epsilon} k(v)[u_0,\ldots,u_{r-1}]/(u_0^p,\ldots,u_{r-1}^p) \xrightarrow{\mu_{v*}} k(v)G\ .$$

So, $\theta_v = \mu_{v*}(u_{r-1}) \in k(v)G$. For a given kG-module M we also let

$$\theta_v : M_{k(v)} \to M_{k(v)}$$

denote the associated p-nilpotent endomorphism. Thus, $\mathrm{JType}(\epsilon^*\mu_v^*(M_{k(v)}))$ is the Jordan type of θ_v on $M_{k(v)}$.

Definition 2.3. Let M be a kG-module of dimension m. We define the *local Jordan type function*

$$\mathrm{JType}_M : V(G) \ \to\ \mathbb{N}^{\times p}, \tag{2.3.1}$$

by sending v to $(a_1,\ldots,a_p)$, where $(\theta_v)^*(M_{k(v)}) \simeq \sum_{i=1}^p a_i[i]$.

Definition 2.4. For a given $\underline{a} = (a_1,\ldots,a_p) \in \mathbb{N}^{\times p}$, we define

$$V^{\underline{a}}(G)_M = \{v \in V(G) \mid \mathrm{JType}_M(v) = \underline{a}\},$$

$$V^{\le \underline{a}}(G)_M = \{v \in V(G) \mid \mathrm{JType}_M(v) \le \underline{a}\}.$$

As we see in the following example, $V^{\underline{a}}(G)_M$ is a generalization of a nilpotent orbit of the adjoint representation (and $V^{\le\underline{a}}(G)_M$ is a generalization of an orbit closure).

Example 2.5. Let $G = \mathrm{GL}_{N(1)}$ and let M be the standard N-dimensional representation of GL_N. Then JType_M sends a p-nilpotent matrix X to its Jordan type as an endomorphism of M. Consequently, JType_M has image inside $\mathbb{N}^{\times p}$ consisting of those p-tuples $\underline{a} = (a_1,\ldots,a_p)$ such that $\sum_i a_i \cdot i = N$. The locally closed subvarieties $V^{\underline{a}}(G)_M \subset \mathcal{N}_p(\mathfrak{gl}_N)$ are precisely the adjoint GL_N-orbits inside the p-nilpotent cone $\mathcal{N}_p(\mathfrak{gl}_N)$ of the Lie algebra $\mathfrak{gl}_N$.

Example 2.6. Let $\zeta \in \mathrm{H}^{2i+1}(G,k)$ be a non-zero cohomology class of odd degree. Let L_ζ be the Carlson module defined as the kernel of the map $\Omega^{2i+1}(k) \to k$ corresponding to ζ (see [3, II.5.9]). The module $\Omega^{2i+1}(k)$ has constant Jordan type $m[p]+[p-1]$. Let $\underline{a} = m[p]+[p-2]$ and $\underline{b} = (m-1)[p]+2[p-1]$. Then the image of JType_{L_ζ} equals $\{\underline{a},\underline{b}\} \subset \mathbb{N}^{\times p}$. Moreover, $V^{\underline{a}}(G)_{L_\zeta}$ is open in $V(G)$, with complement $V^{\underline{b}}(G)_{L_\zeta}$.

Remark 2.7. An explicit determination of the global p-nilpotent operator $\Theta_M : M \otimes k[V(G)] \to M \otimes k[V(G)]$ of [17, 2.4] immediately determines the local Jordan type function JType_M. Namely, to any $v \in V(G)$ we associate the nilpotent linear operator $\theta_v : M_{k(v)} \to M_{k(v)}$ defined by $\theta_v = \Theta_M \otimes_{k(v)[V(G)]} k(v)$. The local Jordan type of M at the point v is precisely the Jordan type of the linear operator θ_v.
The reader should consult [17] for many explicit examples of kG-modules M for each of the four families of examples of infinitesimal group schemes: (i.) G of height 1, so that M is a p-restricted module for $\mathrm{Lie}(G)$; (ii.) $G = \mathbb{G}_{a(r)}$; (iii.) $\mathrm{GL}_{n(r)}$; and (iv.) $\mathrm{SL}_{2(2)}$.

We provide a few elementary properties of these refined support varieties.

Proposition 2.8. *Let M be a kG-module of dimension m and let $\underline{a} = (a_1, \ldots, a_p)$ such that $\sum_{i=1}^p a_i \cdot i = m$.*

1. *If $m = p \cdot m'$, then $V(G) \setminus V(G)_M = V^{(0,\ldots,0,m')}(G)_M$; otherwise, $V(G) = V(G)_M$.*
2. *M has constant Jordan type if and only if $V(G)_M = V^{\underline{a}}(G)_M$ for some $\underline{a} \in \mathbb{N}^{\times p}$ (in which case $\underline{a}$ is the Jordan type of M).*
3. *$V^{\leq \underline{a}}(G)_M = \{v \in V(G) \mid \mathrm{JType}_M(v) \leq \underline{a}\}$ is a closed subvariety of $V(G)$.*
4. *$V^{\underline{a}}(G)_M$ is a locally closed subvariety of $V(G)$, open in $V^{\leq \underline{a}}(G)_M$.*
5. *$V^{\leq \underline{b}}(G)_M \subseteq V^{\leq \underline{a}}(G)_M$, if $\underline{b} \leq \underline{a}$, where " $\leq$ " is the dominance order on Jordan types.*

Proof. Properties (1) and (2) follow immediately from the definitions of $V(G)_M$ and of constant Jordan type. Property (5) is immediate.
To prove (3) we utilize $\theta_v = \Theta_M \otimes_{k(v)[V(G)]} k(v) : M_{k(v)} \to M_{k(v)}$ described in Remark 2.7. Applying Nakayama's Lemma as in [17, 4.11] to $\mathrm{Ker}\{\Theta_M^j\}$, $1 \leq j < p$, we conclude that $\mathrm{rk}(\theta_v^j, M), 1 \leq j \leq p-1$, is lower semi-continuous. Consequently, (1.3.1) and Lemma 1.4 imply that $V^{\leq \underline{a}}(G)_M$ is closed.
Property (4) follows from the observation that $V^{\underline{a}}(G)_M$ is the complement inside $V^{\leq \underline{a}}(G)_M$ of the finite union $V^{<\underline{a}}(G)_M = \cup_{\underline{a}' < \underline{a}} V^{\leq \underline{a}'}$, which is closed by (3). □

It is often convenient to consider the *stable Jordan type* of a $k[t]/t^p$-module M: if $a_1[1] + \ldots + a_p[p]$ is the Jordan type of M, then the stable Jordan type of M is $a_1[1] + \ldots + a_{p-1}[p-1]$ (equivalently, the isomorphism class of M in the stable module category $\mathrm{stmod}\, k[u]/u^p$). We define the stable local Jordan type function

$$\underline{\mathrm{JType}}_M : V(G) \to \mathbb{N}^{\times p-1}, \quad v \mapsto (a_1, \ldots, a_{p-1})$$

by sending v to the stable Jordan type of $\theta_v^*(M_{k(v)})$.

The following proposition relates the Jordan type function for a module M and its Heller twist.

Proposition 2.9. *For a stable Jordan type* $\underline{a} = \sum_{i=1}^{p-1} a_i[i]$*, denote by* $\underline{a}^\perp$ *the "flip" of* $\underline{a}$*,*

$$\underline{a}^\perp = \sum_{i=1}^{p-1} a_{p-i}[i].$$

Then

$$\underline{\mathrm{JType}}_{\Omega(M)}(v) = \underline{\mathrm{JType}}_M(v)^\perp, \quad v \in V(G).$$

Proof. For any $v \in V(G)$, $\mu_v^* : (k(v)G - \mathrm{mod}) \to (k(v)\mathbb{G}_{a(r)} - \mathrm{mod})$ is exact. Moreover, $\epsilon^* : (k\mathbb{G}_{a(r)} - \mathrm{mod}) \to (k[u]/u^p - \mathrm{mod})$ is also exact. Consequently, the existence of a short exact sequence of the form $0 \to \Omega M \to P \to M \to 0$ with $\mathrm{JType}_P(v) = N[p]$ for some N implies the assertion. □

Example 2.10. Let g be a restricted Lie algebra with restricted enveloping algebra $u(g)$ (which is isomorphic to the group algebra of an infinitesimal group scheme of height 1). Let ζ be an even dimensional cohomology class in $\mathrm{H}^\bullet(u(g), k)$, and L_ζ be the Carlson module defined by ζ. Then L_ζ has two local Jordan types: it is generically projective (that is, the local Jordan type is $m[p]$ on a dense open set), and has the type $r[p] + [p-1] + [1]$ on the hypersurface $\langle \zeta = 0 \rangle$ in $\mathrm{Spec}\, \mathrm{H}^\bullet(u(g), k)$. Let M be a g-module of constant Jordan type $\underline{a}$. Then the module $L_\zeta \otimes M$ has two local Jordan types: it is generically projective, and has the "stably palindromic" type $\underline{a} + \underline{a}^\perp + [\mathrm{proj}]$ on $\langle \zeta = 0 \rangle$.

We conclude this section with the following cautionary example which shows why the construction of our local Jordan type function does not apply to kG-modules M for finite groups G.

Example 2.11. ([18, 2.3]) Let $E = \mathbb{Z}/p \times \mathbb{Z}/p$, and write $kE = k[x, y]/(x^p, y^p)$. Let $M = kE/(x - y^2)$. Then

$$\alpha : k[t]/t^p \to kE, \quad t \mapsto x$$

and

$$\alpha' : k[t]/t^p \to kE, \quad t \mapsto x - y^2$$

are equivalent as π-points of E. However, the Jordan type of $\alpha^*(M)$ equals $[\frac{p-1}{2}] + [\frac{p+1}{2}]$, whereas the Jordan type of $\alpha'^*(M)$ is $p[1]$.

3. Maximal j–rank for arbitrary finite group schemes

We begin with the following definition.

Definition 3.1. Let G be a finite group scheme, $\alpha_K : K[t]/t^p \to KG$ be a π-point of G, and j a positive integer with $1 \leq j < p$. Then α_K is said to be of maximal j-rank for some finite-dimensional kG-module M provided that the rank of $\alpha_K(t^j) = \alpha_K(t)^j : M_K \to M_K$ is greater or equal to the rank of $\beta_L(t^j) : M_L \to M_L$ for any π-point $\beta_L : L[t]/t^p \to LG$.

The purpose of this section is to establish in Theorem 3.6 that maximality of j-rank at α_K implies maximal j-rank at β_L for any $\beta_L \sim \alpha_K$. The proof consists of repeating almost verbatim the proof by A. Suslin and the authors in [18] of Theorem 1.5, so that we merely indicate here the explicit places at which the proof of Theorem 1.5 should be modified in order to prove Theorem 3.6.

The following theorem provides the key step.

Theorem 3.2. *Let k be an infinite field, M be a finite-dimensional k-vector space, and $\alpha, \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ be a family of commuting nilpotent k-linear endomorphisms of M. Let $1 \leq j \leq p-1$, and assume that*

$$\operatorname{rk} \alpha^j \geq \operatorname{rk}(\alpha + \lambda_1\alpha_1 + \ldots + \lambda_n\alpha_n)^j$$

for any field extension K/k and any n-tuple $(\lambda_1, \ldots, \lambda_n) \in K^n$. Then

$$\operatorname{rk} \alpha^j = \operatorname{rk}(\alpha + \alpha_1\beta_1 + \ldots + \alpha_n\beta_n)^j.$$

In particular, if $p(x, x_1, \ldots, x_n)$ is any polynomial without constant or linear term then

$$\operatorname{rk} \alpha^j = \operatorname{rk}(\alpha + p(\alpha, \alpha_1, \ldots, \alpha_n))^j.$$

Proof. For $j = 1$, this is [18, 1.9]. For general j, the statement follows by applying Corollary 1.11 of [18]. □

For any π-point $\alpha_K : K[t]/t^p \to KG$, we denote by $\operatorname{rk}(\alpha_K(t^j), M_K)$ the rank of the K-linear endomorphism $\alpha_K(t^j) : M_K \to M_K$.

In the next 3 propositions, we consider the special cases in which G is an elementary abelian p-group, an abelian finite group scheme, and an infinitesimal finite group scheme. In this manner, we follow the strategy of the proof of Theorem 1.5.

Proposition 3.3. *Let E be an elementary abelian p-group of rank r, let M be a finite dimensional kE-module, and let α_K be a π-point of E which is of maximal j-rank for M. Then for any $\beta_L \sim \alpha_K$,*

$$\operatorname{rk}(\alpha_K(t^j), M_K) = \operatorname{rk}(\beta_L(t^j), M_L).$$

Proof. The proof of [18, 2.7] applies verbatim provided one replaces references to [18, 1.12] by references to [18, 1.9]. □

Proposition 3.4. *Let C be an abelian finite group scheme over k, let M be a finite dimensional kC-module, and let α_K be a π-point of C which is of maximal j-rank for M. Then for any $\beta_L \sim \alpha_K$,*

$$\operatorname{rk}(\alpha_K(t^j), M_K) = \operatorname{rk}(\beta_L(t^j), M_L).$$

Proof. The proof of [18, 2.9] applies verbatim provided one replaces references to [18, 2.7] by references to Proposition 3.3 and references to [18, 1.12] by references to Theorem 3.2. □

Proposition 3.5. *Let G be an infinitesimal group scheme over k and let M be a finite dimensional kG-module. Let $\beta_L : L[t]/t^p \to LG$ be a π-point of G with the property that the j-rank of $\beta_L^*(M_L)$ is maximal for M. Then for any π-point $\alpha_K : K[t]/t^p \to KG$ which specializes to β_L,*

$$\mathrm{rk}(\alpha_K(t^j), M_K) = \mathrm{rk}(\beta_L(t^j), M_L).$$

Proof. The proof of [18, 3.5] applies verbatim provided one replaces references to [18, 2.9] by references to Proposition 3.4. □

We now state and prove the assertion that maximality of j-rank at α_K implies maximality of j-rank at β_L for any $\beta_L \sim \alpha_K$. This statement for all $j, 1 \leq j < p$, implies the maximality of Jordan type as asserted in Theorem 1.5.

Theorem 3.6. *Let G be a finite group scheme over k and let M be a finite dimensional kG-module. Let $\alpha_K : K[t]/t^p \to KG$ be a π-point of G which is of maximal j-rank for M. Then for any π-point $\beta_L : L[t]/t^p \to LG$ that specializes to α_K, we have*

$$\mathrm{rk}(\alpha_K(t^j), M_K) = \mathrm{rk}(\beta_L(t^j), M_L).$$

Proof. The proof of [18, 4.10] applies verbatim provided one replaces references to [18, 2.9] by references to Proposition 3.4 and references to [18, 3.5] by references to Proposition 3.5. □

We can now generalize the *modules of constant j-rank* as defined for infinitesimal group schemes in [17] to all finite group schemes.

Definition 3.7. A finite dimensional kG-module M is said to be of *constant j-rank*, $1 \leq j < p$, if for any two π-points $\alpha_K : K[t]/t^p \to KG$, $\beta_L : L[t]/t^p \to LG$, we have

$$\mathrm{rk}(\alpha_K(t^j), M_K) = \mathrm{rk}(\beta_L(t^j), M_L).$$

Remark 3.8. By Theorem 3.6, M has constant j-rank n if and only if for each point of $\Pi(G)$ there is some π-point representative α_K with $\mathrm{rk}(\alpha_K(t^j), M_K) = n$.

Evidently, a kG-module has constant Jordan type if and only if it has constant j-rank for all $j, 1 \leq j < p$ (see (1.3.1)).

We shall say that M is a *module of constant rank* if it has constant 1-rank. Every module of constant Jordan type has, by definition, constant rank. On the other hand, there are numerous examples of modules of constant rank which do not have constant Jordan type. For example, if $\zeta \in \mathrm{H}^{2i+1}(G, k)$ is non-zero and $p > 2$, then the Carlson module L_ζ is a kG-module of constant rank but not of constant Jordan type.

We finish this section with a cautionary example that illustrates that not all properties of maximal or constant Jordan type have natural analogues for maximal or constant rank. Recall that a *generic Jordan type* of a kG-module M is the Jordan type at a π-point which represents a generic point of $\Pi(G)$. By the main theorem of [18], it is well-defined. If $\Pi(G)$ is irreducible, we can therefore refer to the generic Jordan type of M. We can similarly define a *generic*

j-rank of a kG-module to be $\mathrm{rk}(\alpha_K(t^j), M_K)$ for a π-point α of G representing a generic point of $\Pi(G)$. By [18, 4.2], generic j-rank is well-defined.

Example 3.9. Throughout this example we are using the formula for the tensor product of Jordan types (see, for example, [8, Appendix]).

(1). Let $\underline{a} = \sum a_i[i], \underline{b} = \sum b_i[i]$ be two Jordan types (or partitions) such that $\sum a_i \cdot i = \sum b_i \cdot i$. In [8, 4.1] the authors showed that $\underline{a} \geq \underline{b}$ implies $\underline{a} \otimes \underline{c} \geq \underline{b} \otimes \underline{c}$ for any Jordan type $\underline{c}$. The analogous statement is not true for ranks.
Indeed, let $\underline{a} = 3[2]$, $\underline{b} = [3] + 3[1]$, and $\underline{c} = [2]$. Then

$$\mathrm{rk}\,\underline{a} = 3 > \mathrm{rk}\,\underline{b} = 2.$$

Since $\underline{a} \otimes \underline{c} = 3[3] + 3[1]$ and $\underline{b} \otimes \underline{c} = [4] + 4[2]$, we have

$$\mathrm{rk}\,\underline{a} \otimes \underline{c} = 6 < \mathrm{rk}\,\underline{b} \otimes \underline{c} = 7.$$

(2). Part (1) of this example illustrates a common failure of the upper semi-continuity property of the ranks of partitions with respect to tensor product. Since this fails for partitions, it is reasonable to expect the same property to fail for maximal ranks of modules. The following is an explicit realization by kG-modules of this failure of upper semi-continuity. This example also shows that $M \otimes N$ can fail to have maximal rank at a π-point at which both M and N have maximal rank. This should be contrasted with the situation for maximal Jordan types ([8, 4.2]).
Let $G = \mathbb{G}_{a(1)}^{\times 2}$ so that $kG \simeq k[x,y]/(x^p, y^p)$. Consider the kG-module M of Example [8, 2.4], pictured as follows:

$$\begin{tikzcd}
& \bullet \arrow[dl, "y"'] \arrow[dr, "x"] & & \bullet \arrow[dl, "y"'] \arrow[dr, "x"] & & \bullet \arrow[dl, "y"'] \arrow[dr, "x"] & & \bullet \arrow[dl, "y"'] \arrow[dr, "x"] & \\
\bullet \arrow[dr, "x"'] & & \bullet \arrow[dl, "y"] \arrow[dr, "x"'] & & \bullet \arrow[dl, "y"] \arrow[dr, "x"'] & & \bullet \arrow[dl, "y"] \arrow[dr, "x"'] & & \bullet \arrow[dl, "y"] \\
& \bullet & & \bullet & & \bullet & & \bullet &
\end{tikzcd}$$

Recall that $\Pi(G) \simeq \mathrm{Proj}\,\mathrm{H}^\bullet(G,k) \simeq \mathbb{P}^1$. A point $[\lambda_1 : \lambda_2]$ on $\mathbb{P}^1$ is represented by a π-point $\alpha : k[t]/t^p \to kG$ such that $\alpha(t) = \lambda_1 x + \lambda_2 y$.
For $p > 5$, the module M has two Jordan types: the generic type $4[3] + 1[1]$ and the singular type $3[3] + 2[2]$, which occurs at $[1 : 0]$ and $[0 : 1]$ (see [8, 2.4]). Hence, M has constant rank. We compute possible local Jordan types of $M \otimes M$ using the fact that $\mu_{v*} : k(v)[t]/(t^p) \to k(v)G$ is a map of Hopf algebras for any $v \in V(G)$:

(i) $(4[3] + 1[1])^{\otimes 2} = 16[5] + 24[3] + 17[1]$,
(ii) $(3[3] + 2[2])^{\otimes 2} = 9[5] + 12[4] + 13[3] + 12[2] + 13[1]$.

By [18, 4.4], the first type is the generic Jordan type of $M \otimes M$. Hence, the generic (and maximal) rank of $M \otimes M$ is 112. On the other hand, the rank of the second type is 110. Hence, the rank of M at the points $[1 : 0], [0 : 1]$ is maximal, but the rank of $M \otimes M$ is not.

(3). Yet another result in [8], a direct consequence of the result on the tensor products of maximal types mentioned in (2), states that a tensor product of modules of constant Jordan type is a module of constant Jordan type. This distinguishes the family of modules of constant Jordan type from the modules of constant rank, for which this property fails. Let M be the same as in (2). The calculation above shows that M is of constant rank but $M \otimes M$ is not.

We also give an example of a different nature, avoiding point by point calculations of Jordan types. This example was pointed out to us by the referee. Let M be a cyclic kG-module of dimension less than p (e.g., $M = k[x,y]/(x^2,y)$). We have a short exact sequence $0 \to \Omega M \to kG \to M \to 0$. This implies that the Jordan type of ΩM at any π-point necessarily has p blocks, and, hence, ΩM has constant rank. Since $\Omega M \otimes \Omega^{-1} k \simeq M \oplus [\mathrm{proj}]$, we conclude that the tensor product of two modules of constant rank produces a module which is not of constant rank.

4. Non-maximal rank varieties for arbitrary finite group schemes

In this section, we introduce the non-maximal rank varieties $\Gamma^j(G)_M$ for an arbitrary finite group scheme, finite dimensional kG-module M, and integer $j, 1 \leq j < p$. The non-maximal rank varieties, a type of generalized support variety defined for any finite dimensional module over any finite group scheme, are defined in terms of ranks of local p-nilpotent operators. These are well defined thanks to Theorem 3.6. After verifying a few simple properties of these varieties, we investigate various explicit examples.

Definition 4.1. Let G be a finite group scheme, and let M be a finite dimensional kG-module. Set

$$\Gamma^j(G)_M = \{[\alpha_K] \in \Pi(G) \mid \mathrm{rk}(\alpha_K(t^j), M_K) \text{ is not maximal}\},$$

the non-maximal j-rank variety of M.

Our first example demonstrates that $\{\Gamma^j(G)_M\}$ is a finer collection of geometric invariants than $\Pi(G)_M$.

Example 4.2. Let $G = \mathrm{GL}(3, \mathbb{F}_p)$ with $p > 3$. By [21], the irreducible components of $\Pi(G)$ are indexed by the conjugacy classes of maximal elementary p-subgroups of G which are represented by subgroups of the unipotent group $U(3, \mathbb{F}_p)$ of strictly upper triangular matrices. There are 3 such conjugacy classes, represented by the following subgroups:

$$\left\{\begin{pmatrix} 1 & a & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} a, b \in \mathbb{F}_p\right\}\left\{\begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} a, b \in \mathbb{F}_p\right\}\left\{\begin{pmatrix} 1 & 0 & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} a, b \in \mathbb{F}_p\right\}$$

Let M be the second symmetric power of the standard 3-dimensional (rational) representation of G. Then the generic Jordan type of M indexed by the first of these maximal elementary abelian subgroups of G is $[3] + 3[1]$, whereas

the Jordan types indexed by each of the other conjugacy classes of maximal elementary abelian p–subgroups are $[2]+4[1]$.
Thus, $\Pi(G)_M = \Pi(G)$ provides no information about M.
On the other hand, $\Gamma(G)_M = \Gamma^1(G)_M = \Gamma^2(G)_M$ equals the union of the two irreducible components of $\Pi(G)$ corresponding to the second and third maximal elementary abelian p–subgroups, whereas $\Gamma^i(G)_M = \emptyset$ for $i > 2$.

Our second example shows that $\Gamma^i(G)_M$ and $\Gamma^j(M)$ can be different, proper subsets of $\Pi(G)$.

Example 4.3. In [18, 4.13] A. Suslin and the authors constructed an example of a finite group G and a finite dimensional G-module M, such that $\Pi(G) = X \cup Y$ has two irreducible components and the generic Jordan types of M at the generic points of X and Y respectively are incomparable. Let G and M satisfy this property, and let α_K and β_L be generic π-points of X and Y respectively. If $\mathrm{JType}(\alpha_K^*(M_K))$ and $\mathrm{JType}(\beta_L^*(M_L))$ are incomparable, then Lemma 1.4 implies that there exist $i \neq j$ such that $\mathrm{rk}(\alpha_K(t^i), M_K) > \mathrm{rk}(\beta_L(t^i), M_L)$ but $\mathrm{rk}(\alpha_K(t^j), M_K) < \mathrm{rk}(\beta_L(t^j), M_L)$. Hence, $\Gamma^i(G)_M$ is a proper subvariety that contains the irreducible component Y whereas $\Gamma^j(G)_M$ is a proper subvariety that contains the irreducible component X.

Our third example is a simple computation for a general finite group scheme. It provides another possible "pattern" for the varieties $\Gamma^i(G)_M$.

Example 4.4. Let $\zeta_1 \in \mathrm{H}^{n_1}(G,k)$ be an even dimensional class, and $\zeta_2 \in \mathrm{H}^{n_2}(G,k)$ be an odd dimensional class. Consider $L_{\underline{\zeta}} = L_{\zeta_1,\zeta_2}$, the kernel of the map
$$\zeta_1 + \zeta_2 : \Omega^{n_1}k \oplus \Omega^{n_2}k \to k$$
The local Jordan type of $L_{\underline{\zeta}}$ at a π-point α is given in the following table:
$$\begin{cases} r[p]+[p-1], & \alpha^*(\zeta_1) \neq 0 \\ r[p]+[p-2]+[1], & \alpha^*(\zeta_1) = 0,\ \alpha^*(\zeta_2) \neq 0 \\ (r-1)[p]+2[p-1]+[1], & \alpha^*(\zeta_1) = \alpha^*(\zeta_2) = 0 \end{cases}$$
Hence, $\Gamma^1(G)_{L_{\underline{\zeta}}} = \ldots = \Gamma^{p-2}(G)_{L_{\underline{\zeta}}} = Z(\zeta_1)$, whereas $\Gamma^{p-1}(G)_{L_{\underline{\zeta}}} = Z(\zeta_1) \cap Z(\zeta_2)$, where $Z(\zeta_1)$ denotes the zero locus of a class $\zeta_1 \in \mathrm{H}^\bullet(G,k)$ and $Z(\zeta_2)$ for $\zeta_2 \in \mathrm{H}^{\mathrm{odd}}(G,k)$ is defined in (5.3).

We next verify a few elementary properties of $M \mapsto \Gamma^j(G)_M$. Some of them are analogous to the properties of $\Gamma(G)_M$ stated in Prop 1.8.

Proposition 4.5. *Let G be a finite group scheme and M a finite dimensional kG-module.*

1. *$\Gamma^j(G)_M$ is a proper closed subset of $\Pi(G)$ for $1 \leq j < p$.*
2. *$\Gamma^j(G)_M = \emptyset$ if and only if M has constant j-rank.*
3. *If M and N are stably isomorphic, then $\Gamma^j(G)_M = \Gamma^j(G)_N$*
4. *If M is a module of constant j-rank, then $\Gamma^j(G)_{M\oplus N} = \Gamma^j(G)_N$.*
5. *$\Gamma^j(G)_M = \Gamma^j(G)_{\Omega^2(M)}$.*

(6) $\Gamma(G)_M = \cup_{1\leq j<p}\Gamma^j(G)_M$.
(7) *If M has the Jordan type $m[p]$ at some generic π-point, then* $\Gamma^1(G)_M = \ldots = \Gamma^{p-1}(G)_M = \Pi(G)_M$.

Proof. By definition, $\Gamma^j(G)_M \subset \Pi(G)$ can never equal $\Pi(G)$, so it is a proper subvariety. Moreover, assertions (2) and (6) also immediately follow from definitions and Lemma 1.4. Assertion (4) follows from the additivity of ranks and of the functor $\alpha_K^* : KG - \mathrm{mod} \to K[t]/t^p - \mathrm{mod}$ induced by a π-point α_K. Property (3) is proved exactly as in the proof of Proposition 1.8(1).
For (5), observe that a π-point α_K induces an exact functor on the module categories and hence commutes with the Heller operator Ω. The statement now follows from the observation that for $K[t]/t^p$-modules, applying Ω^2 does not change the stable Jordan type.
To prove that $\Gamma^j(G)_M \subset \Pi(G)$ is closed as asserted in (1), we repeat the proof of [18, 5.2] establishing that $\Gamma(G)_M$ is closed. Indeed, the reduction in that proof to the special case in which G is infinitesimal applies without change. The proof in the special case of G infinitesimal uses the affine scheme of 1-parameter subgroups; this proof applies with only one minor change: the set of equations on the ranks of powers of $f_A : A[t]/t^p \to \mathrm{End}_A(M)$ (in the notation of that proof) is replaced by the set of equations on rank of only one, the j-th, power of f_A.
If M is generically projective as in (7), then $\Gamma(G)_M = \Pi(G)_M$. Let $\alpha_K \notin \Gamma(G)_M$ so that the Jordan type of $\alpha_K^*(M)$ is $m[p]$, and let $\beta_L \in \Gamma(G)_M$. Let $\sum b_i[i]$ be the Jordan type of $\beta_L^*(M_L)$. The statement follows easily from the formula (1.4.1): we have

$$\mathrm{rk}(\alpha_K(t^j), M_K) = m(p-j) > \sum_{i=j+1}^{p} b_i(i-j) = \mathrm{rk}(\beta_L(t^j), M_L),$$

where the inequality in the middle follows by downward induction on j from the assumption $mp = \dim M = \sum\limits_{i=1}^{p} b_i i$. Thus, $\Gamma^j(G)_M = \Gamma(G)_M$ for each $j, 1 \leq j < p$.

□

Example 4.6. We point out that the "natural" analog of 1.8(5) is not true for modules of constant rank. Namely, $\Gamma^1(G)_{M\otimes N}$ does not have to be equal to $\Gamma^1(G)_N$ for M of constant rank. Indeed, let M be as in Example 3.9. Then M has constant rank and $\Gamma^1(E)_M = \emptyset$. But $\Gamma^1(E)_{M\otimes M} \neq \emptyset$ since $M \otimes M$ is not a module of constant rank.

Using a recent result of R. Farnsteiner [12, 3.3.2], we verify below that the non-maximal subvarieties $\Gamma^i(G)_M \subset \Pi(G)$ of an indecomposable kG-module M do not change when we replace M by any N in the same component as M of the stable Auslander-Reiten quiver of G. This is a refinement of a result of J. Carlson and the authors [8, 8.7] which asserts that if M is an indecomposable module of constant Jordan type than any N in the same component of the stable Auslander-Reiten quiver of G as M is also of constant Jordan type.

Proposition 4.7. *Let k be an algebraically closed field, and G be a finite group scheme over k. Let $\Theta \subset \Gamma_s(G)$ be a component of the stable Auslander-Reiten quiver of G. For any two modules M, N in Θ, and any $j, 1 \leq j \leq p-1$,*

$$\Gamma^j(G)_M = \Gamma^j(G)_N$$

Proof. Recall that $\Pi(G)$ is connected. If $\dim \Pi(G) = 0$, then $\Pi(G)$ is a single point so that $\Gamma^j(G)_M$ is empty for any kG-module M.
Now, assume that $\Pi(G)$ is positive dimensional. Since k is assumed to be algebraically closed, to show that $\Gamma^j(G)_M = \Gamma^j(G)_N$, it's enough to show that their k-valued points are the same. For this reason, we shall only consider π-points defined over k.
Let M be a kG-module in the component Θ, and write the Jordan type of $\alpha^*(M)$ as $\sum_{i=1}^p \alpha_i(M)[i]$. By [12, 3.1.1], each component Θ determines non-negative integer valued functions d_i on the set of π-points (possibly different on equivalent π-points) and a positive, integer valued function f on the modules occurring in Θ such that

$$\text{(4.7.1)} \qquad \begin{cases} \alpha_i(M) = d_i(\alpha)f(M) \text{ for } 1 \leq i \leq p-1 \\ \alpha_p(M) = \frac{1}{p}(\dim M - d_p(\alpha)f(M)) \end{cases}$$

Assume $[\beta] \in \Gamma^j(G)_M$, so that there exists a π-point $\alpha : k[t]/t^p \to kG$ such that $\mathrm{rk}(\alpha^j(t), M) > \mathrm{rk}(\beta^j(t), M)$. By (1.4.1), this is equivalent to the inequality

$$\sum_{j=i+1}^{p} \alpha_i(M)(i-j) > \sum_{j=i+1}^{p} \beta_i(M)(i-j).$$

Using formula (4.7.1), we rewrite this inequality as

$$\sum_{j=i+1}^{p-1} d_i(\alpha)f(M)(i-j) + \frac{1}{p}(\dim M - d_p(\alpha)f(M))(p-j) >$$

$$\sum_{j=i+1}^{p-1} d_i(\beta)f(M)(i-j) + \frac{1}{p}(\dim M - d_p(\beta)f(M))(p-j).$$

Simplifying, we obtain
(4.7.2)

$$(\sum_{j=i+1}^{p-1} d_i(\alpha)(i-j) - \frac{p-j}{p}d_p(\alpha))f(M) > (\sum_{j=i+1}^{p-1} d_i(\beta)(i-j) - \frac{p-j}{p}d_p(\beta))f(M).$$

Now, let N be any other indecomposable kG-module in the component Θ. Multiplying the inequality (4.7.2) by the positive, rational function $f(N)/f(M)$, we obtain the same inequality as (4.7.2) with M replaced by N. Thus, $[\beta] \in \Gamma^j(G)_N$. Interchanging the roles of M and N, we conclude that $\Gamma^j(G)_M = \Gamma^j(G)_N$. $\square$

For an infinitesimal group scheme G, the closed subvarieties $\Gamma^j(G)_M \subset \Pi(G)$ admit an affine version $V^j(G) \subset V(G)$ defined as follows

Definition 4.8. Let G be an infinitesimal group scheme, M a finite dimensional kG-module, and j a positive integer, $1 \le j < p$. We define

$$V^j(G)_M = \{v \in V(G)| \ \mathrm{rk}(\theta_v^j, M_{k(v)}) \ \text{ is not maximal}\} \cup \{0\} \subset V(G).$$

(see §2 for notations). So defined, $V^j(G)_M - \{0\}$ equals $\mathrm{pr}^{-1}(\Gamma^j(G)_M)$, where $\mathrm{pr} : V(G) - \{0\} \to \Pi(G)$ is the natural (closed) projection (see [16]).

Remark 4.9. We can express $V^j(G)_M$ in terms of the locally closed subvarieties $V^{\underline{a}}(G)_M$ introduced in §2. Namely, $V^j(G)_M$ is the union of $V^{\underline{a}}(G)_M \subset V(G)$ indexed by the Jordan types $\underline{a}$ with $\sum_{i=1}^p a_i \cdot i = \dim(M)$ satisfying the condition that there exists some Jordan type $\underline{b}$ with $V^{\underline{b}}(G)_M \neq \{0\}$ and $\sum_{i>j}^p b_i(i-j) > \sum_{i>j}^p a_i(i-j)$.

Our first representative example of $V^j(G)_M$ is a continuation of (2.5).

Example 4.10. Let $G = \mathrm{GL}_{N(1)}$, let M be the standard representation of GL_N, and assume p does not divide N. Recall that $V(\mathrm{GL}_{N(1)}) \simeq \mathcal{N}_p$, where $\mathcal{N}_p$ is the p-restricted nullcone of the Lie algebra $\mathfrak{gl}_N$ ([24, §6]). The maximal Jordan type of M is $r[p] + [N - rp]$, where rp is the greatest non-negative multiple of p which is less or equal to N (see [18, 4.15]). Hence, the maximal j-rank equals $r(p-j) + (N - rp - j)$ if $N - rp > j$ and $r(p-j)$ otherwise.
For simplicity, assume k is algebraically closed so that we only need to consider k-rational points of $\mathcal{N}_p$. For any $X \in \mathcal{N}_p$, $\theta_X : M \to M$ is simply the endomorphism X itself. Consequently, if $N - rp \le j$, $V^j(G)_M \subset \mathcal{N}_p$ consists of 0 together with those non-zero p-nilpotent $N \times N$ matrices with the property that their Jordan types have strictly fewer than r blocks of size p; if $N - rp > j$, then $V^j(G)_M$ consists of 0 together with $0 \neq X \in \mathcal{N}_p$ whose Jordan type is strictly less than $r[p] + [N - rp]$.
Hence, the pattern for varieties $V^j(M)$ in this case looks like

$$\{0\} \neq V^1(G)_M = \ldots = V^n(G)_M \subset V^{n+1}(G)_M = \ldots = V^{p-1}(G)_M \subset V(G)$$

where $n = N - rp$.

Computing examples of $V^j(G)_M$ is made easier by the presence of other structure. For example, if $G = \mathcal{G}_{(r)}$, the r^{th}-Frobenius kernel of the algebraic group $\mathcal{G}$ and if the kG-module M is the restriction of a rational $\mathcal{G}$-module, then we verify in the following proposition that $V^j(G)_M$ is $\mathcal{G}$-stable, and thus a union of $\mathcal{G}$-orbits inside $V(G)$.

Lemma 4.11. *Let $\mathcal{G}$ be an algebraic group, and let G be the r^{th} Frobenius kernel of $\mathcal{G}$ for some $r \ge 1$. If M is a finite dimensional rational $\mathcal{G}$-module, then each $V^j(G)_M$, $1 \le j < p$, is a $\mathcal{G}$-stable closed subvariety of $V(G)$.*

Proof. Composition with the adjoint action of $\mathcal{G}$ on G determines an action

$$\mathcal{G} \times V(G) \ \to \ V(G).$$

Observe that for any field extension K/k and any $x \in \mathcal{G}(K)$, the pull-back of M_K via the conjugation action $\gamma_x : G_K \to G_K$ is isomorphic to M_K as a KG-module. Thus, the Jordan type of $(\mu \circ \epsilon)^*(M_K)$ equals that of $(\gamma_x \circ \mu \circ \epsilon)^*(M_K)$ for any 1-parameter subgroup $\mu : \mathbb{G}_{a(r),K} \to G_K$. □

Using Lemma 4.11, we carry out our second computation of $V^j(G)_M$ with G infinitesimal, this time for G of height 2.

Example 4.12. Let $G = \mathrm{SL}_{2(2)}$. For simplicity, assume k is algebraically closed. Recall that

$$V(G) = \{(\alpha_0, \alpha_1) \,|\, \alpha_1, \alpha_2 \in sl_2, \alpha_1^p = \alpha_2^p = [\alpha_1, \alpha_2] = 0\},$$

the variety of pairs of commuting p-nilpotent matrices ([23]). The algebraic group SL_2 acts on $V(G)$ by conjugation (on each entry).
Let $e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. An easy calculation shows that the non-trivial orbits of $V(G)$ with respect to the conjugation action are parameterized by $\mathbb{P}^1$, where $[s_0 : s_1] \in \mathbb{P}^1$ corresponds to the orbit represented by the pair (s_0e, s_1e).
Let S_λ be a simple SL_2-module of highest weight λ, $0 \le \lambda \le p^2 - 1$. Since S_λ is a rational SL_2-module, the non-maximal rank varieties $V^j(G)_{S_\lambda}$ are SL_2-stable by Proposition 4.11. Hence, to compute the non-maximal rank varieties for S_λ it suffices to compute the Jordan type of S_λ at the orbit representatives (s_0e, s_1e). By the explicit formula ([17, 2.6.5]), the Jordan type of S_λ at (s_0e, s_1e) is given by the Jordan type of the nilpotent operator $s_1e + s_0^pe^{(p)}$ (here, $e^{(p)}$ is the divided power generator of $k\,\mathrm{SL}_{2(2)}$ as described in [17, 1.4]). The non-maximal rank varieties $V^j(G)_{S_\lambda}$ depend upon which of the following three conditions λ satisfies.

(1) $\boxed{0 \le \lambda \le p-1}$. In this case, the Jordan type of $e \in k\,\mathrm{SL}_{2(2)}$ as an operator on S_λ is $[\lambda+1]$. On the other hand, the action of $e^{(p)}$ is trivial. Hence, if $j \ge \lambda + 1$, then the action $(s_1e + s_0^pe^{(p)})^j$ is trivial for any pair (s_0, s_1). For $1 \le j \le \lambda$, the j-rank is maximal (and equals $\lambda + 1 - j$) whenever $s_1 \ne 0$. We conclude that for $j > \lambda$, we have $V^j(G)_{S_\lambda} = 0$, and for $1 \le j \le \lambda$, $V^j(G)_{S_\lambda}$ is the orbit of $V(G)$ parametrized by $[1 : 0]$.

(2) $\boxed{p \le \lambda < p^2 - 1}$. Let $\lambda = \lambda_0 + p\lambda_1$. By the Steinberg tensor product theorem, we have $S_\lambda = S_{\lambda_0} \otimes S_{\lambda_1}^{(1)}$. Observe that e acts trivially on $S_{\lambda_1}^{(1)}$ and $e^{(p)}$ acts trivially on S_{λ_0}. Moreover, the Jordan type of $e^{(p)}$ as an operator on $S_{\lambda_1}^{(1)}$ is the same as the Jordan type of e as an operator on S_{λ_1}. Hence, the Jordan type of $s_1e + s_0^pe^{(p)}$ as an operator on $S_{\lambda_0} \otimes S_{\lambda_1}^{(1)}$ is $[\lambda_0 + 1] \otimes [\lambda_1 + 1]$ when $s_0s_1 \ne 0$. If $s_0 = 0$ or $s_1 = 0$ we get the types $[\lambda_0 + 1] \otimes (\mathrm{triv})$ or $(\mathrm{triv}) \otimes [\lambda_1 + 1]$ respectively.

(a) For $0 < \lambda_0, \lambda_1 < p-1$, the tensor product formula for Jordan types (see [8, Appendix]) implies that the j-rank of $[\lambda_0+1] \otimes [\lambda_1+1]$ is strictly greater than that of $[\lambda_0+1] \otimes (\mathrm{triv})$ or $(\mathrm{triv}) \otimes [\lambda_1+1]$ for

$j \leq \lambda_1 + \lambda_0$. Hence, the non-maximal j-rank variety in the case when $j \leq \lambda_1 + \lambda_0$ is a union of two orbits, parameterized by $[1:0]$ and $[0:1]$. If $j > \lambda_1 + \lambda_0$, then the non-maximal j-rank variety is trivial since the j-rank is 0 at every point.

(b) If $\lambda_0 = 0$, then $S_\lambda \simeq S_{\lambda_1}^{(1)}$. Hence, the computation for S_λ for $\lambda < p$ implies that the non-maximal j-rank variety in this case is the orbit corresponding to $[0:1]$ for $j \leq \lambda_1$ and is trivial otherwise.

(c) For $\lambda_0 = p-1$ or $\lambda_1 = p-1$, the non-maximal j-rank variety is the same as the support variety for any j, since the support variety is a proper subvariety of $V(G)$ in this case. The support varieties for these modules were computed in [24, §7] (see also [17, 1.17(4)]).

(3) $\boxed{\lambda = p^2 - 1}$. In this case, S_λ is the Steinberg module for $\mathrm{SL}_{2(2)}$. Hence, it is projective, so the non-maximal rank varieties are all trivial.

We summarize our calculations in the table below. Let $\lambda = \lambda_0 + p\lambda_1$, and $\overline{\lambda} = \lambda_0 + \lambda_1$. If $j > \overline{\lambda}$, then $V^j(G)_{S_\lambda} = 0$. For $j \leq \overline{\lambda}$, we have

$$V^j(G)_{S_\lambda} = \begin{cases} \{(\alpha_0, 0)\} \cup \{(0, \alpha_1)\} & \text{if } 0 < \lambda_0, \lambda_1 < p-1 \\ \{(\alpha_0, 0)\} & \text{if } \lambda_0 \neq 0,\ \lambda_1 = 0 \text{ or } \lambda_0 = p-1, \lambda_1 \neq p-1 \\ \{(0, \alpha_1)\} & \text{if } \lambda_0 = 0,\ \lambda_1 \neq 0 \text{ or } \lambda_0 \neq p-1, \lambda_1 = p-1 \\ 0 & \text{if } \lambda_0 = \lambda_1 = p-1. \end{cases}$$

where α_0, α_1 run over all nilpotent matrices in sl_2. In particular, for a given $\lambda = \lambda_0 + p\lambda_1$ we get the following pattern for $M = S_\lambda$:

$$V(G) \supset V^1(G)_M = \cdots = V^{\bar{\lambda}}(G)_M \supset V^{\bar{\lambda}+1}(G)_M = \cdots = V^{p-1}(G)_M = \{0\}.$$

Observe that the only simple modules of constant rank are the trivial module and the Steinberg module. An interested reader may find it instructive to compare this calculation to the calculation of support varieties for $\mathrm{SL}_{2(2)}$ ([17, 1.18(4)], see also [24, §7]).

5. Subvarieties of $\Pi(G)$ associated to individual Ext-classes

For M a kG-module of constant rank, we associate to a cohomology class ζ in $\mathrm{H}^1(G, M)$ a closed subvariety $Z(\zeta) \subset \Pi(G)$ which generalizes the construction of the zero locus $Z(\zeta) \subset \operatorname{Spec} \mathrm{H}^\bullet(G, k)$ of a homogeneous cohomology class. We show that this construction is closely related to the non-maximal rank variety, and establish some "realization" results for non-maximal varieties as an application. Unless otherwise indicated, throughout this section G will denote an arbitrary finite group scheme over k.

Lemma 5.1. *Let M be a finite dimensional kG-module, and let ζ be a cohomology class in $\mathrm{H}^1(G, M)$. Consider the corresponding extension*

$$\tilde{\zeta} : 0 \to M \to E_\zeta \to k \to 0.$$

For any π-point $\alpha_K : K[t]/t^p \to KG$, the following are equivalent:

(i) *the cohomology class $\alpha_K^*(\zeta_K) \in \mathrm{H}^1(K[t]/t^p, M_K)$ is trivial.*

(ii) $\mathrm{rk}(\alpha_K^*(t), E_\zeta) = \mathrm{rk}(\alpha_K^*(t), M)$.

(iii) $\operatorname{JType}(\alpha_K^*(E_{\zeta,K})) \ = \ \operatorname{JType}(\alpha_K^*(M_K)) + 1[1]$.

Proof. Recall that $\alpha_K^*(-)$ is exact (by definition, α_K is flat); moreover, the sequence $\alpha_K^*(\tilde{\zeta})$ splits if and only if $\alpha_K^*(\zeta) = 0$ in $\mathrm{H}^1(K[t]/t^p, K)$. Thus, it suffices to prove that a short exact sequence $0 \to M \to E \to K \to 0$ of $K[t]/t^p$-modules splits if and only if $\operatorname{rk}(t, M) = \operatorname{rk}(t, E)$ if and only if $\operatorname{JType}(E) = \operatorname{JType}(M) + 1[1]$. Let $\underline{b} = \sum_{i=1}^p b_i[i]$ be the Jordan type of E and $\underline{a} = \sum_{i=1}^p a_i[i]$ be the Jordan type of M. Then this short exact sequence splits if and only if the map $E \to k$ factors through the summand $b_1[1]$ of E which occurs if and only if $b_i = a_i, i > 1$ which is equivalent to $\operatorname{rk}(t, M) = \operatorname{rk}(t, E)$. □

Proposition 5.2. *Let M be a kG-module of constant rank, and let ζ be a cohomology class in $\mathrm{H}^1(G, M)$. Consider the corresponding extension*

$$\tilde{\zeta} : 0 \to M \to E_\zeta \to k \to 0.$$

1. *If E_ζ has constant rank equal to that of M, then $\alpha_K^*(\zeta_K) \in \mathrm{H}^1(K[t]/t^p, M)$ is trivial for every π-point $\alpha_K : K[t]/t^p \to KG$.*
2. *If E_ζ has constant rank greater than that of M, then $\alpha_K^*(\zeta_K) \in \mathrm{H}^1(K[t]/t^p, M)$ is non-trivial for every π-point $\alpha_K : K[t]/t^p \to KG$.*
3. *If E_ζ does not have constant rank, then $\alpha_K^*(\zeta)$ is trivial if and only if $[\alpha_K] \in \Gamma^1(G)_{E_\zeta} \subset \Pi(G)$.*
4. *For any two equivalent π-points α_K, β_L of G, $\alpha_K^*(\zeta_K)$ is trivial if and only if $\beta_L^*(\zeta_L)$ is trivial.*

Proof. Assertions (1) and (2) follow immediately from Lemma 5.1. Assertion (3) also follows from Lemma 5.1: if E_ζ does not have constant rank, then the complement of $\Gamma^1(G)_{E_\zeta}$ in $\Pi(G)$ consists of those equivalence classes of π-points α_K satisfying Lemma 5.1(ii.).
To prove that the vanishing of $\alpha_K^*(\zeta_K)$ depends only upon the equivalence class of α_K, we examine each of the three cases considered above. In case (1), $\alpha_K^*(\zeta_K) = 0$ for all π-points α_K: on the other hand, in case (2) $\alpha_K^*(\zeta_K) \neq 0$ for all π-points α_K. Finally, the assertion in case (3) follows immediately from Theorem 3.6. □

Proposition 5.2(4) justifies the following definition.

Definition 5.3. For M a module of constant rank, and $\zeta \in \mathrm{H}^1(G, M)$, we define

$$Z(\zeta) \ \equiv \ \{[\alpha_K] \mid \alpha_K^*(\zeta) = 0\} \ \subset \ \Pi(G). \tag{5.3.1}$$

For $\zeta \in \mathrm{H}^m(G, k)$, we define

$$Z(\zeta) \ \equiv \ \{[\alpha_K] \mid \alpha_K^*(\zeta) = 0\} \ \subset \ \Pi(G). \tag{5.3.2}$$

Since $\mathrm{H}^m(G, k) \simeq \mathrm{H}^1(G, \Omega^{1-m} k)$, the definition of (5.3.2) is a special case of that of (5.3.1). For $m = 2n$ even, $Z(\zeta)$ corresponds under the isomorphism $\Pi(G) \simeq \operatorname{Proj} \mathrm{H}^\bullet(G, k)$ with the hypersurface $\langle \zeta = 0 \rangle$ in $\operatorname{Spec} \mathrm{H}^\bullet(G, k)$.

Remark 5.4. We point out that Definition 5.3 is not as straight-forward as it might appear.

- Let $G = \mathbb{Z}/p \times \mathbb{Z}/p$ with $p > 2$, write $kG = k[x,y]/(x^p, y^p)$ and consider $M = kG/(x - y^2)$ as in Example 2.11. Consider the short exact sequence
$$0 \to \mathrm{Rad}(M) \to M \to k \to 0,$$
with associated extension class $\zeta \in \mathrm{H}^1(G, \mathrm{Rad}(M))$. Consider the equivalent π-points $\alpha, \alpha' : k[t]/t^p \to kG$ of Example 2.11. Then, $\alpha^*(\zeta) \neq 0$, yet $\alpha'^*(\zeta) = 0$. Thus, the "zero locus" of ζ is not a well defined subset of $\Pi(G)$.
- Let $\zeta \in \mathrm{H}^{2n}(G, k)$ represented by $\hat{\zeta} : \Omega^{2n}k \to k$. By definition of L_ζ, we have an extension
$$\tilde{\xi} : \ 0 \to L_\zeta \to \Omega^{2n}k \xrightarrow{\hat{\zeta}} k \to 0,$$
corresponding to a cohomology class $\xi \in \mathrm{H}^1(G, L_\zeta)$. Then for any π-point $\alpha_K : K[t]/t^p \to KG$, $\alpha_K^*(\tilde{\xi})$ splits if and only if $\alpha_K^*(L_\zeta)$ is free if and only if $[\alpha_K] \notin \Pi(G)_{L_\zeta}$ if and only of $\alpha_K^*(\zeta) \neq 0$. Thus, the zero locus of ξ equals the *complement* of the zero locus of ζ (and thus is open in $\Pi(G)$).
- For $\zeta \in \mathrm{H}^{2n+1}(G, k)$, one could define $Z(\zeta)$ as the zero locus of the Bockstein of ζ provided one is in a situation in which the Bockstein is defined and well behaved. See the discussion of the Bockstein following Example 5.6.

We recall from [7] that a short exact sequence of kG modules
$$\tilde{\xi} : \quad 0 \to M \to E \to Q \to 0$$
is said to be *locally split* if $\alpha_K^*(\tilde{\xi})$ splits for every π-point $\alpha_K : K[t]/t^p \to KG$ of G.

Proposition 5.5. *Let M be a module of constant rank, and let ζ be a cohomology class in $\mathrm{H}^1(G, M)$. Consider the corresponding extension*
$$\tilde{\zeta} : 0 \to M \to E_\zeta \to k \to 0.$$
Then
$$Z(\zeta) = \begin{cases} \Pi(G), & \text{if } \tilde{\zeta} \text{ is locally split} \\ \Gamma^1(G)_{E_\zeta}, & \text{if } \tilde{\zeta} \text{ is not locally split.} \end{cases}$$
In particular, $Z(\zeta) \subset \Pi(G)$ is closed.

Proof. Observe that $\tilde{\zeta}$ is split at $[\alpha_K]$ if and only if $\alpha_K^*(\zeta) = 0$. We first consider ζ such that E_ζ has constant rank. Then by Proposition 5.2.1, $Z(\zeta)$ equals $\Pi(G)$ if $\tilde{\zeta}$ is locally split and $Z(\zeta) = \emptyset$ by Proposition 5.2.2 if $\tilde{\zeta}$ is not locally split. Alternatively, if E_ζ does not have constant rank, then Proposition 5.2.3 gives the asserted description of $Z(\zeta)$.
Because $\Gamma^1(G)_{E_\zeta} \subset \Pi(G)$ is closed by Proposition 4.5 and of course $\Pi(G)$ is itself closed in $\Pi(G)$, we conclude that $Z(\zeta)$ is closed inside $\Pi(G)$. $\square$

We remark that $\zeta \in \mathrm{H}^1(G,M)$ can be non-zero and yet $Z(\zeta) = \emptyset$. To say $Z(\zeta) = \emptyset$ is to say that $\alpha_K^*(\zeta) = 0$ for all π-points α_K. Consider, for example, an even dimensional non-trivial cohomology class $\zeta \in \mathrm{H}^{2n}(G,k)$ which is a product of odd dimensional classes. Since the product of any two odd classes in $\mathrm{H}^*(k[t]/t^p,k)$ is zero, $\alpha_K^*(\zeta) = 0$ for all π-points α_K of G. On the other hand, ζ can be identified with a cohomology class in $\mathrm{H}^1(G,\Omega^{1-2n}(k)) \simeq \mathrm{H}^{2n}(G,k)$. Since $\Omega^{1-2n}(k)$ is a module of constant Jordan type (see [8]), the class ζ satisfies the requirements of Proposition 4.5.
A more interesting example is the following.

Example 5.6. Let G be a finite group scheme with the property that the dimension of $\Pi(G)$ is at least 1. Let $\zeta' \in \widehat{\mathrm{H}}^{-i}(G,k)$, $i > 0$, be an element in the negative Tate cohomology of G. As shown in [8, 6.3], $\alpha_K^*(\zeta') = 0$ for any π-point α_K. Then ζ' corresponds to $\zeta \in \mathrm{H}^1(G,\Omega^{i+1}(k))$ under the isomorphism $\widehat{\mathrm{H}}^{-i}(G,k) \simeq \mathrm{H}^1(G,\Omega^{i+1}(k))$; by the naturality of this isomorphism, $\alpha_K^*(\zeta) = 0 \in \widehat{\mathrm{H}}^{-i}(K[t]/t^p,K)$ for any π-point α_K.
Thus, $\zeta \neq 0$, $\tilde{\zeta}$ is locally split, and $Z(\zeta) = \emptyset$ for this choice of $\zeta \in \mathrm{H}^1(G,\Omega^{i+1}(k))$.

For any field extension K/k, let $R_K = W_2(K)$ denote the Witt vectors of length 2 for K. Assume that G over k embeds into an R_k-group scheme G_{R_k} so that $G = G_{R_k} \times_{\operatorname{Spec} R_k} \operatorname{Spec} k \subset G_{R_k}$, thereby inducing by base change $G_K \subset G_{R_K}$. Then we may define the Bockstein $\beta : \mathrm{H}^i(G_K,K) \to \mathrm{H}^{i+1}(G_K,K)$ for $i > 0$ as the connecting homomorphism for the short exact sequence of G_{R_K}-modules

$$0 \to K \to R_K \to K \to 0. \tag{5.6.1}$$

(The reader is referred to [11, 3.4] for a discussion of this Bockstein.) Since any π-point $\alpha_K : K[t]/t^p \to KG$ lifts to a map $\tilde{\alpha}_K : R_K[t]/t^p \to R_K G_{R_K}$ of R-algebras, $\alpha^* : \mathrm{H}^*(G,K) \to \mathrm{H}^*(K[t]/t^p,K)$ commutes with this Bockstein. Since $\beta : \mathrm{H}^{2d-1}(K[t]/t^p,K) \to \mathrm{H}^{2d}(K[t]/t^p,K)$ is an isomorphism, we conclude that if $x \in \mathrm{H}^{2d-1}(G,k)$, then $\alpha_K^*(x)$ vanishes if and only if $\alpha_K^*(\beta(x)) = 0$, where $\beta(x) \in \mathrm{H}^{2d}(G,k)$. Thus, for such G lifting to G_{R_k} and for $p > 2$, when considering $Z(\zeta)$ for homogeneous classes in $\mathrm{H}^*(G,k)$, it suffices to restrict attention to the subalgebra $\mathrm{H}^\bullet(G,k)$ of even dimensional classes.
As we see in the following family of examples, $\Gamma^1(G)_M$ can be an arbitrary closed subset even when the support variety of M is all of $\Pi(G)$.

Proposition 5.7. *Let G be a finite group scheme over k. Let $\zeta_i \in \mathrm{H}^{n_i+1}(G,k) \simeq \mathrm{H}^1(G,\Omega^{-n_i}(k))$, $n_i \geq 0$. Let $M = \oplus_{i=1}^r \Omega^{-n_i}(k)$, and set $\zeta = \oplus_i \zeta_i \in \mathrm{H}^1(G,M) = \oplus_i \mathrm{H}^1(G,\Omega^{-n_i}(k))$. Let*

$$0 \to M \to E_\zeta \to k \to 0$$

be the corresponding extension. Then

(1) *If $Z(\zeta) \neq \Pi(G)$, then $\Gamma^1(G)_{E_\zeta} = Z(\zeta) = Z(\zeta_1) \cap \ldots \cap Z(\zeta_r)$.*
(2) *If each n_i is even so that each $\zeta_i \in \mathrm{H}^{n_i+1}(G,k)$ has odd degree, then $\Pi(G)_{E_\zeta} = \Pi(G)$.*

Proof. (1). If $Z(\zeta) \neq \Pi(G)$, then Proposition 5.5 implies that $\Gamma^1(G)_{E_\zeta} = Z(\zeta)$. Since $\zeta = \oplus\zeta_i$, we further conclude that $Z(\zeta) = \{[\alpha_K] \mid \alpha_K^*(\zeta) = 0\} = \{[\alpha_K] \mid \alpha_K^*(\zeta_i) = 0 \text{ for all } i\} = \bigcap_i Z(\zeta_i)$. Hence, $\Gamma^1(G)_{E_\zeta} = \bigcap_i Z(\zeta_i)$.

(2). Assume now that each n_i is even so that each $\Omega^{-n_i}(k)$ has constant Jordan type of the form $m_i[p] + [1]$. Thus, the generic Jordan type of E_ζ is of the form $m[p] + [2] + (r-1)[1]$ at generic points $[\alpha_K] \in \Pi(G)$ such that $\alpha_K^*(\zeta) \neq 0$ and of the form $m[p] + (r+1)[1]$ otherwise. Therefore, $\Pi(G)_{E_\zeta} = \Pi(G)$. □

As we see below, the construction of E_ζ in Proposition 5.7 above is in fact a generalized Carlson module L_ζ (as defined in [8]) "in disguise". In the Example 5.8 we consider homogeneous classes ζ_i of even degree.

Example 5.8. Let $\zeta = (\zeta_1, \ldots, \zeta_r)$, where $\zeta_i \in \mathrm{H}^{2d_i}(G,k) \simeq \underline{\mathrm{Hom}}(\Omega^{2d_i}(k), k)$, $1 \leq i \leq r$ with $d_i \geq 0$. Let L_ζ be the kernel of the map $\zeta = \sum \zeta_i : \bigoplus \Omega^{2d_i}(k) \to k$, so that we have an exact sequence:

$$0 \longrightarrow L_\zeta \longrightarrow \bigoplus \Omega^{2d_i}(k) \xrightarrow{\zeta_1+\cdots+\zeta_r} k \longrightarrow 0$$

This short exact sequence represents an exact triangle in $\operatorname{stmod} kG$. Shifting the triangle by Ω^{-1} we obtain a triangle

$$k \longrightarrow \Omega^{-1}(L_\zeta) \longrightarrow \bigoplus \Omega^{2d_i-1}(k) \longrightarrow \Omega^{-1}(k)$$

Hence, ζ corresponds to a short exact sequence

$$0 \longrightarrow k \longrightarrow F_\zeta \longrightarrow \bigoplus \Omega^{2d_i-1}(k) \longrightarrow 0$$

with the middle term stably isomorphic to $\Omega^{-1}(L_\zeta)$. Taking the dual of this short exact sequence, we obtain the the short exact sequence which defines E_ζ in Proposition 5.7:

$$0 \longrightarrow \bigoplus \Omega^{1-2d_i} k \longrightarrow E_\zeta \longrightarrow k \longrightarrow 0\ .$$

Hence, E_ζ is stably isomorphic to $\Omega^{-1}(L_\zeta^{\#})$.

Our final result extends the construction of closed zero loci to extension classes $\xi \in \mathrm{Ext}^n_G(N,M)$ with both M, N of constant Jordan type. In other words, Proposition 5.9 introduces the (closed) support variety $Z(\xi)$ of such an extension class.

Proposition 5.9. *Let G be a finite group scheme and N, M finite dimensional kG-modules of constant Jordan type. Let $\xi \in \mathrm{Ext}^n_G(N,M) \simeq \mathrm{Ext}^1(\Omega^{n-1}(N), M)$ for some $n \neq 0$, and consider the corresponding extension*

$$\tilde{\xi}: \quad 0 \to M \to E_\xi \to \Omega^{n-1}(N) \to 0.$$

(1) If α_K, β_L are equivalent π-points of G, then $\alpha_K^*(\tilde{\xi})$ splits if and only if $\beta_L(\tilde{\xi})$ splits.
(2) If

$$Z(\xi) \;\equiv\; \{[\alpha_K] \mid \alpha_K^*(\tilde{\xi}) \text{ splits}\} \;\subset\; \Pi(G),$$

then

$$Z(\xi) = \begin{cases} \Pi(G), & \text{if } \tilde{\xi} \text{ is locally split} \\ \Gamma^1(G)_{E_\xi}, & \text{if } \tilde{\xi} \text{ is not locally split.} \end{cases}$$

Proof. There is a natural isomorphism

$$\mathrm{Ext}^1_G(\Omega^{n-1}(N), M) \;\simeq\; \mathrm{H}^1(G, (\Omega^{n-1}(N))^{\#} \otimes M)$$

sending the extension class ξ to the cohomology class $\zeta \in \mathrm{H}^1(G, (\Omega^{n-1}(N))^{\#} \otimes M)$ (where $(\Omega^{n-1}(N))^{\#}$ is the linear dual of $\Omega^{n-1}(N)$). Hence, $\alpha_K^*(\tilde{\xi})$ splits if and only $\alpha_K^*(\tilde{\zeta})$ splits for any π-point α_K of G.
By [9, 5.2], $(\Omega^{n-1}(N))^{\#}$ has constant Jordan type. Thus, by [9, 4.3], $(\Omega^{n-1}(N))^{\#} \otimes M$ also has constant Jordan type. Consequently, the assertion of the Proposition for ξ follows from Proposition 4.5 for ζ.

□

References

[1] J. Alperin, L. Evens, Representations, resolutions, and Quillen's dimension theorem, *J. Pure & Applied Algebra* 22 (1981), 1-9.

[2] P. Balmer, The spectrum of prime ideals in tensor triangulated categories *J. für die Reine und Ang. Math. (Crelle)*, 588, (2005), 149–168.

[3] D. Benson, Representations and cohomology, Volume I and II, *Cambridge University Press*, (1991).

[4] D. Benson, Modules of constant Jordan type with one non-projective block, *Algebras and Representation Theory* 13 (2010), 315-318..

[5] D. Benson, Modules of constant Jordan type and the Horrocks-Mumford bundle, preprint.

[6] J. Carlson, The varieties and the cohomology ring of a module, *J. Algebra* 85 (1983) 104-143.

[7] J. Carlson and E. Friedlander, Exact category of modules of constant Jordan type, *Manin Festschift, Progress in Mathematics*, 269, Birkhäuser-Boston (2009), 267–290.

[8] J. Carlson, E. Friedlander, J. Pevtsova, Modules of Constant Jordan type, *Journal fúr die reine und angewandte Mathematik (Crelle)* 614 (2008), 191–234.

[9] J. Carlson, E. Friedlander, A. Suslin, Modules for $\mathbb{Z}/p \times \mathbb{Z}/p$. To appear in *Commentarii Helvetica.*

[10] D. Collingwood, M. McGovern, Nilpotent Orbits in Semisimple Lie Algebras, *Van Nostrand Reinhold* (1993).

[11] L. Evens, The Cohomology of Groups, *Oxford Mathematical Monographs, The Clarendon Press*, 1991.

[12] R. Farnsteiner, *Jordan types for indecomposable modules of finite group scheme*, preprint.

[13] E. Friedlander, B. Parshall, Support varieties for restricted Lie algebras, *Invent. Math.* 86 (1986) 553-562.

[14] E. Friedlander, J. Pevtsova, Representation-theoretic support spaces for finite group schemes, *Amer. J. Math.* 127 (2005), 379-420.

[15] E. Friedlander, J. Pevtsova, Erratum: Representation-theoretic support spaces for finite group schemes, *Amer. J. Math.* 128 (2006), 1067-1068.

[16] E. Friedlander, J. Pevtsova, Π-supports for modules for finite group schemes, *Duke. Math. J.* 139 (2007), 317–368.

[17] E. Friedlander, J. Pevtsova, Constructions for infinitesimal group schemes, to appear in Trans. A.M.S.

[18] E. Friedlander, J. Pevtsova, A. Suslin, Generic and Maximal Jordan types, *Invent. Math.* 168 (2007), 485–522.

[19] E. Friedlander, A. Suslin, Cohomology of finite group scheme over a field, *Invent. Math.* 127 (1997), 235–253.

[20] J. Jantzen, Representations of Algebraic Groups, *American Mathematical Society*, 2003.

[21] D. Quillen, The spectrum of an equivariant cohomology ring, I, II, *Ann. of Math.* 94 (1971), 549-572, 573–602.

[22] A. Suslin, The detection theorem for finite group schemes, *J. Pure Appl. Algebra* 206 (2006), 189-221.

[23] A. Suslin, E. Friedlander, C. Bendel, Infinitesimal 1-parameter subgroups and cohomology, *J. Amer. Math. Soc.* 10 (1997) 693-728.

[24] A. Suslin, E. Friedlander, C. Bendel, Support varieties for infinitesimal group schemes, *J. Amer. Math. Soc.* 10 (1997) 729-759.

Eric M. Friedlander
Department of Mathematics
University of Southern California
Los Angeles, CA
USA
ericmf@usc.edu

Julia Pevtsova
Department of Mathematics
University of Washington
Seattle, WA
USA
julia@math.washington.edu

On Suslin's Singular Homology and Cohomology

Dedicated to A. A. Suslin on his 60th birthday

Thomas Geisser[1]

Received: December 10, 2009
Revised: April 4, 2010

Abstract. We study properties of Suslin homology and cohomology over non-algebraically closed base fields, and their p-part in characteristic p. In the second half we focus on finite fields, and consider finite generation questions and connections to tamely ramified class field theory.

2010 Mathematics Subject Classification: 19E15, 14F42, 14G15
Keywords and Phrases: Suslin homology, motivic homology, algebraic cycles, albanese map

1 Introduction

Suslin and Voevodsky defined Suslin homology (also called singular homology) $H_i^S(X,A)$ of a scheme of finite type over a field k with coefficients in an abelian group A as $\operatorname{Tor}_i(\operatorname{Cor}_k(\Delta^*,X),A)$. Here $\operatorname{Cor}_k(\Delta^i,X)$ is the free abelian group generated by integral subschemes Z of $\Delta^i\times X$ which are finite and surjective over Δ^i, and the differentials are given by alternating sums of pull-back maps along face maps. Suslin cohomology $H^i_S(X,A)$ is defined to be $\operatorname{Ext}^i_{\mathrm{Ab}}(\operatorname{Cor}_k(\Delta^*,X),A)$. Suslin and Voevodsky showed in [22] that over a separably closed field in which m is invertible, one has

$$H^i_S(X,\mathbb{Z}/m)\cong H^i_{\mathrm{et}}(X,\mathbb{Z}/m) \qquad (1)$$

(see [2] for the case of fields of characteristic p).
In the first half of this paper, we study both the situation that m is a power of the characteristic of k, and that k is not algebraically closed. In the second

[1]Supported in part by NSF grant No.0901021

half, we focus on finite base fields and discuss a modified version of Suslin homology, which is closely related to etale cohomology on the one hand, but is also expected to be finitely generated. Moreover, its zeroth homology is $\mathbb{Z}^{\pi_0(X)}$, and its first homology is expected to be an integral model of the abelianized tame fundamental group.

We start by discussing the p-part of Suslin homology over an algebraically closed field of characteristic p. We show that, assuming resolution of singularities, the groups $H_i^S(X,\mathbb{Z}/p^r)$ are finite abelian groups, and vanish outside the range $0\le i\le \dim X$. Thus Suslin cohomology with finite coefficients is etale cohomology away from the characteristic, but better behaved than etale cohomology at the characteristic (for example, $H^1_{\mathrm{et}}(\mathbb{A}^1,\mathbb{Z}/p)$ is not finite). Moreover, Suslin homology is a birational invariant in the following strong sense: If X has a resolution of singularities $p: X'\to X$ which is an isomorphism outside of the open subset U, then $H_i^S(U,\mathbb{Z}/p^r)\cong H_i^S(X,\mathbb{Z}/p^r)$. It was pointed out to us by N.Otsubo that this can be applied to generalize a theorem of Spiess-Szamuely [20] to include p-torsion:

Theorem 1.1 *Let X be a smooth, connected, quasi-projective variety over an algebraically closed field and assume resolution of singularities. Then the albanese map*

$$alb_X : H_0^S(X,\mathbb{Z})^0 \to Alb_X(k)$$

from the degree-0-part of Suslin homology to the k-valued points of the Albanese variety induces an isomorphism on torsion groups.

Next we examine the situation over non-algebraically closed fields. We redefine Suslin homology and cohomology by imposing Galois descent. Concretely, if G_k is the absolute Galois group of k, then we define Galois-Suslin homology to be

$$H_i^{GS}(X,A) = H^{-i}R\Gamma(G_k, \mathrm{Cor}_{\bar{k}}(\Delta^*_{\bar{k}},\bar{X})\times A),$$

and Galois-Suslin cohomology to be

$$H^i_{GS}(X,A) = \mathrm{Ext}^i_{G_k}(\mathrm{Cor}_{\bar{k}}(\Delta^*_{\bar{k}},\bar{X}),A).$$

Ideally one would like to define Galois-Suslin homology via Galois homology, but we are not aware of such a theory. With rational coefficients, the newly defined groups agree with the original groups. On the other hand, with finite coefficients prime to the characteristic, the proof of (1) in [22] carries over to show that $H^i_{GS}(X,\mathbb{Z}/m)\cong H^i_{\mathrm{et}}(X,\mathbb{Z}/m)$. As a corollary, we obtain an isomorphism between $H_0^{GS}(X,\mathbb{Z}/m)$ and the abelianized fundamental group $\pi_1^{ab}(X)/m$ for any separated X of finite type over a finite field and m invertible. The second half of the paper focuses on the case of a finite base field. We work under the assumption of resolution of singularities in order to see the picture of the properties which can expected. The critical reader can view our statements as theorems for schemes of dimension at most three, and conjectures in general. A theorem of Jannsen-Saito [11] can be generalized to show that Suslin homology and cohomology with finite coefficients for any X over a finite field is finite. Rationally, $H_0^S(X,\mathbb{Q}) \cong H^0_S(X,\mathbb{Q}) \cong \mathbb{Q}^{\pi_0(X)}$. Most other properties are equivalent to the following Conjecture P_0 considered in [7]: For X smooth and proper over a finite field, $CH_0(X,i)$ is torsion for $i \neq 0$. This is a particular case of Parshin's conjecture that $K_i(X)$ is torsion for $i \neq 0$. Conjecture P_0 is equivalent to the vanishing of $H_i^S(X,\mathbb{Q})$ for $i \neq 0$ and all smooth X. For arbitrary X of dimension d, Conjecture P_0 implies the vanishing of $H_i^S(X,\mathbb{Q})$ outside of the range $0 \leq i \leq d$ and its finite dimensionality in this range. Combining the torsion and rational case, we show that $H_i^S(X,\mathbb{Z})$ and $H^i_S(X,\mathbb{Z})$ are finitely generated for all X if and only if Conjecture P_0 holds.

Over a finite field and with integral coefficients, it is more natural to impose descent by the Weil group G generated by the Frobenius endomorphism φ instead of the Galois group [14, 3, 4, 7]. We define arithmetic homology

$$H_i^{\mathrm{ar}}(X,A) = \mathrm{Tor}_i^G(\mathrm{Cor}_{\bar{k}}(\Delta^*_{\bar{k}},\bar{X}),A)$$

and arithmetic cohomology

$$H^i_{\mathrm{ar}}(X,\mathbb{Z}) = \mathrm{Ext}^i_G(\mathrm{Cor}_{\bar{k}}(\Delta^*_{\bar{k}},\bar{X}),\mathbb{Z}).$$

We show that $H_0^{\mathrm{ar}}(X,\mathbb{Z}) \cong H^0_{\mathrm{ar}}(X,\mathbb{Z}) \cong \mathbb{Z}^{\pi_0(X)}$ and that arithmetic homology and cohomology lie in long exact sequences with Galois-Suslin homology and cohomology, respectively. They are finitely generated abelian groups if and only if Conjecture P_0 holds.

The difference between arithmetic and Suslin homology is measured by a third theory, which we call Kato-Suslin homology, and which is defined as $H_i^{KS}(X,A) = H_i((\mathrm{Cor}_{\bar{k}}(\Delta^*_{\bar{k}},\bar{X}) \otimes A)_G)$. By definition there is a long exact sequence

$$\cdots \to H_i^S(X,A) \to H_{i+1}^{\mathrm{ar}}(X,A) \to H_{i+1}^{KS}(X,A) \to H_{i-1}^S(X,A) \to \cdots.$$

It follows that $H_0^{KS}(X,\mathbb{Z}) = \mathbb{Z}^{\pi_0(X)}$ for any X. As a generalization of the integral version [7] of Kato's conjecture [12], we propose

Conjecture 1.2 *The groups $H_i^{KS}(X,\mathbb{Z})$ vanish for all smooth X and $i > 0$.*

Equivalently, there are short exact sequences

$$0 \to H_{i+1}^S(\bar{X},\mathbb{Z})_G \to H_i^S(X,\mathbb{Z}) \to H_i^S(\bar{X},\mathbb{Z})^G \to 0$$

for all $i \geq 0$ and all smooth X. We show that this conjecture, too, is equivalent to Conjecture P_0. This leads us to a conjecture on abelian tamely ramified class field theory:

Conjecture 1.3 *For every X separated and of finite type over $\mathbb{F}_q$, there is a canonical injection*

$$H_1^{\mathrm{ar}}(X,\mathbb{Z}) \to \pi_1^t(X)^{ab}$$

with dense image.

It might even be true that the relative group $H_1^{\rm ar}(X,\mathbb{Z})^\circ := \ker(H_1^{\rm ar}(X,\mathbb{Z}) \to \mathbb{Z}^{\pi_0(X)})$ is isomorphic to the geometric part of the abelianized fundamental group defined in SGA 3X§6. To support our conjecture, we note that the generalized Kato conjecture above implies $H_0^S(X,\mathbb{Z}) \cong H_1^{\rm ar}(X,\mathbb{Z})$ for smooth X, so that in this case our conjecture becomes a theorem of Schmidt-Spiess [19]. In addition, we show (independently of any conjectures)

Proposition 1.4 *If $1/l \in \mathbb{F}_q$, then $H_1^{\rm ar}(X,\mathbb{Z})^{\wedge l} \cong \pi_1^t(X)^{ab}(l)$ for arbitrary X.*

In particular, the conjectured finite generation of $H_1^{\rm ar}(X,\mathbb{Z})$ implies the conjecture away from the characteristic. We also give a conditional result at the characteristic.
Notation: In this paper, scheme over a field k means separated scheme of finite type over k. The separable algebraic closure of k is denoted by $\bar{k}$, and if X is a scheme over k, we sometimes write $\bar{X}$ or $X_{\bar{k}}$ for $X \times_k \bar{k}$.
We thank Uwe Jannsen for interesting discussions related to the subject of this paper, and Shuji Saito and Takeshi Saito for helpful comments during a series of lectures I gave on the topic of this paper at Tokyo University.

2 Motivic homology

Suslin homology $H_i^S(X,\mathbb{Z})$ of a scheme X over a field k is defined as the homology of the global sections $C_*^X(k)$ of the complex of etale sheaves $C_*^X(-) = \mathrm{Cor}_k(- \times \Delta^*, X)$. Here $\mathrm{Cor}_k(U,X)$ is the group of universal relative cycles of $U\times Y/U$ [23]. If U is smooth, then $\mathrm{Cor}_k(U,X)$ is the free abelian group generated by closed irreducible subschemes of $U\times X$ which are finite and surjective over a connected component of U. Note that $C_*^X(-) = C_*^{X^{red}}(-)$, and we will use that all contructions involving C_*^X agree for X and X^{red} without further notice.
More generally [1], motivic homology of weight n are the extension groups in Voevodsky's category of geometrical mixed motives

$$H_i(X,\mathbb{Z}(n)) = \mathrm{Hom}_{DM_k^-}(\mathbb{Z}(n)[i], M(X)),$$

and are isomorphic to

$$H_i(X,\mathbb{Z}(n)) = \begin{cases} H^{2n-i}_{(0)}(\mathbb{A}^n, C_*^X) & n \geq 0 \\ H_{i-2n-1}(C_*\big(\frac{c_0(X\times(\mathbb{A}^n-\{0\}))}{c_0(X\times\{1\})}\big)(k)) & n<0. \end{cases}$$

Here cohomology is taken for the Nisnevich topology. There is an obvious version with coefficients. Motivic homology is a covariant functor on the category of schemes of finite type over k, and has the following additional properties, see [1] (the final three properties require resolution of singularities)

a) It is homotopy invariant.

b) It satisfies a projective bundle formula

$$H_i(X \times \mathbb{P}^1, \mathbb{Z}(n)) = H_i(X, \mathbb{Z}(n)) \oplus H_{i-2}(X, \mathbb{Z}(n-1)).$$

c) There is a Mayer-Vietoris long exact sequence for open covers.

d) Given an abstract blow-up square

$$\begin{tikzcd} Z' \arrow[r] \arrow[d] & X' \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

there is a long exact sequence

$$\cdots \to H_{i+1}(X, \mathbb{Z}(n)) \to H_i(Z', \mathbb{Z}(n)) \to \\ H_i(X', \mathbb{Z}(n)) \oplus H_i(Z, \mathbb{Z}(n)) \to H_i(X, \mathbb{Z}(n)) \to \cdots \quad (2)$$

e) If X is proper, then motivic homology agrees with higher Chow groups indexed by dimension of cycles, $H_i(X, \mathbb{Z}(n)) \cong CH_n(X, i-2n)$.

f) If X is smooth of pure dimension d, then motivic homology agrees with motivic cohomology with compact support,

$$H_i(X, \mathbb{Z}(n)) \cong H_c^{2d-i}(X, \mathbb{Z}(d-n)).$$

In particular, if Z is a closed subscheme of a smooth scheme X of pure dimension d, then we have a long exact sequence

$$\cdots \to H_i(U, \mathbb{Z}(n)) \to H_i(X, \mathbb{Z}(n)) \to H_c^{2d-i}(Z, \mathbb{Z}(d-n)) \to \cdots . \quad (3)$$

In order to remove the hypothesis on resolution of singularities, it would be sufficient to find a proof of Theorem 5.5(2) of Friedlander-Voevodsky [1] that does not require resolution of singularities. For all arguments in this paper (except the p-part of the Kato conjecture) the sequences (2) and (3) and the existence of a smooth and proper model for every function field are sufficient.

2.1 Suslin cohomology

Suslin cohomology is by definition the dual of Suslin homology, i.e. for an abelian group A it is defined as

$$H^i_S(X, A) := \mathrm{Ext}^i_{\mathrm{Ab}}(C^X_*(k), A).$$

We have $H^i_S(X, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}(H_i^S(X, \mathbb{Z}), \mathbb{Q}/\mathbb{Z})$, and a short exact sequence of abelian groups gives a long exact sequence of cohomology groups, in particular long exact sequences

$$\cdots \to H^i_S(X, \mathbb{Z}) \to H^i_S(X, \mathbb{Z}) \to H^i_S(X, \mathbb{Z}/m) \to H^{i+1}_S(X, \mathbb{Z}) \to \cdots . \quad (4)$$

and

$$\cdots \to H_S^{i-1}(X, \mathbb{Q}/\mathbb{Z}) \to H_S^i(X, \mathbb{Z}) \to H_S^i(X, \mathbb{Q}) \to H_S^i(X, \mathbb{Q}/\mathbb{Z}) \to \cdots .$$

Consequently, $H_S^i(X, \mathbb{Z})_{\mathbb{Q}} \cong H_S^i(X, \mathbb{Q})$ if Suslin-homology is finitely generated. If A is a ring, then $H_S^i(X, A) \cong \operatorname{Ext}_A^i(C_*^X(k) \otimes A, A)$, and we get a spectral sequence

$$E_2^{s,t} = \operatorname{Ext}_A^s(H_t^S(X, A), A) \Rightarrow H_S^{s+t}(X, A). \tag{5}$$

In particular, there are perfect pairings

$$H_i^S(X, \mathbb{Q}) \times H_S^i(X, \mathbb{Q}) \to \mathbb{Q}$$
$$H_i^S(X, \mathbb{Z}/m) \times H_S^i(X, \mathbb{Z}/m) \to \mathbb{Z}/m.$$

Lemma 2.1 *There are natural pairings*

$$H_S^i(X, \mathbb{Z})/\mathrm{tor} \times H_i^S(X, \mathbb{Z})/\mathrm{tor} \to \mathbb{Z}$$

and

$$H_S^i(X, \mathbb{Z})_{\mathrm{tor}} \times H_{i-1}^S(X, \mathbb{Z})_{\mathrm{tor}} \to \mathbb{Q}/\mathbb{Z}.$$

Proof. The spectral sequence (5) gives a short exact sequence

$$0 \to \operatorname{Ext}^1(H_{i-1}^S(X, \mathbb{Z}), \mathbb{Z}) \to H_S^i(X, \mathbb{Z}) \to \operatorname{Hom}(H_i^S(X, \mathbb{Z}), \mathbb{Z}) \to 0. \tag{6}$$

The resulting map $H_S^i(X, \mathbb{Z})/\mathrm{tor} \twoheadrightarrow \operatorname{Hom}(H_i^S(X, \mathbb{Z}), \mathbb{Z})$ induces the first pairing. Since $\operatorname{Hom}(H_i^S(X, \mathbb{Z}), \mathbb{Z})$ is torsion free, we obtain the map

$$H_S^i(X, \mathbb{Z})_{\mathrm{tor}} \hookrightarrow \operatorname{Ext}^1(H_{i-1}^S(X, \mathbb{Z}), \mathbb{Z}) \twoheadrightarrow$$
$$\operatorname{Ext}^1(H_{i-1}^S(X, \mathbb{Z})_{\mathrm{tor}}, \mathbb{Z}) \xleftarrow{\sim} \operatorname{Hom}(H_{i-1}^S(X, \mathbb{Z})_{\mathrm{tor}}, \mathbb{Q}/\mathbb{Z})$$

for the second pairing. □

2.2 Comparison to motivic cohomology

Recall that in the category DM_k^- of bounded above complexes of homotopy invariant Nisnevich sheaves with transfers, the motive $M(X)$ of X is the complex of presheaves with transfers C_*^X. Since a field has no higher Nisnevich cohomology, taking global sections over k induces a canonical map

$$\operatorname{Hom}_{DM_k^-}(M(X), A[i]) \to \operatorname{Hom}_{DM^-(\mathrm{Ab})}(C_*^X(k), A[i]),$$

hence a natural map

$$H_M^i(X, A) \to H_S^i(X, A). \tag{7}$$

If X is a schene over $L \supseteq k$, then even though the cohomology groups do not depend on the base field, the map does. For example, if L/k is an extension of degree d, then the diagram of groups isomorphic to $\mathbb{Z}$,

$$\begin{tikzcd} H^0_M(\operatorname{Spec} k, \mathbb{Z}) \arrow[r, equal] \arrow[d, equal] & H^0_S(\operatorname{Spec} k, \mathbb{Z}) \arrow[d, "\times d"] \\ H^0_M(\operatorname{Spec} L, \mathbb{Z}) \arrow[r] & H^0_S(\operatorname{Spec} L, \mathbb{Z}) \end{tikzcd}$$

shows that the lower horizontal map is multiplication by d. We will see below that conjecturally (7) is a map between finitely generated groups which is rationally an isomorphism, and one might ask if its Euler characteristic has any interpretation.

3 The mod p Suslin homology in characteristic p

We examine the p-part of Suslin homology in characteristic p. We assume that k is perfect and resolution of singularities exists over k in order to obtain stronger results. We first give an auxiliary result on motivic cohomology with compact support:

Proposition 3.1 *Let $d = \dim X$.*
a) We have $H^i_c(X, \mathbb{Z}/p^r(n)) = 0$ for $n > d$.
b) If k is algebraically closed, then $H^i_c(X, \mathbb{Z}/p^r(d))$ is finite, $H^i_c(X, \mathbb{Q}_p/\mathbb{Z}_p(d))$ is of cofinite type, and the groups vanish unless $d \le i \le 2d$.

Proof. By induction on the dimension and the localization sequence, the statement for X and a dense open subset of X are equivalent. Hence replacing X by a smooth subscheme and then by a smooth and proper model, we can assume that X is smooth and proper. Then a) follows from [8]. If k is algebraically closed, then

$$H^i(X, \mathbb{Z}/p(d)) \cong H^{i-d}(X_{Nis}, \nu^d) \cong H^{i-d}(X_{\mathrm{et}}, \nu^d),$$

by [8] and [13]. That the latter group is finite and of cofinite type, respectively, can be derived from [16, Thm.1.11], and it vanishes outside of the given range by reasons of cohomological dimension. □

Theorem 3.2 *Let X be separated and of finite type over k.*
a) The groups $H_i(X, \mathbb{Z}/p^r(n))$ vanish for all $n < 0$.
b) If k is algebraically closed, then the groups $H^S_i(X, \mathbb{Z}/p^r)$ are finite, the groups $H^S_i(X, \mathbb{Q}_p/\mathbb{Z}_p)$ are of cofinite type, and both vanish unless $0 \le i \le d$.

Proof. If X is smooth, then $H_i(X, \mathbb{Z}/p^r(n)) \cong H^{2d-i}_c(X, \mathbb{Z}/p^r(d-n))$ and we conclude by the Proposition. In general, we can assume by (2) and induction on the number of irreducible components that X is integral. Proceeding by induction on the dimension, we choose a resolution of singularities X' of X, let Z be the closed subscheme of X where the map $X' \to X$ is not an isomorphism, and let $Z' = Z \times_X X'$. Then we conclude by the sequence (2). $\square$

EXAMPLE. If X' is the blow up of a smooth scheme X in a smooth subscheme Z, then the strict transform $Z' = X' \times_X Z$ is a projective bundle over Z, hence by the projective bundle formula $H_i^S(Z, \mathbb{Z}/p^r) \cong H_i^S(Z', \mathbb{Z}/p^r)$ and $H_i^S(X, \mathbb{Z}/p^r) \cong H_i^S(X', \mathbb{Z}/p^r)$. More generally, we have

PROPOSITION 3.3 *Assume X has a desingularization $p : X' \to X$ which is an isomorphism outside of the dense open subset U. Then $H_i^S(U, \mathbb{Z}/p^r) \cong H_i^S(X, \mathbb{Z}/p^r)$. In particular, the p-part of Suslin homology is a birational invariant.*

The hypothesis is satisfied if X is smooth, or if U contains all singular points of X and a resolution of singularities exists which is an isomorphism outside of the singular points.

Proof. If X is smooth, then this follows from Proposition 3.1a) and the localization sequence (3). In general, let Z be the set of points where p is not an isomorphism, and consider the cartesian diagram

$$\begin{array}{ccccc} Z' & \longrightarrow & U' & \longrightarrow & X' \\ \downarrow & & \downarrow & & \downarrow \\ Z & \longrightarrow & U & \longrightarrow & X. \end{array}$$

Comparing long exact sequence (2) of the left and outer squares,

$$\begin{array}{ccccc} \to H_i^S(Z', \mathbb{Z}/p^r) & \longrightarrow & H_i^S(U', \mathbb{Z}/p^r) \oplus H_i^S(Z, \mathbb{Z}/p^r) & \longrightarrow & H_i^S(U, \mathbb{Z}/p^r) \to \\ \| & & \| & & \downarrow \\ \to H_i^S(Z', \mathbb{Z}/p^r) & \longrightarrow & H_i^S(X', \mathbb{Z}/p^r) \oplus H_i^S(Z, \mathbb{Z}/p^r) & \longrightarrow & H_i^S(X, \mathbb{Z}/p^r) \to \end{array}$$

we see that $H_i^S(U', \mathbb{Z}/p^r) \cong H_i^S(X', \mathbb{Z}/p^r)$ implies $H_i^S(U, \mathbb{Z}/p^r) \cong H_i^S(X, \mathbb{Z}/p^r)$. $\square$

EXAMPLE. If X is a node, then the blow-up sequence gives $H_i^S(X, \mathbb{Z}/p^r) \cong H_{i-1}^S(k, \mathbb{Z}/p^r) \oplus H_i^S(k, \mathbb{Z}/p^r)$, which is $\mathbb{Z}/p^r$ for $i = 0, 1$ and vanishes otherwise. Reid constructed a normal surface with a singular point whose blow-up is a node, showing that the p-part of Suslin homology is not a birational invariant for normal schemes.

COROLLARY 3.4 *The higher Chow groups $CH_0(X, i, \mathbb{Z}/p^r)$ and the logarithmic de Rham-Witt cohomology groups $H^i(X_{\mathrm{et}}, \nu_r^d)$, for $d = \dim X$, are birational invariants.*

Proof. Suslin homology agrees with higher Chow groups for proper X, and with motivic cohomology for smooth and proper X. □

Note that integrally $CH_0(X)$ is a birational invariant, but the higher Chow groups $CH_0(X,i)$ are generally not.
Suslin and Voevodsky [22, Thm.3.1] show that for a smooth compactification $\bar{X}$ of the smooth curve X, $H_0^S(X,\mathbb{Z})$ is isomorphic to the relative Picard group $\mathrm{Pic}(\bar{X},Y)$ and that all higher Suslin homology groups vanish. Proposition 3.3 implies that the kernel and cokernel of $\mathrm{Pic}(\bar{X},Y) \to \mathrm{Pic}(\bar{X})$ are uniquely p-divisible. Given U with compactification $j : U \to X$, the normalization $X^\sim$ of X in U is the affine bundle defined by the integral closure of $\mathcal{O}_X$ in $j_*\mathcal{O}_U$. We call X normal in U if $X^\sim \to X$ is an isomorphism.

Proposition 3.5 *If X is normal in the curve U, then $H_i^S(U,\mathbb{Z}/p) \cong H_i^S(X,\mathbb{Z}/p)$.*

Proof. This follows by applying the argument of Proposition 3.3 to X' the normalization of X, Z the closed subset where $X' \to X$ is not an isomorphism, $Z' = X' \times_X Z$ and $U' = X' \times_X U$. Since X is normal in U, we have $Z \subseteq U$ and $Z' \subseteq U'$. □

3.1 The albanese map

The following application was pointed out to us by N.Otsubo. Let X be a smooth connected quasi-projective variety over an algeraically closed field k of characteristic p. Then Spiess and Szamuely defined in [20] an albanese map

$$alb_X : H_0^S(X,\mathbb{Z})^0 \to Alb_X(k)$$

from the degree-0-part of Suslin homology to the k-valued points of the Albanese variety in the sense of Serre. They proved that if X is a dense open subscheme in a smooth projective scheme over k, then alb_X induces an isomorphism of the prime-to-p-torsion subgroups. We can remove the last hypothesis:

Theorem 3.6 *Assuming resolution of singularities, the map alb_X induces an isomorphism on torsion groups for any smooth, connected, quasi-projective variety over an algebraically closed field.*

Proof. In view of the result of Spiess and Szamuely, it suffices to consider the p-primary groups. Let T be a smooth and projective model of X. Since both sides are covariantly functorial and alb_X is functorial by construction, we obtain a commutative diagram

$$\begin{array}{ccc} H_0^S(X,\mathbb{Z})^0 & \xrightarrow{alb_X} & Alb_X(k) \\ \downarrow & & \downarrow \\ H_0^S(T,\mathbb{Z})^0 & \xrightarrow{alb_T} & Alb_T(k) \end{array}$$

The lower map is an isomorphism on torsion subgroups by Milne [15]. To show that the left vertical map is an isomorphism, consider the map of coefficient sequences

$$\begin{array}{ccccccc}
H_1^S(X,\mathbb{Z})\otimes\mathbb{Q}_p/\mathbb{Z}_p & \longrightarrow & H_1^S(X,\mathbb{Q}_p/\mathbb{Z}_p) & \longrightarrow & {}_pH_0^S(X,\mathbb{Z}) & \longrightarrow & 0\\
\downarrow & & \downarrow & & \downarrow & & \\
H_1^S(T,\mathbb{Z})\otimes\mathbb{Q}_p/\mathbb{Z}_p & \longrightarrow & H_1^S(T,\mathbb{Q}_p/\mathbb{Z}_p) & \longrightarrow & {}_pH_0^S(T,\mathbb{Z}) & \longrightarrow & 0
\end{array}$$

The right vertical map is an isomorphism because the middle map vertical map is an isomorphism by Proposition 3.3, and because $H_1^S(T,\mathbb{Z})\otimes\mathbb{Q}_p/\mathbb{Z}_p \cong CH_0(T,1)\otimes\mathbb{Q}_p/\mathbb{Z}_p$ vanishes by [6, Thm.6.1]. Finally, the map $Alb_X(k)\to Alb_T(k)$ is an isomorphism on p-torsion groups because by Serre's description [18], the two Albanese varieties differ by a torus, which does not have any p-torsion k-rational points in characteristic p, □

4 Galois properties

Suslin homology is covariant, i.e. a separated map $f: X\to Y$ of finite type induces a map $f_*:\mathrm{Cor}_k(T,X)\to\mathrm{Cor}_k(T,Y)$ by sending a closed irreducible subscheme Z of $T\times X$, finite over T, to the subscheme $[k(Z):k(f(Z))]\cdot f(Z)$ (note that $f(Z)$ is closed in $T\times Y$ and finite over T). On the other hand, Suslin homology is contravariant for finite flat maps $f: X\to Y$, because f induces a map $f^*:\mathrm{Cor}_k(T,Y)\to\mathrm{Cor}_k(T,X)$ by composition with the graph of f in $\mathrm{Cor}_k(Y,X)$ (note that the graph is a universal relative cycle in the sense of [23]). We examine the properties of Suslin homology under change of base-fields.

Lemma 4.1 *Let L/k be a finite extension of fields, X a scheme over k and Y a scheme over L. Then $\mathrm{Cor}_L(Y,X_L)=\mathrm{Cor}_k(Y,X)$ and if X is smooth, then $\mathrm{Cor}_L(X_L,Y)=\mathrm{Cor}_k(X,Y)$. In particular, Suslin homology does not depend on the base field.*

Proof. The first statement follows because $Y\times_L X_L\cong Y\times_k X$. The second statement follows because the map $X_L\to X$ is finite and separated, hence a closed subscheme of $X_L\times_L Y\cong X\times_k Y$ is finite and surjective over X_L if and only if it is finite and surjective over X. □

Given a scheme over k, the graph of the projection $X_L\to X$ in $X_L\times X$ gives elements $\Gamma_X\in\mathrm{Cor}_k(X,X_L)$ and $\Gamma_X^t\in\mathrm{Cor}_k(X_L,X)$.

4.1 Covariance

Lemma 4.2 *a) If X and Y are separated schemes of finite type over k, then the two maps*

$$\mathrm{Cor}_L(X_L,Y_L)\to\mathrm{Cor}_k(X,Y)$$

induced by composition and precomposition, respectively, with Γ_Y^t *and* Γ_X *agree. Both maps send a generator* $Z \subseteq X_L \times_k Y \cong X \times_k Y_L$ *to its image in* $X \times Y$ *with multiplicity* $[k(Z) : k(f(Z))]$*, a divisor of* $[L : k]$*.*
b) If F/k *is an infinite algebraic extension, then* $\lim_{L/k} \mathrm{Cor}_L(X_L, Y_L) = 0$*.*

Proof. The first part is easy. If Z is of finite type over k, then $k(Z)$ is a finitely generated field extension of k. For every component Z_i of Z_F, we obtain a map $F \to F \otimes_k k(Z) \to k(Z_i)$, and since F is not finitely generated over k, neither is $k(Z_i)$. Hence going up the tower of finite extensions L/k in F, the degree of $[k(W_L) : k(Z)]$, for W_L the component of Z_L corresponding to Z_i, goes to infinity. □

4.2 Contravariance

Lemma 4.3 *a) If* X *and* Y *are schemes over* k*, then the two maps*

$$\mathrm{Cor}_k(X, Y) \to \mathrm{Cor}_L(X_L, Y_L)$$

induced by composition and precomposition, respectively with Γ_Y *and* Γ_X^t *agree. Both maps send a generator* $Z \subseteq X \times Y$ *to the cycle associated to* $Z_L \subseteq X \times_k Y_L \cong X_L \times_k Y$*. If* L/k *is separable, this is a sum of the integral subschemes lying over* Z *with multiplicity one. If* L/k *is Galois with group* G*, then the maps induce an isomorphism*

$$\mathrm{Cor}_k(X, Y) \cong \mathrm{Cor}_L(X_L, Y_L)^G.$$

b) Varying L*,* $\mathrm{Cor}_L(X_L, Y_L)$ *forms an etale sheaf on* $\mathrm{Spec}\, k$ *with stalk* $M = \mathrm{colim}_L \mathrm{Cor}_L(X_L, Y_L) \cong \mathrm{Cor}_{\bar{k}}(X_{\bar{k}}, Y_{\bar{k}})$*, where* L *runs through the finite extensions of* k *in a separable algebraic closure* $\bar{k}$ *of* k*. In particular,* $\mathrm{Cor}_L(X_L, Y_L) \cong M^{\mathrm{Gal}(\bar{k}/L)}$*.*

Proof. Again, the first part is easy. If L/k is separable, Z_L is finite and etale over Z, hence $Z_L \cong \sum_i Z_i$, a finite sum of the integral cycles lying over Z with multiplicity one each. If L/k is moreover Galois, then $\mathrm{Cor}_k(X, Y) \cong \mathrm{Cor}_L(X_L, Y_L)^G$ and $\mathrm{Cor}_{\bar{k}}(X_{\bar{k}}, Y_{\bar{k}}) \cong \mathrm{colim}_{L/k} \mathrm{Cor}_L(X_L, Y_L)$ by EGA IV Thm. 8.10.5. □

The proposition suggests to work with the complex C_*^X of etale sheaves on $\mathrm{Spec}\, k$ given by

$$C_*^X(L) := \mathrm{Cor}_L(\Delta_L^*, X_L) \cong \mathrm{Cor}_k(\Delta_L^*, X).$$

Corollary 4.4 *If* $\bar{k}$ *is a separable algebraic closure of* k*, then* $H_i^S(X_{\bar{k}}, A) \cong \mathrm{colim}_{L/k} H_i^S(X_L, A)$*, and there is a spectral sequence*

$$E_2^{s,t} = \lim_{L/k}{}^s H_S^t(X_L, A) \Rightarrow H_S^{s+t}(X_{\bar{k}}, A).$$

The direct and inverse system run through finite separable extensions L/k, and the maps in the systems are induced by contravariant functoriality of Suslin homology for finite flat maps.

Proof. This follows from the quasi-isomorphisms

$$R\operatorname{Hom}_{\mathrm{Ab}}(C_*^X(\bar{k}),\mathbb{Z}) \cong R\operatorname{Hom}_{\mathrm{Ab}}(\operatorname*{colim}_L C_*^X(L),\mathbb{Z}) \cong R\lim_L R\operatorname{Hom}_{\mathrm{Ab}}(C_*^X(L),\mathbb{Z}).$$

□

4.3 Coinvariants

If G_k is the absolute Galois group of k, then $\operatorname{Cor}_{\bar{k}}(\bar{X},\bar{Y})_{G_k}$ can be identified with $\operatorname{Cor}_k(X,Y)$ by associating orbits of points of $\bar{X}\times_{\bar{k}}\bar{Y}$ with their image in $X\times_k Y$. However, this identification is neither compatible with covariant nor with contravariant functoriality, and in particular not with the differentials in the complex $C_*^X(k)$. But the obstruction is torsion, and we can remedy this problem by tensoring with $\mathbb{Q}$: Define an isomorphism

$$\tau : (\operatorname{Cor}_{\bar{k}}(\bar{X},\bar{Y})_{\mathbb{Q}})_{G_k} \to \operatorname{Cor}_k(X,Y)_{\mathbb{Q}}.$$

as follows. A generator $1_{\bar{Z}}$ corresponding to the closed irreducible subscheme $\bar{Z}\subseteq \bar{X}\times\bar{Y}$ is sent to $\frac{1}{g_Z}1_Z$, where Z is the image of $\bar{Z}$ in $X\times Y$ and g the number of irreducible components of $Z\times_k\bar{k}$, i.e. g_Z is the size of the Galois orbit of $\bar{Z}$.

Lemma 4.5 *The isomorphism τ is functorial in both variables, hence it induces an isomorphism of complexes*

$$(C_*^X(\bar{k})_{\mathbb{Q}})_{G_k} \cong C_*^X(k)_{\mathbb{Q}}.$$

Proof. This can be proved by explicit calculation. We give an alternate proof. Consider the composition

$$\operatorname{Cor}_k(X,Y)\to \operatorname{Cor}_{\bar{k}}(\bar{X},\bar{Y})^{G_k} \to \operatorname{Cor}_{\bar{k}}(\bar{X},\bar{Y})_{G_k} \xrightarrow{\tau} \operatorname{Cor}_k(X,Y)_{\mathbb{Q}}.$$

The middle map is induced by the identity, and is multiplication by g_Z on the component corresponding to Z. All maps are isomorphisms upon tensoring with $\mathbb{Q}$. The first map, the second map, and the composition are functorial, hence so is τ. □

5 Galois descent

Let $\bar{k}$ be the algebraic closure of k with Galois group G_k, and let A be a continuous G_k-module. Then $C_*^X(\bar{k})\otimes A$ is a complex of continuous G_k-modules, and if k has finite cohomological dimension we define Galois-Suslin homology to be

$$H_i^{GS}(X,A) = H^{-i}R\Gamma(G_k, C_*^X(\bar{k}) \otimes A).$$

By construction, there is a spectral sequence

$$E_{s,t}^2 = H^{-s}(G_k, H_t^S(\bar{X},A)) \Rightarrow H_{s+t}^{GS}(X,A).$$

The case $X = \operatorname{Spec} k$ shows that Suslin homology does not agree with Galois-Suslin homology, i.e. Suslin homology does not have Galois descent. We define Galois-Suslin cohomology to be

$$H^i_{GS}(X,A) = \operatorname{Ext}^i_{G_k}(C_*^X(\bar{k}), A). \tag{8}$$

This agrees with the old definition if k is algebraically closed. Let τ_* be the functor from G_k-modules to continuous G_k-modules which sends M to $\operatorname{colim}_L M^{G_L}$, where L runs through the finite extensions of k. It is easy to see that $R^i\tau_* M = \operatorname{colim}_H H^i(H,M)$, with H running through the finite quotients of G_k.

Lemma 5.1 *We have $H^i_{GS}(X,A) = H^iR\Gamma G_k R\tau_* \operatorname{Hom}_{\mathrm{Ab}}(C_*^X(\bar{k}), A)$. In particular, there is a spectral sequence*

$$E_2^{s,t} = H^s(G_k, R^t\tau_* \operatorname{Hom}_{\mathrm{Ab}}(C_*^X(\bar{k}), A)) \Rightarrow H^{s+t}_{GS}(X,A). \tag{9}$$

Proof. This is [17, Ex. 0.8]. Since $C_*^X(\bar{k})$ is a complex of free $\mathbb{Z}$-modules, $\operatorname{Hom}_{\mathrm{Ab}}(C_*^X(\bar{k}), -)$ is exact and preserves injectives. Hence the derived functor of $\tau_* \operatorname{Hom}_{\mathrm{Ab}}(C_*^X(\bar{k}), -)$ is $R^t\tau_*$ applied to $\operatorname{Hom}_{\mathrm{Ab}}(C_*^X(\bar{k}), -)$. □

Lemma 5.2 *For any abelian group A, the natural inclusion $C_*^X(k) \otimes A \to (C_*^X(\bar{k}) \otimes A)^{G_k}$ is an isomorphism.*

Proof. Let Z be a cycle corresponding to a generator of $C_*(k)$. If $Z \otimes_k \bar{k}$ is the union of g irreducible components, then the corresponding summand of $C_*(\bar{k})$ is a free abelian group of rank g on which the Galois group permutes the summands transitively. The claim is now easy to verify. □

Proposition 5.3 *We have*

$$H_i^{GS}(X,\mathbb{Q}) \cong H_i^S(X,\mathbb{Q})$$
$$H^i_{GS}(X,\mathbb{Q}) \cong H^i_S(X,\mathbb{Q}).$$

Proof. By the Lemma, $H_i^S(X,\mathbb{Q}) = H_i(C_*^X(k) \otimes \mathbb{Q}) \cong H_i((C_*^X(\bar{k}) \otimes \mathbb{Q})^{G_k})$. But the latter agrees with $H_i^{GS}(X,\mathbb{Q})$ because higher Galois cohomology is torsion. Similarly, we have $R^t\tau_* \operatorname{Hom}(C_i^X(\bar{k}), \mathbb{Q}) = 0$ for $t > 0$, and

$H^s(G_k, \tau_* \operatorname{Hom}(C_*^X(\bar{k}), \mathbb{Q})) = 0$ for $s > 0$. Hence $H^i_{GS}(X, \mathbb{Q})$ is isomorphic to the ith cohomology of

$$\operatorname{Hom}_{G_k}(C_*^X(\bar{k}), \mathbb{Q}) \cong \operatorname{Hom}_{\mathrm{Ab}}(C_*^X(\bar{k})_{G_k}, \mathbb{Q}) \cong \operatorname{Hom}_{\mathrm{Ab}}(C_*^X(k), \mathbb{Q}).$$

The latter equality follows with Lemma 4.5. □

Theorem 5.4 *If m is invertible in k and A is a finitely generated m-torsion G_k-module, then*

$$H^i_{GS}(X, A) \cong H^i_{\mathrm{et}}(X, A).$$

Proof. This follows with the argument of Suslin-Voevodsky [22]. Indeed, let $f : (Sch/k)_h \to Et_k$ be the canonical map from the large site with the h-topology of k to the small etale site of k. Clearly $f_* f^* \mathcal{F} \cong \mathcal{F}$, and the proof of Thm.4.5 in loc.cit. shows that the cokernel of the injection $f^* f_* \mathcal{F} \to \mathcal{F}$ is uniquely m-divisible, for any homotopy invariant presheaf with transfers (like, for example, $C_i^X : U \mapsto \operatorname{Cor}_k(U \times \Delta^i, X)$). Hence

$$\operatorname{Ext}^i_h(\mathcal{F}_h^\sim, f^*A) \cong \operatorname{Ext}^i_h(f^* f_* \mathcal{F}_h^\sim, f^*A) \cong \operatorname{Ext}^i_{Et_k}(f_* \mathcal{F}_h^\sim, A) \cong \operatorname{Ext}^i_{G_k}(\mathcal{F}(\bar{k}), A).$$

Then the argument of section 7 in loc.cit. together with Theorem 6.7 can be descended from the algebraic closure of k to k. □

Duality results for the Galois cohomology of a field k lead via theorem 5.4 to duality results between Galois-Suslin homology and cohomology over k.

Theorem 5.5 *Let k be a finite field, A a finite G_k-module, and $A^* = \operatorname{Hom}(A, \mathbb{Q}/\mathbb{Z})$. Then there is a perfect pairing of finite groups*

$$H^{GS}_{i-1}(X, A) \times H^i_{GS}(X, A^*) \to \mathbb{Q}/\mathbb{Z}.$$

Proof. According to [17, Example 1.10] we have

$$\operatorname{Ext}^r_{G_k}(M, \mathbb{Q}/\mathbb{Z}) \cong \operatorname{Ext}^{r+1}_{G_k}(M, \mathbb{Z}) \cong H^{1-r}(G_k, M)^*$$

for every finite G_k-module M, and the same holds for any torsion module by writing it as a colimit of finite modules. Hence

$$\begin{aligned} \operatorname{Ext}^r_{G_k}(C_*^X(\bar{k}), \operatorname{Hom}(A, \mathbb{Q}/\mathbb{Z})) &\cong \operatorname{Ext}^r_{G_k}(C_*^X(\bar{k}) \otimes A, \mathbb{Q}/\mathbb{Z}) \cong \\ &H^{1-r}(G_k, C_*^X(\bar{k}) \otimes A)^* = H^{GS}_{r-1}(X, A)^*. \end{aligned}$$

□

The case of non-torsion sheaves is discussed below.

Theorem 5.6 *Let k be a local field with finite residue field and separable closure k^s. For a finite G_k-module A let $A^D = \mathrm{Hom}(A,(k^s)^\times)$. Then we have isomorphisms*

$$H^i_{GS}(X, A^D) \cong \mathrm{Hom}(H^{GS}_{i-2}(X, A), \mathbb{Q}/\mathbb{Z}).$$

Proof. According to [17, Thm.2.1] we have

$$\mathrm{Ext}^r_{G_k}(M, (k^s)^\times) \cong H^{2-r}(G_k, M)^*$$

for every finite G_k-module M. This implies the same statement for torsion modules, and the rest of the proof is the same as above. □

6 Finite base fields

From now on we fix a finite field $\mathbb{F}_q$ with algebraic closure $\bar{\mathbb{F}}_q$. To obtain the following results, we assume resolution of singularities. This is needed to use the sequences (2) and (3) to reduce to the smooth and projective case on the one hand, and the proof of Jannsen-Saito [11] of the Kato conjecture on the other hand (however, Kerz and Saito announced a proof of the prime to p-part of the Kato conjecture which does not require resolution of singularities). The critical reader is invited to view the following results as conjectures which are theorems in dimension at most 3.
We first present results on finite generation in the spirit of [11] and [7].

Theorem 6.1 *For any $X/\mathbb{F}_q$ and any integer m, the groups $H^S_i(X, \mathbb{Z}/m)$ and $H^i_S(X, \mathbb{Z}/m)$ are finitely generated.*

Proof. It suffices to consider the case of homology. If X is smooth and proper of dimension d, then $H^S_i(X, \mathbb{Z}/m) \cong CH_0(X, i, \mathbb{Z}/m) \cong H^{2d-i}_c(X, \mathbb{Z}/m(d))$, and the result follows from work of Jannsen-Saito [11]. The usual devisage then shows that $H^j_c(X, \mathbb{Z}/m(d))$ is finite for all X and $d \geq \dim X$, hence $H^S_i(X, \mathbb{Z}/m)$ is finite for smooth X. Finally, one proceeds by induction on the dimension of X with the blow-up long-exact sequence to reduce to the case X smooth. □

6.1 Rational Suslin-homology

We have the following unconditional result:

Theorem 6.2 *For every connected X, the map $H^S_0(X, \mathbb{Q}) \to H^S_0(\mathbb{F}_q, \mathbb{Q}) \cong \mathbb{Q}$ is an isomorphism.*

Proof. By induction on the number of irreducible components and (2) we can first assume that X is irreducible and then reduce to the situation where X is smooth. In this case, we use (3) and the following Proposition to reduce to the smooth and proper case, where $H^S_0(X, \mathbb{Q}) = CH_0(X)_{\mathbb{Q}} \cong CH_0(\mathbb{F}_q)_{\mathbb{Q}}$. □

Proposition 6.3 *If $n > \dim X$, then $H^i_c(X,\mathbb{Q}(n)) = 0$ for $i \geq n + \dim X$.*

Proof. By induction on the dimension and the localization sequence for motivic cohomology with compact support one sees that the statement for X and a dense open subscheme of X are equivalent. Hence we can assume that X is smooth and proper of dimension d. Comparing to higher Chow groups, one sees that this vanishes for $i > d + n$ for dimension (of cycles) reasons. For $i = d + n$, we obtain from the niveau spectral sequence a surjection

$$\bigoplus_{X_{(0)}} H^{n-d}_M(k(x),\mathbb{Q}(n-d)) \twoheadrightarrow H^{d+n}_M(X,\mathbb{Q}(n)).$$

But the summands vanish for $n > d$ because higher Milnor K-theory of finite fields is torsion. $\square$

By definition, the groups $H_i(X,\mathbb{Q}(n))$ vanish for $i < n$. We will consider the following conjecture P_n of [5]:

Conjecture P_n: *For all smooth and projective schemes X over the finite field $\mathbb{F}_q$, the groups $H_i(X,\mathbb{Q}(n))$ vanish for $i \neq 2n$.*

This is a special case of Parshin's conjecture: If X is smooth and projective of dimension d, then

$$H_i(X,\mathbb{Q}(n)) \cong H^{2d-i}_M(X,\mathbb{Q}(d-n)) \cong K_{i-2n}(X)^{(d-n)}$$

and, according to Parshin's conjecture, the latter group vanishes for $i \neq 2n$. By the projective bundle formula, P_n implies P_{n-1}.

Proposition 6.4 *a) Let U be a curve. Then $H^S_i(U,\mathbb{Q}) \cong H^S_i(X,\mathbb{Q})$ for any X normal in U.*
b) Assume conjecture P_{-1}. Then $H_i(X,\mathbb{Q}(n)) = 0$ for all X and $n < 0$, and if X has a desingularization $p : X' \to X$ which is an isomorphism outside of the dense open subset U, then $H^S_i(U,\mathbb{Q}) \cong H^S_i(X,\mathbb{Q})$. In particular, Suslin homology and higher Chow groups of weight 0 are birational invariant.
c) Under conjecture P_0, the groups $H^S_i(X,\mathbb{Q})$ are finite dimensional and vanish unless $0 \leq i \leq d$.
d) Conjecture P_0 is equivalent to the vanishing of $H^S_i(X,\mathbb{Q})$ for all $i \neq 0$ and all smooth X.

Proof. The argument is the same as in Theorem 3.2. To prove b), we have to show that $H^i_c(X,\mathbb{Q}(n)) = 0$ for $n > d = \dim X$ under P_{-1}, and for c) we have to show that $H^i_c(X,\mathbb{Q}(d))$ is finite dimensional and vanishes unless $d \leq i \leq 2d$ under P_0. By induction on the dimension and the localization sequence we can assume that X is smooth and projective. In this case, the statement is Conjecture P_{-1} and P_0, respectively, plus the fact that $H^S_0(X,\mathbb{Q}) \cong CH_0(X)_{\mathbb{Q}}$ is a finite dimensional vector space. The final statement follows from the exact sequence (3) and the vanishing of $H^i_c(X,\mathbb{Q}(n))=0$ for $n>d=\dim X$ under P_{-1}. □

Proposition 6.5 *Conjecture P_0 holds if and only if the map $H^i_M(X,\mathbb{Q})\to H^i_S(X,\mathbb{Q})$ of (7) is an isomorphism for all $X/\mathbb{F}_q$ and i.*

Proof. The second statement implies the first, because if the map is an isomorphism, then $H^i_S(X,\mathbb{Q})=0$ for $i\neq 0$ and X smooth and proper, and hence so is the dual $H_i^S(X,\mathbb{Q})$. To show that P_0 implies the second statement, first note that because the map is compatible with long exact blow-up sequences, we can by induction on the dimension assume that X is smooth of dimension d. In this case, motivic cohomology vanishes above degree 0, and the same is true for Suslin cohomology in view of Proposition 6.4d). To show that for connected X the map (7) is an isomorphism of $\mathbb{Q}$ in degree zero, we consider the commutative diagram induced by the structure map

$$\begin{array}{ccc} H^0_M(\mathbb{F}_q,\mathbb{Q}) & \longrightarrow & H^0_S(\mathbb{F}_q,\mathbb{Q}) \\ \downarrow & & \downarrow \\ H^0_M(X,\mathbb{Q}) & \longrightarrow & H^0_S(X,\mathbb{Q}) \end{array}$$

This reduces the problem to the case $X=\operatorname{Spec}\mathbb{F}_q$, where it can be directly verified. □

6.2 Integral coefficients

Combining the torsion results [11] with the rational results, we obtain the following

Proposition 6.6 *Conjecture P_0 is equivalent to the finite generation of $H_i^S(X,\mathbb{Z})$ for all $X/\mathbb{F}_q$.*

Proof. If X is smooth and proper, then according to the main theorem of Jannsen-Saito [11], the groups $H_i^S(X,\mathbb{Q}/\mathbb{Z})=CH_0(X,i,\mathbb{Q}/\mathbb{Z})$ are isomorphic to etale homology, and hence finite for $i>0$ by the Weil-conjectures. Hence finite generation of $H_i^S(X,\mathbb{Z})$ implies that $H_i^S(X,\mathbb{Q})=0$ for $i>0$.
Conversely, we can by induction on the dimension assume that X is smooth and has a smooth and proper model. Expressing Suslin homology of smooth schemes in terms of motivic cohomology with compact support and again using induction, it suffices to show that $H^i_M(X,\mathbb{Z}(n))$ is finitely generated for smooth and proper X and $n\geq\dim X$. Using the projective bundle formula we can assume that $n=\dim X$, and then the statement follows because $H^i_M(X,\mathbb{Z}(n))\cong CH_0(X,2n-i)$ is finitely generated according to [7, Thm 1.1]. □

Recall the pairings of Lemma 2.1. We call them perfect if they identify one group with the dual of the other group. In the torsion case, this implies that the groups are finite, but in the free case this is not true: For example, $\oplus_I \mathbb{Z}$ and $\prod_I \mathbb{Z}$ are in perfect duality.

PROPOSITION 6.7 *Let X be a separated scheme of finite type over a finite field. Then the following statements are equivalent:*

a) The groups $H_i^S(X, \mathbb{Z})$ are finitely generated for all i.

b) The groups $H^i_S(X, \mathbb{Z})$ are finitely generated for all i.

c) The groups $H^i_S(X, \mathbb{Z})$ are countable for all i.

d) The pairings of Lemma 2.1 are perfect for all i.

Proof. a) $\Rightarrow$ b) $\Rightarrow$ c) are clear, and c) $\Rightarrow$ a) follows from [9, Prop.3F.12], which states that if A is not finitely generated, then either $\mathrm{Hom}(A, \mathbb{Z})$ or $\mathrm{Ext}(A, \mathbb{Z})$ is uncountable.
Going through the proof of Lemma 2.1 it is easy to see that a) implies d). Conversely, if the pairing is perfect, then ${}_{\mathrm{tor}}H_i^S(X, \mathbb{Z})$ is finite. Let $A = H^i_S(X, \mathbb{Z})/\mathrm{tor}$ and fix a prime l. Then A/l is a quotient of $H^i_S(X, \mathbb{Z})/l \subseteq H^i_S(X, \mathbb{Z}/l)$, and which is finite by Theorem 6.1. Choose lifts $b_i \in A$ of a basis of A/l and let B be the finitely generated free abelian subgroup of A generated by the b_i. By construction, A/B is l-divisible, hence $H_i^S(X, \mathbb{Z})/\mathrm{tor} = \mathrm{Hom}(A, \mathbb{Z}) \subseteq \mathrm{Hom}(B, \mathbb{Z})$ is finitely generated. □

6.3 THE ALGEBRAICALLY CLOSURE OF A FINITE FIELD

Suslin homology has properties similar to a Weil-cohomology theory. Let X_1 be separated and of finite type over $\mathbb{F}_q$, $X_n = X \times_{\mathbb{F}_q} \mathbb{F}_{q^n}$ and $X = X_1 \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q$. From Corollary 4.4, we obtain a short exact sequence

$$0 \to \lim{}^1 H_S^{t+1}(X_n, \mathbb{Z}) \to H_S^t(X, \mathbb{Z}) \to \lim H_S^t(X_n, \mathbb{Z}) \to 0.$$

The outer terms can be calculated with the 6-term lim-lim^1-sequence associated to (6). The theorem of Suslin and Voevodsky implies that

$$\lim H_S^i(X, \mathbb{Z}/l^r) \cong H^i_{\mathrm{et}}(X, \mathbb{Z}_l)$$

for $l \neq p = \mathrm{char}\, \mathbb{F}_q$. For X is proper and $l = p$, we get the same result from [6]

$$H^i_S(X, \mathbb{Z}/p^r) \cong \mathrm{Hom}(CH_0(X, i, Z/p^r), \mathbb{Z}/p^r) \cong H^i_{\mathrm{et}}(X, \mathbb{Z}/p^r).$$

We show that this is true integrally:

Proposition 6.8 *Let X be a smooth and proper curve over the algebraic closure of a finite field k of characteristic p. Then the non-vanishing cohomology groups are*

$$H^i_S(X,\mathbb{Z})\cong\begin{cases}\mathbb{Z} & i=0\\ \lim_r \operatorname{Hom}_{GS}(\mu_{p^r},\operatorname{Pic}X)\times\prod_{l\neq p}T_l\operatorname{Pic}X(-1) & i=1\\ \prod_{l\neq p}\mathbb{Z}_l(-1) & i=2.\end{cases}$$

Here Hom_{GS} *denotes homomorphisms of group schemes.*

Proof. By properness and smoothness we have

$$H_i^S(X,\mathbb{Z})\cong H_M^{2-i}(X,\mathbb{Z}(1))\cong\begin{cases}\operatorname{Pic}X & i=0;\\ k^\times & i=1;\\ 0 & i\neq 0,1.\end{cases}$$

Now

$$\operatorname{Ext}^1(k^\times,\mathbb{Z})=\operatorname{Hom}(\operatorname*{colim}_{p\nmid m}\mu_m,\mathbb{Q}/\mathbb{Z})\cong\prod_{l\neq p}\mathbb{Z}_l(-1)$$

and since $\operatorname{Pic}X$ is finitely generated by torsion,

$$\operatorname{Ext}^1(\operatorname{Pic}X,\mathbb{Z})\cong\operatorname{Hom}(\operatorname*{colim}_m {}_m\operatorname{Pic}X,\mathbb{Q}/\mathbb{Z})\cong \lim\operatorname{Hom}_{GS}({}_m\operatorname{Pic}X,\mathbb{Z}/m)\cong\lim_m\operatorname{Hom}_{GS}(\mu_m,{}_m\operatorname{Pic}X)$$

by the Weil-pairing. □

Proposition 6.9 *Let X be smooth, projective and connected over the algebraic closure of a finite field. Assuming conjecture P_0, we have*

$$H^i_S(X,\mathbb{Z})\cong\begin{cases}\mathbb{Z} & i=0\\ \prod_l H^i_{\text{et}}(X,\mathbb{Z}_l) & i\geq 1.\end{cases}$$

In particular, the l-adic completion of $H^i_S(X,\mathbb{Z})$ is l-adic cohomology $H^i_{\text{et}}(X,\mathbb{Z}_l)$ for all l.

Proof. Let $d=\dim X$. By properness and smoothness we have

$$H_i^S(X,\mathbb{Z})\cong H_M^{2d-i}(X,\mathbb{Z}(d)).$$

Under hypothesis P_0, the groups $H_i^S(X,\mathbb{Z})$ are torsion for $i>0$, and $H_0^S(X,\mathbb{Z})=CH_0(X)$ is the product of a finitely generated group and a torsion group. Hence for $i\geq 1$ we get by (6) that

$$\begin{aligned}H^i_S(X,\mathbb{Z})&\cong\operatorname{Ext}^1(H^S_{i-1}(X,\mathbb{Z}),\mathbb{Z})\cong\operatorname{Hom}(H^S_{i-1}(X,\mathbb{Z})_{\text{tor}},\mathbb{Q}/\mathbb{Z})\\ &\cong\operatorname{Hom}(H_M^{2d-i+1}(X,\mathbb{Z}(d))_{\text{tor}},\mathbb{Q}/\mathbb{Z})\cong\operatorname{Hom}(H_{\text{et}}^{2d-i}(X,\mathbb{Q}/\mathbb{Z}(d)),\mathbb{Q}/\mathbb{Z})\\ &\cong\operatorname{Hom}(\operatorname*{colim}_m H_{\text{et}}^{2d-i}(X,\mathbb{Z}/m(d)),\mathbb{Q}/\mathbb{Z})\cong\lim_m\operatorname{Hom}(H_{\text{et}}^{2d-i}(X,\mathbb{Z}/m(d)),\mathbb{Z}/m).\end{aligned}$$

By Poincare-duality, the latter agrees with $\lim H^i_{\text{et}}(X,\mathbb{Z}/m) \cong \prod_l H^i_{\text{et}}(X,\mathbb{Z}_l)$. $\square$

7 Arithmetic homology and cohomology

We recall some definitions and results from [3]. Let X be separated and of finite type over a finite field $\mathbb{F}_q$, $\bar{X} = X \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q$ and G be the Weil-group of $\mathbb{F}_q$. Let $\gamma : \mathcal{T}_G \to \mathcal{T}_{\hat{G}}$ be the functor from the category of G-modules to the category of continuous $\hat{G} = \mathrm{Gal}(\mathbb{F}_q)$-modules which associated to M the module $\gamma_* = \mathrm{colim}_m M^{mG}$, where the index set is ordered by divisibility. It is easy to see that the forgetful functor is a left adjoint of γ_*, hence γ_* is left exact and preserves limits. The derived functors γ^i_* vanish for $i > 1$, and $\gamma^1_* M = R^1\gamma_* M = \mathrm{colim}\, M_{mG}$, where the transition maps are given by $M_{mG} \to M_{mnG}, x \mapsto \sum_{s\in mG/mnG} sx$. Consequently, a complex $M^{\cdot}$ of G-modules gives rise to an exact triangle of continuous $\hat{G}$-modules

$$\gamma_* M^{\cdot} \to R\gamma_* M^{\cdot} \to \gamma^1_* M^{\cdot}[-1]. \tag{10}$$

If $M = \gamma^* N$ is the restriction of a continuous $\hat{G}$-module, then $\gamma_* M = N$ and $\gamma^1_* M = N \otimes \mathbb{Q}$. In particular, Weil-etale cohomology and etale cohomology of torsion sheaves agree. Note that the derived functors γ_* restricted to the category of $\hat{G}$-modules does not agree with the derived functors of τ_* considered in Lemma 5.1. Indeed, $R^i\tau_* M = \mathrm{colim}_L H^i(G_L, M)$ is the colimit of Galois cohomology groups, whereas $R^i\gamma_* M = \mathrm{colim}_m H^i(mG, M)$ is the colimit of cohomology groups of the discrete group $\mathbb{Z}$.

7.1 Homology

We define arithmetic homology with coefficients in the G-module A to be

$$H^{\text{ar}}_i(X, A) := \mathrm{Tor}^G_i(C^X_*(\bar{k}), A).$$

A concrete representative is the double complex

$$C^X_*(\bar{k}) \otimes A \xrightarrow{1-\varphi} C^X_*(\bar{k}) \otimes A,$$

with the left and right term in homological degrees one and zero, respectively, and with the Frobenius endomorphism φ acting diagonally. We obtain short exact sequences

$$0 \to H^S_i(\bar{X}, A)_G \to H^{\text{ar}}_i(X, A) \to H^S_{i-1}(\bar{X}, A)^G \to 0. \tag{11}$$

Lemma 7.1 *The groups $H^{\text{ar}}_i(X, \mathbb{Z}/m)$ are finite. In particular, $H^{\text{ar}}_i(X, \mathbb{Z})/m$ and ${}_mH^{\text{ar}}_i(X, \mathbb{Z})$ are finite.*

Proof. The first statement follows from the short exact sequence (11). Indeed, if m is prime to the characteristic, then we apply (1) together with finite generation of etale cohomology, and if m is a power of the characteristic, we apply Theorem 3.2 to obtain finiteness of the outer terms of (11). The final statements follows from the long exact sequence

$$\cdots \to H_i^{\mathrm{ar}}(X,\mathbb{Z}) \xrightarrow{\times m} H_i^{\mathrm{ar}}(X,\mathbb{Z}) \to H_i^{\mathrm{ar}}(X,\mathbb{Z}/m) \to \cdots$$

□

If A is the restriction of a $\hat{G}$-module, then (10), applied to the complex of continuous $\hat{G}$-modules $C_*^X(\bar{k})\otimes A$, gives upon taking Galois cohomology a long exact sequence

$$\cdots \to H_i^{GS}(X,A) \to H_{i+1}^{\mathrm{ar}}(X,A) \to H_{i+1}^{GS}(X,A_{\mathbb{Q}}) \to H_{i-1}^{GS}(X,A) \to \cdots$$

With rational coefficients this sequence breaks up into isomorphisms

$$H_i^{\mathrm{ar}}(X,\mathbb{Q}) \cong H_i^S(X,\mathbb{Q}) \oplus H_{i-1}^S(X,\mathbb{Q}). \tag{12}$$

7.2 Cohomology

In analogy to (8), we define arithmetic cohomology with coefficients in the G-module A to be

$$H^i_{\mathrm{ar}}(X,A) = \mathrm{Ext}^i_G(C_*^X(\bar{k}),A). \tag{13}$$

Note the difference to the definition in [14], which does not give well-behaved (i.e. finitely generated) groups for schemes which are not smooth and proper. A concrete representative is the double complex

$$\mathrm{Hom}(C_*^X(\bar{k}),A) \xrightarrow{1-\varphi} \mathrm{Hom}(C_*^X(\bar{k}),A),$$

where the left and right hand term are in cohomological degrees zero and one, respectively. There are short exact sequences

$$0 \to H_S^{i-1}(\bar{X},A)_G \to H^i_{\mathrm{ar}}(X,A) \to H^i_S(\bar{X},A)^G \to 0. \tag{14}$$

The proof of Lemma 7.1 also shows

Lemma 7.2 *The groups $H^i_{\mathrm{ar}}(X,\mathbb{Z}/m)$ are finite. In particular, ${}_mH^i_{\mathrm{ar}}(X,\mathbb{Z})$ and $H^i_{\mathrm{ar}}(X,\mathbb{Z})/m$ are finite.*

Lemma 7.3 *For every G-module A, we have an isomorphism*

$$H^i_{\mathrm{ar}}(X,A) \cong H^i_{GS}(X,R\gamma_*\gamma^*A).$$

Proof. Since $M^G = (\gamma_* M)^{\hat{G}}$, arithmetic cohomology is the Galois cohomology of the derived functor of $\gamma_* \operatorname{Hom}_{\mathrm{Ab}}(C_*^X(\bar{k}), -)$ on the category of G-modules. By Lemma 5.1, it suffices to show that this derived functor agrees with the derived functor of $\tau_* \operatorname{Hom}_{\mathrm{Ab}}(C_*^X(\bar{k}), \gamma_* -)$ on the category of G-modules. But given a continuous $\hat{G}$-modules M and a G-module N, the inclusion

$$\tau_* \operatorname{Hom}_{\mathrm{Ab}}(M, \gamma_* N) \subseteq \gamma_* \operatorname{Hom}_{\mathrm{Ab}}(\gamma^* M, N)$$

induced by the inclusion $\gamma_* N \subseteq N$ is an isomorphism. Indeed, if $f : M \to N$ is H-invariant and $m \in M$ is fixed by H', then $f(m)$ is fixed by $H \cap H'$, hence f factors through $\gamma_* N$. $\square$

Corollary 7.4 *If A is a continuous $\hat{G}$-module, then there is a long exact sequence*

$$\cdots \to H^i_{GS}(X, A) \to H^i_{\mathrm{ar}}(X, A) \to H^{i-1}_{GS}(X, A_{\mathbb{Q}}) \to H^{i+1}_{GS}(X, A) \to \cdots .$$

Proof. This follows from the Lemma by applying the long exact $\operatorname{Ext}^*_{\hat{G}}(C_*^X(\bar{k}), -)$-sequence to (10). $\square$

7.3 Finite generation and duality

Lemma 7.5 *There are natural pairings*

$$H^i_{\mathrm{ar}}(X, \mathbb{Z})/\mathrm{tor} \times H_i^{\mathrm{ar}}(X, \mathbb{Z})/\mathrm{tor} \to \mathbb{Z}$$

and

$$H^i_{\mathrm{ar}}(X, \mathbb{Z})_{\mathrm{tor}} \times H_{i-1}^{\mathrm{ar}}(X, \mathbb{Z})_{\mathrm{tor}} \to \mathbb{Q}/\mathbb{Z}.$$

Proof. From the adjunction $\operatorname{Hom}_G(M, \mathbb{Z}) \cong \operatorname{Hom}_{\mathrm{Ab}}(M_G, \mathbb{Z})$ and the fact that $L(-)_G = R(-)^G[-1]$, we obtain by deriving a quasi-isomorphism

$$R \operatorname{Hom}_G(C_*^X(\bar{k}), \mathbb{Z}) \cong R \operatorname{Hom}_{\mathrm{Ab}}(C_*^X(\bar{k}) \otimes^L_G \mathbb{Z}, \mathbb{Z}).$$

Now we obtain the pairing as in Lemma 2.1 using the resulting spectral sequence

$$\operatorname{Ext}^s_{\mathrm{Ab}}(H_t^{\mathrm{ar}}(X, \mathbb{Z}), \mathbb{Z}) \Rightarrow H^{s+t}_{\mathrm{ar}}(X, \mathbb{Z}).$$

$\square$

Proposition 7.6 *For a given separated scheme X of finite type over $\mathbb{F}_q$, the following statements are equivalent:*

a) The groups $H_i^{\mathrm{ar}}(X, \mathbb{Z})$ are finitely generated.

b) The groups $H^i_{\mathrm{ar}}(X, \mathbb{Z})$ are finitely generated.

c) *The groups* $H^i_{\mathrm{ar}}(X,\mathbb{Z})$ *are countable.*

d) *The pairings of Lemma 7.5 are perfect.*

Proof. This is proved exactly as Proposition 6.7, with Theorem 6.1 replaced by Lemma 7.1. □

We need a Weil-version of motivic cohomology with compact support. We define $H^i_c(X_W,\mathbb{Z}(n))$ to be the ith cohomology of $R\Gamma(G,R\Gamma_c(\bar{X},\mathbb{Z}(n)))$, where the inner term is a complex defining motivic cohomology with compact support of $\bar{X}$. We use this notation to distinguish it from arithmetic homology with compact support considered in [4], which is the cohomology of $R\Gamma(G,R\Gamma_c(\bar{X}_{\mathrm{et}},\mathbb{Z}(n)))$. However, if $n\geq \dim X$, which is the case of most importance for us, both theories agree.
Similar to (3) we obtain for a closed subscheme Z of a smooth scheme X of pure dimension d with open complement U a long exact sequence

$$\cdots\to H_i^{\mathrm{ar}}(U,\mathbb{Z})\to H_i^{\mathrm{ar}}(X,\mathbb{Z})\to H_c^{2d+1-i}(Z_W,\mathbb{Z}(d))\to\cdots. \tag{15}$$

The shift by 1 in degrees occurs because arithmetic homology is defined using homology of G, whereas cohomology with compact support is defined using cohomology of G.

Proposition 7.7 *The following statements are equivalent:*

a) *Conjecture* P_0.

b) *The groups* $H_i^{\mathrm{ar}}(X,\mathbb{Z})$ *are finitely generated for all* X.

Proof. a) $\Rightarrow$ b): By induction on the dimension of X and the blow-up square, we can assume that X is smooth of dimension d, where

$$H_i^{\mathrm{ar}}(X,\mathbb{Z})\cong H_c^{2d+1-i}(X_W,\mathbb{Z}(d)).$$

By localization for $H^*_c(X_W,\mathbb{Z}(d))$ and induction on the dimension we can reduce the question to X smooth and projective. In this case $\mathbb{Z}(d)$ has etale hypercohomological descent over an algebraically closed field by [6], hence $H^j_c(X_W,\mathbb{Z}(d))$ agrees with the Weil-etale cohomology $H^j_W(X,\mathbb{Z}(d))$ considered in [3]. These groups are finitely generated for $i>2d$ by [3, Thm.7.3,7.5]. By conjecture P_0, and the isomorphism $H^i_W(X,\mathbb{Z}(d))_{\mathbb{Q}}\cong CH_0(X,2d-i)_{\mathbb{Q}}\oplus CH_0(X,2d-i+1)_{\mathbb{Q}}$ of Thm.7.1c) loc.cit., these groups are torsion for $i<2d$, so that the finite group $H^{i-1}(X_{\mathrm{et}},\mathbb{Q}/\mathbb{Z}(d))$ surjects onto $H^i_W(X,\mathbb{Z}(d))$. Finally, $H^{2d}_W(X,\mathbb{Z}(d))$ is an extension of the finitely generated group $CH_0(\bar{X})^G$ by the finite group $H^{2d-1}(\bar{X}_{\mathrm{et}},\mathbb{Z}(d))_G\cong H^{2d-2}(\bar{X}_{\mathrm{et}},\mathbb{Q}/\mathbb{Z}(d))_G$.
b) $\Rightarrow$ a) Consider the special case that X is smooth and projective. Then as above, $H_i^{\mathrm{ar}}(X,\mathbb{Z})\cong H_W^{2d+1-i}(X,\mathbb{Z}(d))$. If this group is finitely generated, then we obtain from the coefficient sequence that $H_W^{2d+1-i}(X,\mathbb{Z}(d))\otimes\mathbb{Z}_l\cong\lim H^{2d+1-i}(X_{\mathrm{et}},\mathbb{Z}/l^r(d))$, and the latter group is torsion for $i>1$ by the Weil-conjectures. Now use (12). □

Theorem 7.8 *For connected X, the map $H_0^{\mathrm{ar}}(X,\mathbb{Z}) \to H_0^{\mathrm{ar}}(\mathbb{F}_q,\mathbb{Z}) \cong \mathbb{Z}$ is an isomorphism. In particular, we have $H_0^{\mathrm{ar}}(X,\mathbb{Z}) \cong \mathbb{Z}^{\pi_0(X)}$.*

Proof. The proof is similar to the proof of Theorem 6.2. Again we use induction on the dimension and the blow-up sequence to reduce to the situation where X is irreducible and smooth. In this case, we can use (15) and the following Proposition to reduce to the smooth and proper case, where we have $H_0^{\mathrm{ar}}(X,\mathbb{Z}) = CH_0(\bar{X})_G \cong \mathbb{Z}$. □

Proposition 7.9 *If $n > \dim X$, then $H^i_c(X_W,\mathbb{Z}(n)) = 0$ for $i > n + \dim X$.*

Proof. By induction on the dimension and the localization sequence for motivic cohomology with compact support one sees that the statement for X and a dense open subscheme of X are equivalent. Hence we can assume that X is smooth and proper of dimension d. In this case, $H^i_c(X_W,\mathbb{Z}(n))$ is an extension of $H^i_M(\bar{X},\mathbb{Z}(n))^G$ by $H^{i-1}_M(\bar{X},\mathbb{Z}(n))_G$. These groups vanish for $i-1 > d+n$ for dimension (of cycles) reasons. For $i = d+n+1$, we have to show that $H^{d+n}_M(\bar{X},\mathbb{Z}(n))_G$ vanishes. From the niveau spectral sequence for motivic cohomology we obtain a surjection

$$\bigoplus_{\bar{X}_{(0)}} H^{n-d}_M(k(x),\mathbb{Z}(n-d)) \twoheadrightarrow H^{d+n}_M(\bar{X},\mathbb{Z}(n)).$$

The summands are isomorphic to $K^M_{n-d}(\bar{\mathbb{F}}_q)$. If $n > d+1$, then they vanish because higher Milnor K-theory of the algebraical closure of a finite field vanishes. If $n = d+1$, then the summands are isomorphic to $(\bar{\mathbb{F}}_q)^\times$, whose coinvariants vanish. □

8 A Kato type homology

We construct a homology theory measuring the difference between Suslin homology and arithmetic homology. The cohomological theory can be defined analogously. Kato-Suslin-homology $H_i^{KS}(X,A)$ with coefficients in the G-module A is defined as the ith homology of the complex of coinvariants $(C_*^X(\bar{k})\otimes A)_G$. If A is trivial as a G-module, then Lemma 5.2 gives a short exact sequence of double complexes

$$\begin{tikzcd}
0 \arrow[r] & C_*^X(k)\otimes A \arrow[r] \arrow[d] & C_*^X(\bar{k})\otimes A \arrow[r] \arrow[d, "1-\varphi"'] & 0 \arrow[d] & \\
& 0 \arrow[r] & C_*^X(\bar{k})\otimes A \arrow[r] & (C_*^X(\bar{k})\otimes A)_G \arrow[r] & 0
\end{tikzcd}$$

and hence a long exact sequence

$$\cdots \to H_i^S(X,A) \to H_{i+1}^{\mathrm{ar}}(X,A) \to H_{i+1}^{KS}(X,A) \to H_{i-1}^S(X,A) \to \cdots .$$

By Theorem 7.8 we have $H_0^{KS}(X,\mathbb{Z}) \cong H_0^{\mathrm{ar}}(X,\mathbb{Z}) \cong \mathbb{Z}^{\pi_0(X)}$. The following is a generalization of the integral version [7] of Kato's conjecture [12].

Conjecture 8.1 *(Generalized integral Kato-conjecture) If X is smooth, then $H_i^{KS}(X,\mathbb{Z}) = 0$ for $i > 0$.*

Equivalently, the canonical map $H_i^S(X,\mathbb{Z}) \cong H_{i+1}^{\mathrm{ar}}(X,\mathbb{Z})$ is an isomorphism for all smooth X and all $i \geq 0$, i.e. there are short exact sequences

$$0 \to H_{i+1}^S(\bar{X},\mathbb{Z})_G \to H_i^S(X,\mathbb{Z}) \to H_i^S(\bar{X},\mathbb{Z})^G \to 0.$$

Theorem 8.2 *Conjecture 8.1 is equivalent to conjecture P_0.*

Proof. If Conjecture 8.1 holds, then

$$H_i^S(X,\mathbb{Q}) \cong H_{i+1}^{\mathrm{ar}}(X,\mathbb{Q}) \cong H_{i+1}^S(X,\mathbb{Q}) \oplus H_i^S(X,\mathbb{Q})$$

implies the vanishing of $H_i^S(X,\mathbb{Q})$ for $i > 0$.
Conversely, we first claim that for smooth and proper Z, the canonical map $H_c^i(Z,\mathbb{Z}(n)) \to H_c^i(Z_W,\mathbb{Z}(n))$ is an isomorphism for all i if $n > \dim Z$, and for $i \leq 2n$ if $n = \dim Z$. Indeed, if $n \geq \dim Z$ then the cohomology of $\mathbb{Z}(n)$ agrees with the etale hypercohomology of $\mathbb{Z}(n)$, see [6], hence satisfies Galois descent. But according to (the proof of) Proposition 6.4b), these groups are torsion groups, so that the derived funtors $R\Gamma G_k$ and $R\Gamma G$ agree.
Using localization for cohomology with compact support and induction on the dimension, we get next that $H_c^i(Z,\mathbb{Z}(n)) \cong H_c^i(Z_W,\mathbb{Z}(n))$ for all i and all Z with $n > \dim Z$. Now choose a smooth and proper compactification C of X. Comparing the exact sequences (3) and (15), we see with the 5-Lemma that the isomorphism $H_i^S(C,\mathbb{Z}) \cong H_c^{2d-i}(C,\mathbb{Z}(d)) \to H_{i+1}^{\mathrm{ar}}(C,\mathbb{Z}) \cong H_c^{2d-i}(C_W,\mathbb{Z}(d))$ for C implies the same isomorphism for X and $i \geq 0$. $\square$

9 Tamely ramified class field theory

We propose the following conjecture relating Weil-Suslin homology to class field theory:

Conjecture 9.1 *(Tame reciprocity) For any X separated and of finite type over a finite field, there is a canonical injection to the tame abelianized fundamental group with dense image*

$$H_1^{\mathrm{ar}}(X,\mathbb{Z}) \to \pi_1^t(X)^{ab}.$$

Note that the group $H_1^{\mathrm{ar}}(X,\mathbb{Z})$ is conjecturally finitely generated. At this point, we do not have an explicit construction (associating elements in the Galois groups to algebraic cycles) of the map. One might even hope that $H_1^{\mathrm{ar}}(X,\mathbb{Z})^\circ := \ker(H_1^{\mathrm{ar}}(X,\mathbb{Z}) \to \mathbb{Z}^{\pi_0(X)}$ is finitely generated and isomorphic to

the abelianized geometric part of the tame fundamental group defined in SGA 3X§6.
Under Conjecture 8.1, $H_0^S(X,\mathbb{Z}) \cong H_1^{\mathrm{ar}}(X,\mathbb{Z})$ for smooth X, and Conjecture 9.1 is a theorem of Schmidt-Spiess [19].

Proposition 9.2 *a) We have $H_1^{\mathrm{ar}}(X,\mathbb{Z})^{\wedge l} \cong \pi_1^t(X)^{ab}(l)$. In particular, the prime to p-part of Conjecture 9.1 holds if $H_1^{\mathrm{ar}}(X,\mathbb{Z})$ is finitely generated.*
b) The analog statement holds for the p-part if X has a compactification T which has a desingularization which is an isomorphism outside of X.

Proof. a) By Theorem 7.8, $H_0^{\mathrm{ar}}(X,\mathbb{Z})$ contains no divisible subgroup. Hence if $l \neq p$, we have by Theorems 5.4 and 5.5

$$\begin{aligned} H_1^{\mathrm{ar}}(X,\mathbb{Z})^{\wedge l} &\cong \lim H_1^{\mathrm{ar}}(X,\mathbb{Z}/l^r) \cong \lim H_0^{GS}(X,\mathbb{Z}/l^r) \\ &\cong \lim H^1_{\mathrm{et}}(X,\mathbb{Z}/l^r)^* \cong \pi_1^t(X)^{ab}(l). \end{aligned}$$

b) Under the above hypothesis, we can use the duality result of [6] for the proper scheme T to get with Proposition 3.3

$$\begin{aligned} H_1^{\mathrm{ar}}(X,\mathbb{Z})\otimes\mathbb{Z}_p &\cong \lim H_0^{GS}(X,\mathbb{Z}/p^r) \cong \lim H_0^{GS}(T,\mathbb{Z}/p^r) \\ &\cong \lim H^1_{\mathrm{et}}(T,\mathbb{Z}/p^r)^* \cong \pi_1(T)^{ab}(p) \cong \pi_1^t(X)^{ab}(p). \end{aligned}$$

□

References

[1] E. Friedlander, V. Voevodsky, Bivariant cycle cohomology. Cycles, transfers, and motivic homology theories, 138–187, Ann. of Math. Stud., 143, Princeton Univ. Press, Princeton, NJ, 2000.

[2] T. Geisser, Applications of de Jong's theorem on alterations. Resolution of singularities (Obergurgl, 1997), 299–314, Progr. Math., 181, Birkhauser, Basel, 2000.

[3] T. Geisser, Weil-etale cohomology over finite fields. Math. Ann. 330 (2004), no. 4, 665–692.

[4] T. Geisser, Arithmetic cohomology over finite fields and special values of ζ-functions. Duke Math. J. 133 (2006), no. 1, 27–57.

[5] T. Geisser, Parshin's conjecture revisited. K-theory and noncommutative geometry, 413–425, EMS Ser. Congr. Rep., Eur. Math. Soc., Zurich, 2008.

[6] T. Geisser, Duality via cycle complexes, to appear in Annals of Math.

[7] T. Geisser, Arithmetic homology and an integral version of Kato's conjecture, to appear in J. reine angew. Math.

[8] T. Geisser, M. Levine, The p-part of K-theory of fields in characteristic p. Inv. Math. 139 (2000), 459–494.

[9] A. Hatcher, Algebraic topology. Cambridge University Press, Cambridge, 2002.

[10] U. Jannsen, A cohomological Hasse principle, Preprint 2006.

[11] U. Jannsen, S. Saito, Kato conjecture and motivic cohomology over finite fields, Preprint 2009.

[12] K. Kato, A Hasse principle for two-dimensional global fields. J. Reine Angew. Math. 366 (1986), 142–183.

[13] K. Kato, T. Kuzumaki, The dimension of fields and algebraic K-theory. J. Number Theory 24 (1986), no. 2, 229–244.

[14] S. Lichtenbaum, The Weil-etale topology on schemes over finite fields. Compos. Math. 141 (2005), no. 3, 689–702.

[15] J. S. Milne, Zero cycles on algebraic varieties in nonzero characteristic: Rojtman's theorem. Compositio Math. 47 (1982), no.3, 271-287.

[16] J. S. Milne, Values of zeta functions of varieties over finite fields. Amer. J. Math. 108 (1986), no. 2, 297–360.

[17] J. S. Milne, Arithmetic duality theorems. Perspectives in Mathematics, 1. Academic Press, 1986.

[18] J-P. Serre, Morphismes universels et variétés d'Albanese. Séminaire Chevalley, année 1958/1959, exposé 11.

[19] A. Schmidt, M. Spiess, Singular homology and class field theory of varieties over finite fields. J. Reine Angew. Math. 527 (2000), 13–36.

[20] M. Spiess, T.Szamuely, On the Albanese map for smooth quasi-projective varieties, Math.Ann. 325, 1-17 (2003).

[21] A. Suslin, Higher Chow groups and etale cohomology. Cycles, transfers, and motivic homology theories, 239–254, Ann. of Math. Stud., 143, Princeton Univ. Press, Princeton, NJ, 2000.

[22] A. Suslin, V. Voevodsky, Singular homology of abstract algebraic varieties. Invent. Math. 123 (1996), no. 1, 61–94.

[23] A. Suslin, V. Voevodsky, Relative cycles and Chow sheaves. Cycles, transfers, and motivic homology theories, 10–86, Ann. of Math. Stud., 143, Princeton Univ. Press, Princeton, NJ, 2000.

[24] V. Voevodsky, Motivic cohomology groups are isomorphic to higher Chow groups in any characteristic. Int. Math. Res. Not. 2002, no. 7, 351–355.

Thomas Geisser
USC Department of Math.
3620 South Vermont Ave.
Los Angeles, CA 90089
geisser@usc.edu

Dimensions of Anisotropic Indefinite Quadratic Forms II

To Andrei Suslin on the occasion of his 60th birthday

Detlev W. Hoffmann

Received: September 29, 2009
Revised: April 15, 2010

Abstract. The u-invariant and the Hasse number $\widetilde{u}$ of a field F of characteristic not 2 are classical and important field invariants pertaining to quadratic forms. They measure the suprema of dimensions of anisotropic forms over F that satisfy certain additional properties. We prove new relations between these invariants and a new characterization of fields with finite Hasse number (resp. finite u-invariant for nonreal fields), the first one of its kind that uses intrinsic properties of quadratic forms and which, conjecturally, allows an 'algebro-geometric' characterization of fields with finite Hasse number.

2010 Mathematics Subject Classification: primary: 11E04; secondary: 11E10, 11E81, 14C25
Keywords and Phrases: quadratic form, Pfister form, Pfister neighbor, real field, ordering, strong approximation property, effective diagonalization, u-invariant, Hasse number, Pythagoras number, Rost correspondence, Rost projector

1. Introduction

Throughout this paper, fields are assumed to be of characteristic different from 2 and quadratic forms over a field are always assumed to be finite-dimensional and nondegenerate. The u-invariant of a field F is one of the most important field invariants pertaining to quadratic forms. The definition as introduced by Elman and Lam [EL1] is as follows:

$$u(F) := \sup\{\dim\varphi \mid \varphi \text{ is an anisotropic torsion form over } F\}\ ,$$

where 'torsion' means torsion when considered as an element in the Witt ring WF. Note that over a formally real field (or real field for short) torsion forms are exactly the forms of total signature zero, whereas over a nonreal field, all forms are torsion.

If F is a real field, for a form φ over F to be isotropic, it is clearly necessary for φ to be indefinite at each ordering of F, i.e., for φ to be *totally indefinite* or *t.i.* for short. This leads to another field invariant, the *Hasse number* $\widetilde{u}$ defined as

$$\widetilde{u}(F) := \sup\{\dim\varphi \,|\, \varphi \text{ is an anisotropic t.i. form}\}\ .$$

One puts $\widetilde{u}(F) = 0$ if there are no anisotropic t.i. forms over F. Clearly, $u(F) \leq \widetilde{u}(F)$, with equality in the case of nonreal fields since being totally indefinite is then an empty condition.

In the present paper, we focus on finiteness criteria for u and $\widetilde{u}$ and on upper bounds on $\widetilde{u}$ in terms of u for fields with finite $\widetilde{u}$. To formulate these results, we need to introduce further properties. We refer to [L3] for all undefined terminology and basic facts about quadratic forms.

Recall that a quadratic form of type $\langle 1, -a_1\rangle \otimes \ldots \otimes \langle 1, -a_n\rangle$ ($a_i \in F^*$) is called an n-fold Pfister form, and we write $\langle\langle a_1, \ldots, a_n\rangle\rangle$ for short. P_nF (resp. GP_nF) denotes the set of all isometry classes of n-fold Pfister forms (resp. of forms similar to n-fold Pfister forms). A form φ is a Pfister neighbor if there exists a Pfister form π and $a \in F^*$ such that $\varphi \subset a\pi$ and $\dim\varphi > \frac{1}{2}\dim\pi$. Pfister forms are either hyperbolic or anisotropic, and if φ is a Pfister neighbor of a Pfister form π then φ is anisotropic iff π is anisotropic. Recall that the n-fold Pfister forms generate additively I^nF, the n-th power of the fundamental ideal IF of classes of even-dimensional forms in the Witt ring WF. The Arason-Pfister Hauptsatz [AP], APH for short, states that if $\varphi \in I^nF$, then $\dim\varphi < 2^n$ implies that φ is hyperbolic, and $\dim\varphi = 2^n$ implies $\varphi \in GP_nF$.

Let F be a real field and let X_F denote its space of orderings. X_F is a compact totally disconnected Hausdorff space with a subbasis of the topology given by the clopen sets $H(a) = \{P \in X_F \,|\, a >_P 0\}$, $a \in F^*$. φ is called positive (resp. negative) definite at $P \in X_F$ if $\mathrm{sgn}_P(\varphi) = \dim\varphi$ (resp. $\mathrm{sgn}_P(\varphi) = -\dim\varphi$), and indefinite at P if it is not definite at P. A totally positive definite (t.p.d.) form is a form that is positive definite at each $P \in X_F$.

If φ is a form over F, we denote by $D_F(\varphi)$ those elements in F^* represented by φ, by $D_F(n)$ ($n \in \mathbb{N}$) those elements in F^* that can be written as a sum of n squares, and we write $D_F(\infty) = \bigcup_{n\in\mathbb{N}} D_F(n)$ for the nonzero sums of squares in F. If F is nonreal then $F^* = D_F(\infty)$, and if F is real then $D_F(\infty)$ is the set of all totally positive elements in F.

The *Pythagoras number* $p(F)$ of a field F is the smallest n such that $D_F(n) = D_F(\infty)$ if such an n exists, otherwise $p(F) = \infty$.

If F is real, then $x \in D_F(\varphi)$ clearly implies that $x >_P 0$ (resp. $x <_P 0$) if φ is positive (resp. negative) definite at P. If the converse also holds, i.e. if

$$\begin{array}{rcll} D_F(\varphi) & = & \{x \in F^* \,| & x >_P 0 \text{ (resp. } x <_P 0) \text{ if } \varphi \text{ is} \\ & & & \text{positive (resp. negative) definite at } P\} \end{array}$$

then φ is called *signature-universal* (*sgn-universal* for short). Over a real field, a form is universal (in the usual sense) if and only if it is t.i. and sgn-universal.

One readily sees that if $\widetilde{u}(F) < \infty$ then any form φ with $\dim\varphi \geq \widetilde{u}(F)$ is sgn-universal.
The following properties of fields will be used repeatedly.

Definition 1.1. (i) F is said to satisfy the *strong approximation property* SAP if given any disjoint closed subsets U, V of X_F there exists $a \in F^*$ such that $U \subset H(a)$ and $V \subset H(-a)$.
(ii) A form φ over a real field F is said to have *effective diagonalization* ED if it has a diagonalization $\langle a_1, \ldots, a_n\rangle$ such that $H(a_i) \subset H(a_{i+1})$ for $1 \leq i \leq n-1$. F is said to be ED if each form over F has ED.
(iii) F is said to have property S_1 if for every binary torsion form β over F one has $D_F(\beta) \cap D_F(\infty) \neq \emptyset$.
(iv) F is said to have property $PN(n)$ for some $n \in \mathbb{N}$ if each form of dimension $2^n + 1$ over F is a Pfister neighbor.

Note that if F is a nonreal field, i.e., F has no orderings, then $F^* = D_F(\infty)$ and all forms over F are torsion, so F is SAP, ED and S_1.
The paper is structured as follows. In §2 we give a new proof of the fact that ED is equivalent to SAP plus S_1, a result originally due to Prestel-Ware [PW]. In §3 we prove that for a field, having finite Hasse number is equivalent to having finite u-invariant plus having property ED. This result is originally due to Elman-Prestel [EP], but we give a proof that also allows us to derive various estimates for $\widetilde{u}$ in terms of u that are better than any previously known such estimates. In §4, we prove that having finite Hasse number is equivalent to having property $PN(n)$ for some $n \geq 2$, in which case we give estimates on $\widetilde{u}$ in terms of n. Since property $PN(2)$ is equivalent to F being linked (see Lemma 4.3), we will thus also recover as corollary a famous result on the u-invariant and the Hasse number of linked fields due to Elman-Lam [EL2], [E] (Corollary 4.12). We also explain how our results, conjecturally, provide an 'algebro-geometric' criterion for the finiteness of $\widetilde{u}$ (resp. u in case of nonreal fields).

Acknowledgment. I am grateful to the referee for various suggestions that helped to streamline the paper considerably. The revised version of this paper has been completed during a stay at Emory University. I thank Skip Garibaldi and Emory University for their hospitality during that stay.

2. ED equals SAP plus S_1

The following theorem is due to Prestel-Ware [PW]. We give a new proof based mainly on the study of binary forms.

Theorem 2.1. *F has ED if and only if F has SAP and S_1.*

To prove this, we use alternative descriptions of the properties involved.

Lemma 2.2. *Let F be a real field.*

(i) *F is SAP if and only if for all $a, b \in F^*$ there exists $c \in F^*$ such that $H(c) = H(a) \cap H(b)$ (or, equivalently, there exists $d \in F^*$ such that $H(d) = H(a) \cup H(b)$).*

(ii) *F is ED if and only if for all $a, b \in F^*$, there exist $c, d \in F^*$ such that $\langle a, b\rangle \cong \langle c, d\rangle$ and $H(c) = H(a) \cap H(b)$ (or, equivalently, $H(d) = H(a) \cup H(b)$).*

(iii) *F has property S_1 if and only if, for all $a \in F^*$, $s \in D_F(\infty)$, and $x \in D_F(\langle 1, as\rangle)$, there exists $t \in D_F(\infty)$ such that $tx \in D_F(\langle 1, a\rangle)$.*

Proof. (i) This is well known, see, e.g., [L1, Prop. 17.2].
(ii) The 'only if' is nothing else but ED for binary forms. As for the converse, we use induction on the dimension n of forms. Forms of dimension ≤ 2 have ED by assumption. So let φ be a form of dimension $n \geq 3$. Then we can write $\varphi = \langle a_1, \ldots, a_n\rangle$ and we may assume that $\langle a_2, \ldots, a_n\rangle$ is already an ED. Write $\langle a_1, a_2\rangle \cong \langle b_1, b_2\rangle$ with $H(b_1) = H(a_1) \cap H(a_2)$ (so $\langle b_1, b_2\rangle$ is an ED of $\langle a_1, a_2\rangle$). Then $\varphi \cong \langle b_1, b_2, a_3, \ldots, a_n\rangle$. Now let $\langle c_2, \ldots, c_n\rangle$ be an ED of $\langle b_2, a_3, \ldots, a_n\rangle$. Then one readily checks that $\langle b_1, c_2, \ldots, c_n\rangle$ is an ED of φ.
(iii) 'if': Let $\langle u, v\rangle \cong u\langle 1, uv\rangle$ be torsion. Then $uv = -s$ with $s \in D_F(\infty)$. Put $a = -s$. Then $\langle 1, -1\rangle \cong \langle 1, as\rangle$ which is hyperbolic and hence represents u. But then, by assumption, there exists $t \in D_F(\infty)$ such that tu is represented by $\langle 1, a\rangle \cong \langle 1, -s\rangle$ and hence t is represented by $u\langle 1, -s\rangle \cong \langle u, v\rangle$.
'only if': $x \in D_F(\langle 1, sa\rangle)$ implies that there exists $y \in F^*$ such that $\langle 1, sa\rangle \cong \langle x, y\rangle$. Now the torsion form $xa\langle s, -1\rangle$ represents some $u \in D_F(\infty)$ by S_1. Hence $\langle sa, -a\rangle \cong \langle xu, -xus\rangle$ and hence

$$\langle 1, sa, -a\rangle \cong \langle 1, xu, -xus\rangle \cong \langle -a, x, y\rangle$$

Thus, $\langle 1, a\rangle = \langle x, xus, -xu, y\rangle$ in WF, so $x\langle 1, us, -u, xy\rangle$ is isotropic and there exists $v \in D_F(\langle 1, us\rangle) \cap D_F(\langle u, -xy\rangle)$. Note that $us \in D_F(\infty)$, so $v \in D_F(\infty)$. Hence, $\langle 1, us\rangle \cong \langle v, vus\rangle$ and $\langle -u, xy\rangle \cong \langle -v, vuxy\rangle$, and we get $\langle 1, a\rangle \cong x\langle vus, vuxy\rangle \cong \langle xvus, vuy\rangle$, thus $xt \in D_F(\langle 1, a\rangle)$ with $t := vus \in D_F(\infty)$. $\square$

Proof of Theorem 2.1. 'only if': Clearly, ED implies SAP. Now let $\langle a, b\rangle$ be any binary torsion form. Then $\mathrm{sgn}_P(\langle a, b\rangle) = 0$, so $H(a) \cap H(b) = \emptyset$, and by ED, there exists $c \in -D_F(\infty)$ and $d \in D_F(\infty)$ such that $\langle a, b\rangle \cong \langle c, d\rangle$, in particular, d is a totally positive element represented by $\langle a, b\rangle$ and we have established S_1.
'if': Let F be SAP and S_1. We will verify the alternative description of ED from Lemma 2.2(ii). Let $\langle a, b\rangle$ be any binary form. By SAP, there exists $d' \in F^*$ such that $H(a) \cup H(b) = H(d')$. Then $\langle a, b, -d'\rangle$ is t.i., thus the form $\varphi \cong \langle a, b, -d', -d'ab\rangle \cong -d'\langle\langle ad', bd'\rangle\rangle$ has total signature zero and is therefore torsion. Hence, there exists some $n \in F$ such that for $\sigma_n \cong \langle\langle -1\rangle\rangle^{\otimes n} \cong \langle 1, 1\rangle^{\otimes n}$, we have that $\sigma_n \otimes \langle a, b, -d', -d'ab\rangle \in GP_{n+2}F$ is hyperbolic. But then its Pfister neighbor $\sigma_n \otimes \langle a, b\rangle \perp \langle -d'\rangle$ is isotropic. It follows that there exist $u, v \in D_F(\sigma_n) \subset D_F(\infty)$ such that $d' \in D_F(\langle ua, vb\rangle)$, and hence $ad'u \in D_F(\langle 1, abuv\rangle)$. Now $uv \in D_F(\infty)$, and by Lemma 2.2(iii), there exists $w \in D_F(\infty)$ such that $ad'uw \in D_F(\langle 1, ab\rangle)$, i.e. $d := d'uw \in D_F(\langle a, b\rangle)$.

In particular, there exists $c \in F^*$ such that $\langle a, b\rangle \cong \langle c, d\rangle$. Since $uw \in D_F(\infty)$, we have $H(d) = H(d') = H(a) \cup H(b)$ as required. $\square$

3. Relations between the Hasse number and the u-invariant

In this section, we will only consider real fields since for nonreal fields $u = \widetilde{u}$, and most of the statements below are trivially true. It is quite possible for a real field F that $u(F)$ is finite but $\widetilde{u}(F)$ is infinite. Elman-Prestel [EP, Th. 2.5] gave the following necessary and sufficient criterion for the finiteness of $\widetilde{u}(F)$:

Theorem 3.1. *$\widetilde{u}(F) < \infty$ if and only if $u(F) < \infty$ and F has ED.*

The main purpose of this section is to give a new and elementary proof of this statement that in the case of ED-fields will allow us at the same time to derive upper bounds for $\widetilde{u}$ in terms of u that considerably improve previous upper bounds obtained by Elman-Prestel [EP, Prop. 2.7] and Hornix [Hor1, Th. 3.9]. The following remark is well known and will be useful.

Remark 3.2. For any field F, if $p(F) > 2^n$ then $\widetilde{u}(F) \geq u(F) \geq 2^{n+1}$. In particular, $p(F) \leq u(F) \leq \widetilde{u}(F)$.

Proposition 3.3. *Suppose that F has ED and that there exists an n-dimensional t.p.d. sgn-universal form ρ. Then*

$$\widetilde{u}(F) \leq \frac{n}{2}(u(F) + 2)\ .$$

Proof. We may clearly assume that $u(F)$ (and hence $p(F)$) is finite. The form $p(F) \times \langle 1\rangle$ is t.p.d. and sgn-universal, so we may assume that $n \leq p(F)$. If $n = 1$ then F is obviously pythagorean and $u(F) = 0$. Since F has ED, any t.i. form φ over F contains a binary torsion form β as a subform. But then β is isotropic as $u(F) = 0$, hence φ is isotropic. It follows that $\widetilde{u}(F) = 0$ and the above inequality is clearly satisfied. So we may assume that $2 \leq n \leq p(F) = p$ and we have $\widetilde{u}(F) \geq u(F) \geq p \geq n$ by Remark 3.2.

It suffices to consider the case $\widetilde{u}(F) > u(F)$. Let φ_0 be any anisotropic t.i. form with $\dim\varphi_0 > u(F)$, and write $\dim\varphi_0 = m = rn + k + 1$ with $r \geq 1$ and $0 \leq k \leq n - 1$. Since F is ED and thus SAP, we may assume after scaling that $0 \leq \operatorname{sgn}_P \varphi_0 \leq \dim\varphi_0 - 2 = rn + k - 1$ for all orderings P on F.

Let $\varphi_1 = a_0(\varphi_0 \perp -\rho)_{\mathrm{an}}$, where a_0 is chosen such that $0 \leq \operatorname{sgn}_P \varphi_1$ for all orderings P.

If i_W denotes the Witt index, we have $i_W(\varphi_0 \perp -\rho) \leq n - 1$, for otherwise one could write $\varphi_0 \cong \rho \perp \tau$ for some form τ. Since φ_0 is t.i. and since F has ED, this implies that there exists $x \in D_F(\infty)$ such that $-x$ is represented by τ. But then the form φ_0 contains the subform $\rho \perp \langle -x\rangle$ which is isotropic as ρ is t.p.d. and sgn-universal, clearly a contradiction. This implies that

$$\dim\varphi_1 \geq \dim\varphi_0 + n - 2(n-1) = (r-1)n + (k+1) + 2\ .$$

Note also that $\operatorname{sgn}_P(\varphi_0 \perp -\rho) = \operatorname{sgn}_P \varphi_0 - n$ for each ordering P. Hence, one obtains

$$\operatorname{sgn}_P \varphi_1 \leq \max\{(r-1)n + k - 1, n\}$$

for each ordering P. Note that if $r \geq 2$, then φ_1 is again t.i. as $0 \leq \mathrm{sgn}_P\,\varphi_1 < \dim \varphi_1$ for all orderings P. Applying this procedure altogether $r-1$ times, we get a form φ_{r-1} which is anisotropic, t.i., and such that

$$\dim \varphi_{r-1} \geq n + (k+1) + 2(r-1) \ ,$$

$$0 \leq \mathrm{sgn}_P\,\varphi_{r-1} \leq \max\{n+k-1, n\} \text{ for all orderings } P.$$

We therefore have

$$\dim \varphi_{r-1} - \mathrm{sgn}_P\,\varphi_{r-1} \geq \min\{2r, k+2r-1\} \ .$$

Since $\dim \varphi_{r-1} - \mathrm{sgn}_P\,\varphi_{r-1}$ is even, this yields $\dim \varphi_{r-1} - \mathrm{sgn}_P\,\varphi_{r-1} \geq 2r$ for all orderings P. By ED, the anisotropic form φ_{r-1} contains a torsion subform φ_t of dimension $\geq 2r$. Hence $u(F) \geq 2r$ and thus $u(F)+2 \geq 2(r+1)$. On the other hand, by assumption $m = rn+k+1 \leq n(r+1)$. These two inequalities together imply $m \leq \frac{n}{2}(u(F)+2)$. It follows readily that $\widetilde{u}(F) \leq \frac{n}{2}(u(F)+2)$. □

Proof of Theorem 3.1. The 'only if' part is easy and left to the reader. As for the 'if' part, we have $\infty > u(F) \geq p(F)$ by Lemma 3.2, and if we put $\rho = p(F) \times \langle 1 \rangle$, then Proposition 3.3 immediately yields $\widetilde{u}(F) \leq \frac{p(F)}{2}(u(F) + 2) < \infty$. □

For a real field F, let $\widetilde{m}(F)$ be the smallest integer $n \geq 1$ such that there exists an n-dimensional t.p.d. sgn-universal form, and $\widetilde{m}(F) = \infty$ if there are no t.p.d. sgn-universal forms (cf. [GV] where an analogous invariant $m(F)$ for anisotropic universal forms was introduced). If $p(F) < \infty$, we have that $p(F) \times \langle 1 \rangle$ is sgn-universal. Hence $\widetilde{m}(F) \leq p(F)$. With this new invariant, Proposition 3.3 immediately implies

Corollary 3.4. *Suppose that $\widetilde{u}(F) < \infty$. Then*

$$\widetilde{u}(F) \leq \frac{\widetilde{m}(F)}{2}(u(F)+2) \ .$$

Next, we give another bound which will lead to further improvements.

Proposition 3.5. *Suppose that $u(F) < \infty$ and that F has ED (or, equivalently, that $\widetilde{u}(F) < \infty$). Let $\rho = \langle 1 \rangle \perp \rho'$ be a t.p.d. m-fold Pfister form, $m \geq 1$, such that its pure part ρ' is sgn-universal. Then*

$$\widetilde{u}(F) \leq 2^{m-2}(u(F)+6) \ .$$

If $m = 2$ then $\widetilde{u}(F) \leq u(F) + 4$.

Proof. If $m = 1$, then $\dim \rho' = 1$ and the assumptions imply that F is pythagorean, hence $\widetilde{u} = u = 0$ and there is nothing to show. So we may assume $m \geq 2$. Furthermore, if d is an integer such that $2^d \leq p(F) = p \leq 2^{d+1}-1$, then we may assume that $m \leq d+1$. For we have that $(2^{d+1}-1) \times \langle 1 \rangle$ is the pure part of $\langle\langle -1, \ldots, -1 \rangle\rangle \in P_{d+1}F$ and it is totally positive definite and sgn-universal. We proceed similarly as before, but this time we put $\widetilde{u} = \widetilde{u}(F) = r2^m + k + 1$ with $r \geq 0$ and $0 \leq k \leq 2^m - 1$.

If $r = 0$ then we have $\widetilde{u} \leq 2^m$. If $2^d + 1 \leq p \leq 2^{d+1} - 1$ then $u \geq 2^{d+1} \geq 2^m$ by Remark 3.2, and thus necessarily $u = \widetilde{u}$ and there is nothing to show.

Suppose that $p = 2^d$ so that in particular $u \geq 2^d$. Our previous bound yields $\widetilde{u} \leq 2^{d-1}(u+2)$. If $m = d+1$, then $2^{d-1}(u+2) < 2^{m-2}(u+6)$ and there is nothing to show. If $m \leq d$, then we have $\widetilde{u} = k+1 \leq 2^m \leq 2^d \leq u$ and thus $\widetilde{u} = u$, again there is nothing to show. So we may assume that $r \geq 1$.
Let φ_0 be an anisotropic t.i. form of dimension $\widetilde{u}$. As before, we may this time assume that $\dim \varphi_0 - 2 = r2^m + k - 1 \geq \operatorname{sgn}_P \varphi_0 \geq 0$ for all orderings P.
We claim that $i_W(\varphi_0 \perp -\rho) \leq 2^m - 2$. Indeed, otherwise φ_0 would contain a subform $\tilde{\rho}$ of dimension $2^m - 1$ with $\tilde{\rho} \subset \rho$. Now it is well known that all codimension 1 subforms of a Pfister form are similar to its pure part. Hence, φ_0 would contain a subform similar to ρ', and since φ_0 is t.i. and by ED, φ_0 would contain a subform similar to $\rho' \perp \langle -x \rangle$ for some $x \in D_F(\infty)$. By assumption, $\rho' \perp \langle -x \rangle$ is isotropic, a contradiction.
Thus, we obtain as in the proof of the previous lemma an anisotropic t.i. form φ_1 such that

$$\dim \varphi_1 \geq (r-1)2^m + k + 1 + 4 \ ,$$

$$0 \leq \operatorname{sgn}_P \varphi_1 \leq \max\{(r-1)2^m + k - 1, 2^m\} \ ,$$

and reiterating this construction $r-1$ times, we get an anisotropic t.i. form φ_{r-1} such that

$$\dim \varphi_{r-1} \geq 2^m + k + 1 + 4(r-1) \ ,$$

$$0 \leq \operatorname{sgn}_P \varphi_{r-1} \leq \max\{2^m + k - 1, 2^m\} \text{ for all orderings } P.$$

This yields $\dim \varphi_{r-1} - \operatorname{sgn}_P \varphi_{r-1} \geq 4r-2$ for all orderings P, and thus, by ED, the existence of an anisotropic torsion subform φ_t of φ_{r-1} with $\dim \varphi_t \geq 4r-2$. In particular, $u + 6 \geq 4(r+1)$. On the other hand, $\widetilde{u} \leq 2^m(r+1)$ and thus $\widetilde{u} \leq 2^{m-2}(u+6)$.
Now if $m = 2$, we have $\dim \varphi_{r-1} \geq 4r + k + 1 = \dim \varphi_0$ and $0 \leq \operatorname{sgn}_P \varphi_{r-1} \leq \max\{4 + k - 1, 4\}$. In particular, since all the forms φ_i are anisotropic and t.i., it follows readily from the construction and the fact that $\widetilde{u} = 4r + k + 1$ that $\dim \varphi_0 = \dim \varphi_1 = \ldots = \varphi_{r-1} = \widetilde{u}$. Note also that $0 \leq k \leq 3$, so that by repeating our construction one more time, we obtain an anisotropic t.i. form φ_r such that $\dim \varphi_r = \widetilde{u}$ and $\operatorname{sgn}_P \varphi_r \leq 4$ for all orderings P. Thus, φ_r contains a torsion subform of dimension $\geq \widetilde{u} - 4$ and therefore $\widetilde{u} \leq u + 4$. □

Proposition 3.6. *Suppose that $I_t^3 F = 0$, and that $u(F) < \infty$ and F has ED (or, equivalently, that $\widetilde{u}(F) < \infty$). If there exists a t.p.d. sgn-universal binary form ρ over F, then $u(F) = \widetilde{u}(F)$.*

Proof. By [ELP, Th. H], $I_t^3 F = 0$ implies that $\widetilde{u} = \widetilde{u}(F)$ is even. By Proposition 3.3, $\widetilde{u} \leq u + 2$. So let us assume that $\widetilde{u} \neq u$, i.e. $\widetilde{u} = u + 2$. The proof of Proposition 3.3 then shows that there exists an anisotropic t.i. form φ (which is nothing but the form φ_{r-1} in the proof) with $\dim \varphi = \widetilde{u}$ and which contains a torsion subform φ_t, $\dim \varphi_t = \dim \varphi - 2 = u$. After scaling, we may assume that $\varphi_t \perp \langle 1 \rangle \subset \varphi$. Let $d = d_\pm \varphi_t$. Then $\varphi_t \perp \langle 1, -d \rangle \in I^2 F$, and since $\operatorname{sgn}_P \varphi_t = 0$ and $\operatorname{sgn}_P \varphi_t \perp \langle 1, -d \rangle \in 4\mathbb{Z}$, it follows that $\varphi_t \perp \langle 1, -d \rangle \in I_t^2 F$. As $\dim \varphi_t \perp \langle 1, -d \rangle = u + 2$, this form must be isotropic. Thus, $\varphi_t \perp \langle 1 \rangle \cong \psi \perp \langle d \rangle$. Comparing discriminants and signatures, it follows that $\psi \in I_t^2F$. So $\langle 1, -x\rangle \otimes \psi \in I_t^3F = 0$ for all $x \in F^*$, thus $\psi \cong x\psi$ which implies that ψ is universal, hence the subform $\psi \perp \langle d\rangle$ of φ is isotropic, a contradiction. □

The following is an immediate consequence.

Corollary 3.7. *Suppose that $p(F) = 2$ and $\widetilde{u}(F) < \infty$. If $I_t^3F = 0$ then $u(F) = \widetilde{u}(F)$. In particular, if $u(F) \leq 6$ or $\widetilde{u}(F) \leq 8$, then $\widetilde{u}(F) = u(F)$.*

Remark 3.8. Let F be a real field with $\widetilde{u}(F) < \infty$. Suppose that d is an integer with $2^d + 1 \leq p \leq 2^{d+1} - 1$. The Pfister form $\langle\langle -1, \ldots, -1\rangle\rangle \in P_{d+1}F$ is t.p.d. and its pure part is sgn-universal, so we can use Proposition 3.5 for $m = d+1$. For $p = 2^d + 1$, $d \geq 1$, we get $2^{d-1}(u+6) - \frac{p}{2}(u+2) = 2^{d+1} - \frac{1}{2}u - 1$. In this case, Proposition 3.3 gives a better bound when $u \leq 2^{d+2} - 4$ (note that we will have $u \geq 2^{d+1}$), the bounds are the same for $u = 2^{d+2} - 2$, and for $u \geq 2^{d+2}$ Proposition 3.5 gives a sharper bound.
Summarizing our best bounds in the various cases, we obtain

(i) $p(F) = 1$ if and only if $\widetilde{u}(F) = u(F) = 0$.
(ii) If $p(F) = 2$ then $\widetilde{u}(F) \leq u(F) + 2$. If in addition $I_t^3F = 0$ then $\widetilde{u}(F) = u(F) = 2n$ for some integer $n \geq 1$.
(iii) If $p(F) = 3$ then $\widetilde{u}(F) \leq u(F) + 4$.
(iv) If $p(F) = 2^m$ then $\widetilde{u}(F) \leq 2^{m-1}(u(F) + 2)$.
(v) If $p(F) = 2^m + 1$ then $\widetilde{u}(F) \leq (2^{m-1} + \frac{1}{2})(u(F) + 2)$ if $u(F) \leq 2^{m+2} - 2$, and $\widetilde{u}(F) \leq 2^{m-1}(u(F) + 6)$ if $u(F) \geq 2^{m+2} - 2$.
(vi) If $2^m + 2 \leq p(F) \leq 2^{m+1} - 1$, then $\widetilde{u}(F) \leq 2^{m-1}(u(F) + 6)$.

Remark 3.9. It is difficult to say at this point how good our bounds really are. In fact, we know extremely little about fields with $u(F) < \widetilde{u}(F) < \infty$. The only values which could be realized so far are fields where $u(F) = 2n$ and $\widetilde{u}(F) = 2n + 2$ for any $n \geq 2$ (see [L2], [Hor2], [H3]), and fields with $u(F) = 8$ and $\widetilde{u}(F) = 12$, see [H2, Cor. 6.4].

For the balance of this section, we finish with stating results about *all* possible pairs of values for $(p(F), u(F))$ for real fields, in particular real fields satisfying SAP but not S_1 or vice versa (such fields will always have $\widetilde{u} = \infty$). The construction of such fields with prescribed values (p, u) uses Merkurjev's method of iterated function fields and is rather technical. We omit the proof and refer the interested reader to [H4].

Theorem 3.10. *Let $\mathcal{N}'$ be the set of pairs of integers (p, u) such that either $p = 1$ and $u = 0$ or $u = 2n \geq 2^m \geq p \geq 2$ for some integers m and n. Let $\mathcal{N} = \mathcal{N}' \cup \{(p, \infty);\ p \geq 2 \text{ or } p = \infty\}$.*

(i) *If F is a real field, then $(p(F), u(F)) \in \mathcal{N}$.*
(ii) *Let E be a real field and let $(p, u) \in \mathcal{N}$. Then there exists a real field extension F/E such that F is non-SAP, F has property S_1 and $(p(F), u(F)) = (p, u)$. In particular, $\widetilde{u}(F) = \infty$.*
(iii) *If F is a real SAP field with $\widetilde{u}(F) = \infty$, then $u(F) \geq 4$ and $(p(F), u(F)) \in \mathcal{N}$.*

(iv) *Let E be a real field and let $(p, u) \in \mathcal{N}$ with $u \geq 4$. Then there exists a real field extension F/E such that F is SAP, F does not have property S_1 and $(p(F), u(F)) = (p, u)$. In particular, $\widetilde{u}(F) = \infty$.*

4. Linkage of fields and the Pfister neighbor property

The purpose of this section is to derive a criterion for the finiteness of the Hasse number. Real fields with finite Hasse number are relatively scarce but interesting nonetheless. But our results are just as valid for nonreal fields, we thus get also a criterion for the finiteness of u for nonreal fields.
Recall that the field F is said to have the Pfister neighbor propery $PN(n)$, $n \geq 0$, if every form of dimension 2^n+1 over F is a Pfister neighbor. This property is a somewhat stronger version of the notion of n-linkage whose definition we now recall:

Definition 4.1. Let $n \geq 1$ be an integer. A field F is called *n-linked* if to any n-fold Pfister forms π_1 and π_2 over F there exist $a_1, a_2 \in F^*$ and an $(n-1)$-fold Pfister form σ such that $\pi_i \cong \langle\langle a_i \rangle\rangle \otimes \sigma$, $i = 1, 2$. F is called *linked* if F is 2-linked.

Remark 4.2. (i) Trivially, every field is 1-linked and satisfies $PN(0)$ and $PN(1)$.
(ii) Let $n \geq 2$. Every isotropic form of dimension 2^n+1 is a Pfister neighbor. In fact, if $\dim \varphi = 2^n+1$ and φ is isotropic, then $\varphi \cong \mathbb{H} \perp \psi$ with $\dim \psi = 2^n - 1$. Then $\varphi \perp -\psi \cong \pi \in P_{n+1}F$, where π denotes the hyperbolic $(n+1)$-fold Pfister form. So in particular, if F is nonreal and $u(F) \leq 2^n$, then F has property $PN(n)$

Lemma 4.3. *Let $n \geq 2$.*

(i) *If F is n-linked then F is m-linked for all $m \geq n$ and $I_t^{n+2}F = 0$.*
(ii) *F is n-linked iff to each form $\varphi \in I^nF$ there exists a form $\pi \in P_nF$ such that $\varphi \equiv \pi \bmod I^{n+1}F$ iff to each anisotropic $\varphi \in I^nF$ there exist $\tau \in P_{n-1}F$ and an even-dimensional form σ such that $\varphi \cong \tau \otimes \sigma$.*
(iii) *F has property $PN(n)$ if and only if there exists to every form φ over F a form ψ such that $\dim \psi \leq 2^n$ if $\dim \varphi$ even (resp. $\dim \psi \leq 2^n - 1$ if $\dim \varphi$ odd) such that $\varphi \equiv \psi \bmod I^{n+1}F$.*
(iv) *If F has property $PN(n)$ then F is n-linked. In particular, $I_t^{n+2}F = 0$. Furthermore, F is ED.*
(v) *F has property $PN(2)$ iff F is linked.*

Proof. (i) and (ii) are well known, see [EL2, § 2], [H1].
(iii) 'only if': If $\dim \varphi \leq 2^n$, then put $\psi \cong \varphi$. So suppose $\dim \varphi \geq 2^n+1$. Write $\varphi \cong \psi \perp \tau$ with $\dim \psi = 2^n+1$. By $PN(n)$, ψ is a Pfister neighbor and there exists ψ', $\dim \psi' = 2^n - 1$ such that $\psi \perp -\psi' \cong \pi \in GP_{n+1}F$. Then, in WF, we have

$$\varphi \equiv \varphi - \pi \equiv \psi' \perp \tau \bmod I^{n+1}F \ .$$

Now $\dim \psi' \perp \tau = \dim \varphi - 2$ and the result follows by an easy induction on the dimension.

'if': Let $\dim\varphi = 2^n+1$. By assumption, there exists a form ψ, $\dim\psi = 2^n-1$ (possibly after adding hyperbolic planes) such that $\varphi\perp -\psi \in I^{n+1}F$. Then $\dim(\varphi\perp-\psi) = 2^{n+1}$ and thus $\varphi\perp-\psi\in GP_{n+1}F$ by APH, which implies that φ is a Pfister neighbor.

(iv) To show that F is n-linked, let $\varphi\in I^nF$. By (iii), there exists ψ such that $\dim\psi = 2^n$ (possibly after adding hyperbolic planes) and $\varphi\equiv\psi \bmod I^{n+1}F$. But clearly $\psi\in I^nF$, and thus $\psi\in GP_nF$ by APH. Let $x\in F^*$ be such that $x\psi\in P_nF$. We then have $\psi\equiv x\psi \bmod I^{n+1}F$, and n-linkage together with $I_t^{n+2}F=0$ follows from (i) and (ii).

Now n-linked fields, $n\geq 2$, are easily seen to be SAP. So to establish ED, it suffices to establish property S_1 by Theorem 2.1. Let $\langle a,b\rangle$ be any torsion form. Let $\gamma\cong\langle\underbrace{1,\ldots,1}_{2^n-1}\rangle$. Then by $PN(n)$, the form $\gamma\perp\langle -a,-b\rangle$ is a t.i. Pfister neighbor of a Pfister form $\pi\in P_{n+1}F$. Since π contains γ which is a Pfister neighbor (and in fact subform) of $\sigma_n\cong\langle 1,1\rangle^{\otimes n}$, one necessarily has that σ_n divides π, so there exists $c\in F^*$ such that $\pi\cong\sigma_n\otimes\langle 1,c\rangle$. Now π contains a t.i. Pfister neighbor and is therefore also t.i. and hence torsion. But then $\rho\cong\langle 1,1\rangle\otimes\sigma_n\otimes\langle 1,c\rangle\in P_{n+2}F$ is torsion as well and therefore hyperbolic by (i). Now $\sigma_n\perp\gamma\perp\langle -a,-b\rangle$ is a Pfister neighbor of ρ. Since ρ is hyperbolic, its neighbor $\sigma_n\perp\gamma\perp\langle -a,-b\rangle$ is isotropic. Hence there exists $x\in D_F(\langle a,b\rangle)\cap D_F(\sigma_n\perp\gamma)$. But clearly, $D_F(\sigma_n\perp\gamma)\subset D_F(\infty)$ which shows that the binary torsion form $\langle a,b\rangle$ represents the totally positive element x.

(v) This follows immediately from the fact that a field is linked iff the classes of quaternion algebras form a subgroup in $\mathrm{Br}(F)$ together with the characterization of 5-dimensional Pfister neighbors by their Clifford invariant (see [Kn, p. 10]). $\square$

The following observation is essentially due to Fitzgerald [F, Lemma 4.5(ii)].

LEMMA 4.4. *Suppose that $\widetilde{u}(F)\leq 2^n$. Let φ be a form over F of dimension 2^n+1. Then φ is a Pfister neighbor. In particular, F has $PN(n)$.*

Proof. By Remark 4.2(ii) the result is clear if φ is isotropic. Thus, we may assume φ anisotropic, so necessarily F must be real. Since $\widetilde{u}(F)<\infty$ implies that F is SAP, we may assume that after scaling, $\mathrm{sgn}_P(\varphi)\geq 0$ for all $P\in X_F$, and that there exists $c\in F^*$ such that $H(c)=\{P\in X_F \mid \mathrm{sgn}_P(\varphi)=\dim\varphi\}$. In particular, the Pfister form $\langle\langle\underbrace{-1,\ldots,-1}_{n},-c\rangle\rangle\in P_{n+1}F$ is positive definite at all those $P\in X_F$ at which φ is positive definite, and it has signature zero at all those $P\in X_F$ at which φ is indefinite. Let $\psi\cong(\pi\perp-\varphi)_{\mathrm{an}}$. It follows that $|\mathrm{sgn}_P(\psi)|\leq 2^n-1$ for all $P\in X_F$. But since $\widetilde{u}(F)\leq 2^n$, the anisotropic form ψ must therefore have $\dim\psi\leq 2^n$, so in particular,

$$i_W(\pi\perp-\varphi)=\tfrac{1}{2}(\dim(\pi\perp-\varphi)-\dim\psi)\geq\tfrac{1}{2}(2^{n+1}+1)\ ,$$

and therefore $i_W(\pi\perp-\varphi)\geq 2^n+1=\dim\varphi$, which implies that $\varphi\subset\pi$. In particular, φ is a Pfister neighbor of π. $\square$

Theorem 4.5. *If a field F has property $PN(n)$, $n \geq 2$, then either $u(F) \leq \widetilde{u}(F) \leq 2^n$, or $2^{n+1} \leq u(F) \leq \widetilde{u}(F) \leq 2^{n+1} + 2^n - 2$.*

Proof. Let F be a field with property $PN(n)$ for some $n \geq 2$. Suppose that $\widetilde{u}(F) > 2^n$, i.e. there exists an anisotropic t.i. φ with $\dim \varphi = m > 2^n$. By Lemma 4.3(iv), F has ED and so φ can be diagonalized as $\varphi \cong \langle a_1, \ldots, a_m \rangle$ with $-a_1, a_m \in D_F(\infty)$. By removing some of the a_i, $2 \leq i \leq m-1$ if necessary, we will retain a t.i. form, so we may assume that φ is t.i. and $\dim \varphi = 2^n + 1$. But then, by $PN(n)$, φ is a Pfister neighbor of some $\pi \in P_{n+1}F$ which in turn is torsion and anisotropic as its Pfister neighbor φ is t.i. and anisotropic. This shows that $2^{n+1} \leq u(F) \leq \widetilde{u}(F)$.
Now suppose that $\widetilde{u}(F) > 2^{n+1} + 2^n - 2$. By a similar argument as above, we conclude that there exists an anisotropic t.i. form φ with $\dim \varphi = 2^{n+1} + 2^n - 1$. By Lemma 4.3(iii), there exists an anisotropic form ψ of dimension $\leq 2^n - 1$ such that $\varphi \equiv \psi \bmod I^{n+1}F$. Let $\pi \cong (\varphi \perp -\psi)_{\mathrm{an}} \in I^{n+1}F$. Then by dimension count and since φ is anisotropic, we have $2^{n+1} \leq \dim \pi \leq 2^{n+2} - 2$. Since F is $(n+1)$-linked, Lemma 4.3(ii) implies $\dim \pi = 2^{n+1}$, and thus, by APH, $\pi \in GP_{n+1}F$. Also, by dimension count, we have $\varphi \cong \pi \perp \psi$.
After scaling, we may assume that $\pi \in P_{n+1}F$, so that $\mathrm{sgn}_P(\pi) \in \{0, 2^{n+1}\}$. Now φ is t.i., and since F has ED by Lemma 4.3(iv), we can write $\psi \cong \langle a, \ldots \rangle$ with $a <_P 0$ whenever $\mathrm{sgn}_P(\pi) = 2^{n+1}$. But then $\pi \perp \langle a \rangle$ is a t.i. subform of φ. On the other hand, $\pi \perp \langle a \rangle$ is also a Pfister neighbor of $\pi \otimes \langle 1, a \rangle \in P_{n+2}F$. Since $\pi \perp \langle a \rangle$ is t.i., this implies that $\pi \otimes \langle 1, a \rangle$ is torsion and therefore hyperbolic since $I_t^{n+2}F = 0$ by Lemma 4.3(ii). But then the Pfister neighbor $\pi \perp \langle a \rangle$ is isotropic and therefore also φ, a contradiction. ☐

Remark 4.6. (i) The above proof also shows that if F has $PN(n)$, $n \geq 2$, then the case $\widetilde{u}(F) \leq 2^n$ occurs iff there are no anisotropic torsion $(n+1)$-fold Pfister forms iff $I_t^{n+1}F = 0$.
(ii) If we were only considering nonreal fields then the proofs could be shortened by essentially deleting arguments referring to or making use of ED, signatures, etc..

Corollary 4.7. *$\widetilde{u}(F) < \infty$ if and only if F has $PN(n)$ for some $n \geq 2$. In particular, if F is nonreal then $u(F) < \infty$ if and only if F has $PN(n)$ for some $n \geq 2$*

Proof. The 'if'-part in the first statement follows from Theorem 4.5, the converse from Lemma 4.4. The statement for nonreal fields is then clear because in that case $u = \widetilde{u}$. ☐

Remark 4.8. If F is real, then we still get a sufficient criterion for the finiteness of $u(F)$ even if $\widetilde{u}(F) = \infty$. Indeed, for real F, one has that if $u(F(\sqrt{-1}))$ is finite then $u(F)$ is finite, more precisely, one has $u(F) < 4u(F(\sqrt{-1}))$ (see [EKM, Th. 37.4]). Thus, we get the following: If $F(\sqrt{-1})$ has property $PN(n)$ for some $n \geq 2$, then $u(F) < 2^{n+3} + 2^{n+2} - 8$.

Conjecture 4.9. If a field F has property $PN(n)$, $n \geq 2$, then $u(F) \leq \widetilde{u}(F) \leq 2^n$, or $u(F) = \widetilde{u}(F) = 2^{n+1}$.

Corollary 4.10. *For $n \geq 2$, $PN(n)$ implies $PN(m)$ for all $m \geq n+2$. Furthermore, the following are equivalent:*

(i) *Conjecture 4.9 holds.*
(ii) *For $n \geq 2$, $PN(n)$ implies $PN(n+1)$.*

Proof. If $n \geq 2$, then $PN(n)$ implies that $\widetilde{u}(F) \leq 2^{n+2}$, and $PN(m)$ for $m \geq n+2$ follows from Lemma 4.4.
Now suppose that F has $PN(n)$ and that Conjecture 4.9 holds. Then $PN(n{+}1)$ follows from Lemma 4.4. Conversely, suppose that $n \geq 2$ and that $PN(n)$ implies $PN(n+1)$. Then we have $u(F) \leq \widetilde{u}(F) \leq 2^n$ or $2^{n+1} \leq u(F) \leq \widetilde{u}(F) \leq 2^{n+1} + 2^n - 2$ because of $PN(n)$, and also $u(F) \leq \widetilde{u}(F) \leq 2^{n+1}$ or $2^{n+2} \leq u(F) \leq \widetilde{u}(F) \leq 2^{n+2} + 2^{n+1} - 2$ because of $PN(n+1)$. Putting the two together, we obtain $u(F) \leq \widetilde{u}(F) \leq 2^n$ or $u(F) = \widetilde{u}(F) = 2^{n+1}$. $\square$

The only evidence we have as to the veracity of Conjecture 4.9 is the following.

Lemma 4.11. $PN(2)$ *implies* $PN(3)$. *In particular, if F has $PN(2)$, then $u(F) \leq \widetilde{u}(F) \leq 4$ or $u(F) = \widetilde{u}(F) = 8$.*

Proof. Suppose F has $PN(2)$ and let φ be any 9-dimensional form over F. Write $\varphi \cong \alpha \perp \beta$ with $\dim \alpha = 5$. Since α is a Pfister neighbor, there exists $\pi \in GP_2F$ such that $\pi \subset \alpha \subset \varphi$ (see, e.g., [L3, Ch. X, Prop. 4.19]). Write $\varphi \cong \pi \perp \gamma$. Then $\dim \gamma = 5$ and γ is also a Pfister neighbor, so there exists $\rho \in GP_2F$ such that $\rho \subset \gamma$. Hence, there exist $a, b, c, d, e, f, g \in F^*$ such that $\varphi \cong a\langle\langle b, c\rangle\rangle \perp d\langle\langle e, f\rangle\rangle \perp \langle g\rangle$.
Since $PN(2)$ implies that F is linked by Lemma 4.3(v), we may assume that $b = e$, and after scaling (which doesn't change the property of being a Pfister neighbor), we may also assume $a = 1$, so

$$\varphi \cong \langle\langle b, c\rangle\rangle \perp d\langle\langle b, f\rangle\rangle \perp \langle g\rangle \subset \langle\langle b\rangle\rangle \otimes (\langle\langle c\rangle\rangle \perp d\langle\langle f\rangle\rangle \perp \langle g\rangle)\ .$$

Now $\delta \cong \langle\langle c\rangle\rangle \perp d\langle\langle f\rangle\rangle \perp \langle g\rangle$ has dimension 5 and is therefore again a Pfister neighbor, so as above there exist $h, k, l, m \in F^*$ such that $\delta \cong h\langle\langle k, l\rangle\rangle \perp \langle m\rangle$. We thus get that

$$\varphi \subset \langle\langle b\rangle\rangle \otimes \delta \cong h\langle\langle b, k, l\rangle\rangle \perp m\langle\langle b\rangle\rangle \subset h\langle\langle b, k, l, -hm\rangle\rangle \in GP_4F\ ,$$

which shows that φ is a Pfister neighbor.
The remaining statement now follows from Corollary 4.10. $\square$

Since linked fields are exactly the fields that have $PN(2)$, one readily recovers the following result due to Elman and Lam [EL2] and Elman [E, Th. 4.7]. We leave it as an exercise to the reader to fill in the details.

Corollary 4.12. *Let F be a linked field. Then $u(F) = \widetilde{u}(F) \in \{0, 1, 2, 4, 8\}$. In particular, $I_t^4F = 0$. Furthermore, let $n \in \{0, 1, 2\}$. Then $\widetilde{u}(F) \leq 2^n$ iff $I_t^{n+1}F = 0$.*

Note that $u(F) = \widetilde{u}(F) = 0$ can only occur when F is real, whereas $u(F) = \widetilde{u}(F) = 1$ implies that F is nonreal.

Remark 4.13. It is not difficult to see that the iterated power series field $F = \mathbb{C}((X_1))((X_2))\dots((X_n))$ is a (nonreal) field with property $PN(n)$ and $u(F) = 2^{n+1}$.
Using Merkurjev's method of iterated function fields, it is also possible to construct to any $n \geq 2$ a real field F with property $PN(n)$ and $\widetilde{u}(F) = 2^{n+1}$. For details, see [H4].

Remark 4.14. Merkurjev [M] constructed to each positive integer n a field F with $I^3F = 0$ and $u(F) = 2n$ (resp. a field F with $I^3F = 0$ and $u(F) = \infty$). Trivially, such a (nonreal) field is 3-linked. So the n-linkage property, $n \geq 3$, does not give any indication on how large u might be, whereas the stronger property $PN(n)$ does.

We finish this paper with some remarks on a possible geometric interpretation of the property $PN(n)$ which can be formulated in the language of Chow groups. We refer to [Kar], [EKM, §80].
Let φ be a (nondegenerate) quadratic form of dimension $n+2 \geq 3$, and let $X = X_\varphi$ be the smooth projective n-dimensional quadric $\{\varphi = 0\}$ over F. We call X (an)isotropic if φ is (an)isotropic. Let $\overline{F}$ denote the algebraic closure of F and let $\overline{X} = X_{\overline{F}}$. Let l_0 be the class of a rational point in $\mathrm{CH}^n(\overline{X})$, the Chow group of 0-dimensional cycles, and let $1 \in \mathrm{CH}^0(X)$ be the class of X. A *Rost correspondence* on X is an element $\rho \in \mathrm{CH}^n(X \times X)$ which, over $\overline{F}$, is equal to $l_0 \times 1 + 1 \times l_0 \in \mathrm{CH}^n(\overline{X} \times \overline{X})$. A *Rost projector* is a Rost correspondence that is also an idempotent in the ring of correspondences on X. It is known that if a quadric has a Rost correspondence, then it has in fact also a Rost projector (see [Kar, Rem. 1.4]). The study of Rost correspondences/projectors has proven to be crucial in the motivic theory of quadrics.
It is known that if X is isotropic, then $l_0 \times 1 + 1 \times l_0$ is actually the unique Rost projector on X (see [Kar, Lem. 5.1]). For anisotropic forms, the situation is much more complicated.
The following is known:

Theorem 4.15. *Let φ be an anisotropic form over F of dimension ≥ 3.*

(i) *If X_φ possesses a Rost projector, then* $\dim\varphi = 2^n + 1$ *for some $n \geq 1$ (see Karpenko* [Kar, Prop. 6.2, 6.4]*).*

(ii) *If φ is a Pfister neighbor of dimension 2^n+1 then X_φ has a unique Rost projector (considered as element in $CH^r(X_\varphi \times X_\varphi)$, $r = 2^n - 1$) (see Izhboldin-Vishik* [IV, Th. 1.12] *for* $\mathrm{char}(F) = 0$*, Elman-Karpenko-Merkurjev* [EKM, Cor. 80.11] *in the general case).*

In view of part (i), it is natural to ask whether or not the converse of part (ii) also holds. This is still an open problem (see also [Kar, Conj. 1.6]):

Conjecture 4.16. If an anisotropic quadric X_φ possesses a Rost correspondence, then φ is a Pfister neighbor of dimension $2^n + 1$ for some $n \geq 1$.

Of course, by Theorem 4.15(ii), to prove the conjecture, one may assume that $\dim\varphi = 2^n+1$ for some $n \geq 1$. Since 3-dimensional forms are always Pfister neighbors, trivially the conjecture holds in that case. The conjecture is also true in the cases $n = 2, 3$ as shown by Karpenko (see [Kar, Prop. 10.8, Th. 1.7]):

Theorem 4.17. *Let φ be an anisotropic form over F of dimension 2^n+1, $n = 2, 3$. If X_φ possesses a Rost correspondence, then φ is a Pfister neighbor.*

It is now natural to introduce the property $RP(n)$ for $n \geq 1$:

$RP(n)$: *F has the property $RP(n)$ for $n \geq 1$ if every form φ over F of dimension 2^n+1 has a Rost projector.*

In view of the above, we immediately get

Proposition 4.18. *Let $n \geq 1$.*

(i) *$PN(n)$ implies $RP(n)$.*
(ii) *If $n \leq 3$, then $RP(n)$ implies $PN(n)$.*
(iii) *If Conjecture 4.16 holds, then $RP(n)$ implies $PN(n)$ for all $n \in \mathbb{N}$.*

Conjecturally and in view of Theorem 4.5, we therefore get an 'algebro-geometric' criterion for the finiteness of the Hasse number:

Corollary 4.19. *If Conjecture 4.16 holds, then $\widetilde{u}(F) < \infty$ (resp. $u(F) < \infty$ for nonreal F) if and only if F has property $RP(n)$ for some $n \geq 2$.*

References

[AP] Arason, J.Kr.; Pfister, A.: *Beweis des Krullschen Durchschnittsatzes für den Wittring*, Invent. Math. 12 (1971), 173–176.

[E] Elman, R.: *Quadratic forms and the u-invariant*, III. Proc. of Quadratic Forms Conference (ed. G. Orzech). Queen's Papers in Pure and Applied Mathematics No. 46 (1977), 422–444.

[EKM] Elman, R.; Karpenko, N.; Merkurjev, A.: The algebraic and geometric theory of quadratic forms. Amer. Math. Soc. Coll. Publ. 56, Providence, RI: Amer. Math. Soc. 2008.

[EL1] Elman, R.; Lam, T.Y.: *Quadratic forms and the u-invariant* I. Math. Z. 131 (1973), 283–304.

[EL2] Elman, R.; Lam, T.Y.: *Quadratic forms and the u-invariant* II. Invent. Math. 21 (1973), 125–137.

[ELP] Elman, R.; Lam, T.Y.; Prestel, A.: *On some Hasse principles over formally real fields.* Math. Z. 134 (1973), 291–301.

[EP] Elman, R.; Prestel, A.: *Reduced stability of the Witt ring of a field and its Pythagorean closure.* Amer. J. Math. 106 (1983), 1237–1260.

[F] Fitzgerald, R.W.: *Witt kernels of function field extensions.* Pacific J. Math. 109 (1983), 89–106.

[GV] Gesquière, N.; Van Geel, J.: *Note on universal quadratic forms.* Bull. Soc. Math. Belg. — Tijdschr. Belg. Wisk. Gen., Ser. B, 44 (1992), 193–205.

[H1] Hoffmann, D.W.: *A note on simple decomposition of quadratic forms over linked fields.* J. Algebra 175 (1995), 728–731.

[H2] Hoffmann, D.W.: *Isotropy of quadratic forms and field invariants.* Quadratic forms and their applications (Dublin, 1999), Contemp. Math. 272, Providence, RI: Amer. Math. Soc. 2000, 73–102.

[H3] Hoffmann, D.W.: *Dimensions of anisotropic indefinite quadratic forms, I.* Proceedings of the Conference on Quadratic Forms and Related Topics (Baton Rouge, LA, 2001). Doc. Math. 2001, Extra Vol., 183–200.

[H4] Hoffmann, D.W.: *Dimensions of anisotropic indefinite quadratic forms, II — The lost proofs.* arxiv:1004.2483v1 (2010), 11 pages.

[Hor1] Hornix, E.A.M.: *Totally indefinite quadratic forms over formally real fields.* Indag. Math. 47 (1985), 305–312.

[Hor2] Hornix, E.A.M.: *Formally real fields with prescribed invariants in the theory of quadratic forms.* Indag. Math., New Ser. 2 (1991), 65–78.

[IV] Izhboldin, O.; Vishik, A.: *Quadratic forms with absolutely maximal splitting.* Quadratic forms and their applications (Dublin, 1999), Contemp. Math. 272, Providence, RI: Amer. Math. Soc. 2000, 103–125.

[Kar] Karpenko, N.A.: *Characterization of minimal Pfister neighbors via Rost projectors.* J. Pure Appl. Algebra 160 (2001), 195–227.

[Kn] Knebusch, M.: *Generic splitting of quadratic forms.* II. Proc. London Math. Soc. (3) 34 (1977), no. 1, 1–31.

[L1] Lam, T.Y.: Orderings, valuations and quadratic forms. CBMS Regional Conference Series in Mathematics 52, Providence, Rhode Island: American Mathematical Society 1983.

[L2] Lam, T.Y.: *Some consequences of Merkurjev's work on function fields.* Unpublished manuscript (1989).

[L3] Lam, T.Y.: Introduction to Quadratic Forms over Fields. Graduate Studies in Mathematics 67, Providence, Rhode Island: American Mathematical Society 2005.

[M] Merkurjev, A.S.: *Simple algebras and quadratic forms.* Izv. Akad. Nauk. SSSR 55 (1991) 218–224. (English translation: Math. USSR Izvestiya 38 (1992) 215–221.)

[PW] Prestel, A; Ware, R.: *Almost isotropic quadratic forms.* J. London Math. Soc. 19 (1979), 241–244.

Detlev Hoffmann
School of Mathematical Sciences
University of Nottingham
University Park
Nottingham NG7 2RD
UK
detlev.hoffmann@nottingham.ac.uk

Homology Stability for the Special Linear Group of a Field and Milnor-Witt K-theory

Dedicated to Andrei Suslin

Kevin Hutchinson, Liqun Tao

Received: May 28, 2009
Revised: April 1, 2010

Abstract. Let F be a field of characteristic zero and let $f_{t,n}$ be the stabilization homomorphism from the nth integral homology of $\mathrm{SL}_t(F)$ to the nth integral homology of $\mathrm{SL}_{t+1}(F)$. We prove the following results: For all n, $f_{t,n}$ is an isomorphism if $t \geq n+1$ and is surjective for $t = n$, confirming a conjecture of C-H. Sah. $f_{n,n}$ is an isomorphism when n is odd and when n is even the kernel is isomorphic to the $(n+1)$st power of the fundamental ideal of the Witt Ring of F. When n is even the cokernel of $f_{n-1,n}$ is isomorphic to the nth Milnor-Witt K-theory group of F. When n is odd, the cokernel of $f_{n-1,n}$ is isomorphic to the square of the nth Milnor K-group of F.

2010 Mathematics Subject Classification: 19G99, 20G10
Keywords and Phrases: K-theory, special linear group, group homology

1. Introduction

Given a family of groups $\{G_t\}_{t\in\mathbb{N}}$ with canonical homomorphisms $G_t \to G_{t+1}$, we say that the family has homology stability if there exist constants $K(n)$ such that the natural maps $\mathrm{H}_n(G_t, \mathbb{Z}) \to \mathrm{H}_n(G_{t+1}, \mathbb{Z})$ are isomorphisms for $t \geq K(n)$. The question of homology stability for families of linear groups over a ring R - general linear groups, special linear groups, symplectic, orthogonal and unitary groups - has been studied since the 1970s in connection with applications to algebraic K-theory, algebraic topology, the scissors congruence problem, and the homology of Lie groups. These families of linear groups are known to have homology stability at least when the rings satisfy some appropriate finiteness condition, and in particular in the case of fields and local rings

([4],[26],[27],[25], [5],[2], [21],[15],[14]). It seems to be a delicate - but interesting and apparently important - question, however, to decide the minimal possible value of $K(n)$ for a particular class of linear groups (with coefficients in a given class of rings) and the nature of the obstruction to extending the stability range further.

The best illustration of this last remark are the results of Suslin on the integral homology of the general linear group of a field in the paper [23]. He proved that, for an infinite field F, the maps $\mathrm{H}_n(\mathrm{GL}_t(F),\mathbb{Z}) \to \mathrm{H}_n(\mathrm{GL}_{t+1}(F),\mathbb{Z})$ are isomorphisms for $t \geq n$ (so that $K(n) = n$ in this case), while the cokernel of the map $\mathrm{H}_n(\mathrm{GL}_{n-1}(F),\mathbb{Z}) \to \mathrm{H}_n(\mathrm{GL}_n(F),\mathbb{Z})$ is naturally isomorphic to the nth Milnor K-group, $K_n^{\mathrm{M}}(F)$. In fact, if we let

$$H_n(F) := \mathrm{Coker}(\mathrm{H}_n(\mathrm{GL}_{n-1}(F),\mathbb{Z}) \to \mathrm{H}_n(\mathrm{GL}_n(F),\mathbb{Z})),$$

his arguments show that there is an isomorphism of graded rings $H_\bullet(F) \cong K_\bullet^{\mathrm{M}}(F)$ (where the multiplication on the first term comes from direct sum of matrices and cross product on homology). In particular, the non-negatively graded ring $H_\bullet(F)$ is generated in dimension 1.

Recent work of Barge and Morel ([1]) suggested that Milnor-Witt K-theory may play a somewhat analogous role for the homology of the special linear group. The Milnor-Witt K-theory of F is a $\mathbb{Z}$-graded ring $K_\bullet^{\mathrm{MW}}(F)$ surjecting naturally onto Milnor K-theory. It arises as a ring of operations in stable motivic homotopy theory. (For a definition see section 2 below, and for more details see [17, 18, 19].) Let $SH_n(F) := \mathrm{Coker}(\mathrm{H}_n(\mathrm{SL}_{n-1}(F),\mathbb{Z}) \to \mathrm{H}_n(\mathrm{SL}_n(F),\mathbb{Z}))$ for $n \geq 1$, and let $SH_0(F) = \mathbb{Z}[F^\times]$ for convenience. Barge and Morel construct a map of graded algebras $SH_\bullet(F) \to K_\bullet^{\mathrm{MW}}(F)$ for which the square

$$\begin{array}{ccc} SH_\bullet(F) & \longrightarrow & K_\bullet^{\mathrm{MW}}(F) \\ \downarrow & & \downarrow \\ H_\bullet(F) & \longrightarrow & K_\bullet^{\mathrm{M}}(F) \end{array}$$

commutes.

A result of Suslin ([24]) implies that the map $\mathrm{H}_2(\mathrm{SL}_2(F),\mathbb{Z}) = SH_2(F) \to K_2^{\mathrm{MW}}(F)$ is an isomorphism. Since positive-dimensional Milnor-Witt K-theory is generated by elements of degree 1, it follows that the map of Barge and Morel is surjective in even dimensions greater than or equal to 2. They ask the question whether it is in fact an isomorphism in even dimensions.

As to the question of the range of homology stability for the special linear groups of an infinite field, as far as the authors are aware the most general result to date is still that of van der Kallen [25], whose results apply to much more general classes of rings. In the case of a field, he proves homology stability for $\mathrm{H}_n(\mathrm{SL}_t(F),\mathbb{Z})$ in the range $t \geq 2n+1$. On the other hand, known results when n is small suggest a much larger range. For example, the theorems of Matsumoto and Moore imply that the maps $\mathrm{H}_2(\mathrm{SL}_t(F),\mathbb{Z}) \to \mathrm{H}_2(\mathrm{SL}_{t+1}(F),\mathbb{Z})$ are isomorphisms for $t \geq 3$ and are surjective for $t = 2$. In the paper [22] (Conjecture 2.6), C-H. Sah conjectured that for an infinite field F (and more generally for a division algebra with infinite centre), the homomorphism $\mathrm{H}_n(\mathrm{SL}_t(F),\mathbb{Z}) \to \mathrm{H}_n(\mathrm{SL}_{t+1}(F),\ )$ is an isomorphism if $t \geq n+1$ and is surjective for $t = n$.

The present paper addresses the above questions of Barge/Morel and Sah in the case of a field of characteristic zero. We prove the following results about the homology stability for special linear groups:

Theorem 1.1. *Let F be a field of characteristic* 0*. For $n, t \geq 1$, let $f_{t,n}$ be the stabilization homomorphism* $\mathrm{H}_n(\mathrm{SL}_t(F),\mathbb{Z}) \to \mathrm{H}_n(\mathrm{SL}_{t+1}(F),\mathbb{Z})$

(1) *$f_{t,n}$ is an isomorphism for $t \geq n+1$ and is surjective for $t = n$.*
(2) *If n is odd $f_{n,n}$ is an isomorphism*
(3) *If n is even the kernel of $f_{n,n}$ is isomorphic to $I^{n+1}(F)$.*
(4) *For even n the cokernel of $f_{n-1,n}$ is naturally isomorphic to $K_n^{\mathrm{MW}}(F)$.*
(5) *For odd $n \geq 3$ the cokernel of $f_{n-1,n}$ is naturally isomorphic to $2K_n^{\mathrm{M}}(F)$.*

Proof. The proofs of these statements can be found below as follows:

(1) Corollary 5.11.
(2) Corollary 6.12.
(3) Corollary 6.13.
(4) Corollary 6.11.
(5) Corollary 6.13

□

Our strategy is to adapt Suslin's argument for the general linear group in [23] to the case of the special linear group. Suslin's argument is an ingenious variation on the method of van der Kallen in [25], in turn based on ideas of Quillen. The broad idea is to find a highly connected simplicial complex on which the group G_t acts and for which the stabilizers of simplices are (approximately) the groups G_r, with $r \leq t$, and then to use this to construct a spectral sequence calculating the homology of the G_n in terms of the homology of the G_r. Suslin constructs a family $\mathcal{E}(n)$ of such spectral sequences, calculating the homology of $\mathrm{GL}_n(F)$. He constructs partially-defined products $\mathcal{E}(n) \times \mathcal{E}(m) \to \mathcal{E}(n+m)$ and then proves some periodicity and decomposabilty properties which allow him to conclude by an easy induction.

Initially, the attempt to extend these arguments to the case of $\mathrm{SL}_n(F)$ does not appear very promising. Two obstacles to extending Suslin's arguments become quickly apparent.

The main obstacle is Suslin's Theorem 1.8 which says that a certain inclusion of a block diagonal linear group in a block triangular group is a homology isomorphism. The corresponding statement for subgroups of the special linear group is emphatically false, as elementary calculations easily show. Much of Suslin's subsequent results - in particular, the periodicity and decomposability properties of the spectral sequences $\mathcal{E}(n)$ and of the graded algebra $S_\bullet(F)$ which plays a central role - depend on this theorem. And, indeed, the analogous spectral sequences and graded algebra which arise when we replace the general linear with the special linear group do not have these periodicity and decomposability properties.

However, it turns out - at least when the characteristic is zero - that the failure of Suslin's Theorem 1.8 is not fatal. A crucial additional structure is available to us in the case of the special linear group; almost everything in sight in a $\mathbb{Z}[F^\times]$-module. In the analogue of Theorem 1.8, the map of homology groups is a split inclusion whose cokernel has a completely different character as a $\mathbb{Z}[F^\times]$-module than the homology of the block diagonal group. The former is 'additive ', while the latter is 'multiplicative ', notions which we define and explore in section 4 below. This leads us to introduce the concept of '$\mathcal{AM}$ modules', which decompose in a canonical way into a direct sum of an additive factor and a multiplicative factor. This decomposition is sufficiently canonical that in our graded ring structures the additive and multiplicative parts are each ideals. By working modulo the messy additive factors and projecting onto multiplicative parts, we recover an analogue of Suslin's Theorem 1.8 (Theorem 4.23 below), which we then use to prove the necessary periodicity (Theorem 5.10) and decomposability (Theorem 6.8) results.

A second obstacle to emulating the case of the general linear group is the vanishing of the groups $\mathrm{H}_1(\mathrm{SL}_n(F),\mathbb{Z})$. The algebra $H_\bullet(F)$, according to Suslin's arguments, is generated by degree 1. On the other hand, $SH_1(F) = 0 = \mathrm{H}_1(\mathrm{SL}_1(F),\mathbb{Z}) = 0$. This means that the best we can hope for in the case of the special linear group is that the algebra $SH_\bullet(F)$ is generated by degrees 2 and 3. This indeed turns out to be essentially the case, but it means we have to work harder to get our induction off the ground. The necessary arguments in degree $n = 2$ amount to the Theorem of Matsumoto and Moore, as well as variations due to Suslin ([24]) and Mazzoleni ([11]). The argument in degree $n = 3$ was supplied recently in a paper by the present authors ([8]).

We make some remarks on the hypothesis of characteristic zero in this paper: This assumption is used in our definition of $\mathcal{AM}$-modules and the derivation of their properties in section 4 below. In fact, a careful reading of the proofs in that section will show that at any given point all that is required is that the prime subfield be sufficiently large; it must contain an element of order not dividing m for some appropriate m. Thus in fact our arguments can easily be adapted to show that our main results on homology stability for the nth homology group of the special linear groups are true provided the prime field is sufficiently large (in a way that depends on n). However, we have not attempted here to make this more explicit. To do so would make the statements of the results unappealingly complicated, and we will leave it instead to a later paper to deal with the case of positive characteristic. We believe that an appropriate extension of the notion of $\mathcal{AM}$-module will unlock the characteristic $p > 0$ case.

As to our restriction to fields rather than more general rings, we note that Daniel Guin [5] has extended Suslin's results to a larger class of rings with many units. We have not yet investigated a similar extension of the results below to this larger class of rings.

2. Notation and Background Results

2.1. Group Rings and Grothendieck-Witt Rings. For a group G, we let $\mathbb{Z}[G]$ denote the corresponding integral group ring. It has an additive $\mathbb{Z}$-basis consisting of the elements $g \in G$, and is made into a ring by linearly extending the multiplication of group elements. In the case that the group G is the multiplicative group, $F^\times$, of a field F, we will denote the basis elements by $\langle a \rangle$, for $a \in F^\times$. We use this notation in order, for example, to distinguish the elements $\langle 1-a \rangle$ from $1 - \langle a \rangle$, or $\langle -a \rangle$ from $-\langle a \rangle$, and also because it coincides, conveniently for our purposes, with the notation for generators of the Grothendieck-Witt ring (see below). There is an augmentation homomorphism $\epsilon : \mathbb{Z}[G] \to \mathbb{Z}$, $\langle g \rangle \mapsto 1$, whose kernel is the augmentation ideal $\mathcal{I}_G$, generated by the elements $g - 1$. Again, if $G = F^\times$, we denote these generators by $\langle\langle a \rangle\rangle := \langle a \rangle - 1$.

The Grothendieck-Witt ring of a field F is the Grothendieck group, $\mathrm{GW}(F)$, of the set of isometry classes of nondgenerate symmetric bilinear forms under orthogonal sum. Tensor product of forms induces a natural multiplication on the group. As an abstract ring, this can be described as the quotient of the ring $\mathbb{Z}\left[F^\times/(F^\times)^2\right]$ by the ideal generated by the elements $\langle\langle a \rangle\rangle \cdot \langle\langle 1-a \rangle\rangle$, $a \neq 0,1$. (This is just a mild reformulation of the presentation given in Lam, [9], Chapter II, Theorem 4.1.) Here, the induced ring homomorphism $\mathbb{Z}[F^\times] \to \mathbb{Z}\left[F^\times/(F^\times)^2\right] \to \mathrm{GW}(F)$, sends $\langle a \rangle$ to the class of the 1-dimensional form with matrix $[a]$. This class is (also) denoted $\langle a \rangle$. $\mathrm{GW}(F)$ is again an augmented ring and the augmentation ideal, $I(F)$, - also called the *fundamental ideal* - is generated by *Pfister* 1-*forms*, $\langle\langle a \rangle\rangle$. It follows that the n-th power, $I^n(F)$, of this ideal is generated by *Pfister* n-forms $\langle\langle a_1, \ldots, a_n \rangle\rangle := \langle\langle a_1 \rangle\rangle \cdots \langle\langle a_n \rangle\rangle$.

Now let $\textsc{h} := \langle 1 \rangle + \langle -1 \rangle = \langle\langle -1 \rangle\rangle + 2 \in \mathrm{GW}(F)$. Then $\textsc{h} \cdot I(F) = 0$, and the *Witt ring* of F is the ring

$$W(F) := \frac{\mathrm{GW}(F)}{\langle \textsc{h} \rangle} = \frac{\mathrm{GW}(F)}{\textsc{h} \cdot \mathbb{Z}}.$$

Since $\textsc{h} \mapsto 2$ under the augmentation, there is a natural ring homomorphism $W(F) \to \mathbb{Z}/2$. The fundamental ideal $I(F)$ of $\mathrm{GW}(F)$ maps isomorphically to the kernel of this ring homomorphism under the map $\mathrm{GW}(F) \to W(F)$, and we also let $I(F)$ denote this ideal.

For $n \leq 0$, we define $I^n(F) := W(F)$. The graded additive group $I^\bullet(F) = \{I^n(F)\}_{n \in \mathbb{Z}}$ is given the structure of a commutative graded ring using the natural graded multiplication induced from the multiplication on $W(F)$. In particular, if we let $\eta \in I^{-1}(F)$ be the element corresponding to $1 \in W(F)$, then multiplication by $\eta : I^{n+1}(F) \to I^n(F)$ is just the natural inclusion.

2.2. Milnor K-theory and Milnor-Witt K-theory. The Milnor ring of a field F (see [12]) is the graded ring $K^{\mathrm{M}}_\bullet(F)$ with the following presentation:
Generators: $\{a\}$, $a \in F^\times$, in dimension 1.
Relations:

(a) $\{ab\} = \{a\} + \{b\}$ for all $a, b \in F^\times$.

(b) $\{a\}\cdot\{1-a\}=0$ for all $a\in F^\times\setminus\{1\}$.

The product $\{a_1\}\cdots\{a_n\}$ in $K_n^{\mathrm{M}}(F)$ is also written $\{a_1,\ldots,a_n\}$. So $K_0^{\mathrm{M}}(F)=\mathbb{Z}$ and $K_1^{\mathrm{M}}(F)$ is an additive group isomorphic to $F^\times$.

We let $k_\bullet^{\mathrm{M}}(F)$ denote the graded ring $K_\bullet^{\mathrm{M}}(F)/2$ and let $i^n(F) := I^n(F)/I^{n+1}(F)$, so that $i^\bullet(F)$ is a non-negatively graded ring.

In the 1990s, Voevodsky and his collaborators proved a fundamental and deep theorem - originally conjectured by Milnor ([13]) - relating Milnor K-theory to quadratic form theory:

Theorem 2.1 ([20]). *There is a natural isomorphism of graded rings $k_\bullet^{\mathrm{M}}(F)\cong i^\bullet(F)$ sending $\{a\}$ to $\langle\langle a\rangle\rangle$.*

In particular for all $n\geq 1$ we have a natural identification of $k_n^{\mathrm{M}}(F)$ and $i^n(F)$ under which the symbol $\{a_1,\ldots,a_n\}$ corresponds to the class of the form $\langle\langle a_1,\ldots,a_n\rangle\rangle$.

The Milnor-Witt K-theory of a field is the graded ring $K_\bullet^{\mathrm{MW}}(F)$ with the following presentation (due to F. Morel and M. Hopkins, see [17]):

Generators: $[a]$, $a\in F^\times$, in dimension 1 and a further generator η in dimension -1.

Relations:

(a) $[ab]=[a]+[b]+\eta\cdot[a]\cdot[b]$ for all $a,b\in F^\times$

(b) $[a]\cdot[1-a]=0$ for all $a\in F^\times\setminus\{1\}$

(c) $\eta\cdot[a]=[a]\cdot\eta$ for all $a\in F^\times$

(d) $\eta\cdot h=0$, where $h=\eta\cdot[-1]+2\in K_0^{\mathrm{MW}}(F)$.

Clearly there is a unique surjective homomorphism of graded rings $K_\bullet^{\mathrm{MW}}(F)\to K_\bullet^{\mathrm{M}}(F)$ sending $[a]$ to $\{a\}$ and inducing an isomorphism

$$\frac{K_\bullet^{\mathrm{MW}}(F)}{\langle\eta\rangle}\cong K_\bullet^{\mathrm{M}}(F).$$

Furthermore, there is a natural surjective homomorphism of graded rings $K_\bullet^{\mathrm{MW}}(F)\to I^\bullet(F)$ sending $[a]$ to $\langle\langle a\rangle\rangle$ and η to η. Morel shows that there is an induced isomorphism of graded rings

$$\frac{K_\bullet^{\mathrm{MW}}(F)}{\langle h\rangle}\cong I^\bullet(F).$$

The main structure theorem on Milnor-Witt K-theory is the following theorem of Morel:

Theorem 2.2 (Morel, [18]). *The commutative square of graded rings*

$$\begin{tikzcd} K_\bullet^{\mathrm{MW}}(F) \arrow[r] \arrow[d] & K_\bullet^{\mathrm{M}}(F) \arrow[d] \\ I^\bullet(F) \arrow[r] & i^\bullet(F) \end{tikzcd}$$

is cartesian.

Thus for each $n \in \mathbb{Z}$ we have an isomorphism

$$K_n^{\mathrm{MW}}(F) \cong K_n^{\mathrm{M}}(F) \times_{i^n(F)} I^n(F).$$

It follows that for all n there is a natural short exact sequence

$$0 \to I^{n+1}(F) \to K_n^{\mathrm{MW}}(F) \to K_n^{\mathrm{M}}(F) \to 0$$

where the inclusion $I^{n+1}(F) \to K_n^{\mathrm{MW}}(F)$ is given by

$$\langle\langle a_1, \ldots, a_{n+1}\rangle\rangle \mapsto \eta[a_1]\cdots[a_n].$$

Similarly, for $n \geq 0$, there is a short exact sequence

$$0 \to 2K_n^{\mathrm{M}}(F) \to K_n^{\mathrm{MW}}(F) \to I^n(F) \to 0$$

where the inclusion $2K_n^{\mathrm{M}}(F) \to K_n^{\mathrm{MW}}(F)$ is given (for $n \geq 1$) by

$$2\{a_1, \ldots, a_n\} \mapsto h[a_1]\cdots[a_n].$$

Observe that, when $n \geq 2$,

$$h[a_1][a_2]\cdots[a_n] = ([a_1][a_2] - [a_2][a_1])[a_3]\cdots[a_n] = [a_1^2][a_2]\cdots[a_n].$$

(The first equality follows from Lemma 2.3 (3) below, the second from the observation that $[a_1^2]\cdots[a_n] \in \mathrm{Ker}(K_n^{\mathrm{MW}}(F) \to I^n(F)) = 2K_n^{\mathrm{M}}(F)$ and the fact, which follows from Morel's theorem, that the composite $2K_n^{\mathrm{M}}(F) \to K_n^{\mathrm{MW}}(F) \to K_n^{\mathrm{M}}(F)$ is the natural inclusion map.)
When $n = 0$ we have an isomorphism of rings

$$\mathrm{GW}(F) \cong W(F) \times_{\mathbb{Z}/2} \mathbb{Z} \cong K_0^{\mathrm{MW}}(F).$$

Under this isomorphism $\langle\langle a\rangle\rangle$ corresponds to $\eta[a]$ and $\langle a\rangle$ corresponds to $\eta[a]+1$. (Observe that with this identification, $h = \eta[-1] + 2 = \langle 1\rangle + \langle -1\rangle \in K_0^{\mathrm{MW}}(F) = \mathrm{GW}(F)$, as expected.)
Thus each $K_n^{\mathrm{MW}}(F)$ has the structure of a $\mathrm{GW}(F)$-module (and hence also of a $\mathbb{Z}[F^\times]$-module), with the action given by $\langle\langle a\rangle\rangle \cdot ([a_1]\cdots[a_n]) = \eta[a][a_1]\cdots[a_n]$.
We record here some elementary identities in Milnor-Witt K-theory which we will need below.

Lemma 2.3. *Let $a, b \in F^\times$. The following identities hold in the Milnor-Witt K-theory of F:*

(1) $[a][-1] = [a][a]$.
(2) $[ab] = [a] + \langle a\rangle[b]$.
(3) $[a][b] = -\langle -1\rangle[b][a]$.

Proof.

(1) See, for example, the proof of Lemma 2.7 in [7].
(2) $\langle a\rangle b = (\eta[a] + 1)[b] = \eta[a][b] + [b] = [ab] - [a]$.
(3) See [7], Lemma 2.7.

□

2.3. Homology of Groups. Given a group G and a $\mathbb{Z}[G]$-module M, $\mathrm{H}_n(G,M)$ will denote the nth homology group of G with coefficients in the module M. $B_\bullet(G)$ will denote the *right bar resolution of* G: $B_n(G)$ is the free right $\mathbb{Z}[G]$-module with basis the elements $[g_1|\cdots|g_n]$, $g_i \in G$. ($B_0(G)$ is isomorphic to $\mathbb{Z}[G]$ with generator the symbol $[\ ]$.) The boundary $d = d_n : B_n(G) \to B_{n-1}(G)$, $n \geq 1$, is given by

$$d([g_1|\cdots|g_n]) = \sum_{i=o}^{n-1}(-1)^i[g_1|\cdots|\hat{g}_i|\cdots|g_n] + (-1)^n[g_1|\cdots|g_{n-1}]\langle g_n\rangle .$$

The augmentation $B_0(G) \to \mathbb{Z}$ makes $B_\bullet(G)$ into a free resolution of the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$, and thus $\mathrm{H}_n(G,M) = H_n(B_\bullet(G)\otimes_{\mathbb{Z}[G]} M)$.
If $C_\bullet = (C_q,d)$ is a non-negative complex of $\mathbb{Z}[G]$-modules, then $E_{\bullet,\bullet} := B_\bullet(G)\otimes_{\mathbb{Z}[G]} C_\bullet$ is a double complex of abelian groups. Each of the two filtrations on $E_{\bullet,\bullet}$ gives a spectral sequence converging to the homology of the total complex of $E_{\bullet,\bullet}$, which is by definition, $\mathrm{H}_\bullet(G,C)$. (see, for example, Brown, [3], Chapter VII).
The first spectral sequence has the form

$$E^2_{p,q} = \mathrm{H}_p(G, H_q(C)) \Longrightarrow \mathrm{H}_{p+q}(G,C).$$

In the special case that there is a weak equivalence $C_\bullet \to \mathbb{Z}$ (the complex consisting of the trivial module $\mathbb{Z}$ concentrated in dimension 0), it follows that $\mathrm{H}_\bullet(G,C) = \mathrm{H}_\bullet(G,\mathbb{Z})$.
The second spectral sequence has the form

$$E^1_{p,q} = \mathrm{H}_p(G, C_q) \Longrightarrow \mathrm{H}_{p+q}(G,C).$$

Thus, if $C_\bullet$ is weakly equivalent to $\mathbb{Z}$, this gives a spectral sequence converging to $\mathrm{H}_\bullet(G,\mathbb{Z})$.
Our analysis of the homology of special linear groups will exploit the action of these groups on certain permutation modules. It is straightforward to compute the map induced on homology groups by a map of permutation modules. We recall the following basic principles (see, for example, [6]): If G is a group and if X is a G-set, then Shapiro's Lemma says that

$$\mathrm{H}_p(G,\mathbb{Z}[X]) \cong \bigoplus_{y\in X/G} \mathrm{H}_p(G_y,\mathbb{Z}),$$

the isomorphism being induced by the maps

$$\mathrm{H}_p(G_y,\mathbb{Z}) \to \mathrm{H}_p(G,\mathbb{Z}[X])$$

described at the level of chains by

$$B_p \otimes_{\mathbb{Z}[G_y]} \mathbb{Z} \to B_p \otimes_{\mathbb{Z}[G]} \mathbb{Z}[X], \quad z\otimes 1 \mapsto z\otimes y.$$

Let X_i, $i=1,2$ be transitive G-sets. Let $x_i \in X_i$ and let H_i be the stabiliser of x_i, $i=1,2$. Let $\phi : \mathbb{Z}[X_1] \to \mathbb{Z}[X_2]$ be a map of $\mathbb{Z}[G]$-modules with

$$\phi(x_1) = \sum_{g\in G/H_2} n_g g x_2, \qquad \text{with } n_g \in \mathbb{Z}.$$

Then the induced map $\phi_\bullet : \mathrm{H}_\bullet(H_1, \mathbb{Z}) \to \mathrm{H}_\bullet(H_2, \mathbb{Z})$ is given by the formula

$$(1) \qquad \phi_\bullet(z) = \sum_{g \in H_1 \backslash G/H_2} n_g \mathrm{cor}^{H_2}_{g^{-1}H_1 g \cap H_2} \mathrm{res}^{g^{-1}H_1 g}_{g^{-1}H_1 g \cap H_2} \left(g^{-1} \cdot z\right)$$

There is an obvious extension of this formula to non-transitive G-sets.

2.4. Homology of $\mathrm{SL}_n(F)$ and Milnor-Witt K-theory. Let F be an infinite field.

The theorem of Matsumoto and Moore ([10], [16]) gives a presentation of the group $\mathrm{H}_2(\mathrm{SL}_2(F), \mathbb{Z})$. It has the following form: The generators are symbols $\langle a_1, a_1 \rangle$, $a_i \in F^\times$, subject to the relations:

(i) $\langle a_1, a_2 \rangle = 0$ if $a_i = 1$ for some i
(ii) $\langle a_1, a_2 \rangle = \langle a_2^{-1}, a_1 \rangle$
(iii) $\langle a_1, a_2 b_2 \rangle + \langle a_2, b_2 \rangle = \langle a_1 a_2, b_2 \rangle + \langle a_1, a_2 \rangle$
(iv) $\langle a_1, a_2 \rangle = \langle a_1, -a_1 a_2 \rangle$
(v) $\langle a_1, a_2 \rangle = \langle a_1, (1 - a_1) a_2 \rangle$

It can be shown that for all $n \geq 2$, $K_n^{\mathrm{MW}}(F)$ admits a (generalised) Matsumoto-Moore presentation:

Theorem 2.4 ([7], Theorem 2.5). *For $n \geq 2$, $K_n^{\mathrm{MW}}(F)$ admits the following presentation as an additive group:*
Generators: The elements $[a_1][a_2] \cdots [a_n]$, $a_i \in F^\times$.
Relations:

(i) $[a_1][a_2] \cdots [a_n] = 0$ *if* $a_i = 1$ *for some* i.
(ii) $[a_1] \cdots [a_{i-1}][a_i] \cdots [a_n] = [a_1] \cdots [a_i^{-1}][a_{i-1}] \cdots [a_n]$
(iii) $[a_1] \cdots [a_{n-1}][a_n b_n] + [a_1] \cdots \widehat{[a_{n-1}]}[a_n][b_n] = [a_1] \cdots [a_{n-1} a_n][b_n] + [a_1] \cdots [a_{n-1}][a_n]$
(iv) $[a_1] \cdots [a_{n-1}][a_n] = [a_1] \cdots [a_{n-1}][-a_{n-1} a_n]$
(v) $[a_1] \cdots [a_{n-1}][a_n] = [a_1] \cdots [a_{n-1}][(1 - a_{n-1}) a_n]$

In particular, it follows when $n = 2$ that there is a natural isomorphism $K_2^{\mathrm{MW}}(F) \cong \mathrm{H}_2(\mathrm{SL}_2(F), \mathbb{Z})$. This last fact is essentially due to Suslin ([24]). A more recent proof, which we will need to invoke below, has been given by Mazzoleni ([11]).

Recall that Suslin ([23]) has constructed a natural surjective homomorphism $\mathrm{H}_n(\mathrm{GL}_n(F), \mathbb{Z}) \to K_n^{\mathrm{M}}(F)$ whose kernel is the image of $\mathrm{H}_n(\mathrm{GL}_{n-1}(F), \mathbb{Z})$.

In [8], the authors proved that the map $\mathrm{H}_3(\mathrm{SL}_3(F), \mathbb{Z}) \to \mathrm{H}_3(\mathrm{GL}_3(F), \mathbb{Z})$ is injective, that the image of the composite $\mathrm{H}_3(\mathrm{SL}_3(F), \mathbb{Z}) \to \mathrm{H}_3(\mathrm{GL}_3(F), \mathbb{Z}) \to K_3^{\mathrm{M}}(F)$ is $2K_3^{\mathrm{M}}(F)$ and that the kernel of this composite is precisely the image of $\mathrm{H}_3(\mathrm{SL}_2(F), \mathbb{Z})$.

In the next section we will construct natural homomorphisms $T_n \circ \epsilon_n$: $\mathrm{H}_n(\mathrm{SL}_n(F), \mathbb{Z}) \to K_n^{\mathrm{MW}}(F)$, in a manner entirely analogous to Suslin's construction. In particular, the image of $\mathrm{H}_n(\mathrm{SL}_{n-1}(F), \mathbb{Z})$ is contained in the kernel of $T_n \circ \epsilon_n$ and the diagrams

$$\begin{tikzcd} \mathrm{H}_n(\mathrm{SL}_n(F),\mathbb{Z}) \arrow[r] \arrow[d] & K_n^{\mathrm{MW}}(F) \arrow[d] \\ \mathrm{H}_n(\mathrm{GL}_n(F),\mathbb{Z}) \arrow[r] & K_n^{\mathrm{M}}(F) \end{tikzcd}$$

commute. It follows that the image of $T_3 \circ \epsilon_3$ is $2K_3^{\mathrm{M}}(F) \subset K_3^{\mathrm{MW}}(F)$, and its kernel is the image of $\mathrm{H}_3(\mathrm{SL}_2(F),\mathbb{Z})$.

3. The algebra $\tilde{S}(F^\bullet)$

In this section we introduce a graded algebra functorially associated to F which admits a natural homomorphism to Milnor-Witt K-theory and from the homology of $\mathrm{SL}_n(F)$. It is the analogue of Suslin's algebra $S_\bullet(F)$ in [24], which admits homomorphisms to Milnor K-theory and from the homology of $\mathrm{GL}_n(F)$. However, we will need to modify this algebra in the later sections below, by projecting onto the 'multiplicative ' part, in order to derive our results about the homology of $\mathrm{SL}_n(F)$.

We say that a finite set of vectors $v_1,\dots,v_q$ in an n-dimensional vector space V *are in general position* if every subset of size $\min(q,n)$ is linearly independent. If $v_1,\dots,v_q$ are elements of the n-dimensional vector space V and if $\mathcal{E}$ is an ordered basis of V, we let $[v_1|\cdots|v_q]_{\mathcal{E}}$ denote the $n\times q$ matrix whose i-th column is the components of v_i with respect to the basis $\mathcal{E}$.

3.1. Definitions. For a field F and finite-dimensional vector spaces V and W, we let $X_p(W,V)$ denote the set of all ordered p-tuples of the form

$$((w_1,v_1),\dots,(w_p,v_p))$$

where $(w_i,v_i)\in W\oplus V$ and the v_i are in general position. We also define $X_0(W,V) := \emptyset$. $X_p(W,V)$ is naturally an $\mathrm{A}(W,V)$-module, where

$$\mathrm{A}(W,V) := \begin{pmatrix} \mathrm{Id}_W & \mathrm{Hom}(V,W) \\ 0 & \mathrm{GL}(V)\end{pmatrix} \subset \mathrm{GL}(W\oplus V)$$

Let $C_p(W,V) = \mathbb{Z}[X_p(W,V)]$, the free abelian group with basis the elements of $X_p(W,V)$. We obtain a complex, $C_\bullet(W,V)$, of $\mathrm{A}(W,V)$-modules by introducing the natural simplicial boundary map

$$\begin{array}{rcl} d_{p+1}: C_{p+1}(W,V) & \to & C_p(W,V) \\ ((w_1,v_1),\dots,(w_{p+1},v_{p+1})) & \mapsto & \\ & & \displaystyle\sum_{i=1}^{p+1}(-1)^{i+1}((w_1,v_1),\dots,\widehat{(w_i,v_i)},\dots,(w_{p+1},v_{p+1})) \end{array}$$

Lemma 3.1. *If F is infinite, then $\mathrm{H}_p(C_\bullet(W,V)) = 0$ for all p.*

Proof. If

$$z = \sum_i n_i((w_1^i, v_1^i), \ldots, (w_p^i, v_p^i)) \in C_p(W, V)$$

is a cycle, then since F is infinite, it is possible to choose $v \in V$ such that $v, v_1^i, \ldots, v_p^i$ are in general position for all i. Then $z = d_{p+1}((-1)^p s_v(z))$ where s_v is the 'partial homotopy operator' defined by $s_v((w_1, v_1), \ldots, (w_p, v_p)) =$

$$\begin{cases} ((w_1, v_1), \ldots, (w_p, v_p), (0, v)), & \text{if } v, v_1, \ldots v_p \text{ are in general position,} \\ 0, & \text{otherwise} \end{cases}$$

□

We will assume our field F is infinite for the remainder of this section. (In later sections, it will even be assumed to be of characteristic zero.)
If $n = \dim_F(V)$, we let $H(W, V) := \mathrm{Ker}(d_n) = \mathrm{Im}(d_{n+1})$. This is an $\mathrm{A}(W, V)$-submodule of $C_n(W, V)$. Let $\tilde{\mathrm{S}}(W, V) := \mathrm{H}_0(\mathrm{SA}(W, V), H(W, V)) = H(W, V)_{\mathrm{SA}(W,V)}$ where $\mathrm{SA}(W, V) := \mathrm{A}(W, V) \cap \mathrm{SL}(W \oplus V)$.
If $W' \subset W$, there are natural inclusions $X_p(W', V) \to X_p(W, V)$ inducing a map of complexes of $\mathrm{A}(W', V)$-modules $C_\bullet(W', V) \to C_\bullet(W, V)$.
When $W = 0$, we will use the notation, $X_p(V)$, $C_p(V)$, $H(V)$ and $\tilde{\mathrm{S}}(V)$ instead of $X_p(0, V)$, $C_p(0, V)$, $H(0, V)$ and $\tilde{\mathrm{S}}(0, V)$
Since, $\mathrm{A}(W, V)/\mathrm{SA}(W, V) \cong F^\times$, any homology group of the form

$$\mathrm{H}_i(\mathrm{SA}(W, V), M), \text{ where } M \text{ is a } \mathrm{A}(W, V)\text{-module,}$$

is naturally a $\mathbb{Z}[F^\times]$-module: If $a \in F^\times$ and if $g \in \mathrm{A}(W, V)$ is any element of determinant a, then the action of a is the map on homology induced by conjugation by g on $\mathrm{A}(W, V)$ and multiplication by g on M. In particular, the groups $\tilde{\mathrm{S}}(W, V)$ are $\mathbb{Z}[F^\times]$-modules.
Let $e_1, \ldots, e_n$ denote the standard basis of F^n. Given $a_1, \ldots, a_n \in F^\times$, we let $\lfloor a_1, \ldots, a_n \rceil$ denote the class of $d_{n+1}(e_1, \ldots, e_n, a_1e_1 + \cdots + a_ne_n)$ in $\tilde{\mathrm{S}}(F^n)$. If $b \in F^\times$, then $\langle b \rangle \cdot \lfloor a_1, \ldots, a_n \rceil$ is represented by

$$d_{n+1}(e_1, \ldots, be_i, \ldots, e_n, a_1e_1 + \cdots a_ibe_i \cdots + a_ne_n)$$

for any i. (As a lifting of $b \in F^\times$, choose the diagonal matrix with b in the (i, i)-position and 1 in all other diagonal positions.)

Remark 3.2. Given $x = (v_1, \ldots, v_v, v) \in X_{n+1}(F^n)$, let $A = [v_1 | \cdots | v_n] \in \mathrm{GL}_n(F)$ of determinant $\det A$ and let $A' = \mathrm{diag}(1, \ldots, 1, \det A)$. Then $B = A'A^{-1} \in \mathrm{SL}_n(F)$ and thus x is in the $\mathrm{SL}_n(F)$-orbit of

$$(e_1, \ldots, e_{n-1}, \det Ae_n, A'w) \text{ with } w = A^{-1}v,$$

and hence $d_{n+1}(x)$ represents the element $\langle \det A \rangle \lfloor w \rceil$ in $\tilde{S}(F^n)$.

Theorem 3.3. $\tilde{\mathrm{S}}(F^n)$ *has the following presentation as a* $\mathbb{Z}[F^\times]$*-module:*
Generators: The elements $\lfloor a_1, \ldots, a_n \rceil$, $a_i \in F^\times$

Relations: For all $a_1, \ldots, a_n \in F^\times$ and for all $b_1, \ldots, b_n \in F^\times$ with $b_i \neq b_j$ for $i \neq j$

$$\lfloor b_1a_1, \ldots, b_na_n \rceil - \lfloor a_1, \ldots, a_n \rceil =$$
$$\sum_{i=1}^{n} (-1)^{n+i} \left\langle (-1)^{n+i} a_i \right\rangle \lfloor a_1(b_1 - b_i), \ldots, a_i\widehat{(b_i - b_i)}, \ldots, a_n(b_n - b_i), b_i \rceil.$$

Proof. Taking $\mathrm{SL}_n(F)$-coinvariants of the exact sequence of $\mathbb{Z}[\mathrm{GL}_n(F)]$-modules

$$C_{n+2}(F^n) \xrightarrow{d_{n+2}} C_{n+1}(F^n) \xrightarrow{d_{n+1}} H(F^n) \longrightarrow 0$$

gives the exact sequence of $\mathbb{Z}[F^\times]$-modules

$$C_{n+2}(F^n)_{\mathrm{SL}_n(F)} \xrightarrow{d_{n+2}} C_{n+1}(F^n)_{\mathrm{SL}_n(F)} \xrightarrow{d_{n+1}} \tilde{S}(F^n) \longrightarrow 0.$$

It is straightforward to verify that

$$X_{n+1}(F^n) \cong \coprod_{a=(a_1,\ldots,a_n), a_i \neq 0} \mathrm{GL}_n(F) \cdot (e_1, \ldots, e_n, a)$$

as a $\mathrm{GL}_n(F)$-set. It follows that

$$C_{n+1}(F^n) \cong \bigoplus_a \mathbb{Z}[\mathrm{GL}_n(F)] \cdot (e_1, \ldots, e_n, a)$$

as a $\mathbb{Z}[\mathrm{GL}_n(F)]$-module, and thus that

$$C_{n+1}(F^n)_{\mathrm{SL}_n(F)} \cong \bigoplus_a \mathbb{Z}[F^\times] \cdot (e_1, \ldots, e_n, a)$$

as a $\mathbb{Z}[F^\times]$-module.

Similarly, every element of $X_{n+2}(F^n)$ is in the $\mathrm{GL}_n(F)$-orbit of a unique element of the form $(e_1, \ldots, e_n, a, b \cdot a)$ where $a = (a_1, \ldots, a_n)$ with $a_i \neq 0$ for all i and $b = (b_1, \ldots, b_n)$ with $b_i \neq 0$ for all i and $b_i \neq b_j$ for all $i \neq j$, and $b \cdot a := (b_1a_1, \ldots, b_na_n)$. Thus

$$X_{n+2}(F^n) \cong \coprod_{(a,b)} \mathrm{GL}_n(F) \cdot (e_1, \ldots, e_n, a, b \cdot a)$$

as a $\mathrm{GL}_n(F)$-set and

$$C_{n+2}(F^n)_{\mathrm{SL}_n(F)} \cong \bigoplus_{(a,b)} \mathbb{Z}[F^\times] \cdot (e_1, \ldots, e_n, a, b \cdot a)$$

as a $\mathbb{Z}[F^\times]$-module.

So d_{n+1} induces an isomorphism

$$\frac{\oplus \mathbb{Z}[F^\times] \cdot (e_1, \ldots, e_n, a)}{\langle d_{n+2}(e_1, \ldots, e_n, a, b \cdot a) | (a, b) \rangle} \cong \tilde{S}(F^n).$$

Now $d_{n+2}(e_1, \ldots, e_n, a, b \cdot a) =$

$$\sum_{i=1}^{n} (-1)^{i+1} (e_1, \ldots, \hat{e_i}, \ldots, e_n, a, b \cdot a) + (-1)^i \big((e_1, \ldots, e_n, b \cdot a) - (e_1, \ldots, e_n, a) \big).$$

Applying the idea of Remark 3.2 to the terms $(e_1,\ldots,\hat{e_i},\ldots,e_n,a,b\cdot a)$ in the sum above, we let $M_i(a) := [e_1|\cdots|\hat{e_i}|\cdots|e_n|a]$ and $\delta_i = \det M_i(a) = (-1)^{n-i}a_i$. Since

$$M_i(a)^{-1} = \begin{pmatrix} 1 & \ldots & 0 & -a_1/a_i & 0 & \ldots & 0 \\ 0 & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 1 & -a_{i-1}/a_i & 0 & \ldots & 0 \\ 0 & \ldots & 0 & -a_{i+1}a_i & 1 & \ldots & 0 \\ 0 & \ldots & 0 & \vdots & 0 & \ddots & 0 \\ 0 & \ldots & 0 & -a_n/a_i & 0 & \ldots & 1 \\ 0 & \ldots & 0 & 1/a_i & 0 & \ldots & 0 \end{pmatrix}$$

it follows that $d_{n+1}(e_1,\ldots,\hat{e_i},\ldots,e_n,a,b\cdot a)$ represents $\langle\delta_i\rangle\lfloor w_i\rceil \in \tilde{S}(F^n)$ where $w_i = M_i(a)^{-1}(b\cdot a) = (a_1(b_1-b_i),\ldots,a_i\widehat{(b_i-b_i)},\ldots,a_n(b_n-b_i),b_i)$. This proves the theorem. □

3.2. Products. If $W' \subset W$, there is a natural bilinear pairing

$$C_p(W',V)\times C_q(W)\to C_{p+q}(W\oplus V), \qquad (x,y)\mapsto x*y$$

defined on the basis elements by

$$((w_1',v_1),\ldots,(w_p',v_p))*(w_1,\ldots,w_q) := \left((w_1',v_1),\ldots,(w_p',v_p),(w_1,0),\ldots,(w_q,0)\right).$$

This pairing satisfies $d_{p+q}(x*y) = d_p(x)*y+(-1)^p x*d_q(y)$.
Furthermore, if $\alpha\in \mathrm{A}(W',V)\subset \mathrm{GL}(W\oplus V)$ then $(\alpha x)*y = \alpha(x*y)$, and if $\alpha\in\mathrm{GL}(V)\subset \mathrm{A}(W',V)\subset\mathrm{GL}(W\oplus V)$ and $\beta\in\mathrm{GL}(W)\subset\mathrm{GL}(W\oplus V)$, then $(\alpha x)*(\beta y) = (\alpha\cdot\beta)(x*y)$. (However, if $W'\neq 0$ then the images of $\mathrm{A}(W',V)$ and $\mathrm{GL}(W)$ in $\mathrm{GL}(W\oplus V)$ don't commute.)
In particular, there are induced pairings on homology groups

$$H(W',V)\otimes H(W)\to H(W\oplus V),$$

which in turn induce well-defined pairings

$$\tilde{\mathrm{S}}(W',V)\otimes H(W)\to\tilde{\mathrm{S}}(W,V) \text{ and } \tilde{\mathrm{S}}(V)\otimes\tilde{\mathrm{S}}(W)\to\tilde{\mathrm{S}}(W\oplus V).$$

Observe further that this latter pairing is $\mathbb{Z}[F^\times]$-balanced: If $a\in F^\times$, $x\in\tilde{\mathrm{S}}(W)$ and $y\in\tilde{\mathrm{S}}(V)$, then $(\langle a\rangle x)*y = x*(\langle a\rangle y) = \langle a\rangle(x*y)$. Thus there is a well-defined map

$$\tilde{\mathrm{S}}(V)\otimes_{\mathbb{Z}[F^\times]}\tilde{\mathrm{S}}(W)\to\tilde{\mathrm{S}}(W\oplus V).$$

In particular, the groups $\{H(F^n)\}_{n\geq 0}$ form a natural graded (associative) algebra, and the groups $\{\tilde{\mathrm{S}}(F^n)\}_{n\geq 0} = \tilde{S}(F^\bullet)$ form a graded associative $\mathbb{Z}[F^\times]$-algebra.
The following explicit formula for the product in $\tilde{S}(F^\bullet)$ will be needed below:

Lemma 3.4. *Let $a_1,\ldots,a_n$ and $a_1',\ldots,a_m'$ be elements of $F^\times$. Let $b_1,\ldots,b_n$, $b_1',\ldots,b_m'$ be any elements of $F^\times$ satisfying $b_i\neq b_j$ for $i\neq j$ and $b_s'\neq b_t'$ for $s\neq t$.*

Then

$$\lfloor a_1,\ldots,a_n\rceil * \lfloor a'_1,\ldots,a'_m\rceil =$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{m}(-1)^{m+n+i+j}\left\langle(-1)^{i+j}a_ia'_j\right\rangle \times$$
$$\times\lfloor a_1(b_1-b_i),\ldots,a_i\widehat{(b_i-b_i)},\ldots,b_i,a'_1(b'_1-b'_j),\ldots,a'_j\widehat{(b'_j-b'_j)},\ldots,b'_j\rceil$$
$$+(-1)^n\sum_{i=1}^{n}(-1)^{i+1}\left\langle(-1)^{i+1}a_i\right\rangle\lfloor a_1(b_1-b_i),\ldots,a_i\widehat{(b_i-b_i)},\ldots,b_i,b'_1a'_1,\ldots,b'_ma'_m\rceil$$
$$+(-1)^m\sum_{j=1}^{m}(-1)^{j+1}\left\langle(-1)^{j+1}a'_j\right\rangle\lfloor b_1a_1,\ldots,b_na_n,a'_1(b'_1-b'_j),\ldots,a'_j\widehat{(b'_j-b'_j)},\ldots,b'_j\rceil$$
$$+\lfloor b_1a_1,\ldots,b_na_n,b'_1a'_1,\ldots,b'_ma'_m\rceil$$

Proof. This is an entirely straightforward calculation using the definition of the product, Remark 3.2, the matrices $M_i(a)$, $M_j(a')$ as in the proof of Theorem 3.3, and the partial homotopy operators s_v with $v=(a_1b_1,\ldots,a_nb_n,a'_1b'_1,\ldots,a'_mb'_m)$. □

3.3. THE MAPS ϵ_V. If $\dim_F(V)=n$, then the exact sequence of $\mathrm{GL}(V)$-modules

$$0\longrightarrow H(V)\longrightarrow C_n(V)\xrightarrow{d_n}C_{n-1}(V)\xrightarrow{d_{n-1}}\cdots\xrightarrow{d_1}C_0(V)=\mathbb{Z}\longrightarrow 0$$

gives rise to an iterated connecting homomorphism

$$\epsilon_V:\mathrm{H}_n(\mathrm{SL}(V),\mathbb{Z})\to\mathrm{H}_0(\mathrm{SL}(V),H(V))=\tilde{\mathrm{S}}(V).$$

Note that the collection of groups $\{\mathrm{H}_n(\mathrm{SL}_n(F),\mathbb{Z})\}$ form a graded $\mathbb{Z}[F^\times]$-algebra under the graded product induced by exterior product on homology, together with the obvious direct sum homomorphism $\mathrm{SL}_n(F)\times\mathrm{SL}_m(F)\to\mathrm{SL}_{n+m}(F)$.

LEMMA 3.5. *The maps $\epsilon_n:\mathrm{H}_n(\mathrm{SL}_n(F),\mathbb{Z})\to\tilde{\mathrm{S}}(F^n)$, $n\geq 0$, give a well-defined homomorphism of graded $\mathbb{Z}[F^\times]$-algebras; i.e.*

(1) *If $a\in F^\times$ and $z\in\mathrm{H}_n(\mathrm{SL}_n(F),\mathbb{Z})$, then $\epsilon_n(\langle a\rangle z)=\langle a\rangle\epsilon_n(z)$ in $\tilde{\mathrm{S}}(F^n)$, and*
(2) *If $z\in\mathrm{H}_n(\mathrm{SL}_n(F),\mathbb{Z})$ and $w\in\mathrm{H}_m(\mathrm{SL}_m(F),\mathbb{Z})$ then*
$$\epsilon_{n+m}(z\times w)=\epsilon_n(z)*\epsilon_m(w)\ \text{in}\ \tilde{\mathrm{S}}(F^{n+m}).$$

Proof.

(1) The exact sequence above is a sequence of $\mathrm{GL}(V)$-modules and hence all of the connecting homomorphisms $\delta_i:\mathrm{H}_{n-i+1}(\mathrm{SL}(V),\mathrm{Im}(d_i))\to\mathrm{H}_{n-i}(\mathrm{SL}(V),\mathrm{Ker}(d_i))$ are $F^\times$-equivariant.
(2) Let $\mathcal{C}^\tau_\bullet(V)$ denote the truncated complex.
$$\mathcal{C}^\tau_p(V)=\begin{cases}C_p(V), & p\leq\dim_F(V)\\ 0, & p>\dim_F(V)\end{cases}$$

Thus $H(V) \to \mathcal{C}^{\mathcal{T}}_\bullet(V)$ is a weak equivalence of complexes (where we regard $H(V)$ as a complex concentrated in dimension $\dim(V)$). Since the complexes $\mathcal{C}^{\mathcal{T}}_\bullet(V)$ are complexes of free abelian groups, it follows that for two vector spaces V and W, the map $H(V) \otimes_{\mathbb{Z}} H(W) \to T_\bullet(V,W)$ is an equivalence of complexes, where $T_\bullet(V,W)$ is the total complex of the double complex $\mathcal{C}^{\mathcal{T}}_\bullet(V) \otimes_{\mathbb{Z}} \mathcal{C}^{\mathcal{T}}_\bullet(W)$. Now $T_\bullet(V,W)$ is a complex of $\mathrm{SL}(V) \times \mathrm{SL}(W)$-modules, and the product $*$ induces a commutative diagram of complexes of $\mathrm{SL}(V) \times \mathrm{SL}(W)$-complexes:

$$\begin{tikzcd} H(V) \otimes_{\mathbb{Z}} H(W) \arrow[r] \arrow[d, "*"] & \mathcal{C}^{\mathcal{T}}_\bullet(V) \otimes \mathcal{C}^{\mathcal{T}}_\bullet(W) \arrow[d, "*"] \\ H(V \oplus W) \arrow[r] & \mathcal{C}^{\mathcal{T}}_\bullet(V \oplus W) \end{tikzcd}$$

which, in turn, induces a commutative diagram

$$\begin{tikzcd} \mathrm{H}_n(\mathrm{SL}(V),\mathbb{Z}) \otimes \mathrm{H}_m(\mathrm{SL}(W),\mathbb{Z}) \arrow[r, "\epsilon_V \otimes \epsilon_W"] \arrow[d, "\times"] & \mathrm{H}_0(\mathrm{SL}(V),H(V)) \otimes \mathrm{H}_0(\mathrm{SL}(W),H(W)) \arrow[d, "\times"] \\ \mathrm{H}_{n+m}(\mathrm{SL}(V) \times \mathrm{SL}(W),\mathbb{Z} \otimes \mathbb{Z}) \arrow[r, "\epsilon_{T_\bullet}"] \arrow[d] & \mathrm{H}_0(\mathrm{SL}(V) \times \mathrm{SL}(W),H(V) \otimes H(W)) \arrow[d] \\ \mathrm{H}_{n+m}(\mathrm{SL}(V \oplus W),\mathbb{Z}) \arrow[r, "\epsilon_{V \oplus W}"] & \mathrm{H}_0(\mathrm{SL}(V \oplus W),H(V \oplus W)) \end{tikzcd}$$

(where $n = \dim(V)$ and $m = \dim(W)$).

$\square$

Lemma 3.6. *If $V = W \oplus W'$ with $W' \neq 0$, then the composite*

$$\mathrm{H}_n(\mathrm{SL}(W),\mathbb{Z}) \longrightarrow \mathrm{H}_n(\mathrm{SL}(V),\mathbb{Z}) \xrightarrow{\epsilon_V} \tilde{\mathrm{S}}(V)$$

is zero.

Proof. The exact sequence of $\mathrm{SL}(V)$-modules

$$0 \to \mathrm{Ker}(d_1) \to C_1(V) \to \mathbb{Z} \to 0$$

is split as a sequence of $\mathrm{SL}(W)$-modules via the map $\mathbb{Z} \to C_1(V), m \mapsto m \cdot e$ where e is any nonzero element of W'. It follows that the connecting homomorphism $\delta_1 : \mathrm{H}_n(\mathrm{SL}(W),\mathbb{Z}) \to \mathrm{H}_{n-1}(\mathrm{SL}(W),\mathrm{Ker}(d_1))$ is zero. $\square$

Let $\mathrm{SH}_n(F)$ denote the cokernel of the map $\mathrm{H}_n(\mathrm{SL}_{n-1}(F),\mathbb{Z}) \to \mathrm{H}_n(\mathrm{SL}_n(F),\mathbb{Z})$. It follows that the maps ϵ_n give well-defined homomorphisms $\mathrm{SH}_n(F) \to \tilde{\mathrm{S}}(F^n)$, which yield a homomorphism of graded $\mathbb{Z}[F^\times]$-algebras $\epsilon_\bullet : \mathrm{SH}_\bullet(F) \to \tilde{\mathrm{S}}(F^\bullet)$.

3.4. The maps D_V. Suppose now that W and V are vector spaces and that $\dim(V) = n$. Fix a basis $\mathcal{E}$ of V. The group $\mathrm{A}(W,V)$ acts transitively on $X_n(W,V)$ (with trivial stabilizers), while the orbits of $\mathrm{SA}(W,V)$ are in one-to-one correspondence with the points of $F^\times$ via the correspondence

$$X_n(W,V) \to F^\times, \quad ((w_1,v_1),\ldots,(w_n,v_n)) \mapsto \det\left([v_1|\cdots|v_n]_{\mathcal{E}}\right).$$

Thus we have an induced isomorphism

$$\mathrm{H}_0(\mathrm{SA}(W,V),C_n(W,V)) \xrightarrow[\cong]{\det} \mathbb{Z}[F^\times].$$

Taking $\mathrm{SA}(W,V)$-coinvariants of the inclusion $H(W,V) \to C_n(W,V)$ then yields a homomorphism of $\mathbb{Z}[F^\times]$-modules

$$D_{W,V} : \tilde{\mathrm{S}}(W,V) \to \mathbb{Z}[F^\times].$$

In particular, for each $n \geq 1$ we have a homomorphism of $\mathbb{Z}[F^\times]$-modules $D_n : \tilde{\mathrm{S}}(F^n) \to \mathbb{Z}[F^\times]$.
We will also set $D_0 : \tilde{\mathrm{S}}(F^0) = \mathbb{Z} \to \mathbb{Z}$ equal to the identity map. Here $\mathbb{Z}$ is a trivial $F^\times$-module.
We set

$$\mathcal{A}_n = \begin{cases} \mathbb{Z}, & n = 0 \\ \mathcal{I}_{F^\times}, & n \text{ odd} \\ \mathbb{Z}[F^\times], & n > 0 \text{ even} \end{cases}$$

We have $\mathcal{A}_n \subset \mathbb{Z}[F^\times]$ for all n and we make $\mathcal{A}_\bullet$ into a graded algebra by using the multiplication on $\mathbb{Z}[F^\times]$.

Lemma 3.7.

(1) *The image of D_n is $\mathcal{A}_n$.*
(2) *The maps $D_\bullet : \tilde{\mathrm{S}}(F^\bullet) \to \mathcal{A}_\bullet$ define a homomorphism of graded $\mathbb{Z}[F^\times]$-algebras.*
(3) *For each $n \geq 0$, the surjective map $D_n : \tilde{\mathrm{S}}(F^n) \to \mathcal{A}_n$ has a $\mathbb{Z}[F^\times]$-splitting.*

Proof.

(1) Consider a generator $\lfloor a_1, \ldots, a_n \rceil$ of $\tilde{\mathrm{S}}(F^n)$.

Let $e_1, \ldots, e_n$ be the standard basis of F^n. Let $a := a_1e_1 + \cdots + a_ne_n$. Then

$$\begin{aligned} \lfloor a_1, \ldots, a_n \rceil &= d_{n+1}(e_1, \ldots, e_n, a) \\ &= \sum_{i=1}^{n} (-1)^{i+1}(e_1, \ldots, \widehat{e_i}, \ldots, e_n, a) + (-1)^n(e_1, \ldots, e_n). \end{aligned}$$

Thus

$$\begin{aligned} D_n(\lfloor a_1, \ldots, a_n \rceil) &= \sum_{i=1}^{n} (-1)^{i+1} \left\langle \det\left([e_1|\cdots|\widehat{e_i}|\cdots|e_n|a]\right)\right\rangle + (-1)^n \langle 1 \rangle \\ &= \begin{cases} \langle a_1 \rangle - \langle -a_2 \rangle + \cdots + \langle a_n \rangle - \langle 1 \rangle, & n \text{ odd} \\ \langle -a_1 \rangle - \langle a_2 \rangle + \cdots - \langle a_n \rangle + \langle 1 \rangle, & n > 0 \text{ even} \end{cases} \end{aligned}$$

Thus, when n is even, $D_n(\lfloor -1, 1, -1, \ldots, -1, 1 \rceil) = \langle 1 \rangle$ and D_n maps onto $\mathbb{Z}[F^\times]$.

When n is odd, clearly, $D_n(\lfloor a_1, \ldots, a_n \rceil) \in \mathcal{I}_{F^\times}$. However, for any $a \in F^\times$, $D_n(\lfloor a, -1, 1, \ldots, -1, 1 \rceil) = \langle\langle a \rangle\rangle \in \mathcal{A}_n = \mathcal{I}_{F^\times}$.

(2) Note that $C_n(F^n) \cong \mathbb{Z}[\mathrm{GL}_n(F)]$ naturally. Let μ be the homomorphism of additive groups

$$\begin{aligned}\mu : \mathbb{Z}[\mathrm{GL}_n(F)] \otimes \mathbb{Z}[\mathrm{GL}_m(F)] &\to \mathbb{Z}[\mathrm{GL}_{n+m}(F)],\\ A \otimes B &\mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\end{aligned}$$

The formula $D_{m+n}(x * y) = D_n(x) \cdot D_m(y)$ now follows from the commutative diagram

$$\begin{tikzcd}
H(F^n) \otimes H(F^m) \arrow[r, "*"] \arrow[d] & H(F^{n+m}) \arrow[d] \\
C_n(F^n) \otimes C_m(F^m) \arrow[r, "*"] \arrow[d, "\cong"] & C_{n+m}(F^{n+m}) \arrow[d, "\cong"] \\
\mathbb{Z}[\mathrm{GL}_n(F)] \otimes \mathbb{Z}[\mathrm{GL}_m(F)] \arrow[r, "\mu"] \arrow[d, "\det \otimes \det"] & \mathbb{Z}[\mathrm{GL}_{n+m}(F)] \arrow[d, "\det"] \\
\mathbb{Z}[F^\times] \otimes \mathbb{Z}[F^\times] \arrow[r, "\cdot"] & \mathbb{Z}[F^\times]
\end{tikzcd}$$

(3) When n is even the maps D_n are split surjections, since the image is a free module of rank 1.

It is easy to verify that the map $D_1 : \tilde{\mathrm{S}}(F) \to \mathcal{A}_1 = \mathcal{I}_{F^\times}$ is an isomorphism. Now let $E \in \tilde{\mathrm{S}}(F^2)$ be any element satisfying $D_2(E) = \langle 1 \rangle$ (eg. we can take $E = \lfloor -1, 1 \rceil$). Then for $n = 2m+1$ odd, the composite $\tilde{\mathrm{S}}(F) * E^{*m} \to \tilde{\mathrm{S}}(F^n) \to \mathcal{I}_{F^\times} = \mathcal{A}_n$ is an isomorphism.

□

We will let $\tilde{S}(W, V)^+ = \mathrm{Ker}(D_{W,V})$. Thus $\tilde{S}(F^n) \cong \tilde{S}(F^n)^+ \oplus \mathcal{A}_n$ as a $\mathbb{Z}[F^\times]$-module by the results above.

Observe that it follows directly from the definitions that the image of ϵ_V is contained in $\tilde{S}(V)^+$ for any vector space V.

3.5. THE MAPS T_n.

LEMMA 3.8. *If $n \geq 2$ and $b_1, \ldots, b_n$ are distinct elements of $F^\times$ then*

$$[b_1][b_2]\cdots[b_n] = \sum_{i=1}^{n} [b_1 - b_i]\cdots[b_{i-1} - b_i][b_i][b_{i+1} - b_i]\cdots[b_n - b_i] \text{ in } K_n^{\mathrm{MW}}(F).$$

Proof. We will use induction on n starting with $n=2$: Suppose that $b_1 \neq b_2 \in F^\times$. Then

$$\begin{aligned}
&[b_1-b_2]([b_1]-[b_2])\\
&\quad= \left([b_1]+\langle b_1\rangle\left[1-\frac{b_2}{b_1}\right]\right)\left(-\langle b_1\rangle\left[\frac{b_2}{b_1}\right]\right) \text{ by Lemma 2.3 (2)}\\
&\quad= -\langle b_1\rangle[b_1]\left[\frac{b_2}{b_1}\right] \text{ since } [x][1-x]=0\\
&\quad= [b_1]([b_1]-[b_2]) \text{ by Lemma 2.3(2) again}\\
&\quad= [b_1]([-1]-[b_2]) \text{ by Lemma 2.3 (1)}\\
&\quad= [b_1](-\langle -1\rangle[-b_2])\\
&\quad= [-b_2][b_1] \text{ by Lemma 2.3 (3).}
\end{aligned}$$

Thus

$$\begin{aligned}
[b_1][b_2-b_1]+[b_1-b_2][b_2] &= -\langle -1\rangle[b_2-b_1][b_1]+[b_1-b_2][b_2]\\
&= -([b_1-b_2]-[-1])[b_1]+[b_1-b_2][b_2]\\
&= -[b_1-b_2]([b_1]-[b_2])+[-1][b_1]\\
&= -[-b_2][b_1]+[-1][b_1]=([-1]-[-b_2])[b_1]\\
&= -\langle -1\rangle[b_2][b_1]=[b_1][b_2]
\end{aligned}$$

proving the case $n=2$.

Now suppose that $n>2$ and that the result holds for $n-1$. Let $b_1,\ldots,b_n$ be distinct elements of $F^\times$.We wish to prove that

$$\left(\sum_{i=1}^{n-1}[b_1-b_i]\cdots[b_i]\cdots[b_{n-1}-b_i]\right)[b_n]=\sum_{i=1}^{n}[b_1-b_i]\cdots[b_i]\cdots[b_n-b_i].$$

We re-write this as:

$$\sum_{i=1}^{n-1}[b_1-b_i]\cdots[b_i]\cdots[b_{n-1}-b_i]([b_n]-[b_n-b_i])=[b_1-b_n]\cdots[b_{n-1}-b_n][b_n].$$

Now

$$\begin{aligned}
&[b_1-b_i]\cdots[b_i]\cdots[b_{n-1}-b_i]([b_n]-[b_n-b_i])\\
&\quad= (-\langle -1\rangle)^{n-i}[b_1-b_i]\cdots[b_{n-1}-b_i]\Big([b_i]([b_n]-[b_n-b_i])\Big)\\
&\quad= (-\langle -1\rangle)^{n-i}[b_1-b_i]\cdots[b_{n-1}-b_i]\Big([b_i-b_n][b_n]\Big)\\
&\quad= [b_1-b_i]\cdots[b_i-b_n]\cdots[b_{n-1}-b_i][b_n].
\end{aligned}$$

So the identity to be proved reduces to

$$\left(\sum_{i=1}^{n-1}[b_1-b_i]\cdots[b_i-b_n]\cdots[b_{n-1}-b_i]\right)[b_n]=[b_1-b_n]\cdots[b_{n-1}-b_n][b_n].$$

Letting $b'_i = b_i - b_n$ for $1 \leq i \leq n-1$, then $b_j - b_i = b'_j - b'_i$ for $i, j \leq n-1$ and this reduces to the case $n-1$. $\square$

Theorem 3.9.

(1) *For all $n \geq 1$, there is a well-defined homomorphism of $\mathbb{Z}[F^\times]$-modules*

$$T_n : \tilde{\mathrm{S}}(F^n) \to K_n^{\mathrm{MW}}(F)$$

sending $\lfloor a_1, \ldots, a_n \rceil$ to $[a_1] \cdots [a_n]$.

(2) *The maps $\{T_n\}$ define a homomorphism of graded $\mathbb{Z}[F^\times]$-algebras $\tilde{\mathrm{S}}(F^\bullet) \to K_\bullet^{\mathrm{MW}}(F)$: We have*

$$T_{n+m}(x * y) = T_n(x) \cdot T_m(y), \qquad \text{for all } x \in \tilde{\mathrm{S}}(F^n), y \in \tilde{\mathrm{S}}(F^m).$$

Proof.

(1) By Theorem 3.3, in order to show that T_n is well-defined we must prove the identity

$$[b_1 a_1] \cdots [b_n a_n] - [a_1] \cdots [a_n] =$$
$$\sum_{i=1}^{n} (-\langle -1 \rangle)^{n+i} \langle a_i \rangle [a_1(b_1 - b_i)] \cdots [a_i \widehat{(b_i - b_i)}] \cdots [a_n(b_n - b_i][b_i]$$

in $K_n^{\mathrm{MW}}(F)$.

Writing $[b_i a_i] = [a_i] + \langle a_i \rangle [b_i]$ and $[a_j(b_j - b_i)] = [a_j] + \langle a_j \rangle [b_j - b_i]$ and expanding the products on both sides and using (3) of Lemma 2.3 to permute terms, this identity can be rewritten as

$$\sum_{\emptyset \neq I \subset \{1,\ldots,n\}} (-\langle -1 \rangle)^{\mathrm{sgn}(\sigma_I)} \langle a_{i_1} \cdots a_{i_k} \rangle [a_{j_1}] \cdots [a_{j_s}][b_{i_1}] \cdots [b_{i_k}] =$$
$$\sum_{\emptyset \neq I \subset \{1,\ldots,n\}} (-\langle -1 \rangle)^{\mathrm{sgn}(\sigma_I)} \langle a_{i_1} \cdots a_{i_k} \rangle [a_{j_1}] \cdots [a_{j_s}] \times$$
$$\times \left(\sum_{t=1}^{k} [b_{i_1} - b_{i_t}] \cdots [b_{i_t}] \cdots [b_{i_k} - b_{i_t}] \right)$$

where $I = \{i_1 < \cdots < i_k\}$ and the complement of I is $\{j_1 < \cdots < j_s\}$ (so that $k + s = n$) and σ_I is the permutation

$$\begin{pmatrix} 1 & \ldots & s & s+1 & \ldots & n \\ j_1 & \ldots & j_s & i_1 & \ldots & i_k \end{pmatrix}.$$

The result now follows from the identity of Lemma 3.8.

(2) We can assume that $x = \lfloor a_1, \ldots, a_n \rceil$ and $y = \lfloor a'_1, \ldots, a'_m \rceil$ with $a_i, a'_j \in F^\times$. From the definition of T_{n+m} and the formula of Lemma 3.4,

$$
\begin{aligned}
&T_{n+m}(x*y) = \\
&\sum_{i=1}^{n}\sum_{j=1}^{m}(-1)^{n+m+i+j}\left\langle(-1)^{i+j}a_ia'_j\right\rangle \times \\
&\qquad\times [a_1(b_1-b_i)]\cdots[a_i\widehat{(b_i-b_i)}]\cdots[b_i][a'_1(b'_1-b'_j)]\cdots[a'_j\widehat{(b'_j-b'_j)}]\cdots[b'_j] \\
&\quad +(-1)^n\sum_{i=1}^{n}(-1)^{i+1}\left\langle(-1)^{i+1}a_i\right\rangle[a_1(b_1-b_i)]\cdots[a_i\widehat{(b_i-b_i)}]\cdots[b_i][b'_1a'_1]\cdots[b'_ma'_m] \\
&\quad +(-1)^m\sum_{j=1}^{m}(-1)^{j+1}\left\langle(-1)^{j+1}a'_j\right\rangle[b_1a_1]\cdots[b_na_n][a'_1(b'_1-b'_j)]\cdots[a'_j\widehat{(b'_j-b'_j)}]\cdots[b'_j] \\
&\qquad +[b_1a_1]\cdots[b_na_n][b_i][b'_1a'_1]\cdots[b'_ma'_m]
\end{aligned}
$$

which factors as $X \cdot Y$ with $X =$

$$
\begin{aligned}
&\sum_{i=1}^{n}(-1)^{n+i+1}\left\langle(-1)^{i+1}a_i\right\rangle[a_1(b_1-b_i)]\cdots[a_i\widehat{(b_i-b_i)}]\cdots[b_i]+[b_1a_1]\cdots[b_na_n] \\
&\qquad\qquad = [a_1]\cdots[a_n] = T_n(x) \text{ by part (1)}
\end{aligned}
$$

and $Y =$

$$
\begin{aligned}
&\sum_{j=1}^{m}(-1)^{m+j+1}\left\langle(-1)^{j+1}a'_j\right\rangle[a'_1(b'_1-b'_j)]\cdots[a'_j\widehat{(b'_j-b'_j)}]\cdots[b'_j]+[b'_1a'_1]\cdots[b'_ma'_m] \\
&\qquad\qquad = [a'_1]\cdots[a'_m] = T_m(y) \text{ by (1) again.}
\end{aligned}
$$

□

Note that T_1 is the natural surjective map $\tilde{S}(F) \cong \mathcal{I}_{F^\times} \to K_1^{\mathrm{MW}}(F)$, $\lfloor a\rceil \leftrightarrow \langle\langle a\rangle\rangle \mapsto [a]$. It has a nontrivial kernel in general.
Note furthermore that $\mathrm{SH}_2(F) = \mathrm{H}_2(\mathrm{SL}_2(F),\mathbb{Z})$. It is well-known ([24],[11], and [7]) that $\mathrm{H}_2(\mathrm{SL}_2(F),\mathbb{Z}) \cong K_2^{\mathrm{M}}(F)\times_{k_2^{\mathrm{M}}(F)} I^2(F) \cong K_2^{\mathrm{MW}}(F)$.
In fact we have:

Theorem 3.10. *The composite $T_2\circ\epsilon_2 : \mathrm{H}_2(\mathrm{SL}_2(F),\mathbb{Z}) \to K_2^{\mathrm{MW}}(F)$ is an isomorphism.*

Proof. For $p \geq 1$, let $\bar{X}_p(F)$ denote the set of all p-tuples $(x_1,\ldots,x_p)$ of points of $\mathbb{P}^1(F)$ and let $\bar{X}_0(F) = \emptyset$. We let $\bar{C}_p(F)$ denote the $\mathrm{GL}_2(F)$ permutation module $\mathbb{Z}[\bar{X}_p(F)]$ and form a complex $\bar{C}_\bullet(F)$ using the natural simplicial boundary maps, $\bar{d}_p$. This complex is acyclic and the map $F^2\setminus\{0\}\to\mathbb{P}^1(F)$, $v\mapsto\overline{v}$ induces a map of complexes $C_\bullet(F^2)\to\bar{C}_\bullet(F)$.
Let $\bar{H}_2(F) := \mathrm{Ker}(\bar{d}_2 : \bar{C}_2(F)\to\bar{C}_1(F))$ and let $\bar{S}_2(F) = \mathrm{H}_0(\mathrm{SL}_2(F),\bar{H}_2(F))$. We obtain a commutative diagram of $\mathrm{SL}_2(F)$-modules with exact rows:

$$
\begin{tikzcd}
C_4(F^2) \arrow[r, "d_4"] \arrow[d] & C_3(F^2) \arrow[r, "d_3"] \arrow[d] & H(F^2) \arrow[r] \arrow[d] & 0 \\
\bar{C}_4(F) \arrow[r, "\bar{d}_4"] & \bar{C}_3(F) \arrow[r, "\bar{d}_3"] & \bar{H}_2(F) \arrow[r] & 0
\end{tikzcd}
$$

Taking $\mathrm{SL}_2(F)$-coinvariants gives the diagram

$$\begin{tikzcd}
\mathrm{H}_0(\mathrm{SL}_2(F), C_4(F^2)) \arrow[r, "d_4"] \arrow[d] & \mathrm{H}_0(\mathrm{SL}_2(F), C_3(F^2)) \arrow[r, "d_3"] \arrow[d] & \tilde{S}(F^2) \arrow[r] \arrow[d, "\phi"] & 0 \\
\mathrm{H}_0(\mathrm{SL}_2(F), \bar{C}_4(F)) \arrow[r, "\bar{d}_4"] & \mathrm{H}_0(\mathrm{SL}_2(F), \bar{C}_3(F)) \arrow[r, "\bar{d}_3"] & \bar{S}_2(F) \arrow[r] & 0
\end{tikzcd}$$

Now the calculations of Mazzoleni, [11], show that $\mathrm{H}_0(\mathrm{SL}_2(F), \bar{C}_3(F)) \cong \mathbb{Z}[F^\times/(F^\times)^2]$ via

$$\text{class of } (\infty, 0, a) \mapsto \langle a \rangle \in \mathbb{Z}[F^\times/(F^\times)^2],$$

where $a \in \mathbb{P}^1(F) = \overline{e_1 + ae_2}$ and $\infty := \overline{e_1}$. Furthermore $\bar{S}_2(F) \cong \mathrm{GW}(F)$ in such a way that the induced map $\mathbb{Z}[F^\times/(F^\times)^2] \to \mathrm{GW}(F)$ is the natural one. Since $\lfloor a, b \rceil = d_3(e_1, e_2, ae_1 + be_2)$, it follows that $\phi(\lfloor a, b \rceil) = \langle a/b \rangle = \langle ab \rangle$ in $\mathrm{GW}(F)$.
Associated to the complex $\bar{C}_\bullet(F)$ we have an iterated connecting homomorphism $\omega : \mathrm{H}_2(\mathrm{SL}_2(F), \mathbb{Z}) \to \bar{S}_2(F) = \mathrm{GW}(F)$. Observe that $\omega = \phi \circ \epsilon_2$. In fact, (Mazzoleni, [11], Lemma 5) the image of ω is $I^2(F) \subset \mathrm{GW}(F)$.
On the other hand, the module $\tilde{S}(F^2)^+$ is generated by the elements $[[a, b]] := \lfloor a, b \rceil - D_2(\lfloor a, b \rceil) \cdot E$ (where E, as above, denotes the element $\lfloor -1, 1 \rceil$). Note that $T_2([[a, b]]) = T_2(\lfloor a, b \rceil) = [a][b]$ since $T_2(E) = [-1][1] = 0$ in $K_2^{\mathrm{MW}}(F)$.
Furthermore,

$$\begin{aligned}
\phi([[a, b]]) &= \phi(\lfloor a, b \rceil) - D_2(\lfloor a, b \rceil)\phi(E) \\
&= \langle ab \rangle - (\langle -a \rangle - \langle b \rangle + \langle 1 \rangle)\langle -1 \rangle \\
&= \langle ab \rangle - \langle a \rangle + \langle -b \rangle - \langle -1 \rangle \\
&= \langle ab \rangle - \langle a \rangle - \langle b \rangle + \langle 1 \rangle \\
&= \langle \langle a, b \rangle \rangle
\end{aligned}$$

(using the identity $\langle b \rangle + \langle -b \rangle = \langle 1 \rangle + \langle -1 \rangle$ in $\mathrm{GW}(F)$).
Using these calculations we thus obtain the commutative diagram

$$\begin{tikzcd}
\mathrm{H}_2(\mathrm{SL}_2(F), \mathbb{Z}) \arrow[r, "\epsilon_2"] \arrow[dr, "\omega"'] & \tilde{S}(F^2)^+ \arrow[r, "T_2"] \arrow[d, "\phi"] & K_2^{\mathrm{MW}}(F) \arrow[dl] \\
& I^2(F) &
\end{tikzcd}$$

Now, the natural embedding $F^\times \to \mathrm{SL}_2(F)$, $a \mapsto \mathrm{diag}(a, a^{-1}) := \tilde{a}$ induces a homomorphism, μ:

$$\begin{aligned}
\bigwedge^2 (F^\times) \cong \mathrm{H}_2(F^\times, \mathbb{Z}) &\to \mathrm{H}_2(\mathrm{SL}_2(F), \mathbb{Z}), \\
a \wedge b &\mapsto \Big([\tilde{a}|\tilde{b}] - [\tilde{b}|\tilde{a}] \Big) \otimes 1 \in B_2(\mathrm{SL}_2(F)) \otimes_{\mathbb{Z}[\mathrm{SL}_2(F)]} \mathbb{Z}.
\end{aligned}$$

Mazzoleni's calculations (see [11], Lemma 6) show that $\mu(\bigwedge^2(F^\times)) = \mathrm{Ker}(\omega)$ and that there is an isomorphism $\mu(\bigwedge^2(F^\times)) \cong 2\cdot K_2^{\mathrm{M}}(F)$ given by $\mu(a\wedge b)\mapsto 2\{a,b\}$.

On the other hand, a straightforward calculation shows that $\epsilon_2\left(\mu(a\wedge b)\right) =$

$$\langle a\rangle \lfloor b,\frac{1}{ab}\rceil - \lfloor b,\frac{1}{b}\rceil - \langle a\rangle \lfloor 1,\frac{1}{a}\rceil + \langle b\rangle \lfloor 1,\frac{1}{b}\rceil + \lfloor a,\frac{1}{a}\rceil - \langle b\rangle \lfloor a,\frac{1}{ab}\rceil := C_{a,b}$$

Now by the diagram above,

$$T_2(C_{a,b}) = T_2(\epsilon_2\left(\mu(a\wedge b)\right)) \in \mathrm{Ker}(K_2^{\mathrm{MW}}(F)\to I^2(F)) \cong 2K_2^{\mathrm{M}}(F).$$

Recall that the natural embedding $2K_2^{\mathrm{M}}(F)\to K_2^{\mathrm{MW}}(F)$ is given by $2\{a,b\}\mapsto [a^2][b] = [a][b]-[b][a]$ and the composite

$$2K_2^{\mathrm{M}}(F) \longrightarrow K_2^{\mathrm{MW}}(F) \xrightarrow{\kappa_2} K_2^{\mathrm{M}}(F)$$

is the natural inclusion map. Since

$$\begin{aligned}\kappa_2\left(T_2(C_{a,b})\right) &= \left\{b,\frac{1}{ab}\right\} - \left\{b,\frac{1}{b}\right\} - \left\{1,\frac{1}{a}\right\} + \left\{1,\frac{1}{b}\right\} + \left\{a,\frac{1}{a}\right\} - \left\{a,\frac{1}{ab}\right\}\\ &= \{a,b\}-\{b,a\} = 2\{a,b\},\end{aligned}$$

it follows that we have a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mu(\bigwedge^2(F^\times)) \arrow[r] \arrow[d, "\cong"] & \mathrm{H}_2(\mathrm{SL}_2(F),\mathbb{Z}) \arrow[r, "\omega"] \arrow[d, "T_2\circ\epsilon_2"] & I^2(F) \arrow[r] \arrow[d, "="] & 0\\ 0 \arrow[r] & 2K_2^{\mathrm{M}}(F) \arrow[r] & K_2^{\mathrm{MW}}(F) \arrow[r] & I^2(F) \arrow[r] & 0 \end{tikzcd}$$

proving the theorem. □

4. $\mathcal{AM}$-modules

From the results of the last section, it follows that there is a $\mathbb{Z}[F^\times]$-decomposition

$$\tilde{S}(F^2) \cong K_2^{\mathrm{MW}}(F)\oplus\mathbb{Z}[F^\times]\oplus ?$$

It is not difficult to determine that the missing factor is isomorphic to the 1-dimensional vector space F (with the tautological $F^\times$-action). However, as we will see, this extra term will not play any role in the calculations of $\mathrm{H}_n(\mathrm{SL}_k(F),\mathbb{Z})$.

As $\mathbb{Z}[F^\times]$-modules, our main objects of interest (Milnor-Witt K-theory, the homology of the special linear group, the powers of the fundamental ideal in the Grothendieck-Witt ring) are what we call below 'multiplicative '; there exists $m\geq 1$ such that, for all $a\in F^\times$, $\langle a^m\rangle$ acts trivially. This is certainly not true of the vector space F above. In this section we formalise this difference, and use this formalism to prove an analogue of Suslin's Theorem 1.8 ([23]) (see Theorem 4.23 below).

Throughout the remainder of this article, F will denote a field of characteristic 0.

Let $\mathcal{S}_F \subset \mathbb{Z}[F^\times]$ denote the multiplicative set generated by the elements $\{\langle\langle a\rangle\rangle = \langle a\rangle - 1 \mid a \in F^\times \setminus \{1\}\}$. Note that $0 \notin \mathcal{S}_F$, since the elements of $\mathcal{S}_F$ map to units under the natural ring homomorphism $\mathbb{Z}[F^\times] \to F$. We will also let $\mathcal{S}_{\mathbb{Q}}^+ \subset \mathbb{Z}[\mathbb{Q}^\times]$ denote the multiplicative set generated by $\{\langle\langle a\rangle\rangle = \langle a\rangle - 1 \mid a \in \mathbb{Q}^\times \setminus \{\pm 1\}\}$.

Definition 4.1. A $\mathbb{Z}[F^\times]$-module M is said to be *multiplicative* if there exists $s \in \mathcal{S}_{\mathbb{Q}}^+$ with $sM = 0$.

Definition 4.2. We will say that a $\mathbb{Z}[F^\times]$-module is *additive* if every $s \in \mathcal{S}_{\mathbb{Q}}^+$ acts as an automorphism on M.

Example 4.3. Any trivial $\mathbb{Z}[F^\times]$-module M is multiplicative , since $\langle\langle a\rangle\rangle$ annihilates M for all $a \neq 1$.

Example 4.4. $\mathrm{GW}(F)$, and more generally $I^n(F)$, is multiplicative since $\langle\langle a^2\rangle\rangle$ annihilates these modules for all $a \in F^\times$.

Example 4.5. Similarly, the groups $\mathrm{H}_n(\mathrm{SL}_n(F), \mathbb{Z})$ are multiplicative since they are annihilated by the elements $\langle\langle a^m\rangle\rangle$.

Example 4.6. Any vector space over F, with the induced action of $\mathbb{Z}[F^\times]$, is additive since all elements of $\mathcal{S}_F$ act as automorphisms.

Example 4.7. More generally, if V is a vector space over F, then for all $r \geq 1$, the rth tensor power $\mathrm{T}^r_{\mathbb{Z}}(V) = \mathrm{T}^r_{\mathbb{Q}}(V)$ is an additive module since, if $a \in \mathbb{Q} \setminus \{\pm 1\}$, $\langle a\rangle$ acts as multiplication by a^r and hence $\langle\langle a\rangle\rangle$ acts as multiplication by $a^r - 1$. For the same reasons, the rth exterior power, $\bigwedge^r_{\mathbb{Z}}(V)$, is an additive module.

Remark 4.8. Observe that if $\langle\langle a^m\rangle\rangle$ acts as an automorphism of the $\mathbb{Z}[F^\times]$-module M for some $a \in F^\times$, $m > 1$, then so does $\langle\langle a\rangle\rangle$, since $\langle\langle a^m\rangle\rangle = \langle\langle a\rangle\rangle(\left\langle a^{m-1}\right\rangle + \cdots + \langle a\rangle + 1) = (\left\langle a^{m-1}\right\rangle + \cdots + \langle a\rangle + 1)\langle\langle a\rangle\rangle$ in $\mathbb{Z}[F^\times]$.

Lemma 4.9. *Let*

$$0 \to M_1 \to M \to M_2 \to 0$$

be a short exact sequence of $\mathbb{Z}[F^\times]$*-modules.*
Then M is multiplicative if and only if M_1 and M_2 are.

Proof. Suppose M is multiplicative . If $s \in \mathcal{S}_{\mathbb{Q}}^+$ satisfies $sM = 0$, it follows that $sM_1 = sM_2 = 0$.
Conversely, if M_1 and M_2 are multiplicative then there exist $s_1, s_2 \in \mathcal{S}_{\mathbb{Q}}^+$ with $s_iM_i = 0$ for $i = 1, 2$. It follows that $sM = 0$ for $s = s_1s_2 \in \mathcal{S}_{\mathbb{Q}}^+$. □

Lemma 4.10. *Let*

$$0 \to A_1 \to A \to A_2 \to 0$$

be a short exact sequence of $\mathbb{Z}[F^\times]$*-modules. If A_1 and A_2 are additive modules, then so is A.*

Proof. This is immediate from the definition. □

Lemma 4.11. *Let $\phi : M \to N$ be a homomorphism of $\mathbb{Z}[F^\times]$-modules.*

1. *If M and N are multiplicative , then so are* $\mathrm{Ker}(\phi)$ *and* $\mathrm{Coker}(\phi)$*.*
2. *If M and N are additive , then so are* $\mathrm{Ker}(\phi)$ *and* $\mathrm{Coker}(\phi)$*.*

Proof. (1) This follows from Lemma 4.9 above.

(2) If $s \in \mathcal{S}_{\mathbb{Q}}^+$, then s acts as an automorphism of M and N, and hence of $\mathrm{Coker}(\phi)$ and $\mathrm{Ker}(\phi)$.

□

Corollary 4.12. *Let $C = (C_\bullet, d)$ be a complex of $\mathbb{Z}[F^\times]$-modules. If $C_\bullet$ is additive (i.e. if each C_n is an additive module), then each $H_n(C)$ is an additive module. If each C_n is multiplicative then each $H_n(C)$ is a multiplicative module.*

Lemma 4.13. *Let M be a multiplicative $\mathbb{Z}[F^\times]$-module and A an additive $\mathbb{Z}[F^\times]$-module. Then* $\mathrm{Hom}_{\mathbb{Z}[F^\times]}(M, A) = 0$ *and* $\mathrm{Hom}_{\mathbb{Z}[F^\times]}(A, M) = 0$*.*

Proof. Let $f : M \to A$ be a $\mathbb{Z}[F^\times]$-homomorphism. Every $s \in \mathcal{S}_{\mathbb{Q}}^+$ acts as an automorphism of A. However, there exists $s \in \mathcal{S}_{\mathbb{Q}}^+$ with $sM = 0$. Thus, for $m \in M$, $0 = f(sm) = sf(m) \Longrightarrow f(m) = 0$.
Let $g : A \to M$ be a $\mathbb{Z}[F^\times]$-homomorphism. Again, choose $s \in \mathcal{S}_{\mathbb{Q}}^+$ acting as an automorphism of A and annihilating M. If $a \in A$, then there exists $b \in a$ with $a = sb$. Hence $g(a) = sg(b) = 0$ in M. □

Lemma 4.14. *If P is a $\mathbb{Z}[F^\times]$-module and if A is an additive submodule and M a multiplicative submodule, then $A \cap M = 0$.*

Proof. There exists $s \in \mathbb{Z}[\mathbb{Q}^\times]$ which annihilates any submodule of M but is injective on any submodule of A. □

Lemma 4.15.

1. *If*
$$0 \longrightarrow M \longrightarrow H \xrightarrow{\pi} A \longrightarrow 0$$
is an exact sequence of $\mathbb{Z}[F^\times]$-modules with M multiplicative and A additive then the sequence splits (over $\mathbb{Z}[F^\times]$).
2. *Similarly, if*
$$0 \longrightarrow A \longrightarrow H \longrightarrow M \longrightarrow 0$$
is an exact sequence of $\mathbb{Z}[F^\times]$-modules with M multiplicative and A additive then the sequence splits.

Proof. As above we can find $s \in \mathbb{Z}[\mathbb{Q}^\times]$ such that $s \cdot M = 0$ and s acts as an automorphism of A.

1. Then sH is a $\mathbb{Z}[F^\times]$-submodule of H and π induces an isomorphism $sH \cong A$, since $\pi(sH) = s\pi(H) = sA = A$ and if $\pi(sh) = 0$ then $s\pi(h) = 0$ in A, so that $\pi(h) = 0$ and $h \in M$.
2. We have $sH = A$ and multiplication by s gives an automorphism, α, of A. Thus the $\mathbb{Z}[F^\times]$-homomorphism $H \to A, h \mapsto \alpha^{-1}(s \cdot h)$ splits the sequence.

$\square$

DEFINITION 4.16. We will say that a $\mathbb{Z}[F^\times]$-module H is *an $\mathcal{AM}$ module* if there exists a multiplicative $\mathbb{Z}[F^\times]$-module M and an additive $\mathbb{Z}[F^\times]$ module A and an isomorphism of $\mathbb{Z}[F^\times]$-modules $H \cong A \oplus M$.

LEMMA 4.17. *Let H be an $\mathcal{AM}$ module and let $\phi : H \to A \oplus M$ be an isomorphism of $\mathbb{Z}[F^\times]$-modules, with M multiplicative and A additive .*
Then

$$\phi^{-1}(A) = \bigcup_{A' \subset H, A' \text{additive}} A' \qquad \text{and} \qquad \phi^{-1}(M) = \bigcup_{M' \subset H, M' \text{multiplicative}} M'$$

Proof. Let $M' \subset H$ be multiplicative. Then the composite

$$M' \longrightarrow H \xrightarrow{\phi} A \oplus M \longrightarrow A$$

is zero by Lemma 4.13, and thus $M' \subset \phi^{-1}(M)$.
An analogous argument can be applied to $\phi^{-1}(A)$. $\square$

It follows that the submodules $\phi^{-1}(A)$ and $\phi^{-1}(M)$ are independent of the choice of ϕ, A and M. We will denote the first as $H_{\mathcal{A}}$ and the second as $H_{\mathcal{M}}$. Thus if H is an $\mathcal{AM}$ module then there is a canonical decomposition $H = H_{\mathcal{A}} \oplus H_{\mathcal{M}}$, where $H_{\mathcal{A}}$ (resp. $H_{\mathcal{M}}$) is the maximal additive (resp. multiplicative) submodule of H. We have canonical projections

$$\pi_{\mathcal{A}} : H \to H_{\mathcal{A}}, \qquad \pi_{\mathcal{M}} : H \to H_{\mathcal{M}}.$$

LEMMA 4.18. *Let H be a $\mathcal{AM}$ module. Suppose that H is also a module over a ring R and that the action of R commutes with that of $\mathbb{Z}[F^\times]$. Then $H_{\mathcal{A}}$ and $H_{\mathcal{M}}$ are R-submodules of H.*

Proof. Let $r \in R$. Then the composite

$$H_{\mathcal{A}} \xrightarrow{r\cdot} H \xrightarrow{\pi_{\mathcal{M}}} H_{\mathcal{M}}$$

is a $\mathbb{Z}[F^\times]$-homomorphism and thus is 0 by Lemma 4.13. It follows that $r \cdot H_{\mathcal{A}} \subset \mathrm{Ker}(\pi_{\mathcal{M}}) = H_{\mathcal{A}}$. $\square$

LEMMA 4.19. *Let $f : H \to H'$ be a $\mathbb{Z}[F^\times]$-homomorphism of $\mathcal{AM}$ modules. Then there exist $\mathbb{Z}[F^\times]$-homomorphisms $f_{\mathcal{A}} : H_{\mathcal{A}} \to H'_{\mathcal{A}}$ and $f_{\mathcal{M}} : H_{\mathcal{M}} \to H'_{\mathcal{M}}$ such that $f = f_{\mathcal{A}} \oplus f_{\mathcal{M}}$.*
Suppose that H and H' are modules over a ring R and that the R-action commutes with the $\mathbb{Z}[F^\times]$-action in each case. If f is an R-homomorphism, then so are $f_{\mathcal{A}}$ and $f_{\mathcal{M}}$.

Proof. This is immediate from Lemmas 4.13 and 4.18. $\square$

LEMMA 4.20. *If*

$$0 \longrightarrow L \xrightarrow{j} H \xrightarrow{\pi} K \longrightarrow 0$$

is a short exact sequence of $\mathbb{Z}[F^\times]$-modules and if L and K are $\mathcal{AM}$ modules, then so is H.

Proof. Let $\tilde{H} = \pi^{-1}(K_{\mathcal{M}})$. Then the exact sequence

$$0 \to L \to \tilde{H} \to K_{\mathcal{M}} \to 0$$

gives the exact sequence

$$0 \to \frac{L}{L_{\mathcal{M}}} \to \frac{\tilde{H}}{j(L_{\mathcal{M}})} \to K_{\mathcal{M}} \to 0.$$

Since $L/L_{\mathcal{M}} \cong L_{\mathcal{A}}$ is additive , this latter sequence is split, by Lemma 4.15 (2).
So $\tilde{H}/j(L_{\mathcal{M}})$ is a $\mathcal{AM}$ module, and there is a $\mathbb{Z}[k^\times]$-isomorphism

$$\tilde{H}/j(L_{\mathcal{M}}) \xrightarrow[\cong]{\phi} L_{\mathcal{A}} \oplus K_{\mathcal{M}}.$$

Let $\bar{\phi}$ be the composite

$$\tilde{H} \longrightarrow \tilde{H}/j(L_{\mathcal{M}}) \xrightarrow{\phi} L_{\mathcal{A}} \oplus K_{\mathcal{M}}.$$

Let $H_m = \bar{\phi}^{-1}(K_{\mathcal{M}}) \subset \tilde{H} \subset H$. Then, we have an exact sequence

$$0 \to L_{\mathcal{M}} \to H_m \to K_{\mathcal{M}} \to 0$$

so that H_m is multiplicative .
On the other hand, since $\tilde{H}/H_m \cong L_{\mathcal{A}}$ and $H/\tilde{H} \cong K_{\mathcal{A}}$, we have a short exact sequence

$$0 \to L_{\mathcal{A}} \to \frac{H}{H_m} \to K_{\mathcal{A}} \to 0.$$

This implies that H/H_m is additive , and thus H is $\mathcal{AM}$ by Lemma 4.15 (1). $\square$

Lemma 4.21. *Let $(C_\bullet, d)$ be a complex of $\mathbb{Z}[k^\times]$-modules. If each C_n is $\mathcal{AM}$, then $\mathrm{H}_\bullet(C)$ is $\mathcal{AM}$, and furthermore*

$$\mathrm{H}_\bullet(C_{\mathcal{A}}) = \mathrm{H}_\bullet(C)_{\mathcal{A}}$$
$$\mathrm{H}_\bullet(C_{\mathcal{M}}) = \mathrm{H}_\bullet(C)_{\mathcal{M}}$$

Proof. The differentials d decompose as $d = d_{\mathcal{A}} \oplus d_{\mathcal{M}}$ by Lemma 4.19. $\square$

Theorem 4.22. *Let (E^r, d^r) be a first quadrant spectral sequence of $\mathbb{Z}[k^\times]$-modules converging to the $\mathbb{Z}[k^\times]$-module $H_\bullet = \{H_n\}_{n\geq 0}$.*
If for some $r_0 \geq 1$ all of the modules $E^{r_0}_{p,q}$ are $\mathcal{AM}$, then the same holds for all the modules $E^r_{p,q}$ for all $r \geq r_0$ and hence for the modules $E^\infty_{p,q}$.
Furthermore, $H_\bullet$ is $\mathcal{AM}$ and the spectral sequence decomposes as a direct sum $E^r = E^r{}_{\mathcal{A}} \oplus E^r{}_{\mathcal{M}}$ $(r \geq r_0)$ with $E^r{}_{\mathcal{A}}$ converging to $H_{\bullet\mathcal{A}}$ and $E^r{}_{\mathcal{M}}$ converging to $H_{\bullet\mathcal{M}}$.

Proof. Since $E^{r+1} = \mathrm{H}(E^r, d^r)$ for all r, the first statement follows from Lemma 4.21.
Since E^r is a first quadrant spectral sequence (and, in particular, is bounded), it follows that for any fixed (p, q), $E^{\infty}_{p,q} = E^r_{p,q}$ for all sufficiently large r. Thus E^{∞} is also $\mathcal{AM}$.
Now H_n admits a filtration $0 = F_0H_n \subset \cdots \subset F_nH_n = H_n$ with corresponding quotients $\mathrm{gr}_p H_n \cong E^{\infty}_{p,n-p}$.
Since all the quotients are $\mathcal{AM}$, it follows by Lemma 4.20, together with an induction on the filtration length, that H_n is $\mathcal{AM}$.
The final two statements follow again from Lemma 4.21. □

If G is a subgroup of $\mathrm{GL}(V)$, we let SG denote $G \cap \mathrm{SL}(V)$.

THEOREM 4.23. *Let V, W be finite-dimensional vector spaces over F and let $G_1 \subset \mathrm{GL}(W)$, $G_2 \subset \mathrm{GL}(V)$ be subgroups and suppose that G_2 contains the group $F^{\times}$ of scalar matrices.*
Let M be a subspace of $\mathrm{Hom}_F(V, W)$ *for which $G_1M = M = MG_2$.*
Let
$$G = \begin{pmatrix} G_1 & M \\ 0 & G_2 \end{pmatrix} \subset \mathrm{GL}(W \oplus V).$$
Then, for $i \geq 1$, the groups $\mathrm{H}_i(SG, \mathbb{Z})$ are $\mathcal{AM}$ and the natural embedding $j : S(G_1 \times G_2) \to SG$ induces an isomorphism
$$\mathrm{H}_i(S(G_1 \times G_2), \mathbb{Z}) \cong \mathrm{H}_i(SG, \mathbb{Z})_{\mathcal{M}}.$$

Proof. We begin by noting that the groups $\mathrm{H}_i(SG, \mathbb{Z})$ are $\mathbb{Z}[F^{\times}]$-modules: The action of $F^{\times}$ is derived from the short exact sequence
$$1 \longrightarrow SG \longrightarrow G \xrightarrow{\det} F^{\times} \longrightarrow 1$$
We have a split extension of groups (split by the map j) which is $F^{\times}$-stable:
$$0 \longrightarrow M \longrightarrow SG \xrightarrow{\pi} S(G_1 \times G_2) \longrightarrow 1.$$
The resulting Hochschild-Serre spectral sequence has the form
$$E^2_{p,q} = \mathrm{H}_p(S(G_1 \times G_2), \mathrm{H}_q(M, \mathbb{Z})) \Longrightarrow \mathrm{H}_{p+q}(SG, \mathbb{Z}).$$
This spectral sequence exists in the category of $\mathbb{Z}[F^{\times}]$-modules and all differentials and edge homomorphisms are $\mathbb{Z}[F^{\times}]$-maps.
Since the map π is split by j it induces a split surjection on integral homology groups. Thus
$$\mathrm{H}_n(S(G_1 \times G_2), \mathbb{Z}) = E^2_{n,0} = E^{\infty}_{n,0} \quad \text{for all } n \geq 0.$$
Observe furthermore that the $\mathbb{Z}[F^{\times}]$-module $\mathrm{H}_n(S(G_1 \times G_2), \mathbb{Z})$ is multiplicative : Given $a \in F^{\times}$, the element
$$\rho_a := \begin{pmatrix} \mathrm{Id}_W & 0 \\ 0 & a \cdot \mathrm{Id}_V \end{pmatrix} \in G$$

has determinant a^m ($m = \dim_F(V)$) and centralizes $S(G_1 \times G_2)$. It follows that $\langle a^m \rangle$ acts trivially on $\mathrm{H}_n(S(G_1 \times G_2), \mathbb{Z})$ for all n; i.e. $\langle\langle a^m \rangle\rangle$ annihilates $\mathrm{H}_n(S(G_1 \times G_2), \mathbb{Z})$.

Recall (Example 4.7 above) that for $q \geq 1$, the modules $\mathrm{H}_q(M, \mathbb{Z}) = \bigwedge_{\mathbb{Z}}^q(M)$, with the $\mathbb{Z}[F^\times]$-action derived from the action of F by scalars on M, are additive modules.

Now if $a \in F^\times$, then conjugation by ρ_a is trivial on $S(G_1 \times G_2)$ but acts on M as scalar multiplication by a. It follows that for $q > 0$, $\langle\langle a^m \rangle\rangle$ acts as an automorphism on $\mathrm{H}_p(S(G_1 \times G_2), \mathrm{H}_q(M, \mathbb{Z}))$ for all $a \in \mathbb{Q} \setminus \{\pm 1\}$. Thus, for $q > 0$, the groups $\mathrm{H}_p(S(G_1 \times G_2), \mathrm{H}_q(M, \mathbb{Z}))$ are additive $\mathbb{Z}[F^\times]$-modules; i.e., all $E^2_{p,q}$ are additive for $q > 0$. It follows at once that the groups $E^\infty_{p,q}$ are additive for all $q > 0$. Thus, from the convergence of the spectral sequence, we have a short exact sequence

$$0 \to H \to \mathrm{H}_n(SG, \mathbb{Z}) \to E^\infty_{n,0} = j\left(\mathrm{H}_n(S(G_1 \times G_2), \mathbb{Z})\right) \to 0$$

and H has a filtration whose graded quotients are all additive .

So $\mathrm{H}_n(SG, \mathbb{Z})$ is $\mathcal{AM}$ as claimed, and $\mathrm{H}_n(SG, \mathbb{Z})_{\mathcal{M}} \cong \mathrm{H}_n(S(G_1 \times G_2), \mathbb{Z})$. □

Corollary 4.24. *Suppose that $W' \subset W$. Then there is a corresponding inclusion* $\mathrm{SA}(W', V) \to \mathrm{SA}(W, V)$. *This inclusion induces an isomorphism*

$$\mathrm{H}_n(\mathrm{SA}(W', V), \mathbb{Z})_{\mathcal{M}} \xrightarrow[\cong]{} \mathrm{H}_n(\mathrm{SA}(W, V), \mathbb{Z})_{\mathcal{M}} \cong \mathrm{H}_n(\mathrm{SL}(V), \mathbb{Z})$$

for all $n \geq 1$.

5. The spectral sequences

Recall that F is a field of characteristic 0 throughout this section.

In this section we use the complexes $C_\bullet(W, V)$ to construct spectral sequences converging to 0 in dimensions less than $n = \dim_F(V)$, and to $\tilde{S}(W, V)$ in dimension n. By projecting onto the multiplicative part, we obtain spectral sequences with good properties: the terms in the E^1-page are just the kernels and cokernels of the stabilization maps $f_{t,n} : \mathrm{H}_n(\mathrm{SL}_t(F), \mathbb{Z}) \to \mathrm{H}_n(\mathrm{SL}_{t+1}(F), \mathbb{Z})$. We then prove that the higher differentials are all zero. Since the spectral sequences converge to 0 in low degrees, this already implies the main stability result (Corollary 5.11); the maps $f_{t,n}$ are isomorphisms for $t \geq n+1$ and are surjective for $t = n$. The remainder of the paper is devoted to an analysis of the case $t = n - 1$, which requires some more delicate calculations.

Let $\mathcal{C}^\tau_\bullet(W, V)$ denote the truncated complex.

$$\mathcal{C}^\tau_p(W, V) = \begin{cases} C_p(W, V), & p \leq \dim_F(V) \\ 0, & p > \dim_F(V) \end{cases}$$

Thus

$$\mathrm{H}_p(\mathcal{C}^\tau_\bullet(W, V)) = \begin{cases} 0, & p \neq n \\ H(W, V), & p = n \end{cases}$$

where $n = \dim_F(V)$.

Thus the natural action of $\mathrm{SA}(W,V)$ on $\mathcal{C}^\tau_\bullet(W,V)$ gives rise to a spectral sequence $\mathcal{E}(W,V)$ which has the form

$$E^1_{p,q} = \mathrm{H}_p(\mathrm{SA}(W,V), \mathcal{C}^\tau_q(W,V)) \Longrightarrow \mathrm{H}_{p+q-n}(\mathrm{SA}(W,V), H(W,V)).$$

The groups $\mathcal{C}^\tau_q(W,V)$ are permutation modules for $\mathrm{SA}(W,V)$ and thus the E^1-terms (and the differentials d^1) can be computed in terms of the homology of stabilizers.

Fix a basis $\{e_1,\ldots,e_n\}$ of V. Let V_r be the span of $\{e_1,\ldots,e_r\}$ and let V'_s be the span of $\{e_{n-s},\ldots,e_n\}$, so that $V = V_r \oplus V'_{n-r}$ if $0 \le r \le n$.

For any $0 \le q \le n-1$, the group $\mathrm{SA}(W,V)$ acts transitively on the basis of $\mathcal{C}^\tau_q(W,V)$ and the stabilizer of

$$\big((0,e_1),\ldots,(0,e_q)\big)$$

is $\mathrm{SA}(W \oplus V_q, V'_{n-q})$.

Thus, for $q \le n-1$,

$$E^1_{p,q} = \mathrm{H}_p(\mathrm{SA}(W,V), \mathcal{C}^\tau_q(W,V)) \cong \mathrm{H}_p(\mathrm{SA}(W \oplus V_q, V'_{n-q}), \mathbb{Z})$$

by Shapiro's Lemma.

By the results in section 4 we have:

Lemma 5.1. *The terms $E^1_{p,q}$ in the spectral sequence $\mathcal{E}(W,V)$ are $\mathcal{AM}$ for $q>0$, and*

$$\left(E^1_{p,q}\right)_{\mathcal{M}} = \mathrm{H}_p(\mathrm{SL}(V'_{n-q}), \mathbb{Z}) \cong \mathrm{H}_p(\mathrm{SL}_{n-q}(F), \mathbb{Z}).$$

For $q=n$, the orbits of $\mathrm{SA}(W,V)$ on the basis of $\mathcal{C}^\tau_n(W,V)$ are in bijective correspondence with $F^\times$ via

$$\big((w_1,v_1),\ldots,(w_n,v_n)\big) \mapsto \det\left([v_1|\cdots|v_n]_{\mathcal{E}}\right).$$

The stabilizer of any basis element of $\mathcal{C}^\tau_n(W,V)$ is trivial. Thus

$$E^1_{p,n} = \begin{cases} \mathbb{Z}[F^\times], & p=0 \\ 0, & p>0 \end{cases}$$

Of course, $E^1_{p,q} = 0$ for $q>n$.

The first column of the E^1-page of the spectral sequence $\mathcal{E}(W,V)$ has the form

$$E^1_{0,q} = \begin{cases} \mathbb{Z}, & q<n \\ \mathbb{Z}[F^\times], & q=n \\ 0, & q>n \end{cases}$$

and the differentials are easily computed: For $q<n$

$$d^1_{0,q} : E^1_{0,q} \to E^1_{0,q} = \begin{cases} \mathrm{Id}_{\mathbb{Z}}, & q \text{ is odd} \\ 0, & q \text{ is even} \end{cases}$$

and

$$d^1_{0,n} : \mathbb{Z}[F^\times] \to \mathbb{Z} = \begin{cases} \text{augmentation}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

It follows that $E^2_{0,q} = 0$ for $q \neq n$ and

$$E^2_{0,n} = \begin{cases} \mathcal{I}_{F^\times}, & n \text{ odd} \\ \mathbb{Z}[F^\times], & n \text{ even} \end{cases}$$

Note that the composite

$$\tilde{\mathrm{S}}(W,V) \xrightarrow{\text{edge}} E^\infty_{0,n} \subset E^2_{0,n} = \mathcal{A}_n$$

is just the map $D_{W,V}$ of section 3 above.

Lemma 5.2. *The map $D_{W,V}$ is a split surjective homomorphism of $\mathbb{Z}[F^\times]$-modules.*

Proof. If $W = 0$, this is Lemma 3.7 (1) and (3), since $V \cong F^n$.
In general the natural map of complexes $\mathcal{C}^\tau_\bullet(V) \to \mathcal{C}^\tau_\bullet(W,V)$ gives rise to a commutative diagram of $\mathbb{Z}[F^\times]$-modules

$$\begin{tikzcd} \tilde{\mathrm{S}}(V) \arrow[rr] \arrow[dr, "D_V"'] & & \tilde{\mathrm{S}}(W,V) \arrow[dl, "D_{W,V}"] \\ & \mathcal{A}_n & \end{tikzcd}$$

□

We let $\tilde{S}(W,V)^+ := \mathrm{Ker}(D_{W,V} : \tilde{\mathrm{S}}(W,V) \to \mathcal{A}_n)$, so that $\tilde{\mathrm{S}}(W,V) \cong \tilde{S}(W,V)^+ \oplus \mathcal{A}_n$ for all W, V.

Corollary 5.3. *In the spectral sequence $\mathcal{E}(W,V)$, we have $E^2_{0,q} = E^\infty_{0,q}$ for all $q \geq 0$.*
All higher differentials $d^r_{0,q} : E^r_{0,q} \to E^r_{r-1,q+r}$ are zero.

It follows that the spectral sequences $\mathcal{E}(W,V)$ decompose as a direct sum of two spectral sequences

$$\mathcal{E}(W,V) = \mathcal{E}^0(W,V) \oplus \mathcal{E}^+(W,V)$$

where $\mathcal{E}^0(W,V)$ is the first column of $\mathcal{E}(W,V)$ and $\mathcal{E}^+(W,V)$ involves only the terms $E^r_{p,q}$ with $q > 0$.
The spectral sequence $\mathcal{E}^0(W,V)$ converges in degree d to

$$\begin{cases} 0, & d \neq n \\ \mathcal{A}_n, & d = n \end{cases}$$

The spectral sequence $\mathcal{E}^+(W,V)$ converges in degree d to

$$\begin{cases} 0, & d < n \\ \tilde{S}(W,V)^+, & d = n \\ \mathrm{H}_{d-n}(\mathrm{SA}(W,V), H(W,V)), & d > n \end{cases}$$

By Lemma 5.1 above, all the terms of the spectral sequence $\mathcal{E}^+(W,V)$ are $\mathcal{AM}$.
We thus have

Corollary 5.4.

(1) *The $\mathbb{Z}[F^\times]$-modules $\tilde{S}(W,V)^+$ are $\mathcal{AM}$.*
(2) *The graded submodule $\tilde{S}(F^\bullet)^+{}_{\mathcal{A}} \subset \tilde{\mathrm{S}}(F^\bullet)$ is an ideal.*

Proof.

(1) This follows from Theorem 4.22.
(2) This follows from Lemma 4.18, since $\tilde{S}(F^\bullet)^+$ is an ideal in $\tilde{\mathrm{S}}(F^\bullet)$ by Lemma 3.7 (2).

$\square$

Corollary 5.5. *The natural embedding $H(V) \to H(W,V)$ induces an isomorphism*

$$\tilde{S}(V)^+{}_{\mathcal{M}} \xrightarrow{\cong} \tilde{S}(W,V)^+{}_{\mathcal{M}}.$$

Proof. The map of complexes of $\mathrm{SL}(V)$-modules $\mathcal{C}^\tau_\bullet(V) \to \mathcal{C}^\tau_\bullet(W,V)$ gives rise to a map of spectral sequences $\mathcal{E}^+(V) \to \mathcal{E}^+(W,V)$ and hence a map $\mathcal{E}^+(V)_{\mathcal{M}} \to \mathcal{E}^+(W,V)_{\mathcal{M}}$. The induced map on the E^1-terms is

$$\begin{array}{ccc} \mathrm{H}_p(\mathrm{SL}_{n-q}(F),\mathbb{Z}) & \xrightarrow{\mathrm{Id}} & \mathrm{H}_p(\mathrm{SL}_{n-q}(F),\mathbb{Z}) \\ \downarrow{\cong} & & \downarrow{\cong} \\ \mathrm{H}_p(\mathrm{SL}(V),\mathcal{C}^\tau_q(V))_{\mathcal{M}} & \longrightarrow & \mathrm{H}_p(\mathrm{SA}(W,V),\mathcal{C}^\tau_q(W,V))_{\mathcal{M}} \end{array}$$

and thus is an isomorphism.
It follows that there is an induced isomorphism of abutments

$$\tilde{S}(V)^+{}_{\mathcal{M}} \cong \tilde{S}(W,V)^+{}_{\mathcal{M}}$$

and

$$\mathrm{H}_k(\mathrm{SL}(V),H(V))_{\mathcal{M}} \cong \mathrm{H}_k(\mathrm{SA}(W,V),H(W,V))_{\mathcal{M}}.$$

$\square$

For convenience, we now *define*

$$\tilde{\mathrm{S}}(W,V)_{\mathcal{M}} := \frac{\tilde{\mathrm{S}}(W,V)}{\tilde{S}(W,V)^+{}_{\mathcal{A}}}$$

(even though $\tilde{\mathrm{S}}(W,V)$ is not an $\mathcal{AM}$ module).
This gives:

Corollary 5.6.

$$\tilde{\mathrm{S}}(W,V)_{\mathcal{M}} \cong \tilde{S}(W,V)^+{}_{\mathcal{M}} \oplus \mathcal{A}_n \cong \tilde{S}(V)^+{}_{\mathcal{M}} \oplus \mathcal{A}_n \cong \tilde{\mathrm{S}}(V)_{\mathcal{M}}$$

as $\mathbb{Z}[F^\times]$-modules, and $\tilde{\mathrm{S}}(F^\bullet)_{\mathcal{M}}$ is a graded $\mathbb{Z}[F^\times]$-algebra.

Lemma 5.7. *For any $k \geq 1$, the corestriction map*

$$\mathrm{cor} : \mathrm{H}_i(\mathrm{SL}_k(F),\mathbb{Z}) \to \mathrm{H}_i(\mathrm{SL}_{k+1}(F),\mathbb{Z})$$

is $F^\times$-invariant;i.e. if $a \in F^\times$ and $z \in \mathrm{H}_i(\mathrm{SL}_k(F),\mathbb{Z})$, then

$$\mathrm{cor}(\langle a\rangle\, z) = \langle a\rangle\,\mathrm{cor}(z) = \mathrm{cor}(z).$$

Proof. Of course, cor is a homomorphism of $\mathbb{Z}[F^\times]$-modules. However, for $a \in F^\times$, $\langle a^k \rangle$ acts trivially on $\mathrm{H}_i(\mathrm{SL}_k(F),\mathbb{Z})$ while $\langle a^{k+1} \rangle$ acts trivially on $\mathrm{H}_i(\mathrm{SL}_{k+1}(F),\mathbb{Z})$ so that

$$\mathrm{cor}(\langle a \rangle z) = \mathrm{cor}(\langle a^{k+1} \rangle z) = \langle a^{k+1} \rangle \mathrm{cor}(z) = \mathrm{cor}(z).$$

□

Lemma 5.8. *For $0 \leq q < n$, the differentials of the spectral sequence $\mathcal{E}^+(W,V)_{\mathcal{M}}$*

$$d^1_{p,q} : (E^1_{p,q})_{\mathcal{M}} \cong \mathrm{H}_p(\mathrm{SL}_{n-q}(F),\mathbb{Z}) \to (E^1_{p,q-1})_{\mathcal{M}} \cong \mathrm{H}_p(\mathrm{SL}_{n-q+1}(F),\mathbb{Z})$$

are zero when q is even and are equal to the corestriction map when q is odd.

Proof. d^1 is derived from the map $d_q : \mathcal{C}^\tau_q(W,V) \to \mathcal{C}^\tau_{q-1}(W,V)$ of permutation modules. Here

$$\begin{aligned} d_q\big((0,e_1),\ldots,(0,e_q)\big) &= \sum_{i=1}^{q}(-1)^{i+1}\big((0,e_1),\ldots,\widehat{(0,e_i)},\ldots,(0,e_q)\big) \\ &= \sum_{i=1}^{q}(-1)^{i+1}\phi_i\big((0,e_1),\ldots,(0,e_{q-1})\big) \end{aligned}$$

where $\phi_i \in \mathrm{SA}(W,V)$ can be chosen to be of the form

$$\phi_i = \begin{pmatrix} \mathrm{Id}_W & 0 \\ 0 & \psi_i \end{pmatrix}, \quad \psi_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \tau_i \end{pmatrix} \in \mathrm{GL}(V)$$

with $\sigma_i \in \mathrm{GL}(V_q)$ a permutation matrix of determinant ϵ_i and $\tau_i \in \mathrm{GL}(V'_{n-q})$ also of determinant ϵ_i.

ϕ_i normalises $\mathrm{SA}(W \oplus V_q, V'_{n-q},)$ and $\mathrm{SL}(V'_{n-q})$. Thus for $z \in \mathrm{H}_p(\mathrm{SL}(V'_{n-q}),\mathbb{Z})$,

$$\begin{aligned} d^1(z) &= \sum_{i=1}^{q}(-1)^{i+1}\mathrm{cor}(\tau_i z) \\ &= \sum_{i=1}^{q}(-1)^{i+1}\mathrm{cor}(\langle \epsilon_i \rangle z) \\ &= \sum_{i=1}^{q}(-1)^{i+1}\mathrm{cor}(z) = \begin{cases} \mathrm{cor}(z), & q \text{ odd} \\ 0, & q \text{ even} \end{cases} \end{aligned}$$

□

Let $E := \lfloor -1, 1 \rceil \in \tilde{\mathrm{S}}(F^2)_{\mathcal{M}}$. E is represented by the element

$\tilde{E} := d_3(e_1,e_2,e_2-e_1) = (e_2,e_2-e_1)-(e_1,e_2-e_1)+(e_1,e_2) \in H(F^2) \subset \mathcal{C}^\tau_2(F^2)$.

Multiplication by $\tilde{E}$ induces a map of complexes of $\mathrm{GL}_{n-2}(F)$-modules

$$\mathcal{C}^\tau_\bullet(F^{n-2})[2] \to \mathcal{C}^\tau_\bullet(F^n)$$

There is an induced map of spectral sequences $\mathcal{E}(F^{n-2})[2] \to \mathcal{E}(F^n)$, which in turn induces a map $\mathcal{E}^+(F^{n-2})[2] \to \mathcal{E}^+(F^n)$, and hence a map $\mathcal{E}^+(F^{n-2})_{\mathcal{M}}[2] \to \mathcal{E}^+(F^n)_{\mathcal{M}}$.

By the work above, the E^1-page of $\mathcal{E}^+(F^n)_{\mathcal{M}}$ has the form

$$E^1_{p,q} = \mathrm{H}_p(\mathrm{SL}_{n-q}(F), \mathbb{Z}) \quad (p > 0)$$

while the E^1-page of $\mathcal{E}^+(F^{n-2})_{\mathcal{M}}[2]$ has the form

$$E'^1_{p,q} = \begin{cases} \mathrm{H}_p(\mathrm{SL}_{(n-2)-(q-2)}(F), \mathbb{Z}) = \mathrm{H}_p(\mathrm{SL}_{n-q}(F), \mathbb{Z}), & q \geq 2, p > 0 \\ 0, & q \leq 1 \text{ or } p = 0 \end{cases}$$

Lemma 5.9. *For $q \geq 2$ (and $p > 0$), the map*

$$E'^1_{p,q} \cong \mathrm{H}_p(\mathrm{SL}_{n-q}(F), \mathbb{Z}) \to E^1_{p,q} = \mathrm{H}_p(\mathrm{SL}_{n-q}(F), \mathbb{Z})$$

induced by $\tilde{E} * -$ is the identity map.

Proof. There is a commutative diagram

$$\begin{tikzcd}
E'^1_{p,q} = \mathrm{H}_p(\mathrm{SL}_{n-q}(F), \mathbb{Z}) \arrow[r] \arrow[d, "(\tilde{E}*-)_{\mathcal{M}}"] & \mathrm{H}_p(\mathrm{SA}(F^{q-2}, F^{n-q}), \mathbb{Z}) \arrow[r, "\cong"] \arrow[d, "\tilde{E}*-"] & \mathrm{H}_p(\mathrm{SL}_{n-2}(F), \mathcal{C}^{\tau}_{q-2}(F^{n-2})) \arrow[d, "\tilde{E}*-"] \\
E^1_{p,q} = \mathrm{H}_p(\mathrm{SL}_{n-q}(F), \mathbb{Z}) \arrow[r] & \mathrm{H}_p(\mathrm{SA}(F^{q}, F^{n-q}), \mathbb{Z}) \arrow[r, "\cong"] & \mathrm{H}_p(\mathrm{SL}_{n}(F), \mathcal{C}^{\tau}_{q}(F^{n}))
\end{tikzcd}$$

We number the standard basis of F^{n-2} $e_3, \ldots, e_n$ so that the inclusion $\mathrm{SL}_{n-2}(F) \to \mathrm{SL}_n(F)$ has the form

$$A \mapsto \begin{pmatrix} I_2 & 0 \\ 0 & A \end{pmatrix}.$$

So we have a commutative diagram of inclusions of groups

$$\begin{tikzcd}
\mathrm{SL}_{n-q}(F) \arrow[r] \arrow[d, "="] & \mathrm{SA}(F^{q-2}, F^{n-q}) \arrow[r] \arrow[d] & \mathrm{SL}_{n-2}(F) \arrow[d] \\
\mathrm{SL}_{n-q}(F) \arrow[r] & \mathrm{SA}(F^{q}, F^{n-q}) \arrow[r] & \mathrm{SL}_{n}(F).
\end{tikzcd}$$

Let $B_\bullet = B_\bullet(\mathrm{SL}_n(F))$ be the right bar resolution of $\mathrm{SL}_n(F)$. We can use it to compute the homology of any of the groups occurring in this diagram.
Suppose now that $q \geq 2$ and we have a class, w, in $E'^1_{p,q} = \mathrm{H}_p(\mathrm{SL}_{n-q}(F), \mathbb{Z})$ represented by a cycle

$$z \otimes 1 \in B_p \otimes_{\mathbb{Z}[\mathrm{SL}_{n-q}(F)]} \mathbb{Z}.$$

Its image in $\mathrm{H}_p(\mathrm{SL}_{n-2}(F), \mathcal{C}^{\tau}_{q-2}(F^{n-2}))$ is represented by $z \otimes (e_3, \ldots, e_q)$. The image of this in $\mathrm{H}_p(\mathrm{SL}_n(F), \mathcal{C}^{\tau}_q(F^n))$ is

$$\begin{aligned}
& z \otimes \left[\tilde{E} * (e_3, \ldots, e_q)\right] \\
= \quad & z \otimes [(e_2, e_2 - e_1, e_3, \ldots) - (e_1, e_2 - e_1, e_3, \ldots) + (e_1, e_2, e_3, \ldots)] \\
= \quad & z \otimes [(g_1 - g_2 + 1)(e_1, e_2, e_3, \ldots)] \in B_p \otimes_{\mathbb{Z}[\mathrm{SL}_n(F)]} \mathcal{C}^{\tau}_q(F^n)
\end{aligned}$$

where

$$g_1 = \begin{pmatrix} 0 & -1 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \in \mathrm{SL}_n(F).$$

This corresponds to the element in $\mathrm{H}_p(\mathrm{SL}_{n-q}(F),\mathbb{Z})$ represented by

$$z(g_1 - g_2 + 1) \otimes 1 \in B_p \otimes_{\mathbb{Z}[\mathrm{SL}_{n-q}(F)]} \mathbb{Z}$$

Since the elements g_i centralize $\mathrm{SL}_{n-q}(F)$ it follows that this is $(g_1 - g_2 + 1)\cdot w = w$. $\square$

Recall that the spectral sequence $\mathcal{E}^+(F^n)_{\mathcal{M}}$ converges in degree n to $\tilde{S}(F^n)^+{}_{\mathcal{M}}$. Thus there is a filtration

$$0 = \mathcal{F}_{n,-1} \subset \mathcal{F}_{n,0} \subset \mathcal{F}_{n,1} \subset \cdots \mathcal{F}_{n,n} = \tilde{S}(F^n)^+{}_{\mathcal{M}}$$

with

$$\frac{\mathcal{F}_{n,i}}{\mathcal{F}_{n,i-1}} \cong E^{\infty}_{n-i,i}.$$

The E^1-page of $\mathcal{E}^+(F^n)_{\mathcal{M}}$ has the form

$$\begin{array}{ccccc}
0 & 0 & 0 & \cdots & 0 \\
 & & & & \\
0 & \mathrm{H}_1(\mathrm{SL}_2(F),\mathbb{Z}) & \mathrm{H}_2(\mathrm{SL}_2(F),\mathbb{Z}) & \cdots & \mathrm{H}_n(\mathrm{SL}_2(F),\mathbb{Z}) \\
 & \downarrow & \downarrow & & \downarrow \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
 & \downarrow{\scriptstyle \mathrm{cor}} & \downarrow{\scriptstyle \mathrm{cor}} & & \downarrow{\scriptstyle \mathrm{cor}} \\
0 & \mathrm{H}_1(\mathrm{SL}_{n-2}(F),\mathbb{Z}) & \mathrm{H}_2(\mathrm{SL}_{n-2}(F),\mathbb{Z}) & \cdots & \mathrm{H}_n(\mathrm{SL}_{n-2}(F),\mathbb{Z}) \\
 & \downarrow{\scriptstyle 0} & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} \\
0 & \mathrm{H}_1(\mathrm{SL}_{n-1}(F),\mathbb{Z}) & \mathrm{H}_2(\mathrm{SL}_{n-1}(F),\mathbb{Z}) & \cdots & \mathrm{H}_n(\mathrm{SL}_{n-1}(F),\mathbb{Z}) \\
 & \downarrow{\scriptstyle \mathrm{cor}} & \downarrow{\scriptstyle \mathrm{cor}} & & \downarrow{\scriptstyle \mathrm{cor}} \\
0 & \mathrm{H}_1(\mathrm{SL}_n(F),\mathbb{Z}) & \mathrm{H}_2(\mathrm{SL}_n(F),\mathbb{Z}) & \cdots & \mathrm{H}_n(\mathrm{SL}_n(F),\mathbb{Z})
\end{array}$$

Theorem 5.10.

(1) *The higher differentials $d^2, d^3, \ldots,$ in the spectral sequence $\mathcal{E}^+(F^n)_{\mathcal{M}}$ are all* 0.

(2) $\tilde{S}(F^{n-2})_{\mathcal{M}} \cong E * \tilde{S}(F^{n-2})_{\mathcal{M}}$ *and this latter is a direct summand of* $\tilde{S}(F^n)_{\mathcal{M}}$.

Proof.

(1) We will use induction on n. For $n \leq 2$ the statement is true for trivial reasons.

On the other hand, if $n > 2$, by Lemma 5.9, the map

$$\tilde{E} * - : \mathcal{E}^+(F^{n-2})_{\mathcal{M}}[2] \to \mathcal{E}^+(F^n)_{\mathcal{M}}$$

induces an isomorphism on E^1-terms for $q \geq 2$. By induction (and the fact that ${E'}^1_{p,q} = 0$ for $q \leq 1$), the result follows for n.

(2) The map of spectral sequences $\mathcal{E}^+(F^{n-2})_{\mathcal{M}}[2] \to \mathcal{E}^+(F^n)_{\mathcal{M}}$ induces a homomorphism on abutments

$$\tilde{S}(F^{n-2})^+{}_{\mathcal{M}} \xrightarrow{E*-} \tilde{S}(F^n)^+{}_{\mathcal{M}}$$

By Lemma 5.9 again, it follows that the composite

$$\tilde{S}(F^{n-2})^+{}_{\mathcal{M}} \xrightarrow{E*-} \tilde{S}(F^n)^+{}_{\mathcal{M}} \longrightarrow \left(\tilde{S}(F^n)^+{}_{\mathcal{M}}\right)/\mathcal{F}_{n,1}$$

is an isomorphism.

Thus $\tilde{S}(F^{n-2})^+{}_{\mathcal{M}} \cong E * \tilde{S}(F^{n-2})^+{}_{\mathcal{M}}$ and

$$\tilde{S}(F^n)^+{}_{\mathcal{M}} \cong \left(E * \tilde{S}(F^{n-2})^+{}_{\mathcal{M}}\right) \oplus \mathcal{F}_{n,1}.$$

□

As a corollary we obtain the following general homology stability result for the homology of special linear groups:

Corollary 5.11.
The corestriction maps $\mathrm{H}_p(\mathrm{SL}_{n-1}(F),\mathbb{Z}) \to \mathrm{H}_p(\mathrm{SL}_n(F),\mathbb{Z})$ are isomorphisms for $p < n-1$ and are surjective when $p = n-1$.

Proof. Using (1) of Theorem 5.10 and Lemma 5.8, we have (for the spectral sequence $\mathcal{E}^+(F^n)_{\mathcal{M}}$) that $E^\infty_{p,q} = E^2_{p,q} =$

$$\frac{\mathrm{Ker}(d^1)}{\mathrm{Im}(d^1)} = \begin{cases} \mathrm{Ker}(\mathrm{H}_p(\mathrm{SL}_{n-q}(F),\mathbb{Z}) \to \mathrm{H}_p(\mathrm{SL}_{n-q+1}(F),\mathbb{Z})) & q \text{ odd} \\ \mathrm{Coker}(\mathrm{H}_p(\mathrm{SL}_{n-q-1}(F),\mathbb{Z}) \to \mathrm{H}_p(\mathrm{SL}_{n-q}(F),\mathbb{Z})) & q \text{ even} \end{cases}$$

But the abutment of the spectral sequence is 0 in dimensions less than n. It follows that $E^\infty_{p,q} = 0$ whenever $p + q \leq n-1$. □

Remark 5.12. Note that in the spectral sequence $\mathcal{E}^+(F^n)_{\mathcal{M}}$,

$$E^\infty_{n,0} = \mathrm{Coker}(\mathrm{H}_n(\mathrm{SL}_{n-1}(F),\mathbb{Z}) \to \mathrm{H}_n(\mathrm{SL}_n(F),\mathbb{Z})) = \mathrm{SH}_n(F).$$

Clearly, the edge homomorphism $\mathrm{H}_n(\mathrm{SL}_n(F),\mathbb{Z}) \to E^\infty_{n,0} \to \tilde{S}(F^n)_{\mathcal{M}}$ is just the iterated connecting homomorphism ϵ_n of section 3 above. Thus we have:

Corollary 5.13. *The maps*

$$\epsilon_\bullet : \mathrm{SH}_\bullet(F) \to \tilde{S}(F^\bullet)_{\mathcal{M}}$$

define an injective homomorphism of graded $\mathbb{Z}[F^\times]$-algebras.

COROLLARY 5.14. *$\tilde{S}(F^2)_{\mathcal{M}} = \mathcal{F}_{2,1} \oplus \mathbb{Z}[F^\times]E$ and for all $n \geq 3$,*

$$\tilde{S}(F^n)_{\mathcal{M}} = (E * \tilde{S}(F^{n-2})_{\mathcal{M}}) \oplus \mathcal{F}_{n,1} \cong \tilde{S}(F^{n-2})_{\mathcal{M}} \oplus \mathcal{F}_{n,1}.$$

Proof. Clearly $\tilde{S}(F^2)^+{}_{\mathcal{M}} = \mathcal{F}_{1,2}$, while for $n \geq 3$ we have

$$\tilde{S}(F^n)_{\mathcal{M}} = \begin{cases} \tilde{S}(F^n)^+{}_{\mathcal{M}} \oplus \mathbb{Z}[F^\times]E^{*\frac{n}{2}} & n \text{ even} \\ \tilde{S}(F^n)^+{}_{\mathcal{M}} \oplus \left(\tilde{S}(F) * E^{*\frac{n-1}{2}}\right) & n \text{ odd} \end{cases}$$

□

COROLLARY 5.15. *For all $n \geq 3$,*

$$\tilde{S}(F^n)_{\mathcal{M}} \cong \begin{cases} \mathcal{F}_{n,1} \oplus \mathcal{F}_{n-2,1} \oplus \cdots \oplus \mathcal{F}_{2,1} \oplus \mathbb{Z}[F^\times] & n \text{ even} \\ \mathcal{F}_{n,1} \oplus \mathcal{F}_{n-2,1} \oplus \cdots \oplus \mathcal{F}_{3,1} \oplus \mathcal{I}_{F^\times} & n \text{ odd} \end{cases}$$

as a $\mathbb{Z}[F^\times]$-module.

Note that $\mathcal{F}_{1,1} = \tilde{S}(F) = \mathcal{I}_{F^\times}$, and for all $n \geq 2$, $\mathcal{F}_{n,1}$ fits into an exact sequence associated to the spectral sequence $\mathcal{E}^+(F^n)_{\mathcal{M}}$:

$$0 \to E^\infty_{n,0} = \mathcal{F}_{n,0} \to \mathcal{F}_{n,1} \to E^\infty_{n-1,1} \to 0.$$

COROLLARY 5.16. *For all $n \geq 2$ we have an exact sequence*

$$\mathrm{H}_n(\mathrm{SL}_{n-1}(F),\mathbb{Z}) \to \mathrm{H}_n(\mathrm{SL}_n(F),\mathbb{Z}) \to \mathcal{F}_{n,1} \to$$
$$\mathrm{H}_{n-1}(\mathrm{SL}_{n-1}(F),\mathbb{Z}) \to \mathrm{H}_{n-1}(\mathrm{SL}_n(F),\mathbb{Z}) \to 0.$$

LEMMA 5.17. *For all $n \geq 2$, the map T_n induces a surjective map $\mathcal{F}_{n,1} \to K_n^{\mathrm{MW}}(F)$.*

Proof. First observe that since $K_n^{\mathrm{MW}}(F)$ is generated by the elements of the form $[a_1]\cdots[a_n]$ it follows from the definition of T_n that $T_n : \tilde{S}(F^n) \to K_n^{\mathrm{MW}}(F)$ is surjective for all $n \geq 1$.
Next, since $K_\bullet^{\mathrm{MW}}(F)$ is multiplicative, $T_\bullet$ factors through an algebra homomorphism $\tilde{S}(F^\bullet)_{\mathcal{M}} \to K_\bullet^{\mathrm{MW}}(F)$. The lemma thus follows from Corollary 5.14 and the fact that $T_2(E) = 0$. □

LEMMA 5.18. *$\mathcal{F}_{2,1} = \mathcal{F}_{2,0}$ and $T_2 : \mathcal{F}_{2,1} \to K_2^{\mathrm{MW}}(F)$ is an isomorphism.*

Proof. Since $\mathrm{H}_1(\mathrm{SL}_1(F),\mathbb{Z}) = 0$, $\mathcal{F}_{2,1} = \mathcal{F}_{2,0} = E^\infty_{2,0} = \epsilon_2(\mathrm{H}_2(\mathrm{SL}_2(F),\mathbb{Z}))$. Now apply Theorem 3.10. □

It is natural to define elements $[a,b] \in \mathcal{F}_{2,0} \subset \tilde{S}(F^2)_{\mathcal{M}}$ by $[a,b] := T_2^{-1}([a][b])$.

LEMMA 5.19. *In $\tilde{S}(F^2)_{\mathcal{M}}$ we have the formula*

$$[a,b] = \lfloor a\rceil * \lfloor b\rceil - \langle\langle a\rangle\rangle\langle\langle b\rangle\rangle E.$$

Proof. The results above show that the maps T_2 and D_2 induce an isomorphism

$$(T_2, D_2) : \tilde{S}(F^2)_{\mathcal{M}} \cong K_2^{\mathrm{MW}}(F) \oplus \mathbb{Z}[F^\times].$$

Since $D_2(\lfloor a\rceil * \lfloor b\rceil) = \langle\langle a\rangle\rangle\langle\langle b\rangle\rangle$, while $D_2(E) = 1$, the result follows. □

THEOREM 5.20.

(1) *The product $*$ respects the filtrations on $\tilde{S}(F^n)$; i.e. for all $n,m \geq 1$ and $i,j \geq 0$*

$$\mathcal{F}_{n,i} * \mathcal{F}_{m,j} \subset \mathcal{F}_{n+m,i+j}.$$

(2) *For $n \geq 1$, let $\epsilon_{n+1,1}$ denote the composite $\mathcal{F}_{n+1,1} \to E^{\infty}_{n,1} = E^2_{n,1} \to \mathrm{H}_n(\mathrm{SL}_n(F),\mathbb{Z})$. For all $a \in F^\times$ and for all $n \geq 1$ the following diagram commutes:*

$$\begin{tikzcd} \mathcal{F}_{n,0} \arrow[rr, "\lfloor a\rceil *"] & & \mathcal{F}_{n+1,1} \arrow[d, "\epsilon_{n+1,1}"] \\ \mathrm{H}_n(\mathrm{SL}_n(F),\mathbb{Z}) \arrow[u, "\epsilon_n"] \arrow[rr, "\langle\langle a\rangle\rangle\cdot"] & & \mathrm{H}_n(\mathrm{SL}_n(F),\mathbb{Z}) \end{tikzcd}$$

Proof.

(1) The filtration on $\tilde{S}(F^n)_{\mathcal{M}}$ is derived from the spectral sequence $\mathcal{E}(F^n)$. This is the spectral sequence of the double complex $B_\bullet \otimes_{\mathrm{SL}_n(F)} \mathcal{C}^\tau_\bullet(F^n)$, regarded as a filtered complex by truncating $\mathcal{C}^\tau_\bullet(F^n)$ at i for $i = 0,1,\ldots$. Since the product $*$ is derived from a graded bilinear pairing on the complexes $\mathcal{C}^\tau_\bullet(F^n)$, the result easily follows.

(2) The spectral sequence $\mathcal{E}(F^{n+1})$ calculates

$$\mathrm{H}_\bullet(\mathrm{SL}_{n+1}(F),\mathcal{C}^\tau(F^{n+1})) \cong \mathrm{H}_\bullet(\mathrm{SL}_{n+1}(F),H(F^{n+1})[n+1]$$

(where $[n+1]$ denotes a degree shift by $n+1$).

Let $C[1,n]$ denote the truncated complex

$$\mathcal{C}^\tau_1(F^{n+1}) \xrightarrow{d_1} \mathcal{C}^\tau_0(F^{n+1})$$

and let Z_1 denote the kernel of d_1. Then

$$\mathrm{H}_\bullet(\mathrm{SL}_{n+1}(F),C[1,n]) \cong \mathrm{H}_\bullet(\mathrm{SL}_{n+1}(F),Z_1)[1].$$

If $\mathcal{F}_i$ denotes the filtration on $\mathrm{H}_\bullet(\mathrm{SL}_{n+1}(F),\mathcal{C}^\tau(F^{n+1}))$ associated to the spectral sequence $\mathcal{E}(F^{n+1})$, then from the definition of this filtration, $\mathcal{F}_1\mathrm{H}_k(\mathrm{SL}_{n+1}(F),\mathcal{C}^\tau(F^{n+1})) =$

$$\mathrm{Im}(\mathrm{H}_k(\mathrm{SL}_{n+1}(F),C[1,n]) \to \mathrm{H}_k(\mathrm{SL}_{n+1}(F),\mathcal{C}^\tau(F^{n+1}))).$$

In particular,

$$\mathcal{F}_{n+1,1} \cong \mathrm{Im}(\mathrm{H}_{n+1}(\mathrm{SL}_{n+1}(F),C[1,n]) \to \mathrm{H}_{n+1}(\mathrm{SL}_{n+1}(F),\mathcal{C}^\tau(F^{n+1})))$$

and with this identification the diagram

$$\begin{tikzcd} \mathrm{H}_{n+1}(\mathrm{SL}_{n+1}(F),C[1,n]) \arrow[rr] & & \mathcal{F}_{n+1,1} \arrow[d, "\epsilon_{n+1,1}"] \\ \mathrm{H}_n(\mathrm{SL}_{n+1}(F),Z_1) \arrow[u, "\cong"] \arrow[rr] & & \mathrm{H}_n(\mathrm{SL}_{n+1}(F),\mathcal{C}^\tau_1(F^{n+1})) \end{tikzcd}$$

commutes (and $\mathrm{H}_n(\mathrm{SL}_{n+1}(F),\mathcal{C}^\tau_1(F^{n+1})) \cong \mathrm{H}_n(\mathrm{SA}(F,F^n),\mathbb{Z})$ by Shapiro's Lemma, of course).

We consider $\mathrm{SL}_n(F) \subset \mathrm{SA}(F, F^n) \subset \mathrm{SL}_{n+1}(F) \subset \mathrm{GL}_{n+1}(F)$ where the first inclusion is obtained by inserting a 1 in the $(1,1)$ position. Let $B_\bullet$ denote a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}[\mathrm{GL}_{n+1}(F)]$. Let $z \in \mathrm{H}_n(\mathrm{SL}_n(F), \mathbb{Z})$ be represented by $x \otimes 1 \in B_n \otimes_{\mathbb{Z}[\mathrm{SL}_n(F)]} \mathbb{Z} = B_n \otimes_{\mathbb{Z}[\mathrm{SL}_n(F)]} \mathcal{C}_0^\tau(F^n)$. Then $\lfloor a \rceil * \epsilon_n(z)$ is represented by $z \otimes [(ae_1) - (e_1)] \in B_n \otimes_{\mathrm{SL}_{n+1}(F)} Z_1$ which maps to the element of $\mathrm{H}_n(\mathrm{SL}_{n+1}(F), \mathcal{C}_1^\tau(F^{n+1}))$ represented by $z(g-1) \otimes (e_1)$ where $g = \mathrm{diag}(a, 1, \ldots, 1, a^{-1})$. But this is just the image of $\langle\langle a \rangle\rangle z$ under the map $\mathrm{H}_n(\mathrm{SL}_n(F), \mathbb{Z}) \to \mathrm{H}_n(\mathrm{SA}(F, F^n), \mathbb{Z}) \cong \mathrm{H}_n(\mathrm{SL}_{n+1}(F), \mathcal{C}_1^\tau(F^{n+1}))$. □

LEMMA 5.21. *The map $T_3 : \mathcal{F}_{3,1} \to K_3^{\mathrm{MW}}(F)$ is an isomorphism.*

Proof. Consider the short exact sequence

$$0 \to E^\infty_{3,0} \to \mathcal{F}_{3,1} \to E^\infty_{2,1} \to 0.$$

Here ϵ_3 induces an isomorphism

$$E^\infty_{3,0} \cong \mathrm{Coker}(\mathrm{H}_3(\mathrm{SL}_2(F), \mathbb{Z}) \to \mathrm{H}_3(\mathrm{SL}_3(F), \mathbb{Z})).$$

By the main result of [8] (Theorem 4.7 - see also section 2.4 of this article), T_3 thus induces an isomorphism $E^\infty_{3,0} \cong 2K_3^{\mathrm{M}}(F) \subset K_3^{\mathrm{MW}}(F)$.
On the other hand,

$$E^\infty_{2,1} \cong \mathrm{Ker}(\mathrm{H}_2(\mathrm{SL}_2(F), \mathbb{Z}) \to \mathrm{H}_2(\mathrm{SL}_3(F), \mathbb{Z})) \cong I^3(F)$$

Thus we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & E^\infty_{3,0} \arrow[r] \arrow[d, "\cong"', "T_3"] & \mathcal{F}_{3,1} \arrow[r, "\rho"] \arrow[d, "T_3"] & I^3(F) \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & 2K_3^{\mathrm{M}}(F) \arrow[r] & K_3^{\mathrm{MW}}(F) \arrow[r, "p_3"] & I^3(F) \arrow[r] & 0
\end{tikzcd}$$

where the vertical arrows are surjections.
Now the inclusion $I^3(F) \to K_2^{\mathrm{MW}}(F)$ is given by $\langle\langle a, b, c \rangle\rangle \mapsto \langle\langle a \rangle\rangle [b][c]$. Thus the inclusion $j : I^3(F) \to \mathrm{H}_2(\mathrm{SL}_2(F), \mathbb{Z})$ is given by $\langle\langle a, b, c \rangle\rangle \mapsto \langle\langle a \rangle\rangle \langle b, c \rangle$ where $\langle b, c \rangle = \epsilon_2^{-1}([b, c])$. Thus for all $a, b, c \in F^\times$ we have

$$j \circ \rho(\lfloor a \rceil * [b, c]) = \epsilon_{3,1}(\lfloor a \rceil * [b, c]) = \langle\langle a \rangle\rangle \langle b, c \rangle$$

using Theorem 5.20 (2), and thus $\rho(\lfloor a \rceil * [b, c]) = \langle\langle a, b, c \rangle\rangle \in I^3(F)$. It follows from the diagram that

$$\alpha(\langle\langle a, b, c \rangle\rangle) = \alpha \circ \rho(\lfloor a \rceil * [b, c]) = p_3 \circ T_3(\lfloor a \rceil * [b, c]) = \langle\langle a, b, c \rangle\rangle$$

so that α is the identity map, and the result follows. □

LEMMA 5.22. *For all $a \in F^\times$, $\lfloor a \rceil * E = E * \lfloor a \rceil$ in $\tilde{S}(F^3)_{\mathcal{M}}$.*

Proof. By the calculations above, $\mathcal{F}_{3,1} = \tilde{S}(F^3)^+{}_{\mathcal{M}} = \mathrm{Ker}(D_3)$. Thus $R_a := \lfloor a \rceil * E - E * \lfloor a \rceil \in \mathcal{F}_{3,1}$. But then $T_3(R_a) = 0$ since $T_2(E) = 0$ and thus $R_a = 0$ by the previous lemma. □

Lemma 5.23.

(1) *For all* $a, b, c \in F^\times$
$$\lfloor a\rceil * [b, c] = [a, b] * \lfloor c\rceil \text{ in } \tilde{S}(F^3)_{\mathcal{M}}.$$
(2) *For all* $a, b, c \in F^\times$
$$\lfloor a\rceil * \lfloor b\rceil * \lfloor c\rceil = \lfloor c\rceil * \lfloor a\rceil * \lfloor b\rceil \text{ in } \tilde{S}(F^3)_{\mathcal{M}}.$$
(3) *For all* $a, b, c, d \in F^\times$
$$[a, b] * [c, d] = [a, c^{-1}] * [b, d] \text{ in } \tilde{S}(F^4)_{\mathcal{M}}.$$

Proof. The calculations above have established that the map
$$(T_3, D_3) : \tilde{S}(F^3)_{\mathcal{M}} \to K_3^{\mathrm{MW}}(F) \oplus \mathcal{I}_{F^\times}$$
is an isomorphism.

(1) This follows from the identities
$$T_3(\lfloor a\rceil * [b, c]) = [a][b][c] = T_3([a, b] * \lfloor c\rceil)$$
and
$$D_3(\lfloor a\rceil * [b, c]) = \langle\langle a, b, c\rangle\rangle = D_3([a, b] * \lfloor c\rceil)$$
(2) This follows from the fact that $[a][b][c] = [c][a][b]$ in $K_3^{\mathrm{MW}}(F)$.

(3) We begin by observing that, since $\tilde{S}(F) \cong \mathcal{I}_{F^\times}$ as a $\mathbb{Z}[F^\times]$-module we have $\langle\langle a\rangle\rangle\lfloor b\rceil = \lfloor ab\rceil - \lfloor a\rceil - \lfloor b\rceil = \langle\langle b\rangle\rangle\lfloor a\rceil$ for all $a, b \in F^\times$.

For $x_1, \ldots, x_n \in F^\times$ and $i, j \geq 1$ with $i + j = n$ we set
$$L_{i,j}(x_1, \ldots, x_n) := \langle\langle x_1\rangle\rangle \cdots \langle\langle x_i\rangle\rangle \left(\lfloor x_{i+1}\rceil * \cdots * \lfloor x_n\rceil\right) \in \tilde{S}(F^j)_{\mathcal{M}}.$$
By the observation just made, we have
$$L_{i,j}(x_1, \ldots, x_n) = L_{i,j}(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$
for any permutation σ of $1, \ldots, n$.

So
$$[a, b] * [c, d] = (\lfloor a\rceil * \lfloor b\rceil - \langle\langle a\rangle\rangle\langle\langle b\rangle\rangle E) * (\lfloor c\rceil * \lfloor d\rceil - \langle\langle c\rangle\rangle\langle\langle d\rangle\rangle E)$$
$$= \lfloor a\rceil * \lfloor b\rceil * \lfloor c\rceil * \lfloor d\rceil - 2L_{2,2}(a, b, c, d) * E + \langle\langle a\rangle\rangle\langle\langle b\rangle\rangle\langle\langle c\rangle\rangle\langle\langle d\rangle\rangle E^{*2}$$
Let $R = [a, b] * [c, d] - [a, c^{-1}] * [b, d]$.

So $R =$
$$\lfloor a\rceil * \lfloor b\rceil * \lfloor c\rceil * \lfloor d\rceil - \lfloor a\rceil * \lfloor c^{-1}\rceil * \lfloor b\rceil * \lfloor d\rceil - 2(L_{2,2}(a, b, c, d) - L_{2,2}(a, c^{-1}, b, d)) * E$$
$$+ \langle\langle a\rangle\rangle\langle\langle d\rangle\rangle \left[(\langle\langle b\rangle\rangle\langle\langle c\rangle\rangle - \langle\langle c^{-1}\rangle\rangle\langle\langle b\rangle\rangle)E\right] * E.$$
However, since $[b, c] = [c^{-1}, b]$ in $\tilde{S}(F^2)_{\mathcal{M}}$ we have (by Lemma 5.19)
$$(\langle\langle b\rangle\rangle\langle\langle c\rangle\rangle - \langle\langle c^{-1}\rangle\rangle\langle\langle b\rangle\rangle)E = \lfloor b\rceil * \lfloor c\rceil - \lfloor c^{-1}\rceil * \lfloor b\rceil.$$
Thus $\langle\langle a\rangle\rangle\langle\langle d\rangle\rangle \left[(\langle\langle b\rangle\rangle\langle\langle c\rangle\rangle - \langle\langle c^{-1}\rangle\rangle\langle\langle b\rangle\rangle)E\right] * E =$
$$(L_{2,2}(a, b, c, d) - L_{2,2}(a, c^{-1}, b, d)) * E$$
and hence $R =$
$$\lfloor a\rceil * \lfloor b\rceil * \lfloor c\rceil * \lfloor d\rceil - \lfloor a\rceil * \lfloor c^{-1}\rceil * \lfloor b\rceil * \lfloor d\rceil - (L_{2,2}(a, b, c, d) - L_{2,2}(a, c^{-1}, b, d)) * E.$$

Now

$$\begin{aligned}(L_{2,2}(a,b,c,d) - L_{2,2}(a,c^{-1},b,d)) * E \\ &= \lfloor a\rceil * \lfloor d\rceil * \left[(\langle\langle b\rangle\rangle\langle\langle c\rangle\rangle - \langle\langle c^{-1}\rangle\rangle\langle\langle b\rangle\rangle)E\right] \\ &= \lfloor a\rceil * \lfloor d\rceil * \left[\lfloor b\rceil * \lfloor c\rceil - \lfloor c^{-1}\rceil * \lfloor b\rceil\right] \\ &= \lfloor a\rceil * (\lfloor d\rceil * \lfloor b\rceil * \lfloor c\rceil) - \lfloor a\rceil * (\lfloor d\rceil * \lfloor c^{-1}\rceil * \lfloor b\rceil) \\ &= \lfloor a\rceil * \lfloor b\rceil * \lfloor c\rceil * \lfloor d\rceil - \lfloor a\rceil * \lfloor c^{-1}\rceil * \lfloor b\rceil * \lfloor d\rceil\end{aligned}$$

using (2) in the last step.

□

Theorem 5.24. *For all $n \geq 2$ there is a homomorphism $\mu_n : K_n^{\mathrm{MW}}(F) \to \mathcal{F}_{n,1}$ such that the composite $T_n \circ \mu_n$ is the identity map.*

Proof. For $n \geq 2$ and $a_1, \ldots, a_n \in F^\times$, let $\{\{a_1, \ldots, a_n\}\} :=$

$$\left\{\begin{array}{ll} [a_1,a_2] * \cdots * [a_{n-1},a_n], & n \text{ even} \\ \lfloor a_1\rceil * [a_2,a_3] * \cdots * [a_{n-1},a_n], & n \text{ odd}\end{array}\right\} \in \mathcal{F}_{n,1} \subset \tilde{S}(F^n)_{\mathcal{M}}.$$

By Lemma 5.23 (1) and (3), as well as the definition of $[x,y]$, the elements $\{\{a_1, \ldots, a_n\}\}$ satisfy the 'Matsumoto-Moore' relations (see Section 2.4 above), and thus there is a well-defined homomorphism of groups

$$\mu_n : K_n^{\mathrm{MW}}(F) \to \mathcal{F}_{n,1}, \qquad [a_1]\cdots[a_n] \mapsto \{\{a_1, \ldots, a_n\}\}.$$

Since $T_n(\{\{a_1, \ldots, a_n\}\}) = [a_1]\cdots[a_n]$, the result follows. □

Corollary 5.25. *The subalgebra of $\mathrm{SH}_{2\bullet}(F)$ generated by $\mathrm{SH}_2(F) = \mathrm{H}_2(\mathrm{SL}_2(F),\mathbb{Z})$ is isomorphic to $K_{2\bullet}^{\mathrm{MW}}(F)$ and is a direct summand of $\mathrm{SH}_{2\bullet}(F)$.*

Proof. This is immediate from Theorems 3.10 and 5.24. □

6. Decomposabilty

Recall that F is a field of characteristic 0 throughout this section.

In [24], Suslin proved that $\mathrm{H}_n(\mathrm{GL}_n(F),\mathbb{Z})/\mathrm{H}_n(\mathrm{GL}_{n-1}(F),\mathbb{Z}) \cong K_n^{\mathrm{M}}(F)$. This is, in particular, a decomposability result. It says that $\mathrm{H}_n(\mathrm{GL}_n(F),\mathbb{Z})$ is generated, modulo the image of $\mathrm{H}_n(\mathrm{GL}_{n-1}(F),\mathbb{Z})$ by products of 1-dimensional cycles. In this section we will prove analogous results for the special linear group, with Milnor-Witt K-theory replacing Milnor K-theory. To do this, we prove the decomposability of the algebra $\tilde{S}(F^\bullet)_{\mathcal{M}}$ (for $n \geq 3$). Theorem 6.2 is an analogue of Suslin's Proposition 3.3.1. The proof is essentially identical, and we reproduce it here for the convenience of the reader. From this we deduce our decomposability result (Theorem 6.8), which requires still a little more work than in the case of the general linear group.

Lemma 6.1. *For any finite-dimensional vector spaces W and V, the image of the pairing*

(2) $$\tilde{\mathrm{S}}(W,V) \otimes H(W) \to \tilde{\mathrm{S}}(W \oplus V)_{\mathcal{M}}$$

coincides with the image of the pairing

$$\tilde{\mathrm{S}}(V)\otimes\tilde{\mathrm{S}}(W)\to\tilde{\mathrm{S}}(W\oplus V)_{\mathcal{M}} \tag{3}$$

Proof. The image of the pairing (2) is equal to the image of

$$\tilde{\mathrm{S}}(W,V)_{\mathcal{M}}\otimes H(W)\to\tilde{\mathrm{S}}(W\oplus V)_{\mathcal{M}}$$

which coincides with the image of

$$\tilde{\mathrm{S}}(V)_{\mathcal{M}}\otimes\tilde{\mathrm{S}}(W)_{\mathcal{M}}\to\tilde{\mathrm{S}}(W\oplus V)_{\mathcal{M}}$$

by the isomorphism of Corollary 5.6. □

Let $\tilde{S}(F^n)^{\mathrm{dec}}\subset\tilde{\mathrm{S}}(F^n)_{\mathcal{M}}$ be *the* $\mathbb{Z}[F^\times]$*-submodule of decomposable elements*; i.e. $\tilde{S}(F^n)^{\mathrm{dec}}$ is the image of

$$\bigoplus\nolimits_{p+q=n,p,q>0}\left(\tilde{\mathrm{S}}(F^p)_{\mathcal{M}}\otimes\tilde{\mathrm{S}}(F^q)_{\mathcal{M}}\right)\xrightarrow{\ *\ }\tilde{\mathrm{S}}(F^n)_{\mathcal{M}}.$$

More generally, note that if $V=V_1\oplus V_2=V_1'\oplus V_2'$ and if $\dim_F(V_i)=\dim_F(V_i')$ for $i=1,2$, then the image of $\tilde{\mathrm{S}}(V_1)\otimes\tilde{\mathrm{S}}(V_2)\to\tilde{\mathrm{S}}(V)$ coincides with $\tilde{\mathrm{S}}(V_1')\otimes\tilde{\mathrm{S}}(V_2')\to\tilde{\mathrm{S}}(V)$. This follows from the fact that there exists $\phi\in\mathrm{SL}(V)$ with $\phi(V_i)=V_i'$ for $i=1,2$.
Therefore $\tilde{S}(F^n)^{\mathrm{dec}}$ is the image of

$$\bigoplus\nolimits_{F^n=V_1\oplus V_2,V_i\neq 0}\left(\tilde{\mathrm{S}}(V_1)_{\mathcal{M}}\otimes\tilde{\mathrm{S}}(V_2)_{\mathcal{M}}\right)\xrightarrow{\ *\ }\tilde{\mathrm{S}}(F^n)_{\mathcal{M}}.$$

If $x=\sum_i n_i(x_1^i,\ldots,x_p^i)\in C_p(V)$ and $y=\sum_j m_j(y_1^j,\ldots,y_q^j)\in C_q(V)$ and if $(x_1^i,\ldots,x_p^i,y_1^j,\ldots,y_q^j)\in X_{p+q}(V)$ for all i,j, then we let

$$x\circledast y:=\sum_{i,j}n_im_j(x_1^i,\ldots,x_p^i,y_1^j,\ldots,y_q^j)\in C_{p+q}(V).$$

Of course, if $x\in C_p(V_1)$ and $y\in C_q(V_2)$ with $V=V_1\oplus V_2$, then $x\circledast y=x*y$. Furthermore, when $x\circledast y$ is defined, we have

$$d(x\circledast y)=d(x)\circledast y+(-1)^px\circledast d(y).$$

Theorem 6.2. *Let $n\geq 1$. For any $a_1,\ldots,a_n,b\in F^\times$ and for any $1\leq i\leq n$*

$$\lfloor a_1,\ldots,ba_i,\ldots,a_n\rceil\cong\langle b\rangle\lfloor a_1,\ldots,a_n\rceil\pmod{\tilde{S}(F^n)^{dec}}.$$

Proof. Let $a=a_1e_1+\cdots+ba_ie_i+\cdots a_ne_n$.
We have

$$\begin{aligned}
&\lfloor a_1,\ldots,ba_i,\ldots,a_n\rceil-\langle b\rangle\lfloor a_1,\ldots,a_n\rceil\\
&\quad= d(e_1,\ldots,e_i,\ldots,e_n,a)-d(e_1,\ldots,b_ie_i,\ldots,e_n,a)\\
&\quad= d\Big((e_1,\ldots,e_{i-1})\circledast((e_i)-(be_i))\circledast(e_{i+1},\ldots,e_n,a)\Big)\\
&\quad= d(e_1,\ldots,e_{i-1})\circledast((e_i)-(be_i))\circledast(e_{i+1},\ldots,e_n,a)\\
&\quad+ (-1)^i(e_1,\ldots,e_{i-1})\circledast((e_i)-(be_i))\circledast d(e_{i+1},\ldots,e_n,a)
\end{aligned}$$

Let $u = a_1e_1 + \cdots + a_{i-1}e_{i-1} + ba_ie_i = a - \sum_{j=i+1}^{n} a_je_j$.Then

$$(-1)^{i-1}(e_1, \dots, e_{i-1}) = d\big((e_1, \dots, e_{i-1}) \circledast (u)\big) - d(e_1, \dots, e_{i-1}) \circledast (u)$$

and

$$(e_{i+1}, \dots, e_n, a) = d\big((u) \circledast (e_{i+1}, \dots, e_n, a)\big) + (u) \circledast d(e_{i+1}, \dots, e_n, a).$$

Thus $\lfloor a_1, \dots, ba_i, \dots, a_n\rceil - \langle b\rangle \lfloor a_1, \dots, a_n\rceil = X_1 - X_2 + X_3$ where

$$\begin{aligned} X_1 &= d(e_1, \dots, e_{i-1}) \circledast \big((e_i) - (be_i)\big) \circledast d(u, e_{i+1}, \dots, e_n, a), \\ X_2 &= d(e_1, \dots, e_{i-1}, u) \circledast \big((e_i) - (be_i)\big) \circledast d(e_{i+1}, \dots, e_n, a), \text{ and} \\ X_3 &= d(e_1, \dots, e_{i-1}) \circledast \Big[\big((e_i) - (be_i)\big) \circledast (u) + (u) \circledast \big((e_i) - (be_i)\big)\Big] \circledast \\ &\quad \circledast d(e_{i+1}, \dots, e_n, a) \end{aligned}$$

We show that each X_i is decomposable: Let $V \subset F^n$ be the span of $u, e_{i+1}, \dots, e_n$ (which is also equal to the span of $a, e_{i+1}, \dots, e_n$), and let V' be the span of $e_1, \dots, e_{i-1}$. Then $F^n = V' \oplus V$ and $d(u, e_{i+1}, \dots, e_n, a) \in H(V)$ while
$d(e_1, \dots, e_{i-1}) \circledast \big((e_i) - (be_i)\big) \in H(V, V')$.
Thus X_1 lies in the image of

$$H(V, V') \otimes H(V) \xrightarrow{*} \tilde{S}(F^n)_{\mathcal{M}}$$

and so is decomposable.
Similarly, if we let W be the span of $e_1, \dots, e_i$ and W' the span of $e_{i+1}, \dots, e_n$, then

$$d(e_1, \dots, e_{i-1}, u) \circledast \big((e_i) - (be_i)\big),\ d(e_1, \dots, e_{i-1}) \circledast \Big[\big((e_i) - (be_i)\big) \circledast (u) + (u) \circledast \big((e_i) - (be_i)\big)\Big]$$

belongs to $H(W)$ and $d(e_{i+1}, \dots, e_n, a) \in H(W, W')$.Thus X_2, X_3 lie in the image of

$$H(W) \otimes H(W, W') \xrightarrow{*} \tilde{S}(F^n)_{\mathcal{M}}$$

and are also decomposable. □

Let $\tilde{S}(F^n)^{\mathrm{ind}} := \tilde{S}(F^n)_{\mathcal{M}}/\tilde{S}(F^n)^{\mathrm{dec}}$.
The main goal of this section is to show that $\tilde{S}(F^n)^{\mathrm{ind}} = 0$ for all $n \geq 3$ (Theorem 6.8 below).

Lemma 6.3. *For all $n \geq 3$, $\tilde{S}(F^n)^{ind}$ is a multiplicative $\mathbb{Z}[F^\times]$-module.*

Proof. We have

$$\mathcal{A}_n \cong \begin{cases} \mathbb{Z}[F^\times]E^{*n/2}, & n \text{ even} \\ \tilde{S}(F) * E^{*(n-1)/2}, & n \text{ odd} \end{cases}$$

and these modules are decomposable for all $n \geq 3$. It follows that the map

$$\tilde{S}(F^n)^+{}_{\mathcal{M}} \to \tilde{S}(F^n)^{\mathrm{ind}}$$

is surjective for all $n \geq 3$. □

REMARK 6.4. Since $E * \tilde{S}(F^{n-2})_{\mathcal{M}} \subset \tilde{S}(F^n)^{\mathrm{dec}}$, in fact we have that $\mathcal{F}_{n,1} \to \tilde{S}(F^n)^{\mathrm{ind}}$ is surjective.

Theorem 6.2 shows that for all $a_1, \ldots, a_n \in F^\times$

$$\lfloor a_1, \ldots, a_n \rceil \cong \left\langle \prod_i a_i \right\rangle \lfloor 1, \ldots, 1 \rceil \pmod{\tilde{S}(F^n)^{\mathrm{dec}}}.$$

In other words the map

$$\mathbb{Z}[F^\times] \to \tilde{S}(F^n)^{\mathrm{ind}}, \quad \alpha \mapsto \alpha \lfloor 1, \ldots, 1 \rceil$$

is a surjective homomorphism of $\mathbb{Z}[F^\times]$-modules. Thus, we are required to establish that $\lfloor 1, \ldots, 1 \rceil \in \tilde{S}(F^n)^{\mathrm{dec}}$ for all $n \geq 3$.

For convenience below, we will let $\tilde{\Sigma}_n(F)$ denote the free $\mathbb{Z}[F^\times]$-module on the symbols $\big[a_1, \ldots, a_n\big]$, $a_1, \ldots, a_n \in F^\times$. Let $p_n : \tilde{\Sigma}_n(F) \to \tilde{S}(F^n)$ be the $\mathbb{Z}[F^\times]$-module homomorphism sending $\big[a_1, \ldots, a_n\big]$ to $\lfloor a_1, \ldots, a_n \rceil$. We will say that $\sigma \in \tilde{S}(F^n)$ is represented by $\tilde{\sigma} \in \tilde{\Sigma}_n(F)$ if $p_n(\tilde{\sigma}) = \sigma$.

Note that $\tilde{\Sigma}_\bullet(F)$ can be given the structure of a graded $\mathbb{Z}[F^\times]$-algebra by setting

$$\big[a_1, \ldots, a_n\big] \cdot \big[a_{n+1}, \ldots, a_{n+m}\big] := \big[a_1, \ldots, a_{n+m}\big];$$

i.e., we can identify $\tilde{\Sigma}_\bullet(F)$ with the tensor algebra over $\mathbb{Z}[F^\times]$ on the free module with basis $\big[a\big]$, $a \in F^\times$.

Let $\Pi_\bullet : \tilde{\Sigma}_\bullet(F) \to \mathbb{Z}[F^\times][x]$ be the homomorphism of graded $\mathbb{Z}[F^\times]$-algebras sending $\big[a\big]$ to $\langle a \rangle\, x$.

For all $n \geq 1$ we have a commutative square of surjective homomorphisms of $\mathbb{Z}[F^\times]$-modules

$$\begin{tikzcd}
\tilde{\Sigma}_n(F) \arrow[r, "\Pi_n"] \arrow[d, "p_n"] & \mathbb{Z}[F^\times] \cdot x^n \arrow[d, "\gamma_n"] \\
\tilde{S}(F^n) \arrow[r] & \tilde{S}(F^n)^{\mathrm{ind}}
\end{tikzcd}$$

where $\gamma_n(x^n) = \lfloor 1, \ldots, 1 \rceil$.

LEMMA 6.5. *If n is odd and $n \geq 3$ then $\tilde{S}(F^n)^{ind} = 0$; i.e.,*

$$\tilde{S}(F^n)_{\mathcal{M}} = \tilde{S}(F^n)^{dec}.$$

Proof. From the fundamental relation in $\tilde{S}(F^n)$ (Theorem 3.3), if $b_1, \ldots, b_n$ are distinct elements of $F^\times$, then $0 \in \tilde{S}(F^n)$ is represented by $R_b :=$

$$[b_1, \ldots, b_n] - [1, \ldots, 1] - \sum_{j=1}^{n} (-1)^{n+j} \left\langle (-1)^{n+j} \right\rangle [b_1 - b_j, \ldots, \widehat{b_j - b_j}, \ldots, b_n - b_j, b_j]$$

in $\in \tilde{\Sigma}_n(F)$.

Now $\Pi_n(R_b) =$

$$\left[\left\langle \prod_i b_i \right\rangle - \langle 1 \rangle - \sum_{j=1}^{n} (-1)^{n+j} \left\langle (b_j - b_1) \cdots (b_j - b_{j-1}) \cdot (b_{j+1} - b_j) \cdots (b_n - b_j) \cdot b_j \right\rangle \right] x^n.$$

We choose $b_i = i$, $i = 1, \ldots, n$. Then

$$\Pi_n(R_b) = \left[\langle n!\rangle - \langle 1\rangle - \sum_{j=1}^{n}(-1)^{n+j}\langle j!(n-j)!\rangle\right] x^n = -\langle 1\rangle x^n \text{ since } n \text{ is odd.}$$

It follows that $-\lfloor 1, \ldots, 1\rceil = 0$ in $\tilde{S}(F^n)^{\mathrm{ind}}$ as required. □

The case n even requires a little more work.
The maps $\{p_n\}_n$ do not define a map of graded algebras. However, we do have the following:

Lemma 6.6. *For $1 \neq a \in F^\times$, let*

$$L(x) := \langle -1\rangle \lfloor 1-x, 1\rceil - \langle x\rangle \lfloor 1-\frac{1}{x}, \frac{1}{x}\rceil + \lfloor 1, 1\rceil \in \tilde{\Sigma}_2(F).$$

Then for all $a_1, \ldots, a_n \in F^\times \setminus \{1\}$, the product

$$\prod_{i=1}^{n}\lfloor 1, a_i\rceil = \lfloor 1, a_1\rceil * \cdots * \lfloor 1, a_n\rceil \in \tilde{S}(F^{2n})$$

is represented by $\prod_i L(a_i) \in \tilde{\Sigma}_{2n}(F)$.

Proof. For convenience of notation, we will represent standard basis elements of $C_q(F^n)$ as $n \times q$ matrices $[v_1|\cdots|v_q]$.
Let $e = (1, \ldots, 1)$ and let $\sigma_i(C)$ denote the sum of the entries in the ith row of the $n \times n$ matrix C. By Remark 3.2, if $A \in \mathrm{GL}_n(F)$ and $[A|e] \in X_{n+1}(F^n)$ then $d_{n+1}([A|e])$ represents $\langle \det A\rangle \lfloor \sigma_1(A^{-1}), \ldots, \sigma_n(A^{-1})\rceil \in \tilde{S}(F^n)$.
Now, for $a \neq 1$, $\lfloor 1, a\rceil$ is represented in $\tilde{S}(F^2)$ by

$$d_3\left(\begin{bmatrix} 1 & 0 & 1\\ 0 & 1 & a\end{bmatrix}\right) = \begin{bmatrix} 0 & 1\\ 1 & a\end{bmatrix} - \begin{bmatrix} 1 & 1\\ 0 & a\end{bmatrix} + \begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix} = T_1(a) - T_2(a) + T_3(a) \in C_2(F^2).$$

From the definition of the product $*$, it follows that $\lfloor 1, a_1\rceil * \cdots * \lfloor 1, a_n\rceil$ is represented by

$$Z := \sum_{j=(j_1,\ldots,j_n)\in(1,2,3)^n} (-1)^{k(j)} \begin{bmatrix} T_{j_1}(a_1) & & \\ & \ddots & \\ & & T_{j_n}(a_n)\end{bmatrix} = \sum_j (-1)^{k(j)} T(j,a).$$

where $k(j) := |\{i \leq n | j_i = 2\}|$
Since $a_i \neq 1$ for all i, the vector $e = (1, \ldots, 1)$ is in general position with respect to the columns of all these matrices. Thus we can use the partial homotopy operator s_e to write this cycle as a boundary:

$$Z = \sum_j (-1)^{k(j)} d_{2n+1}([T(j,a)|e]).$$

By the remarks above

$$d_{2n+1}\left([T(j,a)|e]\right) = \left\langle \prod_i \det T_{j_i}(a_i)\right\rangle \times$$
$$\times \lfloor \sigma_1(T_{j_1}(a_1)), \sigma_2(T_{j_1}(a_1)), \sigma_1(T_{j_2}(a_2)), \ldots, \sigma_1(T_{j_n}(a_n)), \sigma_2(T_{j_n}(a_n))\rceil.$$

This is represented by

$$\left\langle \prod_i \det T_{j_i}(a_i)\right\rangle \times$$
$$\times \left[\sigma_1(T_{j_1}(a_1)), \sigma_2(T_{j_1}(a_1)), \sigma_1(T_{j_2}(a_2)), \ldots, \sigma_1(T_{j_n}(a_n)), \sigma_2(T_{j_n}(a_n))\right]$$
$$= \prod_{i=1}^n \Big(\langle \det T_{j_i}(a_i)\rangle \left[\sigma_1(T_{j_i}(a_i)), \sigma_2(T_{j_i}(a_i))\right]\Big) \in \tilde{\Sigma}_{2n}(F).$$

Thus Z is represented by

$$\sum_j (-1)^{k(j)} \prod_{i=1}^n \Big(\langle \det T_{j_i}(a_i)\rangle \left[\sigma_1(T_{j_i}(a_i)), \sigma_2(T_{j_i}(a_i))\right]\Big) =$$
$$= \prod_{i=1}^n \Big(\sum_{j=1}^3 (-1)^{j+1} \langle \det T_j(a_i)\rangle \left[\sigma_1(T_j(a_i)), \sigma_2(T_j(a_i))\right]\Big) = \prod_{i=1}^n L(a_i) \in \tilde{\Sigma}_{2n}(F).$$

□

Observe that all of our multiplicative modules (and in particular $\tilde{S}(F^n)_{\mathcal{M}}$) have the following property: they admit a finite filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_t = M$ such that each of the associated quotients M_r/M_{r-1} is annihilated by $\mathcal{I}_{(F^\times)^{k_r}}$ for some $k_r \geq 1$. From this observation it easily follows that

Lemma 6.7.

$$\tilde{S}(F^n)^{ind} = 0 \Longleftrightarrow \tilde{S}(F^n)^{ind}/(\mathcal{I}_{(F^\times)^r} \cdot \tilde{S}(F^n)^{ind}) = 0 \textit{ for all } r \geq 1.$$

Theorem 6.8. *$\tilde{S}(F^n)^{ind} = 0$ for all $n \geq 3$.*

Proof. The case n odd has already been dealt with in Lemma 6.5
For the even case, by Lemma 6.7 it will be enough to prove that for all $r \geq 1$

$$\mathbb{Z}[F^\times/(F^\times)^r] \otimes_{\mathbb{Z}[F^\times]} \tilde{S}(F^n)^{\mathrm{ind}} = 0.$$

Fix $r \geq 1$. If $a \in (F^\times)^r \setminus \{1\}$, then

$$\Pi_2(L(a)) = \left(\langle a-1\rangle - \left\langle 1 - \frac{1}{a}\right\rangle + \langle 1\rangle\right) x^2 = \langle 1\rangle\, x^2 \in \mathbb{Z}[F^\times/(F^\times)^r] x^2$$

since

$$1 - \frac{1}{a} = \frac{a-1}{a} \equiv a - 1 \pmod{(F^\times)^r}.$$

Now let $n > 1$ and choose $a_1, \ldots, a_n \in (F^\times)^r \setminus \{1\}$. Let $\sigma = \lfloor 1, a_1 \rceil * \cdots * \lfloor 1, a_n \rceil \in \tilde{S}(F^{2n})$, so that $\sigma \mapsto 0$ in $\tilde{S}(F^{2n})^{\mathrm{ind}}$. By Lemma 6.6, σ is represented by $\tilde{\sigma} = \prod_{i=1}^n L(a_i)$ in $\tilde{\Sigma}_{2n}(F)$ and thus

$$\Pi_{2n}(\tilde{\sigma}) = \prod_{i=1}^n \big(\Pi_2(L(a_i))\big) = \langle 1 \rangle \in \mathbb{Z}[F^\times/(F^\times)^r]x^{2n}$$

so that the image of σ in $\mathbb{Z}[F^\times/(F^\times)^r] \otimes_{\mathbb{Z}[F^\times]} \tilde{S}(F^{2n})^{\mathrm{ind}}$ is $1 \otimes \lfloor 1, \ldots, 1 \rceil$. This proves the theorem. □

Corollary 6.9. *For all $n \geq 2$, the map T_n induces an isomorphism $\mathcal{F}_{n,1} \cong K_n^{\mathrm{MW}}(F)$.*

Proof. Since, by the computations above, $\tilde{S}(F^2)_{\mathcal{M}} = \tilde{S}(F)^{*2} + \mathbb{Z}[F^\times]E$ it follows, using Theorem 6.8 and induction on n, that $\tilde{S}(F^\bullet)_{\mathcal{M}}$ is generated as a $\mathbb{Z}[F^\times]$-algebra by $\{\lfloor a \rceil \in \tilde{S}(F) | 1 \neq a \in F^\times\}$ and E.
Thus E is central in the algebra $\tilde{S}(F^\bullet)_{\mathcal{M}}$ and for all $n \geq 2$,

$$\frac{\tilde{S}(F^n)_{\mathcal{M}}}{E * \tilde{S}(F^{n-2})_{\mathcal{M}}}$$

is generated by the elements of the form $\lfloor a_1 \rceil * \cdots * \lfloor a_n \rceil$, and hence also by the elements $\{\{a_1, \ldots, a_n\}\}$ since $[a, b] \equiv \lfloor a \rceil * \lfloor b \rceil \pmod{\langle E \rangle}$ for all $a, b \in F^\times$.
Since

$$\mathcal{F}_{n,1} \cong \frac{\tilde{S}(F^n)_{\mathcal{M}}}{E * \tilde{S}(F^{n-2})_{\mathcal{M}}}$$

by Corollary 5.14, it follows that $\mathcal{F}_{n,1}$ is generated by the elements $\{\{a_1, \ldots, a_n\}\}$, and thus that the homomorphisms μ_n of Theorem 5.24 are surjective. □

Corollary 6.10. *For all $n \geq 3$,*

$$\tilde{S}(F^n)_{\mathcal{M}} \cong \begin{cases} K_n^{\mathrm{MW}}(F) \oplus K_{n-2}^{\mathrm{MW}}(F) \oplus \cdots \oplus K_2^{\mathrm{MW}}(F) \oplus \mathbb{Z}[F^\times] & n \text{ even} \\ K_n^{\mathrm{MW}}(F) \oplus K_{n-2}^{\mathrm{MW}}(F) \oplus \cdots \oplus K_3^{\mathrm{MW}}(F) \oplus \mathcal{I}_{F^\times} & n \text{ odd} \end{cases}$$

as a $\mathbb{Z}[F^\times]$-module.

Corollary 6.11. *For all even $n \geq 2$ the cokernel of the map*

$$\mathrm{H}_n(\mathrm{SL}_{n-1}(F), \mathbb{Z}) \to \mathrm{H}_n(\mathrm{SL}_n(F), \mathbb{Z})$$

is isomorphic to $K_n^{\mathrm{MW}}(F)$.

Proof. Recall that ϵ_2 induces an isomorphism $\mathrm{H}_2(\mathrm{SL}_2(F), \mathbb{Z}) \cong \mathcal{F}_{2,1} = \mathcal{F}_{2,0}$. Let $\langle a, b \rangle$ denote the generator $\epsilon_2^{-1}([a, b])$ of $\mathrm{H}_2(\mathrm{SL}_2(F), \mathbb{Z})$. Then for even n

$$\begin{aligned} \{\{a_1, \ldots, a_n\}\} &= [a_1, a_2] * \cdots * [a_{n-1}, a_n] \\ &= \epsilon_2(\langle a_1, a_2 \rangle) * \cdots * \epsilon_2(\langle a_{n-1}, a_n \rangle) \\ &= \epsilon_n(\langle a_1, a_2 \rangle \times \cdots \times \langle a_{n-1}, a_n \rangle) \end{aligned}$$

by Lemma 3.5 (2).

Since $\mathcal{F}_{n,1}$ is generated by the elements $\{\{a_1,\ldots,a_n\}\}$, it follows that $\mathcal{F}_{n,1} = \epsilon_n(\mathrm{H}_n(\mathrm{SL}_n(F),\mathbb{Z})) = E^{\infty}_{n,0} = \mathcal{F}_{n,0}$, proving the result. □

Corollary 6.12. *For all odd $n \geq 1$ the maps*

$$\mathrm{H}_n(\mathrm{SL}_k(F),\mathbb{Z}) \to \mathrm{H}_n(\mathrm{SL}_{k+1}(F),\mathbb{Z})$$

are isomorphisms for $k \geq n$.

Proof. In view of Corollary 5.11, the only point at issue is the injectivity of

$$\mathrm{H}_n(\mathrm{SL}_n(F),\mathbb{Z}) \to \mathrm{H}_n(\mathrm{SL}_{n+1}(F),\mathbb{Z}).$$

But the proof of Corollary 6.11 shows that the term

$$\mathcal{F}_{n+1,1}/E^{\infty}_{n+1,0} \cong E^{\infty}_{n,1} = \mathrm{Ker}(\mathrm{H}_n(\mathrm{SL}_n(F),\mathbb{Z}) \to \mathrm{H}_n(\mathrm{SL}_{n+1}(F),\mathbb{Z}))$$

in the spectral sequence $\mathcal{E}^+(F^{n+1})_{\mathcal{M}}$ is zero. □

Corollary 6.13. *If $n \geq 3$ is odd, then*

$$\mathrm{Coker}(\mathrm{H}_n(\mathrm{SL}_{n-1}(F),\mathbb{Z}) \to \mathrm{H}_n(\mathrm{SL}_n(F),\mathbb{Z})) \cong 2K^{\mathrm{M}}_n(F)$$
$$\mathrm{Ker}(\mathrm{H}_{n-1}(\mathrm{SL}_{n-1}(F),\mathbb{Z}) \to \mathrm{H}_{n-1}(\mathrm{SL}_n(F),\mathbb{Z})) \cong I^n(F).$$

Proof. Since we have already proved this result for $n = 3$ above, we will assume that $n \geq 5$ (n odd).

Let $a_1,\ldots,a_n \in F^{\times}$ and let $z \in \mathrm{H}_{n-1}(\mathrm{SL}_{n-1}(F),\mathbb{Z})$ satisfy $\epsilon_{n-1}(z) = \{\{a_2,\ldots,a_n\}\} \in \mathcal{F}_{n-1,0} \cong K^{\mathrm{MW}}_{n-1}(F)$. Thus $\{\{a_1,\ldots,a_n\}\} = \lfloor a_1 \rceil * \epsilon_{n-1}(z)$ and hence $\epsilon_{n,1}(\{\{a_1,\ldots,a_n\}\}) = \langle\langle a_1\rangle\rangle z$ by Theorem 5.20 (2). It follows that the diagram

$$\begin{tikzcd}
\mathcal{F}_{n,1} \arrow[r, "\epsilon_{n,1}"] \arrow[d, "T_n", "\cong"'] & \mathrm{H}_{n-1}(\mathrm{SL}_{n-1}(F),\mathbb{Z}) \arrow[d, "T_{n-1}\circ\epsilon_{n-1}"] \\
K^{\mathrm{MW}}_n(F) \arrow[r, "\eta"] & K^{\mathrm{MW}}_{n-1}(F)
\end{tikzcd}$$

commutes.

Now $\mathrm{Ker}(\epsilon_{n,1}) = \mathrm{Im}(\epsilon_n : \mathrm{H}_n(\mathrm{SL}_n(F),\mathbb{Z}) \to \mathcal{F}_{n,1})$. Since $\mathrm{Im}(\epsilon_3) = T_3^{-1}(2K^{\mathrm{M}}_3(F))$ and $\mathrm{Im}(\epsilon_{n-3}) = \mathcal{F}_{n-3,1} = T^{-1}_{n-3}(K^{\mathrm{MW}}_{n-3}(F))$ we have

$$T_n(\mathrm{Im}(\epsilon_n)) = \mathrm{Im}(T_n \circ \epsilon_n) \supset 2K^{\mathrm{M}}_3(F)\cdot K^{\mathrm{MW}}_{n-3}(F) = 2K^{\mathrm{M}}_n(F) \subset K^{\mathrm{MW}}_n(F)$$

(using the fact that $T_\bullet$ and $\epsilon_\bullet$ are algebra homomorphisms).

Thus we get a commutative diagram

$$\begin{tikzcd}
\frac{K^{\mathrm{MW}}_n(F)}{2K^{\mathrm{M}}_n(F)} \arrow[r, "T_n^{-1}"] \arrow[d, "\eta", "\cong"'] & \frac{\mathcal{F}_{n,1}}{\mathrm{Ker}(\epsilon_{n,1})} \arrow[dl, "T_{n-1}\circ\epsilon_{n-1}\circ\epsilon_{n,1}"] \\
I^n(F) &
\end{tikzcd}$$

from which it follows that the map T_n^{-1} in this diagram is an isomorphism, and hence $\mathrm{Im}(\epsilon_n) = \mathrm{Ker}(\epsilon_{n,1}) \cong 2K^{\mathrm{M}}_n(F)$ and $\mathrm{Im}(\epsilon_{n,1}) \cong I^n(F)$. □

7. Acknowledgements

The work in this article was partially funded by the Science Foundation Ireland Research Frontiers Programme grant 05/RFP/MAT0022.

References

[1] Jean Barge and Fabien Morel. Cohomologie des groupes linéaires, K-théorie de Milnor et groupes de Witt. *C. R. Acad. Sci. Paris Sér. I Math.*, 328(3):191–196, 1999.

[2] Stanislaw Betley. Hyperbolic posets and homology stability for $O_{n,n}$. *J. Pure Appl. Algebra*, 43(1):1–9, 1986.

[3] Kenneth S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1982.

[4] Ruth M. Charney. Homology stability of GL_n of a Dedekind domain. *Bull. Amer. Math. Soc. (N.S.)*, 1(2):428–431, 1979.

[5] Daniel Guin. Stabilité de l'homologie du groupe linéaire et K-théorie algébrique. *C. R. Acad. Sci. Paris Sér. I Math.*, 304(9):219–222, 1987.

[6] Kevin Hutchinson. A new approach to Matsumoto's theorem. *K-Theory*, 4(2):181–200, 1990.

[7] Kevin Hutchinson and Liqun Tao. A note on Milnor-Witt K-theory and a theorem of Suslin. *Comm. Alg*, 36:2710–2718, 2008.

[8] Kevin Hutchinson and Liqun Tao. The third homology of the special linear group of a field. *J. Pure Appl. Algebra*, 213:1665–1680, 2009.

[9] T. Y. Lam. *Introduction to quadratic forms over fields*, volume 67 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2005.

[10] Hideya Matsumoto. Sur les sous-groupes arithmétiques des groupes semi-simples déployés. *Ann. Sci. École Norm. Sup. (4)*, 2:1–62, 1969.

[11] A. Mazzoleni. A new proof of a theorem of Suslin. *K-Theory*, 35(3-4):199–211 (2006), 2005.

[12] John Milnor. Algebraic K-theory and quadratic forms. *Invent. Math.*, 9:318–344, 1969/1970.

[13] John Milnor. *Introduction to algebraic K-theory*. Princeton University Press, Princeton, N.J., 1971. Annals of Mathematics Studies, No. 72.

[14] B. Mirzaii. Homology stability for unitary groups. II. *K-Theory*, 36(3-4):305–326 (2006), 2005.

[15] B. Mirzaii and W. van der Kallen. Homology stability for unitary groups. *Doc. Math.*, 7:143–166 (electronic), 2002.

[16] Calvin C. Moore. Group extensions of p-adic and adelic linear groups. *Inst. Hautes Études Sci. Publ. Math.*, (35):157–222, 1968.

[17] Fabien Morel. An introduction to $\mathbb{A}^1$-homotopy theory. In *Contemporary developments in algebraic K-theory*, ICTP Lect. Notes, XV, pages 357–441 (electronic). Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2004.

[18] Fabien Morel. Sur les puissances de l'idéal fondamental de l'anneau de Witt. *Comment. Math. Helv.*, 79(4):689–703, 2004.

[19] Fabien Morel. $\mathbb{A}^1$-algebraic topology. In *International Congress of Mathematicians. Vol. II*, pages 1035–1059. Eur. Math. Soc., Zürich, 2006.
[20] D. Orlov, A. Vishik, and V. Voevodsky. An exact sequence for $K_*^M/2$ with applications to quadratic forms. *Ann. of Math. (2)*, 165(1):1–13, 2007.
[21] I. A. Panin. Homological stabilization for the orthogonal and symplectic groups. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 160(Anal. Teor. Chisel i Teor. Funktsii. 8):222–228, 301–302, 1987.
[22] Chih-Han Sah. Homology of classical Lie groups made discrete. III. *J. Pure Appl. Algebra*, 56(3):269–312, 1989.
[23] A. A. Suslin. Homology of GL_n, characteristic classes and Milnor K-theory. In *Algebraic K-theory, number theory, geometry and analysis (Bielefeld, 1982)*, volume 1046 of *Lecture Notes in Math.*, pages 357–375. Springer, Berlin, 1984.
[24] A. A. Suslin. Torsion in K_2 of fields. *K-Theory*, 1(1):5–29, 1987.
[25] Wilberd van der Kallen. Homology stability for linear groups. *Invent. Math.*, 60(3):269–295, 1980.
[26] Karen Vogtmann. Homology stability for $\mathrm{O}_{n,n}$. *Comm. Algebra*, 7(1):9–38, 1979.
[27] Karen Vogtmann. Spherical posets and homology stability for $\mathrm{O}_{n,n}$. *Topology*, 20(2):119–132, 1981.

Kevin Hutchinson
School of Mathematical Sciences
University College Dublin
Dublin 4
Ireland
kevin.hutchinson@ucd.ie

Liqun Tao
School of Mathematical Sciences
Luoyang Normal University
Luoyang 47 1022,
P. R. China
lqtao@lynu.edu.cn

Cohomological Approaches to SK_1 and SK_2 of Central Simple Algebras

Bruno Kahn

Received: November 12, 2009
Revised: June 11, 2010

Abstract. We discuss several constructions of homomorphisms from SK_1 and SK_2 of central simple algebras to subquotients of Galois cohomology groups.

2010 Mathematics Subject Classification: 16Kxx, 19Bxx, 20G15.
Keywords and Phrases: Whitehead group, Galois cohomology, motivic cohomology.

To A. Suslin

Contents

Introduction

Given a simple algebra A with centre F, the group $SK_i(A)$ is defined for $i = 1, 2$ as the kernel of the *reduced norm*

$$\mathrm{Nrd}_i : K_i(A) \to K_i(F).$$

The definition of Nrd_1 is classical, and Nrd_2 was defined by Suslin in [47, Cor. 5.7]. For further reference, let us recall these definitions in a uniform way: let X be the Severi-Brauer variety of A. After Quillen [42, Th. 8.4], there is an isomorphism

$$\bigoplus_{r=0}^{d-1} K_i(A^{\otimes r}) \xrightarrow{\sim} K_i(X) \quad (d = \deg(A))$$

for any $i \geq 0$. The reduced norm is then given by the composition

$$K_i(A) \to K_i(X) \to H^0(X, \mathcal{K}_i) \xleftarrow{\sim} K_i(F)$$

where the right isomorphism is obvious for $i = 1$ and is due to Suslin [47, Cor. 5.6] for $i = 2$.

Of course, this definition also makes sense for $i = 0$: in this case, Nrd_0 is simply multiplication by the index of A:

$$K_0(A) \simeq \mathbf{Z} \xrightarrow{\mathrm{ind}(A)} \mathbf{Z} \simeq K_0(F)$$

and $SK_0(A) = 0$.

[For $i > 2$, a reduced norm satisfying reasonable properties cannot exist (Rost, Merkurjev [33, p. 81, Prop. 4]): the right generalisation is in the framework of motivic cohomology, see [22].]

The groups $SK_1(A)$ and $SK_2(A)$ remain mysterious and are known only in very special cases. Here are a few elementary properties they enjoy:

(1) $SK_i(A)$ is Morita-invariant.

(2) $\mathrm{ind}(A)SK_i(A) = 0$ (from Morita invariance, reduce to the case where A is division, and then use a transfer argument thanks to a maximal commutative subfield of A).

(3) The cup-product $K_1(F) \otimes K_1(A) \to K_2(A)$ induces a map

$$K_1(F) \otimes SK_1(A) \to SK_2(A).$$

(4) Let v be a discrete valuation of rank 1 on F, with residue field k, and assume that A spreads as an Azumaya algebra $\mathcal{A}$ over the discrete valuation ring $\mathcal{O}_v$. It can be shown that the map $SK_1(\mathcal{A}) \to SK_1(A)$ is surjective and that, if $K_2(\mathcal{O}_v) \to K_2(F)$ is injective, there is a short exact sequence

$$SK_2(\mathcal{A}) \to SK_2(A) \xrightarrow{\partial} SK_1(\mathcal{A}_k)$$

with

$$\partial(\{f\} \cdot x) = v(f)\bar{x}$$

for $f \in F^*$ and $x \in SK_1(A)$.

(5) Let $A(t) = F(t) \otimes_F A$, and similarly $A(x) = F(x) \otimes_F A$ for any closed point $x \in \mathbf{A}^1_F$. Then there is an isomorphism
$$SK_1(A) \xrightarrow{\sim} SK_1(A(t))$$
due to Platonov and an exact sequence
$$0 \to SK_2(A) \to SK_2(A(t)) \to \bigoplus_{x\in\mathbf{A}^1_F} SK_1(A(x)).$$

From (3) and (4), one deduces that $SK_1(A)$ is a direct summand of $SK_2(A(t))$ via the map $x \mapsto \{t\} \cdot x$: in particular, the latter group is nonzero as soon as the former is. More intriguing is the *Calmès symbol*
$$\begin{aligned} cal : \Lambda^2 \left(\frac{K_1(A)}{\operatorname{ind}(A)K_1(A)}\right) &\to SK_2(A)\\ a\wedge b &\mapsto \operatorname{Nrd}(a)\cdot b - a\cdot \operatorname{Nrd}(b).\end{aligned}$$

The image of this symbol is not detected by residues.
Let us now review known results about SK_1 and SK_2. If F is a global field, then $SK_i(A) = 0$ for $i = 1, 2$: this is classical for $i = 1$ as a consequence of class field theory, while for $i = 2$ it is due to Bak and Rehmann using the Merkurjev-Suslin theorem [2]. In the sequel, I concentrate on more general fields F and always assume that the index of A is invertible in F.

0.A. SK_1. The first one to give an example where $SK_1(A) \neq 0$ was Platonov [41]. In his example, F is provided with a discrete valuation of rank 2 and the Brauer group of the second residue field is nontrivial; in particular, $cd(F) \geq 4$. Over general fields, a striking and early result for SK_1 is Wang's theorem:

Theorem 1 (Wang [58]). *If the index of A is square-free, then $SK_1(A) = 0$.*

The most successful approach to $SK_1(A)$ for other A has been to relate it to Galois cohomology groups. This approach was initiated by Suslin, who (based on Platonov's results) conjectured the existence of a canonical homomorphism
$$SK_1(A) \to H^4(F, \mu_n^{\otimes 3})/[A]\cdot H^2(F, \mu_n^{\otimes 2})$$
where n is the index of A, supposed to be prime to $\operatorname{char} F$ [49, Conj. 1.16]. In [49], Suslin was only able to partially carry over this project: he had to assume that $\mu_{n^3} \subset F$ and then could only construct twice the expected map, assuming the Bloch-Kato conjecture in degree 3.
The next result in this direction is due to Rost in the case of a biquaternion algebra:

Theorem 2 (Rost [33, th. 4]). *If A is a biquaternion algebra, there is an exact sequence*
$$0 \to SK_1(A) \to H^4(F, \mathbf{Z}/2) \to H^4(F(Y), \mathbf{Z}/2)$$
where Y is the quadric defined by an 'Albert form' associated to A.

The surprise here is that Rost gets in particular a finer map than the one expected by Suslin, as he does not have to mod out by multiples of $[A]$.

Merkurjev generalised Rost's theorem to the case of a simple algebra of degree 4 but not necessarily of exponent 2:

Theorem 3 (Merkurjev [35, th. 6.6]). *If A has degree 4, there is an exact sequence*

$$0 \to SK_1(A) \to H^4(F,\mathbf{Z}/2)/2[A]\cdot H^2(F,\mathbf{Z}/2) \to H^4(F(Y),\mathbf{Z}/2)$$

where Y is the generalised Severi-Brauer variety $SB(2,A)$, a twisted form of the Grassmannian $G(2,4)$.

Note that the right map makes sense because $A_{F(Y)}$ has exponent 2.

Merkurjev's exact sequence is obtained from Rost's by descent from $F(Z)$ to F, where $Z = SB(A^{\otimes 2})$. The point is that neither $SK_1(A)$ nor the kernel of the right map in Theorem 3 changes when one passes from F to $F(Z)$.

More recently, Suslin revisited his homomorphism of [49] in [50], where he constructs an (a priori different) homomorphism using motivic cohomology rather than Chern classes in K-theory. He compares it with the one of Rost-Merkurjev and proves the following amazing theorem:

Theorem 4 (Suslin [50, Th. 6]). *For any central simple algebra A of degree 4, there exists a commutative diagram of isomorphisms*

$$\begin{array}{ccc} SK_1(A) & \xrightarrow[\sim]{\varphi} & \dfrac{\operatorname{Ker}(H^4(F,\mu_4^{\otimes 3}) \to H^4(F(X),\mu_4^{\otimes 3}))}{[A]\cdot H^2(F,\mu_4^{\otimes 2})} \\ \| & & \Big\downarrow{\scriptstyle \tau'} \\ SK_1(A) & \xrightarrow[\sim]{\psi} & \dfrac{\operatorname{Ker}(H^4(F,\mu_2^{\otimes 3}) \to H^4(F(Y),\mu_2^{\otimes 3}))}{2[A]\cdot H^2(F,\mu_2^{\otimes 2})} \end{array}$$

where $X = SB(A)$, $Y = SB(2,A)$, φ is Suslin's homomorphism just mentioned and ψ is Merkurjev's isomorphism from Theorem 3.

0.B. SK_2. Concerning $SK_2(A)$, the first result (over an arbitrary base field) was the following theorem of Rost and Merkurjev:

Theorem 5 (Rost [43], Merkurjev [31]). *For any quaternion algebra A, $SK_2(A) = 0$.*

Rost and Merkurjev used this theorem as a step to prove the Milnor conjecture in degree 3; conversely, this conjecture and techniques of motivic cohomology were used in [21, th. 9.3] to give a very short proof of Theorem 5. We revisit this proof in Remark 7.3, in the spirit of the techniques developed here.

The following theorem is more recent. In view of the still fluctuant status of the Bloch-Kato conjecture for odd primes, we assume its validity in the statement. (See §2.A for the Bloch-Kato conjecture.)

Theorem 6 (Kahn-Levine [22, Cor. 2], Merkurjev-Suslin [38, Th. 2.4]). *Assume the Bloch-Kato conjecture in degree ≤ 3. For any central simple algebra A of square-free index, $SK_2(A) = 0$.*

From Theorems 1 and 6, we get by a well-known dévissage argument a refinement of the elementary property (2) given above: for any A and $i = 1, 2$, $\frac{\operatorname{ind}(A)}{\prod l_i} SK_i(A) = 0$, where the l_i are the distinct primes dividing $\operatorname{ind}(A)$.
On the other hand, Baptiste Calmès gave a version of Rost's theorem 2 for SK_2 of biquaternion algebras:

Theorem 7 (Calmès [5]). *Under the assumptions of Theorem 2, assume further that F contains a separably closed field. Then there is an exact sequence*

$$\operatorname{Ker}(A_0(Z, K_2) \to K_2(F)) \to SK_2(A) \to H^5(F, \mathbf{Z}/2) \to H^5(F(Y), \mathbf{Z}/2)$$

where Z is a hyperplane section of Y.

(Note that in the case of SK_1, the corresponding group $\operatorname{Ker}(A_0(Z, K_1) \to K_1(F))$ is 0 by a difficult theorem of Rost.)
Finally, let us mention the construction of homomorphisms à la Suslin

$$SK_1(A) \to H^4(F, \mathbf{Q}/\mathbf{Z}(3))/[A] \cdot K_2(F) \tag{0.1}$$

$$SK_2(A) \to H^5(F, \mathbf{Q}/\mathbf{Z}(4))/[A] \cdot K_3^M(F) \tag{0.2}$$

in [22, §6.9], using an étale version of the Bloch-Lichtenbaum spectral sequence for the motive associated to A. The second map depends on the Bloch-Kato conjecture in degree 3 and assumes, as in Theorem 7, that F contains a separably closed field. This construction goes back to 1999 (correspondence with M. Levine), although the targets of (0.1) and (0.2) were only determined in [22, Prop. 6.9.1].

0.C. The results. Calmès' proof of Theorem 7 is based in part on the methods of [18]. In this paper, I propose to generalise his construction to arbitrary central simple algebras, with the same technique. The methods will also shed some light on the difference between Suslin's conjecture and the theorems of Rost and Merkurjev. The main new results are the following:

Theorem A. *Let F be a field and A a simple algebra with centre F and index e, supposed to be a power of a prime l different from* $\operatorname{char} F$. *Then, for any divisor r of e, there is a complex*

$$0 \to SK_1(A) \xrightarrow{\sigma_r^1} H^4(F, \mathbf{Q}/\mathbf{Z}(3))/r[A] \cdot K_2(F) \to A^0(Y^{[r]}, H^4_{\text{ét}}(\mathbf{Q}/\mathbf{Z}(3)))$$

where $Y^{[r]}$ is the generalised Severi-Brauer variety $SB(r, A)$ and the groups $A^0(Y^{[r]}, -)$ denote unramified cohomology. If the Bloch-Kato conjecture holds in degree 3 for the prime l, these complexes refine into complexes

$$0 \to SK_1(A) \to H^4(F, \mu_{e/r}^{\otimes 3})/r[A] \cdot H^2(F, \mu_{e/r}^{\otimes 2}) \to A^0(Y^{[r]}, H^4_{\text{ét}}(\mu_{e/r}^{\otimes 3})).$$

They are exact for $r = 1, 2$ and $e = 4$.

I don't know, and don't conjecture, that these complexes are exact in general. The map of theorem A coincides with those of Rost and Merkurjev, which is the way we get their nontriviality for $l = 2$ [34].

Theorem B. *Let F, A, e and $Y^{[r]}$ be as in Theorem A; assume the Bloch-Kato conjecture in degree ≤ 3 at the prime l and that F contains a separably closed subfield. Then, for any divisor r of e, there is a complex*

$$0 \to SK_2(A) \xrightarrow{\sigma_r^2} H^5(F, \mathbf{Q}/\mathbf{Z}(4))/r[A] \cdot K_3^M(F) \to A^0(Y^{[r]}, H^5_{\text{ét}}(\mathbf{Q}/\mathbf{Z}(4))).$$

If, moreover, the Bloch-Kato conjecture holds in degree 4 for the prime l, these complexes refine into complexes

$$0 \to SK_2(A) \to H^5(F, \mu_{e/r}^{\otimes 4})/r[A] \cdot H^3(F, \mu_{e/r}^{\otimes 3}) \to A^0(Y^{[r]}, H^5_{\text{ét}}(\mu_{e/r}^{\otimes 4}))).$$

For $l = 2$, the maps starting from $SK_2(A)$ are nontrivial in general for $r = 1, 2$ (unless $\operatorname{ind}(A) \leq 2$).

Theorem C. *For any smooth F-variety X, define*

$$SK_1(X, A) = \varinjlim \operatorname{Hom}_F(X, \mathbf{SL}_n(A))^{\text{ab}}$$

where $\mathbf{SL}_n(A)$ is the reductive group representing the functor $R \mapsto SL_n(A \otimes_F R)$. Then there exists a natural transformation

$$c_A(X) : SK_1(X, A) \to H^5_{\text{ét}}(X, \mathbf{Z}(3)).$$

Restricted to fields, c_A is the universal invariant with values in $H^5_{\text{ét}}(\mathbf{Z}(3)) \simeq H^4_{\text{ét}}(\mathbf{Q}/\mathbf{Z}(3))$ in the sense of Merkurjev [35].

Loosely speaking, c_A is defined out of the "positive" generator of the group $H^5_{\text{ét}}(\mathbf{SL}_1(A), \mathbf{Z}(3))/H^5_{\text{ét}}(F, \mathbf{Z}(3))$ which turns out to be infinite cyclic, much like the Rost invariant is defined out of the "positive" generator of the infinite cyclic group $H^3_{\text{ét}}(\mathbf{SL}_1(A), \mathbf{Z}(2)) \simeq H^4_{\text{ét}}(B\mathbf{SL}_1(A), \mathbf{Z}(2))$ (see [8, App. B]). This replies [35, Rk. 5.8] in the same way as what was done for the Arason invariant in [8].

Theorem D. *Let K be the function field of $\mathbf{SL}_1(A)$. If $\operatorname{ind}(A) = 4$, we have*

$$SK_1(A_K)/SK_1(A) \simeq \mathbf{Z}/2.$$

In Conjecture 10.16 we conjecture that $SK_1(A_K)/SK_1(A)$ is cyclic for any A.

Theorem E. *If $\exp(A) = 2 < \operatorname{ind}(A)$, then*

$$\operatorname{Inv}^4(\mathbf{SL}_1(A), H^*(\mathbf{Q}/\mathbf{Z}(* - 1))) \simeq \mathbf{Z}/2$$

where the former group is Merkurjev's group of invariants of $\mathbf{SL}_1(A)$ with values in $H^4(-, \mathbf{Q}/\mathbf{Z}(3))$ [35]. *In particular the invariant of Theorem C is nontrivial in this case, and equals the invariant σ_2^1 of Theorem A.*

Theorems A, B and C were obtained around 2001/2002, except for the exactness and nontriviality statements for $r = 1$, which follow from the work of Suslin [50]. They were presented at the 2002 Talca-Pucón conference on quadratic forms [20]. Theorems A and C are used by Tim Wouters in recent work [60]. Theorems D and E were obtained while revising this paper for publication.

This paper is organised as follows. We set up notation in Section 1. In Section 2, we recall the slice spectral sequences in the case of geometrically cellular varieties. Sections 3 to 5 are technical. In particular, Section 3 recalls the diagrams of exact sequences from [18, §5], trying to keep track of where the Bloch-Kato conjecture is used; we deduce a simple proof of Suslin's theorem [50, Th. 1], as indicated by himself in the introduction of [50] (see Remark 3.2). In Section 6 we get our first main result, Theorem 6.1, which constructs functorial injections sending a part of lower K-theory of some projective homogeneous varieties into a certain subquotient of the Galois cohomology of the base field. We apply this result in Section 7 to twisted flag varieties, thus getting Theorems A and B (see Corollaries 7.4 and 7.5); in Remark 7.3, we revisit the proof of Theorem 5 given in [21]. In Section 8, we push the main result of [22] one step further. In Section 9, we do some preliminary computations on the slice spectral sequences associated to a reductive group G: the main result is that, if G is simple simply connected of inner type A_r for $r \geq 2$, then the complex $\alpha^* c_3(G)$ of [14] is quasi-isomorphic to $\mathbf{Z}[-1]$ (see Theorem 9.5 for a more complete statement). In section 10, the approach of Merkurjev in [35] plays a central rôle: we prove Theorem C, see Theorem 10.7, Theorem D, see Corollary 10.15 and part of Theorem E, see Proposition 10.11. We conclude with some incomplete computations in Section 11 trying to evaluate the group $SK_1(A_K)/SK_1(A)$ in general, where K is the function field of $\mathbf{SL}_1(A)$: see Theorem 11.9 and Corollary 11.10. At the end of this section we complete the proof of Theorem E, see Corollary 11.12.

This paper contains results which are mostly 8 to 9 years old. The main reason why it was delayed so much is that I tried to compare the 3 ways to construct homomorphisms à la Suslin indicated above: in (0.1)–(0.2), Theorems A and B and Theorem C, and to prove their nontriviality in some new cases. In the first version of this work, I wrote that I had been mostly unsuccessful. Since then the situation has changed a bit with Theorems D and E: they were potentially already in the first version, but Wouters' work [60] was an eye-opener for this. The easy comparisons are, for Theorems A and B, with the Rost and Calmès homomorphisms of Theorems 2 and 7, and with the new Suslin homomorphism of Theorem 4. We can now also compare those of Theorems A and C in certain cases as in Theorem E, see also Corollary 10.10 and [60, §4]. A complete comparison of all invariants still seems challenging[1]: I give some comments on these comparison issues in Subsection 7.F and Remark 10.12.

Acknowledgements. This project was started in 2001. I would like to thank Marc Levine, Baptiste Calmès, Annette Huber, Sasha Merkurjev, Evgueny Schinder, Philippe Gille, Tim Wouters and Nikita Karpenko for helpful discussions and exchanges. Especially, I thank Gille and Wouters for explaining me my own work. Last, I would like to express my admiration for Suslin's work: the inspiration I have drawn from it will be obvious throughout this paper.

[1]Including with the first homomorphism of Suslin in [49], a comparison I had initiated in a preliminary version of this paper. (A vestige remains in §11.C.)

1. Notation

If X is a projective homogeneous variety, we denote as in [18] by E_i the étale F-algebra corresponding to the canonical $\mathbf{Z}$-basis of $CH^i(X_s)$ given by Schubert cycles, where $X_s = X \otimes_F F_s$ and F_s is a separable closure of F.

The motivic cohomology groups used in this paper are (mostly) the Hom groups in Voevodsky's category $DM^{\text{eff}}_{-,\text{ét}}(F)$ of [54, §3.3] (étale topology). In particular, the exponential characteristic p of F is inverted in this category by [54, Prop. 3.3.3 2)], so that those groups are $\mathbf{Z}[1/p]$-modules. Very occasionally we shall use Hom groups in the category $DM^{\text{eff}}_{-}(F)$ (Nisnevich topology).

Let $(\mathbf{Q}/\mathbf{Z})' = \bigoplus_{l \neq p} \mathbf{Q}_l/\mathbf{Z}_l$. We abbreviate the étale cohomology groups $H^i_{\text{ét}}(X, (\mathbf{Q}/\mathbf{Z})'(j))$ with the notation $H^i(X, j)$.

Unless otherwise specified, all cohomology groups appearing are étale cohomology groups, with the exception of cycle cohomology groups in the sense of Rost [44]. The latter are denoted by $A^p(X, M_q)$, where M_* is the relevant cycle module. By Gersten's conjecture [44, Cor. 6.5], these groups are canonically isomorphic to the Zariski cohomology groups $H^p_{\text{Zar}}(X, \mathcal{M}_q)$, where $\mathcal{M}_q$ is the Zariski sheaf on X associated to M_q; we shall occasionally but rarely use this isomorphism, implicitly or explicitly.

2. Motivic cohomology of smooth geometrically cellular varieties updated

2.A. The Bloch-Kato conjecture and the Beilinson-Lichtenbaum conjecture. At the referee's request, I recall these two conjectures and their equivalence:

2.1. Conjecture (Milnor, Bloch, Kato). *Let $n \geq 0$, $m \geq 1$ be two integers. Then, for any field F of characteristic not dividing m, the "norm residue symbol"*

$$K^M_n(F)/m \to H^n(F, \mu_m^{\otimes n})$$

(first defined by Tate in [52]*) is bijective.*

2.2. Conjecture (Suslin-Voevodsky). *Let $n \geq 0$, $m \geq 1$, $i \in \mathbf{Z}$ be three integers. Then, for any field F of characteristic not dividing m and any smooth F-scheme X, the change of topology map*

$$H^i_{\text{Nis}}(X, \mathbf{Z}/m(n)) \to H^i_{\text{ét}}(X, \mathbf{Z}/m(n))$$

is bijective for $i \leq n$ and injective for $i = n+1$, where $\mathbf{Z}/m(n)$ is the mod m version of the n-th motivic complex of Suslin-Voevodsky.

Conjecture 2.2 appears in [51] where (among other places like [54]) the complexes $\mathbf{Z}(n)$ are introduced. It therefore cannot be literally attributed to Beilinson and Lichtenbaum, although it is indeed a common part of conjectures they made in the eighties on the properties of the still conjectural complexes $\mathbf{Z}(n)$. Voevodsky observed in [56] that the special case $X = \operatorname{Spec} F$, $i = n$ of Conjecture 2.2 is a reformulation of Conjecture 2.1. Conversely:

2.3. Theorem ([51, 10], see also [19]). *Conjecture 2.1 (for the pair (n,m)) implies Conjecture 2.2 (for the triples (n,m,i)).*

We shall actually use in this paper the following variant of Conjecture 2.2 with integral coefficients:

2.4. Proposition. *Conjecture 2.2 for m a power of a prime l is equivalent to the following: let $n \geq 0$, $i \in \mathbf{Z}$ be two integers. Then, for any field F of characteristic $\neq l$ and any smooth F-scheme X, the change of topology map*

$$H^i_{\mathrm{Nis}}(X,\mathbf{Z}(n)) \to H^i_{\text{ét}}(X,\mathbf{Z}(n))$$

is bijective for $i \leq n+1$ and injective for $i = n+2$ after localising at l.

The equivalence is an easy consequence of the fact that the map in Proposition 2.4 is an isomorphism after tensoring with $\mathbf{Q}$ for any $i \in \mathbf{Z}$ [53, Prop. 5.28].
The special case $X = \operatorname{Spec} F$, $i = n+1$ of Proposition 2.4 enunciates that $H^{n+1}_{\text{ét}}(F,\mathbf{Z}(n)) \otimes \mathbf{Z}_{(l)} = 0$: this is called "Hilbert's theorem 90 in degree n" and is actually equivalent (for all F) to the above conjectures.
At the time of writing, the status of Conjecture 2.1 is as follows. For $n = 0$ it is trivial, for $n = 1$ it is Kummer theory ($\iff$ Hilbert's theorem 90), for $n = 2$ it is the Merkurjev-Suslin theorem [36], for m a power of 2 it is due to Voevodsky [56]. In general it seems now to be fully proven as a combination of works by several authors, merging in [57] (see [59] for an overview).
In this paper, we use these conjectures for $n = 2$ (*resp.* $n = 3$) when dealing with SK_1 (*resp.* SK_2) and $\mathbf{Q}/\mathbf{Z}$ coefficients, and for $n = 3$ (*resp.* $n = 4$) when dealing with SK_1 (*resp.* SK_2) and finite coefficients.

2.B. The slice spectral sequences. In [18], we constructed spectral sequences for the étale motivic cohomology of smooth geometrically cellular varieties. These results were limited in two respects:

(1) the ground field F was assumed to be of characteristic 0;
(2) the spectral sequences had a strange abutment, which was nevertheless sufficient for applications.

The results of [14] solved both issues. The first one was due to the fact that [18] worked with motives with compact support in Voevodsky's triangulated category of motives [54], which are known to be geometric only in characteristic 0: indeed, it was shown that the motive with compact supports of a cellular variety X is a pure Tate motive in the sense of [14], from which it was deduced by duality that the motive of X (without supports) is also pure Tate if X is smooth. In [14, Prop. 4.11], we prove directly that, over any field, the motive of X is pure Tate if X is smooth and cellular.
The second issue was more subtle and is discussed in [14, Remark 6.3]. The short answer is that by considering a different filtration than the one used in [18], one gets the "right" spectral sequence.
We summarize this discussion by stating the following theorem, which follows from [14, (3.2) and Prop. 4.11] and replaces [18, Th. 4.4]:

2.5. Theorem. *Let X be a smooth, equidimensional, geometrically cellular variety over a perfect field F. For all $n \geq 0$, there is a spectral sequence $E(X,n)$:*

$$E_2^{p,q}(X,n) = H^{p-q}_{\text{ét}}(F, CH^q(X_s) \otimes \mathbf{Z}(n-q)) \Rightarrow H^{p+q}_{\text{ét}}(X, \mathbf{Z}(n)). \tag{2.1}$$

Note that, by cellularity, each $CH^q(X_s)$ is a permutation Galois module. These spectral sequences have the following properties:

(i) Naturality. (2.1) *is covariant in F and contravariant in X (varying among smooth, equidimensional, geometrically cellular varieties) under any maps (even finite correspondences).*

(ii) Products. *There are pairings of spectral sequences*

$$E_r^{p,q}(X,m) \times E_r^{p',q'}(X,n) \to E_r^{p+p',q+q'}(X,m+n)$$

which coincide with the usual cup-product on the E_2-terms and the abutments.

(iii) Transfer. *For any finite extension E/F and any $n \geq 0$, there is a morphism of spectral sequences*

$$E_r^{p,q}(X_E,n) \to E_r^{p,q}(X,n)$$

which coincides with the usual transfer on the E_2-terms and the abutment.

(iv) Covariance for closed equidimensional immersions. *For any closed immersion $i : Y \hookrightarrow X$ of pure codimension c, where X and Y are smooth, geometrically cellular, there is a morphism of spectral sequences*

$$E_r^{p-c,q-c}(Y,n-c) \xrightarrow{i_*} E_r^{p,q}(X,n)$$

"abutting" to the Gysin homomorphisms

$$H^{p+q-2c}_{\text{ét}}(Y, \mathbf{Z}(n-c)) \xrightarrow{i_*} H^{p+q}_{\text{ét}}(X, \mathbf{Z}(n)).$$

If X is split, then (2.1) *degenerates at E_2.*

The only nonobvious point in this theorem is (ii) (products). In [14, p. 915], it is claimed that there are pairings of slice spectral sequences for the tensor product of two arbitrary motives M and N. This is not true in general: I thank Evgeny Shinder for pointing out this issue. However, these pairings certainly exist if M or N is a mixed Tate motive: the argument is essentially the same as the one that proves that the Künneth maps of [14, Cor. 1.6] are isomorphisms in this case [14, Lemma 4.8]. For the reader's convenience, we outline the construction. We take the notation of [14]:

Given the way the slice spectral sequence is constructed in [14, §3] (bottom of p. 914), to get a morphism of filtrations, we need to get morphisms

$$\nu_{\leq q+q'}(M \otimes M') \to \nu_{\leq q} M \otimes \nu_{\leq q'} M'$$

for two motives M, M' and two integers q, q'.

From the canonical maps $M \to \nu_{\leq q} M$ and $M' \to \nu_{\leq q'} M'$, we get a morphism

$$M \otimes M' \to \nu_{\leq q} M \otimes \nu_{\leq q'} M'$$

and we would like to prove that its composition with $\nu^{>q+q'}(M\otimes M') \to M\otimes M'$ is 0. This will be true provided

$$\nu^{>q+q'}(\nu_{\leq q}M \otimes \nu_{\leq q'}M') = \underline{\mathrm{Hom}}(\mathbf{Z}(q+q'+1), \nu_{\leq q}M \otimes \nu_{\leq q'}M')(q+q'+1) = 0.$$

This is false in general (for example $M = M' = h_1(C)$, $q = q' = 0$, where C is a curve of genus > 0 over an algebraically closed field), but it is true if M or M' is a mixed Tate motive. Indeed, we may reduce to $M = \mathbf{Z}(a)$ for some integer a. Then

$$\nu_{\leq q}M = \begin{cases} 0 & \text{if } q < a \\ \mathbf{Z}(a) & \text{if } q \geq a \end{cases}$$

hence $\underline{\mathrm{Hom}}(\mathbf{Z}(q+q'+1), \nu_{\leq q}M \otimes \nu_{\leq q'}M') = 0$ if $q < a$, and if $q \geq a$ we get

$$\begin{aligned} &\underline{\mathrm{Hom}}(\mathbf{Z}(q+q'+1), \nu_{\leq q}M \otimes \nu_{\leq q'}M') \\ &\quad = \underline{\mathrm{Hom}}(\mathbf{Z}(q+q'+1), \mathbf{Z}(a) \otimes \nu_{\leq q'}M') \\ &\quad\quad = \underline{\mathrm{Hom}}(\mathbf{Z}(q+q'+1-a), \nu_{\leq q'}M') = 0 \end{aligned}$$

because $q + q' + 1 - a > q'$.
Dealing with the spectral sequences for étale motivic cohomology, it will suffice that M or N is geometrically mixed Tate in the sense of [14, §5] to have these products.

2.6. *Remark.* As stressed in §1, the spectral sequences of Theorem 2.5 are spectral sequences of $\mathbf{Z}[1/p]$-modules, where p is the exponential characteristic of F. Thus all results of this paper are "away from p". It is nevertheless possible to extend the methods to p-algebras in characteristic p, at some cost: this is briefly discussed in Appendix A. I am grateful to Tim Wouters for a discussion leading to this observation.

2.C. Vanishing of E_2-terms. Since this issue may be confusing, we include here an estimate in the case of the spectral sequences (2.1) and of the coniveau spectral sequences, which will be used in the next section (compare [18, p. 161]). It shows that these two spectral sequences live in somewhat complementary regions of the E_2-plane.

2.7. Proposition. *a) In the spectral sequence* (2.1), *we have* $E_2^{a,b}(X,n) = 0$ *in the following cases:*
(ai) $a \leq b$, $b \geq n-1$, *except* $a = b = n$.
(aii) $a = n+1$ *under the Bloch-Kato conjecture in degree* $n-b$.
Moreover, $E_2^{a,b}(X,n)$ *is uniquely divisible for* $a \leq b$ *and* $b < n-1$.
b) Let X *be a smooth variety. In the coniveau spectral sequence for étale motivic cohomology*

$$E_1^{a,b} = \bigoplus_{x\in X^{(a)}} H^{b-a}(k(x), \mathbf{Z}(n-a)) \Rightarrow H^{a+b}(X, \mathbf{Z}(n))$$

we have $E_1^{a,b} = 0$ *in the following cases:*

(bi) *$a \geq b$, $a \geq n-1$, except $a = b = n$.*
(bii) *$b = n+1$ under the Bloch-Kato conjecture in degree $n-a$.*
Moreover, $E_2^{a,b}(X,n)$ is uniquely divisible for $a \geq b$ and $a < n-1$.
Finally, for $b = n$, the natural map

$$A^a(X, K_n^M)[1/p] \to E_2^{a,n}$$

is surjective under the Bloch-Kato conjecture in degrees $\leq n-a$, and bijective under the Bloch-Kato conjecture in degrees $\leq n-a+1$.

Proof. For (ai), we use that $E_2^{a,b}(X,n) = H_{\text{ét}}^{a-b}(F, CH^b(X_s) \otimes \mathbf{Z}(n-b)) \simeq H_{\text{ét}}^{a-b-1}(F, CH^b(X_s) \otimes \mathbf{Q}/\mathbf{Z}(n-b))$ for $n-b<0$ (by definition of $\mathbf{Z}_{\text{ét}}(n-b)$ for $n-b<0$, see [14, Def. 3.1]), and also that $\mathbf{Z}(0) = \mathbf{Z}$ and $\mathbf{Z}(1) = \mathbb{G}_m[-1]$. (aii) follows from Hilbert 90 in degree $n-b$ (see §2.A after Proposition 2.4). The proofs of (bi) and (bii) are similar. The divisibility claims reduce to the unique divisibility of $H_{\text{ét}}^i(K, \mathbf{Z}(r))$ for $i \leq 0$ ($r > 0$, K/F a function field): this is obvious for $i < 0$, while for $i = 0$ we may reduce to finitely generated fields as in [17, proof of Th. 3.1 a)]. Finally, the last claim follows from a diagram chase in the comparison map between the Gersten complexes for Nisnevich and étale cohomology with $\mathbf{Z}(n)$ coefficients. □

3. Weight 3 and weight 4 étale motivic cohomology

In this section, we examine in more detail the diagrams obtained in [18] by mixing the slice and coniveau spectral sequences, and expand the results in weight 4. In order to stress the irrelevance of Gersten's conjecture, we replace the notation $H^p(X, \mathcal{H}^q)$ or $H^p(X, \mathcal{K}_q)$ used in [18] by the notation $A^p(X, H^q)$ or $A^p(X, K_q)$ (see §1).

3.A. Weight 3. Let X be a projective homogeneous F-variety. In [18, §5.4], we drew a commutative diagram with some exactness properties, by mixing the coniveau spectral sequence and the spectral sequence of [18, Th. 4.4] for étale motivic cohomology in weight 3. We can now use the spectral sequence (2.1) to get the same diagram over any perfect field. To get the diagram of [18, §5.4], we made the blanket assumption in [18] that all groups were localised at 2, because calculations relied on the Bloch-Kato conjecture in degree 3, which was only proven for $l = 2$.
In this paper, we are also interested in making the dependence on this conjecture explicit. How much exactness remains in this diagram if we don't wish to use it in degree 3? Using Proposition 2.7, we see that at least the following part of the diagram of [18, §5.4] remains exact by only using the Bloch-Kato conjecture in degree ≤ 2 (= the Merkurjev-Suslin theorem): the exponential characteristic p is implicitly inverted in this diagram as well as in the next one,

(3.2).

(3.1)

$$\begin{tikzcd}
 & & H^4(F,\mathbf{Z}(3)) \arrow[d] & \\
0 \arrow[r] & A^1(X,K_3^M) \arrow[r] & H^4(X,\mathbf{Z}(3)) \arrow[r] \arrow[d] & A^0(X,H^4(\mathbf{Z}(3))) \\
 & & K_2(E_1) \arrow[d, "d_2^{3,1}(3)"] & \\
 & A^0(X,H^4(\mathbf{Z}(3))) \arrow[d] & H^4(F,3) \arrow[d] \arrow[dr, "\eta^4"] & \\
 & A^2(X,K_3^M) \arrow[r] \arrow[dr, "\xi^4"] & H^5(X,\mathbf{Z}(3)) \arrow[r] \arrow[d] & A^0(X,H^4(3)) \arrow[d] \\
 & & E_2^* & CH^3(X) \arrow[d] \\
 & & & H^6(X,\mathbf{Z}(3)).
\end{tikzcd}$$

The group $A^0(X,H^4(\mathbf{Z}(3)))$, which appears twice in this diagram, is of course torsion, as well as $H^4(F,\mathbf{Z}(3))$, and their l-primary components are 0 under the Bloch-Kato conjecture in degree 3 for the prime l.

3.B. Weight 4. In weight 4, we cannot avoid using the Bloch-Kato conjecture in degree 3. There is a commutative diagram, which was only written down in a special case in [18]:

(3.2)

$$\begin{tikzcd}
H^5(X,\mathbf{Z}(4)) \to K_3(E_2)_{\mathrm{ind}} \arrow[dr, "d_3^{3,2}(4)"] & K_3^M(E_1) \arrow[d, "d_2^{4,1}(4)"] & \\
A^0(X,H^5(\mathbf{Z}(4))) \arrow[d] & H^5(F,4) \arrow[d] \arrow[dr, "\eta^5"] & \\
A^2(X,K_4^M) \arrow[r] \arrow[dr, "\xi^5"] & H^6(X,\mathbf{Z}(4)) \arrow[r] \arrow[d] & A^0(X,H^5(4)) \arrow[d] \\
 & K_2(E_2) \arrow[dl, "d_3^{4,3}(4)"'] \arrow[d, "d_2^{4,3}(4)"] & H^3(X,K_4^M) \arrow[d] \\
H^6(F,4) & H^4(E_1,3) & H^7(X,\mathbf{Z}(4)).
\end{tikzcd}$$

In this diagram, the differentials appearing correspond to the spectral sequence (2.1) in weight 4. The path snaking from $A^0(X,H^5(\mathbf{Z}(4)))$ to $H^7(X,\mathbf{Z}(4))$ is exact (it comes from the coniveau spectral sequence for weight 4 étale motivic cohomology: see Proposition 2.7). The differential $d_3^{4,3}(4)$ is only defined on the kernel of $d_2^{4,3}(4)$ and the differential $d_3^{3,2}(4)$ takes values in the cokernel of $d_2^{3,2}(4)$. The column is a complex, exact at $H^6(X,\mathbf{Z}(4))$; its exactness properties at $H^5(F,4)$ and $K_2(E_2)$ involve the differentials d_3 in an obvious sense.
All these exactness properties depend on the Bloch-Kato conjecture in degree i for any field E and any $i \le 3$, and also on Hilbert's theorem 90 in degree i under the same conditions (which follows from the Bloch-Kato conjecture, see §2.A).
The map η^5 is the natural map from the Galois cohomology of the ground field to the unramified cohomology of X.

3.C. The groups $\overline{\operatorname{Ker}}\ \eta^4$ and $\overline{\operatorname{Ker}}\ \eta^5$.

3.1. Definition. For $i = 1,2$, we denote by $\overline{\operatorname{Ker}}\ \eta^{i+3}$ the homology of the complex

$$K^M_{i+1}(E_1) \xrightarrow{d_2^{i+2,1}(X,i+2)} H^{i+3}(F,i+2) \xrightarrow{\eta^{i+3}} A^0(X,H^{i+3}(i+2)).$$

Diagram (3.1) yields an exact sequence

$$A^0(X,H^4(\mathbf{Z}(3))) \to \operatorname{Ker}\xi^4 \to \overline{\operatorname{Ker}}\ \eta^4 \to 0$$

hence an isomorphism

$$\operatorname{Ker}\xi^4 \xrightarrow{\sim} \overline{\operatorname{Ker}}\ \eta^4 \tag{3.3}$$

under the Bloch-Kato conjecture in degree ≤ 3.
If F contains an algebraically closed subfield, then $K_3(E_2)_{\mathrm{ind}}$ is divisible and the differential $d_3^{3,2}(4)$ is 0 since it is a priori torsion [18, Prop. 4.6]. Then diagram (3.2) yields an exact sequence

$$A^0(X,H^5(\mathbf{Z}(4))) \to \operatorname{Ker}\xi^5 \to \overline{\operatorname{Ker}}\ \eta^5 \to 0$$

under the Bloch-Kato conjecture in degree ≤ 3 and an isomorphism

$$\operatorname{Ker}\xi^5 \xrightarrow{\sim} \overline{\operatorname{Ker}}\ \eta^5 \tag{3.4}$$

under the Bloch-Kato conjecture in degree ≤ 4.

3.2. *Remark.* Let us recover Suslin's theorem [50, Th. 1] from (3.3). The point is simply that the coniveau spectral sequence for Nisnevich motivic cohomology yields an isomorphism

$$A^2(X,K_3^M) \xrightarrow{\sim} H^5_{\mathrm{Nis}}(X,\mathbf{Z}(3))$$

(*cf.* [50, Lemma 9]). The differential $d_2^{3,1}(3)$ was computed in [18, Th. 7.1] for Severi-Brauer varieties.

4. Some K-cohomology groups

4.A. $A^1(X,K_3)$ and $A^0(X,K_3)$. Recall from [18, Prop. 4.5] that

$$A^i(X,K_3^M) \xrightarrow{\sim} A^i(X,K_3) \text{ for } i>0. \tag{4.1}$$

For $A^1(X,K_3)$, we have:

4.1. Proposition. *Let X be a projective homogeneous variety over F, and K/F a regular extension. Under the Bloch-Kato conjecture in degree 3, the map*

$$A^1(X,K_3) \to A^1(X_K,K_3)$$

has p-primary torsion kernel, where p is the exponential characteristic of F. More precisely, the kernel of this map is torsion and its l-primary part vanishes for $l \neq p$ if the Bloch-Kato conjecture holds at the prime l in degree 3.

Proof. Up to passing to its perfect closure, we may assume F perfect. By Diagram (3.1) and (4.1), there is a canonical map

$$A^1(X,K_3) \to K_2(E_1)$$

where E_1 is a certain étale F-algebra associated to X, whose kernel is contained in $H^4_{\text{ét}}(F,\mathbf{Z}(3))$: hence the l-primary part of this kernel vanishes under the condition in Proposition 4.1. The result now follows from [47, th. 3.6]. □

Let still X be a projective homogeneous F-variety. As in [18, §5.1], for all $i \geq 0$ we write E_i for the étale F-algebra determined by the Galois-permutation basis of $CH^i(X_s)$ given by Schubert cycles (see §1).

4.2. Theorem. *a) For $i \leq 2$, the map $K_i(F) \to A^0(X,K_i)$ is bijective.*
b) Under the Bloch-Kato conjecture in degree 3, the cokernel of the homomorphism

$$K_3(F) \to A^0(X,K_3)$$

is torsion, and its prime-to-the-characteristic part is

(1) *finite if F is finitely generated over its prime subfield;*
(2) 0 *in the following cases:*
  (i) *F contains a separably closed subfield;*
  (ii) *the map $CH^1(X_{E_1}) \to CH^1(X_s)$ is surjective.*

More precisely, under the Bloch-Kato conjecture in degree 3 for the prime l, the above is true after localisation at l.

Proof. a) is well-known and is quoted for reference purposes: it is obvious for $i=0,1$ (since X is proper geometrically connected), and for $i=2$ it is a theorem of Suslin [47, Cor. 5.6].
b) After [17, Th. 3 a)] (see also [27, Th. 16.4]), the homomorphism $K_3^M(K) \to K_3(K)$ is injective for any field K. Consider the commutative diagram with

exact rows

$$\begin{tikzcd}
0 \arrow[r] & K_3^M(F) \arrow[r] \arrow[d] & K_3(F) \arrow[r] \arrow[d] & K_3(F)_{\mathrm{ind}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A^0(X, K_3^M) \arrow[r] & A^0(X, K_3) \arrow[r] & A^0(X, K_3^{\mathrm{ind}}).
\end{tikzcd}$$

As X is a rational variety, the right vertical map is bijective [8, lemma 6.2]. It therefore suffices to prove the claims of theorem 4.2 for the left vertical map. Let us first assume F perfect: then we can use Theorem 2.5. Mixing the weight 3 coniveau spectral sequence for étale motivic cohomology with the spectral sequence (2.1) in weight 3, we get modulo the Bloch-Kato conjecture in degree 3 the following commutative diagram with exact rows:

$$\begin{tikzcd}
 & 0 \arrow[d] & \\
 & K_3^M(F) \arrow[d] \arrow[dr, "\alpha"] & \\
0 \to A^1(X, H^2(\mathbf{Z}(3))) \arrow[r] \arrow[dr, "\beta"'] & H^3(X, \mathbf{Z}(3)) \arrow[r] \arrow[d] & A^0(X, K_3^M) \to 0 \\
 & K_3(E_1)_{\mathrm{ind}} \arrow[d] & \\
 & 0. &
\end{tikzcd}$$

For the reader's convenience, let us explain where the Bloch-Kato conjecture in degree 3 is necessary. The weight 3 spectral sequence (2.1) gives a priori an exact sequence

$$\begin{aligned}
H^0(E_1, \mathbf{Z}(2)) \xrightarrow{d_2^{1,1}(X,3)} H^3(F, \mathbf{Z}(3)) \to H^3(X, \mathbf{Z}(3)) \\
\to H^1(E_1, \mathbf{Z}(2)) \to H^4(F, \mathbf{Z}(3)).
\end{aligned}$$

Recall that all groups are étale cohomology groups here. The group $H^0(E_1, \mathbf{Z}(2))$ is conjecturally 0; it is uniquely divisible in any case, see proof of Proposition 2.7. Since the differential $d_2^{1,1}(X,3)$ is torsion (proof as in [18, Prop. 4.6]), it must be 0. The identification of $H^1(E_1, \mathbf{Z}(2))$ with $K_3(E_1)_{\mathrm{ind}}$ only depends on the Merkurjev-Suslin theorem. On the other hand, the bijectivity of $K_3^M(F) \to H^3(F, \mathbf{Z}(3))$ and the vanishing of $H^4(F, \mathbf{Z}(3))$ depend on the Bloch-Kato conjecture in degree 3. This takes care of the vertical exact sequence. Similarly, the Bloch-Kato conjecture in degree 3 is necessary to identify the last term of the horizontal exact sequence (stemming from the coniveau spectral sequence) with $A^0(X, K_3^M)$.

The diagram above gives an isomorphism

$$\operatorname{Coker}\alpha \simeq \operatorname{Coker}\beta.$$

Let us show that $\operatorname{Coker}\beta$ is m-torsion for some $m > 0$. The group $K_3(E_1)_{\mathrm{ind}}$ appearing in the diagram is really

$$H^0(F, CH^1(X_s) \otimes H^1(F_s, \mathbf{Z}(2)))$$

via Shapiro's lemma, the isomorphism $H^1(K, \mathbf{Z}(2)) \simeq K_3(K)_{\mathrm{ind}}$ for any field and Galois descent for $K_3(K)_{\mathrm{ind}}$ [37, 23]. A standard computation shows that the corestriction map

$$H^0(E_1, CH^1(X_s) \otimes H^1(F_s, \mathbf{Z}(2))) \xrightarrow{\mathrm{Cor}} H^0(F, CH^1(X_s) \otimes H^1(F_s, \mathbf{Z}(2)))$$

is split surjective. On the other hand, since $CH^1(X_s)$ is finitely generated, there exists a finite extension E/F such that $CH^1(X_E) \to CH^1(X_s)$ is surjective. Without loss of generality, we may assume that E contains all the residue fields of the étale algebra E_1. A transfer argument then shows that the map $CH^1(X_{E_1}) \to CH^1(X_s)$ has cokernel killed by some integer $m > 0$. Hence the composition

$$\begin{aligned} CH^1(X_{E_1}) \otimes H^1(E_1, \mathbf{Z}(2)) &\to CH^1(X_s) \otimes H^1(E_1, \mathbf{Z}(2)) \\ &\xrightarrow{\sim} CH^1(X_s) \otimes H^0(E_1, H^1(F_s, \mathbf{Z}(2))) \\ &\simeq H^0(E_1, CH^1(X_s) \otimes H^1(F_s, \mathbf{Z}(2))) \end{aligned}$$

has cokernel killed by m, and the same holds for the composition

$$\begin{aligned} CH^1(X_{E_1}) \otimes H^1(E_1, \mathbf{Z}(2)) &\to H^0(E_1, CH^1(X_s) \otimes H^1(F_s, \mathbf{Z}(2))) \\ &\xrightarrow{\mathrm{Cor}} H^0(F, CH^1(X_s) \otimes H^1(F_s, \mathbf{Z}(2))). \end{aligned}$$

But this composition factors via cup-product as

$$\begin{aligned} CH^1(X_{E_1}) \otimes H^1(E_1, \mathbf{Z}(2)) &= A^1(X_{E_1}, \mathcal{H}^2(\mathbf{Z}(1))) \otimes H^1(E_1, \mathbf{Z}(2)) \\ &\to A^1(X_{E_1}, \mathcal{H}^2(\mathbf{Z}(3))) \xrightarrow{\mathrm{Cor}} A^1(X, \mathcal{H}^2(\mathbf{Z}(3))) \\ &\xrightarrow{\beta} H^0(F, CH^1(X_s) \otimes H^1(F_s, \mathbf{Z}(2))) \end{aligned}$$

which proves the claim.

Coming back the the case where F is not necessarily perfect, let F' be its perfect (radicial?) closure and α' the map α "viewed over F'". Then a transfer argument shows that the natural map $\operatorname{Coker}\alpha \to \operatorname{Coker}\alpha'$ has p-primary torsion kernel and cokernel, where p is the exponential characteristic of F. In particular, $\operatorname{Coker}\alpha$ is torsion, and its prime-to-p part is killed by some m.

The integer m equals 1 provided $CH^1(X_{E_1}) \to CH^1(X_s)$ is surjective, which proves 2) (ii) in Theorem 4.2. In general, the map

$$K_3(F_0)_{\mathrm{ind}}/m \to K_3(F)_{\mathrm{ind}}/m$$

is bijective, where F_0 is the field of constants of F [37, 23]. If F_0 is separably closed, then $K_3(F_0)_{\mathrm{ind}}/m = 0$ (ibid.), which proves 2) (i); if F is finitely generated, then F_0 is a finite field or a number field with ring of integers A and $K_3(F_0)_{\mathrm{ind}}$ is a quotient of $K_3(A)$; in both cases it is finitely generated, which proves 1). □

4.3. *Example.* X is a conic curve. Then $\operatorname{Coker}\beta$ is isomorphic to the cokernel of the map

$$\bigoplus_{x\in X^{(1)}} K_3(F(x))_{\mathrm{ind}} \xrightarrow{(N_{F(x)/F})} K_3(F)_{\mathrm{ind}}.$$

Even in the case $F=\mathbf{Q}$, $K_3(\mathbf{Q})_{\mathrm{ind}} \simeq \mathbf{Z}/24$, I am not able either to produce an example where this map is not onto, or to prove that it is always onto. As a first try, one might restrict to points of degree 2 on X. To have an idea of how complex the situation is, the reader may refer to [15, §8]. In particular, Theorem 8.1 (iv) of *loc. cit.* shows that the map is onto provided X has a quadratic splitting field of the form $\mathbf{Q}(\sqrt{-p})$, where p is prime and $\equiv -1 \pmod 8$. If X corresponds to the Hilbert symbol (a,b), with a,b two coprime integers, the theorem of the arithmetic progression shows that there are infinitely many $p\equiv -1 \pmod 8$ such that $p\nmid ab$ and $\left(\frac{-p}{l}\right)=-1$ for all primes $l\mid ab$. Since $-p$ is a square in $\mathbf{Q}_2$, this implies that $(a,b)_{\mathbf{Q}(\sqrt{-p})}=0$ if and only if $(a,b)_{\mathbf{Q}_2}=0$. Thus the above map is surjective if $X(\mathbf{Q}_2)\neq\emptyset$, but I don't know the answer in the other case.

4.B. $A^i(X,K_4^M)$ and $A^i(X,K_4)$.

4.4. Theorem. *a) For any smooth variety X, the natural map*

$$\varphi_i : A^i(X,K_4^M)\to A^i(X,K_4)$$

is bijective for $i\geq 3$ and surjective for $i=2$ with kernel killed by 2.
b) Suppose that F contains a separably closed subfield. Then φ_2 is bijective.

Proof. a) By definition, both groups are cohomology groups of the respective Gersten complexes

$$\dots\to \bigoplus_{x\in X^{(i)}} K_{4-i}^M(F(x))\to\dots$$

$$\dots\to \bigoplus_{x\in X^{(i)}} K_{4-i}(F(x))\to\dots$$

Therefore, Theorem 4.4 is obvious for $i\geq 3$, and φ_2 is surjective. Using the Adams operations on algebraic K-theory, we see that, for any field K, the exact sequence

$$0\to K_3^M(K)\to K_3(K)\to K_3(K)_{\mathrm{ind}}\to 0$$

is split up to 2-torsion. It follows that $2\operatorname{Ker}\varphi_2=0$.
b) We have an exact sequence

$$\bigoplus_{x\in X^{(1)}} K_3(F(x))_{\mathrm{ind}} \xrightarrow{\psi} A^2(X,K_4^M)\xrightarrow{\varphi_2} A^2(X,K_4)\to 0.$$

By assumption, each group $K_3(F(x))_{\mathrm{ind}}$ is divisible (compare the proof of Theorem 4.2). Since their images in $A^2(X, K_4^M)$ are killed by 2, they are 0. $\square$

4.5. *Remark.* I don't know if the condition on F is necessary for the bijectivity of φ_2. Note that ψ factors through the group $A^1(X, H^2(\mathbf{Z}(3)))$ appearing in the proof of Theorem 4.2.

5. An approximation of cycle cohomology

Let M_* be a cycle module in the sense of Rost [44] and let X be projective homogeneous. There are cup-products

$$CH^p(X) \otimes M_{q-p}(F) \to A^p(X, M_q). \tag{5.1}$$

which are isomorphisms when X is split, by [8, Prop. 3.7].

Assume now that X is not necessarily split. Let Y be a splitting variety for X: if $X_s = G_s/P$ where G is a semi-simple F-algebraic group and P is a parabolic subgroup of G_s, we may take Y such that $Y_s = G_s/B$ for B a Borel subgroup contained in P. Then $X_{F(y)}$ is cellular for any point $y \in Y$. It is possible to define a map

$$A^p(X, M_q) \xrightarrow{\tilde{\xi}^{p,q}} A^0(Y_{E_p}, M_{q-p}) \tag{5.2}$$

which is an isomorphism after tensoring with $\mathbf{Q}$ and corresponds to the inverse of (5.1) when X is split. When $q - p \leq 2$ and $M_* = K_*^M$, this map refines into a map

$$A^p(X, K_q^M) \xrightarrow{\xi^{p,q}} K_{q-p}^M(E_p) \tag{5.3}$$

thanks to Suslin's theorem [47, Cor. 5.6] for $q - p = 2$ and trivially for $q - p = 0, 1$. In this paper, we shall only construct such a map in the substantially simpler inner case where all algebras E_p are split, which is sufficient for our needs.

We note that, if X is split, the functor $K \mapsto CH^p(X_K)$ from field extensions of F to abelian groups is constant, with finitely generated free value. When X is arbitrary, we shall authorise ourselves of this to denote by $CH^p(X_s)$ the common value of $CH^p(X_K)$ for all splitting fields K of X.

For Y a splitting variety of X as above, consider the Rost spectral sequence [44, §8]

$$E_2^{p,q} = A^p(Y, R^q\pi_*M_*) \Rightarrow A^{p+q}(X \times Y, M_*)$$

where π is the projection $X \times Y \to Y$ and the $R^q\pi_*M_*$ are the higher direct images of M_* in the sense of Rost [44, §7]. Using the fact that (5.1) is an isomorphism in the split case, we get canonical isomorphisms

$$R^q\pi_*M_* = CH^q(X_s) \otimes M_{*-q}$$

hence an edge homomorphism

$$A^p(X \times Y, M_q) \to E_2^{0,p} = CH^p(X_s) \otimes A^0(Y, M_{q-p}).$$

In the inner case, the composition of this map with the obvious map $A^p(X, M_q) \to A^p(X \times Y, M_q)$ is the desired map $\widetilde{\xi}^{p,q}$ of (5.2).
In the special case $M_* = K_*^M$, a functoriality argument shows that the map $\xi^{2,3}$ (*resp.* $\xi^{2,4}$) of (5.3) coincides with the map ξ^4 of Diagram (3.1) (*resp.* with the map ξ^5 of Diagram (3.2)).

6. A general K-theoretic construction

Let X be projective homogeneous, and let K be a splitting field for X such that K/F is geometrically rational (for example, take for K the function field of the corresponding full flag variety, see beginning of §5). We assume as in the previous section that the associated algebras E_p are split: this is probably not essential. We write $K_*(X)^{(i)}$ for the coniveau filtration on $K_*(X)$, and $K_*(X)^{(i/i+1)}$ for its successive quotients.

6.A. The first steps of the coniveau filtration.

6.1. Theorem. *For $i \leq 2$,*
a) The map

$$K_i(F) \oplus K_i(X)^{(1)} \to K_i(X)$$

is an isomorphism.
b) The maps

$$\operatorname{Ker}(K_i(X)^{(2)} \to K_i(X_K)^{(2)}) \to \operatorname{Ker}(K_i(X)^{(1)} \to K_i(X_K)^{(1)}) \\ \to \operatorname{Ker}(K_i(X) \to K_i(X_K))$$

are isomorphisms. (For $i = 2$, we assume the Bloch-Kato conjecture in degree 3 for the torsion primes of X.)
c) There are canonical monomorphisms

$$\operatorname{Ker}(K_i(X)^{(2/3)} \to K_i(X_K)^{(2/3)}) \hookrightarrow \overline{\operatorname{Ker}}\, \eta^{i+3}$$

where $\overline{\operatorname{Ker}}\, \eta^{i+3}$ was introduced in Definition 3.1. (If $i = 2$, we assume the Bloch-Kato conjecture in degree 3 for the torsion primes of X, and also that F contains a separably closed field.) These homomorphisms are contravariant in X.

Proof. a) By Theorem 4.2 a), the composition

$$K_i(F) \to K_i(X) \to A^0(X, K_i)$$

is bijective; hence this composition yields a splitting to the exact sequence

$$0 \to K_i(X)^{(1)} \to K_i(X) \to A^0(X, K_i).$$

b) It suffices to show that the maps $K_i(X)^{(j/j+1)} \to K_i(X_K)^{(j/j+1)}$ are injective for $j = 0, 1$. For $j = 0$, this is clear from a) (reapplying Theorem 4.2 a)).
For $j = 1$, by the (Brown-Gersten-)Quillen spectral sequence it suffices to show that the map

$$A^1(X, K_{i+1}) \to A^1(X_K, K_{i+1})$$

is injective. For $i = 0$, the statement (concerning Pic) is classical; for $i = 1$, it follows from [32, Theorem] and for $i = 2$ it follows from Proposition 4.1.
c) The BGQ spectral sequence gives a map

$$K_i(X)^{(2/3)} \xrightarrow{\sim} E_\infty^{2,-i-2} \hookrightarrow \operatorname{Coker}(A^0(X, K_{i+1}) \xrightarrow{d_2^{0,-i-1}} A^2(X, K_{i+2})).$$

The differential $d_2^{0,-i-1}$ is 0 by Theorem 4.2. Therefore, we get an injection

$$\operatorname{Ker}(K_i(X)^{(2/3)} \to K_i(X_K)^{(2/3)}) \hookrightarrow \operatorname{Ker}(A^2(X, K_{i+2}) \to A^2(X_K, K_{i+2})).$$

Clearly, the right-hand-side kernel is equal to $\operatorname{Ker} \xi^{2,i+2}$, where $\xi^{2,i+2}$ is the map defined in the previous section. As observed at the end of this section, this map coincides with the map ξ^{i+3} of diagrams (3.1) and (3.2) (for $i = 1, 2$; similarly for $i = 0$). The result then follows from (3.3) and (3.4) (and their analogue for $i = 0$). □

6.B. THE REDUCED NORM AND PROJECTIVE HOMOGENEOUS VARIETIES.

6.2. PROPOSITION. *Let B be a central simple F-algebra, and let $\mathcal{F}$ be a locally free sheaf on X, provided with an action of B. For $i \leq 2$, consider the map*

$$u_{\mathcal{F}} : K_i(B) \to K_i(X)$$

induced by the exact functor

$$\begin{aligned} P(B) &\to P(X) \\ M &\mapsto \mathcal{F} \otimes_B M \end{aligned} \tag{6.1}$$

where $P(B)$ (resp. $P(X)$) denotes the category of finitely generated [projective] B-modules (resp. of locally free $\mathcal{O}_X$-sheaves of finite rank).
a) The composition

$$K_i(B) \xrightarrow{u_{\mathcal{F}}} K_i(X) \to A^0(X, K_i) \xleftarrow{\sim} K_i(F)$$

equals $\operatorname{rk}_B(\mathcal{F}) \operatorname{Nrd}_B$, *where* $\operatorname{rk}_B(\mathcal{F}) := \dfrac{\operatorname{rk}(\mathcal{F})}{\deg(B)}$.
b) The map

$$\tilde{u}_{\mathcal{F}} : K_i(B) \to K_i(X)$$

defined by $x \mapsto u_{\mathcal{F}}(X) - \operatorname{rk}_B(\mathcal{F}) \operatorname{Nrd}_B(x)$ *has image contained in* $K_i(X)^{(1)}$. *The composition*

$$K_i(B) \xrightarrow{\tilde{u}_{\mathcal{F}}} K_i(X)^{(1)} \to A^1(X, K_{i+1}) \xrightarrow{\xi^{1,i+1}} K_i(E_1) = CH^1(X) \otimes K_i(F)$$

where $\xi^{1,i+1}$ is as in Section 5, equals $c_1(\mathcal{F}) \otimes \operatorname{Nrd}_B$.

Proof. Observe that Nrd_B is characterised by the commutation of the diagram

$$\begin{array}{ccc} K_i(B_L) & \xrightarrow{\sim} & K_i(L) \\ \uparrow & & \uparrow \\ K_i(B) & \xrightarrow{\operatorname{Nrd}_B} & K_i(F) \end{array}$$

for any extension L/F that splits B and such that $L = F(Y)$, where Y is a smooth projective geometrically rational F-variety and the upper isomorphism is given by Morita theory. Indeed, this diagram then refines to a diagram of the form

$$\begin{tikzcd} A^0(Y, K_i(B \otimes_F \mathcal{O}_Y)) \arrow[r, "\sim"] & A^0(Y, K_i) \\ K_i(B) \arrow[u] \arrow[r, "\mathrm{Nrd}_B"] & K_i(F) \arrow[u, "\wr"'] \end{tikzcd}$$

see [47, Cor. 5.6] for the right vertical isomorphism.

It is therefore sufficient to check Proposition 6.2 after extending scalars to $L = K(Y)$, where Y is the Severi-Brauer variety of B. Thus, we may assume X and B split.

By Morita, $u_{\mathcal{F}}$ then corresponds to the map $K_i(F) \to K_i(X)$ given by cup-product with $[\mathcal{F} \otimes_B S] \in K_0(X)$, where S is a simple B-module. a) is now obvious, the first statement of b) follows, and the second one is also obvious since $\xi^{i,1}$ commutes with products in the split case. $\square$

From Proposition 6.2 and Theorem 6.1 a), it follows that the restriction of $u_{\mathcal{F}}$ and $\tilde{u}_{\mathcal{F}}$ to $SK_i(B)$ induce the same map: $SK_i(B) \to K_i(X)^{(2)}$, that we shall still denote by $u_{\mathcal{F}}$. If L/F is chosen as in the proof of Proposition 6.2, then clearly the composition $SK_i(B) \to K_i(X)^{(2)} \to K_i(X_L)^{(2)}$ is 0. This yields:

6.3. Definition. Let L/F be a geometrically rational extension splitting both X and B. We denote by $\sigma^i_{\mathcal{F}} : SK_i(B) \to \overline{\mathrm{Ker}}\eta^{i+3}$ the composition

$$SK_i(B) \xrightarrow{u_{\mathcal{F}}} \mathrm{Ker}(K_i(X)^{(2/3)} \to K_i(X_L)^{(2/3)}) \hookrightarrow \overline{\mathrm{Ker}}\,\eta^{i+3}$$

where the second map is that of Theorem 6.1 c).

7. Twisted flag varieties

In this section, we define maps from $SK_i(A)$ to Galois cohomology as promised in Theorems A and B. We use the results of the previous section. In order to get these maps, it is enough to deal with generalised Severi-Brauer varieties (twisted Grassmannians); however, we start with the apparently greater generality of twisted flag varieties. The reason for doing this is the hope to be able to compare the various maps with each other in the future, see Subsection 7.F.

7.A. K-theory of twisted flag varieties. Let A be a simple algebra of degree d, with centre F. For $\underline{r} = (r_1, \dots, r_k)$ with $d \geq r_1 > \dots > r_k \geq 0$, let $Y^{[\underline{r}]} = SB(\underline{r}; A)$ be the twist of the flag variety $G(r_1, \dots, r_k; d)$ by a 1-cocycle defining A: its function field is generic among extensions K/F such that A_K acquires a chain $I_1 \supset \dots \supset I_k$ of left ideals of respective K-dimensions $dr_1, \dots, dr_k$. If $\underline{s}$ is a subset of $\underline{r}$, there is an obvious projection

$$Y^{[\underline{r}]} \to Y^{[\underline{s}]}.$$

The variety $Y^{[\underline{r}]}$ carries a chain of locally free sheaves

$$A_{Y^{[\underline{r}]}} \twoheadrightarrow \mathcal{J}_{r_1} \twoheadrightarrow \dots \twoheadrightarrow \mathcal{J}_{r_k} \tag{7.1}$$

where $A_{Y^{[\underline{r}]}}$ is the constant sheaf with value A: if A is split, (7.1) corresponds by Morita theory to the tautological flag $\mathbf{A}^d_{Y^{[\underline{r}]}} \twoheadrightarrow V_{r_1} \dots \twoheadrightarrow V_{r_k}$ on $G(r_1,\dots,r_k;d)$ ($\mathcal{J}_{r_j}$ is the quotient of $\mathrm{End}(\mathbf{A}^d)_{Y^{[\underline{r}]}}$ by the sheaf of ideals consisting of endomorphisms vanishing on $\mathrm{Ker}(\mathbf{A}^d_{Y^{[\underline{r}]}} \to V_{r_j})$).
There is an action of A on this chain. More generally, for any partition $\alpha = (\alpha_1,\dots,\alpha_m)$ of $|\alpha| = \sum \alpha_i$ with $\alpha_1 \geq \dots \geq \alpha_m \geq 0$, with associated Schur functor S^α, the sheaf $S^\alpha(V_{r_j})$ on $G(r_1,\dots,r_k;d)$ defines by faithfully flat descent a sheaf $S^\alpha(\mathcal{J}_{r_j})$ of $A^{\otimes|\alpha|}$-algebras on $Y^{[\underline{r}]}$ [26, §4].
By Levine-Srinivas-Weyman [26, Th. 4.6], we have an isomorphism

$$\bigoplus_\alpha K_*(A^{\otimes|\alpha|}) \xrightarrow{(u_\alpha)} K_*(Y^{[\underline{r}]}) \tag{7.2}$$

where $\alpha = (\alpha^1,\dots,\alpha^k)$ is a family of partitions, with $0 \leq \alpha^j_i \leq r_i - r_{i+1}$, $|\alpha| = \sum |\alpha^j|$ and u_α is induced by the exact functor

$$\begin{aligned} P(A^{|\alpha|}) &\to P(Y^{[\underline{r}]})\\ M &\mapsto S^\alpha(\mathcal{J}) \otimes_{A^{|\alpha|}} M \end{aligned}$$

with $S^\alpha(\mathcal{J}) = S^{\alpha^1}(\mathcal{J}_1) \otimes \dots \otimes S^{\alpha_k}(\mathcal{J}_k)$. Actually our choice of generators is not the one of [26], but rather the same as in Panin [40, Th. 7.1], who proves the same results by a different method.

7.B. Maps from SK_i to Galois cohomology. We now apply Definition 6.3 with $\mathcal{F} = \mathcal{J}_{r_j}$ for each j: in the above notation, this corresponds to the case $\alpha^{j'} = 0$ for $j' \neq j$ and $\alpha^j = (1,0,\dots)$. We find maps

$$\sigma^i_{r_j} : SK_i(A) \to \overline{\mathrm{Ker}}\eta^{i+1}_{Y^{[\underline{r}]}}. \tag{7.3}$$

We now proceed to compute the differential $d_2^{i+2,1}(Y^{[\underline{r}]}, i+2)$ involved in Definition 3.1. Using the multiplicativity of (2.1) (Th. 2.5 (ii)), we reduce to computing the differential $d_2^{1,1}(Y^{[\underline{r}]},1)$ (*cf.* [18, lemma 6.1]). We have an exact sequence [18, 5.2]

$$CH^1(Y_s^{[\underline{r}]})^{G_F} \xrightarrow{d_2^{1,1}(Y^{[\underline{r}]},1)} Br(F) \to Br(Y^{[\underline{r}]}).$$

The group $CH^1(Y_s^{[\underline{r}]})$ has a basis consisting of the first Chern classes of the bundles V_{r_j}: in particular, G_F acts trivially on it. For $j \in [1,k]$, write $Y^{[r_j]}$ for the twisted Grassmannian (generalised Severi-Brauer variety) corresponding to r_j. Then we have a commutative diagram

$$\begin{array}{ccccc} CH^1(Y_s^{[\underline{r}]}) & \xrightarrow{d_2^{1,1}(Y^{[\underline{r}]},1)} & Br(F) & \longrightarrow & Br(Y^{[\underline{r}]})\\ \uparrow & & \| & & \uparrow\\ \mathbf{Z} = CH^1(Y_s^{[r_j]}) & \xrightarrow{d_2^{1,1}(Y^{[r_j]},1)} & Br(F) & \longrightarrow & Br(Y^{[r_j]}). \end{array} \tag{7.4}$$

This shows that $CH^1(Y_s^{[\underline{r}]})$ is generated by the images of the maps $CH^1(Y_s^{[r_j]}) \to CH^1(Y_s^{[\underline{r}]})$ for $j = 1, \dots, k$, and thus there is no loss of generality in assuming $k = 1$ for the computation of the differential, which we do now. Let us simplify the notation by writing r for r_j. We have the following

7.1. LEMMA ([39, Cor. 2.7]). $\operatorname{Ker}(Br(F) \to Br(Y^{[r]})) = \langle r[A] \rangle$. □

Hence we get $d_2^{1,1}(Y^{[r]}, 1)(1) = r[A]$ (up to a unit), and therefore from Diagram (7.4):

$$d_2^{1,1}(Y^{[\underline{r}]}, 1)(V_{r_j}) = r_j[A] \text{ (up to a unit).}$$

We conclude:

7.2. COROLLARY. *a) The maps* (7.3) *give rise to commutative diagrams of complexes* ($i = 1, 2$)*:*

$$\begin{array}{ccccc}
0 \to SK_i(A) & \xrightarrow{\sigma^i_{r_j}} & \dfrac{H^{i+4}(F, \mathbf{Z}(i+2))}{gcd(r_j)[A] \cdot H^{i+1}(F, \mathbf{Z}(i+1))} & \to & A^0(Y^{[\underline{r}]}), H^{i+4}(\mathbf{Z}(i+2))) \\
|| & & \uparrow & & p^* \uparrow \\
0 \to SK_i(A) & \xrightarrow{\sigma^i_{r_j}} & \dfrac{H^{i+4}(F, \mathbf{Z}(i+2))}{r_j[A] \cdot H^{i+1}(F, \mathbf{Z}(i+1))} & \to & A^0(Y^{[r_j]}), H^{i+4}(\mathbf{Z}(i+2)))
\end{array}$$

where $Y^{[r_j]} = SB(r_j, A)$ is the generalised Severi-Brauer variety of ideals of rank r_j, and the middle vertical map is the natural surjection.
b) If $j = k$ and r_k divides the other r_j, then both vertical maps are isomorphisms.

Proof. The only thing to remain proven is b). The generic fibre of $p : Y^{[\underline{r}]} \to Y^{[r_k]}$ is then easily seen to be the split flag variety $G(r_1 - r_k, \dots, r_{k-1} - r_k; d)$; in particular it is rational and the claim follows. □

7.3. *Remark.* By construction, this homomorphism for $i = 2$ factors through an injection

$$SK_2(A) \hookrightarrow K_2(Y^{[\underline{r}]})^{(2)}.$$

If A is a quaternion algebra, the only choice for $Y^{[\underline{r}]}$ is the conic corresponding to A and $K_2(Y^{[\underline{r}]})^{(2)} = 0$. This is a variant of the proof of Theorem 5 given in [21].

As seen above, for $i = 1$, the definition of $\sigma^i_{r_j}$ only involves the Merkurjev-Suslin theorem, while for $i = 2$ it involves the Bloch-Kato conjecture in degree 3 (for the primes dividing d). If we are ready to grant the Bloch-Kato conjecture one degree further, we get a refinement of these maps:

7.4. COROLLARY. *Assume the Bloch-Kato conjecture in degree $i + 2$ ($i = 1, 2$). Assume also for simplicity that r_j divides d. The the complexes on the bottom row of Corollary 7.2 refine into complexes*

$$SK_1(A) \to H^4(F, \mu_{d/r_j}^{\otimes 3})/r_j[A] \cdot H^2(F, \mu_{d/r_j}^{\otimes 2}) \to A^0(Y^{[r_j]}, H^4(\mu_{d/r_j}^{\otimes 3})) \tag{7.5}$$

(7.6) $\quad SK_2(A) \to H^5(F,\mu_{d/r_j}^{\otimes 4})/r_j[A]\cdot H^3(F,\mu_{d/r_j}^{\otimes 3}) \to A^0(Y^{[r_j]},H^5(\mu_{d/r_j}^{\otimes 4})).$

Proof. Use the fact that $d/r\operatorname{Ker}\eta^i=0$ (transfer argument), and that the map $H^4(F,\mu_{d/r_j}^{\otimes 3})\to H^4(F,\mathbf{Q}/\mathbf{Z}(3))=H^5(F,\mathbf{Z}(3))$ (*resp.* the map $H^5(F,\mu_{d/r_j}^{\otimes 4})\to H^5(F,\mathbf{Q}/\mathbf{Z}(4))=H^6(F,\mathbf{Z}(4)))$ is injective under the Bloch-Kato conjecture in degree 3 (*resp.* 4). $\square$

7.C. Examples: maps à la Suslin and à la Rost-Merkurjev. The case of Suslin corresponds to $r_j=1$ for any A. More precisely, the way Suslin constructs his map in [50, §3] shows that it coincides with the one here for $r_j=1$, compare Remark 3.2. Similarly, the cases of Rost-Merkurjev correspond to $d=4$, $r_j=2$. Using the work of Calmès [5, §2.5], one can check that in the case of a biquaternion algebra we get back Rost's map for SK_1 (*resp.* Calmès' map for SK_2). This implies:

7.5. Corollary. *a) For $i=1$, the bottom sequence in Corollary 7.2 is exact for $r_j=1,2$ and $\deg(A)=4$.*
b) The maps σ_1^1 and σ_2^1 are nonzero in general if $4\mid \operatorname{ind}(A)$.

Proof. a) Let us first assume $r_j=1$. Then, as explained above, the map σ_1^1 coincides with Suslin's map in [50, §3], and the exactness is loc. cit., Th. 3. Suppose now that $r_j=2$. If A is a biquaternion algebra, the exactness is Rost's theorem [33, Th. 4]. If $\exp(A)=4$, we reduce to the biquaternion case by the same argument as in [35, proof of Th. 6.6].
b) This follows from a) by a standard argument, *cf.* [34]. $\square$

7.D. Some properties of the maps σ_r^i. For simplicity, we replace r_j by r; we still assume that r divides d.

7.6. Lemma. *If $r=d$, the maps* (7.5) *and* (7.6) *are* 0.

Proof. In this case the variety $Y^{[r]}$ has a rational point, hence the two kernels are 0. (Alternately, the coefficients of the cohomology groups involved in Corollary 7.4 are 0!) $\square$

7.7. Proposition. *Let $a\in F^*$. Then, for all $r\mid d$, the diagram*

$$\begin{array}{ccc}
SK_1(A) & \xrightarrow{\sigma_r^1} & H^4(F,\mu_{d/r}^{\otimes 3})/r[A]\cdot H^2(F,\mu_{d/r}^{\otimes 2})\\
{\scriptstyle\cdot\{a\}}\downarrow & & {\scriptstyle\cdot\{a\}}\downarrow\\
SK_2(A) & \xrightarrow{\sigma_r^2} & H^5(F,\mu_{d/r}^{\otimes 4})/r[A]\cdot H^3(F,\mu_{d/r}^{\otimes 3})
\end{array}$$

commutes, where the vertical maps are cup-product by $\{a\}$ and the horizontal maps are those of (7.5) *and* (7.6).

Proof. Since the spectral sequences of [18, Th. 4.4] are multiplicative, it suffices to chek that the diagram

$$\begin{array}{ccc} SK_1(A) & \xrightarrow{\sigma_r^1} & \operatorname{Ker}\xi^4_{Y^{[r]}}\\ {\scriptstyle\cdot\{a\}}\Big\downarrow & & {\scriptstyle\cdot\{a\}}\Big\downarrow\\ SK_2(A) & \xrightarrow{\sigma_r^2} & \operatorname{Ker}\xi^5_{Y^{[r]}} \end{array}$$

commutes. This in turn reduces to the compatibility of the BGQ spectral sequence and the isomorphisms (7.2) with products. □

Similarly:

7.8. Proposition. *Let A be a discrete valuation F-algebra, with quotient field K and residue field E. Then the diagrams*

$$\begin{array}{ccc} SK_2(A_K) & \xrightarrow{\sigma_r^2} & H^5(K,\mu_{d/r}^{\otimes 4})/r[A]\cdot H^3(K,\mu_{d/r}^{\otimes 3})\\ {\scriptstyle\partial}\Big\downarrow & & {\scriptstyle\partial}\Big\downarrow\\ SK_1(A_E) & \xrightarrow{\sigma_r^1} & H^4(E,\mu_{d/r}^{\otimes 3})/r[A]\cdot H^2(E,\mu_{d/r}^{\otimes 2}) \end{array}$$

commutes, where the homomorphisms ∂ are induced by the residue maps in K-theory and Galois cohomology respectively.

Proof. Similar. □

Using Corollary 7.5 b), Proposition 7.7 and Proposition 7.8, we find that σ_1^2 and σ_2^2 are nontrivial when $4 \mid \operatorname{ind}(A)$.

7.E. A refinement. In this subsection, where we keep the previous notation, we assume that A is a division algebra, d is a power of a prime l and $r[A]=0$: for r strictly dividing d, this is possible if and only if the exponent ε of A is smaller than d (and then we may choose for r any l-power between ε and d/l). Then we can compute $K_1(X)^{(1/2)}$ and extend the map

$$SK_i(A)\to K_i(X)^{(2)}$$

of the previous section to a map

$$K_i(A)\to K_i(X)^{(2)}.$$

This approach corresponds to that of Rost in the case where A is a biquaternion algebra [33].

Let H be the class of a hyperplane section in $K_0(Y^{[r]})$.

7.9. Proposition. *For $i\le 2$,*

a) The composition

$$K_i(F)\xrightarrow{\cdot H} K_i(Y^{[r]})^{(1)}\to A^1(Y^{[r]},K_{i+1})\xrightarrow{\xi^{1,i+1}} K_i(F)$$

is the identity.
b) The induced map

$$K_i(F) \to K_i(Y^{[r]})^{(1/2)}$$

is an isomorphism.
c) Let $\mathcal{J}$ be the tautological bundle on $Y^{[r]}$. Then the image of the map

$$\begin{aligned}\Phi^{[r]} : K_i(A) &\to K_i(Y^{[r]})^{(1)}\\ x &\mapsto \widetilde{u}_{\mathcal{J}}(x) - \operatorname{Nrd}(x) \cdot H\end{aligned}$$

(see Proposition 6.2 b)) sits into $K_i(X)^{(2)}$.

Proof. By Lemma 7.1, the map

$$CH^1(Y^{[r]}) \to CH^1(Y_s^{[r]})$$

is bijective. In particular, $c_1(H) = h$ in $CH^1(Y_s^{[r]})$. We then get a) by multiplicativity. b) follows from a) and the fact that the maps

$$K_i(Y^{[r]})^{(1/2)} \to H^1(Y^{[r]}, K_{i+1}) \xrightarrow{\xi^{1,i+1}} K_i(F)$$

are injective. c) follows immediately from a). □

7.F. The comparison issue. For $s \mid r \mid d$, let $Y^{[r,s]} = SB(r, s, A)$ be as in 7.A with the two projections

$$\begin{tikzcd} & Y^{[r,s]} \arrow[dl, "p_r"] \arrow[dr, "p_s"] & \\ Y^{[r]} & & Y^{[s]}. \end{tikzcd}$$

We have corresponding diagrams ($i = 1, 2$)

$$\begin{tikzcd} & \operatorname{Ker} \eta^{i+3}_{Y^{[r]}} \arrow[dr, "p_r^*"] & \\ SK_i(A) \arrow[ur, "\sigma_r^i"] \arrow[dr, "\sigma_s^i"'] & & \operatorname{Ker} \eta^{i+3}_{Y^{[r,s]}} \\ & \operatorname{Ker} \eta^{i+3}_{Y^{[s]}} \arrow[ur, "p_s^*"'] & \end{tikzcd}$$

The comparison issue is to know whether this diagram commutes: if this is the case, then the maps σ_r^i and σ_s^i are compatible in an obvious sense thanks to Corollary 7.2 b). In view of Theorem 6.1 c), this commutation is equivalent to the commutation of the diagram

$$\begin{tikzcd}[column sep=small]
& K_i(Y^{[r]})^{(2)} \arrow[dr, "p_r^*"] & \\
SK_i(A) \arrow[ur, "u_{\mathcal{J}_r}"] \arrow[dr, "u_{\mathcal{J}_s}"'] & & K_i(Y^{[r,s]})^{(2)} \\
& K_i(Y^{[s]})^{(2)} \arrow[ur, "p_s^*"'] &
\end{tikzcd}$$

or to the vanishing of the map

$$u_{\mathcal{J}_r} - u_{\mathcal{J}_s} : SK_i(A) \to K_i(Y^{[r,s]})^{(2)}.$$

We may also consider the sheaf $\mathcal{I}_{r,s} = \operatorname{Ker}(\mathcal{I}_r \to \mathcal{I}_s)$; then the above amounts to the vanishing of the map

$$u_{\mathcal{I}_{r,s}} : K_i(A) \to K_i(Y^{[r,s]})$$

on the subgroup $SK_i(A)$. In [50, Th. 4], Suslin obtains this commutation (or vanishing) for $(s,r,d) = (1,2,4)$ in a very sophisticated and roundabout way. I have no idea how to prove it in general.

8. Motivic cohomology of some Severi-Brauer varieties

In this section, unlike in the rest of the paper, we write $H^*(X, \mathbf{Z}(n))$ (*resp.* $H^*_{\text{ét}}(X, \mathbf{Z}(n))$ for motivic cohomology of some smooth variety X computed in the Nisnevich (*resp.* étale) topology. We also use Zariski cohomology with coefficients into sheafified étale cohomology groups instead of cycle cohomology, as those are the groups that come naturally.

8.1. Theorem. *Let A have prime index l, and let X be its Severi-Brauer variety. Let $\mathbf{Z}_A$ be the Nisnevich sheaf with transfers defined in* [22, 5.3]. *Let $n \geq 0$, and assume the Bloch-Kato conjecture in degrees $\leq n+1$. Then:*
a) There is an exact sequence

$$0 \to H^n(F, \mathbf{Z}_A(n)) \xrightarrow{\text{Nrd}} H^n(F, \mathbf{Z}(n)) \xrightarrow{\cdot[A]} H^{n+3}_{\text{ét}}(F, \mathbf{Z}(n+1)) \\ \to H^0(X, \mathcal{H}^{n+3}_{\text{ét}}(\mathbf{Z}(n+1))) \to 0.$$

b) There is a cross of exact sequences

$$\begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ & & H^1(X, \mathcal{H}^{n+3}_{\text{ét}}(\mathbf{Z}(n+1))) & & \\ & & \downarrow & & \\ 0 \to H^{n+4}_{\text{ét}}(F, \mathbf{Z}(n+1)) \to & & H^{n+4}(X, \bar{\mathbf{Z}}(n+1)) & \to H^0(X, \mathcal{H}^{n+2}_{\text{ét}}(\mathbf{Z}(n)) & \\ & & \downarrow & \cdot[A] \downarrow & \\ & & H^0(X, \mathcal{H}^{n+4}_{\text{ét}}(\mathbf{Z}(n+1))) & H^{n+5}_{\text{ét}}(F, \mathbf{Z}(n+1)) & \end{array}$$

where $\bar{\mathbf{Z}}(n)$ is the cone of the morphism $\mathbf{Z}(n) \to R\alpha_*\alpha^*\mathbf{Z}(n)$, with α the projection of the big étale site onto the big Nisnevich site.

Proof. This is an extension of [22, Th. 8.1.4 and 8.2.2], and it is proven by the same method. The exact sequence of a) is part 2 of Theorem 8.1.4 of loc. cit. (where the differential is identified with the cup-product with $[A]$ in 8.2), except that in [22, Th. 8.1.4 (2)], the last term is $H^{n+3}_{\text{ét}}(F(X), \mathbf{Z}(n+3))$ and there is no surjectivity claimed.
To prove a) and b) we look at the spectral sequence (8.4) of [22]. Let $d = \dim X (= l-1)$. In the proof of Theorem 8.1.4 and in 8.2, the following was established:

- $E_2^{p,q} = 0$ for $-q \notin [0,d]$, $p < d-1$, $p = d$ or $(p,q) = (d-1,-d)$.
- The differential

$$\begin{aligned} d_2 : \mathrm{Coker}(H^n(F, \mathbf{Z}_A(n)) \to H^n_{\text{ét}}(F, \mathbf{Z}(n))) \simeq E_2^{d-1,1-d} \\ \to E_2^{d+1,-d} \simeq H^{n+3}_{\text{ét}}(F, \mathbf{Z}(n+1)) \end{aligned}$$

  is injective, and induced by the cup-product $H^n_{\text{ét}}(F, \mathbf{Z}(n)) \xrightarrow{\cdot[A]} H^{n+3}_{\text{ét}}(F, \mathbf{Z}(n+1))$.

The abutment of this spectral sequence on the diagonal $p + q = N$ is

$$\mathrm{Hom}(\mathbf{Z}(d)[2d], \bar{M}(X)(n+1)[n+2+N])$$

computed in $DM^{\text{eff}}(F)$, where

$$\bar{M}(X) = cone(M(X) \to R\alpha_*\alpha^*M(X)).$$

Note that $\bar{M}(X)(n+1) \simeq M(X) \otimes \bar{\mathbf{Z}}(n+1)$ (by a projection formula). Hence the abutment may be rewritten (by Poincaré duality)

$$H^{n+2+N}(X, \bar{\mathbf{Z}}(n+1)).$$

The Bloch-Kato conjecture in degree $n+1$ identifies $\bar{\mathbf{Z}}(n+1)$ with $\tau_{>n+2}(R\alpha_*\alpha^*\mathbf{Z}(n))$. The hypercohomology spectral sequence then gives

$$H^{n+2+N}(X,\bar{\mathbf{Z}}(n+1)) = 0 \text{ for } N \leq 0$$
$$H^{n+3}(X,\bar{\mathbf{Z}}(n+1)) \simeq H^0(X,\mathcal{H}^{n+3}_{\text{ét}}(\mathbf{Z}(n+1)))$$

and for $N=2$ an exact sequence

$$0 \to H^1(X,\mathcal{H}^{n+3}_{\text{ét}}(\mathbf{Z}(n+1))) \to H^{n+4}(X,\bar{\mathbf{Z}}(n+1)) \to H^0(X,\mathcal{H}^{n+4}_{\text{ét}}(\mathbf{Z}(n+1))).$$

Consider the differentials $d_2^{d-1,q}: E_2^{d-1,q} \to E_2^{d+1,q-1}$ for $-q \leq d-1$. We have

$$E_2^{p,q} = \operatorname{Hom}(\mathbf{Z},\bar{\mathbf{Z}}_{A^{\otimes(-q+1)}}(n+1-d-q)[n+2-2d+p-q])$$

where $\bar{\mathbf{Z}}_{A^{\otimes(-q+1)}} = cone(\mathbf{Z}_{A^{\otimes(-q+1)}} \to R\alpha_*\alpha^*\mathbf{Z}_{A^{\otimes(-q+1)}})$. Therefore

$$E_2^{d-1,q} = \operatorname{Hom}(\mathbf{Z},\bar{\mathbf{Z}}_{A^{\otimes(-q+1)}}(n+1-d-q)[n+1-d-q])$$
$$= \operatorname{Coker}(H^{n+1-d-q}(F,\mathbf{Z}_A(n+1-d-q)) \to H^{n+1-d-q}(F,\mathbf{Z}(n+1-d-q)))$$

and

$$E_2^{d+1,q-1} = \operatorname{Hom}(\mathbf{Z},\bar{\mathbf{Z}}_{A^{\otimes(-q+2)}}(n+2-d-q)[n+4-d-q]) = H^{n+4-d-q}_{\text{ét}}(F,\mathbf{Z}(n+2-d-q))).$$

The computation of [22, 8.2] identifies $d_2^{d-1,q}$ with the map induced by cup-product by $[A]$. By the above, we get that $d_2^{d-1,q}$ is *injective*. The computation of [22, 8.2] also identifies $d_2^{d+1,q-1}$ with the cup-product by $[A]$. This gives both a) and b). □

9. Étale motivic cohomology of reductive groups

9.A. The slice spectral sequence for a reductive group. Let X be a smooth F-variety. There are spectral sequences [14, (3.1), (3.2)], similar to those of Theorem 2.5:

(9.1) $$E_2^{p,q}(X,n)_{\text{Nis}} = \operatorname{Hom}_{DM_-^{\text{eff}}(F)}(c_q(X),\mathbf{Z}(n-q)[p-q]) \Rightarrow H^{p+q}_{\text{Nis}}(X,\mathbf{Z}(n))$$

(9.2) $$E_2^{p,q}(X,n)_{\text{ét}} = \operatorname{Hom}_{DM_{-,\text{ét}}^{\text{eff}}(F)}(\alpha^*c_q(X),\mathbf{Z}(n-q)[p-q]) \Rightarrow H^{p+q}_{\text{ét}}(X,\mathbf{Z}(n))$$

where $c_q(X)$ are complexes of Nisnevich sheaves with transfers associated to X (canonically in the derived category) and α is the projection from the étale site of smooth F-varieties to the Nisnevich site. These spectral sequences have the same formal properties as (2.1): transfers, and products if the motive of X is mixed Tate (*resp.* geometrically mixed Tate), *cf.* discussion in the proof of Th. 2.5 (ii).

Let $X=G$ be a connected reductive group over F, with maximal torus T defined over F. Set $Y=G/T$. Assume first G and T split. In [14, Prop. 9.3], it was shown that $c_q(G)$ is dual, in the derived category, to the complex of constant Nisnevich sheaves $c^q(G)$ (denoted by $K(G,q)$ in *loc. cit.*) given by

$$(9.3)\quad 0 \to \Lambda^q(T^*) \to \Lambda^{q-1}(T^*) \otimes CH^1(Y) \to \dots \\ \dots \to T^* \otimes CH^{q-1}(Y) \to CH^q(Y) \to 0$$

in which T^* is the group of characters of T, $CH^q(Y)$ is in degree 0 and the maps are induced by intersection products and the characteristic map $\gamma : T^* \to CH^1(X)$ (compare [8, 3.14]). Thus (9.1) may be rewritten in this case as

$$E_2^{p,q}(G,n)_{\mathrm{Nis}} = H^{p-q}_{\mathrm{Nis}}(F, c^q(G) \otimes \mathbf{Z}(n-q)) \Rightarrow H^{p+q}_{\mathrm{Nis}}(G, \mathbf{Z}(n)).$$

Since $c^q(G)$ is concentrated in degrees ≤ 0, $c^q(G)\otimes\mathbf{Z}(n-q)$ is concentrated in degrees $\le n-q$ and $E_2^{p,q}(G,n)_{\mathrm{Nis}} = 0$ for $p > n$. We also have $E_2^{p,q}(G,n)_{\mathrm{Nis}} = 0$ for $q > n$, since $\mathbf{Z}(n-q) = 0$ in this case. For $(p,q) = (n,n)$ this yields

9.1. Lemma (*cf.* Grothendieck [13, p. 21, Rem. 2]). *If G is split, we have isomorphisms $E_2^{n,n}(G,n)_{\mathrm{Nis}} \simeq E_\infty^{p,q}(G,n)_{\mathrm{Nis}} \simeq H^{2n}(G,\mathbf{Z}(n))$, hence an exact sequence*

$$T^* \otimes CH^{n-1}(Y) \to CH^n(Y) \to CH^n(G) \to 0.$$

We shall also use:

9.2. Lemma. *Suppose G split, simply connected and absolutely simple. Then, for all $n > 0$, $CH^n(G)$ is killed by $(n-1)!$ and by the torsion index t_G of G* [7, §5]. *In particular, $CH^i(G) = 0$ for $i = 1, 2$. If G is of type A_r or C_r, $CH^i(G) = 0$ for all $i > 0$.*

Proof. The first fact follows from $K_0(G) = \mathbf{Z}$, *cf.* [8, Proof of Prop. 3.20 (iii)]. For the second one, Demazure proves in [7, Prop. 5] that the cokernels of the characteristic maps $\gamma^n : \mathbf{S}^n(T^*) \to CH^n(Y)$ are killed by t_G: the claim then follows from Lemma 9.1 and a small diagram chase. The last fact follows from [7, Lemme 5], which says that $t_G = 1$ for G of type A_r or C_r. (This also follows from Suslin [48, Th. 2.7 and 2.12].) □

We now relax the assumption that G is split, and would like to study the spectral sequences (9.2). If we knew that

$$(9.4)\qquad \alpha^* c_q(G) \simeq c_q(G_s)$$

in the derived category of complexes of étale sheaves (or G_F-modules), this would allow us to rewrite (9.2) in the form

$$E_2^{p,q}(G,n)_{\text{ét}} = H^{p-q}_{\text{ét}}(F, c^q(G_s) \otimes \mathbf{Z}(n-q)) \Rightarrow H^{p+q}_{\text{ét}}(G, \mathbf{Z}(n))$$

as for the split case, in the Nisnevich topology.

I don't know how to prove (9.4), but at least the proof of [14, Prop. 9.3] shows that the two complexes have isomorphic cohomology sheaves. Hence they are quasi-isomorphic at least in the case where the cohomology of $c^p(G_s)$ is concentrated in at most one degree. We shall therefore make-do with (9.2) and be saved by the fact that, for low values of q and for the groups G we are interested in, the latter fact is true. For simplicity, we shall write

$$\operatorname{Hom}_{DM^{\mathrm{eff}}_{-,\text{ét}}(F)}(\alpha^* c_q(G), \mathbf{Z}(n-q)[p-q]) = \operatorname{Ext}^{p-q}_{\text{ét}}(\alpha^* c_q(G), \mathbf{Z}(n-q)).$$

We always have $c^0(G_s) = CH^0(Y_s) = \mathbf{Z}^{\pi_0(G_s)}$. Suppose that G is semi-simple, simply connected. Then c is bijective and one finds [8]

$$c^1(G_s) = 0 \tag{9.5}$$

$$c^2(G_s) = \mathbf{S}^2(T_s^*)^W[1] \tag{9.6}$$

where W is the Weyl group of G_s. If G is absolutely simple, then $\operatorname{rk} \mathbf{S}^2(T_s^*)^W = 1$ (with trivial Galois action).

We note that the unit section of G splits off from (9.2) spectral sequences

$$\widetilde{E}_2^{p,q}(G, n) \Rightarrow \widetilde{H}^{p+q}_{\text{ét}}(G, \mathbf{Z}(n))$$

with

$$\widetilde{E}_2^{p,q}(G,n) = \begin{cases} \operatorname{Ext}^{p-q}_{\text{ét}}(\alpha^* c_q(G), \mathbf{Z}(n-q)) & \text{for } q > 0 \\ 0 & \text{for } q = 0 \end{cases}$$

and $H^{p+q}_{\text{ét}}(G, \mathbf{Z}(n)) = H^{p+q}_{\text{ét}}(F, \mathbf{Z}(n)) \oplus \widetilde{H}^{p+q}_{\text{ét}}(G, \mathbf{Z}(n))$ via the unit section. These spectral sequences are modules over (9.2).

From the above spectral sequence in weight 3, the corresponding coniveau spectral sequence, (9.5) and (9.6), we get a commutative diagram analogous to (3.1):

(9.7)

$$\begin{tikzcd}
 & & 0 \arrow[d] & \\
0 \arrow[r] & A^2(G, K_3^M) \arrow[r] & \widetilde{H}^5(G, \mathbf{Z}(3)) \arrow[r] \arrow[d] & \widetilde{A}^0(G, H^4(3)) \arrow[d, "\alpha"'] \\
 & & \operatorname{Ext}^{-1}_{\text{ét}}(\alpha^* c_3(G), \mathbf{Z}) \arrow[d, "\widetilde{d}_2^{2,3}(G,3)"'] & CH^3(G) \arrow[d] \\
 & H^2(F, \mathbb{G}_m \otimes \mathbf{S}^2(T_s^*)^W) \arrow[r, "\sim"] & \operatorname{Ext}^2_{\text{ét}}(\alpha^* c_2(G), \mathbf{Z}(1)) \arrow[d] & \widetilde{H}^6(G, \mathbf{Z}(3)) \\
 & & \widetilde{H}^6(G, \mathbf{Z}(3)) \arrow[d] & \\
 & & \operatorname{Ext}^0_{\text{ét}}(\alpha^* c_3(G), \mathbf{Z}) &
\end{tikzcd}$$

In this diagram, the column and the row forking downwards are both exact. The groups marked with a $\widetilde{\ }$ are, as above, the direct summands of the corresponding groups without a $\widetilde{\ }$ defined by the unit section of G.

9.B. AN INVARIANT COMPUTATION. In this subsection, we want to compute $\mathbf{S}^3(T_s^*)^W$ when G is absolutely simple simply connected. We start with the case of type A_r. It is then convenient to think of G_s as $\mathbf{SL}_{r+1}$ embedded into $\mathbf{GL}_{r+1}$. The maximal torus T_s of G_s is then a subtorus of a maximal torus S of $\mathbf{GL}_{r+1}$, conjugate to its canonical maximal subtorus. The character group S^* is free of rank $r+1$, with basis $(e_1,\dots,e_{r+1})$, and T_s^* is the quotient of S^* by $\mathbf{Z}\sigma_1$, with $\sigma_1 = \sum e_i$.
The Weyl group W of G_s coincides with that of $\mathbf{GL}_{r+1}$; it is isomorphic to $\mathfrak{S}_{r+1}$ and permutes the e_i. Let σ_i be the i-th symmetric function in the e_i: by the symmetric functions theorem, we have

$$\mathbf{S}(S^*)^W = \mathbf{Z}[\sigma_1,\dots,\sigma_{r+1}].$$

It is clear that the sequence

$$0 \to \sigma_1\mathbf{S}(S^*) \to \mathbf{S}(S^*) \to \mathbf{S}(T_s^*) \to 0 \tag{9.8}$$

is exact.

9.3. LEMMA. *If $r \geq 2$, the map $\mathbf{S}^3(S^*)^W \to \mathbf{S}^3(T_s^*)^W$ is surjective; $\mathbf{S}^3(T_s^*)^W$ is free of rank 1, with basis the image $\bar\sigma_3$ of σ_3. If $r=1$, $\mathbf{S}^3(T_s^*)^W = 0$.*

Proof. Suppose first $r \geq 2$. In view of (9.8), for the first assertion it suffices to check that $H^1(W,\mathbf{S}^2(S^*)) = 0$. A basis of $\mathbf{S}^2(S^*)$ is given by $(e_1^2,\dots,e_{r+1}^2, e_1e_2,\dots)$. The group W permutes the squares and the rectangular products transitively; the isotropy group of e_1^2 is $\mathfrak{S}_r$ while the isotropy group of e_1e_2 is $\mathfrak{S}_{r-1}$. By Shapiro lemma, we get

$$H^1(W,\mathbf{S}^2(S^*)) \simeq H^1(\mathfrak{S}_r,\mathbf{Z}) \oplus H^1(\mathfrak{S}_{r-1},\mathbf{Z}) = 0.$$

For the second assertion, we use (9.8) again and get an exact sequence (thanks to the symmetric functions theorem)

$$0 \to \sigma_1\langle\sigma_1^2,\sigma_2\rangle \to \langle\sigma_1^3,\sigma_1\sigma_2,\sigma_3\rangle \to \mathbf{S}(T_s^*)^W \to 0.$$

If $r=1$, the same calculation gives the result. □

In the other cases, an application of the theory of exponents [4, V.6.2, Prop. 3 and tables of Ch. VI] gives

9.4. LEMMA. *If G is not of type A_r, $\mathbf{S}^3(T_s^*)^W = 0$.* □

9.C. SOME FACTS ABOUT THE $c^q(G_s)$. Part a) of the following theorem is a version of S. Gille's theorem [11, th. 1.5][2]:

9.5. THEOREM. *Let G be semi-simple and simply connected. Then:*
a) For $q \geq 3$, $H^r(c^q(G_s)) = 0$ for $r = -q, -q+1$, and $H^{-q+2}(c^q(G_s))$ is torsion-free.
b) Suppose G simple. For $q=3$, $H^{-1}(c^3(G_s)) \simeq \mathbf{S}^3(T_s^*)^W$ and $H^0(c^3(G_s)) \simeq CH^3(G_s)$.
c) If G is simple of type A_r, with $r \geq 2$, then $c^3(G_s) \simeq \mathbf{Z}(\chi)[1]$, generated by

[2]For $q=3$ and G of type A_r, it was obtained in 2001/2002. The general case was inspired by Gille's work.

$\bar{\sigma}_3$ (see Lemma 9.3) where χ is the quadratic character of G_F corresponding to its (possibly trivial) outer action on the Dynkin diagram of G. If G is of type A_1, $c^3(G_s) = 0$. If G is not of type A_r, $c^3(G_s) = CH^3(G_s)[0]$.

Proof. a) For two split reductive groups G, H and $n \geq 0$, we have the Künneth formula

$$c^n(G \times H) \simeq \bigoplus_{p+q=n} c^p(G) \overset{L}{\otimes} c^q(H) \tag{9.9}$$

in the derived category [14, Lemma 4.8], since $M(G)$ and $M(H)$ are mixed Tate motives. Thus we may assume G to be simple. Consider now the commutative diagram

$$\begin{tikzcd}
 & & \Lambda^{q-2}(T_s^*)\otimes\mathbf{S}^2(T_s^*)^W \arrow[r, "e"] \arrow[d] & \Lambda^{q-3}(T_s^*)\otimes\mathbf{S}^2(T_s^*)^W\otimes T_s^* \arrow[d, "f"'] \\
\Lambda^q(T_s^*) \arrow[r] \arrow[d, "||"] & \Lambda^{q-1}(T_s^*)\otimes T_s^* \arrow[r] \arrow[d, "1\otimes\gamma"', "\wr"] & \Lambda^{q-2}(T_s^*)\otimes\mathbf{S}^2(T_s^*) \arrow[r] \arrow[d, "1\otimes\gamma^2"'] & \Lambda^{q-3}(T_s^*)\otimes\mathbf{S}^3(T_s^*) \arrow[d, "1\otimes\gamma^3"'] \\
\Lambda^q(T_s^*) \arrow[r] & \Lambda^{q-1}(T_s^*)\otimes CH^1(Y_s) \arrow[r] & \Lambda^{q-2}(T_s^*)\otimes CH^2(Y_s) \arrow[r] & \Lambda^{q-3}(T_s^*)\otimes CH^3(Y_s)
\end{tikzcd}$$

where the bottom row is the beginning of $c^q(G_s)$, the middle row is the q-th Koszul complex for T_s^*, γ^i are induced by the characteristic map, the top row is $\mathbf{S}^2(T_s^*)^W$ tensored with the beginning of the $(q-2)$-nd Koszul complex for T_s^*, the middle column is obtained by tensoring the exact sequence of free abelian groups

$$0 \to \mathbf{S}^2(T_s^*)^W \to \mathbf{S}^2(T_s^*) \to CH^2(Y_s) \to 0$$

with $\Lambda^{q-2}(T_s^*)$ and, finally, f is induced by the product $\mathbf{S}^2(T_s^*)^W \otimes T_s^* \to \mathbf{S}^3(T_s^*)$. The middle row is universally exact as the Koszul complex of a free module, and the middle column is (split) short exact.
Since G is simple, $\mathbf{S}^2(T_s^*)^W$ is a rank 1 direct summand of $\mathbf{S}^2(T_s^*)$, which implies that f is injective and remains so after tensoring with $\mathbf{Z}/m$ for any m. The same is true for e by the acyclicity of Koszul complexes. A diagram chase then gives the result.
b) For $q = 3$, let us rewrite part of the above diagram, for clarity:

$$\begin{tikzcd}
0 \arrow[r] & \Lambda^3(T_s^*) \arrow[r] \arrow[d, "||"'] & \Lambda^2(T_s^*)\otimes T_s^* \arrow[r] \arrow[d, "1\otimes\gamma"'] & T_s^*\otimes\mathbf{S}^2(T_s^*) \arrow[r] \arrow[d, "1\otimes\gamma^2"'] & \mathbf{S}^3(T_s^*) \arrow[r] \arrow[d, "\gamma^3"'] & 0 \\
0 \arrow[r] & \Lambda^3(T_s^*) \arrow[r] & \Lambda^2(T_s^*)\otimes CH^1(Y_s) \arrow[r] & T_s^*\otimes CH^2(Y_s) \arrow[r] & CH^3(Y_s) \arrow[r] & 0.
\end{tikzcd}$$

The two left vertical maps are isomorphisms; by (9.6), $1\otimes\gamma^2$ is surjective, with kernel $T_s^*\otimes\mathbf{S}^2(T_s^*)^W$; also, by [7, p. 292, Cor. 2] $\operatorname{Ker}\gamma^3$ is the $\mathbf{Q}$-span of

$T_s^*\mathbf{S}^2(T_s^*)^W + \mathbf{S}^3(T_s^*)^W$ in $\mathbf{S}^3(T_s)^*$. Using Lemma 9.1, it follows that

$$H^i(c^3(G_s)) = \begin{cases} 0 & \text{for } i=-3 \\ \operatorname{Ker}\varphi & \text{for } i=-2 \\ \operatorname{Coker}\varphi & \text{for } i=-1 \\ CH^3(G_s) & \text{for } i=0 \end{cases}$$

where φ is the map

$$T_s^* \otimes \mathbf{S}^2(T_s^*)^W \to \langle T_s^*\mathbf{S}^2(T_s^*)^W + \mathbf{S}^3(T_s^*)^W\rangle_{\mathbf{Q}},$$

$\langle - \rangle_{\mathbf{Q}}$ denoting the $\mathbf{Q}$-span. We have seen in a) that $\operatorname{Ker}\varphi = 0$ and $\operatorname{Coker}\varphi$ is torsion-free. We may factor φ as a composition

$$T_s^* \otimes \mathbf{S}^2(T_s^*)^W \xrightarrow{\tilde\varphi} T_s^*\mathbf{S}^2(T_s^*)^W + \mathbf{S}^3(T_s^*)^W \hookrightarrow \langle T_s^*\mathbf{S}^2(T_s^*)^W + \mathbf{S}^3(T_s^*)^W\rangle_{\mathbf{Q}}.$$

Thus $\operatorname{Coker}\varphi$ is an extension of the finite group

$$\frac{\langle T_s^*\mathbf{S}^2(T_s^*)^W + \mathbf{S}^3(T_s^*)^W\rangle_{\mathbf{Q}}}{T_s^*\mathbf{S}^2(T_s^*)^W + \mathbf{S}^3(T_s^*)^W}$$

by a group isomorphic to $\mathbf{S}^3(T_s^*)^W/\mathbf{S}^3(T_s^*)^W \cap T_s^*\mathbf{S}^2(T_s^*)^W$; but

$$\mathbf{S}^3(T_s^*)^W \cap T_s^*\mathbf{S}^2(T_s^*)^W \subseteq (T_s^*\mathbf{S}^2(T_s^*)^W)^W = T_s^{*W}\mathbf{S}^2(T_s^*)^W = 0.$$

Thus, the map $\mathbf{S}^3(T_s^*)^W \to \operatorname{Coker}\widetilde{\varphi}$ is bijective. To conclude, we use the fact that $\mathbf{S}^3(T_s^*)^W$ is pure in $\mathbf{S}^3(T_s^*)$ (the quotient is torsion-free), which follows from Lemmas 9.3 and 9.4: since $\operatorname{Coker}\varphi$ is torsion-free, this implies that it is isomorphic to $\mathbf{S}^3(T_s^*)^W$.

c) now follows from b), Lemmas 9.3, 9.4 and 9.2. For G of type A_r with $r \geq 2$, the claim on the Galois action follows from the well-known fact that the nontrivial outer automorphism of the Dynkin diagram of G_s maps $\bar e_i$ to $-\bar e_{r+1-i}$, where $\bar e_i$ is the image of e_i in T_s^*. □

Here is a complement to Theorem 9.5:

9.6. Lemma. *Let $r \geq 2$, and consider the embedding $\iota : \mathbf{SL}_{r+1} \hookrightarrow \mathbf{SL}_{r+2}$ given by $u \mapsto \left(\begin{smallmatrix} u & 0 \\ 0 & 1\end{smallmatrix}\right)$. Then the induced morphism $\iota^* : c^i(\mathbf{SL}_{r+2}) \to c^i(\mathbf{SL}_{r+1})$ is a quasi-isomorphism for $i = 2, 3$.*

Proof. Let T_{r+1}, T_{r+2} be the diagonal tori of $\mathbf{SL}_{r+1}$ and $\mathbf{SL}_{r+2}$ respectively. It suffices to check that $\mathbf{S}^i(T_{r+2}^*)^{\mathfrak{S}_{r+2}} \xrightarrow{\sim} \mathbf{S}^i(T_{r+1}^*)^{\mathfrak{S}_{r+1}}$ for $i = 2, 3$. This follows from the computations in the proof of Lemma 9.3. □

9.7. *Remark.* For G of type C_r, $CH^i(G_s) = 0$ for all $i > 0$, and for general G, $CH^3(G_s)$ is a 2-torsion group (see Lemma 9.2). Marlin computed $CH^*(G_s)$ for G of type B_r, D_r, G_2 or F_4 in [29]: he finds $CH^3(G_s) = \mathbf{Z}/2$ in each case. I don't know the value of $CH^3(G_s)$ for G of type E_6, E_7, E_8: is it also $\mathbf{Z}/2$?

10. The generic element

In this section we prove Theorem C, see (10.2), (10.3) and Theorem 10.7, Theorem D, see Corollary 10.15, and part of Theorem E, see Proposition 10.11.

10.A. The cohomological generic element. Let G be an absolutely simple simply connected group. From Theorem 9.5 and Diagram (9.7), we first deduce:

10.1. Corollary. *If G is not of inner type A_r for $r \geq 2$, we have $A^2(G, K_3^M) = \widetilde{H}^5(G, \mathbf{Z}(3)) = 0$; the group $\widetilde{A}^0(G, H^4(3))$ is isomorphic to the kernel of the étale motivic cycle map $CH^3(G) \to H^6(G, \mathbf{Z}(3))$ (hence is at most $\mathbf{Z}/2$ except perhaps for types E_6, E_7, E_8, see Remark 9.7).*

Proof. All claims follow from the diagram and the fact that we have $H^{-1}(F, c^3(G_s)) = 0$ in these cases (note that obviously

$$\operatorname{Ker}(CH^3(G) \to H^6(G, \mathbf{Z}(3)) = \operatorname{Ker}(CH^3(G) \to \widetilde{H}^6(G, \mathbf{Z}(3))).$$

□

10.2. Proposition. *If G is of inner type A_r with $r \geq 2$, the map α in Diagram (9.7) is 0.*

Proof. We have $G = \mathbf{SL}_1(A)$ for some central simple algebra A. If $CH^3(G) = 0$, there is nothing to prove; by Merkurjev [35, Prop. 4.3], this happens if and only if $\operatorname{ind}(A)$ is odd. Suppose now $\operatorname{ind}(A)$ even. If A is a quaternion algebra, we have $\widetilde{A}^0(G, H^4(3)) = 0$ by [35, Lemma 5.1]. In general, we proceed as in [35, proof of Prop. 4.3]. Note that α really comes from a map $\alpha' : A^0(G, H^4(3)) \to CH^3(G)$ and that $\alpha = 0$ if and only if $\alpha' = 0$. Let K/F be a field extension such that $\operatorname{ind}(A_K) = 2$, so that $A_K = M_n(Q)$ for some quaternion division algebra Q over K and $G_K = \mathbf{SL}_n(Q)$. Set $H = \mathbf{SL}_1(Q)$ and $X = G_K/H$. By loc. cit., the generic fibre of the projection $G_K \to X$ is H_E, with $E = K(X)$. We then have a commutative diagram

$$\begin{array}{ccccc}
A^0(G, H^4(3)) & \longrightarrow & A^0(G_K, H^4(3)) & \longrightarrow & A^0(H_E, H^4(3)) \\
{\scriptstyle \alpha'}\downarrow & & {\scriptstyle \alpha'}\downarrow & & {\scriptstyle \alpha'=0}\downarrow \\
CH^3(G) & \longrightarrow & CH^3(G_K) & \longrightarrow & CH^3(H_E)
\end{array}$$

and the bottom horizontal maps are isomorphisms by loc. cit. (see [35, Rk 4.4]). □

From now on, we suppose G of inner type A_r with $r \geq 2$, *i.e.* $\deg(A) > 2$ if $G = \mathbf{SL}_1(A)$. Then $H^{-1}(F, c^3(G_s))$ is canonically isomorphic to $\mathbf{Z}$, $H^2(F, \mathbb{G}_m \otimes \mathbf{S}^2(T_s^*)^W) \simeq Br(F)$ and $H^0(F, c^3(G_s)) = 0$. For the reader's convenience, let us redraw Diagram (9.7) in this case, taking Proposition 10.2 into account:

$$
\begin{tikzcd}
 & & 0 \arrow[d] & & \\
0 \arrow[r] & A^2(G,K_3^M) \arrow[r] & \widetilde{H}^5(G,\mathbf{Z}(3)) \arrow[r] \arrow[d] & \widetilde{A}^0(G,H^4(3)) \arrow[r] & 0 \\
 & & \mathbf{Z} \arrow[d, "\tilde{d}_2^{2,3}(G,3)"'] & & \\
 & & Br(F) \arrow[d] & & \\
 & & \widetilde{H}^6(G,\mathbf{Z}(3)) \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}
\tag{10.1}
$$

Since $A^0(G,H^4(3))$ and $Br(F)$ are torsion, we recover Merkurjev's result that $A^2(G,K_3^M)=A^2(G,K_3)$ is infinite cyclic [35, Lemma 5.7]. We also find

10.3. Theorem. *The group $\widetilde{H}^5(G,\mathbf{Z}(3))$ is infinite cyclic and the group $\widetilde{A}^0(G,H^4(3))$ is cyclic of order $(\widetilde{H}^5(G,\mathbf{Z}(3)):A^2(G,K_3^M))$.*

10.4. Definition. Let $G=\mathbf{SL}_1(A)$. We denote by c_A the "positive" generator of $\widetilde{H}^5(G,\mathbf{Z}(3))\subset H^5(G,\mathbf{Z}(3))$, that is, the generator that maps to a positive multiple of $1\in\mathbf{Z}$, and by $\bar{c}_A$ its image in $\widetilde{A}^0(G,H^4(3))\subset A^0(G,H^4(3))$ ($\bar{c}_A$ generates $\widetilde{A}^0(G,H^4(3))$).

10.5. Lemma. *Let still $G=\mathbf{SL}_1(A)$, and let $p_1,p_2,\mu: G\times_F G\to G$ be repectively the first projection, the second projection and the multiplication map. Then*

$$\mu^* c_A = p_1^* c_A + p_2^* c_A.$$

Proof. Since $\widetilde{H}^5(G,\mathbf{Z}(3))\to H^{-1}(F,c^3(G_s))$ is injective for any group G, it is sufficient to show that the maps μ^* and $p_1^*+p_2^*$ from $c^3(G_s)$ to $c^3(G_s\times_{F_s}G_s)$ are equal.[3]

The Künneth formula (9.9) gives an isomorphism

$$c^3(G_s)\oplus c^3(G_s)\xrightarrow{\sim} c^3(G_s\times_{F_s}G_s)$$

induced by $p_1^*\oplus p_2^*$, since $c^1(G_s)=0$.

Let $C=c^3(G_s)$. The inclusion $\iota_1: G\times\{1\}\to G\times G$ induces a map $\iota_1^*: C\oplus C\to C$; since $p_1\circ\iota_1=Id$ and $p_1\circ\iota_1$ is the trivial map, ι_1^* is the first

[3]Note that morphisms between reductive groups preserving the unit sections act on the spectral sequences (9.2) by preserving the spectral sequences $\widetilde{E}_r^{p,q}$. This applies to μ and to the maps ι_1 and ι_2 further below.

projection. Similarly, $\iota_2 : \{1\} \times G \to G \times G$ induces the second projection. We conclude that $\mu^* : C \to C \oplus C$ is the diagonal map, using the left and right unit formulas $\mu \circ \iota_1 = \mu \circ \iota_2 = Id$. □

Let X be a smooth F-variety. To any morphism $f : X \to \mathbf{SL}_1(A)$, we associate the pull-back of c_A:

$$c_A(f) = f^* c_A \in H^5(X, \mathbf{Z}(3)).$$

Lemma 10.5 shows that we have

$$c_A(fg) = c_A(f) + c_A(g)$$

for two maps f, g, where $fg := \mu \circ (f, g)$.
Recall that $\deg(A) > 2$. Consider the embedding $\iota_n : \mathbf{SL}_1(A) \hookrightarrow \mathbf{SL}_n(A)$ given by $u \mapsto \left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Noting that $\mathbf{SL}_n(A) = \mathbf{SL}_1(M_n(A))$, Lemma 9.6 shows that

$$c_{M_n(A)}(\iota_n) = c_A.$$

In particular, $\iota_n^* : \widetilde{H}^5(\mathbf{SL}_n(A), \mathbf{Z}(3)) \to \widetilde{H}^5(\mathbf{SL}_1(A), \mathbf{Z}(3))$ is an isomorphism. So, if f is a morphism from X to $\mathbf{SL}_n(A)$, we may define $c_A(f) = (\iota_n^*)^{-1} c_{M_n(A)}(f)$, and this definition is "stable". We record this as:

10.6. PROPOSITION. *If* $\deg(A) > 2$*, the maps*

$$\widetilde{H}^5(\mathbf{SL}_n(A), \mathbf{Z}(3)) \to \widetilde{H}^5(\mathbf{SL}_1(A), \mathbf{Z}(3))$$
$$\widetilde{A}^0(\mathbf{SL}_n(A), H^5(\mathbf{Z}(3))) \to \widetilde{A}^0(\mathbf{SL}_1(A), H^5(\mathbf{Z}(3)))$$

induced by the inclusion $\mathbf{SL}_1(A) \hookrightarrow \mathbf{SL}_n(A)$ *are isomorphisms.* □

In particular, suppose $X = \operatorname{Spec} R$ affine. Then $\operatorname{Hom}_F(X, \mathbf{SL}_n(A)) = SL_n(A \otimes_F R)$. Define $SL(A \otimes_F R) = \varinjlim SL_n(A \otimes_F R)$ as usual, and

$$SK_1(X, A) = SL(A \otimes_F R)^{\mathrm{ab}}.$$

For X smooth in general, we may similarly define

$$SL(X, A) = \varinjlim \operatorname{Hom}_F(X, \mathbf{SL}_n(A)), \quad SK_1(X, A) = SL(X, A)^{\mathrm{ab}}.$$

The above discussion then yields a homomorphism

$$SK_1(X, A) \to H^5(X, \mathbf{Z}(3)) \tag{10.2}$$

which is contravariant in X.
In particular, for $X = \operatorname{Spec} L$, L/F a function field, we get a homomorphism

$$\bar{c}_A(L) : SK_1(A_L) \to H^5(L, \mathbf{Z}(3)) \xleftarrow{\sim} H^4(L, 3). \tag{10.3}$$

The following theorem was (embarrassingly) pointed out by Philippe Gille, whom I thank here.

10.7. THEOREM. *In* (10.3)*,* $L \mapsto \bar{c}_A(L)$ *defines the universal invariant of* $\mathbf{SL}_1(A)$ *of degree* 4 *with values in* $H^4(3)$*, in the sense of Merkurjev* [35, Def. 2.1].

Proof. Let G be an algebraic group and let M be a cycle module of bounded exponent as in [35, p. 133]. By [35, Th. 2.3], we have an isomorphism

$$\mathrm{Inv}^d(G, M) \xrightarrow{\sim} A^0(G, M_d)_{\mathrm{mult}}, \quad d \in \mathbf{Z}$$

induced by evaluation on the generic point of G, where the left group is the group of invariants of G with values in M_d as in [35, Def. 2.1] and the right group is the multipicative part of $A^0(G, M_d)$ as in [35, 1.3].
We cannot apply this directly to $M_d(K) = H^d(K, d-1)$, which is not of bounded exponent. However, any cycle module M_* such that $M_d(K)$ is torsion prime to $\operatorname{char} F$ for any $d \in \mathbf{Z}$ and any function field K/F may be written as the filtering direct limit of its torsion sub-cycle modules ${}_mM_*$, $m \geq 1$. Then the maps

$$\varinjlim \mathrm{Inv}^d(G, {}_mM) \to \mathrm{Inv}^d(G, M)_{\mathrm{tors}}$$
$$\varinjlim A^0(G, {}_mM_d)_{\mathrm{mult}} \to A^0(G, M_d)_{\mathrm{mult}}$$

are bijective, so that [35, Th. 2.3] extends to an isomorphism

$$\mathrm{Inv}^d(G, M)_{\mathrm{tors}} \xrightarrow{\sim} A^0(G, M_d)_{\mathrm{mult}} \tag{10.4}$$

for any torsion cycle module M (excluding the characteristic of F) and any $d \in \mathbf{Z}$.
In the case $G = \mathbf{SL}_1(A)$, any invariant of G, evaluated at a function field K, factors through $G(K)^{\mathrm{ab}} = SK_1(A_K)$, which is of exponent bounded by $\mathrm{ind}(A)$ (see introduction), so any invariant is a torsion invariant.
(This argument extends to any simply connected semisimple group G by [35, Cor. 2.6] and a transfer argument. On the other hand, $\mathrm{Inv}^1(\mathbb{G}_m, K_*^M) = \mathbf{Z}$ as the construction of [35, beg. of 2.3] shows.)
Thanks to Theorem 10.3, the only thing which remains to be proven is that $\widetilde{A}^0(G, H^4(3)) = A^0(G, H^4(3))_{\mathrm{mult}}$ (notation as in [35, 1.3]): this follows from Lemma 10.5. □

10.8. *Remark.* The above proof yields a little more: if $eSK_1(A_K) = 0$ for all K/F, then $e\,\mathrm{Inv}^d(\mathbf{SL}_1(A), M) = 0$ for any cycle module M and any $d \in \mathbf{Z}$. In particular, $e\widetilde{A}^0(G, H^4(3)) = 0$. This will be amplified in Lemma 10.13 below.

A delicate issue is the behaviour of c_A and $\bar{c}_A$ under extension of scalars: in other words, the universal invariant of Theorem 10.7 might cease to be universal after extending the base field. This is directly related to the differential $\tilde{d}_2^{2,3}(G, 3)$ in Diagram (10.1). Here is at least one case where this does not happen:

10.9. LEMMA. *Let L/F be an extension such that* $\exp(A_L) = \exp(A)$. *Then*

$$\widetilde{H}^5(G, \mathbf{Z}(3)) \xrightarrow{\sim} \widetilde{H}^5(G_L, \mathbf{Z}(3))$$
$$\widetilde{A}^0(G, H^5(\mathbf{Z}(3))) \twoheadrightarrow \widetilde{A}^0(G_L, H^5(\mathbf{Z}(3))).$$

In particular, the image of c_A (resp. $\bar{c}_A$) under extension of scalars equals c_{A_L} (resp. $\bar{c}_{A_L}$).

Proof. We shall show in Corollary 11.3 that $\widetilde{d}_2^{2,3}(G,3)(1)$ is a multiple of $[A] \in Br(F)$. The claim then follows from a diagram chase with (10.1). □

The following corollary to Theorem 10.7 is a special case of [60, Prop. 4.1].

10.10. Corollary. *Assume A of degree $d = l^n$ (l prime) and of exponent $\varepsilon < d$. Let r be such that $\varepsilon \mid r \mid d/l$. Then there is an integer $m(A,r)$ such that*

$$\sigma_r^1 = m(A,r)\bar{c}_A$$

where σ_r^1 is the invariant in (7.3) *(see § 7.D).* □

As in [60, proof of Prop. 4.3], one might learn more on $m(A,r)$ by considering the generic algebra of degree d and exponent ε. We shall content ourselves with

10.11. Proposition (*cf.* [34]). *For $\varepsilon = 2 < \operatorname{ind}(A)$, $\widetilde{A}^0(\mathbf{SL}_1(A), H^4(3)) \neq 0$ and $m(A,2)$ is odd.*

Proof. 1) If A is a biquaternion algebra, the Rost invariant of Theorem 2 is nontrivial [34, proof of Cor.] and, by Remark 10.8 and the remark after Theorem 6 in the introduction, $\widetilde{A}^0(\mathbf{SL}_1(A), H^4(3))$ is cyclic of order ≤ 2. Hence this group is cyclic of order 2 and the Rost invariant coincides with $\bar{c}_A$ (recovering [35, Th. 5.4]). Thus $m(A,2) = 1$ in this case.
2) If $\operatorname{ind}(A) = 4$, let D be the division algebra similar to A, so that $A = M_n(D)$ for some $n \geq 1$. By Morita invariance of algebraic K-theory, the invariant σ_2^1 is the same for A and D. On the other hand, Proposition 10.6 yields an isomorphism

$$\widetilde{A}^0(\mathbf{SL}_1(A), H^4(3)) \xrightarrow{\sim} \widetilde{A}^0(\mathbf{SL}_1(D), H^4(3))$$

so 1) extends to this case.
3) In general, let $L = F(SB(4,A))$, so that $\operatorname{ind}(A_L) = 4$. By 2), $\bar{c}_{A_L} \neq 0$, hence $\bar{c}_A \neq 0$ by Lemma 10.9. Since σ_r^1 commutes with any extension of scalars by construction, we have $m(A,2) = m(A_L,2)$ in $\mathbf{Z}/2$, which shows that $m(A,2)$ is odd. □

We shall show in Corollary 11.12 that actually $\widetilde{A}^0(\mathbf{SL}_1(A), H^4(3)) \simeq \mathbf{Z}/2$ in Proposition 10.11.

10.12. *Remark.* Let r be a divisor of $d = \deg(A)$. Let us write $H^4(3)/r[A]$ for the degree 4 part of the cycle module given by

$$K \mapsto H^n(K, n-1)/r[A] := \operatorname{Coker}(H^{n-2}(K, \mu_r^{\otimes n-2}) \xrightarrow{\cdot r[A]} H^n(K, \mathbf{Q}/\mathbf{Z}(n-1))).$$

It is tempting to conjecture that the map

$$A^0(\mathbf{SL}_1(A), H^4(3))_{\text{mult}} \to A^0(\mathbf{SL}_1(A), H^4(3)/r[A])_{\text{mult}}$$

is surjective, which would provide a relationship between the invariant c_A and the invariant σ_r^1 of Corollary 7.2 in general.[4] However, since $A^0(-)_{\text{mult}}$ is left

[4] Since this was written, Wouters has resolved this question in the negative, [60, Prop. 4.2].

exact rather than right exact, this does not look straightforward at all. A description of the kernel of cup-product with $r[A]$ seems a major issue to solve (*cf.* (3.3)).

10.B. The K-theoretic generic element. In the universal case $X = \mathbf{SL}_1(A)$, we may write $SK_1(\mathbf{SL}_1(A), A) = SK_1(A) \oplus \widetilde{SK}_1(\mathbf{SL}_1(A), A)$ using the unit section of $\mathbf{SL}_1(A)$. The induced morphism

$$\widetilde{SK}_1(\mathbf{SL}_1(A), A) \to \widetilde{H}^5(\mathbf{SL}_1(A), \mathbf{Z}(3))$$

is surjective, hence split surjective since $\widetilde{H}^5(\mathbf{SL}_1(A), \mathbf{Z}(3)) = \mathbf{Z}$. An explicit splitting sends c_A to the class of the inclusions $\iota_n : \mathbf{SL}_1(A) \hookrightarrow \mathbf{SL}_n(A)$.

10.13. Lemma. *a) For any smooth F-variety Y, the map*

$$H^0(Y, SK_1(\mathcal{O}_Y \otimes_F A)) \to SK_1(F(Y) \otimes_F A)$$

is surjective; the image of $c_{F(Y)\otimes_F A}$ is contained in $A^0(Y, H^4(3))$.
b) For $Y = \mathbf{SL}_1(A)$ and $K = F(Y)$, the map c_{A_K} induces a surjection

$$SK_1(A_K)/SK_1(A) \twoheadrightarrow \widetilde{A}^0(\mathbf{SL}_1(A), H^4(3)) \tag{10.5}$$

sending the generic element to $\bar{c}_A$.

Proof. The first assertion of a) is classical (Rost, *cf.* [6, p. 38]), and the second one follows from this and the construction of c_A. For b), let $\eta = \operatorname{Spec} K$ be the generic point of $\mathbf{SL}_1(A)$. It defines an element $\bar{\eta} \in SK_1(A_K)$: the *generic element.* By construction, we have

$$c_{A_K}(\bar{\eta}) = \bar{c}_A$$

from which b) follows. □

We want to better understand the map (10.5). This is possible if A is biquaternion:

10.14. Theorem. *If A is a biquaternion algebra,* (10.5) *is an isomorphism and both sides are isomorphic to $\mathbf{Z}/2$.*

Proof. By Lemma 10.9 and Proposition 10.11, we have a commutative diagram of injections

$$\begin{array}{ccccc}
0 & \longrightarrow & SK_1(A) & \xrightarrow{\bar{c}_A} & H^5(F, \mathbf{Z}(3)) \\
 & & {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow \\
0 & \longrightarrow & SK_1(A_K) & \xrightarrow{\bar{c}_{A_K}} & H^5(K, \mathbf{Z}(3)).
\end{array}$$

Since $\mathbf{SL}_1(A)$ has a rational point, a and b have compatible retractions and this diagram induces a third injection

$$0 \longrightarrow SK_1(A_K)/SK_1(A) \longrightarrow H^5(K, \mathbf{Z}(3))/H^5(F, \mathbf{Z}(3)) \tag{10.6}$$

which is obviously also induced by (10.5). This proves the first claim. The second one follows from [35, Th. 5.4] (or part 1) of the proof of Proposition 10.11). □

10.15. COROLLARY. *If* $\operatorname{ind}(A) = 4$, *then* $SK_1(A_K)/SK_1(A) \simeq \mathbf{Z}/2$.

Proof. If A is biquaternion, this follows from Theorem 10.14. In general, let $L = F(SB(A^{\otimes 2}))$. By [35, Prop. 6.3], the maps $SK_1(A) \to SK_1(A_L)$ and $SK_1(A_K) \to SK_1(A_{KL})$ are isomorphisms, so we are reduced to the biquaternion case. □

In an earlier version of this paper, I had conjectured that (10.5) is always an isomorphism. In the light of the proof of Theorem 10.14, this seems a bit optimistic unless all primes factors of $\operatorname{ind}(A)$ occur at most with exponent 2. In general, a computation of $c^q(\mathbf{SL}_1(A))$ for all $q > 1$ will yield higher cohomological invariants for $SK_1(A)$. A still optimistic but more reasonable conjecture is that these future invariants will detect all of $SK_1(A)$. Based on this expectation, we propose

10.16. CONJECTURE. *If* $K = F(\mathbf{SL}_1(A))$, *the group* $SK_1(A_K)/SK_1(A)$ *is cyclic, generated by the generic element.*

10.17. *Remark.* The homomorphism

$$c_A : \operatorname{Hom}(\mathbf{SL}_1(A), \mathbf{SL}_1(A)) \to H^5(\mathbf{SL}_1(A), \mathbf{Z}(3))$$

also behaves well with respect to composition: for $f \in \operatorname{Hom}(\mathbf{SL}_1(A), \mathbf{SL}_1(A))$, we have $c_A(f) \in \widetilde{H}^5(\mathbf{SL}_1(A), \mathbf{Z}(3))$ if and only if $f(1) = 1$. If this is the case, set $c_A(f) = n(f)c_A$. Then, clearly, $n(g \circ f) = n(g)n(f)$. Can one describe this "degree" map in a more naïve fashion?

11. SOME COMPUTATIONS

We now try and evaluate the groups $SK_1(A_K)/SK_1(A)$, where K is the function field of $\mathbf{SL}_1(A)$, and $\widetilde{A}^0(\mathbf{SL}_1(A), H^4(3))$: our main results in this direction are Theorem 11.9 and Corollaries 11.10 and 11.12, the latter completing the proof of Theorem E. Unfortunately we are not able to prove the nontriviality of either of these groups when $\operatorname{ind}(A)$ is odd (not squarefree) by the present methods.
We assume that $n = \deg(A)$ is of the form l^m, l prime.

11.A. COMPARING SOME QUOTIENTS. First we have already noted:

11.1. LEMMA. $|\widetilde{A}^0(\mathbf{SL}_1(A), H^4(3))| \leq \operatorname{ind}(A)/l$.

Proof. This follows from Lemma 10.13 b) and the fact that $SK_1(A_K)$ has exponent $\leq \operatorname{ind}(A)/l$. □

See Corollary 11.12 for a refinement of this lemma when A is of exponent l.
Let $G = \mathbf{SL}_1(A)$. We note the isomorphisms

$$A^2(G, K_3^M) \xrightarrow{\sim} A^2(G, K_3)$$
$$K_2(F) \xrightarrow{\sim} A^0(G, K_2).$$

The first one is trivial and the second one is [8, Cor. B.3]. By the second one, the BGQ spectral sequence yields an injection

$$K_1(G)^{(2/3)} \hookrightarrow A^2(G, K_3). \tag{11.1}$$

11.2. Proposition. *If G is split, with $r \geq 2$, the maps $\widetilde{H}^5(G, \mathbf{Z}(3)) \to \mathbf{Z}$ and $A^2(G, K_3^M) \to \widetilde{H}^5(G, \mathbf{Z}(3))$ from* (10.1) *are both bijective. The same is true of the map* (11.1).

Proof. Mixing the coniveau spectral sequence for *Nisnevich* motivic cohomology with the slice spectral sequence (9.1) (also for Nisnevich motivic cohomology) yields a diagram similar to (10.1) and mapping to it:

$$\begin{tikzcd} A^2(G, K_3^M) \arrow[r, "\sim"] & \widetilde{H}^5_{\mathrm{Zar}}(G, \mathbf{Z}(3)) \arrow[d, "\wr"] \\ & \mathbf{Z} \end{tikzcd} \tag{11.2}$$

This proves the first two claims of Proposition 11.2 at once. For the last one, we notice that if G is split then all its Chow groups are 0 by Lemma 9.2, hence all differentials leaving from $A^2(G, K_3)$ in the BGQ spectral sequence vanish. □

Note that the horizontal map in (11.2) is an isomorphism for any G, whether split or not.

11.3. Corollary. *In Diagram* (10.1) *for $G = \mathbf{SL}_1(A)$, we have*

$$\widetilde{d}_2^{2,3}(G, 3)(1) = t[A]$$

for some integer t, where $[A]$ is the class of A in $Br(F)$. In particular, $(\mathbf{Z} : \widetilde{H}^5(G, \mathbf{Z}(3)))$ divides the exponent of $[A]$.

Proof. Let K be the function field of the Severi-Brauer variety of A. Then A splits over K. The first statement now follows from Proposition 11.2 and Amitsur's theorem [1] that $\mathrm{Ker}(Br(F) \to Br(K)) = \langle [A] \rangle$. The second statement is obvious. □

11.4. Corollary. *In general,*

$$(\mathbf{Z} : A^2(G, K_3^M)) = (A^2(G_s, K_3^M) : A^2(G, K_3^M)) \\ \mid (K_1(G_s)^{(2/3)} : K_1(G)^{(2/3)}).$$

Proof. This follows immediately from Proposition 11.2. □

The following diagram is a little more precise and may be helpful to the reader ($G = \mathbf{SL}_1(A)$):

$$\begin{array}{cccc} \dfrac{SK_1(A_K)}{SK_1(A)} \twoheadrightarrow \widetilde{A}^0(G, H^4(3)) & & \dfrac{K_1(G_s)^{(2/3)}}{K_1(G)^{(2/3)}} & \\ \wr\uparrow & & \text{onto}\downarrow & \\ 0 \to \dfrac{\widetilde{H}^5(G, \mathbf{Z}(3))}{A^2(G, K_3^M)} & \to & \mathbf{Z}/A^2(G, K_3^M) \to \mathbf{Z}/t\mathbf{Z} \to 0 \end{array} \tag{11.3}$$

where t is as in Corollary 11.3.

11.B. THE MAP $Br(F) \to \widetilde{H}^6(G, \mathbf{Z}(3))$. In order to better understand the differential $\widetilde{d}_2^{2,3}(G, 3)$ in the future, we note:

11.5. PROPOSITION. *Let $G = \mathbf{SL}_1(A)$.*
a) We have an exact sequence

$$0 \to A^1(G, H^4(3)) \to \widetilde{H}^6(G, \mathbf{Z}(3))/CH^3(G) \to \widetilde{A}^0(G, H^5(3)).$$

b) The composition

$$Br(F) \to \widetilde{H}^6(G, \mathbf{Z}(3)) \to \widetilde{H}^6(G, \mathbf{Z}(3))/CH^3(G) \to \widetilde{A}^0(G, H^5(3))$$

from Diagram (10.1) *is* 0*, and so is the map* $\widetilde{H}^6(G, \mathbf{Z}(3))/CH^3(G) \to \widetilde{A}^0(G, H^5(3))$. *Hence we have in fact an exact sequence*

$$0 \to CH^3(G) \to \widetilde{H}^6(G, \mathbf{Z}(3)) \to A^1(G, H^4(3)) \to 0.$$

Proof. a) follows from the coniveau spectral sequence for the étale motivic cohomology of G. b) The second vanishing follows from the first, since $Br(F) \to \widetilde{H}^6(G, \mathbf{Z}(3))$ is surjective. For the first vanishing, given the definition of the homomorphism $Br(F) \to \widetilde{H}^6(G, \mathbf{Z}(3))$, it suffices to show that the map $\alpha^* c_i(V) \to \alpha^* c_i(G)$ induces 0 on homology sheaves for $i = 1, 2, 3$ if V is a suitable open subset V of G.
Let B be a Borel subgroup containing $T_s \subset G_s$. Consider the big cell $\bar{U}_0 \subset G_s/B$: it is an affine space, hence all its Chow groups are 0. Observe that U_0 is defined over a finite extension of F, hence it has only a finite number of Galois conjugates: then their intersection $\bar{U}$ is defined over F, and its geometric Chow groups are still 0. Let U be the inverse image of $\bar{U}$ in Y_s: then U is defined over F and all its geometric Chow groups are 0. Hence, for all $p > 0$, the étale complex $\alpha^* c_p(U)$ is concentrated in degrees < 0.
We now take for V the inverse image of U (viewed as an open subset of Y) in G. As in [14, Prop. 9.3], we have for all $N \geq 0$ a spectral sequence

$$E_1^{p,q}(V_s) = H^q(c_{N-p}(U_s)) \otimes \Lambda^p(T_s^*) \Rightarrow H^{p+q}(c_N(V_s))$$

which maps to the corresponding spectral sequence $E_r^{p,q}(G_s)$ for G_s (that yields the complexes (9.3)). For $N > 0$, we have $E_1^{p,q}(G_s) = 0$ for $q \neq 0$ and $E_1^{p,q}(V_s) = 0$ for $q = 0$, hence all maps $H^i(c_N(V_s)) \to H^i(c_N(G_s))$ are 0. This completes the proof of b). □

11.C. A CHERN CLASS COMPUTATION. We use Gillet's convention for higher Chern classes [12].

11.6. LEMMA. *For a smooth variety X, consider the higher Chern class*

$$c_{3,1} : K_1(X) \to A^2(X, K_3).$$

Then $2d_2^{0,-2} = 0$ and the diagram

$$\begin{tikzcd} K_1(X)^{(2)} \arrow[r, "c_{3,1}"] \arrow[d] & A^2(X,K_3) & A^2(X,K_3)/d_2^{0,-2}A^0(X,K_2) \arrow[l, "2"'] \\ K_1(X)^{(2/3)} \arrow[rr, "\sim"] & & E_\infty^{2,-3} \arrow[u] \end{tikzcd}$$

commutes, where $d_2^{0,-2}$ and $E_\infty^{2,-3}$ are relative to the BGQ spectral sequence for X.

Proof. The BGQ spectral sequence for X may be considered as the coniveau spectral sequence for X relative to algebraic K-theory. For a given $i \geq 0$, consider the corresponding coniveau spectral sequence $'E_r^{p,q}$ relative to $U \mapsto H^*(U, \mathcal{K}_i)$ (for U running through open subsets of X). By [12, pp. 239–240], the i-th Chern class C_i defines a morphism of spectral sequences $E_r^{p,q} \to {}'E_r^{p,q}$ ($r \geq 1$) converging to the higher Chern classes $c_{i,-p-q} : K_{-p-q}(X) \to H^{p+q+i}(X, \mathcal{K}_i)$.
The group $'E_1^{p,q}$ is 0 for $q \neq -i$ and $'E_1^{p,-i} = \bigoplus_{x \in X^{(p)}} K_{i-p}(F(x))$. Hence $'E_2^{p,q} = 0$ for $q \neq -i$ and $'E_2^{p,-i} = H^p(X, \mathcal{K}_i) = {}'E_\infty^{p,-i}$. By [12, Th. 3.9], the map from $E_1^{p,-i}$ to $'E_1^{p,-i}$ induced by C_i equals $\frac{(-1)^p(i-1)!}{(i-p-1)!} c_{i-p,i-p}$ on each summand $K_{i-p}(F(x))$. In particular, for $i = 3$, $c_{1,1}$ is the identity for fields and we get a commutative diagram

$$\begin{tikzcd} E_2^{0,-2} \arrow[r, "d_2^{0,-2}"] \arrow[d] & E_2^{2,-3} \arrow[d, "2"] \\ 0 \arrow[r] & {}'E_2^{2,-3} = E_2^{2,-3} \end{tikzcd}$$

which proves the first claim of the lemma; the second one follows from the morphism of spectral sequences. □

11.D. SOME COMPUTATIONS, CONTINUED. The group $A^1(G, H^4(3))$ of Proposition 11.5 is mysterious and would require a further analysis: we shall refrain from starting it in this paper and will concentrate on computing the index $(K_1(G_s)^{(2/3)} : K_1(G)^{(2/3)})$, which can be done in some interesting cases.
For this, we may try and look at the map $K_1(G) \to K_1(G_s)$ and use the results of Levine [25] and Suslin [48]. In particular, we have an isomorphism [25, Th. 4.3]

$$K_1(G) \simeq K_1(F) \oplus \bigoplus_{i=1}^{r} K_0(A^{\otimes i})$$

where $r = \operatorname{rk} G = \deg A - 1$. If G (equivalently A) is split, the summand $K_0(A^{\otimes i}) \simeq \mathbf{Z}$ is generated by the class of $\Lambda^i(\rho_r)$, where ρ_r is the standard representation of $G = \mathbf{SL}_{r+1}$ into $\mathbf{GL}_{r+1}$. While Levine thinks of ρ_r as a representation, Suslin thinks of it as the generic matrix and denotes it by α_{r+1}: the two viewpoints are of course equivalent.

If we pass to the separable closure, we get a commutative diagram

$$\begin{array}{ccc} K_1(G)^{(2/3)} & \xrightarrow{\gamma_3} & A^2(G, K_3) \\ \downarrow & & \downarrow \\ K_1(G_s)^{(2/3)} & \xrightarrow{\gamma_3} & A^2(G_s, K_3) \simeq \mathbf{Z}. \end{array}$$

11.7. Lemma. *Suppose $G = \mathbf{SL}_n$, with $n = r+1$.*
a) All $[\Lambda^i(\rho_r)]$ belong to $K_1(G)^{(1)}$ and the image of $[\Lambda^i(\rho_r)]$ in $A^1(G, K_2) = \mathbf{Z}$ is $\binom{n-2}{i-1}$.
b) For all i, $[\Lambda^i(\rho_r)] - \binom{n-2}{i-1}[\rho_r] \in K_1(G)^{(2)}$ and its image in $A^2(G, K_3) = \mathbf{Z}$ is $\binom{n-3}{i-2}$.

Proof. (It may not be the most direct, but it works.) For the first assertion of a), we need to show that $[\Lambda^i(\rho_r)]_{|F(SL_n))} = 0$ or, which amounts to the same, that $\Lambda^i(\alpha_n)$ is a product of commutators, where α_n is the generic matrix with determinant 1. For this, it suffices to see that $\det \Lambda^i(\alpha_n) = 1$. But, for any matrix u, $\det \Lambda^i(u)$ is a certain power of $\det(u)$, hence the claim.

For the second assertion of a) and for b), we first do a Chern class computation. Let $\bar\gamma_j = \gamma_j([\rho_r]) = \gamma_j([\alpha_n])$, where γ_j is the j-th gamma operation in K-theory. Note the formula (*cf.* [48, p. 65])

$$\sum [\Lambda^i(\alpha_n)]u^i = \sum \bar\gamma_i u^i (1+u)^{n-i}.$$

Also, from [46, 1.3.4 a) p. 277 and Remark p. 297] (see also [45, IV.6]), we find

$$c_{2,1}(\bar\gamma_j) = \begin{cases} 0 & \text{for } j > 2 \\ -c_{2,1}(\alpha_n) & \text{for } j = 2 \\ c_{2,1}(\alpha_n) & \text{for } j = 1 \end{cases}$$

and

$$c_{3,1}(\bar\gamma_j) = \begin{cases} 0 & \text{for } j > 3 \\ 2c_{3,1}(\alpha_n) & \text{for } j = 3 \\ -3c_{3,1}(\alpha_n) & \text{for } j = 2 \\ c_{3,1}(\alpha_n) & \text{for } j = 1 \end{cases}$$

from which we deduce

$$(11.4) \quad \sum c_{2,1}([\Lambda^i(\alpha_n)])u^i = c_{2,1}(\alpha_n)(u(1+u)^{n-1} - u^2(1+u)^{n-2})$$
$$= c_{2,1}(\alpha_n)u(1+u)^{n-2} = c_{2,1}(\alpha_n)\sum \binom{n-2}{i-1} u^i =: c_{2,1}(\alpha_n)\varphi(u)$$

and

$$\sum c_{3,1}([\Lambda^i(\alpha_n)])u^i$$
$$= c_{3,1}(\alpha_n)(u(1+u)^{n-1} - 3u^2(1+u)^{n-2} + 2u^3(1+u)^{n-3})$$
$$= c_{3,1}(\alpha_n)u(1+u)^{n-3}(1+u)$$

hence

$$\sum c_{3,1}([\Lambda^i(\alpha_n)])u^i - c_{3,1}(\alpha_n)\varphi(u) = -2c_{3,1}(\alpha_n)u^2(1+u)^{n-3} = -2c_{3,1}(\alpha_n)\sum \binom{n-3}{i-2}u^i. \tag{11.5}$$

We now use the fact that, for $i \geq 1$, $A^i(\mathbf{SL}_n, K_{i+1})$ is generated by $c_{i+1,1}([\alpha_n])$ [48, Th. 2.9]. By an analogue of Lemma 11.6, the edge homomorphism $K_1(X)^{(1)} \to A^1(X, K_2)$ of the BGQ spectral sequence coincides with $-c_{2,1}$ for any smooth variety X. With (11.4), this proves the second part of a) and the first part of b). Then the second part of b) follows from Lemma 11.6 and (11.5). □

Let G not be necessarily split anymore. Let e_i be the positive generator of the summand $K_0(A^{\otimes i})$: $e_i \mapsto \operatorname{ind}(A^{\otimes i})[\Lambda^i(\alpha_n)]$. Lemma 11.7 shows that $\frac{\operatorname{ind}(A)}{\operatorname{ind}(A^{\otimes i})}e_i - \binom{n-2}{i-1}e_1 \in K_1(G)^{(2)}$ and that its image in $A^2(G_s, K_3) = \mathbf{Z}$ is $\operatorname{ind}(A)\binom{n-3}{i-2}$.

11.8. Lemma. $v_l(\binom{n-2}{i-1}) = v_l(i)$. *(Recall that $n = l^m$.)*

Proof. For an integer e, let $s_l(e)$ be the sum of the digits of e written in base l. It is well-known that

$$v_l\binom{a}{b} = \frac{s_l(b) + s_l(a-b) - s_l(a)}{l-1}.$$

Clearly, we have $s_l(l^m-2) = m(l-1)-1$. Let $t = v_l(i)$ and write $i-1 = \sum a_j l^j$, with $0 \leq a_j \leq l-1$, $a_j = l-1$ for $j < t$ and $a_t < l-1$. Then $l^m - i - 1 = \sum b_j l^j$ with $b_j = l-1$ for $j < t$, $b_t = l-2-a_t$ and $b_j = l-1-a_j$ for $t < j \leq m$. Hence

$$s_l(i-1) + s_l(l^m-i-1) - s_l(l^m-2) =$$
$$2t(l-1) + (m-t)(l-1) - 1 - (m(l-1)-1) = t(l-1).$$

□

11.9. Theorem. *We have*

$$(K_1(G_s)^{(2/3)} : K_1(G)^{(2/3)}) = \begin{cases} \inf(l^{2t}\operatorname{ind}(A^{\otimes l^t})) & \text{if } l > 2 \\ \inf(l^{2t-1}\operatorname{ind}(A^{\otimes l^t})) & \text{if } l = 2. \end{cases}$$

Proof. Since the index $(K_1(G_s)^{(2/3)} : K_1(G)^{(2/3)})$ *a priori* divides $\mathrm{ind}(A)$ (transfer argument), to evaluate it we may tensor both groups with $\mathbf{Z}_l$, as well as $A^2(G,K_3)$ and $A^2(G_s,K_3)$. Note also that, since $K_1(G)^{(1/2)} \hookrightarrow A^1(G,K_2) \simeq \mathbf{Z}$ is torsion-free, $x \in K_1(G)^{(1)} \otimes \mathbf{Z}_l$ and $mx \in K_1(G)^{(2)} \otimes \mathbf{Z}_l$ for some $m \in \mathbf{Z}_l - \{0\}$ implies $x \in K_1(G)^{(2)} \otimes \mathbf{Z}_l$. This will allow us to divide freely by l-units below.
By Lemma 11.7, we have

$$\frac{n}{\mathrm{ind}(A^{\otimes i})\binom{n-2}{i-1}} e_i - e_1 \mapsto n\frac{\binom{n-3}{i-2}}{\binom{n-2}{i-1}} = \frac{n(i-1)}{n-2} \tag{11.6}$$

under the composite map $K_1(G)^{(2/3)} \otimes \mathbf{Z}_l \to K_1(G_s)^{(2/3)} \otimes \mathbf{Z}_l \xrightarrow{\sim} A^2(G_s,K_3) \otimes \mathbf{Z}_l \xrightarrow{\sim} \mathbf{Z}_l$ (note that the coefficient of e_i is an l-integer by Lemma 11.8).
Let $x = \sum \lambda_i e_i \in K_1(G)^{(2)} \otimes \mathbf{Z}_l$ (with $\lambda_i \in \mathbf{Z}_l$). In $\mathbf{Q}_l$, write

$$\lambda_i = \mu_i \frac{n}{\mathrm{ind}(A^{\otimes i})\binom{n-2}{i-1}}$$

so that

$$x = \sum \mu_i \left(\frac{n}{\mathrm{ind}(A^{\otimes i})\binom{n-2}{i-1}} e_i - e_1\right) + \sum \mu_i e_1 \tag{11.7}$$

hence $x \in K_1(G)^{(2)} \otimes \mathbf{Z}_l$ if and only if $\sum \mu_i = 0$. Note that

$$x \mapsto \sum \mu_i \frac{n(i-1)}{n-2} = \sum \mu_i \frac{n(i-1)}{n-2} + \frac{n}{n-2}\sum \mu_i = \sum i\mu_i \frac{n}{n-2}.$$

Since $v_l(\mu_i) \geq -v_l\left(\frac{n}{\mathrm{ind}(A^{\otimes i})\binom{n-2}{i-1}}\right)$, we have

$$v_l(i\mu_i \frac{n}{n-2}) \geq \begin{cases} 2v_l(i) + v_l(\mathrm{ind}(A^{\otimes i})) & \text{if } l > 2 \\ 2v_l(i) + v_l(\mathrm{ind}(A^{\otimes i})) - 1 & \text{if } l = 2 \end{cases}$$

(see Lemma 11.8).
This proves the inequality $\geq$ in Theorem 11.9. To get equality, let $s = \inf\{t \mid l^{2t}\mathrm{ind}(A^{\otimes l^t}) \text{ is minimum}\}$. Suppose first that $l > 2$. Choose $\lambda_{l^s} = 1$, $\mu_{2l^s} = -\mu_{l^s}$ and $\lambda_i = 0$ otherwise, and we are done.
Suppose now that $l = 2$. We can then argue as above by taking $\mu_{3\cdot 2^s} = -\mu_{2^s}$ provided $3 \cdot 2^s < n = 2^m$, *i.e.* $s \leq m-2$; $s = m$ is clearly impossible and $s = m-1$ may occur only when $2^{2m-3}\mathrm{ind}(A^{\otimes 2^{m-1}}) < 2^m$, *i.e.* when $2^m\mathrm{ind}(A^{\otimes 2^{m-1}}) \leq 4$. This means $m = 1$ or $m = 2$, $\exp(A) = 2$. In the first case we clearly have equality. In the second one we may compute directly

$$2e_2 - e_1 \mapsto 2$$
$$e_3 - e_1 \mapsto 4$$

which shows that $(K_1(G_s)^{(2/3)} : K_1(G)^{(2/3)}) = 2$. So equality still holds in this case. ∎

11.10. Corollary. *a) If* $\mathrm{ind}(A) = \exp(A)$*, then* $(K_1(G_s)^{(2/3)} : K_1(G)^{(2/3)}) = \mathrm{ind}(A)$.
b) Suppose $\exp(A) = l$*. If* $l > 2$ *we have*

$$(K_1(G_s)^{(2/3)} : K_1(G)^{(2/3)}) = \begin{cases} l & \text{if } \mathrm{ind}(A) = l \\ l^2 & \text{if } \mathrm{ind}(A) > l \end{cases}$$

while if $l = 2$ *we always have* $(K_1(G_s)^{(2/3)} : K_1(G)^{(2/3)}) = 2$.

Proof. a) is obvious, since in this case necessarily $\mathrm{ind}(A^{\otimes l^t}) = l^{q-t}$ for all $t \leq q$, if $\mathrm{ind}(A) = l^q$. For b), we have $(K_1(G_s)^{(2/3)} : K_1(G)^{(2/3)}) = \inf(\mathrm{ind}(A), l^2)$ (for $l = 2$) or $\inf(\mathrm{ind}(A), 2)$ (for $l = 2$) and the result immediately follows. □

11.11. *Remark.* An easier computation gives $(K_1(G_s)^{(1/2)} : K_1(G)^{(1/2)}) = lcm(i \cdot \mathrm{ind}(A^{\otimes i})) = \mathrm{ind}(A)$. Since $A^1(G, K_2) \xrightarrow{\sim} A^1(G_s, K_2)$ [8, Cor. B.3], this yields $(A^1(G, K_2) : K_1(G)^{(1/2)}) = \mathrm{ind}(A)$.

The first part of the following corollary was (embarrassingly) pointed out by Wouters [60, 2.4 (c)]:

11.12. Corollary. *If A is of exponent l, then* $\widetilde{A}^0(\mathbf{SL}_1(A), H^4(3))$ *is cyclic of order dividing* 2 *if* $l = 2$ *and dividing* l^2 *if* $l > 2$*. If moreover* $l = 2$ *and* $\mathrm{ind}(A) > 2$*, then* $\widetilde{A}^0(\mathbf{SL}_1(A), H^4(3)) \simeq \mathbf{Z}/2$ *and the invariants* c_A *of Theorem 10.7 and* σ_2^1 *of* §*7.D coincide. In general*

$$|\widetilde{A}^0(\mathbf{SL}_1(A), H^4(3))| \leq \begin{cases} \exp(A)^2 & \text{if } l \text{ is odd} \\ \exp(A)^2/2 & \text{if } l = 2. \end{cases}$$

Proof. The first statement follows from Corollary 11.10, Diagram (11.3) and Theorem 10.3. The second one then follows from Proposition 10.11. The last one follows from taking $l^t = \exp(A)$ in Theorem 11.9. □

11.13. *Question.* Let l be odd. Is it true that $\widetilde{A}^0(\mathbf{SL}_1(A), H^4(3)) \simeq \mathbf{Z}/l$ if A is of exponent l and index $> l$?

Appendix A. A cancellation theorem over imperfect fields

A.1. Theorem. *Let F be a field and* $M, N \in DM_-^{\mathrm{eff}}(F)$ *where N is a mixed Tate motive (see* [14, Def. 4.1]*). Then the map* $- \otimes \mathbf{Z}(1)$ *induces an isomorphism*

$$\mathrm{Hom}_{DM}(M, N) \xrightarrow{\sim} \mathrm{Hom}_{DM}(M(1), N(1)).$$

Proof. It is enough to prove this for $M = C_*(X)[i]$, X a smooth variety and $i \in \mathbf{Z}$, and $N = \mathbf{Z}(n)$, $n \geq 0$. By [54, Prop. 3.2.3] and [55], the left hand side is functorially isomorphic to Bloch's higher Chow group $CH^n(X, 2n+i)$. By [30, Th. 15.12] (projective bundle formula in DM), the right hand side is a direct summand of $CH^{n+1}(X \times \mathbf{P}^1, 2n+2+i)$. By the projective bundle formula for higher Chow groups ([3, Th. 7.1], [24, Cor. 5.4]), the latter decomposes as a direct sum

$$CH^{n+1}(X \times \mathbf{P}^1, 2n+2+i) \simeq CH^{n+1}(X, 2n+2+i) \oplus CH^n(X, 2n+i).$$

Moreover, the constructions of the projective bundle isomorphisms in [30] and [3, 24] show that the latter two are compatible via the isomorphism between motivic cohomology and higher Chow groups in [55]. This proves the theorem. □

Theorem A.1 is sufficient to extend to imperfect fields the construction of the slice spectral sequences in the form of (9.1), *i.e.* for motivic cohomology computed in the Nisnevich topology (= Bloch's higher Chow groups). It is not sufficient, however, to obtain a version of the étale spectral sequences of (9.2) which is interesting at p, since p is automatically inverted in $DM^{\text{eff}}_{-,\text{ét}}(F)$ (see Remark 2.6). In order to achieve this, one may presumably proceed by working directly on Bloch's cycle complexes, as follows:
By the work of Geisser-Levine [9], the étale hypercohomology of Bloch's cycle complexes provides an interesting theory modulo p. The first thing to do is to find a version of the slice filtration directly on the cycle complexes of a given smooth F-variety X: this can be achieved by using the "homotopy coniveau filtration" (which is at the basis of the construction of the Bloch-Lichtenbaum spectral sequence), see [28] and [22, §4].
This will give spectral sequences comparable to those of Theorem 2.5 and (9.2). The issue is then to identify the E_2-terms. This can presumably be done by a slightly tedious imitation of the computations in [14] and §9, where the tediousness comes from the fact that one is limited to work with smooth varieties rather than general motives.
In the course of the computation, the following ingredients will certainly appear: étale versions of the localisation theorem for higher Chow groups (see e.g. the proof of [14, Prop. 4.11]) and of Bloch's projective bundle theorem. They should be obtained much as in [16, Th. 4.2 and Th. 5.1]. Hopefully a partial purity statement similar to [16, Th. 4.2] will be sufficient for the applications. We leave this programme to the interested reader.

References

[1] S. A. Amitsur, *Generic splitting fields of central simple algebras*, Ann. of Math. 68 (1955), 8–43.

[2] A. Bak, U. Rehmann *K_2-analogs of Hasse's norm theorem*, Comm. Math. Helv. 59 (1984), 1–11.

[3] S. Bloch *Algebraic cycles and higher K-theory*, Adv. in Math. 61 (1986), 267–304.

[4] N. Bourbaki Groupes et algèbres de Lie, Ch. 4, 5 et 6, Masson, 1981.

[5] B. Calmès *Le groupe SK_2 d'une algèbre de biquaternions*, J. reine angew. Mathematik 605 (2007), 193–235.

[6] J.-L. Colliot-Thélène *Quelques résultats de finitude pour le groupe SK_1 d'une algèbre de biquaternions*, K-Theory 10 (1996), 31–48.

[7] M. Demazure *Invariants symétriques du groupe de Weyl et torsion*, Invent. Math. 21 (1973), 287–301.

[8] H. Esnault, B. Kahn, M. Levine, E. Viehweg *The Arason invariant and mod* 2 *algebraic cycles*, J. Amer. Math. Soc. 11 (1998), 73–118.
[9] T. Geisser, M. Levine *The K-theory of fields in characteristic* p, Invent. Math. 139 (2000), 459–493.
[10] T. Geisser, M. Levine *The Bloch-Kato conjecture and a theorem of Suslin-Voevodsky*, J. reine angew. Math. 530 (2001), 55–103.
[11] S. Gille *The first Suslin homology group of a split simply connected semisimple algebraic group*, preprint, 2008 (rev. 2009).
[12] H. Gillet *Riemann-Roch theorems in higher algebraic K-theory*, Adv. Math. 40 (1981), 203–289.
[13] A. Grothendieck *Torsion homologique et sections rationnelles*, Exp. 5 *in* Anneaux de Chow et applications, Sém. Chevalley, Paris, 1958.
[14] A. Huber, B. Kahn *The slice filtration and mixed Tate motives*, Compositio Math. 142 (2006), 907–936.
[15] B. Kahn *Descente galoisienne et K_2 des corps de nombres*, K-theory 7 (1993), 55–100.
[16] B. Kahn *Applications of weight-two motivic cohomology*, Doc. Math. 1 (1996), 395–416.
[17] B. Kahn *The Quillen-Lichtenbaum conjecture at the prime* 2, `http://www.math.uiuc.edu/K-theory/0208/`.
[18] B. Kahn *Motivic cohomology of smooth geometrically cellular varieties*, Proc. Symp. Pure Math. 67 (1999), AMS, 149–174.
[19] B. Kahn *The Geisser-Levine method revisited and algebraic cycles over a finite field*, Math. Ann. 324 (2002), 581–617.
[20] B. Kahn *Cohomological approaches to SK_1 and SK_2 of central simple algebras*, notes of a talk in the quadratic forms conference, Talca-Pucón, Chile, Dec. 2002, `http://people.math.jussieu.fr/~kahn/preprints/exp.html #9`.
[21] B. Kahn *Cohomologie non ramifiée des quadriques*, *in* Geometric methods in the algebraic theory of quadratic forms, 1–23, Lecture Notes in Math., 1835, Springer, Berlin, 2004.
[22] B. Kahn, M. Levine *Motives of Azumaya algebras*, J. Inst. Math. Jussieu 9 (2010), 481–599.
[23] M. Levine *The indecomposable K_3 of fields*, Ann. Sci. Éc. Norm. Sup. 22 (1989), 255–344.
[24] M. Levine *Bloch's higher Chow groups revisited*, *in* K-theory (Strasbourg, 1992), Astérisque 226 (1994), 235–320.
[25] M. Levine *The algebraic K-theory of the classical groups and some twisted forms*, Duke Math. J. 70 (1993), 405–443.
[26] M. Levine, V. Srinivas, J. Weyman *K-theory of twisted Grassmannians*, K-Theory 3 (1989), 99–121.
[27] M. Levine *K-theory and motivic cohomology of schemes*, `http://www.math.uiuc.edu/K-theory/0336/`.
[28] M. Levine *Chow's moving lemma and the homotopy coniveau tower*, K-Theory 37 (2006), 129–209.

[29] R. Marlin *Anneaux de Chow des groupes algébriques* SO(n), Spin(n), G_2 *et* F_4, Publications Mathématiques d'Orsay, 95 (1974), 74–119. `http://users.polytech.unice.fr/~marlin/anneauxChow.pdf`.

[30] C. Mazza, V. Voevodsky, C. Weibel Lecture notes on motivic cohomology. Clay Mathematics Monographs 2, Amer. Math. Soc., 2006.

[31] A.S. Merkurjev *The group SK_2 for quaternion algebras* (in Russian), Izv. Akad. Nauk SSSR 32 (1988), 310–335. English translation: Math. USSR Izv. 32 (1989), 313–337.

[32] A.S. Merkurjev *The group $H^1(X, \mathcal{K}_2)$ for projective homogeneous varieties*, Algebra-i-analiz. English translation: Leningrad (Saint-Petersburg) Math. J. 7 (1995), 136–164.

[33] A.S. Merkurjev *K-theory of simple algebras*, in: W. Jacob and A. Rosenberg (ed.), K-theory and algebraic geometry: connections with quadratic forms and division algebras, Proceedings of Symposia in Pure Mathematics 58 (I) (1995), 65–83.

[34] A.S. Merkurjev *Generic element in SK_1 for simple algebras*, K-theory 7 (1993), 1–3.

[35] A.S. Merkurjev *Invariants of algebraic groups*, J. reine angew. Mathematik 508 (1999), 127–156.

[36] A.S. Merkurjev, A.A. Suslin *K-cohomology of Severi-Brauer varieties and the norm residue homomorphism*, Izv. Akad. Nauk SSSR Ser. Mat. 46 (1982), 1011–1046, 1135–1136. English transl.: Math. USSR-Izv. 21 (1983), 307–340.

[37] A.S. Merkurjev, A.A. Suslin *The group K_3 for a field* (in Russian), Izv. Akad. Nauk SSSR 54 (1990). Engl. transl.: Math. USSR Izv. 36 (1991), 541–565.

[38] A.S. Merkurjev, A.A. Suslin *Motivic cohomology of the simplicial motive of a Rost variety*, J. pure appl. Algebra 214 (2010), 2017–2026.

[39] A.S. Merkurjev, J.-P. Tignol *The multiplicers of similitudes and the Brauer group of homogeneous varieties*, J. reine angew. Math. 461 (1995), 13–47.

[40] I.A. Panin *On algebraic K-theory of generalized flag fiber bundles and some of their twisted forms*, Adv. in Soviet Math. 4 (1991), AMS, 21–46.

[41] V. P. Platonov *The Tannaka-Artin problem and reduced K-theory*, Izv. Akad. Nauk SSSR 40 (1976), 227-261. Engl. translation: Math. USSR Izv. 10 (1976).

[42] D. Quillen *Higher algebraic K-theory, I*, Lect. Notes in Math. 341, Springer, 1973, 85–147.

[43] M. Rost *Injectivity of the reduced norm for K_2 of quaternion algebras*, preprint, Regensburg, 1986.

[44] M. Rost *Chow groups with coefficients*, Doc. Math. 1 (1996), 319–393.

[45] C. Soulé *K-théorie des anneaux d'entiers de corps de nombres et cohomologie étale*, Invent. Math. 55 (1979), 251–295.

[46] C. Soulé *Operations on etale K-theory. Applications*, *in* Lect. Notes in Math. 966, Springer, 1982, 271–303.

[47] A.A. Suslin *Torsion in K_2 of fields*, K-theory 1 (1987), 5–29.

[48] A. Suslin *K-theory and $\mathcal{K}$-cohomology of certain group varieties*, Adv. in Soviet Math. 4 (1991), AMS, 53–74.

[49] A. Suslin *SK_1 of division algebras and Galois cohomology*, Adv. in Soviet Math. 4 (1991), AMS, 75–99.

[50] A. Suslin *SK_1 of division algebras and Galois cohomology revisited*, Amer. Math. Soc. Transl. (2) 219 (2006), 125–147.

[51] A. Suslin, V. Voevodsky *Bloch-Kato conjecture and motivic cohomology with finite coefficients*, The arithmetic and geometry of algebraic cycles (Banff, AB, 1998), 117–189, NATO Sci. Ser. C Math. Phys. Sci., 548, Kluwer Acad. Publ., Dordrecht, 2000.

[52] J. Tate *Relations between K_2 and Galois cohomology*, Invent. Math. 36 (1976), 257–274.

[53] V. Voevodsky *Cohomological theory of presheaves with transfers*, *in* Cycles, transfers and motivic cohomology theories, Annals of Math. Studies 143, 2000, 87–137.

[54] V. Voevodsky *Triangulated categories of motives over a field*, *in* Cycles, transfers and motivic cohomology theories, Annals of Math. Studies 143, 2000, 188–238.

[55] V. Voevodsky *Motivic cohomology groups are isomorphic to higher Chow groups in any characteristic*, Int. Math. Res. Not. 2002, 351–355.

[56] V. Voevodsky *Motivic cohomology with $\mathbf{Z}/2$ coefficients*, Publ. Math. IHÉS 98 (2003), 59–104.

[57] V. Voevodsky *On motivic cohomology with $\mathbf{Z}/l$ coefficients*, arXiv:0805.4430v2 [math.AG].

[58] S. Wang *On the commutator group of a simple algebra*, Amer. J. Math. 72 (1950), 323–334.

[59] C. Weibel *Fall 2006 Lectures on the proof of the Bloch-Kato Conjecture*, `http://www.math.rutgers.edu/~weibel/motivic2006.html`.

[60] T. Wouters *Comparing invariants of SK_1*, `http://www.math.uiuc.edu/K-theory/0955/`.

Bruno Kahn
Institut de Mathématiques
de Jussieu
UPMC
UFR 929
Mathématiques
4 place Jussieu
75005 Paris
France
kahn@math.jussieu.fr

Hyperbolicity of Orthogonal Involutions

К дню рождения Андрея Александровича

Nikita A. Karpenko[1]
with an Appendix by Jean-Pierre Tignol

Received: April 21, 2009
Revised: March 6, 2010

Abstract. We show that a non-hyperbolic orthogonal involution on a central simple algebra over a field of characteristic $\neq 2$ remains non-hyperbolic over some splitting field of the algebra.

2010 Mathematics Subject Classification: 14L17; 14C25
Keywords and Phrases: Algebraic groups, involutions, projective homogeneous varieties, Chow groups and motives, Steenrod operations.

1. Introduction

Throughout this note (besides of §3 and §4) F is a field of characteristic $\neq 2$. The basic reference for the material related to involutions on central simple algebras is [13]. The *degree* $\deg A$ of a (finite-dimensional) central simple F-algebra A is the integer $\sqrt{\dim_F A}$; the *index* $\operatorname{ind} A$ of A is the degree of a central division algebra Brauer-equivalent to A. An orthogonal involution σ on A is *hyperbolic*, if the hermitian form $A \times A \to A$, $(a,b) \mapsto \sigma(a) \cdot b$ on the right A-module A is so. This means that the variety $X\big((\deg A)/2; (A,\sigma)\big)$ of §2 has a rational point.
The main result of this paper is as follows (the proof is given in §7):

Theorem 1.1 (Main theorem). *A non-hyperbolic orthogonal involution σ on a central simple F-algebra A remains non-hyperbolic over the function field of the Severi-Brauer variety of A.*

[1]Partially supported by the Collaborative Research Centre 701 of the Bielefeld University and by the Max-Planck-Institut für Mathematik in Bonn

To explain the statement of Abstract, let us note that the function field L of the Severi-Brauer variety of a central simple algebra A is a *splitting field* of A, that is, the L-algebra A_L is Brauer-trivial.

A stronger version of Theorem 1.1, where the word "non-hyperbolic" (in each of two appearances) is replaced by "anisotropic", is, in general, an open conjecture, cf. [11, Conjecture 5.2].

Let us recall that the index of a central simple algebra possessing an orthogonal involution is a power of 2. Here is the complete list of indices $\operatorname{ind} A$ and coindices $\operatorname{coind} A = \deg A / \operatorname{ind} A$ of A for which Theorem 1.1 is known (over arbitrary fields of characteristic $\neq 2$), given in the chronological order:

- $\operatorname{ind} A = 1$ — trivial;
- $\operatorname{coind} A = 1$ (the stronger version) — [11, Theorem 5.3];
- $\operatorname{ind} A = 2$ — [5] and independently (the stronger version) [16, Corollary 3.4];
- $\operatorname{coind} A$ odd — [7, appendix by Zainoulline] and independently [12, Theorem 3.3];
- $\operatorname{ind} A = 4$ and $\operatorname{coind} A = 2$ — [19, Proposition 3];
- $\operatorname{ind} A = 4$ — [8, Theorem 1.2].

Let us note that Theorem 1.1 for any given (A, σ) with $\operatorname{coind} A = 2$ implies the stronger version of Theorem 1.1 for this (A, σ): indeed, by [12, Theorem 3.3], if $\operatorname{coind} A = 2$ and σ becomes isotropic over the function field of the Severi-Brauer variety, then σ becomes hyperbolic over this function field and the weaker version applies. Therefore we get

Theorem 1.2. *An anisotropic orthogonal involution on a central simple F-algebra of coindex 2 remains anisotropic over the function field of the Severi-Brauer variety of the algebra.* □

Sivatski's proof of the case with $\deg A = 8$ and $\operatorname{ind} A = 4$, mentioned above, is based on the following theorem, due to Laghribi:

Theorem 1.3 ([14, Théorème 4]). *Let φ be an anisotropic quadratic form of dimension 8 and of trivial discriminant. Assume that the index of the Clifford algebra C of φ is 4. Then φ remains anisotropic over the function field $F(X_1)$ of the Severi-Brauer variety X_1 of C.*

The following alternate proof of Theorem 1.3, given by Vishik, is a prototype of our proof of Main theorem (Theorem 1.1). Let Y be the projective quadric of φ and let X_2 be the Albert quadric of a biquaternion division algebra Brauer-equivalent to C. Assume that $\varphi_{F(X_1)}$ is isotropic. Then for any field extension E/F, the Witt index of φ_E is at least 2 if and only if $X_2(E) \neq \emptyset$. By [21, Theorem 4.15] and since the Chow motive $M(X_2)$ of X_2 is indecomposable, it follows that the motive $M(X_2)(1)$ is a summand of the motive of Y. The complement summand of $M(Y)$ is then given by a *Rost projector* on Y in the sense of Definition 5.1. Since $\dim Y + 1$ is not a power of 2, it follows that Y is isotropic (cf. [6, Corollary 80.11]).

After introducing some notation in §2 and discussing some important general principles concerning Chow motives in §3, we produce in §4 a replacement of [21, Theorem 4.15] (used right above to split off the summand $M(X_2)(1)$ from the motive of Y) valid for more general (as projective quadrics) algebraic varieties (see Proposition 4.6). In §5 we reproduce some recent results due to Rost concerning the modulo 2 Rost correspondences and Rost projectors on more general (as projective quadrics) varieties. In §6 we apply some standard motivic decompositions of projective homogeneous varieties to certain varieties related to a central simple algebra with an isotropic orthogonal involution. We also reproduce (see Theorem 6.1) some results of [9] which contain the needed generalization of indecomposability of the motive of an Albert quadric used in the previous paragraph. Finally, in §7 we prove Main theorem (Theorem 1.1) following the strategy of [8] and using results of [9] which were not available at the time of [8].

Acknowledgements. Thanks to Anne Quéguiner for asking me the question and to Alexander Vishik for telling me the alternate proof of Theorem I am also grateful to the referee for finding several insufficiently explained points in the manuscript.

2. Notation

We understand under a *variety* a separated scheme of finite type over a field.
Let D be a central simple F-algebra. The F-dimension of any right ideal in D is divisible by $\deg D$; the quotient is the *reduced dimension* of the ideal. For any integer i, we write $X(i;D)$ for the generalized Severi-Brauer variety of the right ideals in D of reduced dimension i. In particular, $X(0;D) = \operatorname{Spec} F = X(\deg D; D)$ and $X(i,D) = \emptyset$ for $i < 0$ and for $i > \deg D$.
More generally, let V be a right D-module. The F-dimension of V is then divisible by $\deg D$ and the quotient $\operatorname{rdim} V = \dim_F V/\deg D$ is called the *reduced dimension* of V. For any integer i, we write $X(i;V)$ for the projective homogeneous variety of the D-submodules in V of reduced dimension i (non-empty iff $0 \le i \le \operatorname{rdim} V$). For a finite sequence of integers $i_1, \dots, i_r$, we write $X(i_1 \subset \dots \subset i_r; V)$ for the projective homogeneous variety of flags of the D-submodules in V of reduced dimensions $i_1, \dots, i_r$ (non-empty iff $0 \le i_1 \le \dots \le i_r \le \operatorname{rdim} V$).
Now we additionally assume that D is endowed with an orthogonal involution τ. Then we write $X(i;(D,\tau))$ for the variety of the totally isotropic right ideals in D of reduced dimension i (non-empty iff $0 \le i \le \deg D/2$).
If moreover V is endowed with a hermitian (with respect to τ) form h, we write $X(i;(V,h))$ for the variety of the totally isotropic D-submodules in V of reduced dimension i.
We refer to [10] for a detailed construction and basic properties of the above varieties. We only mention here that for the central simple algebra $A := \operatorname{End}_D V$ with the involution σ adjoint to the hermitian form h, the varieties $X(i;(A,\sigma))$

and $X(i;(V,h))$ (for any $i \in \mathbb{Z}$) are canonically isomorphic. Besides, $\deg A = \operatorname{rdim} V$, and the following four conditions are equivalent:

(1) σ is hyperbolic;
(2) $X((\deg A)/2;(A,\sigma))(F) \neq \emptyset$;
(3) $X((\operatorname{rdim} V)/2;(V,h))(F) \neq \emptyset$;
(4) h is hyperbolic.

3. Krull-Schmidt principle

The characteristic of the base field F is arbitrary in this section.

Our basic reference for Chow groups and Chow motives (including notation) is [6]. We fix an associative unital commutative ring Λ (we shall take $\Lambda = \mathbb{F}_2$ in the application) and for a variety X we write $\mathrm{CH}(X;\Lambda)$ for its Chow group with coefficients in Λ. Our category of motives is the category $\mathrm{CM}(F,\Lambda)$ of *graded Chow motives with coefficients in* Λ, [6, definition of §64]. By a *sum* of motives we always mean the *direct* sum.

We shall often assume that our coefficient ring Λ is finite. This simplifies significantly the situation (and is sufficient for our application). For instance, for a finite Λ, the endomorphism rings of finite sums of Tate motives are also finite and the following easy statement applies:

LEMMA 3.1. *An appropriate power of any element of any* finite *associative (not necessarily commutative) ring is idempotent.*

Proof. Since the ring is finite, any its element x satisfies $x^a = x^{a+b}$ for some $a \geq 1$ and $b \geq 1$. It follows that x^{ab} is an idempotent. □

Let X be a smooth complete variety over F. We call X *split*, if its *integral* motive $M(X) \in \mathrm{CM}(F,\mathbb{Z})$ (and therefore its motive with any coefficients) is a finite sum of Tate motives. We call X *geometrically split*, if it splits over a field extension of F. We say that X satisfies the *nilpotence principle*, if for any field extension E/F and any coefficient ring Λ, the kernel of the change of field homomorphism $\operatorname{End}(M(X)) \to \operatorname{End}(M(X)_E)$ consists of nilpotents. Any projective homogeneous variety is geometrically split and satisfies the nilpotence principle, [3, Theorem 8.2].

COROLLARY 3.2 ([9, Corollary 2.2]). *Assume that the coefficient ring* Λ *is finite. Let* X *be a geometrically split variety satisfying the nilpotence principle. Then an appropriate power of any endomorphism of the motive of* X *is a projector.*

We say that the *Krull-Schmidt principle* holds for a given pseudo-abelian category, if every object of the category has one and unique decomposition in a finite direct sum of indecomposable objects. In the sequel, we are constantly using the following statement:

COROLLARY 3.3 ([4, Corollary 35], see also [9, Corollary 2.6]). *Assume that the coefficient ring* Λ *is finite. The Krull-Schmidt principle holds for the pseudo-abelian Tate subcategory in* $\mathrm{CM}(F,\Lambda)$ *generated by the motives of the geometrically split* F*-varieties satisfying the nilpotence principle.* □

Remark 3.4. Replacing the Chow groups $\mathrm{CH}(-;\Lambda)$ by the *reduced* Chow groups $\overline{\mathrm{CH}}(-;\Lambda)$ (cf. [6, §72]) in the definition of the category $\mathrm{CM}(F,\Lambda)$, we get a "simplified" motivic category $\overline{\mathrm{CM}}(F,\Lambda)$ (which is still sufficient for the main purpose of this paper). Working within this category, we do not need the nilpotence principle any more. In particular, the Krull-Schmidt principle holds (with a simpler proof) for the pseudo-abelian Tate subcategory in $\overline{\mathrm{CM}}(F,\Lambda)$ generated by the motives of the geometrically split F-varieties.

4. Splitting off a motivic summand

The characteristic of the base field F is still arbitrary in this section.
In this section we assume that the coefficient ring Λ is connected. We shall often assume that Λ is finite.
Before climbing to the main result of this section (which is Proposition 4.6), let us do some warm up.
The following definition of [9] extends some terminology of [20]:

Definition 4.1. Let $M \in \mathrm{CM}(F,\Lambda)$ be a summand of the motive of a smooth complete irreducible variety of dimension d. The summand M is called *upper*, if $\mathrm{CH}^0(M;\Lambda) \neq 0$. The summand M is called *lower*, if $\mathrm{CH}_d(M;\Lambda) \neq 0$. The summand M is called *outer*, if it is simultaneously upper and lower.

For instance, the whole motive of a smooth complete irreducible variety is an outer summand of itself. Another example of an outer summand is the motive given by a *Rost projector* (see Definition 5.1).
Given a correspondence $\alpha \in \mathrm{CH}_{\dim X}(X \times Y;\Lambda)$ between some smooth complete irreducible varieties X and Y, we write $\operatorname{mult}\alpha \in \Lambda$ for the *multiplicity* of α, [6, definition of §75]. Multiplicity of a composition of two correspondences is the product of multiplicities of the composed correspondences (cf. [11, Corollary 1.7]). In particular, multiplicity of a projector is idempotent and therefore $\in \{0,1\}$ because the coefficient ring Λ is connected.
Characterizations of outer summands given in the two following Lemmas are easily obtained:

Lemma 4.2 (cf. [9, Lemmas 2.8 and 2.9]). *Let X be a smooth complete irreducible variety. The motive (X,p) given by a projector $p \in \mathrm{CH}_{\dim X}(X \times X;\Lambda)$ is upper if and only if $\operatorname{mult} p = 1$. The motive (X,p) is lower if and only if $\operatorname{mult} p^t = 1$, where p^t is the transpose of p.*

Lemma 4.3 (cf. [9, Lemma 2.12]). *Assume that a summand M of the motive of a smooth complete irreducible variety of dimension d decomposes into a sum of Tate motives. Then M is upper if and only if the Tate motive Λ is present in the decomposition; it is lower if and only if the Tate motive $\Lambda(d)$ is present in the decomposition.*

The following statement generalizes (the finite coefficient version of) [21, Corollary 3.9]:

Lemma 4.4. *Assume that the coefficient ring* Λ *is finite. Let* X *and* Y *be smooth complete irreducible varieties such that there exist multiplicity* 1 *correspondences*

$$\alpha \in \mathrm{CH}_{\dim X}(X \times Y; \Lambda) \quad \text{and} \quad \beta \in \mathrm{CH}_{\dim Y}(Y \times X; \Lambda).$$

Assume that X *is geometrically split and satisfies the nilpotence principle. Then there is an upper summand of* $M(X)$ *isomorphic to an upper summand of* $M(Y)$*. Moreover, for any upper summand* M_X *of* $M(X)$ *and any upper summand* M_Y *of* $M(Y)$*, there is an upper summand of* M_X *isomorphic to an upper summand of* M_Y*.*

Proof. By Corollary 3.2, the composition $p := (\beta \circ \alpha)^{\circ n}$ for some $n \geq 1$ is a projector. Therefore $q := (\alpha \circ \beta)^{\circ 2n}$ is also a projector and the summand (X, p) of $M(X)$ is isomorphic to the summand (Y, q) of $M(Y)$. Indeed, the morphisms $\alpha : M(X) \to M(Y)$ and $\beta' := \beta \circ (\alpha \circ \beta)^{\circ(2n-1)} : M(Y) \to M(X)$ satisfy the relations $\beta' \circ \alpha = p$ and $\alpha \circ \beta' = q$.
Since $\operatorname{mult} p = (\operatorname{mult} \beta \cdot \operatorname{mult} \alpha)^n = 1$ and similarly $\operatorname{mult} q = 1$, the summand (X, p) of $M(X)$ and the summand (Y, q) of $M(Y)$ are upper by Lemma 4.2.
We have proved the first statement of Lemma 4.4. As to the second statement, let

$$p' \in \mathrm{CH}_{\dim X}(X \times X; \Lambda) \text{ and } q' \in \mathrm{CH}_{\dim Y}(Y \times Y; \Lambda)$$

be projectors such that $M_X = (X, p')$ and $M_Y = (Y, q')$. Replacing α and β by $q' \circ \alpha \circ p'$ and $p' \circ \beta \circ q'$, we get isomorphic upper motives (X, p) and (Y, q) which are summands of M_X and M_Y. □

Remark 4.5. Assume that the coefficient ring Λ is finite. Let X be a geometrically split irreducible smooth complete variety satisfying the nilpotence principle. Then the complete motivic decomposition of X contains precisely one upper summand and it follows by Corollary 3.3 (or by Lemma 4.4) that an upper indecomposable summands of $M(X)$ is unique up to an isomorphism. (Of course, the same is true for the lower summands.)

Here comes the needed replacement of [21, Theorem 4.15]:

Proposition 4.6. *Assume that the coefficient ring* Λ *is finite. Let* X *be a geometrically split, geometrically irreducible variety satisfying the nilpotence principle and let* M *be a motive. Assume that there exists a field extension* E/F *such that*

1. *the field extension* $E(X)/F(X)$ *is purely transcendental;*
2. *the upper indecomposable summand of* $M(X)_E$ *is also lower and is a summand of* M_E*.*

Then the upper indecomposable summand of $M(X)$ *is a summand of* M*.*

Proof. We may assume that $M = (Y, p, n)$ for some irreducible smooth complete F-variety Y, a projector $p \in \mathrm{CH}_{\dim Y}(Y \times Y; \Lambda)$, and an integer n.
By the assumption (2), we have morphisms of motives $f : M(X)_E \to M_E$ and $g : M_E \to M(X)_E$ with $\operatorname{mult}(g \circ f) = 1$. By [9, Lemma 2.14], in order to

prove Proposition 4.6, it suffices to construct morphisms $f' : M(X) \to M$ and $g' : M \to M(X)$ (over F) with $\operatorname{mult}(g' \circ f') = 1$.
Let $\xi : \operatorname{Spec} F(X) \to X$ be the generic point of the (irreducible) variety X. For any F-scheme Z, we write ξ_Z for the morphism $\xi_Z = (\xi \times \operatorname{id}_Z) : Z_{F(X)} = \operatorname{Spec}_{F(X)} \times Z \to X \times Z$. Note that for any $\alpha \in \operatorname{CH}(X \times Z)$, the image $\xi_Z^*(\alpha) \in \operatorname{CH}(Z_{F(X)})$ of α under the pull-back homomorphism $\xi_Z^* : \operatorname{CH}(X \times Z, \Lambda) \to \operatorname{CH}(Z_{F(X)}, \Lambda)$ coincides with the composition of correspondences $\alpha \circ [\xi]$, [6, Proposition 62.4(2)], where $[\xi] \in \operatorname{CH}_0(X_{F(X)}, \Lambda)$ is the class of the point ξ:

$$(*) \qquad \xi_Z^*(\alpha) = \alpha \circ [\xi].$$

In the commutative square

$$\begin{array}{ccc} \operatorname{CH}(X_E \times Y_E; \Lambda) & \xrightarrow{\xi^*_{Y_E}} & \operatorname{CH}(Y_{E(X)}; \Lambda) \\ {\scriptstyle \operatorname{res}_{E/F}} \uparrow & & {\scriptstyle \operatorname{res}_{E(X)/F(X)}} \uparrow \\ \operatorname{CH}(X \times Y; \Lambda) & \xrightarrow{\xi^*_{Y}} & \operatorname{CH}(Y_{F(X)}; \Lambda) \end{array}$$

the change of field homomorphism $\operatorname{res}_{E(X)/F(X)}$ is surjective[1] because of the assumption (1) by the homotopy invariance of Chow groups [6, Theorem 57.13] and by the localization property of Chow groups [6, Proposition 57.11]. Moreover, the pull-back homomorphism ξ_Y^* is surjective by [6, Proposition 57.11]. It follows that there exists an element $f' \in \operatorname{CH}(X \times Y; \Lambda)$ such that $\xi^*_{Y_E}(f'_E) = \xi^*_{Y_E}(f)$.
Recall that $\operatorname{mult}(g \circ f) = 1$. On the other hand, $\operatorname{mult}(g \circ f'_E) = \operatorname{mult}(g \circ f)$. Indeed, $\operatorname{mult}(g \circ f) = \deg \xi^*_{X_E}(g \circ f)$ by [6, Lemma 75.1], where $\deg : \operatorname{CH}(X_{E(X)}) \to \Lambda$ is the degree homomorphism. Furthermore, $\xi^*_{X_E}(g \circ f) = (g \circ f) \circ [\xi_E]$ by $(*)$. Finally, $(g \circ f) \circ [\xi_E] = g \circ (f \circ [\xi_E])$ and $f \circ [\xi_E] = \xi^*_{Y_E}(f) = \xi^*_{Y_E}(f'_E)$ by the construction of f'.
Replacing f' be the composition $p \circ f'$, we get a morphism $f' : M(X) \to M$. Since the composition $g \circ f'_E$ is not changed, we still have $\operatorname{mult}(g \circ f'_E) = 1$.
Since $\operatorname{mult}(g \circ f'_E) = 1$ and the indecomposable upper summand of $M(X)_E$ is lower, we have $\operatorname{mult}((f'_E)^t \circ g^t) = 1$. Therefore we may apply the above procedure to the dual morphisms

$$g^t : M(X)_E \to (Y, p, \dim X - \dim Y - n)_E$$
$$\text{and} \quad (f'_E)^t : (Y, p, \dim X - \dim Y - n)_E \to M(X)_E.$$

This way we get a morphism $g' : M \to M(X)$ such that $\operatorname{mult}((f')^t \circ (g')^t) = 1$. It follows that $\operatorname{mult}(g' \circ f') = 1$. □

Remark 4.7. Replacing $\operatorname{CM}(F, \Lambda)$ by $\overline{\operatorname{CM}}(F, \Lambda)$ in Proposition 4.6, we get a weaker version of Proposition 4.6 which is still sufficient for our application. The nilpotence principle is no more needed in the proof of the weaker version. Because of that, there is no more need to assume that X satisfies the nilpotence principle.

[1] In fact, $\operatorname{res}_{E(X)/F(X)}$ is even an isomorphism, but we do not need its injectivity (which can be obtained with a help of a specialization).

5. Rost correspondences

In this section, X stands for a smooth complete geometrically irreducible variety of a positive dimension d.
The coefficient ring Λ of the motivic category is $\mathbb{F}_2$ in this section. We write $\mathrm{Ch}(-)$ for the Chow group $\mathrm{CH}(-;\mathbb{F}_2)$ with coefficients in $\mathbb{F}_2$. We write $\deg_{X/F}$ for the degree homomorphism $\mathrm{Ch}_0(X) \to \mathbb{F}_2$.

Definition 5.1. An element $\rho \in \mathrm{Ch}_d(X \times X)$ is called a *Rost correspondence* (on X), if $\rho_{F(X)} = \chi_1 \times [X_{F(X)}] + [X_{F(X)}] \times \chi_2$ for some 0-cycle classes $\chi_1, \chi_2 \in \mathrm{Ch}_0(X_{F(X)})$ of degree 1. A *Rost projector* is a Rost correspondence which is a projector.

Remark 5.2. Our definition of a Rost correspondence differs from the definition of a *special correspondence* in [17]. Our definition is weaker in the sense that a special correspondence on X (which is an element of the *integral* Chow group $\mathrm{CH}_d(X \times X)$) considered modulo 2 is a Rost correspondence but not any Rost correspondence is obtained this way. This difference gives a reason to reproduce below some results of [17]. Actually, some of the results below are formally more general than the corresponding results of [17]; their proofs, however, are essentially the same.

Remark 5.3. Clearly, the set of all Rost correspondences on X is stable under transposition and composition. In particular, if ρ is a Rost correspondence, then its both symmetrizations $\rho^t \circ \rho$ and $\rho \circ \rho^t$ are (symmetric) Rost correspondences. Writing $\rho_{F(X)}$ as in Definition 5.1, we have $(\rho^t \circ \rho)_{F(X)} = \chi_1 \times [X_{F(X)}] + [X_{F(X)}] \times \chi_1$ (and $(\rho \circ \rho^t)_{F(X)} = \chi_2 \times [X_{F(X)}] + [X_{F(X)}] \times \chi_2$).

Lemma 5.4. *Assume that the variety X is projective homogeneous. Let $\rho \in \mathrm{Ch}_d(X \times X)$ be a projector. If there exists a field extension E/F such that $\rho_E = \chi_1 \times [X_E] + [X_E] \times \chi_2$ for some 0-cycle classes $\chi_1, \chi_2 \in \mathrm{Ch}_0(X_E)$ of degree 1, then ρ is a Rost projector.*

Proof. According to [3, Theorem 7.5], there exist some integer $n \geq 0$ and for $i = 1, \dots, n$ some integers $r_i > 0$ and some projective homogeneous varieties X_i satisfying $\dim X_i + r_i < d$ such that for $M = \bigoplus_{i=1}^n M(X_i)(r_i)$ the motive $M(X)_{F(X)}$ decomposes as $\mathbb{F}_2 \oplus M \oplus \mathbb{F}_2(d)$. Since there is no non-zero morphism between different summands of this three terms decomposition, the ring $\mathrm{End}\, M(X)$ decomposes in the product of rings

$$\mathrm{End}\,\mathbb{F}_2 \times \mathrm{End}\, M \times \mathrm{End}\,\mathbb{F}_2(d) = \mathbb{F}_2 \times \mathrm{End}\, M \times \mathbb{F}_2.$$

Let $\chi \in \mathrm{Ch}_0(X_{F(X)})$ be a 0-cycle class of degree 1. We set

$$\begin{aligned}\rho' = \chi \times [X_{F(X)}] + [X_{F(X)}] \times \chi &\in \mathbb{F}_2 \times \mathbb{F}_2 \\ &\subset \mathbb{F}_2 \times \mathrm{End}\, M \times \mathbb{F}_2 = \mathrm{End}\, M(X)_{F(X)} = \mathrm{Ch}_d(X_{F(X)} \times X_{F(X)})\end{aligned}$$

and we show that $\rho_{F(X)} = \rho'$. The difference $\varepsilon = \rho_{F(X)} - \rho'$ vanishes over $E(X)$. Therefore ε is a nilpotent element of $\mathrm{End}\, M$. Choosing a positive integer m

with $\varepsilon^m = 0$, we get

$$\rho_{F(X)} = \rho^m_{F(X)} = (\rho' + \varepsilon)^m = (\rho')^m + \varepsilon^m = (\rho')^m = \rho'. \qquad \square$$

Lemma 5.5. *Let $\rho \in \mathrm{Ch}_d(X \times X)$ be a projector. The motive (X, ρ) is isomorphic to $\mathbb{F}_2 \oplus \mathbb{F}_2(d)$ iff $\rho = \chi_1 \times [X] + [X] \times \chi_2$ for some some 0-cycle classes $\chi_1, \chi_2 \in \mathrm{Ch}_0(X)$ of degree 1.*

Proof. A morphism $\mathbb{F}_2 \oplus \mathbb{F}_2(d) \to (X, \rho)$ is given by some

$$f \in \mathrm{Hom}\big(\mathbb{F}_2, M(X)\big) = \mathrm{Ch}_0(X) \ \text{ and } \ f' \in \mathrm{Hom}\big(\mathbb{F}_2(d), M(X)\big) = \mathrm{Ch}_d(X).$$

A morphism in the inverse direction is given by some

$$g \in \mathrm{Hom}(M(X), \mathbb{F}_2) = \mathrm{Ch}^0(X) \ \text{ and } \ g' \in \mathrm{Hom}(M(X), \mathbb{F}_2(d)) = \mathrm{Ch}^d(X).$$

The two morphisms $\mathbb{F}_2 \oplus \mathbb{F}_2(d) \leftrightarrow (X, \rho)$ are mutually inverse isomorphisms iff $\rho = f \times g + f' \times g'$ and $\deg_{X/F}(fg) = 1 = \deg_{X/F}(f'g')$. The degree condition means that $f' = [X] = g$ and $\deg_{X/F}(f) = 1 = \deg_{X/F}(g')$. $\square$

Corollary 5.6. *If X is projective homogeneous and ρ is a projector on X such that*

$$(X, \rho)_E \simeq \mathbb{F}_2 \oplus \mathbb{F}_2(d)$$

for some field extension E/F, then ρ is a Rost projector. $\square$

A smooth complete variety is called *anisotropic*, if the degree of its any closed point is even.

Lemma 5.7 ([17, Lemma 9.2], cf. [18, proof of Lemma 6.2]). *Assume that X is anisotropic and possesses a Rost correspondence ρ. Then for any integer $i \neq d$ and any elements $\alpha \in \mathrm{Ch}_i(X)$ and $\beta \in \mathrm{Ch}^i(X_{F(X)})$, the image of the product $\alpha_{F(X)} \cdot \beta \in \mathrm{Ch}_0(X_{F(X)})$ under the degree homomorphism $\deg_{X_{F(X)}/F(X)} : \mathrm{Ch}_0(X_{F(X)}) \to \mathbb{F}_2$ is 0.*

Proof. Let $\gamma \in \mathrm{Ch}^i(X \times X)$ be a preimage of β under the surjection

$$\xi^*_X : \mathrm{Ch}^i(X \times X) \to \mathrm{Ch}^i(\operatorname{Spec} F(X) \times X)$$

(where ξ^*_X is as defined in the proof of Proposition 4.6). We consider the 0-cycle class

$$\delta = \rho \cdot ([X] \times \alpha) \cdot \gamma \in \mathrm{Ch}_0(X \times X).$$

Since X is anisotropic, so is $X \times X$, and it follows that $\deg_{(X \times X)/F} \delta = 0$. Therefore it suffices to show that $\deg_{(X\times X)/F} \delta = \deg_{X_{F(X)}/F(X)}(\alpha_{F(X)} \cdot \beta)$. We have $\deg_{(X\times X)/F} \delta = \deg_{(X\times X)_{F(X)}/F(X)}(\delta_{F(X)})$ and

$$\delta_{F(X)} = (\chi_1 \times [X_{F(X)}] + [X_{F(X)}] \times \chi_2) \cdot ([X_{F(X)}] \times \alpha_{F(X)}) \cdot \gamma_{F(X)} = \\ (\chi_1 \times [X_{F(X)}]) \cdot ([X_{F(X)}] \times \alpha_{F(X)}) \cdot \gamma_{F(X)}$$

(because $i \neq d$) where $\chi_1, \chi_2 \in \mathrm{Ch}_0(X_{F(X)})$ are as in Definition 5.1. For the first projection $pr_1 : X_{F(X)} \times X_{F(X)} \to X_{F(X)}$ we have

$$\deg_{(X\times X)_{F(X)}/F(X)} \delta_{F(X)} = \deg_{X_{F(X)}/F(X)}(pr_1)_*(\delta_{F(X)})$$

and by the projection formula

$$(pr_1)_*(\delta_{F(X)}) = \chi_1 \cdot (pr_1)_*\big(([X_{F(X)}] \times \alpha_{F(X)}) \cdot \gamma_{F(X)}\big).$$

Finally,

$$(pr_1)_*\big(([X_{F(X)}] \times \alpha_{F(X)}) \cdot \gamma_{F(X)}\big) = \operatorname{mult}\big(([X_{F(X)}] \times \alpha_{F(X)}) \cdot \gamma_{F(X)}\big) \cdot [X_{F(X)}]$$

and

$$\operatorname{mult}\big(([X_{F(X)}] \times \alpha_{F(X)}) \cdot \gamma_{F(X)}\big) = \operatorname{mult}\big(([X] \times \alpha) \cdot \gamma\big).$$

Since $\operatorname{mult}\chi = \deg_{X_{F(X)}/F(X)} \xi_X^*(\chi)$ for any element $\chi \in \operatorname{Ch}_d(X \times X)$ by [6, Lemma 75.1], it follows that

$$\operatorname{mult}\big(([X] \times \alpha) \cdot \gamma\big) = \deg(\alpha_{F(X)} \cdot \beta). \qquad \square$$

For anisotropic X, we consider the homomorphism $\deg/2 : \operatorname{Ch}_0(X) \to \mathbb{F}_2$ induced by the homomorphism $\operatorname{CH}_0(X) \to \mathbb{Z}$, $\alpha \mapsto \deg(\alpha)/2$.

Corollary 5.8. *Assume that X is anisotropic and possesses a Rost correspondence. Then for any integer $i \neq d$ and any elements $\alpha \in \operatorname{Ch}_i(X)$ and $\beta \in \operatorname{Ch}^i(X)$ with $\beta_{F(X)} = 0$ one has* $(\deg/2)(\alpha \cdot \beta) = 0$.

Proof. Let $\beta' \in \operatorname{CH}^i(X)$ be an integral representative of β. Since $\beta_{F(X)} = 0$, we have $\beta'_{F(X)} = 2\beta''$ for some $\beta'' \in \operatorname{CH}^i(X_{F(X)})$. Therefore

$$(\deg/2)(\alpha \cdot \beta) = \deg_{X_{F(X)}/F(X)} \big(\alpha_{F(X)} \cdot (\beta'' \mod 2)\big) = 0$$

by Lemma 5.7. $\square$

Corollary 5.9. *Assume that X is anisotropic and possesses a Rost correspondence ρ. For any integer $i \notin \{0, d\}$ and any $\alpha \in \operatorname{Ch}_i(X)$ and $\beta \in \operatorname{Ch}^i(X)$ one has*

$$(\deg/2)\big((\alpha \times \beta) \cdot \rho\big) = 0.$$

Proof. Let $\alpha' \in \operatorname{CH}_i(X)$ and $\beta' \in \operatorname{CH}^i(X)$ be integral representatives of α and β. Let $\rho' \in \operatorname{CH}_d(X \times X)$ be an integral representative of ρ. It suffices to show that the degree of the 0-cycle class $(\alpha' \times \beta') \cdot \rho' \in \operatorname{CH}_0(X \times X)$ is divisible by 4.

Let χ_1 and χ_2 be as in Definition 5.1. Let $\chi'_1, \chi'_2 \in \operatorname{CH}_0(X_{F(X)})$ be integral representatives of χ_1 and χ_2. Then $\rho'_{F(X)} = \chi'_1 \times [X_{F(X)}] + [X_{F(X)}] \times \chi'_2 + 2\gamma$ for some $\gamma \in \operatorname{CH}_d(X_{F(X)} \times X_{F(X)})$. Therefore (since $i \notin \{0, d\}$)

$$(\alpha'_{F(X)} \times \beta'_{F(X)}) \cdot \rho'_{F(X)} = 2(\alpha'_{F(X)} \times \beta'_{F(X)}) \cdot \gamma.$$

Applying the projection pr_1 onto the first factor and the projection formula, we get twice the element $\alpha'_{F(X)} \cdot (pr_1)_*\big(([X_{F(X)}] \times \beta'_{F(X)}) \cdot \gamma\big)$ whose degree is even by Lemma 5.7 (here we use once again the condition that $i \neq d$). $\square$

Lemma 5.10. *Assume that X is anisotropic and possesses a Rost correspondence ρ. Then* $(\deg/2)(\rho^2) = 1$.

Proof. Let χ_1 and χ_2 be as in Definition 5.1. Let $\chi_1', \chi_2' \in \mathrm{CH}_0(X_E)$ be integral representatives of χ_1 and χ_2. The degrees of χ_1' and χ_2' are odd. Therefore, the degree of the cycle class

$$(\chi_1' \times [X_{F(X)}] + [X_{F(X)}] \times \chi_2')^2 = 2(\chi_1' \times \chi_2') \in \mathrm{CH}_0(X_{F(X)} \times X_{F(X)})$$

is not divisible by 4.
Let $\rho' \in \mathrm{CH}_d(X \times X)$ be an integral representative of ρ. Since $\rho'_{F(X)}$ is $\chi_1' \times [X_{F(X)}] + [X_{F(X)}] \times \chi_2'$ modulo 2, $(\rho'_{F(X)})^2$ is $(\chi_1' \times [X_{F(X)}] + [X_{F(X)}] \times \chi_2')^2$ modulo 4. Therefore $(\deg/2)(\rho^2) = 1$. □

Theorem 5.11 ([17, Theorem 9.1], see also [18, proof of Lemma 6.2]). *Let X be an anisotropic smooth complete geometrically irreducible variety of a positive dimension d over a field F of characteristic $\neq 2$ possessing a Rost correspondence. Then the degree of the highest Chern class $c_d(-T_X)$, where T_X is the tangent bundle on X, is not divisible by 4.*

Proof. In this proof, we write $c_\bullet(-T_X)$ for the total Chern class $\in \mathrm{Ch}(X)$ in the Chow group with coefficient in $\mathbb{F}_2$. It suffices to show that $(\deg/2)(c_d(-T_X)) = 1$.
Let $\mathrm{Sq}_\bullet^X : \mathrm{Ch}(X) \to \mathrm{Ch}(X)$ be the modulo 2 homological Steenrod operation, [6, §59]. We have a commutative diagram

$$\begin{tikzcd}
 & \mathrm{Ch}_d(X \times X) \arrow[dl, "(pr_1)_*"'] \arrow[dd, "\mathrm{Sq}_d^{X\times X}"] & \\
\mathrm{Ch}_d(X) \arrow[dd, "\mathrm{Sq}_d^X"'] & & \\
 & \mathrm{Ch}_0(X \times X) \arrow[dl, "(pr_1)_*"'] \arrow[dr, "(pr_2)_*"] \arrow[dd, "\deg/2"] & \\
\mathrm{Ch}_0(X) \arrow[dr, "\deg/2"'] & & \mathrm{Ch}_0(X) \arrow[dl, "\deg/2"] \\
 & \mathbb{F}_2 &
\end{tikzcd}$$

Since $(pr_1)_*(\rho) = [X]$ and $\mathrm{Sq}_d^X([X]) = c_d(-T_X)$ [6, formula (60.1)], it suffices to show that

$$(\deg/2)\big(\mathrm{Sq}_d^{X\times X}(\rho)\big) = 1.$$

We have $\mathrm{Sq}_\bullet^{X\times X} = c_\bullet(-T_{X\times X}) \cdot \mathrm{Sq}^\bullet_{X\times X}$, where $\mathrm{Sq}^\bullet$ is the cohomological Steenrod operation, [6, §61]. Therefore

$$\mathrm{Sq}_d^{X\times X}(\rho) = \sum_{i=0}^{d} c_{d-i}(-T_{X\times X}) \cdot \mathrm{Sq}^i_{X\times X}(\rho).$$

The summand with $i = d$ is $\mathrm{Sq}^d_{X\times X}(\rho) = \rho^2$ by [6, Theorem 61.13]. By Lemma 5.10, its image under $\deg/2$ is 1.

Since $c_\bullet(-T_{X\times X}) = c_\bullet(-T_X)\times c_\bullet(-T_X)$ and $\mathrm{Sq}^0 = \mathrm{id}$, the summand with $i=0$ is

$$\left(\sum_{j=0}^{d} c_j(-T_X)\times c_{d-j}(-T_X)\right)\cdot\rho.$$

Its image under deg/2 is 0 because

$$(\deg/2)\Big(\big(c_0(-T_X)\times c_d(-T_X)\big)\cdot\rho\Big) = (\deg/2)(c_d(-T_X)) = (\deg/2)\Big(\big(c_d(-T_X)\times c_0(-T_X)\big)\cdot\rho\Big)$$

while for $j\notin\{0,d\}$, we have $(\deg/2)\Big(\big(c_j(-T_X)\times c_{d-j}(-T_X)\big)\cdot\rho\Big) = 0$ by Corollary 5.9.

Finally, for any i with $0<i<d$ the ith summand is the sum

$$\sum_{j=0}^{d-i}\big(c_j(-T_X)\times c_{d-i-j}(-T_X)\big)\cdot\mathrm{Sq}^i_{X\times X}(\rho).$$

We shall show that for any j the image of the jth summand under deg/2 is 0. Note that the image under deg/2 coincides with the image under the composition $(\deg/2)\circ(pr_1)_*$ and also under the composition $(\deg/2)\circ(pr_2)_*$ (look at the above commutative diagram). By the projection formula we have

$$(pr_1)_*\Big(\big(c_j(-T_X)\times c_{d-i-j}(-T_X)\big)\cdot\mathrm{Sq}^i_{X\times X}(\rho)\Big) = c_j(-T_X)\cdot(pr_1)_*\Big(\big([X]\times c_{d-i-j}(-T_X)\big)\cdot\mathrm{Sq}^i_{X\times X}(\rho)\Big)$$

and the image under deg/2 is 0 for positive j by Corollary 5.8 applied to $\alpha = c_j(-T_X)$ and $\beta = (pr_1)_*\Big(\big([X]\times c_{d-i-j}(-T_X)\big)\cdot\mathrm{Sq}^i_{X\times X}(\rho)\Big)$. Corollary 5.8 can be indeed applied, because since $\rho_{F(X)} = \chi_1\times[X_{F(X)}] + [X_{F(X)}]\times\chi_2$ and $i>0$, we have $\mathrm{Sq}^i_{(X\times X)_{F(X)}}(\rho)_{F(X)} = 0$ and therefore $\beta_{F(X)} = 0$.

For $j=0$ we use the projection formula for pr_2 and Corollary 5.8 with $\alpha = c_{d-i}(-T_X)$ and $\beta = (pr_2)_*\big(\mathrm{Sq}^i_{X\times X}(\rho)\big)$. □

Remark 5.12. The reason of the characteristic exclusion in Theorem 5.11 is that its proof makes use of Steenrod operations on Chow groups with coefficients in $\mathbb{F}_2$ which (the operations) -are not available in characteristic 2.

We would like to mention

Lemma 5.13 ([17, Lemma 9.10]). *Let X be an anisotropic smooth complete equidimensional variety over a field of arbitrary characteristic. If $\dim X+1$ is not a power of 2, then the degree of the integral 0-cycle class $c_{\dim X}(-T_X)\in\mathrm{CH}_0(X)$ is divisible by 4.*

Corollary 5.14 ([17, Corollary 9.12]). *In the situation of Theorem 5.11, the integer $\dim X+1$ is a power of 2.* □

6. Motivic decompositions of some isotropic varieties

The coefficient ring Λ is $\mathbb{F}_2$ in this section. Throughout this section, D is a central division F-algebra of degree 2^r with some positive integer r.
We say that motives M and N are *quasi-isomorphic* and write $M \approx N$, if there exist decompositions $M \simeq M_1 \oplus \cdots \oplus M_m$ and $N \simeq N_1 \oplus \cdots \oplus N_n$ such that

$$M_1(i_1) \oplus \cdots \oplus M_m(i_m) \simeq N_1(j_1) \oplus \cdots \oplus N_n(j_n)$$

for some (shift) integers $i_1, \ldots, i_m$ and $j_1, \ldots, j_n$.
We shall use the following

Theorem 6.1 ([9, Theorems 3.8 and 4.1]). *For any integer $l = 0, 1, \ldots, r$, the upper indecomposable summand M_l of the motive of the generalized Severi-Brauer variety $X(2^l; D)$ is lower. Besides of this, the motive of any finite direct product of any generalized Severi-Brauer varieties of D is quasi-isomorphic to a finite sum of M_l (with various l).*

For the rest of this section, we fix an orthogonal involution on the algebra D.

Lemma 6.2. *Let n be an positive integer. Let h be a hyperbolic hermitian form on the right D-module D^{2n} and let Y be the variety $X(n \deg D; (D^{2n}, h))$ (of the* maximal *totally isotropic submodules). Then the motive $M(Y)$ is isomorphic to a finite sum of several shifted copies of the motives $M_0, M_1, \ldots, M_r$.*

Proof. By [10, §15] the motive of the variety Y is quasi-isomorphic to the motive of the "total" variety

$$X(*; D^n) = \coprod_{i \in \mathbb{Z}} X(i; D^n) = \coprod_{i=0}^{2^r n} X(i; D^n)$$

of D-submodules in D^n (the range limit $2^r n$ is the reduced dimension of the D-module D^n). (Note that in our specific situation we always have $i = j$ in the flag varieties $X(i \subset j; D^n)$ which appear in the general formula of [10, Следствие 15.14].) Furthermore, $M(X(*; D^n)) \approx M(X(*; D))^{\otimes n}$ by [10, Следствие 10.19]. Therefore the motive of Y is a direct sum of the motives of products of generalized Severi-Brauer varieties of D. (One can also come to this conclusion by [2] computing the semisimple anisotropic kernel of the connected component of the algebraic group $\mathrm{Aut}(D^{2n}, h)$.) We finish by Theorem 6.1. □

As before, we write $\mathrm{Ch}(-)$ for the Chow group $\mathrm{CH}(-; \mathbb{F}_2)$ with coefficients in $\mathbb{F}_2$. We recall that a smooth complete variety is called *anisotropic*, if the degree of its any closed point is even (the empty variety is anisotropic). The following statement is a particular case of [9, Lemma 2.21].

Lemma 6.3. *Let Z be an anisotropic F-variety with a projector $p \in \mathrm{Ch}_{\dim Z}(Z \times Z)$ such that the motive $(Z, p)_L \in \mathrm{CM}(L, \mathbb{F}_2)$ for a field extension L/F is isomorphic to a finite sum of Tate motives. Then the number of the Tate summands is even. In particular, the motive in $\mathrm{CM}(F, \mathbb{F}_2)$ of any anisotropic F-variety does not contain a Tate summand.*

Proof. Mutually inverse isomorphisms between $(Z,p)_L$ and a sum of, say, n Tate summands, are given by two sequences of homogeneous elements $a_1,\dots,a_n$ and $b_1,\dots,b_n$ in $\mathrm{Ch}(Z_L)$ with $p_L = a_1\times b_1+\dots+a_n\times b_n$ and such that for any $i,j=1,\dots,n$ the degree $\deg(a_ib_j)$ is 0 for $i\neq j$ and $1\in\mathbb{F}_2$ for $i=j$. The pull-back of p via the diagonal morphism of Z is therefore a 0-cycle class on Z of degree n (modulo 2). □

Lemma 6.4. *Let n be an integer ≥ 0. Let h' be a hermitian form on the right D-module D^n such that h'_L is anisotropic for any finite odd degree field extension L/F. Let h be the hermitian form on the right D-module D^{n+2} which is the orthogonal sum of h' and a hyperbolic D-plane. Let Y' be the variety of totally isotropic submodules of D^{n+2} of reduced dimension 2^r ($=\operatorname{ind} D$). Then the complete motivic decomposition of $M(Y')\in\mathrm{CM}(F,\mathbb{F}_2)$ (cf. Corollary 3.3) contains one summand $\mathbb{F}_2$, one summand $\mathbb{F}_2(\dim Y')$, and does not contain any other Tate motive.*

Proof. Since $Y'(F)\neq\emptyset$, $M(Y')$ contains an exemplar of the Tate motive $\mathbb{F}_2$ and an exemplar of the Tate motive $\mathbb{F}_2(\dim Y')$.
According to [10, Следствие 15.14] (see also [10, Следствие 15.9]), $M(Y')$ is quasi-isomorphic to the sum of the motives of the products

$$X(i\subset j;D)\times X(j-i;(D^n,h'))$$

where i,j run over all integers (the product is non-empty only if $0\leq i\leq j\leq 2^r$). The choices $i=j=0$ and $i=j=2^r$ give two exemplars of the Tate motive $\mathbb{F}_2$ (up to a shift). The variety obtained by any other choice of i,j but $i=0,j=2^r$ is anisotropic because the algebra D is division. The variety with $i=0,j=2^r$ is anisotropic by the assumption involving the odd degree field extensions. Lemma 6.3 terminates the proof. □

7. Proof of Main theorem

We fix a central simple algebra A of index >1 with a non-hyperbolic orthogonal involution σ. Since the involution is an isomorphism of A with its dual, the exponent of A is 2; therefore, the index of A is a power of 2, say, $\operatorname{ind} A=2^r$ for a positive integer r. We assume that σ becomes hyperbolic over the function field of the Severi-Brauer variety of A and we are looking for a contradiction. According to [12, Theorem 3.3], $\operatorname{coind} A=2n$ for some integer $n\geq 1$. We assume that Main theorem (Theorem 1.1) is already proven for all algebras (over all fields) of index $<2^r$ as well as for all algebras of index 2^r and coindex $<2n$. Let D be a central division algebra Brauer-equivalent to A. Let X_0 be the Severi-Brauer variety of D. Let us fix an (arbitrary) orthogonal involution τ on D and an isomorphism of F-algebras $A\simeq\mathrm{End}_D(D^{2n})$. Let h be a hermitian (with respect to τ) form on the right D-module D^{2n} such that σ is adjoint to h. Then $h_{F(X_0)}$ is hyperbolic. Since the anisotropic kernel of h also becomes hyperbolic over $F(X_0)$, our induction hypothesis ensures that h is anisotropic. Moreover, h_L is hyperbolic for any field extension L/F such that h_L is isotropic.

It follows by [1, Proposition 1.2] that h_L is anisotropic for any finite odd degree field extension L/F.
Let Y be the variety of totally isotropic submodules in D^{2n} of reduced dimension $n \deg D$. (The variety Y is a twisted form of the variety of maximal totally isotropic subspaces of a quadratic form studied in [6, Chapter XVI].) It is isomorphic to the variety of totally isotropic right ideals in A of reduced dimension $(\deg A)/2$ $(=n2^r)$. Since σ is hyperbolic over $F(X_0)$ and the field F is algebraically closed in $F(X_0)$ (because the variety X_0 is geometrically integral), the discriminant of σ is trivial. Therefore the variety Y has two connected components Y_+ and Y_- corresponding to the components C_+ and C_- (cf. [6, Theorem 8.10]) of the Clifford algebra $C(A,\sigma)$. Note that the varieties Y_+ and Y_- are projective homogeneous under the connected component of the algebraic group $\operatorname{Aut}(D^{2n},h) = \operatorname{Aut}(A,\sigma)$.
The central simple algebras C_+ and C_- are related with A by the formula [13, (9.14)]:

$$[C_+] + [C_-] = [A] \in \operatorname{Br}(F).$$

Since $[C_+]_{F(X_0)} = [C_-]_{F(X_0)} = 0 \in \operatorname{Br}(F(X_0))$, we have $[C_+],[C_-] \in \{0,[A]\}$ and it follows that $[C_+]=0$, $[C_-]=[A]$ up to exchange of the indices $+,-$.
By the index reduction formula for the varieties Y_+ and Y_- of [15, page 594], we have: $\operatorname{ind} D_{F(Y_+)} = \operatorname{ind} D$, $\operatorname{ind} D_{F(Y_-)} = 1$.
Below we will work with the variety Y_+ and not with the variety Y_-. One reason of this choice is Lemma 7.1. Another reason of the choice is that we need $D_{F(Y_+)}$ to be a division algebra when applying Proposition 4.6 in the proof of Lemma 7.2.

Lemma 7.1. *For any field extension L/F one has:*

a) *$Y_-(L) \neq \emptyset \Leftrightarrow D_L$ is Brauer-trivial $\Leftrightarrow D_L$ is Brauer-trivial and σ_L is hyperbolic;*
b) *$Y_+(L) \neq \emptyset \Leftrightarrow \sigma_L$ is hyperbolic.*

Proof. Since $\sigma_{F(X_0)}$ is hyperbolic, $Y(F(X_0)) \neq \emptyset$. Since the varieties Y_+ and Y_- become isomorphic over $F(X_0)$, each of them has an $F(X_0)$-point. Moreover, X_0 has an $F(Y_-)$-point. □

For the sake of notation simplicity, we write Y for Y_+ (we will not meet the old Y anymore).
The coefficient ring Λ is $\mathbb{F}_2$ in this section. We use the F-motives $M_0,\dots,M_r$ introduced in Theorem 6.1. Note that for any field extension E/F such that D_E is still a division algebra, we also have the E-motives $M_0,\dots,M_r$.

Lemma 7.2. *The motive of Y decomposes as $R_1 \oplus R_2$, where R_1 is quasi-isomorphic to a finite sum of several copies of the motives $M_0,\dots,M_{r-1}$, and where $(R_2)_{F(Y)}$ is isomorphic to a finite sum of Tate motives including one exemplar of $\mathbb{F}_2$.*

Proof. According to Lemma 6.2, the motive $M(Y)_{F(Y)}$ is isomorphic to a sum of several shifted copies of the $F(Y)$-motives $M_0,\dots,M_r$ (introduced in Theorem 6.1). Since $Y_{F(Y)} \neq \emptyset$, a (non-shifted) copy of the Tate motive $\mathbb{F}_2$ shows up. If for some $l = 0, \dots, r-1$ there is at least one copy of M_l (with a shift $j \in \mathbb{Z}$) in the decomposition, let us apply Proposition 4.6 taking as X the variety $X_l = X(2^l; D)$, taking as M the motive $M(Y)(-j)$, and taking as E the function field $F(Y)$.

Since D_E is a division algebra, condition (2) of Proposition 4.6 is fulfilled. Since $\operatorname{ind} D_{F(X)} < 2^r$, the hermitian form $h_{F(X)}$ is hyperbolic by the induction hypothesis; therefore the variety $Y_{F(X)}$ is rational (see Remark 7.1) and condition (1) of Proposition 4.6 is fulfilled as well.

It follows that the F-motive M_l is a summand of $M(Y)(-j)$. Let now M be the complement summand of $M(Y)(-j)$. By Corollary 3.3, the complete decomposition of $M_{F(Y)}$ is the complete decomposition of $M(Y)(-j)_{F(Y)}$ with one copy of M_l erased. If $M_{F(Y)}$ contains one more copy of a shift of M_l (for some $l = 0, \dots, r-1$), we once again apply Proposition 4.6 to the variety X_l and an appropriate shift of M. Doing this until we can, we get the desired decomposition in the end. $\square$

Now let us consider a minimal right D-submodule $V \subset D^{2n}$ such that V becomes isotropic over a finite odd degree field extension of $F(Y)$. We set $v = \dim_D V$. Clearly, $v \geq 2$ (because $D_{F(Y)}$ is a division algebra). For $v > 2$, let Y' be the variety $X(2^r; (V, h|_V))$ of totally isotropic submodules in V of reduced dimension 2^r (that is, of "D-dimension" 1). Writing $\tilde{F}$ for an odd degree field extension of $F(Y)$ with isotropic $V_{\tilde{F}}$, we have $Y'(\tilde{F}) \neq \emptyset$ (because $D_{\tilde{F}}$ is a division algebra). Therefore there exists a correspondence of odd multiplicity (that is, of multiplicity $1 \in \mathbb{F}_2$) $\alpha \in \mathrm{Ch}_{\dim Y}(Y \times Y')$.

If $v = 2$, then $h|_V$ becomes hyperbolic over (an odd degree extension of) $F(Y)$. Therefore $h|_V$ becomes hyperbolic over $F(X_0)$, and our induction hypothesis actually insures that $n = v = 2$. In this case we simply take $Y' := Y$ (our component).

The variety Y' is projective homogeneous (in particular, irreducible) of dimension

$$\dim Y' = 2^{r-1}(2^r - 1) + 2^{2r}(v - 2)$$

which is equal to a power of 2 minus 1 only if $r = 1$ and $v = 2$. Moreover, the variety Y' is anisotropic (because the hermitian form h is anisotropic and remains anisotropic over any finite odd degree field extension of the base field). Surprisingly, we can however prove the following

Lemma 7.3. *There is a Rost projector (Definition 5.1) on Y'.*

Proof. By the construction of Y', there exists a correspondence of odd multiplicity (that is, of multiplicity $1 \in \mathbb{F}_2$) $\alpha \in \mathrm{Ch}_{\dim Y}(Y \times Y')$. On the other hand, since $h_{F(Y')}$ is isotropic, $h_{F(Y')}$ is hyperbolic and therefore there exist a rational map $Y' \dashrightarrow Y$ and a multiplicity 1 correspondence $\beta \in \mathrm{Ch}_{\dim Y'}(Y' \times Y)$ (e.g., the class of the closure of the graph of the rational map). Since the summand R_2 of $M(Y)$ given by Lemma 7.2 is upper (cf. Definition 4.1 and Lemma 4.3), by Lemma 4.4 there is an upper summand of $M(Y')$ isomorphic to a summand of R_2.

Let $\rho \in \mathrm{Ch}_{\dim Y'}(Y' \times Y')$ be the projector giving this summand. We claim that ρ is a Rost projector. We prove the claim by showing that the motive $(Y', \rho)_{\tilde{F}}$ is isomorphic to $\mathbb{F}_2 \oplus \mathbb{F}_2(\dim Y')$, cf. Corollary 5.6, where $\tilde{F}/F(Y)$ is a finite odd degree field extension such that V becomes isotropic over $\tilde{F}$.
Since $(R_2)_{F(Y)}$ is a finite sum of Tate motives, the motive $(Y', \rho)_{\tilde{F}}$ is also a finite sum of Tate motives. Since $(Y', \rho)_{\tilde{F}}$ is upper, the Tate motive $\mathbb{F}_2$ is included (Lemma 4.3). Now, by the minimal choice of V, the hermitian form $(h|_V)_{\tilde{F}}$ satisfies the condition on h in Lemma 6.4: $(h|_V)_{\tilde{F}}$ is an orthogonal sum of a hyperbolic $D_{\tilde{F}}$-plane and a hermitian form h' such that h'_L is anisotropic for any finite odd degree field extension $L/\tilde{F}$ of the base field $\tilde{F}$. Indeed, otherwise – if h'_L is isotropic for some such L, the module V_L contains a totally isotropic submodule W of D-dimension 2; any D-hyperplane $V' \subset V$, considered over L, meets W non-trivially; it follows that V'_L is isotropic and this contradicts to the minimality of V. (This is a very standard argument in the theory of quadratic forms over field which we applied now to a hermitian form over a division algebra.)
Therefore, by Lemma 6.4, the complete motivic decomposition of $Y'_{\tilde{F}}$ has one copy of $\mathbb{F}_2$, one copy of $\mathbb{F}_2(\dim Y')$, and no other Tate summands. By Corollary 3.3 and anisotropy of the variety Y' (see Lemma 6.3), it follows that

$$(Y', \rho)_{\tilde{F}} \simeq \mathbb{F}_2 \oplus \mathbb{F}_2(\dim Y'). \qquad \square$$

If we are away from the case where $r = 1$ and $v = 2$, then Lemma 7.3 contradicts to Corollary 5.14 thus proving Main theorem (Theorem 1.1). Note that Corollary 5.14 is a formal consequence of Theorem 5.11 and Lemma 5.13. We can avoid the use of Lemma 5.13 by showing that $\deg c_{\dim Y'}(-T_{Y'})$ is divisible by 4 for our variety Y'. Indeed, if $v > 2$, then let K be the field $F(t_1, \dots, t_{v2^r})$ of rational functions over F in $v2^r$ variables. Let us consider the (generic) diagonal quadratic form $\langle t_1, \dots, t_{v2^r} \rangle$ on the K-vector space K^{v2^r}. Let Y'' be the variety of 2^r-dimensional totally isotropic subspaces in K^{v2^r}. The degree of any closed point on Y'' is divisible by 2^{2^r}. In particular, the integer $\deg c_{\dim Y''}(-T_{Y''})$ is divisible by 2^{2^r}. Since over an algebraic closure $\bar{K}$ of K the varieties Y' and Y'' become isomorphic, we have

$$\deg c_{\dim Y'}(-T_{Y'}) = \deg c_{\dim Y''}(-T_{Y''}).$$

If $v = 2$ and $r > 1$, we can play the same game, taking as Y'' a component of the variety of 2^r-dimensional totally isotropic subspaces of the (generic) diagonal quadratic form (of trivial discriminant) $\langle t_1, \dots, t_{v2^r-1}, t_1 \dots t_{v2^r-1} \rangle$, because the degree of any closed point on Y'' is divisible by 2^{2^r-1}.
Finally, the remaining case where $r = 1$ and $v = 2$ needs a special argument (or reference). Indeed, in this case, the variety Y' is a conic, and therefore Lemma 7.3 does not provide any information on Y'. Of course, a reference to [16] allows one to avoid consideration of the case of $r = 1$ (and any v) at all. Also, [13, §15.B] covers our special case of $r = 1$ and $v = 2$. Finally, to stay with the methods of this paper, we can do this special case as follows: if the anisotropic conic Y' becomes isotropic over (an odd degree extension of) the

function field of the conic X_0, then X_0 becomes isotropic over the function field of Y' and, therefore, of Y; but this is not the case because the algebra $D_{F(Y)}$ is not split by the very definition of Y (we recall that X_0 is the Severi-Brauer variety of the quaternion algebra D).

References

[1] Bayer-Fluckiger, E., and Lenstra, Jr., H. W. Forms in odd degree extensions and self-dual normal bases. *Amer. J. Math. 112*, 3 (1990), 359–373.

[2] Brosnan, P. On motivic decompositions arising from the method of Białynicki-Birula. *Invent. Math. 161*, 1 (2005), 91–111.

[3] Chernousov, V., Gille, S., and Merkurjev, A. Motivic decomposition of isotropic projective homogeneous varieties. *Duke Math. J. 126*, 1 (2005), 137–159.

[4] Chernousov, V., and Merkurjev, A. Motivic decomposition of projective homogeneous varieties and the Krull-Schmidt theorem. *Transform. Groups 11*, 3 (2006), 371–386.

[5] Dejaiffe, I. Formes antihermitiennes devenant hyperboliques sur un corps de déploiement. *C. R. Acad. Sci. Paris Sér. I Math. 332*, 2 (2001), 105–108.

[6] Elman, R., Karpenko, N., and Merkurjev, A. *The algebraic and geometric theory of quadratic forms*, vol. 56 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2008.

[7] Garibaldi, S. Orthogonal involution on algebras of degree 16 and the killing form of E8. Linear Algebraic Groups and Related Structures (preprint server) 285 (2008, Feb 13) , 28 pages.

[8] Karpenko, N. A. Hyperbolicity of hermitian forms over biquaternion algebras. Linear Algebraic Groups and Related Structures (preprint server) 316 (2008, Dec 16, revised: 2009, Jan 07), 17 pages.

[9] Karpenko, N. A. Upper motives of algebraic groups and incompressibility of Severi-Brauer varieties. Linear Algebraic Groups and Related Structures (preprint server) 333 (2009, Apr 3, revised: 2009, Apr 24), 18 pages.

[10] Karpenko, N. A. Cohomology of relative cellular spaces and of isotropic flag varieties. *Algebra i Analiz 12*, 1 (2000), 3–69.

[11] Karpenko, N. A. On anisotropy of orthogonal involutions. *J. Ramanujan Math. Soc. 15*, 1 (2000), 1–22.

[12] Karpenko, N. A. On isotropy of quadratic pair. In *Quadratic Forms – Algebra, Arithmetic, and Geometry*, vol. 493 of *Contemp. Math.* Amer. Math. Soc., Providence, RI, 2009, pp. 211–218.

[13] Knus, M.-A., Merkurjev, A., Rost, M., and Tignol, J.-P. *The book of involutions*, vol. 44 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1998. With a preface in French by J. Tits.

[14] Laghribi, A. Isotropie de certaines formes quadratiques de dimensions 7 et 8 sur le corps des fonctions d'une quadrique. *Duke Math. J. 85*, 2 (1996), 397–410.

[15] Merkurjev, A. S., Panin, I. A., and Wadsworth, A. R. Index reduction formulas for twisted flag varieties. I. *K-Theory 10*, 6 (1996), 517–596.

[16] Parimala, R., Sridharan, R., and Suresh, V. Hermitian analogue of a theorem of Springer. *J. Algebra 243*, 2 (2001), 780–789.

[17] Rost, M. On the basic correspondence of a splitting variety. Preprint, September-Novermber 2006, 42 pages. Available on the web page of the author.

[18] Semenov, N. Motivic construction of cohomological invariants. K-Theory Preprint Archives (preprint server).

[19] Sivatski, A. S. Applications of Clifford algebras to involutions and quadratic forms. *Comm. Algebra 33*, 3 (2005), 937–951.

[20] Vishik, A. Integral motives of quadrics. Preprint of the Max Planck Institute at Bonn, (1998)-13, 82 pages.

[21] Vishik, A. Motives of quadrics with applications to the theory of quadratic forms. In *Geometric methods in the algebraic theory of quadratic forms*, vol. 1835 of *Lecture Notes in Math.* Springer, Berlin, 2004, pp. 25–101.

Appendix A.

Hyperbolicity of Symplectic and Unitary Involutions

by Jean-Pierre Tignol

The purpose of this note is to show how Karpenko's results in [4] and [6] can be used to prove the following analogues for symplectic and unitary involutions:

Theorem A.1. *Let A be a central simple algebra of even degree over an arbitrary field F of characteristic different from 2 and let L be the function field over F of the generalized Severi–Brauer variety $X_2(A)$ of right ideals of dimension $2\deg A$ (i.e., reduced dimension 2) in A (see* [7, (1.16)]*). If a symplectic involution σ on A is not hyperbolic, then its scalar extension $\sigma_L = \sigma \otimes \mathrm{id}_L$ on $A_L = A \otimes_F L$ is not hyperbolic. Moreover, if A is a division algebra then σ_L is anisotropic.*

By a standard specialization argument, it suffices to find a field extension L'/F such that $A_{L'}$ has index 2 and $\sigma_{L'}$ is not hyperbolic to prove the first part. If A is a division algebra we need moreover $\sigma_{L'}$ anisotropic.

Theorem A.2. *Let B be a central simple algebra of exponent 2 over an arbitrary field K of characteristic different from 2, and let τ be a unitary involution on B. Let F be the subfield of K fixed under τ and let M be the function field over F of the Weil transfer $R_{K/F}(X(B))$ of the Severi–Brauer variety of B. If τ is not hyperbolic, then its scalar extension $\tau_M = \tau \otimes \mathrm{id}_M$ on $B_M = B \otimes_F M$ is not hyperbolic. Moreover, if B is a division algebra, then τ_M is anisotropic.*

Again, by a standard specialization argument, it suffices to find a field extension M'/F such that $B_{M'}$ is split and $\tau_{M'}$ is not hyperbolic ($\tau_{M'}$ anisotropic if B is a division algebra).

A.1. Symplectic involutions. Consider the algebra of iterated twisted Laurent series in two indeterminates

$$\widehat{A} = A((\xi))((\eta; f))$$

where f is the automorphism of $A((\xi))$ that maps ξ to $-\xi$ and is the identity on A. Thus, ξ and η anticommute and centralize A. Let $x = \xi^2$ and $y = \eta^2$; the center of $\widehat{A}$ is the field of Laurent series $\widehat{F} = F((x))((y))$. Moreover, ξ and η generate over $\widehat{F}$ a quaternion algebra $(x, y)_{\widehat{F}}$, and we have $\widehat{A} = A \otimes_F (x, y)_{\widehat{F}}$. Let $\widehat{\sigma}$ be the involution on $\widehat{A}$ extending σ and mapping ξ to $-\xi$ and η to $-\eta$. This involution is the tensor product of σ and the canonical (conjugation) involution on $(x, y)_{\widehat{F}}$. Since σ is symplectic, it follows that $\widehat{\sigma}$ is orthogonal.

Proposition A.3. *If σ is anisotropic (resp. hyperbolic), then $\widehat{\sigma}$ is anisotropic (resp. hyperbolic).*

Proof. If σ is hyperbolic, then A contains an idempotent e such that $\sigma(e) = 1 - e$, see [7, (6.7)]. Since $(A, \sigma) \subset (\widehat{A}, \widehat{\sigma})$, this idempotent also lies in $\widehat{A}$ and satisfies $\widehat{\sigma}(e) = 1 - e$, hence $\widehat{\sigma}$ is hyperbolic. Now, suppose $\widehat{\sigma}$ is isotropic and let $a \in \widehat{A}$ be a nonzero element such that $\widehat{\sigma}(a)a = 0$. We may write

$$a = \sum_{i=z}^{\infty} a_i \eta^i$$

for some $a_i \in A((\xi))$ with $a_z \neq 0$. The coefficient of η^{2z} in $\widehat{\sigma}(a)a$ is $(-1)^z f^z(\widehat{\sigma}(a_z)a_z)$, hence $\widehat{\sigma}(a_z)a_z = 0$. Now, let

$$a_z = \sum_{j=y}^{\infty} a_{jz} \xi^j$$

with $a_{jz} \in A$ and $a_{yz} \neq 0$. The coefficient of ξ^{2y} in $\widehat{\sigma}(a_z)a_z$ is $(-1)^y \sigma(a_{yz})a_{yz}$, hence $\sigma(a_{yz})a_{yz} = 0$, which shows σ is isotropic. □

Proof of Theorem A.1. Substituting for (A, σ) its anisotropic kernel, we may assume σ is anisotropic. Proposition A.3 then shows $(\widehat{A}, \widehat{\sigma})$ is anisotropic. Let L' be the function field over $\widehat{F}$ of the Severi–Brauer variety of $\widehat{A}$. By Karpenko's theorem in [6], the algebra with involution $(\widehat{A}_{L'}, \widehat{\sigma}_{L'})$ is not hyperbolic. Therefore, it follows from Proposition A.3 that $(A_{L'}, \sigma_{L'})$ is not hyperbolic. In particular, $A_{L'}$ is not split since every symplectic involution on a split algebra is hyperbolic. On the other hand, $\widehat{A}_{L'}$ is split, hence $A_{L'}$ is Brauer-equivalent to $(x, y)_{L'}$. We have thus found a field L' such that $A_{L'}$ has index 2 and $\sigma_{L'}$ is not hyperbolic, and the first part of Theorem A.1 follows. If A is a division algebra, then $\widehat{A}$ also is division. Karpenko's theorem in [4] then shows that $\widehat{\sigma}_{L'}$ is anisotropic, hence $\sigma_{L'}$ is anisotropic since $(A_{L'}, \sigma_{L'}) \subset (\widehat{A}_{L'}, \widehat{\sigma}_{L'})$. □

Remark A.4. The last assertion in Theorem A.1 also holds if $\operatorname{char} F = 2$, as a result of another theorem of Karpenko [5][2]: if (A, σ) is a central division algebra with symplectic involution over a field F of characteristic 2 and $Q = [x, y)_{F(x,y)}$ is a "generic" quaternion algebra where x and y are independent indeterminates over F, then $A \otimes_F Q$ is a central division algebra over $F(x, y)$ and we may consider on this algebra the quadratic pair $(\sigma \otimes \gamma, f_\otimes)$ where γ is the conjugation involution on Q and $f_\otimes$ is defined in [7, (5.20)]. By [5, Theorem 3.3], this quadratic pair remains anisotropic over the function field L' of the Severi–Brauer variety of $A \otimes Q$, hence $\sigma_{L'}$ also is anisotropic, while $A_{L'}$ has index 2.

A.2. Unitary involutions. The proof of Theorem A.2 follows a line of argument similar to the proof of Theorem A.1. Since the exponent of B is 2, the algebra B carries an orthogonal involution ν. Let $g = \nu \circ \tau$, which is an outer automorphism of B, and consider the algebra of twisted Laurent series

$$\widetilde{B} = B((\xi; g)).$$

It is readily checked that $\widetilde{B}$ carries an involution $\widetilde{\tau}$ extending τ such that $\widetilde{\tau}(\xi) = \xi$. To describe the center $\widetilde{F}$ of $\widetilde{B}$, pick an element $u \in \widetilde{B}$ such that

$$\nu(u) = \tau(u) = u \quad \text{and} \quad g^2(b) = ubu^{-1} \text{ for all } b \in B,$$

see [2, Lemma 3.1], and let $x = u^{-1}\xi^2$. Then $\widetilde{F} = F((x))$, and $\widetilde{B}$ is central simple over $\widetilde{F}$ by [1, Theorem 11.10]. By computing the dimension of the space of $\widetilde{\tau}$-symmetric elements as in [3, Proposition 1.9], we see that $\widetilde{\tau}$ is orthogonal. The algebra with involution $(\widetilde{B}, \widetilde{\tau})$ can be alternatively described as follows: let β be the Brauer class of the central simple F-algebra $B_1 = B \oplus B\zeta$ where $\zeta^2 = u$ and $\zeta b = g(b)\zeta$ for all $b \in B$. Then $(\widetilde{B}, \widetilde{\tau})$ is the unique orthogonal quadratic extension of $(B, \tau)_{\widetilde{F}}$ with Brauer class $\beta_{\widetilde{F}} + (K, x)_{\widetilde{F}}$, see [3, Proposition 1.9].

Proposition A.5. *If τ is anisotropic (resp. hyperbolic), then $\widetilde{\tau}$ is anisotropic (resp. hyperbolic).*

Proof. If τ is hyperbolic, then $\widetilde{\tau}$ also is hyperbolic because $(B, \tau) \subset (\widetilde{B}, \widetilde{\tau})$. If $\widetilde{\tau}$ is isotropic, a leading term argument as in Proposition A.3 shows that τ is isotropic. □

Proof of Theorem A.2. Substituting for (B, τ) its anisotropic kernel, we may assume τ is anisotropic, hence $\widetilde{\tau}$ also is anisotropic. Let M' be the function field over $\widetilde{F}$ of the Severi–Brauer variety of $\widetilde{B}$. By Karpenko's theorem in [6], the algebra with involution $(\widetilde{B}_{M'}, \widetilde{\tau}_{M'})$ is not hyperbolic, hence $(B_{M'}, \tau_{M'})$ is not hyperbolic. On the other hand, $\widetilde{B}_{M'}$ is split, and $B_{M'}$ is the centralizer of K in $\widetilde{B}_{M'}$, hence $B_{M'}$ is split. We have thus found an extension M'/F such that $B_{M'}$ is split and $\tau_{M'}$ is not hyperbolic, which proves the first part of Theorem A.2. If B is a division algebra, then $\widetilde{B}$ also is a division algebra, and Karpenko's theorem in [4] shows that $\widetilde{\tau}_{M'}$ is anisotropic. Then $\tau_{M'}$ is anisotropic since $(B_{M'}, \tau_{M'}) \subset (\widetilde{B}_{M'}, \widetilde{\tau}_{M'})$. □

[2] I am grateful to N. Karpenko for calling my attention on this reference.

REMARK A.6. As for symplectic involutions, the last assertion in Theorem A.2 also holds if $\operatorname{char} F = 2$, with almost the same proof: take $\ell \in K$ such that $\tau(\ell) = \ell + 1$, and consider the quadratic pair $(\widetilde{\tau}, f)$ on $\widetilde{B}$ where f is defined by $f(s) = \operatorname{Trd}_{\widetilde{B}}(\ell s)$ for any $\widetilde{\tau}$-symmetric element $s \in \widetilde{B}$. If B is a division algebra, then $\widetilde{B}$ is a division algebra, hence Karpenko's Theorem 3.3 in [5] shows that the quadratic pair $(\widetilde{\tau}, f)$ remains anisotropic after scalar extension to M'. Therefore, $\tau_{M'}$ is anisotropic.

REFERENCES

[1] A. A. Albert, *Structure of algebras*, Amer. Math. Soc., Providence, R.I., 1961.

[2] M. A. Elomary and J.-P. Tignol, Classification of quadratic forms over skew fields of characteristic 2, J. Algebra 240 (2001), no. 1, 366–392.

[3] S. Garibaldi and A. Quéguiner-Mathieu, Pfister's theorem for orthogonal involutions of degree 12, Proc. Amer. Math. Soc. 137 (2009), no. 4, 1215–1222.

[4] N. A. Karpenko, On anisotropy of orthogonal involutions, J. Ramanujan Math. Soc. 15 (2000), no. 1, 1–22.

[5] N. A. Karpenko, On isotropy of quadratic pair, in *Quadratic forms—algebra, arithmetic, and geometry*, 211–217, Contemp. Math., 493, Amer. Math. Soc., Providence, RI.

[6] N. A. Karpenko, Hyperbolicity of orthogonal involutions, preprint 330, LAG server, `http://www.math.uni-bielefeld.de/LAG/`

[7] M.-A. Knus et al., *The book of involutions*, Amer. Math. Soc., Providence, RI, 1998.

Nikita A. Karpenko
UPMC Univ Paris 06
UMR 7586
Institut de Mathématiques
de Jussieu
4 place Jussieu
F-75252 Paris
France
www.math.jussieu.fr/~karpenko
karpenko *at* math.jussieu.fr

Jean-Pierre Tignol
Institute for Information and
Communication Technologies,
Electronics and
Applied Mathematics
Université catholique de Louvain
B-1348 Louvain-la-Neuve
Belgium
Jean-Pierre.Tignol at uclouvain.be

Slices and Transfers

Marc Levine[1]

Received: September 28, 2009

Abstract. We study Voevodsky's slice tower for S^1-spectra, and raise a number of questions regarding its properties. We show that the 0th slice does not in general admit transfers, although it does for a $\mathbb{P}^1$-loop-spectrum. We define a new tower for each of the higher slices, and show that the layers in these towers have the structure of Eilenberg-Maclane spectra on effective motives.

2010 Mathematics Subject Classification: Primary 14F42, 19E15; Secondary 19E08, 14C25, 55P42
Keywords and Phrases: Algebraic cycles, Morel-Voevodsky stable homotopy category, slice filtration

Contents

[1]Research supported by the NSF grant DMS-0801220 and the Alexander von Humboldt Foundation

Introduction

Voevodsky [22] has defined an analog of the classical Postnikov tower in the setting of motivic stable homotopy theory by replacing the simplicial suspension $\Sigma_s := - \wedge S^1$ with $\mathbb{P}^1$-suspension $\Sigma_{\mathbb{P}^1} := - \wedge \mathbb{P}^1$; we call this construction the *motivic Postnikov tower*.

Let $\mathcal{SH}(k)$ denote the motivic stable homotopy category of $\mathbb{P}^1$-spectra. One of the main results on motivic Postnikov tower in this setting is

Theorem 1. *Let k be a field of characteristic zero. For $E \in \mathcal{SH}(k)$, the slices s_nE have the natural structure of an $\mathcal{H}\mathbb{Z}$-module, and hence determine objects in the category of motives $DM(k)$.*

The statement is a bit imprecise, as the following expansion will make clear: Röndigs-Østvær [19, 20] have shown that the homotopy category of strict $\mathcal{H}\mathbb{Z}$-modules is equivalent to the category of motives, $DM(k)$. Additionally, Voevodsky [22] and the author [11] have shown that the 0th slice of the sphere spectrum $\mathbb{S}$ in $\mathcal{SH}(k)$ is isomorphic to $\mathcal{H}\mathbb{Z}$. Each $E \in \mathcal{SH}(k)$ has a canonical structure of a module over the sphere spectrum $\mathbb{S}$, and thus the slices s_nE acquire an $\mathcal{H}\mathbb{Z}$-module structure, in $\mathcal{SH}_{S^1}(k)$. This has been refined to the model category level by Pelaez [17], showing that the slices of a $\mathbb{P}^1$-spectrum E have a natural structure of a strict $\mathcal{H}\mathbb{Z}$-module, hence are motives.

Let $\mathbf{Spt}_{S^1}(k)$ denote the category of S^1-spectra, with its homotopy category (for the $\mathbb{A}^1$ model structure) $\mathcal{SH}_{S^1}(k)$. The analog for motives is the category complexes of presheaves with transfer and its $\mathbb{A}^1$-homotopy category $DM^{eff}(k)$, the category of effective motives over k. We consider the motivic Postnikov tower in$\mathcal{SH}_{S^1}(k)$, and ask the questions:

(1) Is there a ring object in $\mathbf{Spt}_{S^1}(k)$, $\mathcal{H}\mathbb{Z}^{eff}$, such that the homotopy category of $\mathcal{H}\mathbb{Z}^{eff}$ modules is equivalent to the category of effective motives $DM^{eff}(k)$?
(2) What properties (if any) need an S^1-spectrum E have so that the slices s_nE have a natural structure of Eilenberg-Maclane spectra of a homotopy invariant complex of presheaves with transfer?

Naturally, if $\mathcal{H}\mathbb{Z}^{eff}$ exists as in (1), we are asking the slices in (2) to be (strict) $\mathcal{H}\mathbb{Z}^{eff}$ modules. Of course, a natural candidate for $\mathcal{H}\mathbb{Z}^{eff}$ would be the 0-S^1-spectrum of $\mathcal{H}\mathbb{Z}$, $\Omega^\infty_{\mathbb{P}^1}\mathcal{H}\mathbb{Z}$, but as far as I know, this property has not yet been investigated.

As we shall see, the 0-S^1-spectrum of a $\mathbb{P}^1$-spectrum does have the property that its (S^1) slices are motives, while one can give examples of S^1-spectra for which the 0th slice does not have this property. This suggests a relation of the question of the structure of the slices of an S^1-spectrum with a motivic version of the recognition problem:

(3) How can one tell if a given S^1-spectrum is an n-fold $\mathbb{P}^1$-loop spectrum?

In this paper, we prove two main results about the "motivic" structure on the slices of S^1-spectra:

Theorem 2. *Suppose* $\operatorname{char} k = 0$. *Let E be an S^1-spectrum. Then for each $n \geq 1$, there is a tower*

$$\ldots \to \rho_{\geq p+1} s_n E \to \rho_{\geq p} s_n E \to \ldots \to s_n E$$

in $\mathcal{SH}_{S^1}(k)$ with the following properties:

(1) *the tower is natural in E.*

(2) *Let $s_{p,n}E$ be the cofiber of $\rho_{\geq p+1} s_n E \to \rho_{\geq p} s_n E$. Then there is a homotopy invariant complex of presheaves with transfers $\hat{\pi}_p((s_n E)^{(n)})^* \in DM_-^{eff}(k)$ and a natural isomorphism in $\mathcal{SH}_{S^1}(k)$,*

$$EM_{\mathbb{A}^1}(\hat{\pi}_p((s_n E)^{(n)})^*) \cong s_{p,n} E,$$

where $EM_{\mathbb{A}^1} : DM_-^{eff}(k) \to \mathcal{SH}_{S^1}(k)$ is the Eilenberg-Maclane spectrum functor.

This result is proven in section 9.

One can say a bit more about the tower appearing in theorem 2. For instance, $\operatorname{holim}_p \operatorname{fib}(\rho_{\geq p} s_n E \to s_n E)$ is weakly equivalent to zero, so the spectral sequence associated to this tower is weakly convergent. If $s_n E$ is globally N-connected (i.e., there is an N such that $s_n E(X)$ is N-connected for all $X \in \mathbf{Sm}/k$) then the spectral sequence is strongly convergent. The "$\hat{\pi}_p$" appears in the notation due to the construction of $\hat{\pi}_p((s_n E)^{(n)})^*(X)$ arising from a "Bloch cycle complex" of codimension n cycles on $X \times \Delta^*$ with coefficients in $\pi_p(\Omega^n s_n E)$.

In other words, the *higher* slices of an arbitrary S^1-spectrum have some sort of transfers "up to filtration". The situation for the 0th slice appears to be more complicated, but for a $\mathbb{P}^1$-loop spectrum we have at least the following result:

Theorem 3. *Suppose* $\operatorname{char} k = 0$. *Take $E \in \mathcal{SH}_{\mathbb{P}^1}(k)$. Then for all m, the homotopy sheaf $\pi_m(s_0 \Omega_{\mathbb{P}^1} E)$ has a natural structure of a homotopy invariant sheaf with transfers.*

We actually prove a more precise result (corollary 8.5) which states that the 0th slice $s_0 \Omega_{\mathbb{P}^1} E$ is itself a presheaf with transfers, with values in the stable homotopy category $\mathcal{SH}$, i.e., $s_0 \Omega_{\mathbb{P}^1} E$ has "transfers up to homotopy". This raises the question:

(4) Is there an operad acting on $s_0 \Omega^n_{\mathbb{P}^1} E$ which shows that $s_0 \Omega^n_{\mathbb{P}^1} E$ admits transfers up to homotopy and higher homotopies up to some level?

Part of the motivation for this paper came out of discussions with Hélène Esnault concerning the (admittedly vague) question: Given a smooth projective variety X over some field k, that admits a 0-cycle of degree 1, are there "motivic" properties of X that lead to the existence of a k-point, or conversely, that give obstructions to the existence of a k-point? The fact that the existence of 0-cycles of degree 1 has something to do with the transfer maps from 0-cycles on X_L to 0-cycles on X, as L runs over finite field extensions of k, while the lack of a transfer map in general appears to be closely related to the subtlety of the existence of k-points led to our inquiry into the "motivic" nature of the spaces $\Omega^n_{\mathbb{P}^1}\Sigma^n_{\mathbb{P}^1}X_+$, or rather, their associated S^1-spectra.

Notation and conventions. Throughout this paper the base-field k will be a field of characteristic zero. $\mathbf{Sm}/k$ is the category of smooth finite type k-schemes. We let $\mathbf{Spc}_\bullet$ denote the category of pointed space, i.e., pointed simplicial sets, and $\mathcal{H}_\bullet$ the homotopy category of $\mathbf{Spc}_\bullet$, the *unstable homotopy category.* Similarly, we let $\mathbf{Spt}$ denote the category of spectra and $\mathcal{SH}$ its homotopy category, the *stable homotopy category.* We let $\mathbf{Spc}_\bullet(k)$ denote the category of *pointed spaces over* k, that is, the category of $\mathbf{Spc}_\bullet$-valued presheaves on $\mathbf{Sm}/k$, and $\mathbf{Spt}_{S^1}(k)$ the category of S^1*-spectra over* k, that is, the category of $\mathbf{Spt}$-valued presheaves on $\mathbf{Sm}/k$. We let $\mathbf{Spt}_{\mathbb{P}^1}(k)$ denote the category of $\mathbb{P}^1$*-spectra over* k, which we take to mean the category of $\Sigma_{\mathbb{P}^1}$-spectrum objects over $\mathbf{Spt}_{S^1}(k)$. Concretely, an object is a sequence $(E_0, E_1, \ldots)$, $E_n \in \mathbf{Spt}_{S^1}(k)$, together with bonding maps $\epsilon_n : \Sigma_{\mathbb{P}^1}E_n \to E_{n+1}$. Regarding the categories $\mathbf{Spt}_{S^1}(k)$, $\mathcal{SH}_{S^1}(k)$ and $\mathcal{SH}(k)$, we will use the notation spelled out in [11]. In addition to this source, we refer the reader to [8, 14, 15, 20, 22]. Relying on these sources for details, we remind the reader that $\mathcal{H}_\bullet(k)$ is the homotopy category of the category of $\mathbf{Spc}_\bullet(k)$, for the so-called $\mathbb{A}^1$-model structure. Similarly $\mathcal{SH}_{S^1}(k)$ and $\mathbf{Spt}_{\mathbb{P}^1}(k)$ have model structures, which we call the $\mathbb{A}^1$-model structures, and $\mathcal{SH}_{S^1}(k)$, $\mathcal{SH}(k)$ are the respective homotopy categories. For details on the category $DM^{eff}(k)$, we refer the reader to [3, 5].

We will be passing from the unstable motivic (pointed) homotopy category over k, $\mathcal{H}_\bullet(k)$, to the motivic homotopy category of S^1-spectra over k, $\mathcal{SH}_{S^1}(k)$, via the infinite (simplicial) suspension functor

$$\Sigma^\infty_s : \mathcal{H}_\bullet(k) \to \mathcal{SH}_{S^1}(k)$$

For a smooth k-scheme $X \in \mathbf{Sm}/k$ and a subscheme Y of X (sometimes closed, sometimes open), we let (X, Y) denote the homotopy push-out in the diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X \\ \operatorname{Spec} k & \end{tikzcd}$$

and as usual write X_+ for $(X \amalg \operatorname{Spec} k, \operatorname{Spec} k)$. We often denote $\operatorname{Spec} k$ by $*$. For an object S of $\mathcal{H}_\bullet(k)$, we often use S to denote $\Sigma^\infty_s S \in \mathcal{SH}_{S^1}(k)$ when the context makes the meaning clear; we also use this convention when passing to various localizations of $\mathcal{SH}_{S^1}(k)$.

We let $[n]$ denote the set $\{0, \ldots, n\}$ with the standard total order, and let $\mathbf{Ord}$ denote the category with objects $[n]$, $n = 0, 1, \ldots$ and morphisms the order-preserving maps of sets. Let Δ^n denote the algebraic n-simplex $\operatorname{Spec} k[t_0, \ldots, t_n]/\sum_i t_i - 1$, with vertices $v^n_0, \ldots, v^n_n$, where v^n_i is defined by $t_j = 0$ for $j \neq i$. As is well-known, sending $g : [n] \to [m]$ to the affine-linear

extension $\Delta(g) : \Delta^n \to \Delta^m$ of the map on the set of vertices, $v_j^n \mapsto v_{g(j)}^m$ defines the cosimplicial k-scheme $n \mapsto \Delta^n$.
We recall that, for a category $\mathcal{C}$, the category of pro-objects in $\mathcal{C}$, pro-$\mathcal{C}$, has as objects functors $f : I \to \mathcal{C}$, $i \in I$, where I is a small left-filtering category, a morphism $(f : I \to \mathcal{C}) \to (g : J \to \mathcal{C})$ is a pair $(\rho : I \to J, \theta : f \to g \circ \rho)$, with the evident composition, and we invert morphisms of the form

$$(\rho : I \to J, \mathrm{id} : f := g \circ \rho \to g \circ \rho)$$

if $\rho : I \to J$ has image a left co-final subcategory of J. In this paper we use categories of pro-objects to allow us to use various localizations of smooth finite type k-schemes. This is a convenience rather than a necessity, as all maps and relations lift to the level of finite type k-schemes.
I am very grateful to the referee for making a number of perceptive and useful comments, which led to the correction of some errors and an improvement of the exposition.
Dedication. This paper is warmly dedicated to Andrei Suslin, who has given me more inspiration than I can hope to tell.

1. Infinite $\mathbb{P}^1$-loop spectra

We first consider the case of the 0-S^1-spectrum of a $\mathbb{P}^1$-spectrum. We recall some constructions and results from [20]. We let

$$\Omega^\infty_{\mathbb{P}^1} : \mathcal{SH}(k) \to \mathcal{SH}_{S^1}(k)$$
$$\Omega^\infty_{\mathbb{P}^1,mot} : DM(k) \to DM^{eff}(k)$$

be the (derived) 0-spectrum (resp. 0-complex) functor, let

$$EM_{\mathbb{A}^1} : DM(k) \to \mathcal{SH}(k)$$
$$EM^{eff}_{\mathbb{A}^1} : DM^{eff}(k) \to \mathcal{SH}_{S^1}(k)$$

the respective Eilenberg-Maclane spectrum functors. The functors $\Omega^\infty_{\mathbb{P}^1}$, $\Omega^\infty_{\mathbb{P}^1,mot}$ are right adjoints to the respective infinite suspension functors

$$\Sigma^\infty_{\mathbb{P}^1} : \mathcal{SH}_{S^1}(k) \to \mathcal{SH}(k)$$
$$\Sigma^\infty_{\mathbb{P}^1,mot} : DM^{eff}(k) \to DM(k)$$

and the functors $EM_{\mathbb{A}^1}$, $EM^{eff}_{\mathbb{A}^1}$ are similarly right adjoints to the "linearization" functors

$$\mathbb{Z}^{tr} : \mathcal{SH}(k) \to DM(k)$$
$$\mathbb{Z}^{tr} : \mathcal{SH}_{S^1}(k) \to DM^{eff}(k)$$

induced by the functor $\mathbb{Z}^{tr}$ from simplicial presheaves on $\mathbf{Sm}/k$ to presheaves with transfer on $\mathbf{Sm}/k$ sending the representable presheaf $\mathrm{Hom}_{\mathbf{Sm}/k}(-, X)$ to the free presheaf with transfers $\mathbb{Z}^{tr}_X := \mathrm{Hom}_{SmCor(k)}(-X)$, and taking the Kan extension. The discussion in [20, §2.2.1] show that both these adjoint pairs arise from Quillen adjunctions on suitable model categories (followed by a chain of Quillen equivalences), where on the model categories, the functors $EM_{\mathbb{A}^1}$,

$EM^{eff}_{\mathbb{A}^1}$ are just forgetful functors and the functors Ω^∞ just take a sequence $E_0, E_1, \ldots$ to E_0. Thus one has

$$EM^{eff}_{\mathbb{A}^1} \circ \Omega^\infty_{\mathbb{P}^1, mot} \cong \Omega^\infty_{\mathbb{P}^1} \circ EM_{\mathbb{A}^1} \tag{1.1}$$

as one has an identity of the two functors on the model categories.

Theorem 1.1. *Fix an integer $n \geq 0$. Then there is a functor*

$$Mot^{eff}(s_n) : \mathcal{SH}(k) \to DM^{eff}(k)$$

and a natural isomorphism

$$\varphi_n : EM^{eff}_{\mathbb{A}^1} \circ Mot^{eff}(s_n) \to s_n^{eff} \circ \Omega^\infty_{\mathbb{P}^1}$$

of functors from $\mathcal{SH}(k)$ to $\mathcal{SH}_{S^1}(k)$.

In other words, for $\mathcal{E} \in \mathcal{SH}(k)$, there is a canonical lifting of the slice $s_n^{eff}(\Omega^\infty_{\mathbb{P}^1}\mathcal{E})$ to a motive $Mot^{eff}(s_n)(\mathcal{E})$.

Proof. By Pelaez [18, theorem 3.3], there is a functor

$$Mot(s_n) : \mathcal{SH}(k) \to DM(k)$$

and a natural isomorphism

$$\Phi_n : EM_{\mathbb{A}^1} \circ Mot(s_n) \to s_n$$

i.e., the slice $s_n\mathcal{E}$ lifts canonically to a motive $Mot(s_n)(\mathcal{E})$. Now apply the 0-complex functor to define

$$Mot^{eff}(s_n) := \Omega^\infty_{\mathbb{P}^1, mot} \circ Mot(s_n).$$

We have canonical isomorphisms

$$\begin{aligned} EM^{eff}_{\mathbb{A}^1} \circ \Omega^\infty_{\mathbb{P}^1, mot} \circ Mot(s_n) &\cong \Omega^\infty_{\mathbb{P}^1} \circ EM_{\mathbb{A}^1} \circ Mot(s_n) \\ &\cong \Omega^\infty_{\mathbb{P}^1} \circ s_n \\ &\cong s_n^{eff} \circ \Omega^\infty_{\mathbb{P}^1}. \end{aligned}$$

Indeed, the first isomorphism is (1.1) and the second is Pelaez's isomorphism Φ_n. For the third, we have given in [11] an explicit model for s_n in terms of the functors s_m^{eff} as follows: given a $\mathbb{P}^1$-spectrum E, represented as a sequence of S^1-spectra $E_0, E_1, \ldots$ together with bonding maps $\Sigma_{\mathbb{P}^1} E_n \to E_{n+1}$, suppose that E is fibrant. In particular, the adjoints $E_n \to \Omega_{\mathbb{P}^1} E_{n+1}$ of the bonding maps are weak equivalences and $E_0 = \Omega^\infty_{\mathbb{P}^1} E$. It follows from [11, theorem 9.0.3] that $s_n E$ is represented by the sequence $(s_n^{eff} E_0, s_{n+1}^{eff} E_1, \ldots, s_{n+m}^{eff} E_m, \ldots)$, with certain bonding maps (defined in [11, §8.3]). In addition, by [11, theorem 4.1.1] this new sequence is termwise weakly equivalent to its fibrant model. This defines the natural isomorphism $\Omega^\infty_{\mathbb{P}^1} s_n E \cong s_n^{eff} E_0 \cong s_n^{eff} \Omega_{\mathbb{P}^1} E$. □

In other words, the slices of an infinite $\mathbb{P}^1$-loop spectrum are effective motives.

2. An example

We now show that the 0th slice of an S^1-spectrum is not always a motive. In fact, we will give an example of an Eilenberg-Maclane spectrum whose 0th slice does not admit transfers.
For this, note the following:

Lemma 2.1. *Let $p : Y \to X$ be a finite Galois cover in $\mathbf{Sm}/k$, with Galois group G. Let $\mathcal{F}$ be a presheaf with transfers on $\mathbf{Sm}/k$. Then the composition*

$$p^* \circ p_* : \mathcal{F}(Y) \to \mathcal{F}(Y)$$

is given by

$$p^* \circ p_*(x) = \sum_{g \in G} g^*(x)$$

Proof. Letting $\Gamma_p \subset Y \times X$ be the graph of p, and $\Gamma_g \subset Y \times Y$ the graph of $g : Y \to Y$ for $g \in G$, one computes that

$$\Gamma_p^t \circ \Gamma_p = \sum_{g \in G} \Gamma_g,$$

whence the result. □

Now let C be a smooth projective curve over k, having no k-rational points. We assume that C has genus $g > 0$, so every map $\mathbb{A}^1_F \to C_F$ over a field $F \supset k$ is constant (C is $\mathbb{A}^1$-*rigid*).
Let $\mathbb{Z}_C$ be the representable presheaf:

$$\mathbb{Z}_C(Y) := \mathbb{Z}[\mathrm{Hom}_{\mathbf{Sm}/k}(Y, C)].$$

$\mathbb{Z}_C$ is automatically a Nisnevich sheaf; since C is $\mathbb{A}^1$-rigid, $\mathbb{Z}_C$ is also homotopy invariant. Furthermore $\mathbb{Z}_C$ is a *birational* sheaf, that is, for each dense open immersion $U \to Y$ in $\mathbf{Sm}/k$, the restriction map $\mathbb{Z}_C(Y) \to \mathbb{Z}_C(U)$ is an isomorphism. To see this, it suffices to show that $\mathrm{Hom}_{\mathbf{Sm}/k}(Y, C) \to \mathrm{Hom}_{\mathbf{Sm}/k}(U, C)$ is an isomorphism, and for this, take a morphism $f : U \to C$. Then the projection to Y of the closure $\bar{\Gamma}$ of the graph of f in $Y \times C$ is proper and birational. But since Y is regular, each fiber of $\bar{\Gamma} \to Y$ is rationally connected, hence maps to a point of C, and thus $\bar{\Gamma} \to Y$ is birational and 1-1. By Zariski's main theorem, $\bar{\Gamma} \to Y$ is an isomorphism, hence f extends to $\bar{f} : Y \to C$, as claimed.
Next, $\mathbb{Z}_C$ satisfies Nisnevich excision. This is just a general property of birational sheaves. In fact, let

$$\begin{tikzcd} V \arrow[r, "j_V"] \arrow[d, "f_{|V}"'] & Y \arrow[d, "f"] \\ U \arrow[r, "j_U"'] & X \end{tikzcd}$$

be an elementary Nisnevich square, i.e., the square is cartesian, f is étale, j_U and j_V are open immersions, and f induces an isomorphism $Y \setminus V \to X \setminus U$.

We may assume that U and V are dense in X and Y. Let $\mathcal{F}$ be a birational sheaf on $\mathbf{Sm}/k$, and apply $\mathcal{F}$ to this diagram. This gives us the square

$$\begin{array}{ccc} \mathcal{F}(X) & \xrightarrow{j_U^*} & \mathcal{F}(U) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f_{|V}^*} \\ \mathcal{F}(Y) & \xrightarrow[j_V^*]{} & \mathcal{F}(V) \end{array}$$

As the horizontal arrows are isomorphisms, we have the exact sequence

$$0 \to \mathcal{F}(X) \to \mathcal{F}(U) \oplus \mathcal{F}(Y) \to \mathcal{F}(V) \to 0.$$

Thus, $\mathcal{F}$ transforms elementary Nisnevich squares to distinguished triangles in $D(\mathbf{Ab})$; by definition, $\mathcal{F}$ therefore satisfies Nisnevich excision.

Let $EM_s(\mathbb{Z}_C)$ denote presheaf of Eilenberg-Maclane spectra on $\mathbf{Sm}/k$ associated to $\mathbb{Z}_C$, that is, for $U \in \mathbf{Sm}/k$, $EM_s(\mathbb{Z}_C)(U) \in \mathbf{Spt}$ is the Eilenberg-Maclane spectrum associated to the abelian group $\mathbb{Z}_C(U)$. Since $\mathbb{Z}_C$ is homotopy invariant and satisfies Nisnevich excision, $EM_s(\mathbb{Z}_C)$ is weakly equivalent as a presheaf on $\mathbf{Sm}/k$ to its fibrant model in $\mathcal{SH}_{S^1}(k)$ ($EM_s(\mathbb{Z}_C)$ is *quasi-fibrant*)[2]. In addition, the canonical map

$$EM_s(\mathbb{Z}_C) \to s_0(EM_s(\mathbb{Z}_C))$$

is an isomorphism in $\mathcal{SH}_{S^1}(k)$. Indeed, since $EM_s(\mathbb{Z}_C)$ is quasi-fibrant, a quasi-fibrant model for $s_0(EM_s(\mathbb{Z}_C))$ may be computed by the method of [11, §5] as follows: Take $Y \in \mathbf{Sm}/k$ and let $F = k(Y)$. Let $\Delta^n_{F,0}$ be the *semi-local* algebraic n-simplex over F, that is,

$$\Delta^n_{F,0} = \mathrm{Spec}(\mathcal{O}_{\Delta^n_F,v});\ v = \{v_0, \ldots, v_n\}.$$

The assignment $n \mapsto \Delta^n_{F,0}$ forms a cosimplicial subscheme of $n \mapsto \Delta^n_F$ and for a quasi-fibrant S^1-spectrum E, there is a natural isomorphism in $\mathcal{SH}$

$$s_0(E)(Y) \cong E(\Delta^*_{F,0}),$$

where $E(\Delta^*_{F,0})$ denotes the total spectrum of the simplicial spectrum $n \mapsto E(\Delta^n_{F,0})$. If now E happens to be a birational S^1-spectrum, meaning that $j^* : E(Y) \to E(U)$ is a weak equivalence for each dense open immersion $j : U \to Y$ in $\mathbf{Sm}/k$, then the restriction map

$$j^* : E(\Delta^*_Y) \to E(\Delta^*_{F,0}) \cong s_0(E)(Y)$$

is a weak equivalence. Thus, as E is quasi-fibrant and hence homotopy invariant, we have the sequence of isomorphisms in $\mathcal{SH}$

$$E(Y) \to E(\Delta^*_Y) \to E(\Delta^*_{F,0}) \cong s_0(E)(Y),$$

and hence $E \to s_0(E)$ is an isomorphism in $\mathcal{SH}_{S^1}(k)$. Taking $E = EM_s(\mathbb{Z}_C)$ verifies our claim.

[2]The referee has pointed out that, using the standard model $(\mathbb{Z}_C, B\mathbb{Z}_C, \ldots, B^n\mathbb{Z}_C, \ldots)$ for $EM_s(\mathbb{Z}_C)$, $EM_s(\mathbb{Z}_C)$ is actually fibrant in the projective model structure.

Finally, $\mathbb{Z}_C$ does not admit transfers. Indeed, suppose $\mathbb{Z}_C$ has transfers. Let $k \to L$ be a Galois extension such that $C(L) \neq \emptyset$; let G be the Galois group. Since $\mathbb{Z}_C(k) = \{0\}$ (as we have assumed that $C(k) = \emptyset$), the push-forward map

$$p_* : \mathbb{Z}_C(L) \to \mathbb{Z}_C(k)$$

is the zero map, hence $p^* \circ p_* = 0$. But for each L-point x of C, lemma 2.1 tells us that

$$p^* \circ p_*(x) = \sum_{g \in G} x^g \neq 0,$$

a contradiction.

Thus the homotopy sheaf

$$\pi_0(s_0 EM_s(\mathbb{Z}_C)) = \pi_0(EM_s(\mathbb{Z}_C)) = \mathbb{Z}_C$$

does not admit transfers, giving us the example we were seeking.

Even if we ask for transfers in a weaker sense, namely, that there is a functorial separated filtration $F^*\mathbb{Z}_C$ admitting transfers on the associated graded $gr_F^*\mathbb{Z}_C$, a slight extension of the above argument shows that this is not possible as long as the filtration on $\mathbb{Z}_C(L)$ is finite. Indeed, p^*p_* would send $F^n\mathbb{Z}_C(L)$ to $F^{n+1}\mathbb{Z}_C(L)$, so $(p^*p_*)^N = 0$ for some $N \geq 1$, and hence $N \cdot \sum_{g \in G} x^g = 0$, a contradiction.

3. Co-transfer

In this section, k will be an arbitrary perfect field. We recall how one uses the deformation to the normal bundle to define the "co-transfer"

$$(\mathbb{P}^1_F, 1_F) \to (\mathbb{P}^1_F(x), 1_F)$$

for a closed point $x \in \mathbb{A}^1_F \subset \mathbb{P}^1_F$, with chosen generator $f \in m_x/m_x^2$. For later use, we work in a somewhat more general setting: Let S be a smooth finite type k-scheme and x a regular closed subscheme of $\mathbb{P}^1_S \setminus \{1\} \subset \mathbb{P}^1_S$, such that the projection $x \to S$ is finite. Let $m_x \subset \mathcal{O}_{\mathbb{P}^1_S}$ be the ideal sheaf of x. We assume that the invertible sheaf m_x/m_x^2 on x is isomorphic to the trivial invertible sheaf $\mathcal{O}_x$, and we choose a generator $f \in \Gamma(x, m_x/m_x^2)$ over $\mathcal{O}_x$.

We will eventually replace S with a semi-local affine scheme, $S = \operatorname{Spec} R$, for R a smooth semi-local k-algebra, essentially of finite type over k, for instance, $R = F$ a finitely generated separable field extension of k. Although this will take us out of the category $\mathcal{H}(k)$, this will not be a problem: when we work with a smooth scheme Y which is essentially of finite type over k, we will consider Y as a pro-object in $\mathcal{H}(k)$, and we will be interested in functors on $\mathcal{H}(k)$ of the form $\operatorname{Hom}_{\mathcal{H}(k)}(Y, -)$, which will then be a well-defined filtered colimit of co-representable functors.

Let $(X_0 : X_1)$ be the standard homogeneous coordinates on $\mathbb{P}^1$. We let $s := X_1/X_0$ be the standard parameter on $\mathbb{P}^1$, and as usual, write $0 = (1 : 0)$, $\infty = (0 : 1)$, $1 = (1 : 1)$. We often write $0, 1, \infty$ for the subschemes $0_X, 1_X, \infty_X$ of $\mathbb{P}^1_X$.

Let $\mu : W_x \to \mathbb{P}^1_S \times \mathbb{A}^1$ be the blow-up of $\mathbb{P}^1_S \times \mathbb{A}^1$ along $(x, 0)$ with exceptional divisor E. Let s_x, C_0 be the proper transforms $s_x = \mu^{-1}[x \times \mathbb{A}^1]$, $C_0 = \mu^{-1}[\mathbb{P}^1 \times 0]$. Let t be the standard parameter on $\mathbb{A}^1$ and let $\tilde{f}$ be a local lifting of f to a section of m_x; the rational function $\tilde{f}/t$ restricts to a well-defined rational parameter on E, independent of the choice of lifting, and thus defines a globally defined isomorphism

$$f/t : E \to \mathbb{P}^1_x.$$

We identify E with $\mathbb{P}^1_x$ by sending $s_x \cap E$ to 0, $C_0 \cap E$ to 1 and the section on E defined by $f/t = 1$ to ∞. Denote this isomorphism by

$$\varphi_f : \mathbb{P}^1_x \to E;$$

we write $(0, 1, \infty)$ for $(s_x \cap E, C_0 \cap E, f/t = 1)$, when the context makes it clear we are referring to subschemes of E.

We let $W_x^{(s_x)}$, $E^{(0)}$, $(\mathbb{P}^1_F)^{(0)}$ be following homotopy push-outs

$$\begin{aligned} W_x^{(s_x)} &:= (W_x, W_x \setminus s_x),\\ E^{(0)} &:= (E, E \setminus 0),\\ (\mathbb{P}^1_S)^{(0)} &:= (\mathbb{P}^1_S, \mathbb{P}^1_S \setminus 0). \end{aligned}$$

Since $(\mathbb{A}^1_x, 0) \cong *$ in $\mathcal{H}_\bullet(k)$, the respective identity maps induce isomorphisms

$$\begin{aligned} &(E, 1) \to E^{(0)},\\ &(\mathbb{P}^1_S, 1) \to (\mathbb{P}^1_S)^{(0)}. \end{aligned}$$

Composing with the isomorphism $\varphi_f : (\mathbb{P}^1_x, 1) \to (E, 1)$, the inclusion $E \to W_x$ induces the map

$$i_{0,f} : (\mathbb{P}^1_x, 1) \to W_x^{(s_x)}.$$

The proof of the homotopy purity theorem of Morel-Voevodsky [15, theorem 2.23] yields as a special case that $i_{0,f}$ is an isomorphism in $\mathcal{H}_\bullet(k)$. This enables us to define the "co-transfer map" as follows:

Definition 3.1. Let $x \subset \mathbb{P}^1_S \setminus 1_S$ be a closed subscheme, smooth over k and finite over S, and suppose that m_x/m_x^2 is a free $\mathcal{O}_x$-module with generator f. The map

$$\textit{co-tr}_{x,f} : (\mathbb{P}^1_S, 1) \to (\mathbb{P}^1_x, 1)$$

in $\mathcal{H}_\bullet(k)$ is defined to be the composition

$$(\mathbb{P}^1_S, 1) \xrightarrow{i_1} W_x^{(s_x)} \xrightarrow{i_{0,f}^{-1}} (\mathbb{P}^1_x, 1).$$

Let $\mathcal{X} \in \mathcal{H}_\bullet(k)$ be a $\mathbb{P}^1$-loop space, i.e., $\mathcal{X} \cong \Omega^1_{\mathbb{P}}\mathcal{Y} := Maps_\bullet(\mathbb{P}^1, \mathcal{Y})$ for some $\mathcal{Y} \in \mathcal{H}_\bullet(k)$. For $x \subset \mathbb{P}^1_S$ and f as above, one has the transfer map

$$\mathcal{X}(x) \to \mathcal{X}(S)$$

in $\mathcal{H}_\bullet$ defined by pre-composing with the co-transfer map

$$\textit{co-tr}_{x,f} : (\mathbb{P}^1_S, 1) \to (\mathbb{P}^1_x, 1).$$

We will find modification of this construction useful in the sequel, namely, in the proof of lemma 5.9 and lemma 5.11. Let $s_1 := 1_S \times \mathbb{A}^1 \subset \mathbb{P}^1_S \times \mathbb{A}^1$; as $W_x \to \mathbb{P}^1_S \times \mathbb{A}^1$ is an isomorphism over a neighborhood of s_1, we view s_1 as a closed subscheme of W_x. We write W for W_x, etc., when the context makes the meaning clear.

Lemma 3.2. *Let $x \subset \mathbb{P}^1_S \setminus 1_S$ be a closed subscheme, smooth over k. Suppose that $x \to S$ is finite and étale. Then the identity on W induces an isomorphism*

$$(W, C_0 \cup s_1) \to W^{(s_x)}$$

in $\mathcal{H}_\bullet(k)$.

Proof. As $s_1 \cong \mathbb{A}^1_S$, with $C_0 \cap s_1 = 0_S$, the inclusion $C_0 \to C_0 \cup s_1$ is an isomorphism in $\mathcal{H}(k)$. Thus, we need to show that $(W, C_0) \to W^{(s_x)}$ is an isomorphism in $\mathcal{H}_\bullet(k)$. As $W^{(s_x)} = (W, W \setminus s_x)$, we need to show that $C_0 \to W \setminus s_x$ is an isomorphism in $\mathcal{H}(k)$. To aid in the proof, we will prove a more general result, namely, let $U \subset \mathbb{P}^1_S$ be a open subscheme containing x. We consider W as a scheme over $\mathbb{P}^1_S$ via the composition

$$W \xrightarrow{\mu} \mathbb{P}^1_S \times \mathbb{A}^1_S \xrightarrow{p_1} \mathbb{P}^1_S$$

and for a subscheme Z of W, let Z_U denote the pull-back $Z \times_{\mathbb{P}^1_S} U$. Then we will show that

$$C_{0U} \to W_U \setminus s_x$$

is an isomorphism in $\mathcal{H}(k)$.

We first reduce to the case in which $x \to S$ is an isomorphism (in $\mathbf{Sm}/k$). For this, we have the étale map $q : \mathbb{P}^1_x \to \mathbb{P}^1_S$ and the canonical x-point of $\mathbb{P}^1_x$, which we write as $\tilde{x}$. Let $U(x) \subset \mathbb{P}^1_x$ be a Zariski open neighborhood of $\tilde{x}$ such that $q^{-1}(x) \cap U(x) = \{\tilde{x}\}$. This gives us the elementary Nisnevich square

$$\mathcal{U}(x) := \begin{array}{ccc} U(x) \setminus \tilde{x} & \longrightarrow & U(x) \\ \downarrow & & \downarrow \\ \mathbb{P}^1_S \setminus x & \longrightarrow & \mathbb{P}^1_S; \end{array}$$

for each $\mathbb{P}^1_S$-scheme $Z \to \mathbb{P}^1_S$ we thus have the elementary Nisnevich square $\mathcal{U}(x) \times_{\mathbb{P}^1_S} Z$, giving a Nisnevich cover of Z.

Let $V \subset \mathbb{P}^1_S$ be an open subscheme with $x \cap V = \emptyset$. Then $W_V \to V \times_S \mathbb{A}^1_S$ is an isomorphism and $W_V \cap s_x = \emptyset$. Similarly $C_{0V} \to V$ is an isomorphism, and thus $C_{0V} \to W_V$ is an isomorphism in $\mathcal{H}(k)$. Replacing S with x, and considering the map of elementary Nisnevich squares

$$\mathcal{U}(x) \times_{\mathbb{P}^1_S} C_{0U} \to \mathcal{U}(x) \times_{\mathbb{P}^1_S} (W_U \setminus s_x)$$

induced by $C_0 \to W \setminus s_x$, we achieve the desired reduction. A similar Mayer-Vietoris argument allows us to replace S with a Zariski open cover of S, so, changing notation, we may assume that x is the point $0 := (1 : 0)$ of $\mathbb{P}^1_S$.

Using the open cover of $\mathbb{P}^1_S$ by the affine open subsets $U_0 := \mathbb{P}^1_S \setminus 1$, $U_1 := \mathbb{P}^1_S \setminus 0$ and arguing as above, we may assume that U is a subset of U_0, which we identify with $\mathbb{A}^1_S$ by sending $(0,\infty)$ to $(0,1)$. We may also assume that $0_S \subset U$. Using coordinates (t_1,t_2) for $\mathbb{A}^2$, (t_1,t_2,t_3) for $\mathbb{A}^3$, the scheme $W_{U_0} \setminus s_0$ is isomorphic to the closed subscheme of $\mathbb{A}^3_S$ defined by $t_2 = t_1t_3$, with μ being the projection $(t_1,t_2,t_3) \mapsto (t_1,t_2)$. C_{0U_0} is the subscheme of $W_{U_0} \setminus s_0$ defined by $t_3 = 0$. The projection $p_{13} : W_{U_0} \setminus s_0 \to \mathbb{A}^2_S$ is thus an isomorphism, sending C_0 to $\mathbb{A}^1_S \times 0$.
Let $y = U \setminus U_0$, so y is a closed subset disjoint from 0_S. Then

$$p_{13}(\mu^{-1}(y \times \mathbb{A}^1)) = y \times \mathbb{A}^1 \subset \mathbb{A}^2_S,$$

hence $p_{13} : W_U \setminus s_0 \to \mathbb{A}^2_S$ identifies $W_U \setminus s_0$ with $U \times \mathbb{A}^1$ and identifies C_{0U} with $U \times 0$. Thus $C_{0U} \to W_U \setminus s_0$ is an isomorphism in $\mathcal{H}(k)$, completing the proof. □

Lemma 3.3. *With hypotheses as in lemma 3.2, the inclusion $E \to W$ and isomorphism $\varphi_f : \mathbb{P}^1 \to E$ induces an isomorphism*

$$\tilde{i}_{0,f} : (\mathbb{P}^1_x, 1) \to (W_x, C_0 \cup s_1)$$

in $\mathcal{H}_\bullet(k)$.

Proof. We have the commutative diagram

$$\begin{tikzcd} (\mathbb{P}^1_x, 1) \arrow[r, "\tilde{i}_{0,f}"] \arrow[dr, "i_{0,f}"'] & (W, C_0 \cup s_1) \arrow[d] \\ & W^{(s_x)}. \end{tikzcd}$$

The diagonal arrow is an isomorphism in $\mathcal{H}_\bullet(k)$ by Morel-Voevodsky; the vertical arrow is an isomorphism by lemma 3.2. □

Definition 3.4. Let $x \subset \mathbb{P}^1_S \setminus 1_S$ be a closed subscheme, smooth over k and finite and étale over S. Suppose that m_x/m_x^2 is a free $\mathcal{O}_x$-module with generator f. The map

$$\widetilde{co\text{-}tr}_{x,f} : (\mathbb{P}^1_S, 1) \to (\mathbb{P}^1_x, 1)$$

in $\mathcal{H}_\bullet(k)$ is defined to be the composition

$$(\mathbb{P}^1_S, 1) \xrightarrow{i_1} (W, C_0 \cup s_1) \xrightarrow{\tilde{i}_{0,f}^{-1}} (\mathbb{P}^1_x, 1).$$

Remark 3.5. Given S, x, f as in definition 3.4, we have

$$\widetilde{co\text{-}tr}_{x,f} = co\text{-}tr_{x,f}.$$

This follows directly from the commutative diagram

$$\begin{tikzcd}
(\mathbb{P}^1_S,1) \arrow[r, "i_1"] \arrow[d, equal] & (W, C_0\cup s_1) \arrow[d, "\mathrm{id}"'] & (\mathbb{P}^1_x,1) \arrow[l, "\tilde{i}_{0,f}"'] \arrow[d, equal] \\
(\mathbb{P}^1_S,1) \arrow[r, "i_1"'] & W^{(s_x)} & (\mathbb{P}^1_x,1). \arrow[l, "i_{0,f}"]
\end{tikzcd}$$

We examine some properties of $co\text{-}tr_{x,f}$. For any ordering (a,b,c) of $\{0,1,\infty\}$, we let $\tau^a_{b,c}$ denote the automorphism of $\mathbb{P}^1$ that fixes a and exchanges b and c. For $u\in k^\times$, we let $\mu(u)$ be the automorphism of $\mathbb{P}^1$ that fixes 0 and ∞ and sends 1 to u. We first prove the following elementary result

Lemma 3.6. *The automorphism* $\rho := \tau^0_{1,\infty}\circ\mu(-1)\circ\tau^0_{1,\infty}\circ\tau^1_{0,\infty}$ *of* $(\mathbb{P}^1,1)$ *is the identity in* $\mathcal{H}_\bullet(k)$.

Proof. $\tau^0_{1,\infty}\rho\tau^0_{1,\infty}$ is the automorphism of $(\mathbb{P}^1,\infty)$ given by the matrix

$$\begin{pmatrix} 1 & 0\\ -1 & 1\end{pmatrix}\in \mathrm{GL}_2(k).$$

Noting that elementary matrices of the form

$$\begin{pmatrix} 1 & 0\\ \lambda & 1\end{pmatrix}$$

all fix ∞ and thus define automorphisms of $(\mathbb{P}^1,\infty)$ that are $\mathbb{A}^1$-homotopic to the identity, we see that $\tau^0_{1,\infty}\rho\tau^0_{1,\infty}=\mathrm{id}$ on $(\mathbb{P}^1,\infty)$ in $\mathcal{H}_\bullet(k)$, and thus $\rho=\mathrm{id}$ on $(\mathbb{P}^1,1)$ in $\mathcal{H}_\bullet(k)$. $\square$

Lemma 3.7. *1. The map* $co\text{-}tr_{0,-s}:(\mathbb{P}^1,1)\to(\mathbb{P}^1,1)$ *is the identity.*

2. The map $co\text{-}tr_{\infty,-s^{-1}}:(\mathbb{P}^1,1)\to(\mathbb{P}^1,1)$ *is the map in* $\mathcal{H}_\bullet(k)$ *induced by the automorphism* $\tau^1_{0,\infty}$.

3. The map $co\text{-}tr_{\infty,s^{-1}}:(\mathbb{P}^1,1)\to(\mathbb{P}^1,1)$ *is the identity.*

Proof. Since $\tau^{1*}_{0,\infty}(-s)=-s^{-1}$, (2) follows from (1) by applying $\tau^1_{0,\infty}$. Next, we show that (2) implies (3). It follows directly from the definition of $co\text{-}tr_{*,*}$ that

$$co\text{-}tr_{\infty,s^{-1}}=\tau^0_{1,\infty}\circ\mu(-1)\circ\tau^0_{1,\infty}\circ co\text{-}tr_{\infty,-s^{-1}}.$$

Thus, assuming (2), we have

$$co\text{-}tr_{\infty,s^{-1}}=\tau^0_{1,\infty}\circ\mu(-1)\circ\tau^0_{1,\infty}\circ\tau^1_{0,\infty}$$

in $\mathcal{H}_\bullet(k)$; (3) then follows from lemma 3.6.
We now prove (1). Identify $\mathbb{A}^1$ with $\mathbb{P}^1\setminus\{1\}$ sending 0 to 0 and 1 to ∞. The blow-up $W:=W_0$ is thus identified with an open subscheme of the blow-up $\bar\mu:\bar W\to\mathbb{P}^1\times\mathbb{P}^1$ of $\mathbb{P}^1\times\mathbb{P}^1$ at $(0,0)$.

The curve C_0 on $\bar{W}$ has self-intersection -1, and can thus be blown down via a morphism

$$\bar{\rho} : \bar{W} \to \bar{W}'.$$

Letting $q : \bar{W} \to \mathbb{P}^1$ be the composition

$$\bar{W} \xrightarrow{\bar{\mu}} \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{p_2} \mathbb{P}^1,$$

the fact that $q(C_0) = 0$ implies that q descends to a morphism

$$\bar{q}' : \bar{W}' \to \mathbb{P}^1.$$

As the complement $\bar{W}_\infty := \bar{W} \setminus W$ is disjoint from C_0 and $\bar{\rho}$ is proper, we have the open subscheme $W' := \bar{W}' \setminus \bar{\rho}(\bar{W}_\infty)$ of $\bar{W}'$ and the proper birational morphism

$$\rho : W \to W',$$

with $\rho(C_0) \cong \operatorname{Spec} k$ and with the restriction $W \setminus C_0 \to W' \setminus \rho(C_0)$ an isomorphism. In addition, $\bar{q}'$ restricts to the proper morphism

$$q' : W' \to \mathbb{P}^1 \setminus 1.$$

In addition, q' is a smooth and projective morphism with geometric fibers isomorphic to $\mathbb{P}^1$. Finally, we have

$$q'^{-1}(0) = \rho(E).$$

Let $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1 \setminus \{1\}$ be the restriction of the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$, giving us the proper transform $\mu^{-1}[\Delta]$ on W and the image $\Delta' = \rho(\mu^{-1}[\Delta])$ on W'. Similarly, let $s_0' = \rho(s_0)$, $s_1' = \rho(s_1)$; note that $\rho(C_0) \subset s_1'$. It is easy to check that s_0', Δ' and s_1' give disjoint sections of $q' : W' \to \mathbb{P}^1 \setminus 1$, hence there is a unique isomorphism (over $\mathbb{P}^1 \setminus 1$) of W' with $\mathbb{P}^1 \times \mathbb{P}^1 \setminus 1$ sending (s_0', s_1', Δ') to $(0, 1, \infty) \times \mathbb{P}^1 \setminus 1$. We have in addition the commutative diagram

$$\begin{tikzcd}
(\mathbb{P}^1, 1) \arrow[r, "i_{0,-s}"] \arrow[dr, "i_0'"'] & (W, W \setminus s_0) \arrow[d, "\rho"] & (\mathbb{P}^1, 1) \arrow[l, "i_1"'] \arrow[dl, "i_1'"] \\
& (W', W' \setminus s_0') &
\end{tikzcd} \tag{3.1}$$

where i_0' is the canonical identification of $\mathbb{P}^1$ with the fiber of W' over 0, sending $(0, 1, \infty)$ into (s_0', s_1', Δ'), and i_1' is defined similarly.

We claim that the isomorphism $\rho : E \to q'^{-1}(0)$ is a pointed isomorphism

$$\rho : (E, 0, 1, \infty) \to (q'^{-1}(0), q'^{-1}(0) \cap s_0', q'^{-1}(0) \cap s_1', q'^{-1}(0) \cap \Delta').$$

Indeed, by definition $0 = E \cap s_0$ and $1 = E \cap C_0$. Since $\rho(s_0) = s_0'$ and $\rho(C_0) \subset s_1'$, we need only show that $\rho(\infty) \subset \Delta'$. To distinguish the two factors of $\mathbb{P}^1$, we write

$$x_1 = p_1^*(s), x_2 = p_2^*(s)$$

where s is the standard parameter on $\mathbb{P}^1$. Using this notation, ∞ is the subscheme of E defined by the equation $-x_1/t = 1$, where t is the standard parameter on $\mathbb{A}^1_S = \mathbb{P}^1_S \setminus 1_S$. As our identification of $\mathbb{P}^1 \setminus 1_S$ with $\mathbb{A}^1$ sends $0 \in \mathbb{A}^1$ to $0 \in \mathbb{P}^1$, $1 \in \mathbb{A}^1$ to ∞ in $\mathbb{P}^1$, the standard parameter t goes over to

the rational function $x_2/(x_2-1)$ on $\mathbb{P}^1$. As the image of $x_2/(x_2-1)$ in m_0/m_0^2 is the same as the image of $-x_2$ in m_0/m_0^2, ∞ is defined by $x_1/x_2 = 1$ on E, which is clearly the subscheme defined by $E \cap \Delta$. Via the isomorphism ρ, this goes over to $q'^{-1}(0) \cap \Delta'$, as desired.
It follows from the proof of [15, theorem 2.2.3] that all the morphisms in the diagram (3.1) are isomorphisms in $\mathcal{H}_\bullet(k)$; as $i_0'^{-1} \circ i_1'$ is clearly the identity, the lemma is proved. □

The proof of the next result is easy and is left to the reader.

LEMMA 3.8. *Let $S' \to S$ be a morphism of smooth finite type k-schemes. Let x be a closed subscheme of $\mathbb{P}^1_S \setminus \{1\}$, finite over S. Let $x' = x \times_S S' \subset \mathbb{P}^1_{S'}$. We suppose we have a generator f for m_x/m_x^2, and let f' be the extension to $m_{x'}/m_{x'}^2$. If either $S' \to S$ is smooth, or $S \to S'$ is flat and $x \to S$ is étale, then the diagram*

$$\begin{tikzcd}
(\mathbb{P}^1_{S'}, 1) \arrow[r, "co\text{-}tr_{x',f'}"] \arrow[d] & (\mathbb{P}^1_{x'}, 1) \arrow[d] \\
(\mathbb{P}^1_S, 1) \arrow[r, "co\text{-}tr_{x,f}"'] & (\mathbb{P}^1_x, 1)
\end{tikzcd}$$

is defined and commutes.

4. CO-GROUP STRUCTURE ON $\mathbb{P}^1$

In this section, k will be an arbitrary perfect field. Let $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}$, which we consider as a pointed scheme with base-point 1. We recall the Mayer-Vietoris square for the standard cover of $\mathbb{P}^1$:

$$\begin{tikzcd}
\mathbb{G}_m \arrow[r, "t_\infty"] \arrow[d, "t_0"'] & \mathbb{A}^1 \arrow[d, "j_\infty"] \\
\mathbb{A}^1 \arrow[r, "j_0"'] & \mathbb{P}^1.
\end{tikzcd}$$

Here j_0, j_∞, t_0, t_∞ are defined by $j_0(t) = (1 : t)$, $j_\infty(t) = (t : 1)$, $t_0(t) = t$ and $t_\infty(t) = t^{-1}$. This gives us the isomorphism in $\mathcal{H}_\bullet(k)$ of $\mathbb{P}^1$ with the homotopy push-out in the diagram

(4.1)
$$\begin{tikzcd}
\mathbb{G}_m \arrow[r, "t_\infty"] \arrow[d, "t_0"'] & \mathbb{A}^1 \\
\mathbb{A}^1;
\end{tikzcd}$$

the contractibility of $\mathbb{A}^1$ gives us the canonical isomorphism

(4.2)
$$\alpha : S^1 \wedge \mathbb{G}_m \xrightarrow{\sim} (\mathbb{P}^1, 1).$$

This, together with the standard co-group structure on S^1, $\sigma : S^1 \to S^1 \vee S^1$, makes $(\mathbb{P}^1, 1)$ a co-group object in $\mathcal{H}_\bullet(k)$; let

$$\sigma_{\mathbb{P}^1} := \sigma \wedge \mathrm{id}_{\mathbb{G}_m} : (\mathbb{P}^1, 1) \to (\mathbb{P}^1, 1) \vee (\mathbb{P}^1, 1)$$

be the co-multiplication. In this section, we discuss a more algebraic description of this structure.

Let $f := (f_0, f_\infty)$ be a pair of generators for m_0/m_0^2, m_∞/m_∞^2. giving us the collapse map

$$\textit{co-tr}_{\{0,\infty\},f} : (\mathbb{P}^1, 1) \to (\mathbb{P}^1, 1) \vee (\mathbb{P}^1, 1).$$

LEMMA 4.1. *Let s be the standard parameter X_1/X_0 on $\mathbb{P}^1$. For $f = (-s, s^{-1})$, we have $\sigma_{\mathbb{P}^1} = \textit{co-tr}_{\{0,\infty\},f}$ in $\mathcal{H}_\bullet(k)$.*

Proof. We first unwind the definition of $\sigma_{\mathbb{P}^1}$ in some detail. As above, we identify $\mathbb{P}^1$ with the push-out in the diagram (4.1) and thus $(\mathbb{P}^1, 1)$ is isomorphic to the push-out in the diagram

$$\begin{array}{ccc}
(\mathbb{G}_m, 1) \vee (\mathbb{G}_m, 1) & \xrightarrow{(\mathrm{id},\mathrm{id})} & (\mathbb{G}_m, 1) \\
{\scriptstyle (t_0 \vee t_\infty)}\downarrow & & \\
(\mathbb{A}^1, 1) \vee (\mathbb{A}^1, 1). & &
\end{array}$$

Let I denote a simplicial model of the interval admitting a "mid-point" $1/2$, for example, we can take $I = \Delta^1{}_1\vee_0 \Delta^1$. The isomorphism $\alpha : S^1 \wedge \mathbb{G}_m \to (\mathbb{P}^1, 1)$ in $\mathcal{H}_\bullet(k)$ arises via a sequence of comparison maps between push-outs in the following diagrams (we point $\mathbb{P}^1$, $\mathbb{A}^1$ and $\mathbb{G}_m$ with 1):

$$\begin{array}{ccccccc}
\mathbb{G}_m \vee \mathbb{G}_m & \xrightarrow{(\mathrm{id},\mathrm{id})} & \mathbb{G}_m & & 0_+ \wedge \mathbb{G}_m \vee 1_+ \wedge \mathbb{G}_m & \xrightarrow{(\iota_0,\iota_1)} & I_+ \wedge \mathbb{G}_m \\
{\scriptstyle (t_0 \vee t_\infty)}\downarrow & & & \leftarrow & {\scriptstyle t_0 \vee t_\infty}\downarrow & & \\
\mathbb{A}^1 \vee \mathbb{A}^1 & & & & 0_+ \wedge \mathbb{A}^1 \vee 1_+ \wedge \mathbb{A}^1 & & \\
 & & & & & \downarrow & \\
 & & & & 0_+ \wedge \mathbb{G}_m \vee 1_+ \wedge \mathbb{G}_m & \xrightarrow{(\iota_0,\iota_1)} & I_+ \wedge \mathbb{G}_m \\
 & & & & \downarrow & & \\
 & & & & * & &
\end{array} \tag{4.3}$$

the first map is induced by the evident projections and the second by contracting $\mathbb{A}^1$ to $*$. Thus, the open immersion $\mathbb{G}_m \to \mathbb{P}^1$, $t \mapsto (1 : t)$, goes over to the map

$$\{1/2\}_+ \wedge \mathbb{G}_m \to I_+ \wedge \mathbb{G}_m \to S^1 \wedge \mathbb{G}_m,$$

the second map given by the bottom diagram in (4.3). This gives us the isomorphism

$$\rho : (\mathbb{P}^1, \mathbb{G}_m) \to S^1 \wedge \mathbb{G}_m \vee S^1 \wedge \mathbb{G}_m$$

in $\mathcal{H}_\bullet(k)$, yielding the commutative diagram

$$\begin{tikzcd}
(\mathbb{P}^1, 1) \arrow[r, "\alpha", "\sim"'] \arrow[d, "\pi"'] & S^1 \wedge \mathbb{G}_m \arrow[d, "\sigma\wedge\mathrm{id}"] \\
(\mathbb{P}^1, \mathbb{G}_m) \arrow[r, "\sim", "\rho"'] & S^1 \wedge \mathbb{G}_m \vee S^1 \wedge \mathbb{G}_m,
\end{tikzcd}$$

where π is the canonical quotient map and α is the isomorphism (4.2).
If we consider the middle diagram in (4.3), we find a similarly defined isomorphism (in $\mathcal{H}_\bullet(k)$)

$$\epsilon : (\mathbb{P}^1, \mathbb{G}_m) \to (\mathbb{A}^1, \mathbb{G}_m)_{t_0} \vee (\mathbb{A}^1, \mathbb{G}_m)_{t_\infty},$$

where the subscripts t_0, t_∞ refer to the morphism $\mathbb{G}_m \to \mathbb{A}^1$ used.
The map from the middle diagram to the last diagram in (4.3) furnishes the commutative diagram of isomorphisms in $\mathcal{H}_\bullet(k)$:

$$\begin{tikzcd}
(\mathbb{A}^1, \mathbb{G}_m)_{t_0} \vee (\mathbb{A}^1, \mathbb{G}_m)_{t_\infty} \arrow[r, "\beta"] \arrow[dr, "\vartheta"'] & S^1 \wedge \mathbb{G}_m \vee S^1 \wedge \mathbb{G}_m \arrow[d, "\alpha\vee\alpha"] \\
& (\mathbb{P}^1, 1) \vee (\mathbb{P}^1, 1).
\end{tikzcd}$$

Putting this all together gives us the commutative diagram in $\mathcal{H}_\bullet(k)$:

(4.4)
$$\begin{tikzcd}
(\mathbb{P}^1, 1) \arrow[rr, "\sim", "\alpha"'] \arrow[d, "\gamma"'] & & S^1 \wedge \mathbb{G}_m \arrow[d, "\sigma\wedge\mathrm{id}"] \\
(\mathbb{P}^1, \mathbb{G}_m) \arrow[rr, "\sim", "\rho"'] \arrow[d, "\epsilon"', "\sim"] & & S^1 \wedge \mathbb{G}_m \vee S^1 \wedge \mathbb{G}_m \arrow[d, "\alpha\vee\alpha"] \\
(\mathbb{A}^1, \mathbb{G}_m)_{t_0} \vee (\mathbb{A}^1, \mathbb{G}_m)_{t_\infty} \arrow[rr, "\sim", "\vartheta"'] \arrow[urr, "\beta", "\sim"'] & & (\mathbb{P}^1, 1) \vee (\mathbb{P}^1, 1).
\end{tikzcd}$$

Letting $\delta := \vartheta \circ \epsilon$, we thus need to show that the map $\delta \circ \gamma : (\mathbb{P}^1, 1) \to (\mathbb{P}^1, 1) \vee (\mathbb{P}^1, 1)$ is given by $\mathit{co\text{-}tr}_{\{0,1\},f}$.
Write $(\mathbb{A}^1, \mathbb{G}_m) := (\mathbb{A}^1, \mathbb{G}_m)_{t_0}$. Letting η be the inverse on $\mathbb{G}_m$, $\eta(t) = t^{-1}$, we identify $(\mathbb{A}^1, \mathbb{G}_m)$ with $(\mathbb{A}^1, \mathbb{G}_m)_{t_\infty}$ via the isomorphism (id, η). The maps $j_0 : \mathbb{A}^1 \to \mathbb{P}^1$, $j_\infty : \mathbb{A}^1 \to \mathbb{P}^1$ induce the isomorphisms in $\mathcal{H}_\bullet(k)$

$$\bar{j}_0 : (\mathbb{A}^1, \mathbb{G}_m) \to (\mathbb{P}^1, j_\infty(\mathbb{A}^1))$$
$$\bar{j}_\infty : (\mathbb{A}^1, \mathbb{G}_m) \to (\mathbb{P}^1, j_0(\mathbb{A}^1))$$

giving together the isomorphism $\tau : (\mathbb{P}^1, 1) \vee (\mathbb{P}^1, 1) \to (\mathbb{A}^1, \mathbb{G}_m) \vee (\mathbb{A}^1, \mathbb{G}_m)$, defined as the composition:

$$(\mathbb{P}^1, 1) \vee (\mathbb{P}^1, 1) \xrightarrow{\mathrm{id}\vee\mathrm{id}} (\mathbb{P}^1, j_\infty(\mathbb{A}^1)) \vee (\mathbb{P}^1, j_0(\mathbb{A}^1)) \xrightarrow{\bar{j}_0^{-1} \vee \bar{j}_\infty^{-1}} (\mathbb{A}^1, \mathbb{G}_m) \vee (\mathbb{A}^1, \mathbb{G}_m).$$

By comparing with the push-out diagrams in (4.3), we see that τ is the inverse to ϑ. As $\tau^1_{0,\infty}$ exchanges j_0 and j_∞, this gives the identity

$$\vartheta = \vartheta_0 \vee \tau^1_{0,\infty} \circ \vartheta_0, \tag{4.5}$$

where ϑ_0 is the composition

$$(\mathbb{A}^1, \mathbb{G}_m) \xrightarrow{j_0} (\mathbb{P}^1, j_\infty(\mathbb{A}^1)) \xleftarrow[\sim]{\mathrm{id}} (\mathbb{P}^1, 1).$$

Let $W \to \mathbb{P}^1 \times \mathbb{A}^1$ be the blow-up at $(\{0,\infty\}, 0)$ with exceptional divisor E. Let g be the trivialization $g := (-s, -s^{-1})$ of $m_0/m_0^2 \times m_\infty/m_\infty^2$. We have the composition of isomorphisms in $\mathcal{H}_\bullet(k)$

$$(\mathbb{P}^1, \mathbb{G}_m) \xrightarrow{i_1} (W, W \setminus s_{\{0,\infty\}}) \xleftarrow{i_0} (E, C_0 \cap E) \xleftarrow{\varphi_g} (\mathbb{P}^1, 1) \vee (\mathbb{P}^1, 1). \tag{4.6}$$

The open cover $(j_0, j_\infty) : \mathbb{A}^1 \amalg \mathbb{A}^1 \to \mathbb{P}^1$ of $\mathbb{P}^1$ gives rise to an open cover of W: Let $\mu' : W' \to \mathbb{A}^1 \times \mathbb{A}^1$ be the blow-up at $(0,0)$, then we have the lifting of (j_0, j_∞) to the open cover

$$(j'_0, j'_\infty) : W' \amalg W' \to W.$$

The cover (j_0, j_∞) induces the excision isomorphism in $\mathcal{H}_\bullet(k)$

$$(\hat{j}_0, \hat{j}_\infty) : (\mathbb{A}^1, \mathbb{G}_m) \vee (\mathbb{A}^1, \mathbb{G}_m) \to (\mathbb{P}^1, \mathbb{G}_m);$$

it is easy to see that $(\hat{j}_0, \hat{j}_\infty)$ is inverse to the isomorphism ϵ in diagram (4.4). Similarly, letting $s' \subset W'$ be the proper transform of $0 \times \mathbb{A}^1$ to W', the cover (j'_0, j'_∞) induces the excision isomorphism in $\mathcal{H}_\bullet(k)$

$$(\tilde{j}'_0, \tilde{j}'_\infty) : (W', W' \setminus s') \vee (W', W' \setminus s') \to (W, W \setminus s_{\{0,\infty\}}).$$

This extends to a commutative diagram of isomorphisms in $\mathcal{H}_\bullet(k)$

$$\begin{tikzcd}
(\mathbb{A}^1, \mathbb{G}_m) \vee (\mathbb{A}^1, \mathbb{G}_m) \arrow[r, "(\hat{j}_0, \hat{j}_\infty)"] \arrow[d, "i_1 \vee i_1"'] & (\mathbb{P}^1, \mathbb{G}_m) \arrow[d, "i_1"] \\
(W', W' \setminus s') \vee (W', W' \setminus s') \arrow[r, "(\tilde{j}'_0, \tilde{j}'_\infty)"] & (W, W \setminus s_{\{0,\infty\}}) \\
(E', E' \cap C'_0) \vee (E', E' \cap C'_0) \arrow[r, "(\tilde{j}'_{E0}, \tilde{j}'_{E\infty})"] \arrow[u, "i_0 \vee i_0"] & (E, E \cap C_0) \arrow[u, "i_0"'] \\
(\mathbb{P}^1, 1) \vee (\mathbb{P}^1, 1) \arrow[r, equal] \arrow[u, "\varphi_{-s} \vee \varphi_{-s}"] & (\mathbb{P}^1, 1) \vee (\mathbb{P}^1, 1). \arrow[u, "\varphi_g"']
\end{tikzcd} \tag{4.7}$$

Indeed, the commutativity is obvious, except on the bottom square. On the first summand $(\mathbb{P}^1, 1)$, the commutativity is also obvious, since both φ_{-s} and φ_g are defined on this factor using the generator $-s$ for m_0/m_0^2, and on the second factor, the map $\tilde{j}_\infty$ sends $-s$ to $-s^{-1}$, which gives the desired commutativity. Examining the push-out diagram (4.3), we see that the map

$$(\hat{j}_0, \hat{j}_\infty) : (\mathbb{A}^1, \mathbb{G}_m) \vee (\mathbb{A}^1, \mathbb{G}_m) \to (\mathbb{P}^1, \mathbb{G}_m)$$

is inverse to the map ϵ in diagram (4.4).

Let $W_0 \to \mathbb{P}^1 \times \mathbb{A}^1$ be the blow-up along $(0,0)$, E^0 the exceptional divisor, C_0^0 the proper transform of $\mathbb{P}^1 \times 0$. The inclusion j_0 induces the excision isomorphism in $\mathcal{H}_\bullet(k)$

$$j : (\mathbb{A}^1, \mathbb{G}_m) \to (\mathbb{P}^1, j_\infty(\mathbb{A}^1))$$

and gives us the commutative diagram

$$\begin{tikzcd}
 & (\mathbb{P}^1,1) \vee (\mathbb{P}^1,1) \arrow[d] \\
(\mathbb{A}^1,\mathbb{G}_m) \vee (\mathbb{A}^1,\mathbb{G}_m) \arrow[ur, "\vartheta_0 \vee \vartheta_0"] \arrow[r, "(j \vee j)"] \arrow[d, "i_1 \vee i_1"'] & (\mathbb{P}^1, j_\infty(\mathbb{A}^1)) \vee (\mathbb{P}^1, j_\infty(\mathbb{A}^1)) \arrow[d, "i_1 \vee i_1"'] \\
(W', W' \setminus s') \vee (W', W' \setminus s') \arrow[r, "(\tilde{j} \vee \tilde{j})"] & (W_0, W_0 \setminus s_0) \vee (W_0, W_0 \setminus s_0) \\
(E', E' \cap C_0') \vee (E', E' \cap C_0') \arrow[u, "i_0 \vee i_0"] \arrow[r, "(\tilde{j}_{E'} \vee \tilde{j}_{E'})"] & (E^0, E^0 \cap C_0^0) \vee (E^0, E^0 \cap C_0^0) \arrow[u, "i_0 \vee i_0"'] \\
(\mathbb{P}^1,1) \vee (\mathbb{P}^1,1) \arrow[u, "\varphi_{-s} \vee \varphi_{-s}"] \arrow[r, equal] & (\mathbb{P}^1,1) \vee (\mathbb{P}^1,1). \arrow[u, "\varphi_{-s} \vee \varphi_{-s}"]
\end{tikzcd}$$

By lemma 3.7 the composition along the right-hand side of this diagram is the identity on $(\mathbb{P}^1,1) \vee (\mathbb{P}^1,1)$, and thus the composition along the left-hand side is $\vartheta_0 \vee \vartheta_0 : (\mathbb{A}^1, \mathbb{G}_m) \vee (\mathbb{A}^1, \mathbb{G}_m) \to (\mathbb{P}^1,1) \vee (\mathbb{P}^1,1)$. Referring to diagram (4.4), as $\epsilon = (\hat{j}_0, \hat{j}_\infty)^{-1}$, it follows from (4.5) that the composition along the right-hand side of (4.7) is the map $(\mathrm{id} \vee \tau^1_{0,\infty}) \circ \delta$. As the right-hand side of (4.7) is the deformation diagram used to define $co\text{-}tr_{\{0,\infty\},g}$, we see that

$$co\text{-}tr_{\{0,\infty\},g} = (\mathrm{id} \vee \tau^1_{0,\infty}) \circ \sigma_{\mathbb{P}^1}.$$

Noting that f and g differ only by the trivialization at ∞, changing s^{-1} to $-s^{-1}$, we thus have

$$co\text{-}tr_{\{0,\infty\},f} = (\mathrm{id} \vee \tau^0_{1,\infty} \circ \mu(-1) \circ \tau^0_{1,\infty}) \circ co\text{-}tr_{\{0,\infty\},g}.$$

By lemma 3.6, we have

$$co\text{-}tr_{\{0,\infty\},f} = (\mathrm{id} \vee \tau^1_{0,\infty}) \circ co\text{-}tr_{\{0,\infty\},g} = \sigma_{\mathbb{P}^1}.$$

□

5. Slice localizations and co-transfer

In general, the co-transfer maps do not have the properties necessary to give a loop-spectrum $\Omega_{\mathbb{P}^1} E$ an action by correspondences. However, if we pass to a certain localization of $\mathcal{SH}_{S^1}(k)$ defined by the slice filtration, the co-transfer maps both extend to arbitrary correspondences and respect the composition of correspondences. This will lead to the action of correspondences on $s_0\Omega_{\mathbb{P}^1}E$ we wish to construct. In this section, k will be an arbitrary perfect field.

We have the localizing subcategory $\Sigma^n_{\mathbb{P}^1}\mathcal{SH}_{S^1}(k)$, generated (as a localizing subcategory) by objects of the form $\Sigma^n_{\mathbb{P}^1}E$, for $E \in \mathcal{SH}_{S^1}(k)$. We let $\mathcal{SH}_{S^1}(k)/f_n$ denote the localization of $\mathcal{SH}_{S^1}(k)$ with respect to $\Sigma^n_{\mathbb{P}^1}\mathcal{SH}_{S^1}(k)$:

$$\mathcal{SH}_{S^1}(k)/f_n = \mathcal{SH}_{S^1}(k)/\Sigma^n_{\mathbb{P}^1}\mathcal{SH}_{S^1}(k).$$

Remark 5.1. Pelaez [17, corollary 3.2.40] has shown that there is a model structure on $\mathbf{Spt}_{S^1}(k)$ with homotopy category equivalent to $\mathcal{SH}_{S^1}(k)/f_n$; in particular, this localization of $\mathcal{SH}_{S^1}(k)$ does exist.

Remark 5.2. In the proofs of some of the next few results we will use the following fact, which relies on our ground field k being perfect: Let $V \subset U$ be a Zariski open subset of some $U \in \mathbf{Sm}/k$. Then we can filter U by open subschemes

$$V = U^{N+1} \subset U^N \subset \ldots \subset U^0 = U$$

such that $U^{i+1} = U^i \setminus C_i$, with $C_i \subset U^i$ smooth and having trivial normal bundle in U^i for $i = 0, \ldots, N$. Indeed, let $C = U \setminus V$, with reduced scheme structure. As k is perfect, there is a dense open subscheme C_{sm} of C which is smooth over k, and there is a non-empty open subscheme $C_1 \subset C_{sm}$ such that the restriction of $\mathcal{I}_C/\mathcal{I}_C^2$ to C_1 is a free sheaf of rank equal to the codimension of C_1 in U. We let $U^1 = U \setminus C_1$, and then proceed by noetherian induction.

Lemma 5.3. *Let $V \to U$ be a dense open immersion in $\mathbf{Sm}/k$, $n \geq 1$ an integer. Then the induced map*

$$\Sigma^n_{\mathbb{P}^1}V_+ \to \Sigma^n_{\mathbb{P}^1}U_+$$

is an isomorphism in $\mathcal{SH}_{S^1}(k)/f_{n+1}$.

Proof. Filter U by open subschemes

$$V = U^{N+1} \subset U^N \subset \ldots \subset U^0 = U$$

as in remark 5.2. Write $U^{i+1} = U^i \setminus C_i$, with C_i having trivial normal bundle in U_i, of rank say r_i, for $i = 0, \ldots, N$.

By the Morel-Voevodsky purity theorem [15, theorem 2.23], the cofiber of $U^{i+1} \to U^i$ is isomorphic in $\mathcal{H}_\bullet(k)$ to $\Sigma^{r_i}_{\mathbb{P}^1}C_{i+}$, and thus the cofiber of $\Sigma^n_{\mathbb{P}^1}U^{i+1}_+ \to \Sigma^n_{\mathbb{P}^1}U^i_+$ is isomorphic to $\Sigma^{r_i+n}_{\mathbb{P}^1}C_{i+}$. Since V is dense in U, we have $r_i \geq 1$ for all i, proving the lemma. □

Take $W \in \mathbf{Sm}/k$. By excision and homotopy invariance, we have a canonical isomorphism

$$\psi_{W,r} : \mathbb{A}^r_W/\mathbb{A}^r_W \setminus 0_W \to \Sigma^r_{\mathbb{P}^1}W_+$$

in $\mathcal{H}_\bullet(k)$. The action of the group-scheme GL_r/k on $\mathbb{A}^r$ gives an action of the group of sections $\mathrm{GL}_r(W)$ on $\mathbb{A}^r_W/\mathbb{A}^r_W \setminus 0_W$, giving us for each $g \in \mathrm{GL}_r(W)$ the isomorphism

$$\psi^g_{W,r} :=: \psi_{W,r} \circ g : \mathbb{A}^r_W/\mathbb{A}^r_W \setminus 0_W \to \Sigma^r_{\mathbb{P}^1}W_+.$$

Lemma 5.4. *For each $g \in \mathrm{GL}_r(W)$, we have $\psi^g_{W,r} = \psi_{W,r}$ in $\mathcal{SH}_{S^1}/f_{r+1}$.*

Proof. The action $\mathrm{GL}_r \times \mathbb{A}^r \to \mathbb{A}^r$ composed with $\psi_{W,r}$ gives us the morphism in $\mathcal{H}_\bullet(k)$

$$\Psi_W : (W \times \mathrm{GL}_r)_+ \wedge (\mathbb{A}^r/\mathbb{A}^r \setminus 0) \to \Sigma^r_{\mathbb{P}^1} W_+;$$

for each section $g \in \mathrm{GL}_r(W)$, composing with the corresponding section $s_g : W \to W \times \mathrm{GL}_r$ gives the map

$$\Psi_W \circ s_g : W_+ \wedge (\mathbb{A}^r/\mathbb{A}^r \setminus 0) \to \Sigma^r_{\mathbb{P}^1} W_+$$

which is clearly equal to $\psi^g_{W,r}$.

The open immersion $j : W \times \mathrm{GL}_r \to W \times \mathbb{A}^{r^2}$ is by lemma 5.3 an isomorphism in $\mathcal{SH}_{S^1}(k)/f_1$; as $(\mathbb{A}^r/\mathbb{A}^r \setminus 0) \cong \Sigma^r_{\mathbb{P}^1} \operatorname{Spec} k_+$, we see that the induced map

$$j \wedge \mathrm{id} : (W \times \mathrm{GL}_r)_+ \wedge (\mathbb{A}^r/\mathbb{A}^r \setminus 0) \to (W \times \mathbb{A}^{r^2})_+ \wedge (\mathbb{A}^r/\mathbb{A}^r \setminus 0)$$

is an isomorphism in $\mathcal{SH}_{S^1}(k)/f_{r+1}$, and thus the projection

$$(W \times \mathrm{GL}_r)_+ \wedge (\mathbb{A}^r/\mathbb{A}^r \setminus 0) \to W_+ \wedge (\mathbb{A}^r/\mathbb{A}^r \setminus 0)$$

is also an isomorphism in $\mathcal{SH}_{S^1}(k)/f_{r+1}$. From this it follows that the maps

$$s_g \wedge \mathrm{id}, s_{\mathrm{id}} \wedge \mathrm{id} : W_+ \wedge (\mathbb{A}^r/\mathbb{A}^r \setminus 0) \to (W \times \mathrm{GL}_r)_+ \wedge (\mathbb{A}^r/\mathbb{A}^r \setminus 0)$$

are equal in $\mathcal{SH}_{S^1}(k)/f_{r+1}$, hence $\psi^g_{W,r} = \psi_{W,r}$ in $\mathcal{SH}_{S^1}/f_{r+1}$. □

As application we have the following result

PROPOSITION 5.5. *1. Let S be in $\mathbf{Sm}/k$. Let $x \subset \mathbb{P}^1_S \setminus 1_S$ be a closed subscheme, smooth over k and finite over S, such that the co-normal bundle m_x/m_x^2 is trivial. Then the maps*

$$\textit{co-tr}_{x,f} : (\mathbb{P}^1_S, 1) \to (\mathbb{P}^1_x, 1)$$

in $\mathcal{SH}_{S^1}(k)/f_2$ are independent of the choice of generator f for m_x/m_x^2. If $S = \operatorname{Spec} \mathcal{O}_{X,x}$ for x a finite set of points on some $X \in \mathbf{Sm}/k$, the analogous independence holds, this time as morphisms in pro-$\mathcal{SH}_{S^1}(k)/f_2$.

2. Let $g : \mathbb{P}^1 \to \mathbb{P}^1$ be a k-automorphism, with $g(1) = 1$. Then $g : (\mathbb{P}^1, 1) \to (\mathbb{P}^1, 1)$ is the identity in $\mathcal{SH}_{S^1}(k)/f_2$.

3. Take $a, b \in \mathbb{P}^1(k)$, with $a \neq b$ and $a, b \neq 1$. The canonical isomorphism $a \amalg b \to \operatorname{Spec} k \amalg \operatorname{Spec} k$ gives the canonical identification $(\mathbb{P}^1_{a,b}, 1) \cong (\mathbb{P}^1, 1) \vee (\mathbb{P}^1, 1)$. Then for each choice of generator f for $m_{a,b}/m_{a,b}^2$, the map

$$\textit{co-tr}_{a,b,f} : (\mathbb{P}^1, 1) \to (\mathbb{P}^1_{a,b}, 1) \cong (\mathbb{P}^1, 1) \vee (\mathbb{P}^1, 1)$$

is equal in $\mathcal{SH}_{S^1}(k)/f_2$ to the co-multiplication $\sigma_{\mathbb{P}^1}$.

Proof. (1) Suppose that we have generators f, f' for m_x/m_x^2. There is thus a unit $a \in \mathcal{O}_x^*$ with $f' = af$. Note that $\textit{co-tr}_{x,f'} = g \circ \textit{co-tr}_{x,f}$, where $g : \mathbb{P}^1_x \to \mathbb{P}^1_x$ is the automorphism $\tau^0_{1,\infty}\mu(a)\tau^1_{1,\infty}$. By lemma 5.4, the map

$$\mu(a) = \psi^a_{\operatorname{Spec} k,1} \circ \psi^{-1}_{\operatorname{Spec} k,1} : (\mathbb{P}^1, \infty) \to (\mathbb{P}^1, \infty)$$

is the identity in $\mathcal{SH}_{S^1}(k)/f_2$, whence (1).

For (2), we may replace 1 with ∞. The affine group of isomorphisms $g : \mathbb{P}^1 \to \mathbb{P}^1$ with $g(\infty) = \infty$ is generated by the matrices of the form

$$\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix},$$

with $u \in k^\times$ and $\lambda \in k$. Clearly the automorphisms of the second type act as the identity on $(\mathbb{P}^1, \infty)$ in $\mathcal{H}_\bullet(k)$; the automorphisms of the first type act by the identity on $(\mathbb{P}^1, \infty)$ in $\mathcal{SH}_{S^1}(k)/f_2$ by lemma 5.4.
(3). Let $g : \mathbb{P}^1 \to \mathbb{P}^1$ be the automorphism sending $(0, 1, \infty)$ to $(a, 1, b)$. Choose a generator f for $m_{a,b}/m_{a,b}^2$, then g^*f gives a generator for $m_{0,\infty}/m_{0,\infty}^2$. The automorphism g extends to an isomorphism $\tilde{g} : W_{0,\infty} \to W_{a,b}$, giving us a commutative diagram

$$\begin{tikzcd}
(\mathbb{P}^1, 1) \arrow[r, "i_1"] \arrow[d, "g"'] & (W_{0,\infty}, W_{0,\infty} \setminus s_{0,\infty}) \arrow[d, "\tilde{g}"'] & (\mathbb{P}^1_{0,\infty}, 1) \arrow[l, "i_{0,g^*f}"'] \arrow[d, "\beta"] \\
(\mathbb{P}^1, 1) \arrow[r, "i_1"'] & (W_{a,b}, W_{a,b} \setminus s_{a,b}) & (\mathbb{P}^1_{a,b}, 1) \arrow[l, "i_{0,f}"]
\end{tikzcd}$$

where $\beta : \mathbb{P}^1_{0,\infty} \to \mathbb{P}^1_{a,b}$ is canonical isomorphism over $(0, \infty) \to (a, b)$. This gives us the identity in $\mathcal{H}_\bullet(k)$:

$$co\text{-}tr_{a,b,f} \circ g = \beta \circ co\text{-}tr_{0,\infty,g^*f}.$$

By (1), the maps $co\text{-}tr_{a,b,g^*f}$ and $co\text{-}tr_{0,\infty,f}$ are independent (in $\mathcal{SH}_{S^1}(k)/f_2$) of the choice of f and by (2), g is the identity in $\mathcal{SH}_{S^1}(k)/f_2$. For suitable f, lemma 4.1 tells us $co\text{-}tr_{0,\infty,f} = \sigma_{\mathbb{P}^1}$, completing the proof of (3). $\square$

As the map

$$co\text{-}tr_{x,f} : (\mathbb{P}^1_S, 1) \to (\mathbb{P}^1_x, 1)$$

in $\mathcal{SH}_{S^1}(k)/f_2$ is independent of the choice of generator $f \in m_x/m_x^2$; we denote this map by $co\text{-}tr_x$.
We have one additional application of lemma 5.4.

Lemma 5.6. *Let $W \subset U$ be a codimension $\geq r$ closed subscheme of $U \in \mathbf{Sm}/k$, let $w_1, \ldots, w_m$ be the generic points of W of codimension $= r$ in U. Then there is a canonical isomorphism of pro-objects in in $\mathcal{SH}_{S^1}/f_{r+1}$*

$$(U, U \setminus W) \cong \oplus_{i=1}^m \Sigma^r_{\mathbb{P}^1} w_{i+}.$$

Specifically, letting $m_i \subset \mathcal{O}_{U,w_i}$ be the maximal ideal, this isomorphism is independent of any choice of isomorphism $m_i/m_i^2 \cong k(w_i)^r$.

Proof. Let $w = \{w_1, \ldots .w_m\}$ and let $\mathcal{O}_{U,w}$ denote the semi-local ring of w in U. Consider the projective system $\mathcal{V} := \{V_\alpha\}$ consisting of open subschemes of U of the form $V_\alpha = U \setminus C_\alpha$, where C_α is a closed subset of W containing no generic point w_i of W.

Take $V_\alpha \in \mathcal{V}$. By applying remark 5.2, and noting that $U \setminus V_\alpha$ has codimension $\geq r+1$ in U, the argument used in the proof of lemma 5.3 shows that the cofiber of

$$(V_\alpha, V_\alpha \setminus W) \to (U, U \setminus W)$$

is in $\Sigma^{r+1}_{\mathbb{P}^1}\mathcal{SH}_{S^1}(k)$. On the other hand, the collection of $V_\alpha \in \mathcal{V}$ such that $V_\alpha \cap W$ is smooth and has on each connected component a trivial normal bundle in V_α forms a cofinal subsystem $\mathcal{V}'$ in $\mathcal{V}$. For each $V_\alpha \in \mathcal{V}'$, we have $V_\alpha \cap W = \amalg_{i=1}^m W_i^\alpha$, with w_i the unique generic point of W_i^α, and we have the isomorphism

$$(V_\alpha, V_\alpha \setminus W) \cong \bigvee_{i=1}^m \Sigma^r_{\mathbb{P}^1} W^\alpha_{i+}$$

in $\mathcal{H}_\bullet(k)$. Since w_i is equal to the projective limit of the W_i^α, we have the desired isomorphism of pro-objects in $\mathcal{SH}_{S^1}(k)/f_{r+1}$.

We need only verify that the resulting isomorphism $(U, U \setminus W) \cong \oplus_{r=1}^m \Sigma^r_{\mathbb{P}^1} w_{i+}$ is independent of any choices. Let $V = \operatorname{Spec} \mathcal{O}_{U,W}$, and let $\mathcal{O}$ denote the henselization of w in V. We have the canonical excision isomorphism (of pro-objects in $\mathcal{H}_\bullet(k)$)

$$(V, V \setminus V \cap W) \cong (\operatorname{Spec} \mathcal{O}, \operatorname{Spec} \mathcal{O} \setminus w).$$

A choice of isomorphism $m_w/m_w^2 \cong k(w)^r$ gives the isomorphism in pro-$\mathcal{H}_\bullet(k)$

$$\Sigma^r_{\mathbb{P}^1} w_+ \cong (\operatorname{Spec} \mathcal{O}, \operatorname{Spec} \mathcal{O} \setminus w);$$

this choice of isomorphism is thus the only choice involved in constructing our isomorphism $(U, U \setminus W) \cong \oplus_{i=1}^m \Sigma^r_{\mathbb{P}^1} w_{i+}$. Explicitly, the choice of isomorphism $m_w/m_w^2 \cong k(w)^r$ is reflected in the isomorphism $(\operatorname{Spec} \mathcal{O}, \operatorname{Spec} \mathcal{O} \setminus w) \cong \Sigma^r_{\mathbb{P}^1} w_+$ through the identification of the exceptional divisor of the blow-up of $V \times \mathbb{A}^1$ along $w \times 0$ with $\mathbb{P}^r_w$. The desired independence now follows from lemma 5.4. $\square$

The computation which is crucial for enabling us to introduce transfers on the higher slices of S^1-spectra is the following:

LEMMA 5.7. *Let $\mu_n : (\mathbb{P}^1, \infty) \to (\mathbb{P}^1, \infty)$ be the map $\mu_n(t_0 : t_1) = (t_0^n : t_1^n)$. Assume the characteristic of k is prime to $n!$. Then in $\mathcal{SH}_{S^1}(k)/f_2$, μ_n is multiplication by n.*

Proof. The proof goes by induction on n, starting with $n = 1, 2$. The case $n = 1$ is trivial. For $n = 2$, lemma 5.6 gives us the canonical isomorphisms in $\mathcal{SH}_{S^1}(k)/f_2$

$$(\mathbb{P}^1, \mathbb{P}^1 \setminus \{\pm 1\}) \xrightarrow{\alpha_{\pm 1}} (\mathbb{P}^1, \infty) \vee (\mathbb{P}^1, \infty); \quad (\mathbb{P}^1, \mathbb{P}^1 \setminus \{1\}) \xrightarrow{\alpha_1} (\mathbb{P}^1, \infty).$$

In addition, we have the commutative diagram

$$\begin{tikzcd}
(\mathbb{P}^1, \infty) \arrow[r] \arrow[d, "\mu_2"'] & (\mathbb{P}^1, \mathbb{P}^1 \setminus \{\pm 1\}) \arrow[d, "\mu_2"] \\
(\mathbb{P}^1, \infty) \arrow[r] & (\mathbb{P}^1, \mathbb{P}^1 \setminus \{1\})
\end{tikzcd}$$

The bottom horizontal arrow is an isomorphism in $\mathcal{H}_\bullet(k)$. We claim the diagram

$$\begin{tikzcd}
(\mathbb{P}^1, \mathbb{P}^1 \setminus \{\pm 1\}) \arrow[r, "\alpha_{\pm 1}"] \arrow[d, "\mu_2"] & (\mathbb{P}^1, \infty) \vee (\mathbb{P}^1, \infty) \arrow[d, "(\mathrm{id},\mathrm{id})"] \\
(\mathbb{P}^1, \mathbb{P}^1 \setminus \{1\}) \arrow[r, "\alpha_1"'] & (\mathbb{P}^1, \infty)
\end{tikzcd}$$

commutes in $\mathcal{SH}_{S^1}(k)/f_2$. Indeed, the isomorphism $\alpha_{\pm 1}$ arises from the Morel-Voevodsky homotopy purity isomorphism identifying $(\mathbb{P}^1, \mathbb{P}^1 \setminus \{\pm 1\})$ canonically with the Thom space of the tangent space $T(\mathbb{P}^1)_{\pm 1}$ of $\mathbb{P}^1$ at ± 1, followed by the isomorphism

$$Th(T(\mathbb{P}^1)_{\pm 1}) \cong \Sigma_{\mathbb{P}^1}(\pm 1_+) = (\mathbb{P}^1, \infty) \vee (\mathbb{P}^1, \infty)$$

induced by a choice of basis for $T(\mathbb{P}^1)_{\pm 1}$ (which plays no role in $\mathcal{SH}_{S^1}(k)/f_2$). Similarly the map α_1 arises from a canonical isomorphism of $(\mathbb{P}^1, \mathbb{P}^1 \setminus \{1\})$ with $Th(T(\mathbb{P}^1)_1)$ followed by the isomorphism

$$Th(T(\mathbb{P}^1)_1) \to (\mathbb{P}^1, \infty)$$

induced by a choice of basis. As the map μ_2 is étale over 1, the differential

$$d\mu_2 : T(\mathbb{P}^1)_{\pm 1} \to T(\mathbb{P}^1)_1$$

is isomorphic to the sum map

$$\mathbb{A}^1 \oplus \mathbb{A}^1 \to \mathbb{A}^1.$$

As this sum map induces $(\mathrm{id}, \mathrm{id})$ on the Thom spaces, we have verified our claim.

Using proposition 5.5 and we see that this diagram together with the isomorphisms $\alpha_{\pm 1}$ and α_1 gives us the factorization of μ_2 (in $\mathcal{SH}_{S^1}(k)/f_2$) as

$$(\mathbb{P}^1, \infty) \xrightarrow{\sigma} (\mathbb{P}^1, \infty) \vee (\mathbb{P}^1, \infty) \xrightarrow{(\mathrm{id},\mathrm{id})} (\mathbb{P}^1, \infty).$$

Here σ is the co-multiplication (using ∞ instead of 1 as base-point). Since $(\mathrm{id}, \mathrm{id}) \circ \sigma$ is multiplication by 2, this takes care of the case $n = 2$.

In general, we consider the map $\rho_n : (\mathbb{P}^1, \infty) \to (\mathbb{P}^1, \infty)$ sending $(t_0 : t_1)$ to $(w_0 : w_1) := (t_0^n : t_1^n - t_0 t_1^{n-1} + t_0^n)$. We may form the family of morphisms

$$\rho_n(s) : (\mathbb{P}^1 \times \mathbb{A}^1, \infty \times \mathbb{A}^1) \to (\mathbb{P}^1 \times \mathbb{A}^1, \infty \times \mathbb{A}^1)$$

sending $(t_0 : t_1, s)$ to $(t_0^n : t_1^n - s t_0 t_1^{n-1} + s t_0^n)$. By homotopy invariance, we have $\rho_n(0) = \rho_n(1)$, and thus $\rho_n = \mu_n$ in $\mathcal{H}_\bullet(k)$.

As above, we localize around $w := w_1/w_0 = 1$. Note that $\rho_n^{-1}(1) = \{0, 1\}$. We replace the target $\mathbb{P}^1$ with the henselization $\mathcal{O}$ at $w = 1$, and see that $\mathbb{P}^1 \times_{\rho_n} \mathcal{O}$ breaks up into two components via the factorization $w - 1 = t(t^{n-1} - 1)$, $t = t_1/t_0$. On the component containing 1, the map ρ_n is isomorphic to a hensel local version of μ_{n-1}, and on the component containing 0, the map ρ_n is isomorphic to the identity.

Using Nisnevich excision and proposition 5.5(1), we thus have the following commutative diagram (in $\mathcal{SH}_{S^1}(k)/f_2$)

$$\begin{tikzcd}
(\mathbb{P}^1,\infty) \arrow[r] \arrow[d, "\rho_n"'] & (\mathbb{P}^1,\mathbb{P}^1\setminus\{0,1\}) \arrow[r, "\sim"] \arrow[d, "\rho_n"'] & (\mathbb{P}^1,\infty)\vee(\mathbb{P}^1,\infty) \arrow[d, "\mu_{n-1}\vee\mathrm{id}"] \\
(\mathbb{P}^1,\infty) \arrow[r] & (\mathbb{P}^1,\mathbb{P}^1\setminus\{1\}) \arrow[r, "\sim"'] & (\mathbb{P}^1,\infty)
\end{tikzcd}$$

By proposition 5.5(2), the upper row is the co-multiplication (in $\mathcal{SH}_{S^1}(k)/f_2$), and thus

$$\rho_n = \mu_{n-1} + \mathrm{id}$$

in $\mathcal{SH}_{S^1}(k)/f_2$. As $\rho_n = \mu_n$ in $\mathcal{H}_\bullet(k)$, our induction hypothesis gives $\mu_n = n\cdot\mathrm{id}$, and the induction goes through. $\square$

While we are on the subject, we might as well note that

Remark 5.8. The co-group $((\mathbb{P}^1,1),\sigma_{\mathbb{P}^1})$ in $\mathcal{SH}_{S^1}(k)/f_2$ is co-commutative.

As pointed out by the referee, every object in $\mathcal{SH}_{S^1}(k)/f_2$ is a co-commutative co-group, since $\mathcal{SH}_{S^1}(k)/f_2$ is a triangulated category and hence each object is a double suspension. In addition, the co-group structure $((\mathbb{P}^1,1),\sigma_{\mathbb{P}^1})$ is isomorphic in $\mathcal{H}_\bullet(k)$ to the co-group structure on $S^1\wedge\mathbb{G}_m$ induced by the co-group structure on S^1, so the "triangulated" co-group structure on $\mathbb{P}^1$ agrees with the one we have given.
One should, however, be able to reproduce our entire theory "modulo $\Sigma^2_{\mathbb{P}^1}$" in the unstable category. We have not done this here, as we do not at present have available a theory of the motivic Postnikov tower in the $\mathcal{H}_\bullet(k)$. We expect that, given such a theory, the results of this section would hold in the unstable setting and in particular, that the co-group $((\mathbb{P}^1,1),\sigma_{\mathbb{P}^1})$ would be co-commutative "modulo $\Sigma^2_{\mathbb{P}^1}$".

We now return to our study of properties of the co-transfer map in $\mathcal{SH}_{S^1}(k)/f_2$. We will find it convenient to work in the setting of smooth schemes essentially of finite type over k; as mentioned at the beginning of §3, we consider schemes Y essentially of finite type over k as pro-objects in $\mathcal{H}(k)$, $\mathcal{SH}_{S^1}(k)$, etc. In the end, we use scheme essentially of finite type over k only as a tool to construct maps in pro-$\mathcal{SH}_{S^1}(k)/f_{n+1}$ between objects of $\mathcal{SH}_{S^1}(k)/f_{n+1}$; this will in the end give us morphisms in $\mathcal{SH}_{S^1}(k)/f_{n+1}$, as the functor $\mathcal{SH}_{S^1}(k)/f_{n+1}\to$ pro-$\mathcal{SH}_{S^1}(k)/f_{n+1}$ is fully faithful,
Suppose we have a semi-local smooth k-algebra A, essentially of finite type, and a finite extension $A\to B$, with B smooth over k. Suppose further that B is generated as an A-algebra by a single element $x\in B$:

$$B = A[x].$$

We say in this case that B is a *simply generated* A-algebra.
Let $\tilde f\in A[T]$ be the monic minimal polynomial of x, giving us the point x' of $\mathbb{A}^1_A = \mathrm{Spec}\,A[T]$ with ideal $(\tilde f)$. We identify $\mathbb{A}^1_A$ with $\mathbb{P}^1_A\setminus\{1\}$ as usual, giving

us the subscheme x of $\mathbb{P}^1_A \setminus \{1\}$, smooth over k and finite over $\operatorname{Spec} A$, in fact, canonically isomorphic to $\operatorname{Spec} B$ over $\operatorname{Spec} A$ via the choice of generator x. Let

$$\varphi_x : x \to \operatorname{Spec} B$$

be this isomorphism. We let f be the generator of m_x/m_x^2 determined by $\tilde{f}$. Via the composition

$$(\mathbb{P}^1_A, 1) \xrightarrow{co\text{-}tr_{x,f}} (\mathbb{P}^1_x, 1) \xrightarrow{\varphi_x \times \mathrm{id}} (\mathbb{P}^1_B, 1)$$

we have the morphism

$$co\text{-}tr_x : (\mathbb{P}^1_A, 1) \to (\mathbb{P}^1_B, 1)$$

in pro-$\mathcal{H}_\bullet(k)$.

Lemma 5.9. *Suppose that* $\operatorname{Spec} B \to \operatorname{Spec} A$ *is étale over each generic point of* $\operatorname{Spec} A$. *Then the map* $co\text{-}tr_x : (\mathbb{P}^1_A, 1) \to (\mathbb{P}^1_B, 1)$ *in pro-*$\mathcal{SH}_{S^1}(k)/f_2$ *is independent of the choice of generator* x *for* B *over* A.

Via this result, we may write $co\text{-}tr_{B/A}$ for $co\text{-}tr_x$.

Proof. We use a deformation argument; we first localize to reduce to the case of an étale extension $A \to B$. For this, let $a \in A$ be a non-zero divisor, and let x be a generator for B as an A-algebra. Then x is a generator for $B[a^{-1}]$ as an $A[a^{-1}]$-algebra and by lemma 3.8 we have the commutative diagram

$$\begin{tikzcd} \mathbb{P}^1_{A[a^{-1}]} \arrow[r] \arrow[d, "co\text{-}tr_x"'] & \mathbb{P}^1_A \arrow[d, "co\text{-}tr_x"] \\ \mathbb{P}^1_{B[a^{-1}]} \arrow[r] & \mathbb{P}^1_B, \end{tikzcd}$$

with horizontal arrows isomorphisms in pro-$\mathcal{SH}_{S^1}(k)/f_2$. Thus, we may assume that $A \to B$ is étale.

Suppose we have generators $x \neq x'$ for B over A; let $d = [B : A]$. Let s be an indeterminate, let $x(s) = sx + (1-s)x' \in B[s]$, and consider the extension $\tilde{B}_s := A[s][x(s)]$ of $A[s]$, considered as a subalgebra of $B[s]$. Clearly $\tilde{B}_s$ is finite over $A[s]$.

Let $m_A \subset A$ be the Jacobson radical, and let $A(s)$ be the localization of $A[s]$ at the ideal $(m_A A[s] + s(s-1))$. In other words, $A(s)$ is the semi-local ring of the set of closed points $\{(0,a), (1,a)\}$ in $\mathbb{A}^1 \times \operatorname{Spec} A$, as a runs over the closed points of $\operatorname{Spec} A$. Define $B(s) := B \otimes_A A(s)$ and $B_s := \tilde{B}_s \otimes_A A(s) \subset B(s)$. Let $y = (1,a)$ be a closed point of $A(s)$, with maximal ideal m_y, and let x_y be the image of x in $B(s)/m_y B(s)$. Clearly x_y is in the image of $B_s \to B(s)/m_y B(s)$, hence $B_s \to B(s)/m_y B(s)$ is surjective. Similarly, $B_s \to B(s)/m_y B(s)$ is surjective for all y of the form $(0,a)$; by Nakayama's lemma $B_s = B(s)$. Also, $B(s)$ and $A(s)$ are regular and $B(s)$ is finite over $A(s)$, hence $B(s)$ is flat over $A(s)$ and thus $B(s)$ is a free $A(s)$-module of rank d. Finally, $B(s)$ is clearly unramified over $A(s)$, hence $A(s) \to B(s)$ is étale.

Using Nakayama's lemma again, we see that $B(s)$ is generated as an $A(s)$ module by $1, x(s), x(s)^2, \ldots, x(s)^{d-1}$. It follows that $x(s)$ satisfies a monic polynomial equation of degree d over $A(s)$, thus $x(s)$ admits a monic minimal polynomial f_s of degree d over $A(s)$. Sending T to $x(s)$ defines an isomorphism

$$\varphi_s : A(s)[T]/(f_s) \to B(s).$$

We let $x_s \subset \mathbb{A}^1_{A(s)} = \mathbb{P}^1_{A(s)} \setminus \{1\}$ be the closed subscheme of $\mathbb{P}^1_{A(s)}$ corresponding to f_s; the isomorphism φ_s gives us the isomorphism

$$\varphi_s : x_s \to \operatorname{Spec} B(s).$$

Thus, we may define the map

$$\textit{co-tr}_{x(s)} : (\mathbb{P}^1_{A(s)}, 1) \to (\mathbb{P}^1_{B(s)}, 1)$$

giving us the commutative diagram

$$\begin{tikzcd}
(\mathbb{P}^1_A, 1) \arrow[r, "i_0"] \arrow[d, "\textit{co-tr}_{x'}"'] & (\mathbb{P}^1_{A(s)}, 1) \arrow[d, "\textit{co-tr}_{x(s)}"'] & (\mathbb{P}^1_A, 1) \arrow[l, "i_1"'] \arrow[d, "\textit{co-tr}_x"] \\
(\mathbb{P}^1_B, 1) \arrow[r, "i_0"'] & (\mathbb{P}^1_{B(s)}, 1) & (\mathbb{P}^1_B, 1) \arrow[l, "i_1"]
\end{tikzcd}$$

By lemma 5.3, the map $(\mathbb{P}^1_{A(s)}, 1) \to (\mathbb{P}^1_{A[s]}, 1)$ is an isomorphism in pro-$\mathcal{SH}_{S^1}(k)/f_2$. By homotopy invariance, it follows that the maps i_0, i_1 are isomorphisms in pro-$\mathcal{SH}_{S^1}(k)/f_2$, inverse to the map $(\mathbb{P}^1_{A(s)}, 1) \to (\mathbb{P}^1_A, 1)$ induced by the projection $\operatorname{Spec} A(s) \to \operatorname{Spec} A$. Therefore $\textit{co-tr}_{x'} = \textit{co-tr}_x$, as desired. □

Lemma 5.10. $\textit{co-tr}_{A/A} = \mathrm{id}_{(\mathbb{P}^1_A, 1)}$.

Proof. We may choose 0 as the generator for A over A, which gives us the point $x = 0 \in \mathbb{P}^1_A$. The result now follows from lemma 3.7. □

Lemma 5.11. *Let $A \to C$ be a finite simply generated extension and $A \subset B \subset C$ a sub-extension, with B also simply generated over A. We suppose that A, B and C are smooth over k, that $A \to B$ and $A \to C$ are étale over each generic point of* $\operatorname{Spec} A$, *and $B \to C$ is étale over each generic point of* $\operatorname{Spec} B$. *Then*

$$\textit{co-tr}_{C/A} = \textit{co-tr}_{C/B} \circ \textit{co-tr}_{B/A}.$$

Proof. This is another deformation argument. As in the proof of lemma 5.9, we may assume that $A \to B$, $B \to C$ and $A \to C$ are étale extensions; we retain the notation from the proof of lemma 5.9. Let y be a generator for C over A, x a generator for B over A. These generators give us corresponding closed subschemes $y, x \subset \mathbb{P}^1_A$ and $y_B \subset \mathbb{P}^1_B$. Let $y(s) = sy + (1-s)x$, giving $y(s) \subset \mathbb{P}^1_{A(s)}$. Note that $y(1) = y$, $y(0)_{\mathrm{red}} = x$

As in the proof of lemma 5.9, the element $y(s)$ of $C(s)$ is a generator over $A(s)$ after localizing at the points of $\operatorname{Spec} A(s)$ lying over $s = 1$. The subscheme $y(s)$ in a neighborhood of $s = 0$ is not in general regular, hence $y(s)$ is not a generator of $C(s)$ over $A(s)$. However, let $\mu : W := W_x \to \mathbb{P}^1 \times \mathbb{A}^1$ be the blow-up along

$\{(x,0)\}$, and let $\tilde{y} \subset W_{A(s)}$ be the proper transform $\mu^{-1}[y]$. An elementary local computation shows that this blow-up resolves the singularities of $y(s)$, and that $\tilde{y}$ is étale over $A(s)$; the argument used in the proof of lemma 5.9 goes through to show that $A(s)(\tilde{y}) \cong C(s)$. In addition, let C_0 be the proper transform to $W_{A(s)}$ of $\mathbb{P}^1 \times 0$ and E the exceptional divisor, then $\tilde{y}(0)$ is disjoint from C_0. Finally, after identifying E with $\mathbb{P}^1_{A[x]} = \mathbb{P}^1_B$ (using the monic minimal polynomial of x as a generator for m_x), we may consider $\tilde{y}(0)$ as a closed subscheme of $\mathbb{P}^1_B$; the isomorphism $A(s)(\tilde{y}) \cong C(s)$ leads us to conclude that $A(\tilde{y}(1)) = B(\tilde{y}(0)) = C$. By lemma 5.9, we may use $\tilde{y}(0)$ to define $\textit{co-tr}_{C/B}$. The map $\textit{co-tr}_{C/A}$ in pro-$\mathcal{SH}_{S^1}(k)/f_2$ is defined via the diagram

$$(\mathbb{P}^1_A, 1) \to (\mathbb{P}^1_A, \mathbb{P}^1_A \setminus y) \cong (\mathbb{P}^1_C, 1)$$

where the various choices involved lead to equal maps. By lemma 5.3, $W_{A(s)} \to W_{A[s]}$ is an isomorphism in pro-$\mathcal{SH}_{S^1}(k)/f_2$; by homotopy invariance, the projection $W_{A(s)} \to W$ is also an isomorphism pro-$\mathcal{SH}_{S^1}(k)/f_2$. The inclusions $i_1 : \mathbb{P}^1_A \to W_{A(s)}$, $i_0 : \mathbb{P}^1_{A[x]} \to W_{A(s)}$ induce isomorphisms (in pro-$\mathcal{SH}_{S^1}(k)/f_2$)

$$(\mathbb{P}^1_A, \mathbb{P}^1_A \setminus y) = (\mathbb{P}^1_A, \mathbb{P}^1_A \setminus y(1)) \cong (W_{A(s)}, W_{A(s)} \setminus \tilde{y}(s)) \cong (\mathbb{P}^1_{A[x]}, \mathbb{P}^1_{A[x]} \setminus \tilde{y}(0)).$$

As in the proof of lemma 5.9, we can use homotopy invariance to see that $\textit{co-tr}_{C/A}$ is also equal to the composition

$$\begin{aligned}(\mathbb{P}^1_A, 1) \to (\mathbb{P}^1_A, \mathbb{P}^1_A \setminus y) &\xrightarrow{i_1} (W_{A(s)}, W_{A(s)} \setminus \tilde{y}(s)) \\ &\xrightarrow{i_0^{-1}} (\mathbb{P}^1_{A[x]}, \mathbb{P}^1_{A[x]} \setminus \tilde{y}(0)) \cong (\mathbb{P}^1_C, 1).\end{aligned}$$

Now let $s_{1A(s)}$ be the transform to $W_{A(s)}$ of the 1-section. By lemma 3.3, the inclusion $i_0 : (\mathbb{P}^1_{A[x]}, 1) \to (W_{A(s)}, C_0 \cup s_{1A(s)})$ is an isomorphism in pro-$\mathcal{H}_\bullet(k)$. The above factorization of $\textit{co-tr}_{C/A}$ shows that $\textit{co-tr}_{C/A}$ is also equal to the composition

$$(\mathbb{P}^1_A, 1) \xrightarrow{i_1} (W_{A(s)}, C_0 \cup s_{1A(s)}) \xrightarrow{i_0^{-1}} (\mathbb{P}^1_{A[x]}, 1) \to (\mathbb{P}^1_{A[x]} \setminus \tilde{y}(0)) \cong (\mathbb{P}^1_C, 1).$$

Using remark 3.5, this latter composition is $\textit{co-tr}_{C/B} \circ (\textit{co-tr}_{B/A})$, as desired. □

Remark 5.12. 1. Suppose we have simply generated finite generically étale extensions $A_1 \to B_1$, $A_2 \to B_2$, with A_i smooth, semi-local and essentially of finite type over k. Then

$$\textit{co-tr}_{B_1\times B_2/A_1\times A_2} = \textit{co-tr}_{B_1/A_1} \vee \textit{co-tr}_{B_2/A_2}$$

where we make the evident identification $(\mathbb{P}^1_{B_1\times B_2}, 1) = (\mathbb{P}^1_{B_1}, 1) \vee (\mathbb{P}^1_{B_2}, 1)$ and similarly for A_1, A_2.

2. Let B_1, B_2 be simply generated finite generically étale A algebras and let $B = B_1 \times B_2$. As a special case of lemma 5.11, we have

$$\textit{co-tr}_{B/A} = (\textit{co-tr}_{B_1/A} \vee \textit{co-tr}_{B_2/A}) \circ \sigma_{\mathbb{P}^1_A}$$

Indeed, we may factor the extension $A \to B$ as $A \xrightarrow{\delta} A \times A \to B_1 \times B_2 = B$. We then use (1) and note that $\sigma_{\mathbb{P}^1_A} = co\text{-}tr_{A\times A/A}$ by lemma 4.1.

Next, we make a local calculation. Let (A, m) be a local ring of essentially finite type and smooth over k. Let $s \in m$ be a parameter, let $B = A[T]/T^n - s$ and let $t \in B$ be the image of T. Set $Y = \operatorname{Spec} B$, $X = \operatorname{Spec} A$, $Z = \operatorname{Spec} A/(s)$, $W = \operatorname{Spec} B/(t)$; the extension $A \to B$ induces an isomorphism $\alpha : W \xrightarrow{\sim} Z$. We write $co\text{-}tr_{Y/X}$ for $co\text{-}tr_{B/A}$, etc. This gives us the diagram in pro-$\mathcal{SH}_{S^1}(k)/f_2$

$$\begin{tikzcd}
\mathbb{P}^1_Z \arrow[r, "i_Z"] & \mathbb{P}^1_X \arrow[d, "co\text{-}tr_{Y/X}"] \\
\mathbb{P}^1_W \arrow[u, "\alpha"] \arrow[r, "i_W"'] & \mathbb{P}^1_Y.
\end{tikzcd}$$

Lemma 5.13. *Suppose that $n!$ is prime to $\operatorname{char} k$. In pro-$\mathcal{SH}_{S^1}(k)/f_2$ we have*

$$co\text{-}tr_{Y/X} \circ i_Z \circ \alpha = n \times i_W.$$

Proof. First, suppose we have a Nisnevich neighborhood $f : X' \to X$ of Z in X, giving us the Nisnevich neighborhood $g : Y' := Y \times_X X' \to Y$ of W in Y. As

$$co\text{-}tr_{Y/X} \circ f = g \circ co\text{-}tr_{Y'/X'}$$

we may replace X with X', Y with Y'. Similarly, we reduce to the case of A a hensel DVR, i.e., the henselization of $0 \in \mathbb{A}^1_F$ for some field F, $Z = W = 0$, with s the image in A of the canonical coordinate on $\mathbb{A}^1_F$.
The map $co\text{-}tr_{Y/X}$ is defined by the closed immersion

$$Y \xrightarrow{i_Y} \mathbb{A}^1_X = \mathbb{P}^1_X \setminus 1_X \subset \mathbb{P}^1_X$$

where i_Y is the closed subscheme of $\mathbb{A}^1 = \operatorname{Spec} A[T]$ defined by $T^n - s$, together with the isomorphism

$$(\mathbb{P}^1_X, \mathbb{P}^1_X \setminus Y) \cong \mathbb{P}^1_Y$$

furnished by the blow-up $\mu : W_Y \to \mathbb{A}^1_X \times \mathbb{A}^1$ of $\mathbb{A}^1_X \times \mathbb{A}^1$ along $(Y, 0)$. The composition $co\text{-}tr_{Y/X} \circ i_Z \circ \alpha$ is given by the composition

$$\begin{aligned}
(\mathbb{P}^1_W, 1) \cong (\mathbb{P}^1_W, \mathbb{P}^1_W \setminus 0_W) &\xrightarrow{\alpha} (\mathbb{P}^1_Z, \mathbb{P}^1_Z \setminus 0_Z) \\
&\xrightarrow{i_Z} (\mathbb{P}^1_X, \mathbb{P}^1_X \setminus 0_X) \xleftarrow{\mathrm{id}} (\mathbb{P}^1_X, 1) \to (\mathbb{P}^1_X, \mathbb{P}^1_X \setminus Y) \cong (\mathbb{P}^1_Y, 1).
\end{aligned}$$

In both cases, the isomorphisms (in pro-$\mathcal{SH}_{S^1}(k)/f_2$) are independent of a choice of trivialization of the various normal bundles. Let $U \to \mathbb{P}^1_X$ be the hensel local neighborhood of 0_Z in $\mathbb{P}^1_X$, $\operatorname{Spec} \mathcal{O}^h_{\mathbb{P}^1_X, 0_Z}$. Let $p : U \to X$ be the map induced by the projection $p_X : \mathbb{P}^1_X \to X$ and let $U_Z = p^{-1}(Z)$, with inclusion $i_Z : U_Z \to U$. We may use excision to rewrite the above description of $co\text{-}tr_{Y/X} \circ i_Z \circ \alpha$ as a composition as

$$(\mathbb{P}^1_W, 1) \cong (\mathbb{P}^1_Z, 1) \cong (U_Z, U_Z \setminus 0_Z) \xrightarrow{i_Z} (U, U \setminus Y) \cong (\mathbb{P}^1_Y, 1).$$

Similarly, letting $i_0 : X \to X \times \mathbb{P}^1$ be the 0-section, the map i_W may be given by the composition

$$(\mathbb{P}^1_W, 1) \cong (X, X \setminus Z) \xrightarrow{i_0} (U, U \setminus Y) \cong (\mathbb{P}^1_Y, 1);$$

again, the isomorphisms in pro-$\mathcal{SH}_{S^1}(k)/f_2$ are independent of choice of trivializations of the various normal bundles.

We write (s, t) for the parameters on U induced by the functions s, T on $\mathbb{A}^1_X$. We change coordinates in U by the isomorphism $(s, t) \mapsto (s - t^n, t)$. This transforms Y to the subscheme $s = 0$, is the identity on the 0-section, and transforms $s = 0$ to $t^n + s = 0$. Replacing s with $-s$, we have just switched the roles of Y and U_Z. Let

$$\varphi : U_Z \to U$$

be the map $\varphi(t) = (t^n, t)$. After making our change of coordinates, the map $co\text{-}tr_{Y/X} \circ i_Z \circ \alpha$ is identified with

$$(\mathbb{P}^1_W, 1) \cong (U_Z, U_Z \setminus 0_Z) \xrightarrow{\varphi} (U, U \setminus U_Z) \cong (\mathbb{P}^1_Y, 1)$$

while the description of i_W becomes

$$(\mathbb{P}^1_W, 1) \cong (X, X \setminus Z) \xrightarrow{i_0} (U, U \setminus U_Z) \cong (\mathbb{P}^1_Y, 1);$$

here we are using lemma 5.4 to conclude that the automorphism $(x_0 : x_1) \mapsto (-x_0 : x_1)$ of $\mathbb{P}^1_W$ induces the identity on $(\mathbb{P}^1_W, \infty)$ in pro-$\mathcal{SH}_{S^1}(k)/f_2$.

We now construct an $\mathbb{A}^1$-family of maps $(U_Z, U_Z \setminus 0_Z) \to (U, U \setminus U_Z)$. Let

$$\Phi : U_Z \times \mathbb{A}^1 \to U$$

be the map $\Phi(t, v) = (t^n, vt)$. Note that Φ defines a map of pairs

$$\Phi : (U_Z, U_Z \setminus 0_Z) \times \mathbb{A}^1 \to (U, U \setminus U_Z).$$

Clearly $\Phi(-, 1) = \varphi$ while $\Phi(-, 0)$ factors as

$$U_Z \xrightarrow{\mu_n} U_Z \xrightarrow{\beta} X \xrightarrow{i_0} U$$

where μ_n is the map $t \mapsto t^n$ and β is the isomorphism $\beta(t) = s$. Thus, we can rewrite $co\text{-}tr_{Y/X} \circ i_Z \circ \alpha$ as

$$(\mathbb{P}^1_W, 1) \cong (X, X \setminus Z) \xrightarrow{\mu_n} (X, X \setminus Z) \xrightarrow{i_0} (U, U \setminus U_Z) \cong (\mathbb{P}^1_Y, 1)$$

We identify X with the hensel neighborhood of 0_Z in $\mathbb{P}^1_Z$. Using excision again, we have the commutative diagram in pro-$\mathcal{H}_\bullet(k)$

$$\begin{tikzcd}
(X, X \setminus Z) \arrow[r, "\mu_n"] \arrow[d] & (X, X \setminus Z) \arrow[d] \\
(\mathbb{P}^1_Z, \mathbb{P}^1_Z \setminus 0_Z) \arrow[r, "\mu_n"] & (\mathbb{P}^1_Z, \mathbb{P}^1_Z \setminus 0_Z) \\
(\mathbb{P}^1_Z, \infty) \arrow[r, "\mu_n^Z"] \arrow[u] & (\mathbb{P}^1_Z, \infty) \arrow[u]
\end{tikzcd}$$

where the vertical arrows are all isomorphisms. By lemma 5.7 the bottom map is multiplication by n, which completes the proof. □

Lemma 5.14. *Let $A \to B$ be a finite simple étale extension, A as above. Let $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, let $i_x : x \to X$ be the closed point of X and $i_y : y \to Y$ the inclusion of $y := x \times_X Y$. Then*

$$co\text{-}tr_{Y/X} \circ i_x = i_y \circ co\text{-}tr_{y/x}.$$

Proof. Take an embedding of Y in $\mathbb{A}^1_X = \mathbb{P}^1_X \setminus 1_X \subset \mathbb{P}^1_X$; the fiber of $Y \to \mathbb{A}^1_X$ over $x \to X$ is thus an embedding $y \to \mathbb{A}^1_x = \mathbb{P}^1_x \setminus 1_x \subset \mathbb{P}^1_x$. The result follows easily from the commutativity of the diagram

$$\begin{tikzcd} \mathbb{P}^1_x \setminus y \arrow[r] \arrow[d] & \mathbb{P}^1_x \arrow[d] \\ \mathbb{P}^1_X \setminus Y \arrow[r] & \mathbb{P}^1_X \end{tikzcd}$$

□

Proposition 5.15. *Let $A \to B$ be a finite generically étale extension, with A a DVR and B a semi-local principal ideal ring. Let $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, let $i_x : x \to X$ be the closed point of X and $i_y : y \to Y$ the inclusion of $y := x \times_X Y$. Write $y = \{y_1, \ldots, y_r\}$, with each y_i irreducible. Let n_i denote the ramification index of y_i; suppose that $n_i!$ is prime to* char k *for each i. Then*

$$co\text{-}tr_{Y/X} \circ i_x = \sum_{i=1}^{r} n_i \cdot i_{y_i} \circ co\text{-}tr_{y_i/x}.$$

Proof. We note that every such extension is simple. By passing to the henselization $A \to A^h$, we may assume A is hensel. By remark 5.12(2), we may assume that $r = 1$. Let $A \to B_0 \subset B$ be the maximal unramified subextension. As $co\text{-}tr_{B/A} = co\text{-}tr_{B/B_0} \circ co\text{-}tr_{B_0/B}$, we reduce to the two cases $A = B_0$, $B = B_0$. We note that a finite separable extension of hensel DVRs $A \to B$ with trivial residue field extension degree and ramification index prime to the characteristic is isomorphic to an extension of the form $t^n = s$ for some $s \in m_A \setminus m_A^2$. Thus, the first case is lemma 5.13, the second is lemma 5.14. □

Consider the functor

$$(\mathbb{P}^1_?, 1) : \mathbf{Sm}/k \to \mathcal{SH}_{S^1}(k)/f_2$$

sending X to $(\mathbb{P}^1_X, 1) \in \mathcal{SH}_{S^1}(k)/f_2$, which we consider as a $\mathcal{SH}_{S^1}(k)/f_2$-valued presheaf on $\mathbf{Sm}/k^{\mathrm{op}}$ (we could also write this functor as $X \mapsto \Sigma^\infty_{\mathbb{P}^1} X_+$). We proceed to extend $(\mathbb{P}^1_?, 1)$ to a presheaf on $SmCor(k)^{\mathrm{op}}$; we will assume that char $k = 0$, so we do not need to worry about inseparability.
We first define the action on the generators of $\operatorname{Hom}_{SmCor}(X, Y)$, i.e., on irreducible $W \subset X \times Y$ such that $W \to X$ is finite and surjective over some component of X. As $\mathcal{SH}_{S^1}(k)/f_2$ is an additive category, it suffices to consider the case of irreducible X. Let $U \subset X$ be a dense open subscheme. Then the map

$(\mathbb{P}^1_U, 1) \to (\mathbb{P}^1_X, 1)$ induced by the inclusion is an isomorphism in $\mathcal{SH}_{S^1}(k)/f_2$. We may therefore define the morphism

$$(\mathbb{P}^1_?, 1)(W) : (\mathbb{P}^1_X, 1) \to (\mathbb{P}^1_Y, 1)$$

in $\mathcal{SH}_{S^1}(k)/f_2$ as the composition (in pro-$\mathcal{SH}_{S^1}(k)/f_2$)

$$(\mathbb{P}^1_X, 1) \cong (\mathbb{P}^1_{k(X)}, 1) \xrightarrow{co\text{-}tr_{k(W)/k(X)}} (\mathbb{P}^1_{k(W)}, 1) \xrightarrow{p_2} (\mathbb{P}^1_Y, 1).$$

We extend linearly to define $(\mathbb{P}^1_?, 1)$ on $\mathrm{Hom}_{SmCor}(X, Y)$.
Suppose that $\Gamma_f \subset X \times Y$ is the graph of a morphism $f : X \to Y$. It follows from lemma 5.10 that $co\text{-}tr_{k(\Gamma_f)/k(X)}$ is the inverse to the isomorphism $p_1 : (\mathbb{P}^1_{k(\Gamma_f)}, 1) \to (\mathbb{P}^1_{k(X)}, 1)$. Thus, the composition

$$(\mathbb{P}^1_{k(X)}, 1) \xrightarrow{co\text{-}tr_{k(\Gamma_f)/k(X)}} (\mathbb{P}^1_{k(\Gamma_f)}, 1) \xrightarrow{p_2} (\mathbb{P}^1_Y, 1)$$

is the map induced by the restriction of f to $\mathrm{Spec}\, k(X)$. Since $(\mathbb{P}^1_{k(X)}, 1) \to (\mathbb{P}^1_X, 1)$ is an isomorphism in pro-$\mathcal{SH}_{S^1}(k)/f_2$, it follows that $(\mathbb{P}^1_?, 1)(\Gamma_f) = f$, i.e., our definition of $(\mathbb{P}^1_?, 1)$ on $\mathrm{Hom}_{SmCor}(X, Y)$ really is an extension of its definition on $\mathrm{Hom}_{\mathbf{Sm}/k}(X, Y)$.
The main point is to check functoriality.

LEMMA 5.16. *Suppose* $\mathrm{char}\, k = 0$. *For* $\alpha \in \mathrm{Hom}_{SmCor}(X, Y)$, $\beta \in \mathrm{Hom}_{SmCor}(Y, Z)$, *we have*

$$(\mathbb{P}^1_?, 1)(\beta \circ \alpha) = (\mathbb{P}^1_?, 1)(\beta) \circ (\mathbb{P}^1_?, 1)(\alpha)$$

Proof. It suffices to consider the case of irreducible finite correspondences $W \subset X \times Y$, $W' \subset Y \times Z$. If W is the graph of a flat morphism, the result follows from lemma 3.8.
As the action of correspondences is defined at the generic point, we may replace X with $\eta := \mathrm{Spec}\, k(X)$. Then W becomes a closed point of Y_η and the correspondence $W_\eta : \eta \to Y$ factors as $p_2 \circ i_{W_\eta} \circ p_1^t$, where $p_1 : W_\eta \to \eta$ and $p_2 : Y_\eta \to Y$ are the projections.
Let $W'_\eta \subset Y_\eta \times Z$ be the pull-back of W'. As we have already established naturality with respect to pull-back by flat maps, we reduce to showing

$$(\mathbb{P}^1_?, 1)(W'_\eta \circ i_{W_\eta}) = (\mathbb{P}^1_?, 1)(W'_\eta) \circ (\mathbb{P}^1_?, 1)(i_{W_\eta}).$$

Since Y is quasi-projective, we can find a sequence of closed subschemes of Y_η

$$W_\eta = W_0 \subset W_1 \subset \ldots \subset W_{d-1} \subset W_d = Y_\eta$$

such that W_i is smooth of codimension $d-i$ on Y_η. Using again the fact the *co-tr* is defined at the generic point, and that we have already proven functoriality with respect to composition of morphisms, we reduce to the case of $Y = \mathrm{Spec}\, \mathcal{O}$ for some DVR $\mathcal{O}$, and i_η the inclusion of the closed point η of Y.
Let $W'' \to W'$ be the normalization of W'. Using functoriality with respect to morphisms in $\mathbf{Sm}/k$ once more, we may replace Z with W'' and W' with the transpose of the graph of the projection $W'' \to Y$. Changing notation, we may assume that W' is the transpose of the graph of a finite morphism $Z \to Y$.

This reduces us to the case considered in proposition 5.15; this latter result completes the proof. □

We will collect the results of this section, generalized to higher loops, in theorem 6.1 of the next section.

6. HIGHER LOOPS

The results of these last sections carry over immediately to statements about the n-fold smash product $(\mathbb{P}^1, 1)^{\wedge n}$ for $n \geq 1$. For clarity and completeness, we list these explicitly in an omnibus theorem.

Let R be a semi-local k-algebra, smooth and essentially of finite type over k, and let $x \subset \mathbb{P}^1_R$ and f be as in section 3. For $n \geq 1$, define

$$co\text{-}tr^n_{x,f} : \Sigma^n_{\mathbb{P}^1} \operatorname{Spec} R_+ \to \Sigma^n_{\mathbb{P}^1} x_+$$

to be the map $\Sigma^{n-1}_{\mathbb{P}^1}(co\text{-}tr_{x,f})$ (in pro-$\mathcal{H}_\bullet(k)$).

Similarly, let A be a semi-local k-algebra, smooth and essentially of finite type over k. Let $B = A[x]$ be a simply generated finite generically étale A-algebra. For $n \geq 1$, define

$$co\text{-}tr^n_x : \Sigma^n_{\mathbb{P}^1} \operatorname{Spec} A_+ \to \Sigma^n_{\mathbb{P}^1} \operatorname{Spec} B_+$$

to be the map $\Sigma^{n-1}_{\mathbb{P}^1}(co\text{-}tr_x)$ (in pro-$\mathcal{SH}_{S^1}(k)/f_{n+1}$).

THEOREM 6.1. *1.* $co\text{-}tr^n_{0,-s} = \mathrm{id}$.

2. Let $R \to R'$ be a flat extension of smooth semi-local k-algebras, essentially of finite type over k. Let x be a smooth closed subscheme of $\mathbb{P}^1_R \setminus \{1\}$, finite and generically étale over R. Let $x' = x \times_R R' \subset \mathbb{P}^1_{R'}$. Let f be a generator for m_x/m_x^2, and let f' be the extension to $m_{x'}/m^2_{x'}$. Suppose that either $R \to R'$ is smooth or that $x \to \operatorname{Spec} R$ is étale. Then the diagram

$$\begin{tikzcd}
\Sigma^n_{\mathbb{P}^1} \operatorname{Spec} R'_+ \arrow[r, "co\text{-}tr^n_{x',f'}"] \arrow[d] & \Sigma^n_{\mathbb{P}^1} x'_+ \arrow[d] \\
\Sigma^n_{\mathbb{P}^1} \operatorname{Spec} R_+ \arrow[r, "co\text{-}tr^n_{x,f}"'] & \Sigma^n_{\mathbb{P}^1} x_+
\end{tikzcd}$$

is well-defined and commutes.

3. The co-group structure $\Sigma^{n-1}_{\mathbb{P}^1}(\sigma_{\mathbb{P}^1})$ on $(\mathbb{P}^1, 1)^{\wedge n}$ is given by the map

$$co\text{-}tr^n_{\{0,\infty\},(-s,s^{-1})} : (\mathbb{P}^1, 1)^{\wedge n} \to (\mathbb{P}^1, 1)^{\wedge n} \vee (\mathbb{P}^1, 1)^{\wedge n}.$$

4. The co-group $((\mathbb{P}^1, 1)^{\wedge n}, \Sigma^{n-1}_{\mathbb{P}^1}(\sigma_{\mathbb{P}^1}))$ in $\mathcal{SH}_{S^1}(k)/f_{n+1}$ is co-commutative.

5. For an extension $A \to B$ as above, the map $co\text{-}tr^n_x : \Sigma^n_{\mathbb{P}^1} \operatorname{Spec} A_+ \to \Sigma^n_{\mathbb{P}^1} \operatorname{Spec} B_+$ is independent of the choice of x, and is denoted $co\text{-}tr^n_{B/A}$.

6. Suppose that $\operatorname{char} k = 0$*. The* $\mathcal{SH}_{S^1}(k)/f_{n+1}$*-valued presheaf on* $\mathbf{Sm}/k^{\mathrm{op}}$

$$\Sigma^n_{\mathbb{P}^1} ?_+ : \mathbf{Sm}/k \to \mathcal{SH}_{S^1}(k)/f_{n+1}$$

extends to an $\mathcal{SH}_{S^1}(k)/f_{n+1}$*-valued presheaf on* $SmCor(k)^{\mathrm{op}}$*, by sending a generator* $W \subset X \times Y$ *of* $\mathrm{Hom}_{SmCor}(X, Y)$ *to the morphism* $\Sigma^n_{\mathbb{P}^1} X_+ \to \Sigma^n_{\mathbb{P}^1} Y_+$ *in* $\mathcal{SH}_{S^1}(k)/f_{n+1}$ *determined by the diagram*

$$\begin{array}{ccc}
\Sigma^n_{\mathbb{P}^1} \operatorname{Spec} k(X)_+ & \xrightarrow{\sim} & \Sigma^n_{\mathbb{P}^1} X_+ \\
\Big\downarrow {\scriptstyle co\text{-}tr^n_{k(W)/k(X)}} & & \\
\Sigma^n_{\mathbb{P}^1} \operatorname{Spec} k(W)_+ & & \\
\Big\downarrow {\scriptstyle p_2} & & \\
\Sigma^n_{\mathbb{P}^1} Y_+ & &
\end{array}$$

in pro-$\mathcal{SH}_{S^1}(k)/f_{n+1}$*. The assertion that*

$$\Sigma^n_{\mathbb{P}^1} \operatorname{Spec} k(X)_+ \to \Sigma^n_{\mathbb{P}^1} X_+$$

is an isomorphism in pro-$\mathcal{SH}_{S^1}(k)/f_{n+1}$ *is part of the statement. We write the map in* $\mathcal{SH}_{S^1}(k)/f_{n+1}$ *associated to* $\alpha \in \mathrm{Hom}_{SmCor}(X, Y)$ *as*

$$co\text{-}tr^n(\alpha) : \Sigma^n_{\mathbb{P}^1} X_+ \to \Sigma^n_{\mathbb{P}^1} Y_+.$$

7. Supports and co-transfers

In this section, we assume that $\operatorname{char} k = 0$. We consider the following situation. Let $i : Y \to X$ be a codimension one closed immersion in $\mathbf{Sm}/k$, and let $Z \subset X$ be a pure codimension n closed subset of X such that $i^{-1}(Z) \subset Y$ also has pure codimension n. We let $T = i^{-1}(Z)$, $X^{(Z)} = (X, X \setminus Z)$, $Y^{(T)} = (Y, Y \setminus T)$, so that i induces the map of pointed spaces

$$i : Y^{(T)} \to X^{(Z)}.$$

Let z be the set of generic points of Z, $\mathcal{O}_{X,z}$ the semi-local ring of z in X, $X_z = \operatorname{Spec} \mathcal{O}_{X,z}$ and $X_z^{(z)} = (X_z, X_z \setminus z)$. We let t be the set of generic points of T, and let $\mathcal{O}_{X,t}$ be the semi-local ring of t in X, $X_t = \operatorname{Spec} \mathcal{O}_{X,t}$. Set $Y_t := X_t \times_X Y$ and let $Y_t^{(t)} = (Y_t, Y_t \setminus t)$.

LEMMA 7.1. *There are canonical isomorphisms in pro-*$\mathcal{SH}_{S^1}(k)/f_{n+1}$

$$X^{(Z)} \cong X_z^{(z)} \cong \Sigma^n_{\mathbb{P}^1} z_+; \quad Y^{(T)} \cong Y_t^{(t)} \cong \Sigma^n_{\mathbb{P}^1} t_+.$$

Proof. This follows from lemma 5.6. □

Thus, the map $i : Y^{(T)} \to X^{(Z)}$ gives us the map in pro-$\mathcal{SH}_{S^1}(k)/f_{n+1}$:

$$i : \Sigma^n_{\mathbb{P}^1} t_+ \to \Sigma^n_{\mathbb{P}^1} z_+. \tag{7.1}$$

On the other hand, we can define a map

$$i_{co\text{-}tr} : \Sigma^n_{\mathbb{P}^1} t_+ \to \Sigma^n_{\mathbb{P}^1} z_+ \tag{7.2}$$

as follows: Let $Z_t = Z \cap X_t \subset X_t$. Since Y has codimension one in X and intersects Z properly, t is a collection of codimension one points of Z, and thus Z_t is a semi-local reduced scheme of dimension one. Let $p : \tilde{Z}_t \to Z_t$ be the normalization, and let $\tilde{t} \subset \tilde{Z}_t$ be the set of points lying over $t \subset Z_t$. Write $\tilde{t}$ as a union of closed points, $\tilde{t} = \amalg_j \tilde{t}_j$. For each j, we let n_j denote the multiplicity at $\tilde{t}_j$ of the pull-back Cartier divisor $Y_t \times_{X_t} \tilde{Z}_t$, and let $t_j = p(\tilde{t}_j)$. This gives us the commutative diagram

$$\begin{tikzcd}
\amalg_j \tilde{t}_j \arrow[r, equal] & \tilde{t} \arrow[r, "\tilde{i}"] \arrow[d, "p"] & \tilde{Z}_t \arrow[d, "p"] & z \arrow[l, "j"'] \arrow[dl] \\
& t \arrow[r, "i"'] & Z. &
\end{tikzcd}$$

Note that j is an isomorphism in pro-$\mathcal{SH}_{S^1}(k)/f_1$. We define $i_{co\text{-}tr}$ to be the composition

$$\Sigma^n_{\mathbb{P}^1} t_+ \xrightarrow{\prod_j n_j co\text{-}tr^n_{\tilde{t}_j/t}} \oplus_j \Sigma^n_{\mathbb{P}^1} \tilde{t}_{j+} = \Sigma^n_{\mathbb{P}^1} \tilde{t}_+ \xrightarrow{\Sigma^n_{\mathbb{P}^1} \tilde{i}} \Sigma^n_{\mathbb{P}^1} \tilde{Z}_+ \xrightarrow{\Sigma^n_{\mathbb{P}^1} j^{-1}} \Sigma^n_{\mathbb{P}^1} z_+$$

in pro-$\mathcal{SH}_{S^1}(k)/f_{n+1}$.

Lemma 7.2. *The morphisms* (7.1) *and* (7.2)*are equal in pro-$\mathcal{SH}_{S^1}(k)/f_{n+1}$.*

Proof. Using Nisnevich excision, we may replace X with the henselization of X along t; we may also assume that t is a single point. Via a limit argument, we may then replace X with a smooth affine scheme of dimension $n+1$ over $k(t)$; Z is thus a reduced closed subscheme of X of pure dimension one over $k(t)$. We may also assume that Y is the fiber over 0 of a morphism $X \to \mathbb{A}^1_{k(t)}$ for which the restriction to Z is finite.

As we are working in pro-$\mathcal{SH}_{S^1}(k)/f_{n+1}$, we may replace (X, Z) with (X', Z') if there is a morphism $f : X \to X'$ over $\mathbb{A}^1_{k(t)}$ which makes (X, t) a hensel neighborhood of $(X', f(t))$ and such that the restriction of f to $f_Z : Z \to Z'$ is birational. Using Gabber's presentation lemma [6, lemma 3.1], we may assume that $X = \mathbb{A}^{n+1}_{k(t)}$, that t is the origin 0 and that Y is the coordinate hyperplane $X_{n+1} = 0$. We write F for $k(t)$ and write simply 0 for t.

After a suitable linear change of coordinates in $\mathbb{A}^{n+1}_F$, we may assume that each coordinate projection

$$q : \mathbb{A}^{n+1}_F \to \mathbb{A}^r_F$$
$$q(x_1, \ldots, x_{n+1}) = (x_{i_1}, \ldots, x_{i_r}),$$

$r = 1, \ldots, n$, restricts to a finite morphism on Z, and that $Z \to q(Z)$ is birational if $r \geq 2$.

We now reduce to the case in which Z is contained in the coordinate subspace $X' = \mathbb{A}^2_F$ defined by $X_1 = \ldots = X_{n-1} = 0$. For this, consider the map

$$m : \mathbb{A}^1 \times \mathbb{A}^{n+1}_F \to \mathbb{A}^1 \times \mathbb{A}^{n+1}_F$$
$$m(t, x_1, \ldots, x_{n+1}) = (t, tx_1, \ldots, tx_{n-1}, x_n, x_{n+1})$$

Let $\mathcal{Z} = m(\mathbb{A}^1 \times Z) \subset \mathbb{A}^1 \times \mathbb{A}^{n+1}_F$. By our finiteness assumptions, $\mathcal{Z}$ is a (reduced) closed subscheme of $\mathbb{A}^1 \times \mathbb{A}^{n+1}_F$, and each fiber $\mathcal{Z}_t \subset t \times \mathbb{A}^{n+1}_F$ is birationally isomorphic to $Z \times_F F(t)$. Consider the inclusion map

$$(\mathbb{A}^1 \times Y)^{(\mathbb{A}^1 \times 0)} \to (\mathbb{A}^1 \times X)^{(\mathcal{Z})}$$

The maps

$$i_0, i_1 : Y^{(0)} \to (\mathbb{A}^1 \times Y)^{(\mathbb{A}^1 \times 0)}$$

are clearly isomorphisms in pro-$\mathcal{H}_\bullet(k)$, and the maps

$$i_1 : X^{(Z)} \to (\mathbb{A}^1 \times X)^{(\mathcal{Z})}$$
$$i_0 : X^{(\mathcal{Z}_0)} \to (\mathbb{A}^1 \times X)^{(\mathcal{Z})}$$

are easily seen to be isomorphisms in pro-$\mathcal{SH}_{S^1}(k)/f_{n+1}$. Combining this with the commutative diagram

$$\begin{tikzcd} Y^{(0)} \arrow[r] \arrow[d, "i_1"'] & X^{(Z)} \arrow[d, "i_1"] \\ (\mathbb{A}^1 \times Y)^{(\mathbb{A}^1 \times 0)} \arrow[r] & X^{(\mathcal{Z})} \\ Y^{(0)} \arrow[r] \arrow[u, "i_0"] & X^{(\mathcal{Z}_0)} \arrow[u, "i_0"'] \end{tikzcd}$$

shows that we can replace Z with $\mathcal{Z}_0 \subset X'$.
Having done this, we see that the map $Y^{(0)} \to X^{(Z)}$ is just the $n-1$-fold $\mathbb{P}^1$ suspension of the map

$$(Y \cap X')^{(0)} \to (X')^{(Z)}$$

This reduces us to the case $n = 1$.
Since $p_2 : Z \to \mathbb{A}^1_F$ is finite, we may take $X = \mathbb{P}^1 \times \mathbb{A}^1_F$ instead of $\mathbb{A}^1 \times \mathbb{A}^1_F$. Then the map $Y^{(0)} \to X^{(Z)}$ is isomorphic to $(\mathbb{P}^1 \times 0, \infty \times 0) \to X^{(Z)}$. We extend this to the isomorphic map

$$(\mathbb{P}^1 \times \mathbb{A}^1_F, \infty \times \mathbb{A}^1_F) \to X^{(Z)} = (\mathbb{P}^1 \times \mathbb{A}^1_F, \mathbb{P}^1 \times \mathbb{A}^1_F \setminus Z).$$

Let s be the generic point of $\mathbb{A}^1_F$, Z_s the fiber of p_2 over s. Then the inclusions

$$(\mathbb{P}^1 \times 0, \infty \times 0) \xrightarrow{j_0} (\mathbb{P}^1 \times \mathbb{A}^1_F, \infty \times \mathbb{A}^1_F) \xleftarrow{j_s} (\mathbb{P}^1 \times s, \infty \times s)$$
$$(\mathbb{P}^1 \times \mathbb{A}^1_F, \mathbb{P}^1 \times \mathbb{A}^1_F \setminus Z) \xleftarrow{j_s} (\mathbb{P}^1 \times s, \mathbb{P}^1_s \setminus Z_s)$$

are isomorphisms in pro-$\mathcal{SH}_{S^1}(k)/f_2$, and thus the map

$$i_0 : Y^{(0)} \cong (\mathbb{P}^1 \times 0, \infty \times 0) \to X^{(Z)} = (\mathbb{P}^1 \times \mathbb{A}^1_F, \mathbb{P}^1 \times \mathbb{A}^1_F \setminus Z)$$

is isomorphic in pro-$\mathcal{SH}_{S^1}(k)/f_2$ to the collapse map

$$(\mathbb{P}^1 \times s, \infty \times s) \to (\mathbb{P}^1 \times s, \mathbb{P}^1_s \setminus Z_s).$$

Therefore, the map

$$i : \Sigma_{\mathbb{P}^1} 0_+ \to \Sigma_{\mathbb{P}^1} z_+$$

we need to consider is equal to the co-transfer map

$$\textit{co-tr}_{Z_s/s} : \Sigma_{\mathbb{P}^1} s_+ \to \Sigma_{\mathbb{P}^1} z_{s+}$$

composed with the (canonical) isomorphisms

$$\Sigma_{\mathbb{P}^1} 0_+ \xrightarrow{i_0} \Sigma_{\mathbb{P}^1} \mathbb{A}^1_+ \cong \Sigma_{\mathbb{P}^1} s_+, \quad \Sigma_{\mathbb{P}^1} z_{s+} \cong \Sigma_{\mathbb{P}^1} z_+;$$

the latter isomorphism arising by noting that z_s is a generic point of Z over F. The result now follows directly from proposition 5.15. □

DEFINITION 7.3. 1. Take $X, X' \in \mathbf{Sm}/k$, and let $Z \subset X$, $Z' \subset X'$ be pure codimension n closed subsets. Take a generator $A \in \mathrm{Hom}_{SmCor}(X, X')$, $A \subset X \times X'$. Let $q : A^N \to A$ be the normalization of A. Let z be the set of generic points of Z, let a be the set of generic points of $A \cap X \times Z'$ and let $a' = q^{-1}(a)$. Suppose that

(1) $A^N \to X$ is étale on a neighborhood of a'
(2) $p_X(a)$ is contained in Z.

Let $\mathcal{O}_{A^N,a}$ be the semi-local ring of a' in A^N, and let $A^N_{a'} = \mathrm{Spec}\, \mathcal{O}_{A^N,a'}$; define X_z similarly. Define

$$\textit{co-tr}^n(W) : X^{(Z)} \to X'^{(Z')}$$

to be the map in $\mathcal{SH}_{S^1}(k)/f_{n+1}$ given by the following composition:

$$X^{(Z)} \cong X^{(z)}_z \cong \Sigma^n_{\mathbb{P}^1} z_+ \xrightarrow{\textit{co-tr}^n_{a'/z}} \Sigma^n_{\mathbb{P}^1} a'_+ \cong A^{N(a')}_{a'} \xrightarrow{p_{X'}} X'^{(Z')}.$$

2. Let $\mathrm{Hom}_{SmCor}(X, X')_{Z,Z'} \subset \mathrm{Hom}_{SmCor}(X, X')$ be the subgroup generated by A satisfying (a) and (b). We extend the definition of the morphism $\textit{co-tr}^n(A)$ to $\mathrm{Hom}_{SmCor}(X, X')_{Z,Z'}$ by linearity.

Note that we implicitly invoke lemma 7.1 to ensure that the isomorphisms used in the definition of $\textit{co-tr}^n(A)$ exist and are canonical; condition (1) implies in particular that A^N is smooth in a neighborhood of a', so we may use lemma 7.1 for the isomorphism $\Sigma^n_{\mathbb{P}^1} a'_+ \cong A^{N(a')}_{a'}$.

LEMMA 7.4. *Take $X, X', X'' \in \mathbf{Sm}/k$, and let $Z \subset X$, $Z' \subset X'$ and $Z'' \subset X''$ be pure codimension n closed subsets. Take $\alpha \in \mathrm{Hom}_{SmCor}(X, X')_{Z,Z'}$, $\alpha' \in \mathrm{Hom}_{SmCor}(X', X'')_{Z',Z''}$. Then*

1. $\alpha' \circ \alpha$ is in $\mathrm{Hom}_{SmCor}(X, X'')_{Z,Z''}$

2. $\textit{co-tr}^n(\alpha') \circ \textit{co-tr}^n(\alpha) = \textit{co-tr}^n(\alpha' \circ \alpha)$.

Proof. For (1), we may assume that α and α' are generators A and A'. We may replace X, X' and X'' with the respective strict henselizations along z, z' and z''. Write $z = \{z_1, \ldots, z_r\}$, $z' = \{z'_1, \ldots, z'_s\}$, $z'' = \{z''_1, \ldots, z''_t\}$. Then the normalizations A^N and A'^N break up as a disjoint union of graphs of morphisms

$$f_{jk} : X_{z_k} \to X'_{z'_j}; \quad g_{ij} : X'_{z'_j} \to X''_{z''_i}$$

and $A' \circ A$ is thus the sum of the graphs of the compositions $g_{ij} \circ f_{jk}$. Therefore, each irreducible component of the normalization of the support of $A' \circ A$ is étale over X. This verifies condition (1) of definition 7.3; the condition (2) is easy and is left to the reader.
(2) follows directly from theorem 6.1(6). □

Proposition 7.5. *Let $i : \Delta_1 \to \Delta$ be a closed immersion of quasi-projective schemes in $\mathbf{Sm}/k$, take $X, X' \in \mathbf{Sm}/k$ and $\alpha \in \mathrm{Hom}_{SmCor}(X, X')$. Let $Z \subset X \times \Delta$, $Z' \subset X' \times \Delta$ be closed codimension n subsets. Suppose that*

1. *$Z_1 := Z \cap X \times \Delta_1$ and $Z'_1 := Z' \cap X' \times \Delta_1$ have codimension n in $X \times \Delta_1$, $X' \times \Delta_1$, respectively.*
2. *$\alpha \times \mathrm{id}_\Delta$ is in $\mathrm{Hom}_{SmCor}(X \times \Delta, X' \times \Delta)_{Z,Z'}$*
3. *$\alpha \times \mathrm{id}_{\Delta_1}$ is in $\mathrm{Hom}_{SmCor}(X \times \Delta_1, X' \times \Delta_1)_{Z_1,Z'_1}$*

Then the diagram in $\mathcal{SH}_{S^1}(k)/f_{n+1}$

$$\begin{CD}
(X \times \Delta_1)^{(Z_1)} @>{co\text{-}tr^n(\alpha\times\mathrm{id})}>> (X' \times \Delta_1)^{(Z'_1)} \\
@V{\mathrm{id}\times i}VV @VV{\mathrm{id}\times i}V \\
(X \times \Delta)^{(Z)} @>>{co\text{-}tr^n(\alpha\times\mathrm{id})}> (X' \times \Delta)^{(Z')}
\end{CD}$$

commutes.

Proof. Since Δ is by assumption quasi-projective, we may factor $\Delta_1 \to \Delta$ as a sequence of closed codimension 1 immersions

$$\Delta_1 = \Delta^d \to \Delta^{d-1} \to \ldots \to \Delta^1 \to \Delta^0 = \Delta$$

such that each closed immersion $\Delta^i \to \Delta$ satisfies the conditions of the proposition. This reduces us to the case of a codimension one closed immersion.
We may replace $X \times \Delta$, $X' \times \Delta$, etc., with the respective semi-local schemes about the generic points of Z_1 and Z'_1. As Δ_1 has codimension one on Δ, it follows that the normalizations Z^N, Z'^N of Z and Z' are smooth over k. Let $\tilde{i} : \tilde{z} \to Z^N$, $\tilde{i}' : \tilde{z}' \to Z'^N$ be the points of Z^N, Z'^N lying over Z_1, Z'_1, respectively, which we write as a disjoint union of closed points

$$\tilde{z} = \amalg_j \tilde{z}_j; \quad \tilde{z}' = \amalg_j \tilde{z}'_j.$$

By lemma 7.1 and lemma 7.2, we may rewrite the diagram in the statement of the proposition as

$$\begin{tikzcd}
\Sigma^n_{\mathbb{P}^1} Z_{1+} \arrow[r, "co\text{-}tr^n(\alpha\times\mathrm{id}_{Z_1^N})"] \arrow[d, "\sum_j m_j co\text{-}tr^n_{\tilde{z}_j/Z_1}"'] & \Sigma^n_{\mathbb{P}^1} Z'_{1+} \arrow[d, "\sum_j m'_j co\text{-}tr^n_{\tilde{z}'_j/Z'_1}"] \\
\Sigma^n_{\mathbb{P}^1} \tilde{z}_+ \arrow[d, "\tilde{i}"'] & \Sigma^n_{\mathbb{P}^1} \tilde{z}'_+ \arrow[d, "\tilde{i}'"] \\
\Sigma^n_{\mathbb{P}^1} Z^N \arrow[r, "co\text{-}tr^n(\alpha\times\mathrm{id}_{Z^N})"'] & \Sigma^n_{\mathbb{P}^1} Z'^N
\end{tikzcd}$$

where $\alpha \times \mathrm{id}_{Z^N}$, $\alpha \times \mathrm{id}_{Z_1}$ denote the correspondences induced by $\alpha \times \mathrm{id}_\Delta$ and $\alpha \times \mathrm{id}_{\Delta_1}$, and the m_j, m'_j are the relevant intersection multiplicities. The commutativity of this diagram follows from theorem 6.1(6). □

8. Slices of loop spectra

Take $E \in \mathcal{SH}_{S^1}(k)$. Following Voevodsky's remarks in [22], Neeman's version of Brown representability [16] gives us the motivic Postnikov tower

$$\ldots \to f_{n+1}E \to f_nE \to \ldots \to f_0E = E,$$

where $f_nE \to E$ is universal for morphisms from an object of $\Sigma^n_{\mathbb{P}^1}\mathcal{SH}_{S^1}(k)$ to E. The layer s_nE is the *nth slice* of E, and is characterized up to unique isomorphism by the distinguished triangle

$$f_{n+1}E \to f_nE \to s_nE \to \Sigma_s f_{n+1}E. \tag{8.1}$$

The fact that this distinguished triangle determines s_nE up to unique isomorphism rather than just up to isomorphism follows from

$$\mathrm{Hom}_{\mathcal{SH}_{S^1}(k)}(\Sigma^{n+1}_{\mathbb{P}^1}\mathcal{SH}_{S^1}(k), s_nE) = 0 \tag{8.2}$$

To see this, just use the universal property of $f_{n+1}E \to E$ and the long exact sequence of Homs associated to the distinguished triangle (8.1). In particular, using the description of $\mathrm{Hom}_{\mathcal{SH}_{S^1}(k)/f_{n+1}}(-,-)$ via right fractions we have

Lemma 8.1. *For all $F, E \in \mathcal{SH}_{S^1}(k)$ and $n \geq 0$, the natural map*

$$\mathrm{Hom}_{\mathcal{SH}_{S^1}(k)}(F, s_nE) \to \mathrm{Hom}_{\mathcal{SH}_{S^1}(k)/f_{n+1}}(F, s_nE)$$

is an isomorphism.

See also [21, proposition 5-3]
We recall the *de-looping formula* [11, theorem 7.4.2]

$$s_n(\Omega_{\mathbb{P}^1}E) \cong \Omega_{\mathbb{P}^1}(s_{n+1}E)$$

for $n \geq 0$.
Take $F \in \mathbf{Spc}_\bullet(k)$. For $E \in \mathbf{Spt}_{S^1}(k)$, we have $\mathcal{H}om^{int}(F, E) \in \mathcal{SH}$, which for $F = X_+$ is just $E(X)$, and in general is formed as the homotopy limit associated to the description of F as a homotopy colimit of representable objects.

This gives us the "internal Hom" functor

$$\mathcal{H}om_{\mathcal{SH}_{S^1}(k)}(F,-):\mathcal{SH}_{S^1}(k)\to\mathcal{SH}_{S^1}(k)$$

and more generally

$$\mathcal{H}om_{\mathcal{SH}_{S^1}(k)/f_{n+1}}(F,-):\mathcal{SH}_{S^1}(k)/f_{n+1}\to\mathcal{SH}_{S^1}(k),$$

with natural transformation

$$\mathcal{H}om_{\mathcal{SH}_{S^1}(k)}(F,-)\to\mathcal{H}om_{\mathcal{SH}_{S^1}(k)/f_{n+1}}(F,-).$$

These have value on $E\in\mathbf{Spt}_{S^1}(k)$ defined by taking a fibrant model $\tilde{E}$ of E (in $\mathcal{SH}_{S^1}(k)$ or $\mathcal{SH}_{S^1}(k)/f_{n+1}$, as the case may be) and forming the presheaf on $\mathbf{Sm}/k$

$$X\mapsto\mathcal{H}om^{int}(F\wedge X_+,\tilde{E}).$$

Putting the de-looping formula together with lemma 8.1 gives us

Proposition 8.2. *For $E\in\mathcal{SH}_{S^1}(k)$ we have natural isomorphisms*

$$s_0(\Omega^n_{\mathbb{P}^1}E)\cong\Omega^n_{\mathbb{P}^1}s_nE\cong\mathcal{H}om_{\mathcal{SH}_{S^1}(k)/f_{n+1}}((\mathbb{P}^1,1)^{\wedge n},s_nE)$$

Proof. Indeed, the first isomorphism is just the de-looping isomorphism repeated n times. For the second, we have

$$\begin{aligned}\Omega^n_{\mathbb{P}^1}s_nE&\cong\mathcal{H}om_{\mathcal{SH}_{S^1}(k)}((\mathbb{P}^1,1)^{\wedge n},s_nE)\\&\cong\mathcal{H}om_{\mathcal{SH}_{S^1}(k)/f_{n+1}}((\mathbb{P}^1,1)^{\wedge n},s_nE)\end{aligned}$$

the second isomorphism following from lemma 8.1. □

Definition 8.3. Suppose that $\operatorname{char}k=0$. Take $E\in\mathcal{SH}_{S^1}(k)$, take $\alpha\in\operatorname{Hom}_{SmCor}(X,Y)$ and let $n\ge1$ be an integer. The *transfer*

$$\operatorname{Tr}_{Y/X}(\alpha):(\Omega^n_{\mathbb{P}^1}s_nE)(Y)\to(\Omega^n_{\mathbb{P}^1}s_nE)(X)$$

is the map in $\mathcal{SH}$ defined as follows:

$$\begin{aligned}(\Omega^n_{\mathbb{P}^1}s_nE)(Y)&=\mathcal{H}om_{\mathcal{SH}_{S^1}(k)}(Y_+,\Omega^n_{\mathbb{P}^1}s_nE)\\&\cong\mathcal{H}om_{\mathcal{SH}_{S^1}(k)}(\Sigma^n_{\mathbb{P}^1}Y_+,s_nE)\\&\cong\mathcal{H}om_{\mathcal{SH}_{S^1}(k)/f_{n+1}}(\Sigma^n_{\mathbb{P}^1}Y_+,s_nE)\\&\xrightarrow{co\text{-}tr^n(\alpha)^*}\mathcal{H}om_{\mathcal{SH}_{S^1}(k)/f_{n+1}}(\Sigma^n_{\mathbb{P}^1}X_+,s_nE)\\&\cong\mathcal{H}om_{\mathcal{SH}_{S^1}(k)}(X_+,\Omega^n_{\mathbb{P}^1}s_nE)\\&\cong(\Omega^n_{\mathbb{P}^1}s_nE)(X).\end{aligned}$$

Theorem 8.4. *Suppose that* $\operatorname{char}k=0$. *For $E\in\mathcal{SH}_{S^1}(k)$, the maps* $\operatorname{Tr}(\alpha)$ *extend the presheaf*

$$\Omega^n_{\mathbb{P}^1}s_nE:\mathbf{Sm}/k^{\mathrm{op}}\to\mathcal{SH}$$

to an $\mathcal{SH}$-valued presheaf with transfers

$$\Omega^n_{\mathbb{P}^1}s_nE:SmCor(k)^{\mathrm{op}}\to\mathcal{SH}$$

Proof. This follows from the definition of the maps $\mathrm{Tr}(\alpha)$ and theorem 6.1, the main point being that the maps $\mathrm{Tr}(\alpha)$ factor through the internal Hom in $\mathcal{SH}_{S^1}(k)/f_{n+1}$. □

COROLLARY 8.5. *Suppose that* $\operatorname{char} k = 0$. *For* $E \in \mathcal{SH}_{S^1}(k)$*, there is an extension of the presheaf*

$$s_0\Omega_{\mathbb{P}^1}E : \mathbf{Sm}/k^{\mathrm{op}} \to \mathcal{SH}$$

to an $\mathcal{SH}$*-valued presheaf with transfers*

$$s_0\Omega_{\mathbb{P}^1}E : SmCor(k)^{\mathrm{op}} \to \mathcal{SH}.$$

Proof. This is just the case $n = 1$ of theorem 8.4, together with the de-looping isomorphism

$$s_0\Omega_{\mathbb{P}^1}E \cong \Omega_{\mathbb{P}^1}s_1E.$$

□

Remark 8.6. The corollary is actually the main result, in that one can deduce theorem 8.4 from corollary 8.5 (applied to $\Omega^{n-1}_{\mathbb{P}^1}E$) and the de-looping formula

$$\Omega^n_{\mathbb{P}^1}s_nE \cong s_0\Omega^n_{\mathbb{P}^1}E = s_0\Omega_{\mathbb{P}^1}(\Omega^{n-1}_{\mathbb{P}^1}E).$$

As the maps $co\text{-}tr^n(\alpha)$ are defined by smashing $co\text{-}tr^1(\alpha)$ with an identity map, this procedure does indeed give back the maps

$$\mathrm{Tr}(\alpha) : \Omega^n_{\mathbb{P}^1}s_nE(Y) \to \Omega^n_{\mathbb{P}^1}s_nE(X)$$

as defined above.

proof of theorem 3. The weak transfers defined above give rise to homotopy invariant sheaves with transfers in the usual sense by taking the sheaves of homotopy groups of the motivic spectrum in question. □

For instance, corollary 8.5 gives the sheaf $\pi_m(s_0\Omega_{\mathbb{P}^1}E)$ the structure of a homotopy invariant sheaf with transfers, in particular, an effective motive. In fact, these are *birational motives* in the sense of Kahn-Huber-Sujatha [7, 10], as s_0F is a birational S^1-spectrum for each S^1-spectrum F. The classical Postnikov tower thus gives us a spectral sequence

$$E^2_{p,q} := H^{-p}(X_{\mathrm{Nis}}, \pi_q(s_0\Omega_{\mathbb{P}^1}E)) \Longrightarrow \pi_{p+q}(s_0\Omega_{\mathbb{P}^1}E(X))$$

with E^2 term a "generalized motivic cohomology" of X. As the sheaves $\pi_q(s_0\Omega_{\mathbb{P}^1}E)$ are motives, we may replace Nisnevich cohomology with Zariski cohomology; as the sheaves $\pi_q(s_0\Omega_{\mathbb{P}^1}E)$ are birational, i.e., Zariski locally constant, the higher Zariski cohomology vanishes, giving us

$$\pi_n(s_0\Omega_{\mathbb{P}^1}E(X)) \cong H^0(X_{\mathrm{Zar}}, \pi_n(s_0\Omega_{\mathbb{P}^1}E)) = \pi_n(s_0\Omega_{\mathbb{P}^1}E(k(X)).$$

In short, we have shown that the 0th slice of a $\mathbb{P}^1$-loop spectrum has transfers in the weak sense. We have already seen in section 2 that this does not hold for an arbitrary object of $\mathcal{SH}_{S^1}(k)$; in the next section we will see that the higher slices of an arbitrary S^1-spectrum do have transfers, albeit in an even weaker sense than the one used above.

9. Transfers on the generalized cycle complex

We begin by recalling from [11, theorem 7.1.1] models for $f_nE(X)$ and $s_nE(X)$ that are reminiscent of Bloch's higher cycle complex [1]. To simplify the notation, we will always assume that we have taken a model $E \in \mathbf{Spt}_{S^1}(k)$ which is quasi-fibrant. For W a closed subset of some $Y \in \mathbf{Sm}/k$, $E^{(W)}(Y)$ will denote the homotopy fiber of the restriction map $E(Y) \to E(Y \setminus W)$.
We make use of the cosimplicial scheme $n \mapsto \Delta^n := \operatorname{Spec} k[t_0, \ldots, t_n]/\sum_i t_i - 1$.
A *face* F of Δ^n is a subscheme defined by $t_{i_1} = \ldots = t_{i_r} = 0$.
For a scheme X of finite type and locally equi-dimensional over k, let $\mathcal{S}_X^{(n)}(m)$ be the set of closed subsets W of $X \times \Delta^m$ of codimension $\geq n$, such that, for each face F of Δ^n, $W \cap X \times F$ has codimension $\geq n$ on $X \times F$ (or is empty). We order $\mathcal{S}_X^{(n)}(m)$ by inclusion.
For $X \in \mathbf{Sm}/k$, we let

$$E^{(n)}(X, m) := \varinjlim_{W \in \mathcal{S}_X^{(n)}(m)} E^{(W)}(X \times \Delta^m).$$

Similarly, for $0 \leq n \leq n'$, we define

$$E^{(n/n')}(X, m) := \varinjlim_{W \in \mathcal{S}_X^{(n)}(m), W' \in \mathcal{S}_X^{(n')}(m)} E^{(W \setminus W')}(X \times \Delta^m \setminus W')$$

The conditions on the intersections of W with $X \times F$ for faces F means that $m \mapsto \mathcal{S}_X^{(n)}(m)$ form a cosimplicial set, denoted $\mathcal{S}_X^{(n)}$, for each n and that $\mathcal{S}_X^{(n')}$ is a cosimplicial subset of $\mathcal{S}_X^{(n)}$ for $n \leq n'$. Thus the restriction maps for E make $m \mapsto E^{(n)}(X, m)$ and $m \mapsto E^{(n/n')}(X, m)$ simplicial spectra, denoted $E^{(n)}(X, -)$ and $E^{(n/n')}(X, -)$. We denote the associated total spectra by $|E^{(n)}(X, -)|$ and $|E^{(n/n')}(X, -)|$.
The inclusion $\mathcal{S}_X^{(n')}(m) \to \mathcal{S}_X^{(n)}(m)$ for $n \leq n'$ and the evident restriction maps give the sequence

$$|E^{(n')}(X, -)| \to |E^{(n)}(X, -)| \to |E^{(n/n')}(X, -)|$$

which is easily seen to be a weak homotopy fiber sequence.
We note that $|E^{(0)}(X, -)| = E(X \times \Delta^*)$; as E is homotopy invariant, the canonical map

$$E(X) \to |E^{(0)}(X, -)|$$

is thus a weak equivalence. We therefore have the tower in $\mathcal{SH}$

$$\ldots \to |E^{(n+1)}(X, -)| \to |E^{(n)}(X, -)| \to \ldots \to |E^{(0)}(X, -)| \cong E(X) \tag{9.1}$$

with nth layer isomorphic to $|E^{(n/n+1)}(X, -)|$. We call this tower the *homotopy coniveau tower* for $E(X)$. In this regard, one of the main results from [11] states

Theorem 9.1 ([11, theorem 7.1.1]). *There is a canonical isomorphism of the tower* (9.1) *with the motivic Postnikov tower evaluated at* X*:*

$$\ldots \to f_{n+1}E(X) \to f_nE(X) \to \ldots \to f_0E(X) = E(X),$$

giving a canonical isomorphism

$$s_nE(X) \cong |E^{(n/n+1)}(X,-)|.$$

We can further modify this description of $s_nE(X)$ as follows: Since s_n is an idempotent functor, we have

$$s_nE(X) \cong s_n(s_nE)(X) \cong |(s_nE)^{(n/n+1)}(X,-)|.$$

Note that $|(s_nE)^{(n/n+1)}(X,-)|$ fits into a weak homotopy fiber sequence

$$|(s_nE)^{(n+1)}(X,-)| \to |(s_nE)^{(n)}(X,-)| \to |(s_nE)^{(n/n+1)}(X,-)|.$$

Using theorem 9.1 in reverse, we have the isomorphism in $\mathcal{SH}$

$$|(s_nE)^{(n+1)}(X,-)| \cong f_{n+1}(s_nE)(X).$$

But as $f_{n+1} \circ f_n \cong f_{n+1}$, we see that $f_{n+1}(s_nE) \cong 0$ in $\mathcal{SH}_{S^1}(k)$ and thus

$$|(s_nE)^{(n)}(X,-)| \cong |(s_nE)^{(n/n+1)}(X,-)| \cong s_nE(X).$$

We may therefore use the simplicial model $|(s_nE)^{(n)}(X,-)|$ for $s_nE(X)$.
We will need a refinement of this construction, which takes into account the interaction of the support conditions with a given correspondence.

DEFINITION 9.2. Let $A \subset Y \times X$ be a generator in $\mathrm{Hom}_{SmCor}(Y,X)$; for each m, we let $A(m) \in \mathrm{Hom}_{SmCor}(Y \times \Delta^m, X \times \Delta^m)$ denote the correspondence $A \times \mathrm{id}_{\Delta^m}$. Let $\mathcal{S}^{(n)}_{X,A}(m)$ be the subset of $\mathcal{S}^{(n)}_X(m)$ consisting of those $W' \in \mathcal{S}^{(n)}_X(m)$ such that

(1) $W := p_{Y\times\Delta^m}(A \times \Delta^m \cap Y \times W')$ is in $\mathcal{S}^{(n)}_Y(m)$.
(2) $A(m)$ is in $\mathrm{Hom}_{SmCor}(Y \times \Delta^m, X \times \Delta^m)_{W,W'}$.

For an arbitrary $\alpha \in \mathrm{Hom}_{SmCor}(Y,X)$, write

$$\alpha = \sum_{i=1}^r n_iA_i$$

with the A_i generators and the n_i non-zero integers and define

$$\mathcal{S}^{(n)}_{X,\alpha}(m) := \cap_{i=1}^r \mathcal{S}^{(n)}_{X,A_i}(m).$$

If we have in addition to α a finite correspondence $\beta \in \mathrm{Hom}_{SmCor}(Z,Y)$, we let $\mathcal{S}^{(n)}_{X,\alpha,\beta}(m) \subset \mathcal{S}^{(n)}_{X,\alpha}(m)$ be the set of $W \subset X \times \Delta^m$ such that W is in $\mathcal{S}^{(n)}_{X,\alpha}(m)$ and $p^{Y\times X\times\Delta^m}_{Y\times\Delta^m}(Y \times W \cap |\alpha| \times \Delta^m)$ is in $\mathcal{S}^{(n)}_{Y,\beta}(m)$.

For $f : Y \to X$ a flat morphism, one has

$$\mathcal{S}^{(n)}_{X,\Gamma_f}(m) = \mathcal{S}^{(n)}_X(m)$$

and for $g : Z \to Y$ a flat morphism, and $\alpha \in \mathrm{Hom}_{SmCor}(Y,X)$, one has

$$\mathcal{S}^{(n)}_{X,\alpha,\Gamma_g}(m) = \mathcal{S}^{(n)}_{X,\alpha}(m)$$

Note that $m \mapsto \mathcal{S}^{(n)}_{X,\alpha}(m)$ and $m \mapsto \mathcal{S}^{(n)}_{X,\alpha,\beta}(m)$ define cosimplicial subsets of $m \mapsto \mathcal{S}^{(n)}_X(m)$. We define the simplicial spectra $E^{(n)}(X,-)_\alpha$ and $E^{(n)}(X,-)_{\alpha,\beta}$ using the support conditions $\mathcal{S}^{(n)}_{X,\alpha}(m)$ and $\mathcal{S}^{(n)}_{X,\alpha,\beta}(m)$ instead of $\mathcal{S}^{(n)}_X(m)$:

$$E^{(n)}(X,m)_\alpha := \varinjlim_{W\in\mathcal{S}^{(n)}_{X,\alpha}(m)} E^{(W)}(X\times\Delta^m)$$

$$E^{(n)}(X,m)_{\alpha,\beta} := \varinjlim_{W\in\mathcal{S}^{(n)}_{X,\alpha,\beta}(m)} E^{(W)}(X\times\Delta^m),$$

giving us the sequence of simplicial spectra

$$E^{(n)}(X,-)_{\alpha,\beta} \to E^{(n)}(X,-)_\alpha \to E^{(n)}(X,-).$$

The main "moving lemma" [12, theorem 2.6.2(2)] yields

Proposition 9.3. *For $X \in \mathbf{Sm}/k$ affine, and $E \in \mathbf{Spt}_{S^1}(k)$ quasi-fibrant, the maps*

$$|E^{(n)}(X,-)_{\alpha,\beta}| \to |E^{(n)}(X,-)_\alpha| \to |E^{(n)}(X,-)|$$

are weak equivalences.

We proceed to the main construction of this section. Consider the simplicial model $|(s_nE)^{(n)}(X,-)|$ for $s_nE(X)$. For each m, we may consider the classical Postnikov tower (or rather, its dual version) for the spectrum $(s_nE)^{(n)}(X,m)$, which we write as

$$\ldots \to \tau_{\geq p+1}(s_nE)^{(n)}(X,m) \to \tau_{\geq p}(s_nE)^{(n)}(X,m) \to \ldots \to (s_nE)^{(n)}(X,m),$$

where

$$\tau_{\geq p+1}(s_nE)^{(n)}(X,m) \to (s_nE)^{(n)}(X,m)$$

is the p-connected cover of $(s_nE)^{(n)}(X,m)$. The pth layer in this tower is of course the pth suspension of the Eilenberg-Maclane spectrum on $\pi_p((s_nE)^{(n)}(X,m))$. Taking a functorial model for the p-connected cover, we have for each p the simplicial spectrum

$$m \mapsto \tau_{\geq p+1}(s_nE)^{(n)}(X,m)$$

giving us the tower of total spectra

(9.2)
$$\ldots \to |\tau_{\geq p+1}(s_nE)^{(n)}(X,-)| \to |\tau_{\geq p}(s_nE)^{(n)}(X,-)| \to \ldots \to |(s_nE)^{(n)}(X,-)|.$$

The pth layer in this tower are then (up to suspension) the Eilenberg-Maclane spectrum on the chain complex $\pi_p(s_nE)^{(n)}(X,*)$, with differential as usual the alternating sum of the face maps.

The chain complexes $\pi_p(s_nE)^{(n)}(X,*)$ are evidently functorial for smooth maps and inherit the homotopy invariance property from $(s_nE)^{(n)}(X,*)$ (see [12, theorem 3.3.5]). Somewhat more surprising is

Lemma 9.4. *The complexes $\pi_p(s_nE)^{(n)}(X,*)$ satisfy Nisnevich excision.*

Proof. Let $W \subset X \times \Delta^m$ be a closed subset in $\mathcal{S}_X^{(n)}(m)$, and let w be the set of generic points of W having codimension exactly n on $X \times \Delta^m$. Then

$$s_n E^{(W)}(X \times \Delta^m) \cong s_n E(\Sigma^n_{\mathbb{P}^1} w_+) \cong \Omega^n_{\mathbb{P}^1}(s_n E)(w) \cong s_0(\Omega^n_{\mathbb{P}^1} E)(w).$$

This gives us the following description of $\pi_p((s_n E)^{(n)}(X, m))$:

$$\pi_p((s_n E)^{(n)}(X, m)) \cong \oplus_w \pi_p(s_0(\Omega^n_{\mathbb{P}^1} E)(w))$$

where the direct sum is over the set $\mathcal{T}_X^{(n)}(m)$ of generic points of the irreducible $W \in \mathcal{S}_X^{(n)}(m)$ having codimension exactly n in $X \times \Delta^m$.
Now let $i : Z \to X$ be a closed subset with open complement $j : U \to X$. For each m, we thus have the exact sequence

$$\begin{aligned} 0 \to \oplus_{w \in Z \times \Delta^m \cap \mathcal{T}_X^{(n)}(m)} \pi_p(s_0(\Omega^n_{\mathbb{P}^1} E)(w)) \to \oplus_{w \in \mathcal{T}_X^{(n)}(m)} \pi_p(s_0(\Omega^n_{\mathbb{P}^1} E)(w)) \\ \to \oplus_{w \in \mathcal{T}_X^{(n)}(m) \cap U \times \Delta^m} \pi_p(s_0(\Omega^n_{\mathbb{P}^1} E)(w)) \to 0 \end{aligned}$$

Define the subcomplex $\pi_p(s_n E)^{(n)}(X, *)_Z$ of $\pi_p(s_n E)^{(n)}(X, *)$ and quotient complex $\pi_p(s_n E)^{(n)}(U_X, *)$ of $\pi_p(s_n E)^{(n)}(X, *)$ by taking supports in

$$\{W \in \mathcal{S}_X^{(n)}(m) \mid W \subset Z \times \Delta^m\}, \text{ resp. } \{W \cap U \times \Delta^m \mid W \in \mathcal{S}_X^{(n)}(m)\}.$$

We thus have the term-wise exact sequence of complexes

$$0 \to \pi_p(s_n E)^{(n)}(X, *)_Z \to \pi_p(s_n E)^{(n)}(X, *) \to \pi_p(s_n E)^{(n)}(U_X, *) \to 0$$

CLAIM. *The inclusion*

$$\pi_p(s_n E)^{(n)}(U_X, *) \xrightarrow{\iota} \pi_p(s_n E)^{(n)}(U, *)$$

is a quasi-isomorphism.

Proof of the claim. This follows using the localization technique [13, theorem 8.10] (for details, see [11, theorem 3.2.1]). In a few words, one takes an integer N and a $W \in \mathcal{S}_U^{(n)}(N)$. We assume that $(\mathrm{id}_U \times \Delta(\sigma))(W) = W$ for each permutation σ of the vertices of Δ^N, where $\Delta(\sigma) : \Delta^N \to \Delta^N$ is the affine-linear extension of σ. For $m \le N$, let $\mathcal{T}_X^{(n)}(m)_W \subset \mathcal{T}_X^{(n)}(m)$ be the set of points w such that $w \in (\mathrm{id}_U \times \Delta(g))^*(W)$ for some injective $g : [m] \to [N]$, and set

$$\pi_p(s_n E)^{(n)}(U, m)_W := \oplus_{w \in \mathcal{T}_X^{(n)}(m)_W} \pi_p(s_n E)^{(n)}(U, m) \subset \pi_p(s_n E)^{(n)}(U, m).$$

For $m > N$ set $\pi_p(s_n E)^{(n)}(U, m)_W = 0$. This gives us the subcomplex

$$\pi_p(s_n E)^{(n)}(U, *)_W \subset \pi_p(s_n E)^{(n)}(U, *);$$

clearly $\pi_p(s_n E)^{(n)}(U, *)$ is the colimit of the subcomplexes $\pi_p(s_n E)^{(n)}(U, *)_W$. Similarly, we have the subcomplex $\pi_p(s_n E)^{(n)}(U_X, *)_W$ of $\pi_p(s_n E)^{(n)}(U_X, *)$ and the inclusion

$$\iota_W : \pi_p(s_n E)^{(n)}(U_X, *)_W \to \pi_p(s_n E)^{(n)}(U, *)_W,$$

with $\pi_p(s_n E)^{(n)}(U_X, *)$ the colimit of the $\pi_p(s_n E)^{(n)}(U_X, *)_W$.

In [11, theorem 3.2.1], we have the formal sums of maps of simplices

$$\psi_W(m) = \sum_i n_i \psi_W(m)_i; \quad \psi_W(m)_i : \Delta^m \to \Delta^m$$
$$\Psi_W(m) = \sum_i m_i \Psi_W(m)_i; \quad \Psi_W(m)_i : \Delta^{m+1} \to \Delta^m$$

for $m = 0, \ldots, N$, such that the pull-back by the maps $\psi_W(m)$ define a map of complexes

$$\psi_W^* : \pi_p(s_n E)^{(n)}(U, *)_W \to \pi_p(s_n E)^{(n)}(U_X, *).$$

Additionally, the pull-back by the $\Psi_W(m)$ define homotopies of the map $\iota \circ \psi_W$ with the inclusion $\pi_p(s_n E)^{(n)}(U, *)_W \to \pi_p(s_n E)^{(n)}(U, *)$ and similarly of $\psi_W \circ \iota_W$ with the inclusion $\pi_p(s_n E)^{(n)}(U_X, *)_W \to \pi_p(s_n E)^{(n)}(U_X, *)$. The claim follows easily from this. □

We therefore have the quasi-isomorphism
(9.3)
$$\pi_p(s_n E)^{(n)}(X, *)_Z \to \mathrm{cone}\left(\pi_p(s_n E)^{(n)}(X, *) \xrightarrow{j^*} \pi_p(s_n E)^{(n)}(U, *)\right)[-1].$$

Now let

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & X' \arrow[d, "p"] \\ U \arrow[r] & X \end{tikzcd}$$

be an elementary Nisnevich square, i.e., the square is cartesian, p is étale and induces an isomorphism $p : Z' := X' \setminus U' \to Z$. Clearly p induces an isomorphism

$$p^* : \pi_p(s_n E)^{(n)}(X, *)_Z \to \pi_p(s_n E)^{(n)}(X', *)_{Z'};$$

using the localization quasi-isomorphism (9.3), it follows that p^* induces a quasi-isomorphism

$$\mathrm{cone}\left(\pi_p(s_n E)^{(n)}(X, *) \xrightarrow{j^*} \pi_p(s_n E)^{(n)}(U, *)\right)[-1]$$
$$\xrightarrow{p^*} \mathrm{cone}\left(\pi_p(s_n E)^{(n)}(X', *) \xrightarrow{j^*} \pi_p(s_n E)^{(n)}(U', *)\right)[-1],$$

proving the lemma. □

We will use the results of section 7 to give $X \mapsto \pi_p(s_n E)^{(n)}(X, *)$ the structure of a complex of homotopy invariant presheaves with transfer on $\mathbf{Sm}/k$, i.e. a motive.
For this, we consider the complexes $\pi_p(s_n E)^{(n)}(X, *)_\alpha$, $\pi_p(s_n E)^{(n)}(X, *)_{\alpha,\beta}$ constructed above. The refined support condition are constructed so that, for each $W \in \mathcal{S}^{(n)}_{X,\alpha}(m)$, $\alpha \times \mathrm{id}_{\Delta^m}$ is in $\mathrm{Hom}_{SmCor}(Y \times \Delta^m, X \times \Delta^m)_{W',W}$, where

$$W' = p_{Y \times \Delta^m}(Y \times \Delta^m \times W \cap |\alpha \times \mathrm{id}_{\Delta^m}|).$$

We may therefore use the morphism $co\text{-}tr^n(\alpha \times \mathrm{id}_{\Delta^m})$ to define the map

$$\mathrm{Tr}_{Y/X}(\alpha)(m) : \pi_p((s_nE)^{(n)}(X,m))_\alpha \to \pi_p(s_nE)^{(n)}(Y,m).$$

By proposition 7.5, the maps $\mathrm{Tr}_{Y/X}(m)$ define a map of complexes

$$\mathrm{Tr}_{Y/X}(\alpha) : \pi_p(s_nE)^{(n)}(X,*)_\alpha \to \pi_p(s_nE)^{(n)}(Y,*).$$

Similarly, given $\beta \in \mathrm{Hom}_{SmCor}(Z,Y)$, we have the map of complexes

$$\mathrm{Tr}_{Y/X}(\alpha)_\beta : \pi_p(s_nE)^{(n)}(X,*)_{\alpha,\beta} \to \pi_p(s_nE)^{(n)}(Y,*)_\beta.$$

Note that, due to possible cancellations occurring when one takes the composition $\alpha \circ \beta$, we have only an inclusion

$$\mathcal{S}^{(n)}_{X,\alpha,\beta}(m) \subset \mathcal{S}^{(n)}_{X,\alpha\circ\beta}(m)$$

giving us a natural comparison map

$$\iota_{\alpha,\beta} : \pi_p(s_nE)^{(n)}(X,*)_{\alpha,\beta} \to: \pi_p(s_nE)^{(n)}(X,*)_{\alpha\circ\beta}.$$

Using our moving lemma again, we see that $\iota_{\alpha,\beta}$ is a quasi-isomorphism in case X is affine.

Lemma 9.5. *Suppose* $\mathrm{char}\, k = 0$. *For*

$$\alpha \in \mathrm{Hom}_{SmCor}(Z,Y),\ \beta \in \mathrm{Hom}_{SmCor}(Z,Y),$$

we have

$$\mathrm{Tr}_{Z/Y}(\beta) \circ \mathrm{Tr}_{Y/X}(\alpha)_\beta = \mathrm{Tr}_{Z/X}(\alpha \circ \beta) \circ \iota_{\alpha,\beta}.$$

Proof. This follows from lemma 7.4. □

We have already noted that complexes $\pi_p(s_nE)^{(n)}(X,*)$ are functorial in X for flat morphisms in $\mathbf{Sm}/k$, in particular for smooth morphisms in $\mathbf{Sm}/k$. Let $\widetilde{\mathbf{Sm}}/k$ denote the subcategory of $\mathbf{Sm}/k$ with the same objects and with morphisms the smooth morphisms. The transfer maps we have defined on the refined complexes, together with the moving lemma 7.4 yield the following result:

Theorem 9.6. *Suppose* $\mathrm{char}\, k = 0$. *Consider the presheaf*

$$\pi_p(s_nE)^{(n)}(-,*)) : \widetilde{\mathbf{Sm}}/k^{\mathrm{op}} \to C^-(\mathbf{Ab})$$

on $\widetilde{\mathbf{Sm}}/k^{\mathrm{op}}$. *Let*

$$\iota : \widetilde{\mathbf{Sm}}/k \to SmCor(k)$$

be the evident inclusion and let

$$Q : C^-(\mathbf{Ab}) \to D^-(\mathbf{Ab})$$

be the evident additive functor. There is a complex of presheaves with transfers

$$\hat{\pi}_p((s_nE)^{(n)})^* : SmCor(k)^{\mathrm{op}} \to C^-(\mathbf{Ab})$$

and an isomorphism of functors from $\widetilde{\mathbf{Sm}}/k^{\mathrm{op}}$ *to* $D^-(\mathbf{Ab})$

$$Q \circ \pi_p(s_nE)^{(n)}(-,*)) \cong Q \circ \hat{\pi}_p((s_nE)^{(n)})^* \circ \iota.$$

Proof. We give a rough sketch of the construction here; for details we refer the reader to [9, proposition 2.2.3], which in turn is an elaboration of [12, theorem 7.4.1]. The construction of $\hat{\pi}_p((s_nE)^{(n)})^*$ is accomplished by first taking a homotopy limit over the complexes $\pi_p(s_nE)^{(n)}(X,*)_\alpha$. These are then functorial on $SmCor(k)^{\mathrm{op}}$, up to homotopy equivalences arising from the replacement of the index category for the homotopy limit with a certain cofinal subcategory. One then forms a regularizing homotopy colimit that is strictly functorial on $SmCor(k)^{\mathrm{op}}$, and finally, one replaces this presheaf with a fibrant model. The moving lemma for affine schemes (proposition 9.3) implies that the homotopy limit construction yields for each affine $X \in \mathbf{Sm}/k$ a complex canonically quasi-isomorphic to $\pi_p(s_nE)^{(n)}(X,*)$; this property is inherited by the regularizing homotopy colimit. As the complexes $\pi_p(s_nE)^{(n)}(X,*)$ satisfy Nisnevich excision (lemma 9.4) and are homotopy invariant for *all* X, this implies that $\hat{\pi}_p((s_nE)^{(n)})^*(X)$ is canonically isomorphic to $\pi_p(s_nE)^{(n)}(X,*)$ in $D^-(\mathbf{Ab})$ for all $X \in \mathbf{Sm}/k$. By construction, this isomorphism is natural with respect to smooth morphisms in $\mathbf{Sm}/k$. □

Corollary 9.7. *Suppose* $\operatorname{char} k = 0$. $\hat{\pi}_p((s_nE)^{(n)})^*$ *is a homotopy invariant complex of presheaves with transfer.*

Proof. By theorem 9.6, we have the isomorphism in $D^-(\mathbf{Ab})$

$$\hat{\pi}_p((s_nE)^{(n)})^*(X) \cong \pi_p(s_nE)^{(n)}(X,*).$$

for all $X \in \mathbf{Sm}/k$, natural with respect to smooth morphisms. As the presheaf $\pi_p(s_nE)^{(n)}(-,*)$ is homotopy invariant, so is $\hat{\pi}_p((s_nE)^{(n)})^*$. □

proof of theorem 2. As in the proof of theorem 9.6, the method of [12, theorem 7.4.1], shows that the tower (9.2) extends to a tower

$$\ldots \to \rho_{\geq p+1}s_nE \to \rho_{\geq p}s_nE \to \ldots \to s_nE \tag{9.4}$$

in $\mathcal{SH}_{S^1}(k)$ with value (9.2) at $X \in \mathbf{Sm}/k$, and with the cofiber of $\rho_{\geq p+1}s_nE \to \rho_{\geq p}s_nE$ naturally isomorphic to $EM^{eff}_{\mathbb{A}^1}(\hat{\pi}_p((s_nE)^{(n)})^*)$. By lemma 9.4 and corollary 9.7, the presheaves $\hat{\pi}_p((s_nE)^{(n)})^*$ define objects in $DM^{eff}_-(k)$. Thus, we have shown that the layers in the tower (9.4) have a motivic structure, proving theorem 2. □

10. The Friedlander-Suslin tower

As the reader has surely noticed, the lack of functoriality for the simplicial spectra $E^{(n)}(X,-)$ creates annoying technical problems when we wish to extend the construction of the homotopy coniveau tower to a tower in $\mathcal{SH}_{S^1}(k)$. In their work on the spectral sequence from motivic cohomology to K-theory, Friedlander and Suslin [4] have constructed a completely functorial version of the homotopy coniveau tower, using "quasi-finite supports". Unfortunately, the comparison between the Friedlander-Suslin version and $E^{(n)}(X,-)$ is proven in [4] only for K-theory and motivic cohomology. In this last section, we recall the

Friedlander-Suslin construction and form the conjecture that the Friedlander-Suslin tower is naturally isomorphic to the homotopy coniveau tower.

Let $\mathcal{Q}_X^{(n)}(m)$ be the set of closed subsets W of $\mathbb{A}^n \times X \times \Delta^m$ such that, for each irreducible component W' of W, the projection $W' \to X \times \Delta^m$ is quasi-finite. For $E \in \mathbf{Spt}_{S^1}(k)$, we let

$$E_{FS}^{(n)}(X, m) := \varinjlim_{W \in \mathcal{Q}_X^{(n)}(m)} E^{(W)}(\mathbb{A}^n \times X \times \Delta^m)$$

As the condition defining $\mathcal{Q}_X^{(n)}(m)$ are preserved under maps

$$\mathrm{id}_{\mathbb{A}^n} \times f \times g : \mathbb{A}^n \times X' \times \Delta^{m'} \to \mathbb{A}^n \times X \times \Delta^m,$$

where $f : X' \to X$ is an arbitrary map in $\mathbf{Sm}/k$, and $g : \Delta^{m'} \to \Delta^m$ is a structure map in Δ^*, the spectra $E_{FS}^{(n)}(X, m)$ define a simplicial spectrum $E_{FS}^{(n)}(X, -)$ and these simplicial spectra, for $X \in \mathbf{Sm}/k$, extend to a presheaf of simplicial spectra on $\mathbf{Sm}/k$:

$$E_{FS}^{(n)}(?, -) : \mathbf{Sm}/k^{\mathrm{op}} \to \Delta^{\mathrm{op}}\mathbf{Spt}.$$

Similarly, if we take the linear embedding $i_n : \mathbb{A}^n \to \mathbb{A}^{n+1} = \mathbb{A}^n \times \mathbb{A}^1$, $x \mapsto (x, 0)$, the pull-back by $i_n \times \mathrm{id}$ preserves the support conditions, and thus gives a well-defined map of simplicial spectra

$$i_n^* : E_{FS}^{(n+1)}(X, -) \to E_{FS}^{(n)}(X, -),$$

forming the tower of presheaves on $\mathbf{Sm}/k$

$$\ldots \to E_{FS}^{(n+1)}(?, -) \to E_{FS}^{(n)}(?, -) \to \ldots \;. \tag{10.1}$$

We may compare $E_{FS}^{(n)}(X, -)$ and $E^{(n)}(X, -)$ using the method of [4] as follows: The simplicial spectra $E^{(n)}(X, -)$ are functorial for flat maps in $\mathbf{Sm}/k$, in the evident manner. They satisfy homotopy invariance, in that the pull-back map

$$p^* : E^{(n)}(X, -) \to E^{(n)}(\mathbb{A}^1 \times X, -)$$

induces a weak equivalence on the total spectra. We have the evident inclusion of simplicial sets

$$\mathcal{Q}_X^{(n)}(-) \hookrightarrow \mathcal{S}_{\mathbb{A}^n \times X}^{(n)}(-)$$

inducing the map

$$\varphi_{X,n} : E_{FS}^{(n)}(X, -) \to E^{(n)}(\mathbb{A}^n \times X, -).$$

Together with the weak equivalence $p^* : |E^{(n)}(X, -)| \to |E^{(n)}(\mathbb{A}^n \times X, -)|$, the maps $\varphi_{X,n}$ induce a map of towers of total spectra in $\mathcal{SH}$

$$\varphi_{X,*} : |E_{FS}^{(*)}(X, -)| \to |E^{(*)}(X, -)|. \tag{10.2}$$

Conjecture 10.1. *For each $X \in \mathbf{Sm}/k$ and each quasi-fibrant $E \in \mathbf{Spt}_{S^1}(k)$, the map* (10.2) *induces an isomorphism in $\mathcal{SH}$ of the towers of total spectra.*

Combined with the weak equivalence given by homotopy invariance and the results of [11], this would give us an isomorphism in $\mathcal{SH}_{S^1}(k)$:

$$f_nE \cong |E^{(n)}_{FS}(?, -)|.$$

As transfers in some form or other are used in the arguments relating the Friedlander-Suslin complex to the Bloch-type complexes in the known cases, a weaker form of the conjecture might be more reasonable:

Conjecture 10.2. *For each $X \in \mathbf{Sm}/k$ and each quasi-fibrant $E \in \mathbf{Spt}_{S^1}(k)$ with $s_0E \cong 0$ in $\mathcal{SH}_{S^1}(k)$, the map* (10.2) *induces an isomorphism in $\mathcal{SH}$ of the towers of total spectra.*

References

[1] Bloch, S., *Algebraic cycles and higher K-theory*, Adv. in Math. 61 (1986), no. 3, 267–304.

[2] Bloch, S. and Lichtenbaum, S., *A spectral sequence for motivic cohomology*, preprint (1995).

[3] Cisinski, D.-C. and Déglise, F. *Local and stable homological algebra in Grothendieck abelian categories*, Homology, Homotopy and Applications 11 (2009), 219-260.

[4] Friedlander, E. and Suslin, A., *The spectral sequence relating algebraic K-theory to motivic cohomology*, Ann. Sci. École Norm. Sup (4) 35, No. 6 (2002), 773-875.

[5] Friedlander, E., Suslin, A. and Voevodsky, V., Cycles, Transfers and Motivic Homology Theories, Annals of Math. Studies 143, Princeton Univ. Press, 2000.

[6] Gabber, O. *Gersten's conjecture for some complexes of vanishing cycles.* Manuscripta Math. 85 (1994), no. 3-4, 323–343.

[7] Huber, A. and Kahn, B., *The slice filtration and mixed Tate motives*, Compositio Math. 142 (2006), 907–936.

[8] Jardine, J.F., *Motivic symmetric spectra*, *Doc. Math.* 5 (2000), 445–553.

[9] Kahn, B. and Levine, M. *Motives of Azumaya algebras*, Journal of the Inst. of Math. Jussieu 9(3), 2010, 481599.

[10] Kahn, B. and Sujatha, R., *Birational motives*, revised version, in preparation.

[11] Levine, M., *The homotopy coniveau tower.* J Topology 1 (2008) 217–267.

[12] Levine, M., *Chow's moving lemma in $\mathbb{A}^1$-homotopy theory.* K-theory 37 (1-2) (2006) 129-209.

[13] Levine, M. *Techniques of localization in the theory of algebraic cycles*, J. Alg. Geom. 10 (2001) 299-363.

[14] Morel, F., *An introduction to $\mathbb{A}^1$-homotopy theory. Contemporary developments in algebraic K-theory* 357–441, ICTP Lect. Notes, XV, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2004.

[15] Morel, F. and Voevodsky, V., *$\mathbb{A}^1$-homotopy theory of schemes*, Inst. Hautes Études Sci. Publ. Math. 90 (1999), 45–143.

[16] Neeman, A, Triangulated categories. Annals of Mathematics Studies, 148. Princeton University Press, Princeton, NJ, 2001.

[17] Pelaez-Menaldo, J.P. *Multiplicative structure on the motivic Postnikov tower.* preprint, 2008. Math arXiv:0806.1704v3

[18] Pelaez-Menaldo, J.P. *Mixed motives and the slice filtration.* C. R. Math. Acad. Sci. Paris 347 (2009), no. 9-10, 541–544.

[19] Röndigs, O. and Østvær, P.A. *Motives and modules over motivic cohomology.* C. R. Math. Acad. Sci. Paris 342 (2006), no. 10, 751–754.

[20] Röndigs, O. and Østvær, P.A. *Modules over motivic cohomology*, Adv. Math. 219 (2008), 689–727.

[21] Verdier, J.L., *Categorie derivees, état 0.* in Cohomologie étale. Séminaire de Géométrie Algébrique du Bois-Marie SGA 4-1/2 . P. Deligne, avec la collaboration de J. F. Boutot, A. Grothendieck, L. Illusie et J. L. Verdier. pp. 262–311. Lecture Notes in Mathematics, 569. Springer-Verlag, Berlin-New York, 1977.

[22] Voevodsky, V. *A possible new approach to the motivic spectral sequence for algebraic K-theory*, *Recent progress in homotopy theory (Baltimore, MD, 2000)* 371–379, Contemp. Math., 293 (Amer. Math. Soc., Providence, RI, 2002).

[23] Voevodsky, V. *Cohomological theory of presheaves with transfers.* Cycles, transfers and motivic cohomology theories, Ann. of Math. Studies 143, Princeton Univ. Press, 2000.

Marc Levine
Northeastern University
Department of Mathematics
Boston
MA 02115
U.S.A.
marc@neu.edu

Fakultät Mathematik,
Campus Essen
Universität Duisburg-Essen
45117 Essen
Germany
marc.levine@uni-due.de

Some Consequences of the Karpenko-Merkurjev Theorem

To Andrei Alexandrovich Suslin
on the occasion of his 60th birthday

Aurel Meyer[1], ZinovyReichstein[2]

Received: December 12, 2008
Revised: May 1, 2009

Abstract. We use a recent theorem of N. A. Karpenko and A. S. Merkurjev to settle several questions in the theory of essential dimension.

2010 Mathematics Subject Classification: 20D15, 20C15, 20G15
Keywords and Phrases: Essential dimension, linear representation, p-group, algebraic torus

1. Introduction

Let k be a field, Fields_k be the category of field extensions K/k, Sets be the category of sets, and $F\colon \text{Fields}_k \to \text{Sets}$ be a covariant functor. Given a tower of field extensions $k \subset K \subset L$, we will denote the image of $a \in F(K)$ under the natural map $F(K) \to F(L)$ by a_L. Conversely, if $b \in F(L)$ lies in the image of this map, we will say that b *descends* to K.

Given a field extension K/k and $b \in F(L)$, the *essential dimension* $\text{ed}_k(b)$ of b is defined as the minimal transcendence degree $\text{trdeg}_k(K)$, as K ranges over all intermediate subfields $k \subset K \subset L$ such that b descends to K. Informally speaking, this is the minimal number of parameters one needs to define b. The essential dimension $\text{ed}_k(F)$ of the functor F is the maximal value of $\text{ed}_k(b)$, as L ranges over all field extensions of k and b ranges over $F(L)$. Informally speaking, this is the minimal number of parameters required to define any object of F.

The *essential dimension* $\text{ed}_k(b; p)$ at a prime p is defined as the minimum of $\text{ed}_k(b_{L'})$, taken over all finite field extensions L'/L such that the degree $[L' : L]$

[1]Partially supported by a University Graduate Fellowship at the University of British Columbia.

[2]Partially supported by NSERC Discovery and Accelerator Supplement grants.

is prime to p. The essential dimension of $\mathrm{ed}_k(F;p)$ of F at a prime p is the supremum of $\mathrm{ed}_k(b;p)$ taken over all $b \in F(L)$ and over all field extensions L/k. An important example where the above notions lead to a rich theory is the nonabelian cohomology functor $F_G = H^1(*, G)$, sending a field K/k to the set $H^1(K, G)$ of isomorphism classes of G-torsors over $\mathrm{Spec}(K)$, in the fppf topology. Here G is an algebraic group defined over k. The essential dimension of this functor can be thought of as a numerical measure of complexity of G-torsors over fields or, alternatively, as the minimal number of parameters required to define a versal G-torsor. In the case where G is a finite (constant) group defined over k, which will be the main focus of this paper, $\mathrm{ed}_k(G)$ is the minimal number of parameters required to describe all G-Galois extensions.
For details on the notion of essential dimension of a finite group we refer the reader to [BuR], [Re] or [JLY, Chapter 8], on the notion of essential dimension of a functor to [BF] or [BRV$_2$] and on essential dimension at a prime p to [Me].
N. Karpenko and A. Merkurjev [KM] recently proved the following formula for the essential dimension of a (finite) p-group.

Theorem 1.1. *Let G be a p-group and k be a field of characteristic $\neq p$ containing a primitive pth root of unity. Then*

$$\mathrm{ed}_k(G;p) = \mathrm{ed}_k(G) = \min \dim(V)\,,$$

where the minimum is taken over all faithful k-representations $G \hookrightarrow \mathrm{GL}(V)$.

The purpose of this paper is to explore some of the consequences of this theorem. The following notation will be used throughout.
We will fix a prime p and a base field k such that

$$\mathrm{char}(k) \neq p \text{ and } k \text{ contains } \zeta, \tag{1}$$

where ζ is a primitive pth root of unity if $p \geq 3$ and a primitive 4th root of unity if $p = 2$.
For a finite group H, we will denote the intersection of the kernels of all multiplicative characters $\chi \colon H \to k^*$ by H'. In particular, if k contains an eth root of unity, where e is the exponent of H, then $H' = [H, H]$ is the commutator subgroup of H.
All p-groups in this paper will be assumed to be finite. Given a p-group G, we set $C(G)$ to be the center of G and

$$C(G)_p := \{g \in C(G) \mid g^p = 1\} \tag{2}$$

to be the p-torsion subgroup of $C(G)$. We will view $C(G)_p$ and its subgroups as $\mathbb{F}_p$-vector spaces, and write "$\dim_{\mathbb{F}_p}$" for their dimensions. We further set

$$K_i := \bigcap_{[G:H]=p^i} H' \quad \text{and} \quad C_i := K_i \cap C(G)_p\,. \tag{3}$$

for every $i \geq 0$, $K_{-1} := G$ and $C_{-1} := K_{-1} \cap C(G)_p = C(G)_p$.
Our first main result is following theorem. Part (b) may be viewed as a variant of Theorem 1.1.

Theorem 1.2. *Let G be a p-group, k be a base field satisfying* (1) *and* $\rho\colon G \hookrightarrow \mathrm{GL}(V)$ *be a faithful linear k-representation of G. Then*

(a) ρ has minimal dimension among the faithful linear representations of G defined over k if and only if for every $i \geq 0$ the irreducible decomposition of ρ has exactly

$$\dim_{\mathbb{F}_p} C_{i-1} - \dim_{\mathbb{F}_p} C_i$$

irreducible components of dimension p^i, each with multiplicity 1.

(b) $\mathrm{ed}_k(G;p) = \mathrm{ed}_k(G) = \sum_{i=0}^{\infty}(\dim_{\mathbb{F}_p} C_{i-1} - \dim_{\mathbb{F}_p} C_i)p^i$.

Note that $K_i = C_i = \{1\}$ for large i (say, if $p^i \geq |G|$), so only finitely many terms in the above infinite sum are non-zero. We also remark that the minimal number of irreducible components in a faithful representations of a finite group (but not necessarily a p-group) was studied in [Ta, Na], see also [Lo, Section 4].

We will prove Theorem 1.2 in section 2; the rest of the paper will be devoted to its applications. The main results we will obtain are summarized below.

Classification of p-groups of essential dimension $\leq p$.

Theorem 1.3. *Let p be a prime, k be as in* (1) *and G be a p-group such that $G' \neq \{1\}$. Then the following conditions are equivalent.*

(a) $\mathrm{ed}_k(G) \leq p$,

(b) $\mathrm{ed}_k(G) = p$,

(c) The center $C(G)$ is cyclic and G has a subgroup H of index p such that $H' = \{1\}$.

Note that the assumption that $G' \neq \{1\}$ is harmless. Indeed, if $G' = \{1\}$ then by Theorem 1.2(b) $\mathrm{ed}_k(G) = \mathrm{rank}\,(G)$; cf. also [BuR, Theorem 6.1] or [BF, section 3].

Essential dimension of p-groups of nilpotency class 2.

Theorem 1.4. *Let G be a p-group of exponent e and k be a field of characteristic $\neq p$ containing a primitive e-th root of unity. Suppose the commutator subgroup $[G,G]$ is central in G. Then*

(a) $\mathrm{ed}_k(G;p) = \mathrm{ed}_k(G) \leq \mathrm{rank}\ C(G) + \mathrm{rank}\,[G,G](p^{\lfloor m/2 \rfloor} - 1)$, *where p^m is the order of $G/C(G)$.*

(b) Moreover, if $[G,G]$ is cyclic then $|G/C(G)|$ is a complete square and equality holds in (a). That is, in this case

$$\mathrm{ed}_k(G;p) = \mathrm{ed}_k(G) = \sqrt{|G/C(G)|} + \mathrm{rank}\ C(G) - 1\,.$$

Essential dimension of a quotient group. C. U. Jensen, A. Ledet and N. Yui asked if $\mathrm{ed}_k(G) \geq \mathrm{ed}_k(G/N)$ for every finite group G and normal subgroup $N \lhd G$; see [JLY, p. 204]. The following theorem shows that this inequality is false in general.

Theorem 1.5. *Let p be a prime and k be a field of characteristic $\neq p$ containing a primitive pth root of unity. For every real number $\lambda > 0$ there exists a p-group G and a central subgroup H of G such that $\mathrm{ed}_k(G/H) > \lambda \, \mathrm{ed}_k(G)$.*

Essential dimension of $\mathrm{SL}_n(\mathbb{Z})$. G. Favi and M. Florence [FF] showed that $\mathrm{ed}_k(\mathrm{GL}_n(\mathbb{Z})) = n$ for every $n \geq 1$ and $\mathrm{ed}_k(\mathrm{SL}_n(\mathbb{Z})) = n - 1$ for every odd n. For details, including the definitions of $\mathrm{ed}_k(\mathrm{GL}_n(\mathbb{Z}))$ and $\mathrm{ed}_k(\mathrm{SL}_n(\mathbb{Z}))$, see Section 5. For even n Favi and Florence showed that $\mathrm{ed}_k(\mathrm{SL}_n(\mathbb{Z})) = n - 1$ or n and left the exact value of $\mathrm{ed}_k(\mathrm{SL}_n(\mathbb{Z}))$ as an open question. In this paper we will answer this question as follows.

Theorem 1.6. *Suppose k is a field of characteristic $\neq 2$. Then*

$$\mathrm{ed}_k(\mathrm{SL}_n(\mathbb{Z}); 2) = \mathrm{ed}_k(\mathrm{SL}_n(\mathbb{Z})) = \begin{cases} n-1, \textit{ if } n \textit{ is odd,} \\ n, \textit{ if } n \textit{ is even} \end{cases}$$

for any $n \geq 3$.

Acknowledgement. Theorems 1.4(b) and 1.5 first appeared in the unpublished preprint [BRV$_1$] by P. Brosnan, the second author and A. Vistoli. We thank P. Brosnan and A. Vistoli for allowing us to include them in this paper. Theorem 1.4(b) was, in fact, a precursor to Theorem 1.1; the techniques used in [BRV$_1$] were subsequently strengthened and refined by Karpenko and Merkurjev [KM] to prove Theorem 1.1. The proof of Theorem 1.4(b) in Section 4 may thus be viewed as a result of reverse engineering. We include it here because it naturally fits into the framework of this paper, because Theorem 1.4(b) is used in a crucial way in [BRV$_2$], and because a proof of this result has not previously appeared in print.
We are also grateful to R. Lötscher for pointing out and helping us correct an inaccuracy in the proof of Lemma 2.1.

2. Proof of Theorem 1.2

Throughout this section we assume k to be as in (1). An important role in the proof will be played by the p-torsion subgroup $C(G)_p$ of the center of G and by the descending sequences

$$\begin{aligned} K_{-1} = G \supset K_0 \supset K_1 \supset K_2 \supset \dots \quad \text{and} \\ C_{-1} = C(G)_p \supset C_0 \supset C_1 \supset C_2 \supset \dots \end{aligned}$$

of characteristic subgroups of G defined in (3). To simplify the notation, we will write C for $C_{-1} = C(G)_p$ for the rest of this section. We will repeatedly use the well-known fact that

(4) A normal subgroup N of G is trivial if and only if $N \cap C$ is trivial.

We begin with three elementary lemmas.

Lemma 2.1. $K_i = \bigcap_{\dim(\rho)\le p^i} \ker(\rho)$, *where the intersection is taken over all irreducible representations* ρ *of* G *of dimension* $\le p^i$.

Proof. Let $j \le i$. Recall that every irreducible representation ρ of G of dimension p^j is induced from a 1-dimensional representation χ of a subgroup $H \subset G$ of index p^j; see [LG-P, (II.4)] for $p \ge 3$ (cf. also [Vo]) and [LG-P, (IV.2)] for $p = 2$. (Note that our assumption (1) on the base field k is crucial here. In the case where $k = \mathbb{C}$ a more direct proof can be found in [Se, Section 8.5]).
Thus $\ker(\rho) = \ker(\operatorname{ind}_H^G \chi) = \bigcap_{g\in G} g\ker(\chi)g^{-1}$, and since each $g\ker(\chi)g^{-1}$ contains $(gHg^{-1})'$, we see that $\ker(\rho) \supset K_j \supset K_i$. The opposite inclusion is proved in a similar manner. $\Box$

Lemma 2.2. *Let* $\rho\colon G \to \mathrm{GL}(V)$ *an irreducible representation of a* p*-group* G. *Then*

(a) $\rho(C)$ *consists of scalar matrices. In other words, the restriction of* ρ *to* C *decomposes as* $\chi \oplus \ldots \oplus \chi$ $(\dim(V)$ *times), for some multiplicative character* $\chi\colon C \to \mathbb{G}_m$. *We will refer to* χ *as the* character associated to ρ.

(b) $C_i = \bigcap_{\dim(\psi)\le p^i} \ker(\chi_\psi)$, *where the intersection is taken over all irreducible* G*-representations* ψ *of dimension* $\le p^i$ *and* $\chi_\psi\colon C \to \mathbb{G}_m$ *denotes the character associated to* ψ. *In particular, if* $\dim(\rho) \le p^i$ *then* χ_ρ *vanishes on* C_i.

Proof. (a) follows from Schur's lemma. (b) By Lemma 2.1

$$C_i = C \cap \bigcap_{\dim(\psi)\le p^i} \ker(\psi) = \bigcap_{\dim(\psi)\le p^i} (C \cap \ker(\psi)) = \bigcap_{\dim(\psi)\le p^i} \ker(\chi_\psi)\,.$$

$\Box$

Lemma 2.3. *Let* G *be a* p*-group and* $\rho = \rho_1 \oplus \ldots \oplus \rho_m$ *be the direct sum of the irreducible representations* $\rho_i\colon G \to \mathrm{GL}(V_i)$. *Let* $\chi_i := \chi_{\rho_i}\colon C \to \mathbb{G}_m$ *be the character associated to* ρ_i.

(a) ρ *is faithful if and only if* $\chi_1, \ldots, \chi_m$ *span* C^* *as an* $\mathbb{F}_p$*-vector space.*

(b) Moreover, if ρ *is of minimal dimension among the faithful representations of* G *then* $\chi_1, \ldots, \chi_m$ *form an* $\mathbb{F}_p$*-basis of* C^*.

Proof. (a) By (4), $\mathrm{Ker}(\rho)$ is trivial if and only if $\mathrm{Ker}(\rho) \cap C = \cap_{i=1}^m \mathrm{Ker}(\chi_i)$ is trivial. On the other hand, $\cap_{i=1}^m \mathrm{Ker}(\chi_i)$ is trivial if and only if $\chi_1, \ldots, \chi_m$ span C^*.

(b) Assume the contrary, say χ_m is a linear combination of $\chi_1, \ldots, \chi_{m-1}$. Then part (a) tells us that $\rho_1 \oplus \ldots \oplus \rho_{m-1}$ is a faithful representation of G, contradicting the minimality of $\dim(\rho)$. $\Box$

We are now ready to proceed with the proof of Theorem 1.2. Part (b) is an immediate consequence of part (a) and Theorem 1.1. We will thus focus on proving part (a). In the sequel for each $i \ge 0$ we will set

$$\delta_i := \dim_{\mathbb{F}_p} C_{i-1} - \dim_{\mathbb{F}_p} C_i$$

and

$$\Delta_i := \delta_0 + \delta_1 + \cdots + \delta_i = \dim_{\mathbb{F}_p} C - \dim_{\mathbb{F}_p} C_i \,,$$

where the last equality follows from $C_{-1} = C$.

Our proof will proceed in two steps. In Step 1 we will construct a faithful representation μ of G such that for every $i \geq 0$ exactly δ_i irreducible components of μ have dimension p^i. In Step 2 we will show that $\dim(\rho) \geq \dim(\mu)$ for any other faithful representation ρ of G, and moreover equality holds if and only if ρ has exactly δ_i irreducible components of dimension p^i, for every $i \geq 0$.

Step 1: We begin by constructing μ. By definition,

$$C = C_{-1} \supset C_0 \supset C_1 \supset \dots \,,$$

where the inclusions are not necessarily strict. Dualizing this flag of $\mathbb{F}_p$-vector spaces, we obtain a flag

$$(0) = (C^*)_{-1} \subset (C^*)_0 \subset (C^*)_1 \subset \dots$$

of $\mathbb{F}_p$-subspaces of C^*, where

$$(C^*)_i := \{\chi \in C^* \mid \chi \text{ is trivial on } C_i\} \simeq (C/C_i)^*.$$

Let $\mathrm{Ass}(C) \subset C^*$ be the set of characters of C associated to irreducible representations of G, and let $\mathrm{Ass}_i(C)$ be the set of characters associated to irreducible representations of dimension p^i. Lemma 2.2(b) tells us that

$$\mathrm{Ass}_0(C) \cup \mathrm{Ass}_1(C) \cup \cdots \cup \mathrm{Ass}_i(C) \text{ spans } (C^*)_i$$

for every $i \geq 0$. Hence, we can choose a basis $\chi_1, \dots, \chi_{\Delta_0}$ of $(C^*)_0$ from $\mathrm{Ass}_0(C)$, then complete it to a basis $\chi_1, \dots, \chi_{\Delta_1}$ of $(C^*)_1$ by choosing the last $\Delta_1 - \Delta_0$ characters from $\mathrm{Ass}_1(C)$, then complete this basis of $(C^*)_1$ to a basis of $(C^*)_2$ by choosing $\Delta_2 - \Delta_1$ additional characters from $\mathrm{Ass}_2(C)$, etc. We stop when $C_i = (0)$, i.e., $\Delta_i = \dim_{\mathbb{F}_p} C$.

By the definition of $\mathrm{Ass}_i(C)$, each χ_j is the associated character of some irreducible representation μ_j of G. By our construction

$$\mu = \mu_1 \oplus \cdots \oplus \mu_{\dim_{\mathbb{F}_p}(C)}$$

has the desired properties. Indeed, since $\chi_1, \dots, \chi_{\dim_{\mathbb{F}_p}(C)}$ form a basis of C^*, Lemma 2.3 tells us that μ is faithful. On the other hand, by our construction exactly

$$\delta_i - \delta_{i-1} = \dim_{\mathbb{F}_p} C_i^* - \dim_{\mathbb{F}_p} C_{i-1}^* = \dim_{\mathbb{F}_p} C_{i-1} - \dim_{\mathbb{F}_p} C_i$$

of the characters $\chi_1, \dots, \chi_c$ come from $\mathrm{Ass}_i(C)$. Equivalently, exactly

$$\dim_{\mathbb{F}_p} C_{i-1} - \dim_{\mathbb{F}_p} C$$

of the irreducible representations $\mu_1, \dots, \mu_c$ are of dimension p^i.

Step 2: Let $\rho \colon G \to \mathrm{GL}(V)$ be a faithful linear representation of G of the smallest possible dimension,

$$\rho = \rho_1 \oplus \dots \oplus \rho_c$$

be its irreducible decomposition, and $\chi_i \colon C \to \mathbb{G}_m$ be the character associated to ρ_i. By Lemma 2.3(b), $\chi_1, \dots, \chi_c$ form a basis of C^*. In particular, $c = \dim_{\mathbb{F}_p} C$ and at most $\dim_{\mathbb{F}_p} C - \dim_{\mathbb{F}_p} C_i$ of the characters $\chi_1, \dots, \chi_c$ can vanish on C_i. On the other hand, by Lemma 2.2(b) every representation of dimension $\leq p^i$ vanishes on C_i. Thus if exactly d_i of the irreducible representations $\rho_1, \dots, \rho_c$ have dimension p^i then

$$d_0 + d_1 + d_2 + \ldots + d_i \leq \dim_{\mathbb{F}_p} C - \dim_{\mathbb{F}_p} C_i$$

for every $i \geq 0$. For $i \geq 0$, set $D_i := d_0 + \cdots + d_i =$ number of representations of dimension $\leq p^i$ among $\rho_1, \dots, \rho_c$. We can now write the above inequality as

$$D_i \leq \Delta_i \text{ for every } i \geq 0. \tag{5}$$

Our goal is to show that $\dim(\rho) \geq \dim(\mu)$ and that equality holds if and only if exactly δ_i of the irreducible representations $\rho_1, \dots, \rho_{\dim_{\mathbb{F}_p}(C)}$ have dimension p^i. The last condition translates into $d_i = \delta_i$ for every $i \geq 0$, which is, in turn equivalent to $D_i = \Delta_i$ for every $i \geq 0$.
Indeed, setting $D_{-1} := 0$ and $\Delta_{-1} := 0$, we have,

$$\begin{aligned}\dim(\rho) - \dim(\mu) = \sum_{i=0}^{\infty}(d_i - \delta_i)p^i &= \sum_{i=0}^{\infty}(D_i - \Delta_i)p^i - \sum_{i=0}^{\infty}(D_{i-1} - \Delta_{i-1})p^i \\ &= \sum_{i=0}^{\infty}(D_i - \Delta_i)(p^i - p^{i+1}) \geq 0\,,\end{aligned}$$

where the last inequality follows from (5). Moreover, equality holds if and only if $D_i = \Delta_i$ for every $i \geq 0$, as claimed. This completes the proof of Step 2 and thus of Theorem 1.2. $\square$

3. Proof of Theorem 1.3

Since $K_0 = G'$ is a non-trivial normal subgroup of G, we see that $K_0 \cap C(G)$ and thus $C_0 = K_0 \cap C(G)_p$ is non-trivial. This means that in the summation formula of Theorem 1.2(b) at least one of the terms

$$(\dim_{\mathbb{F}_p} C_{i-1} - \dim_{\mathbb{F}_p} C_i)p^i$$

with $i \geq 1$ will be non-zero. Hence, $\mathrm{ed}_k(G) \geq p$; this shows that (a) and (b) are equivalent. Moreover, equality holds if and only if (i) $\dim_{\mathbb{F}_p} C_{-1} = 1$, (ii) $\dim_{\mathbb{F}_p} C_0 = 1$ and (iii) C_1 is trivial. Since we are assuming $K_0 = G' \neq \{1\}$ and hence, $C_0 = K_0 \cap C(G)_p \neq \{1\}$ by (4), (ii) follows from (i) and thus can be dropped.
It now suffices to prove that (i) and (iii) are equivalent to condition (c) of the theorem. Since $C_{-1} = C(G)_p$, (i) is equivalent to $C(G)$ being cyclic. On the other hand, (iii) means that

$$K_1 = \bigcap_{[G:H]=p} H' \tag{6}$$

intersects $C(G)_p$ trivially. Since K_1 is a normal subgroup of G, (4) tells us that (iii) holds if and only if $K_1 = \{1\}$.

It remains to show that $K_1 = \{1\}$ if and only if $H' = \{1\}$ for some subgroup H of G of index p. One direction is obvious: if $H' = \{1\}$ for some H of index p then the intersection (6) is trivial. To prove the converse, assume the contrary: the intersection (6) is trivial but $H' \neq \{1\}$ for every subgroup H of index p. Since every such H is normal in G (and so is H'), (4) tells us that that $H' \neq \{1\}$ if and only if $H' \cap C(G) \neq \{1\}$. Since $C(G)$ is cyclic, the latter condition is equivalent to $C(G)_p \subset H'$. Thus

$$C(G)_p \subset K_1 = \bigcap_{[G:H]=p} H' \, ,$$

contradicting our assumption that $K_1 = \{1\}$.

To sum up, we have shown that (c) is equivalent to conditions (i) and (iii) above, and that these conditions are in turn, equivalent to (a) (or to (b)). This completes the proof of Theorem 1.3.

Remark 3.1. p-groups that have a faithful representation of degree p over a field k, satisfying (1) are described in [LG-P, II.4, III.4, IV.2]; see also [Vo]. Combining this description with Theorem 1.1 yields the following variant of Theorem 1.3.

Let k be a field satisfying (1) *and G be a p-group such that $G' \neq \{1\}$. Then the following conditions are equivalent:*

(a) $\mathrm{ed}_k(G) \leq p$,

(b) $\mathrm{ed}_k(G) = p$,

(c) G is isomorphic to a subgroup of $\mathbb{Z}/p^\alpha \wr \mathbb{Z}/p = (\mathbb{Z}/p^\alpha)^p \rtimes \mathbb{Z}/p$, for some $\alpha \geq 1$ such that k contains a primitive root of unity of degree p^α. $\square$

4. Proof of Theorems 1.4 and 1.5

Proof of Theorem 1.4. Since the commutator $K_0 = [G, G]$ is central, $C_0 = K_0 \cap C(G)_p$ is of dimension rank $[G, G]$ and the p^0 term in the formula of Theorem 1.2 is (rank $C(G) -$ rank $[G, G]$).

Let $Q = G/C(G)$ which is abelian by assumption. Let $h_1, ..., h_s$ be generators of $[G, G]$, where $s = \mathrm{rank}\,[G, G]$, so that

$$[G, G] = \mathbb{Z}/p^{e_1} h_1 \oplus \cdots \oplus \mathbb{Z}/p^{e_1} h_1 \, ,$$

written additively. For $g_1, g_2 \in G$ the commutator can then be expressed as

$$[g_1, g_2] = \beta_1(g_1, g_2)h_1 + \ldots + \beta_s(g_1, g_2)h_s \, .$$

Note that each $\beta_i(g_1, g_2)$ depends on g_1, g_2 only modulo the center $C(G)$. Thus each β_i descends to a skew-symmetric bilinear form

$$Q \times Q \to \mathbb{Z}/p^{e_i}$$

which, by a slight abuse of notation, we will continue to denote by β_i. Let p^m be the order of Q. For each form β_i there is an isotropic subgroup Q_i of Q of order at least $p^{\lfloor (m+1)/2 \rfloor}$ (or equivalently, of index at most $p^{\lfloor m/2 \rfloor}$ in Q); see [AT, Corollary 3]. Pulling these isotropic subgroups back to G, we obtain subgroups $G_1, \dots, G_s$ of G of index $\le p^{\lfloor m/2 \rfloor}$ with the property that $G_i' = [G_i, G_i]$ lies in the subgroup of $C(G)$ generated by $h_1, \dots, h_{i-1}, h_{i+1}, \dots, h_s$. In particular, $G_1' \cap \dots \cap G_s' = \{1\}$. Thus, all K_i (and hence, all C_i) in (3) are trivial for $i \ge \lfloor m/2 \rfloor$, and Theorem 1.2 tells us that

$$\mathrm{ed}_k(G) = \dim_{\mathbb{F}_p} C_{-1} - \dim_{\mathbb{F}_p} C_0 + \sum_{j=1}^{\lfloor m/2 \rfloor} (\dim_{\mathbb{F}_p} C_{j-1} - \dim_{\mathbb{F}_p} C_j) p^j \le$$

$$\dim_{\mathbb{F}_p} C_{-1} - \dim_{\mathbb{F}_p} C_0 + \sum_{j=1}^{\lfloor m/2 \rfloor} (\dim_{\mathbb{F}_p} C_{j-1} - \dim_{\mathbb{F}_p} C_j) \cdot p^{\lfloor m/2 \rfloor} =$$

$$\mathrm{rank}\ C(G) + \mathrm{rank}\ [G,G](p^{\lfloor m/2 \rfloor} - 1)\,.$$

(b) In general, the skew-symmetric bilinear forms β_i may be degenerate. However, if $[G,G]$ is cyclic, i.e., $s = 1$, then we have only one form, β_1, which is easily seen to be non-degenerate. For notational simplicity, we will write β instead of β_1. To see that β is non-degenerate, suppose $\overline{g} := g$ (modulo $C(G)$) lies in the kernel of β for some $g \in G$. Then by definition

$$\beta(g, g_1) = g g_1 g^{-1} g_1^{-1} = 1$$

for every $g_1 \in G$. Hence, g is central in G, i.e., $\overline{g} = 1$ in $Q = G/C(G)$, as claimed.

We conclude that the order of $Q = G/C(G)$ is a perfect square, say p^{2i}, and Q contains a maximal isotropic subgroup $I \subset Q$ of order $p^i = \sqrt{|G/C(G)|}$; see [AT, Corollary 4]. The preimage of I in G is a maximal abelian subgroup of index p^i. Consequently, $K_0 = [G,G], K_1, \dots, K_{i-1}$ are all of rank 1 and K_i is trivial, where $p^i = \sqrt{|G/C(G)|}$. Moreover, since all of these groups lie in $[G,G]$ and hence, are central, we have $C_i = (K_i)_p$ and thus

$$\dim_{\mathbb{F}_p}(C_0) = \dim_{\mathbb{F}_p}(C_1) = \ldots = \dim_{\mathbb{F}_p}(C_{i-1}) = 1 \text{ and } \dim_{\mathbb{F}_p}(C_i) = 0\,.$$

Specializing the formula of Theorem 1.4 to this situation, we obtain part (b). □

Proof of Theorem 1.5. Let Γ be the non-abelian group of order p^3 given by generators x, y, z and relations $x^p = y^p = z^p = [x,z] = [y,z] = 1$, $[x,y] = z$. Choose a multiplicative character $\chi\colon H \to k^*$ of the subgroup $A = \langle x, z \rangle \simeq (\mathbb{Z}/p\mathbb{Z})^2$ which is non-trivial on the center $\langle z \rangle$ of Γ and consider the p-dimensional induced representation $\mathrm{Ind}_A^\Gamma(\chi)$. Since the center $\langle z \rangle$ of Γ does not lie in the kernel of $\mathrm{Ind}_A^\Gamma(\chi)$, we conclude that $\mathrm{Ind}_A^\Gamma(\chi)$ is faithful. Thus we have constructed a faithful p-dimensional representation of Γ defined over k. Consequently

$$\mathrm{ed}_k(\Gamma) \le p\,. \tag{7}$$

Taking the direct sum of n copies of this representation, we obtain a faithful representation of Γ^n of dimension np. Thus for any $n \geq 1$ we have

$$\mathrm{ed}_k \Gamma^n \leq np\,. \tag{8}$$

(We remark that both (7) and (8) are in fact equalities. Indeed, if ζ_{p^2} is a primitive root of unity of degree p^2 then

$$\mathrm{ed}_k(\Gamma) \geq \mathrm{ed}_{k(\zeta_{p^2})}(\Gamma) = \sqrt{p^2} + 1 - 1 = p\,,$$

where the middle equality follows from Theorem 1.4(b). Hence, we have $\mathrm{ed}_k(\Gamma) = p$. Moreover, by [KM, Theorem 5.1], $\mathrm{ed}_k \Gamma^n = n \cdot \mathrm{ed}_k(\Gamma) = np$. However, we will only need the upper bound (8) in the sequel.)

The center of Γ is $\langle z \rangle$; denote it by C. The center of Γ^n is then isomorphic to C^n. Let H_n be the subgroup of C^n consisting of n-tuples $(c_1, \dots, c_n)$ such that $c_1 \cdots c_n = 1$. The center $C(\Gamma^n/H_n)$ of Γ^n/H_n is clearly cyclic of order p (it is generated by the class of the element $(z, 1, \dots, 1)$ modulo H_n), and the commutator $[\Gamma^n/H_n, \Gamma^n/H_n]$ is central. Hence,

$$\mathrm{ed}_k(\Gamma^n/H_n) \geq \mathrm{ed}_{k(\zeta^2)}(\Gamma^n/H_n) = \sqrt{p^{2n}} + 1 - 1 = p^n\ , \tag{9}$$

where the middle equality follows from Theorem 1.4(b). Setting $G = \Gamma^n$ and $H = H_n$, and comparing (8) with (9), we see that the desired inequality $\mathrm{ed}_k(G/H) > \lambda\, \mathrm{ed}_k(G)$ holds for suitably large n. □

5. Proof of Theorem 1.6

Recall that the essential dimension of the group $\mathrm{GL}_n(\mathbb{Z})$ over a field k, or $\mathrm{ed}_k(\mathrm{GL}_n(\mathbb{Z}))$ for short, is defined as the essential dimension of this functor

$$H^1(*, \mathrm{GL}_n(\mathbb{Z}))\colon K \to \{K\text{-isomorphism classes of } n\text{-dimensional } K\text{-tori}\}\,,$$

where K/k is a field extension. Similarly $\mathrm{ed}_k(\mathrm{SL}_n(\mathbb{Z}))$ is defined as the essential dimension of the functor

$$H^1(*, \mathrm{SL}_n(\mathbb{Z}))\colon K \to \{K\text{-isomorphism classes of } n\text{-dimensional } K\text{-tori with } \phi_T \subset \mathrm{SL}_n(\mathbb{Z})\ \}\,,$$

where $\phi_T\colon\ \mathrm{Gal}(K) \to \mathrm{GL}_n(\mathbb{Z})$ is the natural representation of the Galois group of K on the character lattice of T. The essential dimensions $\mathrm{ed}_k(\mathrm{GL}_n(\mathbb{Z}); p)$ and $\mathrm{ed}_k(\mathrm{SL}_n(\mathbb{Z}); p)$ are respectively the essential dimensions of the above functors at a prime p.

G. Favi and M. Florence [FF] showed that for $\Gamma = \mathrm{GL}_n(\mathbb{Z})$ or $\mathrm{SL}_n(\mathbb{Z})$,

$$\mathrm{ed}_k(\Gamma) = \max\{\mathrm{ed}_k(F) | F \text{ finite subgroup of } \Gamma\}. \tag{10}$$

From this they deduced that

$$\mathrm{ed}_k(\mathrm{GL}_n(\mathbb{Z})) = n, \quad \text{and} \quad \mathrm{ed}_k(\mathrm{SL}_n(\mathbb{Z})) = \begin{cases} n-1, \text{ if } n \text{ is odd,} \\ n-1 \text{ or } n, \text{ if } n \text{ is even.} \end{cases}$$

For details, see [FF, Theorem 5.4].

Favi and Florence also proved that $\mathrm{ed}_k(\mathrm{SL}_2(\mathbb{Z})) = 1$ if k contains a primitive 12th root of unity and asked whether $\mathrm{ed}_k(\mathrm{SL}_n(\mathbb{Z})) = n - 1$ or n in the case where $n \geq 4$ is even; see [FF, Remark 5.5]. In this section we will prove Theorem 1.6 which shows that the answer is always n.

A minor modification of the arguments in [FF] shows that (10) holds also for essential dimension at a prime p:

$$\mathrm{ed}_k(\Gamma; p) = \max\{\mathrm{ed}_k(F; p) | F \text{ a finite subgroup of } \Gamma\}, \tag{11}$$

where $\Gamma = \mathrm{GL}_n(\mathbb{Z})$ or $\mathrm{SL}_n(\mathbb{Z})$. The finite groups F that Florence and Favi used to find the essential dimension of $\mathrm{GL}_n(\mathbb{Z})$ and $\mathrm{SL}_n(\mathbb{Z})$ (n odd) are $(\mathbb{Z}/2\mathbb{Z})^n$ and $(\mathbb{Z}/2\mathbb{Z})^{n-1}$ respectively. Thus $\mathrm{ed}_k(\mathrm{GL}_n(\mathbb{Z}); 2) = \mathrm{ed}_k(\mathrm{GL}_n(\mathbb{Z})) = n$ for every $n \geq 1$ and $\mathrm{ed}_k(\mathrm{SL}_n(\mathbb{Z}); 2) = \mathrm{ed}_k(\mathrm{SL}_n(\mathbb{Z})) = n - 1$ if n is odd.

Our proof of Theorem 1.6 will rely on part (b) of the following easy corollary of Theorem 1.2.

Corollary 5.1. *Let G be a p-group, and k be as in* (1).

(a) If $C(G)_p \subset K_i$ then $\mathrm{ed}_k(G)$ is divisible by p^{i+1}.

(b) If $C(G)_p \subset G'$ then $\mathrm{ed}_k(G)$ is divisible by p.

(c) If $C(G)_p \subset G^{(i)}$, where $G^{(i)}$ denotes the ith derived subgroup of G, then $\mathrm{ed}_k(G)$ is divisible by p^i.

Proof. (a) $C(G)_p \subset K_i$ implies $C_{-1} = C_0 = \cdots = C_i$. Hence, in the formula of Theorem 1.2(b) the $p^0, p^1, \ldots, p^i$ terms appear with coefficient 0. All other terms are divisible by p^{i+1}, and part (a) follows.

(b) is an immediate consequence of (a), since $K_0 = G'$.

(c) By [H, Theorem V.18.6] $G^{(i)}$ is contained in the kernel of every p^{i-1}-dimensional representation of G. Lemma 2.1 now tells us that $G^{(i)} \subset K_{i-1}$ and part (c) follows from part (a). □

Proof of Theorem 1.6. We assume that $n = 2d \geq 4$ is even. To prove Theorem 1.6 it suffices to find a 2-subgroup F of $\mathrm{SL}_n(\mathbb{Z})$ of essential dimension n.

Diagonal matrices and permutation matrices generate a subgroup of $\mathrm{GL}_n(\mathbb{Z})$ isomorphic to $\mu_2^n \rtimes \mathrm{S}_n$. The determinant function restricts to a homomorphism

$$\det \colon \mu_2^n \rtimes \mathrm{S}_n \to \mu_2$$

sending $((\epsilon_1, \ldots, \epsilon_n), \tau)) \in \mu_2^n \rtimes \mathrm{S}_n$ to the product $\epsilon_1 \epsilon_2 \cdots \epsilon_n \cdot \mathrm{sign}(\tau)$. Let P_n be a Sylow 2-subgroup of S_n and F_n be the kernel of $\det \colon \mu_2^n \rtimes P_n \to \mu_2$. By construction F_n is a finite 2-group contained in $\mathrm{SL}_n(\mathbb{Z})$. Theorem 1.6 is now a consequence of the following proposition.

Proposition 5.2. *If $\mathrm{char}(k) \neq 2$ then $\mathrm{ed}_k(F_{2d}) = 2d$ for any $d \geq 2$.*

To prove the proposition, let

$$D_{2d} = \{\mathrm{diag}(\epsilon_1, \ldots, \epsilon_{2d}) \,|\, \text{each } \epsilon_i = \pm 1 \text{ and } \epsilon_1 \epsilon_2 \cdots \epsilon_{2d} = 1\}$$

be the subgroup of "diagonal" matrices contained in F_{2d}.

Since $D_{2d} \simeq \mu_2^{2d-1}$ has essential dimension $2d - 1$, we see that $\mathrm{ed}_k(F_{2d}) \geq \mathrm{ed}_k(D_{2d}) = 2d - 1$. On the other hand the inclusion $F_{2d} \subset \mathrm{SL}_{2d}(\mathbb{Z})$ gives rise to a $2d$-dimensional representation of F_{2d}, which remains faithful over any field k of characteristic $\neq 2$. Hence, $\mathrm{ed}_k(F_{2d}) \leq 2d$. We thus conclude that

$$\mathrm{ed}_k(F_{2d}) = 2d - 1 \text{ or } 2d. \tag{12}$$

Using elementary group theory, one easily checks that

$$C(F_{2d}) \subset [F_{2d}, F_{2d}] \subset F'_{2d}\,. \tag{13}$$

Thus, if $k' \supset k$ is a field as in (1), $\mathrm{ed}_{k'}(F_{2d})$ is even by Corollary 5.1; since $\mathrm{ed}_k(F_{2d}) \geq \mathrm{ed}_{k'}(F_{2d})$, (12) now tells us that $\mathrm{ed}_k(F_{2d}) = 2d$. This completes the proof of Proposition 5.2 and thus of Theorem 1.6. □

Remark 5.3. The assumption that $d \geq 2$ is essential in the proof of the inclusion (13). In fact, $F_2 \simeq \mathbb{Z}/4\mathbb{Z}$, so (13) fails for $d = 1$.

Remark 5.4. Note that for any integers $m, n \geq 2$, F_{m+n} contains the direct product $F_m \times F_n$. Thus

$$\mathrm{ed}_k(F_{m+n}) \geq \mathrm{ed}_k(F_m \times F_n) = \mathrm{ed}_k(F_m) + \mathrm{ed}_k(F_n)\,,$$

where the last equality follows from [KM, Theorem 5.1]. Thus Proposition 5.2 only needs to be proved for $d = 2$ and 3 (or equivalently, $n = 4$ and 6); all other cases are easily deduced from these by applying the above inequality recursively, with $m = 4$. In particular, the group-theoretic inclusion (13) only needs to be checked for $d = 2$ and 3. Somewhat to our surprise, this reduction does not appear to simplify the proof of Proposition 5.2 presented above to any significant degree.

Remark 5.5. It is interesting to note that while the value of $\mathrm{ed}_k(\mathrm{SL}_2(\mathbb{Z}))$ depends on the base field k (see [FF, Remark 5.5]), for $n \geq 3$, the value of $\mathrm{ed}_k(\mathrm{SL}_n(\mathbb{Z}))$ does not (as long as $\mathrm{char}(k) \neq 2$).

References

[AT] A. Amitsur, J.-P. Tignol, *Symplectic modules*, Israel J. Math. 54 (1986), 267-290.

[BF] G. Berhuy, G. Favi, *Essential dimension: a functorial point of view (after A. Merkurjev)*, Doc. Math. 8 (2003), 279-330.

[BRV$_1$] P. Brosnan, Z. Reichstein, A. Vistoli, *Essential dimension and algebraic stacks*, 2007, `arXiv:math/0701903v1 [math.AG]`.

[BRV$_2$] P. Brosnan, Z. Reichstein, A. Vistoli, *Essential dimension, spinor groups and quadratic forms*, 10 pages, Annals of Math. 171 (2010), no. 1, 533–544.

[BuR] J. Buhler, Z. Reichstein, *On the essential dimension of a finite group*, Compositio Math. 106 (1997), no. 2, 159-179.

[FF] G. Favi, M. Florence, *Tori and essential dimension*, J. Algebra, 319 (2008), no. 9, 3885-3900.

[H] B. Huppert, Endliche Gruppen. I, Die Grundlehren der Mathematischen Wissenschaften, 134 Springer-Verlag, 1967.

[JLY] C. U. Jensen, A. Ledet, and N. Yui, *Generic polynomials*, Mathematical Sciences Research Institute Publications, vol. 45, Cambridge University Press, Cambridge, 2002, Constructive aspects of the inverse Galois problem.

[KM] N. A. Karpenko and A. S. Merkurjev, *Essential dimension of finite p-groups*, Inventiones Math., 172, no. 3 (2008), pp. 491-508.

[LG-P] C. R. Leedham-Green, W. Plesken, *Some remarks on Sylow subgroups of general linear groups*, Math. Z. 191 (1986), no. 4, 529–535.

[Lo] R. Lötscher, *Application of multihomogeneous covariants to the essential dimension of finite groups*, `http://arxiv.org/PS_cache/arxiv/pdf/0811/0811.3852v1.pdf`

[Na] T. Nakayama, *Finite groups with faithful irreducible and directly indecomposable modular representations*, Proc. Japan Acad. 23 (1947), no. 3, 22–25.

[Me] A. Merkurjev, *Essential dimension*, in Quadratic forms – algebra, arithmetic, and geometry (R. Baeza, W.K. Chan, D.W. Hoffmann, and R. Schulze-Pillot, eds.), Contemporary Mathematics 493 (2009), 299–326.

[Re] Z. Reichstein, *On the notion of essential dimension for algebraic groups*, Transform. Groups 5 (2000), no. 3, 265–304.

[Se] J.-P. Serre, *Linear representations of finite groups*, Graduate Texts in Mathematics, 42. Springer-Verlag, New York–Heidelberg, 1977.

[Ta] M. Tazawa, *Über die isomorphe Darstellung der endliehen Gruppe*, Tôhoku Math. J. 47 (1940), 87–93.

[Vo] R. T. Vol'vacev, *Sylow p-subgroups of the full linear group* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. 27 (1963), 1031–1054.

Aurel Meyer
Department of Mathematics
University
of British Columbia
Vancouver
Canada, V6T 1Z2

Zinovy Reichstein
Department of Mathematics
University
of British Columbia
Vancouver
Canada, V6T 1Z2

K-Theory and the Enriched Tits Building

To A. A. Suslin with admiration, on his sixtieth birthday.

M. V. Nori and V. Srinivas

Received: December 31, 2009
Revised: May 22, 2010

Abstract. Motivated by the splitting principle, we define certain simplicial complexes associated to an associative ring A, which have an action of the general linear group $GL(A)$. This leads to an exact sequence, involving Quillen's algebraic K-groups of A and the symbol map. Computations in low degrees lead to another view on Suslin's theorem on the Bloch group, and perhaps show a way towards possible generalizations.

2010 Mathematics Subject Classification: 19D06, 51A05, 55P10
Keywords and Phrases: $K_i(A)$, geometries, homotopy types

The homology of $GL_n(A)$ has been studied in great depth by A.A. Suslin. In some of his works ([20] and [21] for example), the action of $GL_n(A)$ on certain simplicial complexes facilitated his homology computations.
We introduce three simplicial complexes in this paper. They are motivated by the splitting principle. The description of these spaces is given below. This is followed by the little information we possess on their homology. After that comes the connection with K-theory.
These objects are defined quickly in the context of affine algebraic groups as follows. Let G be a connected algebraic group[1] defined over a field k. The collection of minimal parabolic subgroups $P \subset G$ is denoted by $FL(G)$ and the collection of maximal k-split tori $T \subset G$ is denoted by $SPL(G)$. The simplicial complex $\mathbb{FL}(G)$ has $FL(G)$ as its set of vertices. Minimal parabolics $P_0, P_1, ..., P_r$ of G form an r-simplex if their intersection contains a maximal k-split torus[2]. The dimension of $\mathbb{FL}(G)$ is one less than the order of the Weyl

[1]Gopal Prasad informed us that we should take G reductive or k perfect. The standard classification, via root systems, of all parabolic subgroups containing a maximal split torus, requires this hypothesis.
[2]John Rognes has an analogous construction with maximal parabolics replacing minimal parabolics. His spaces, different homotopy types from ours, are connected with K-theory as well, see [15] .

group of any $T \in SPL(G)$. Dually, we define $\mathbb{SPL}(G)$ as the simplicial complex with $SPL(G)$ as its set of vertices, and $T_0, T_1, ..., T_r$ forming an r-simplex if they are all contained in a minimal parabolic. In general, $\mathbb{SPL}(G)$ is infinite dimensional.

That both $\mathbb{SPL}(G)$ and $\mathbb{FL}(G)$ have the same homotopy type can be deduced from corollary 7, which is a general principle. A third simplicial complex, denoted by $\mathbb{ET}(G)$, which we refer to as the *enriched Tits building*, is better suited for homology computations. This is the simplicial complex whose simplices are (nonempty) chains of the partially ordered set $\mathcal{E}(G)$ whose definition follows. For a parabolic subgroup $P \subset G$, we denote by $U(P)$ its unipotent radical and by $j(P) : P \to P/U(P)$ the given morphism. Then $\mathcal{E}(G)$ is the set of pairs (P, T) where $P \subset G$ is a parabolic subgroup and $T \subset P/U(P)$ is a maximal k-split torus. We say $(P', T') \leq (P, T)$ in $\mathcal{E}(G)$ if $P' \subset P$ and $j(P)^{-1}(T) \subset j(P')^{-1}T'$. Note that $\dim \mathcal{E}(G)$ is the split rank of the quotient of $G/U(G)$ by its center. Assume for the moment that this quotient is a simple algebraic group. Then $(P, T) \mapsto P$ gives a map to the cone of the Tits building. The topology of $\mathbb{ET}(G)$ is more complex than the topology of the Tits building, which is well known to be a bouquet of spheres.

When $G = GL(V)$, we denote the above three simplicial complexes by $\mathbb{FL}(V)$, $\mathbb{SPL}(V)$ and $\mathbb{ET}(V)$. These constructions have simple analogues even when one is working over an arbitrary associative ring A. Their precise definition is given with some motivation in section 2. Some basic properties of these spaces are also established in section 2. Amongst them is Proposition 11 which shows that $\mathbb{ET}(V)$ and $\mathbb{FL}(V)$ have the same homotopy type.

$\mathbb{ET}(A^n)$ has a polyhedral decomposition (see lemma 21). This produces a spectral sequence (see Theorem 2) that computes its homology. The $E^1_{p,q}$ terms and the differentials $d^1_{p,q}$ are recognisable since they involve only the homology groups of $\mathbb{ET}(A^a)$ for $a < n$. The differentials $d^r_{p,q}$ for $r > 1$ are not understood well enough, however.

There are natural inclusions $\mathbb{ET}(A^n) \hookrightarrow \mathbb{ET}(A^d)$ for $d > n$, and the induced map on homology factors through

$$H_m(\mathbb{ET}(A^n)) \to H_0(E_n(A), H_m(\mathbb{ET}(A^n))) \to H_m(\mathbb{ET}(A^d)).$$

where $E_n(A)$ is the group of elementary matrices (see Corollary 9).

For the remaining statements on the homology of $\mathbb{ET}(A^n)$, we assume that A is a commutative ring with many units, in the sense of Van der Kallen. See [12] for a nice exposition of the definition and its consequences. Commutative local rings A with infinite residue fields are examples of such rings. Under this assumption, $E_n(A)$ can be replaced by $GL_n(A)$ in the above statement.

We have observed that $\mathbb{ET}(A^{m+1})$ has dimension m. Thus it is natural question to ask whether

$$H_0(GL_{m+1}(A), H_m(\mathbb{ET}(A^{m+1}))) \otimes \mathbb{Q} \to H_m(\mathbb{ET}(A^d)) \otimes \mathbb{Q}$$

is an isomorphism when $d > m + 1$. Theorem 3 asserts that this is true for $m = 1, 2, 3$. The statement is true in general (see Proposition 29) if a certain

Compatible Homotopy Question has an affirmative answer. The higher differentials of the spectral sequence can be dealt with if this is true. Proposition 22 shows that this holds in some limited situations.
The computation of $H_0(GL_3(A), H_2(\mathbb{ET}(A^3))) \otimes \mathbb{Q}$ is carried out at in the last lemma of the paper. This is intimately connected with Suslin's result (see [21]) connecting K_3 and the Bloch group. A closed form for $H_0(GL_4(A), H_3(\mathbb{ET}(A^4))$ is awaited. This should impact on the study of $K_4(A)$.
We now come to the connection with the Quillen K-groups $K_i(A)$ as obtained by his plus construction.
$GL(A)$ acts on the geometric realisation $|\mathbb{SPL}(A^\infty)|$ and thus we have the Borel construction, namely the quotient of $|\mathbb{SPL}(A^\infty)| \times EGL(A)$ by $GL(A)$, a familiar object in the study of equivariant homotopy. We denote this space by $\mathbb{SPL}(A^\infty)//GL(A)$. We apply Quillen's plus construction to $\mathbb{SPL}(A^\infty)//GL(A)$ and a suitable perfect subgroup of its fundamental group to obtain a space $Y(A)$. Proposition 17 shows that $Y(A)$ is an H-space and that the natural map $Y(A) \to BGL(A)^+$ is an H-map. Its homotopy fiber, denoted by $\mathbb{SPL}(A^\infty)^+$, is thus also a H-space. The n-th homotopy group of $\mathbb{SPL}(A^\infty)^+$ at its canonical base point is denoted by $L_n(A)$. There is of course a natural map $\mathbb{SPL}(A^\infty) \to \mathbb{SPL}(A^\infty)^+$. That this map is a homology isomorphism is shown in lemma 16. This assertion is easy, but not tautological: it relies once again on the triviality of the action of $E(A)$ on $H_*(\mathbb{SPL}(A^\infty))$. As a consequence of this lemma, $L_n(A) \otimes \mathbb{Q}$ is identified with the primitive rational homology of $\mathbb{SPL}(A^\infty)$, or equivalently, that of $\mathbb{ET}(A^\infty)$.
We have the inclusion $N_n(A) \hookrightarrow GL_n(A)$, where $N_n(A)$ is the semidirect product of the permutation group S_n with $(A^\times)^n$. Taking direct limits over $n \in \mathbb{N}$, we obtain $N(A) \subset GL(A)$. Let $H' \subset N(A)$ be the infinite alternating group and let H be the normal subgroup generated by H'. Applying Quillen's plus construction to the space $BN(A)$ with respect to H, we obtain $BN(A)^+$. Its n-th homotopy group is defined to be $\mathcal{H}_n(A^\times)$. From the Dold-Thom theorem, it is easy to see that $\mathcal{H}_n(A^\times) \otimes \mathbb{Q}$ is isomorphic to the group homology $H_n(A^\times) \otimes \mathbb{Q}$. When A is commutative, this is simply $\wedge^n_{\mathbb{Q}}(A^\times \otimes \mathbb{Q})$. Proposition 20 identifies the groups $\mathcal{H}_n(A^\times)$ with certain stable homotopy groups. Its proof was shown to us by J. Peter May. It is sketched in the text of the paper after the proof of the Theorem below.

Theorem 1. *Let A be a Nesterenko-Suslin ring. Then there is a long exact sequence, functorial in A:*

$$\cdots \to L_2(A) \to \mathcal{H}_2(A^\times) \to K_2(A) \to L_1(A) \to \mathcal{H}_1(A^\times) \to K_1(A) \to L_0(A) \to 0.$$

We call a ring A Nesterenko-Suslin if it satisfies the hypothesis of Remark 1.13 of their paper [13]. The precise requirement is that for every finite set F, there is a function $f_F : F \to$ the *center* of A so that the sum $\Sigma\{f_F(s) : s \in S\}$ is a *unit* of A for every nonempty $S \subset F$. If k is an infinite field, every associative k-algebra is Nesterenko-Suslin, and so is every commutative ring with many units in the sense of Van der Kallen. Remark 1.13 of Nesterenko-Suslin [13]

permits us to ignore unipotent radicals. This is used crucially in the proof of Theorem 1 (see also Proposition 12).
In the first draft of the paper, we conjectured that this theorem is true without any hypothesis on A. Sasha Beilinson then brought to our attention Suslin's paper [22] on the equivalence of Volodin's K-groups and Quillen's. From Suslin's description of Volodin's spaces, it is possible to show that these spaces are homotopy equivalent to the total space of the $N_n(A)$-torsor on $\mathbb{FL}(A^n)$ given in section 2 of this paper. This requires proposition 1 and a little organisation. Once this is done, Corollary 9 can also be obtained from Suslin's set-up. The statement "$X(R)$ is acyclic" stated and proved by Suslin in [22] now validates Proposition 12 at the infinite level, thus showing that Theorem 1 is true without any hypothesis on A. The details have not been included here.
R. Kottwitz informed us that the maximal simplices of $\mathbb{FL}(V)$ are referred to as "regular stars" in the work of Langlands(see [5]).
We hope that this paper will eventually connect with mixed Tate motives (see[3],[1]).
The arrangement of the paper is a follows. Section 1 has some topological preliminaries used through most of the paper. The proofs of Corollary 7 and Proposition 11 rely on Quillen's Theorem A. Alternatively, they can both be proved directly by repeated applications of Proposition 1. The definitions of $\mathbb{SPL}(A^n), \mathbb{FL}(A^n), \mathbb{ET}(A^n)$ and first properties are given in section two. The next four sections are devoted to the proof of Theorem 1. The last four sections are concerned with the homology of $\mathbb{ET}(A^n)$.
The lemmas, corollaries and propositions are labelled sequentially. For instance, corollary 9 is followed by lemma 10 and later by proposition 11; there is no proposition 10 or corollary 10. The other numbered statements are the three theorems. Theorems 2 and 3 are stated and proved in sections 7 and 9 respectively. Section 0 records some assumptions and notation, some perhaps non-standard, that are used in the paper. The reader might find it helpful to glance at this section for notation regarding elementary matrices and the Borel construction and the use of "simplicial complexes".

0. Assumptions and Notation

Rings, Elementary matrices, $\mathrm{Elem}(W \hookrightarrow V), \mathrm{Elem}(V, q)$*,* $\mathcal{L}(V)$ *and* $\mathcal{L}_p(V)$

We are concerned with the Quillen K-groups of a ring A.
We assume that A has the following property: if $A^m \cong A^n$ as left A-modules, then $m = n$. The phrase "A-module" always means left A-module.
For a finitely generated free A-module V, the collection of A-submodules $L \subset V$ so that (i) V/L is free and (ii) L is free of rank one , is denoted by $\mathcal{L}(V)$.
$\mathcal{L}_p(V)$ is the collection of subsets q of cardinality $(p+1)$ of $\mathcal{L}(V)$ so that $\oplus\{L : L \in q\} \to V$ is a monomorphism whose cokernel is free.
Given an A-submodule W of a A-module V so that the short exact sequence

$$0 \to W \to V \to V/W \to 0$$

is split, we have the subgroup $\mathrm{Elem}(W \hookrightarrow V) \subset Aut_A(V)$, defined as follows. Let $H(W)$ be the group of automorphisms h of V so that
$(id_V - h)V \subset W \subset ker(id_V - h)$. Let $W' \subset V$ be a submodule that is complementary to W. Define $H(W')$ in the same manner. The subgroup of $Aut_A(V)$ generated by $H(W)$ and $H(W')$ is $\mathrm{Elem}(W \hookrightarrow V)$. It does not depend on the choice of W' because $H(W)$ acts transitively on the collection of such W'.
For example, if $V = A^n$, and W is the A-submodule generated by any r members of the given basis of A^n, then $\mathrm{Elem}(W \hookrightarrow A^n)$ equals $E_n(A)$, the subgroup of elementary matrices in $GL_n(A)$, provided of course that $0 < r < n$.
If V is finitely generated free and if $q \in \mathcal{L}_p(V)$, the above statement implies that $\mathrm{Elem}(L \hookrightarrow V)$ does not depend on the choice of $L \in q$. Thus we denote this subgroup by $\mathrm{Elem}(V, q) \subset GL(V)$.

The Borel Construction

Let X be a topological space equipped with the action of group G. Let EG be the principal G-bundle on BG (as in [14]). The Borel construction, namely the quotient of $X \times EG$ by the G-action, is denoted by $X//G$ throughout the paper.

Categories, Geometric realisations, Posets

Every category $\mathcal{C}$ gives rise to a simplicial set, namely its nerve (see [14]). Its geometric realisation is denoted by $B\mathcal{C}$.
A poset (partially ordered set) P gives rise to a category. The B-construction of this category, by abuse of notation, is denoted by BP. Associated to P is the simplicial complex with P as its set of vertices; the simplices are finite non-empty chains in P. The geometric realisation of this simplicial complex coincides with BP.

Simplicial Complexes, Products and Internal Hom, Barycentric subdivision

Simplicial complexes crop up throughout this paper. We refer to Chapter 3,[18], for the definition of a simplicial complex and its barycentric subdivision. $\mathcal{S}(K)$ and $\mathcal{V}(K)$ denote the sets of vertices and simplices respectively of a simplicial complex K. The geometric realisation of K is denoted by $|K|$. The set $\mathcal{S}(K)$ is a partially ordered set (with respect to inclusion of subsets). Note that $B\mathcal{S}(K)$ is simply the (geometric realisation of) the barycentric subdivision $sd(K)$. The geometric realisations of K and $sd(K)$ are canonically homeomorphic to each other, but not by a simplicial map.
Given simplicial complexes K_1 and K_2, the product $|K_1| \times |K_2|$ (in the compactly generated topology) is canonically homeomorphic to $B(\mathcal{S}(K_1) \times \mathcal{S}(K_2))$.

The category of simplicial complexes and simplicial maps has a *categorical product*:
$\mathcal{V}(K_1 \times K_2) = \mathcal{V}(K_1) \times \mathcal{V}(K_2)$. A non-empty subset of $\mathcal{V}(K_1 \times K_2)$ is a simplex of $K_1 \times K_2$ if and only if it is contained in $S_1 \times S_2$ for some $S_i \in \mathcal{S}(K_i)$ for $i = 1, 2$. The geometric realisation of the product is not homeomorphic to the product of the geometric realisations, but they do have the same homotopy

type. In fact Proposition 1 of section 1 provides a contractible collection of homotopy equivalences $|K_1| \times |K_2| \to |K_1 \times K_2|$. For most purposes, it suffices to note that there is a *canonical* map $P(K_1, K_2) : |K_1| \times |K_2| \to |K_1 \times K_2|$. This is obtained in the following manner. Let $C(K)$ denote the $\mathbb{R}$-vector space with basis $\mathcal{V}(K)$ for a simplicial complex K. Recall that $|K|$ is a subset of $C(K)$. For simplicial complexes K_1 and K_2, we have the evident isomorphism

$$j : C(K_1) \otimes_{\mathbb{R}} C(K_2) \to C(K_1 \times K_2).$$

For $c_i \in |K_i|$ for $i = 1, 2$ we put $P(K_1, K_2)(c_1, c_2) = j(c_1 \otimes c_2) \in C(K_1 \times K_2)$. We note that $j(c_1 \otimes c_2)$ belongs to the subset $|K_1 \times K_2|$. This gives the canonical $P(K_1, K_2)$.

Given simplicial complexes K, L there is a simplicial complex $\mathcal{H}om(K, L)$ with the following property: if M is a simplicial complex, then the set of simplicial maps $K \times M \to L$ is naturally identified with the set of simplicial maps $M \to \mathcal{H}om(K, L)$. This simple verification is left to the reader.

Simplicial maps $f : K_1 \times K_2 \to K_3$ occur in sections 2 and 5 of this paper.

$$|f| \circ P(K_1, K_2) : |K_1| \times |K_2| \to |K_3|$$

is the map we employ on geometric realisations. Maps $|K_1| \times |K_2| \to |K_3|$ associated to simplicial maps f_1 and f_2 are seen (by contiguity) to be homotopic to each other if $\{f_1, f_2\}$ is a simplex of $\mathcal{H}om(K_1 \times K_2, K_3)$. This fact is employed in Lemma 8.

Simplicial maps $f : K_1 \times K_2 \to K_3$ are in reality maps $\mathcal{V}(f) : \mathcal{V}(K_1) \times \mathcal{V}(K_2) \to \mathcal{V}(K_3)$ with the property that $\mathcal{V}(f)(S_1 \times S_2)$ is a simplex of K_3 whenever S_1 and S_2 are simplices of K_1 and K_2 respectively. One should note that such an f induces a map of posets $\mathcal{S}(K_1) \times \mathcal{S}(K_2) \to \mathcal{S}(K_3)$, which in turn induces a continuous map $B(\mathcal{S}(K_1) \times \mathcal{S}(K_2)) \to B\mathcal{S}(K_3)$. In view of the natural identifications, this is the same as giving a map $|K_1| \times |K_2| \to |K_3|$. This map coincides with the $|f| \circ P(K_1, K_2)$ considered above.

The homotopy assertion of maps $|K_1| \times |K_2| \to |K_3|$ associated to f_1, f_2 where $\{f_1, f_2\}$ is an edge of $\mathcal{H}om(K_1 \times K_2, K_3)$ cannot be proved by the quick poset definition of the maps (for $|K_3|$ has been subdivided and contiguity is not available any more). This explains our preference for the longwinded $|f| \circ P(K_1, K_2)$ definition.

1. Some preliminaries from topology

We work with the category of compactly generated weakly Hausdorff spaces. A good source is Chapter 5 of [11]. This category possesses products. It also possesses an internal Hom in the following sense: for compactly generated Hausdorff X, Y, Z, continuous maps $Z \to \mathcal{H}om(X, Y)$ are the same as continuous maps $Z \times X \to Y$, where $Z \times X$ denotes the product in this category. This internal Hom property is required in the proof of Proposition 1 stated below.

$\mathcal{H}om(X,Y)$ is the space of continuous maps from X to Y. This space of maps has the compact-open topology, which is then replaced by the inherited compactly generated topology. This space $\mathcal{H}om(X,Y)$ is referred to frequently as $\mathrm{Map}(X,Y)$, and some times even as $\mathrm{Maps}(X,Y)$, in the text.

Now consider the following set-up. Let Λ be a partially ordered set assumed to be Artinian: (i) every non-empty subset in Λ has a minimal element with respect to the partial order, or equivalently (ii) there are no infinite strictly descending chains $\lambda_1 > \lambda_2 > \cdots$ in Λ. The poset Λ will remain fixed throughout the discussion below.

We consider topological spaces X equipped with a family of closed subsets $X_\lambda, \lambda \in \Lambda$ with the property that $X_\mu \subset X_\lambda$ whenever $\mu \leq \lambda$.

Given another $Y, Y_\lambda, \lambda \in \Lambda$ as above, the collection of Λ-compatible continuous $f : X \to Y$ (i.e. satisfying $f(X_\lambda) \subset Y_\lambda, \forall \lambda \in \Lambda$) will be denoted by $Map_\Lambda(X,Y)$. $Map_\Lambda(X,Y)$ is a closed subset of $\mathcal{H}om(X,Y)$, and this topologises $Map_\Lambda(X,Y)$.

We say that $\{X_\lambda\}$ is a *weakly admissible covering* of X if the three conditions listed below are satisfied. It is an *admissible covering* if in addition, each X_λ is contractible.

(1) For each pair of indices $\lambda, \mu \in \Lambda$, we have
$$X_\lambda \cap X_\mu = \underset{\nu \leq \lambda, \nu \leq \mu}{\cup} X_\nu$$

(2) If
$$\partial X_\lambda = \underset{\nu < \lambda}{\cup} X_\nu,$$
then $\partial X_\lambda \hookrightarrow X_\lambda$ is a cofibration

(3) The topology on X is coherent with respect to the family of subsets $\{X_\lambda\}_{\lambda \in \Lambda}$, that is, $X = \cup_\lambda X_\lambda$, and a subset $Z \subset X$ is closed precisely when $Z \cap X_\lambda$ is closed in X_λ in the relative topology, for all λ.

Proposition 1. *Assume that $\{X_\lambda\}$ is a weakly admissible covering of X. Assume also that each Y_λ is contractible.*
Then the space $\mathrm{Map}_\Lambda(X,Y)$ of Λ-compatible maps $f : X \to Y$ is contractible. In particular, it is non-empty and path-connected.

Corollary 2. *If both $\{X_\lambda\}$ and $\{Y_\lambda\}$ are admissible, then X and Y are homotopy equivalent.*

With assumptions as in the above corollary, the proposition yields the existence of Λ-compatible maps $f : X \to Y$ and $g : Y \to X$. Because $g \circ f$ and $f \circ g$ are also Λ-compatible, that they are homotopic to id_X and id_Y respectively is deduced from the path-connectivity of $\mathrm{Map}_\Lambda(X,X)$ and $\mathrm{Map}_\Lambda(Y,Y)$.

Corollary 3. *If $\{X_\lambda\}$ is admissible, then there is a homotopy equivalence $X \to B\Lambda$.*

Here, recall that $B\Lambda$ is the geometric realization of the simplicial complex associated to the set of nonempty finite chains (totally ordered subsets) in Λ; equivalently, regarding Λ as a category, $B\Lambda$ is the geometric realization of its nerve. We put $Y = B\Lambda$ and $Y_\lambda = B\{\mu \in \Lambda : \mu \leq \lambda\}$ in Corollary 2 to deduce Corollary 3.

The proof of Proposition 1 is easily reduced to the following extension lemma.

Lemma 4. *Let $\{X_\lambda\}, \{Y_\lambda\}$ etc. be as in the above proposition. Let $\Lambda' \subset \Lambda$ be a subset, with induced partial order, so that for any $\lambda \in \Lambda'$, $\mu \in \Lambda$ with $\mu \leq \lambda$, we have $\mu \in \Lambda'$. Let $X' = \cup_{\lambda \in \Lambda'} X_\lambda$, $Y' = \cup_{\lambda\in\Lambda'} Y_\lambda$. Assume given a continuous map $f' : X' \to Y'$ with $f'(X_\lambda) \subset Y_\lambda$ for all $\lambda \in \Lambda'$. Then f' extends to a continuous map $f : X \to Y$ with $f(X_\lambda) \subset Y_\lambda$ for all $\lambda \in \Lambda$.*

Proof. Consider the collection of pairs (Λ'', f'') satisfying:
(a) $\Lambda' \subset \Lambda'' \subset \Lambda$
(b) $\mu \in \Lambda, \lambda \in \Lambda'', \mu \leq \lambda$ implies $\mu \in \Lambda''$
(c) $f'' : \cup\{X_\mu | \mu \in \Lambda''\} \to Y$ is a continuous map
(d) $f''(X_\mu) \subset Y_\mu$ for all $\mu \in \Lambda''$
(e) $f'|X_\mu = f''|X_\mu$ for all $\mu \in \Lambda'$
This collection is partially ordered in a natural manner. The coherence condition on the topology of X ensures that every chain in this collection has an upper bound. The presence of (Λ', f') shows that it is non-empty. By Zorn's lemma, there is a maximal element (Λ'', f'') in this collection. The Artinian hypothesis on Λ shows that if $\Lambda'' \neq \Lambda$, then its complement possesses a minimal element μ. Let D'' be the domain of f''. The minimality of μ shows that $D'' \cap X_\mu = \partial X_\mu$. By condition (d) above, we see that $f''(\partial X_\mu)$ is contained in the contractible space Y_μ. Because $\partial X_\mu \hookrightarrow X_\mu$ is a cofibration, it follows that $f''|\partial X_\mu$ extends to a map $g : X_\mu \to Y_\mu$. The f'' and g patch together to give a continuous map $h : D'' \cup X_\mu \to Y$. Since the pair $(\Lambda'' \cup \{\mu\}, h)$ evidently belongs to this collection, the maximality of (Λ'', f'') is contradicted. Thus $\Lambda'' = \Lambda$ and this completes the proof.

□

The proof of the Proposition follows in three standard steps.
Step 1: Taking $\Lambda' = \emptyset$ in Lemma 4 we deduce that $\mathrm{Map}_\Lambda(X, Y)$ is nonempty.
Step 2: For the path-connectivity of $\mathrm{Map}_\Lambda(X, Y)$, we replace X by $X \times [0, 1]$ and replace the original poset Λ by the product $\Lambda \times \{\{0\}, \{1\}, \{0, 1\}\}$, with the product partial order, where the second factor is partially ordered by inclusion. The subsets of $X \times I$ (resp.Y) indexed by $(\lambda, 0), (\lambda, 1), (\lambda, \{0, 1\})$ are $X_\lambda \times \{0\}, X_\lambda \times \{1\}$ and $X_\lambda \times [0, 1]$ (resp. Y_λ in all three cases).
We then apply the lemma to the sub-poset $\Lambda \times \{\{0\}, \{1\}\}$.
Step 3: Finally, for the contractibility of $\mathrm{Map}_\Lambda(X, Y)$, we first choose $f_0 \in \mathrm{Map}_\Lambda(X, Y)$ and then consider the two maps $\mathrm{Map}_\Lambda(X, Y) \times X \to Y$ given by $(f, x) \mapsto f(x)$ and $(f, x) \mapsto f_0(x)$. Putting $(\mathrm{Map}_\Lambda(X, Y) \times X)_\lambda = \mathrm{Map}_\Lambda(X, Y) \times X_\lambda$ for all $\lambda \in \Lambda$, we see that both the above maps are Λ-compatible. The path-connectivity assertion in Step 2 now gives a homotopy between the identity map of $\mathrm{Map}_\Lambda(X, Y)$ and the constant map $f \mapsto f_0$. This completes the proof of Proposition 1.
We now want to make some remarks about equivariant versions of the above statements.

Given $X, \{X_\lambda; \lambda \in \Lambda\}$ as above, an action of a group G on X is called Λ-compatible if G also acts on the poset Λ so that for all $g \in G$, $\lambda \in \Lambda$, we have $g(X_\lambda) = X_{g\lambda}$.

Under the conditions of Proposition 1, suppose $\{X_\lambda\}$, $\{Y_\lambda\}$ admit Λ-compatible G-actions. There is no G-equivariant $f \in Map_\Lambda(X,Y)$ in general. However, if $f \in Map_\Lambda(X,Y)$ and g_X, g_Y denote the actions of $g \in G$ on X and Y respectively, we see that , then $g_Y \circ f \circ g_X^{-1}$ is also a Λ-compatible map. By Proposition 1, we see that this map is homotopic to f. Thus $g_Y \circ f$ and $f \circ g_X$ are homotopic to each other. *In particular, $H_n(f) : H_n(X) \to H_n(Y)$ is a homomorphism of G-modules.*

In the sequel a better version of this involving the Borel construction is needed. We recall the *Borel construction* of equivariant homotopy quotient spaces. Let EG denote a contractible CW complex on which G has a proper free cellular action; for our purposes, it suffices to fix a choice of this space EG to be the geometric realization of the nerve of the translation category of G (the category with vertices $[g]$ indexed by the elements of G, and unique morphisms between ordered pairs of vertices $([g],[h])$, thought of as given by the left action of hg^{-1}). The classifying space BG is the quotient space EG/G.

If X is any G-space, let $X//G$ denote the *homotopy quotient of X by G*, obtained using the Borel construction, i.e.,

$$X//G = (X \times EG)/G, \tag{1}$$

where EG is as above, and G acts diagonally. Note that the natural quotient map

$$q_X : X \times EG \to X//G$$

is a Galois covering space, with covering group G.

If X and Y are G-spaces, then considering G-equivariant maps $\widetilde{f} : X \times EG \to Y \times EG$ compatible with the projections to EG, giving a commutative diagram

$$\begin{tikzcd} X \times EG \arrow[rr, "\widetilde{f}"] \arrow[dr] & & Y \times EG \arrow[dl] \\ & EG & \end{tikzcd}$$

is equivalent to considering maps $\overline{f} : X//G \to Y//G$ compatible with the maps $q_X : X//G \to BG$, $q_Y : Y//G \to BG$, giving a commutative diagram

$$\begin{tikzcd} X//G \arrow[rr, "\overline{f}"] \arrow[dr, "q_X"'] & & Y//G \arrow[dl, "q_Y"] \\ & BG & \end{tikzcd}$$

Proposition 5. *Assume that, in the situation of proposition 1, there are Λ-compatible G-actions on X and Y. Let EG be as above, and consider the Λ-compatible families $\{X_\lambda \times EG\}$, which is a weakly admissible covering family for $X \times EG$, and $\{Y_\lambda \times EG\}$, which is an admissible covering family for $Y \times EG$.*

Then there is a G-equivariant map $\widetilde{f} : X \times EG \to Y \times EG$, compatible with the projections to EG, such that

(i) $\widetilde{f}(X_\lambda \times EG) \subset (Y_\lambda \times EG)$ *for all* $\lambda \in \Lambda$

(ii) *if* $\widetilde{g} : X \times EG \to Y \times EG$ *is another such equivariant map, then there is a G-equivariant homotopy between $\widetilde{f}$ and $\widetilde{g}$, compatible with the projections to EG*

(iii) *The space of such equivariant maps $X \times EG \to Y \times EG$, as in (i), is contractible.*

Proof. We show the existence of the desired map, and leave the proof of other properties, by similar arguments, to the reader.
Let $\mathrm{Map}_\Lambda(X, Y)$ be the contractible space of Λ-compatible maps from X to Y; note that it comes equipped with a natural G-action, so that the canonical evaluation map $X \times \mathrm{Map}_\Lambda(X, Y) \to Y$ is equivariant. This induces $X \times \mathrm{Map}_\Lambda(X, Y) \times EG \to Y \times EG$. There is also a natural G-equivariant map $\pi : X \times \mathrm{Map}_\Lambda(X, Y) \times EG \to X \times EG$. This map π has equivariant sections, since the projection $\mathrm{Map}_\Lambda(X, Y) \times EG \to EG$ is a G-equivariant map between weakly contractible spaces, so that the map on quotients modulo G is a weak homotopy equivalence (i.e., $(\mathrm{Map}_\Lambda(X, Y) \times EG)/G$ is another "model" for the classifying space $BG = EG/G$). However BG is a CW complex, so the map has a section. □

As another preliminary, we note some facts (see lemma 6 below) which are essentially corollaries of Quillen's Theorem A (these are presumably well-known to experts, though we do not have a specific reference).
If P is any poset, let $C(P)$ be the poset consisting of non-empty finite chains (totally ordered subsets) of P. If $f : P \to Q$ is a morphism between posets (an order preserving map) there is an induced morphism $C(f) : C(P) \to C(Q)$. If S is a simplicial complex (literally, a collection of finite non-empty subsets of the vertex set), we may regard S as a poset, partially ordered with respect to inclusion; then the classifying space BS is naturally homeomorphic to the geometric realisation $|S|$ (and gives the barycentric subdivision of $|S|$). A simplicial map $f : S \to T$ between simplicial complexes (that is, a map on vertex sets which sends simplices to simplices, not necessarily preserving dimension) is also then a morphism of posets. We say that a poset P is contractible if its classifying space BP is contractible.

Lemma 6. *(i) Let $f : P \to Q$ be a morphism between posets. Suppose that for each $X \in C(Q)$, the fiber poset $C(f)^{-1}(X)$ is contractible. Then $Bf : BP \to BQ$ is a homotopy equivalence.*
(ii) Let $f : S \to T$ be a simplicial map between simplicial complexes. Suppose that for any simplex $\sigma \in T$, the fiber $f^{-1}(\sigma)$, considered as a poset, is contractible. Then $|f| : |S| \to |T|$ is a homotopy equivalence.

Proof. We first prove (i). For any poset P, there is morphism of posets $\varphi_P : C(P) \to P$, sending a chain to its first (smallest) element. If $a, b \in P$ with $a \leq b$, and C is a chain in $\varphi_P^{-1}(b)$, then $\{a\} \cup C$ is a chain in $\varphi_P^{-1}(a)$. This

gives an order preserving map of posets $\varphi_P^{-1}(b) \to \varphi_P^{-1}(a)$ (i.e., a "base-change" functor). This makes $C(P)$ prefibred over P, in the sense of Quillen (see page 96 in [19], for example). Also, $\varphi_P^{-1}(a)$ has the minimal element (initial object) $\{a\}$, and so its classifying space is contractible.
Hence Quillen's Theorem A (see [19], page 96) implies that $B(\varphi_P)$ is a homotopy equivalence, for any P.
Now let $f : P \to Q$ be a morphism between posets. Let $C(f) : C(P) \to C(Q)$ be the corresponding morphism on the posets of (finite, nonempty) chains. If $A \subset B$ are two chains in $C(Q)$, there is an obvious order preserving map $C(f)^{-1}(B) \to C(f)^{-1}(A)$. Again, this makes $C(f) : C(P) \to C(Q)$ prefibred. Since we assumed that $BC(f)^{-1}(A)$ is contractible, for all $A \in C(Q)$, Quillen's Theorem A implies that $BC(f)$ is a homotopy equivalence.
We thus have a commutative diagram of posets and order preserving maps

$$\begin{array}{ccc} C(P) & \stackrel{C(f)}{\to} & C(Q) \\ \varphi_P \downarrow & & \downarrow \varphi_Q \\ P & \stackrel{f}{\to} & Q \end{array}$$

where three of the four sides yield homotopy equivalences on passing to classifying spaces. Hence $Bf : BP \to BQ$ is a homotopy equivalence, proving (i).
The proof of (ii) is similar. This is equivalent to showing that $Bf : BS \to BT$ is a homotopy equivalence. Since $f : S \to T$, regarded as a morphism of posets, is naturally prefibered, and by assumption, $Bf^{-1}(\sigma)$ is contractible for each $\sigma \in T$, Quillen's Theorem A implies that Bf is a homotopy equivalence. □

We make use of Propositions 1 and 5 in the following way.
Let A, B be sets, $Z \subset A \times B$ a subset such that the projections $p : Z \to A$, $q : Z \to B$ are both surjective. Consider simplicial complexes $S_Z(A)$, $S_Z(B)$ on vertex sets A, B respectively, with simplices in $S_Z(A)$ being finite nonempty subsets of fibers $q^{-1}(b)$, for any $b \in B$, and simplices in $S_Z(B)$ being finite, nonempty subsets of fibers $p^{-1}(a)$, for any $a \in A$.
Consider also a third simplicial complex $S_Z(A, B)$ with vertex set Z, where a finite non-empty subset $Z' \subset Z$ is a simplex if and only it satisfies the following condition:

$$(a_1, b_1), (a_2, b_2) \in Z' \ \Rightarrow (a_1, b_2) \in Z.$$

Note that the natural maps on vertex sets $p : Z \to A$, $q : Z \to B$ induce canonical simplicial maps on geometric realizations

$$|p| : |S_Z(A, B)| \to |S_Z(A)|, \ \ |q| : |S_Z(A, B)| \to |S_Z(B)|.$$

Corollary 7. (1) *With the above notation, the simplicial maps*

$$|p| : |S_Z(A, B)| \to |S_Z(A)|, \ \ |q| : |S_Z(A, B)| \to |S_Z(B)|$$

are homotopy equivalences. In particular, $|S_Z(A)|$, $|S_Z(B)|$ *are homotopy equivalent.*

(2) *If a group G acts on A and on B, so that Z is stable under the diagonal G action on $A \times B$, then the homotopy equivalences $|p|$, $|q|$ are G-equivariant homotopy equivalences. Hence there exists a natural G-equivariant homotopy equivalence between $|S_Z(A)| \times EG$ and $|S_Z(B)| \times EG$.*

Proof. Since the situation is symmetric with respect to the sets A, B, it suffices to show $|p|$ is a homotopy equivalence. Note that in the context of a G-action as stated, the G-equivariance of $|p|$ is clear.
Let Λ be the poset of all simplices of $S_Z(A)$, thought of as subsets of A, and ordered by inclusion. Clearly Λ is Artinian.
Apply Corollary 2 with $X = |S_Z(A, B)|$, $Y = |S_Z(A)|$, Λ as above, and the following Λ-admissible coverings: for $\sigma \in \Lambda$, let Y_σ be the (closed) simplex in $Y = |S_Z(A)|$ determined by σ (clearly $\{Y_\sigma\}$ is admissible); take $X_\sigma = |p|^{-1}(Y_\sigma)$ (this is evidently weakly admissible). For admissibility of $\{X_\sigma\}$, we need to show that each X_σ is contractible.
In fact, regarding the sets of simplices $S_Z(A, B)$ and $S_Z(A)$ as posets, and $S_Z(A, B) \to S_Z(A)$ as a morphism of posets, X_σ is the geometric realization of the simplicial complex determined by $\cup_{\tau \leq \sigma} p^{-1}(\tau)$.
The corresponding map of posets

$$p^{-1}(\{\tau | \tau \leq \sigma\}) \to \{\tau | \tau \leq \sigma\}$$

has contractible fiber posets – if we fix an element $x \in p^{-1}(\tau)$, and $p^{-1}(\tau)(\geq x)$ is the sub-poset of elements bounded below by x, then $y \mapsto y \cup x$ is a morphism of posets $r_x : p^{-1}(\tau) \to p^{-1}(\tau)(\geq x)$ which gives a homotopy equivalence on geometric realizations (it is left adjoint to the inclusion of the sub-poset). But the sub-poset has a minimal element, and so its realization is contractible.
The poset $\{\tau | \tau \leq \sigma\}$ is obviously contractible, since it has a maximal element. Hence, applying lemma 6(ii), X_σ is contractible.
Since the map $|p| : X \to Y$ is Λ-compatible, it is the unique such map upto Λ-compatible homotopy, and is a homotopy equivalence. $\square$

We note that the argument with Quillen's Theorem A in fact implies directly that $S_Z(A, B) \to S_Z(A)$ is a homotopy equivalence; the uniqueness assertion is not, apparently, a formal consequence of Quillen's Theorem A.

2. Flag Spaces

In this section, we discuss various constructions of spaces (generally simplicial complexes) defined using flags of free modules, and various maps, and homotopy equivalences, between these. These are used as building blocks in the proof of Theorem 1.
Let A be a ring, and let V be a free (left) A-module of rank n. Define a simplicial complex $\mathbb{FL}(V)$ as follows.
Its vertex set is

$FL(V) =$

$$= \{F = (F_0, F_1, \ldots, F_n) \mid 0 = F_0 \subset F_1 \subset \cdots \subset F_n = V \text{ are } A\text{-submodules,}$$
$$\text{and each quotient } F_i/F_{i-1} \text{ is a free } A\text{-module of rank } 1\}.$$

We think of this vertex set as the set of "full flags" in V.
To describe the simplices in $\mathbb{FL}(V)$, we need another definition. Let

$$SPL(V) = \{\{L_1, \ldots, L_n\} \mid L_i \subset V \text{ is a free } A\text{-submodule of rank 1, and}$$
$$\text{the induced map } \oplus_{i=1}^n L_i \to V \text{ is an isomorphism}\}.$$

Note that $\{L_1, \ldots, L_n\}$ is regarded as an *unordered* set of free A-submodules of rank 1 of L (i.e., as a subset of cardinality n in the set of all free A-submodules of rank 1 of V). We think of $SPL(V)$ as the "set of unordered splittings of V into direct sums of free rank 1 modules".
Given $\alpha \in SPL(V)$, say $\alpha = \{L_1, \ldots, L_n\}$, we may choose some ordering $(L_1, \ldots, L_n)$ of its elements, and thus obtain a full flag in V (i.e., an element in $FL(V)$), given by

$$(0, L_1, L_1 \oplus L_2, \cdots, L_1 \oplus \cdots \oplus L_n = V) \in FL(V).$$

Let

$$[\alpha] \subset FL(V)$$

be the set of $n!$ such full flags obtained from α.
We now define a simplex in $\mathbb{FL}(V)$ to be any subset of such a set $[\alpha]$ of vertices, for any $\alpha \in SPL(V)$. Thus, $\mathbb{FL}(V)$ becomes a simplicial complex of dimension $n! - 1$, with the sets $[\alpha]$ as above corresponding to maximal dimensional simplices.
Clearly $\mathrm{Aut}\,(V) \cong \mathrm{GL}_n(A)$ acts on the simplical complex $\mathbb{FL}(V)$ through simplicial automorphisms, and thus acts on the homology groups $H_*(\mathbb{FL}(V), \mathbb{Z})$ (and other similar invariants of $\mathbb{FL}(V)$).
Next, remark that if $F \in FL(V)$ is any vertex of $\mathbb{FL}(V)$, we may associate to it the free A-module $\mathrm{gr}_F(V) = \oplus_{i=1}^n F_i/F_{i-1}$. If (F, F') is an ordered pair of distinct vertices, which are joined by an edge in $\mathbb{FL}(V)$, then we obtain a *canonical* isomorphism (determined by the edge)

$$\varphi_{F,F'} : \mathrm{gr}_F(V) \xrightarrow{\cong} \mathrm{gr}_{F'}(V).$$

One way to describe it is by considering the edge as lying in a simplex $[\alpha]$, for some $\alpha = \{L_1, \ldots, L_n\} \in SPL(V)$; this determines an identification of $\mathrm{gr}_F(V)$ with $\oplus_i L_i$, and a similar identification of $\mathrm{gr}_{F'}(V)$, and thereby an identification between $\mathrm{gr}_F(V)$ and $\mathrm{gr}_{F'}(V)$. Note that from this description of the maps $\varphi_{F,F'}$, it follows that if F, F', F'' form vertices of a 2-simplex in $\mathbb{FL}(V)$, i.e., there exists some $\alpha \in SPL(V)$ such that $F, F', F'' \in [\alpha]$, then we also have

$$\varphi_{F,F''} = \varphi_{F',F''} \circ \varphi_{F,F'}.$$

The isomorphism $\varphi_{F,F'}$ depends only on the (oriented) edge in $\mathbb{FL}(V)$ determined by (F, F'), and not on the choice of the simplex $[\alpha]$ in which it lies. One way to see this is to use that, for any two such filtrations F, F' of V there is a canonical isomorphism $\mathrm{gr}^p_F \mathrm{gr}^q_{F'}(V) \cong \mathrm{gr}^q_{F'} \mathrm{gr}^p_F(V)$ (Schur-Zassenhaus lemma)

for each p, q. But in case F, F' are flags which are connected by an edge, then there is also a canonical isomorphism $\mathrm{gr}_F \mathrm{gr}_{F'}(V) \cong \mathrm{gr}_{F'}(V)$ (in fact the F-filtration induced on $\mathrm{gr}^p_{F'}(V)$ has only 1 non-trivial step, for each p), and similarly there is a canonical isomorphism $\mathrm{gr}_{F'} \mathrm{gr}_F(V) \cong \mathrm{gr}_F(V)$. These three canonical isomorphisms combine to give the isomorphism $\varphi_{F,F'}$.
Hence there is a well-defined *local system* $\mathbf{gr}(V)$ of A-modules on the geometric realization $|\mathbb{FL}(V)|$ of the simplicial complex $\mathbb{FL}(V)$, whose fibre over a vertex F is $\mathrm{gr}_F(V)$.
Notice further that this local system $\mathbf{gr}(V)$ comes equipped with a natural $\mathrm{Aut}(V)$ action, compatible with the natural actions on $FL(V)$ and $\mathbb{FL}(V)$. Indeed, any element $g \in \mathrm{Aut}(V)$ gives a bijection on the set of full flags $FL(V)$, with

$$F = (0 = F_0, F_1, \ldots, F_n = V) \in FL(V))$$

mapping to

$$gF = (0 = gF_0, gF_1, \ldots, gF_n = V).$$

This clearly gives an induced isomorphism $\oplus_i F_i/F_{i-1} \cong \oplus_i gF_i/gF_{i-1}$, identifying the fibers of the local system over F and gF in a specific way. It is easy to see that if $\alpha = \{L_1, \ldots, L_n\} \in SPL(V)$, then $g\alpha = \{gL_1, \ldots, gL_n\} \in SPL(V)$, giving the action of $\mathrm{Aut}\,(V)$ on $SPL(V)$, so that if a pair of vertices F, F' of $FL(V)$ lie on an edge contained in $[\alpha]$, then gF, gF' lie on an edge contained in $[g\alpha]$, and so the induced identification $\varphi_{F,F'}$ is compatible with $\varphi_{gF,gF'}$. This induces the desired action of $\mathrm{Aut}(V)$ on the local system.
Further, note that if $F, F' \in FL(V)$ are connected by an edge in $\mathbb{FL}(V)$, then we may realize $\varphi_{F,F'}$ by the action of a suitable element of $\mathrm{Aut}\,(V)$, which preserves a simplex $[\alpha]$ in which the edge lies, and permutes the lines in the splitting $\alpha \in SPL(V)$. Thus, given any edge-path joining vertices F, F' in $\mathbb{FL}(V)$, the induced composite isomorphism $\mathrm{gr}_F(V) \to \mathrm{gr}_{F'}(V)$ is again realized by the action of an element of $\mathrm{Aut}\,(V)$. In particular, given an edge-path loop based as $F \in FL(V)$, the induced automorphism of $\mathrm{gr}_F(V)$ is induced by the action on $\mathrm{gr}_F(V)$ of an element of the isotropy group of F in $\mathrm{Aut}\,(V)$, which is the "Borel subgroup" corresponding to the flag F.
Hence, the monodromy group of the local system $\mathbf{gr}(V)$ is clearly contained in $N_n(A)$, defined as a semidirect product

$$N_n(A) = (A^\times \times \cdots \times A^\times) \ltimes S_n \tag{2}$$

where S_n is the permutation group; we regard $N_n(A)$ as a subgroup of $\mathrm{Aut}\,(\oplus_i L_i)$ in an obvious way.
Now we make infinite versions of the above constructions.
Let A^∞ be the set of sequences $(a_1, a_2, \ldots, a_n, \ldots)$ of elements of A, all but finitely many of which are 0, considered as a free A-module of countable rank. There is a standard inclusion $i_n : A^n \hookrightarrow A^\infty$ of the standard free A-module of rank n as the submodule of sequences with $a_m = 0$ for all $m > n$. The induced inclusion $i : A^n \to A^{n+1}$ is the usual one, given by $i(a_1, \ldots, a_n) = (a_1, \ldots, a_n, 0)$.

We may thus view A^∞ as being given with a tautological flag, consisting of the A-submodules $i_n(A^n)$. We define a simplicial complex $\mathbb{FL}(A^\infty)$, with vertex set $FL(A^\infty)$ equal to the set of flags $0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset A^\infty$ where V_i/V_{i-1} is a free A-module of rank 1, for each $i \geq 1$, and with $V_n = i(A^n)$ for all sufficiently large n. Thus $FL(A^\infty)$ is naturally the union of subsets bijective with $FL(A^n)$. To make $\mathbb{FL}(A^\infty)$ into a simplicial complex, we define a simplex to be a finite set of vertices in some subset $FL(A^n)$ which determines a simplex in the simplicial complex $\mathbb{FL}(A^n)$; this property does not depend on the choice of n, since the natural inclusion $FL(A^n) \hookrightarrow FL(A^{n+1})$, regarded as a map on vertex sets, identifies $\mathbb{FL}(A^n)$ with a subcomplex of $\mathbb{FL}(A^{n+1})$, such that any simplex of $\mathbb{FL}(A^{n+1})$ with vertices in $FL(A^n)$ is already in the subcomplex $\mathbb{FL}(A^n)$.

We consider $GL(A) \subset \mathrm{Aut}(A^\infty)$ as the union of the images of the obvious maps $i_n : GL_n(A) \hookrightarrow \mathrm{Aut}\,(A^\infty)$, obtained by automorphisms which fix all the basis elements of A^∞ beyond the first n. We clearly have an induced action of $GL(A)$ on the simplicial complex $\mathbb{FL}(A^\infty)$, and hence on its geometric realisation $|\mathbb{FL}(A^\infty)|$ through homeomorphisms preserving the simplicial structure. The inclusion $\mathbb{FL}(A^n) \hookrightarrow \mathbb{FL}(A^\infty)$ as a subcomplex is clearly $GL_n(A)$-equivariant.

Next, observe that there is a local system $\mathbf{gr}(A^\infty)$ on $\mathbb{FL}(A^\infty)$ whose fiber over a vertex $F = (F_0 = 0, F_1, \ldots, F_n, \ldots)$ is $\mathrm{gr}_F(V) = \oplus_i F_i/F_{i-1}$. This has monodromy contained in

$$N(A) = \cup_n N_n(A) \subset GL(A),$$

where we may also view $N(A)$ as the semidirect product of

$$(A^\times)^\infty = \text{diagonal matrices in } GL(A)$$

by the infinite permutation group S_∞. This local system also carries a natural $GL(A)$-action, compatible with the $GL(A)$-action on $\mathbb{FL}(A^\infty)$.

Next, we prove a property (Corollary 9) about the action of elementary matrices on homology, which is needed later. The corollary follows immediately from the lemma below.

For the statement and proof of the lemma, we suggest that the reader browse the remarks on $\mathcal{H}om(K_1 \times K_2, K_3)$ in section 0, given simplicial complexes K_i for $i = 1, 2, 3$. The notation $\mathrm{Elem}(V' \hookrightarrow V' \oplus V'')$ that appears in the lemma has also been introduced in section 0 under the heading "elementary matrices".

Lemma 8. *Let V', V'' be two free A-modules of finite rank, $i' : V' \to V' \oplus V''$, $i'' : V'' \to V' \oplus V''$ the inclusions of the direct summands. Consider the two natural maps*

$$\alpha, \beta : FL(V') \times FL(V'') \to FL(V' \oplus V'') \tag{3}$$

given by

$$\begin{aligned}\alpha : &((F_1', \ldots, F_r' = V'), (F_1'', \ldots, F_s'')) \mapsto \\ &(i'(F_1'), \ldots, i'(F_r') = i'(V'), i'(V') + i''(F_1''), \ldots, i'(V') + i''(F_s'') = V' \oplus V''),\end{aligned}$$

$\beta : ((F'_1, \ldots, F'_r = V'), (F''_1, \ldots, F''_s)) \mapsto$
$(i''(F''_1), \ldots, i''(F''_s) = i''(V''), i''(V'') + i'(F'_1), \ldots, i''(V'') + i'(F'_r) = V' \oplus V'')$.

(A) α and β are vertices of a one-simplex of $\mathcal{H}om(\mathbb{FL}(V') \times \mathbb{FL}(V''), \mathbb{FL}(V' \oplus V''))$.
(B) The maps $|\mathbb{FL}(V')| \times |\mathbb{FL}(V'')| \to |\mathbb{FL}(V' \oplus V'')|$ induced by α, β are homotopic to each other.
(C) Let $c : |\mathbb{FL}(V')| \times |\mathbb{FL}(V'')| \to |\mathbb{FL}(V' \oplus V'')|$ denote the map produced by α. Denote the action of $g \in GL(V' \oplus V'')$ on $|\mathbb{FL}(V' \oplus V'')|$ by $|\mathbb{FL}(g)|$. Then c and $|\mathbb{FL}(g)| \circ c$ are homotopic to each other, if $g \in \mathrm{Elem}(V' \hookrightarrow V' \oplus V'') \subset GL(V' \oplus V'')$.

Proof. Part (A). By the definition of $\mathcal{H}om(K_1 \times K_2, K_3)$ in section 0, we only have to check that $\alpha(\sigma' \times \sigma'') \cup \beta(\sigma' \times \sigma'')$ is a simplex of $FL(V' \oplus V'')$ for all simplices σ' of $\mathbb{FL}(V')$ and all simplices σ'' of $\mathbb{FL}(V'')$. Clearly it suffices to prove this for maximal simplices, so we assume that both σ' and σ'' are maximal.
Note that if we consider any maximal simplex σ' in $\mathbb{FL}(V')$, it corresponds to a splitting $\{L'_1, \ldots, L'_r\} \in SPL(V')$. Similarly any maximal simplex σ'' of $\mathbb{FL}(V'')$ corresponds to a splitting $\{L''_1, \ldots, L''_s\} \in SPL(V'')$. This determines the splitting $\{i'(L'_1), \ldots, i'(L'_r), i''(L''_1), \ldots, i''(L''_s)\}$ of $V' \oplus V''$, giving rise to a maximal simplex τ of $\mathbb{FL}(V' \oplus V'')$, and clearly $\alpha(\sigma' \times \sigma'')$ and $\beta(\sigma' \times \sigma'')$ are both contained in τ. Thus their union is a simplex.
(B) follows from (A). We now address (C). We note that $c = g \circ c$ for all $g \in id + Hom_A(V'', V')$. Denoting by d the map produced by β we see that $d = g \circ d$ for all $g \in id + Hom_A(V', V'')$. Because c, d are homotopic to each other, we see that c and $g \circ c$ are in the same homotopy class when g is in either of the two groups above. These groups generate $\mathrm{Elem}(V' \hookrightarrow V' \oplus V'')$, and so this proves (C). $\square$

Corollary 9. *(i) The group $E_{n+1}(A)$ of elementary matrices acts trivially on the image of the natural map*

$$i_* : H_*(\mathbb{FL}(A^{\oplus n}), \mathbb{Z}) \to H_*(\mathbb{FL}(A^{\oplus n+1}), \mathbb{Z}).$$

(ii) The action of the group $E(A)$ of elementary matrices on $H_*(\mathbb{FL}(A^\infty), \mathbb{Z})$ is trivial.

Proof. We put $V' = A^n$ and $V'' = A$ in the previous lemma. The c in part (B) of the lemma is precisely the i being considered here. By (C) of the lemma, $g \circ i$ is homotopic to i for all $g \in \mathrm{Elem}(A^n \hookrightarrow A^{n+1}) = E_{n+1}(A)$. This proves (i). The direct limit of the r-homology of $|\mathbb{FL}(A^n)|$, taken over all n, is the r-th homology of $|\mathbb{FL}(A^\infty|$. Thus (i) implies (ii).

$\square$

We will find it useful below to have other "equivalent models" of the spaces $\mathbb{FL}(V)$, $\mathbb{FL}(A^\infty)$, by which we mean other simplicial complexes, also defined using collections of appropriate A-submodules, such that there are natural homotopy equivalences between the different models of the same homotopy type, compatible with the appropriate group actions, etc.

We apply corollary 7 as follows. Let $V \cong A^n$. We put $A = SPL(V), B = FL(V)$ and $Z = \{\alpha, F) : F \in [\alpha]\}$. The simplicial complex $S_Z(B)$ of corollary 7 is $\mathbb{FL}(V)$ by its definition. The simplicial complex $S_Z(A)$ is our definition of $\mathbb{SPL}(V)$. The homotopy equivalence of $\mathbb{SPL}(V)$ and $\mathbb{FL}(V)$ follows from this corollary.

We define $SPL(A^\infty)$ to be the collection of sets S satisfying
(a) $L \in S$ implies that L is a free rank one A-submodule of A^∞,
(b)$\oplus\{L : L \in S\} \to A^\infty$ is an isomorphism, and
(c) the symmetric difference of S and the standard collection:
$\{A(1,0,0,...), A(0,1,0,...), \cdots\}$ is a finite set.

Corollary 7 is then applied to the subset $Z \subset SPL(A^\infty) \times FL(A^\infty)$ consisting of the pairs (S, F) so that there is a bijection $h : S \to \mathbb{N}$ so that for every $L \in S$,
$L \subset F_{h(L)}$ and $L \to gr^F_{h(L)}$ is an isomorphism.

The above Z defines $\mathbb{SPL}(A^\infty)$. The desired homotopy equivalence of the geometric realisations of $\mathbb{SPL}(A^\infty)$ and $\mathbb{FL}(A^\infty)$ comes from the same corollary.

We also find it useful to introduce a third model of the homotopy types of $\mathbb{FL}(V)$ and $\mathbb{FL}(A^\infty)$, the "enriched Tits buildings" $\mathbb{ET}(V)$ and $\mathbb{ET}(A^\infty)$. The latter is defined in the last remark of this section.

Let $V \cong A^n$ as a left A-module. Let $\mathcal{E}(V)$ be the set consisting of ordered pairs

$$(F, S) = ((0 = F_0 \subset F_1 \subset \cdots \subset F_r = V), (S_1, S_2, \dots, S_r)),$$

where F is a *partial flag* in V, which means that $F_i \subset V$ is an A-submodule, such that F_i/F_{i-1} is a nonzero free module for each i, and $S_i \in SPL(F_i/F_{i-1})$ is an unordered collection of free A-submodules of F_i/F_{i-1} giving rise to a direct sum decomposition $\oplus_{L\in S_i} L \cong F_i/F_{i-1}$. Thus S is a collection of splittings of the quotients F_i/F_{i-1} for each i.

We may put a partial order on the set $\mathcal{E}(V)$ in the following way: $(F, S) \leq (F', T)$ if the filtration F is a refinement of F', and the data S, T of direct sum decompositions of quotients are compatible, in the following natural sense — if $F'_{i-1} = F_{j-1} \subset F_j \subset \cdots F_l = F'_i$, then T_i must be partitioned into subsets, which map to the sets $S_j, S_{j+1}, \dots, S_l$ under the appropriate quotient maps. In particular, (F', T) has only finitely many possible predecessors (F, S) in the partial order.

We have a simplicial complex $ET(V) := N\mathcal{E}(V)$, the nerve of the partially ordered set $\mathcal{E}(V)$ considered as a category, so that simplices are just nonempty finite chains of elements of the vertex (po)set $\mathcal{E}(V)$.

Note that maximal elements of $\mathcal{E}(V)$ are naturally identified with elements of $SPL(V)$, while minimal elements are naturally identified with elements of $FL(V)$. Simplices in $\mathbb{FL}(V)$ are nonempty finite subsets of $FL(V)$ which have a common upper bound in $\mathcal{E}(V)$, and similarly simplices in $\mathbb{SPL}(V)$ are nonempty finite subsets of $SPL(V)$ which have a common lower bound in $\mathcal{E}(V)$.

We now show that $\mathbb{ET}(V) = B\mathcal{E}(V)$, the classifying space of the poset $\mathcal{E}(V)$, is another model of the homotopy type of $|\mathbb{FL}(V)|$.
In a similar fashion, we may define a poset $\mathcal{E}(A^\infty)$, and a space $\mathbb{ET}(A^\infty)$, giving another model of the homotopy type of $|\mathbb{FL}(A^\infty)|$.
We first have a lemma on classifying spaces of certain posets. For any poset $(P, \leq)$, and any $S \subset P$, let

$$L(S) = \{x \in P |\ x \leq s \ \forall\ s \in S\},\ \ U(S) = \{x \in P |\ s \leq x \ \forall\ s \in S\}$$

be the upper and lower sets of S in P, respectively. Let $\mathcal{P}_{min}$ denote the simplicial complex with vertex set P_{min} given by minimal elements of P, and where a nonempty finite subset $S \subset P_{min}$ is a simplex if $U(S) \neq \emptyset$. Let $|\mathcal{P}_{min}|$ denote the geometric realisation of $\mathcal{P}_{min}$.

Lemma 10. *Let $(P, \leq)$ be a poset such that*
(a) $\forall\ \ s \in P$, the set $L(\{s\})$ is finite
(b) if $\emptyset \neq S \subset P$ with $L(S) \neq \emptyset$, then the classifying space $BL(S)$ of $L(S)$ (as a subposet) is contractible.

Then $|\mathcal{P}_{min}|$ is naturally homotopy equivalent to BP.

Proof. We apply Proposition 1. Take

$$\Lambda = \{L(S) |\ \emptyset \neq S \subset P \text{ and } L(S) \neq \emptyset\}.$$

This is a poset with respect to inclusion. All $\lambda \in \Lambda$ are finite subsets of P, so Λ is Artinian. By assumption, the subsets $B(\lambda) \subset BP$, for $\lambda \in \Lambda$, are contractible. On the other hand, the sets $\lambda \cap P_{min}$ give simplices in $|\mathcal{P}_{min}|$. Thus, both the spaces BP and $|\mathcal{P}_{min}|$ have Λ-admissible coverings, and are thus homotopy equivalent. □

Remark. If a poset P has g.c.d. in the sense that $\emptyset \neq S \subset P$ and $\emptyset \neq L(S)$ implies $L(S) = L(t)$ for some $t \in P$, then condition (b) of the lemma is immediately satisfied. However $\mathcal{E}(V)$ does not enjoy the latter property.
For example, if $V = A^3$ with basis e_1, e_2, e_3, let $s = \{Ae_1, Ae_2, Ae_3\}$ and $t = \{Ae_1, A(e_1 + e_2), Ae_3\}$ and let $S = \{s, t\} \subset SPL(V) \subset \mathcal{E}(V)$. Then $L(S)$ has three minimal elements and two maximal elements. In particular, g.c.d. (s, t) does not exist. In this example, $B(L(S))$ is an oriented graph in the shape of the letter M.

Proposition 11. *If V is a free A module of finite rank, the poset $\mathcal{E}(V)$ satisfies the hypotheses of lemma 10. Thus, $|\mathbb{FL}(V)|$ is naturally homotopy equivalent to $\mathbb{ET}(V) = B(\mathcal{E}(V))$.*

Proof. Clearly the condition (a) of lemma 10 holds, so it suffices to prove (b). We now make a series of observations.

(i) Regard $SPL(V)$ as the set of maximal elements of the poset $\mathcal{E}(V)$. We observe that for any $s \in \mathcal{E}(V)$, if $H(s) = SLP(V) \cap U(\{s\})$, then we have that $L(\{s\}) = L(H(s))$. This is easy to see, once one has unravelled the definitions.

Thus, it suffices to show that for sets S of the type $\emptyset \neq S \subset SPL(V) \subset \mathcal{E}(V)$, we have that $B(L(S)$ is contractible. We assume henceforth that $S \subset SPL(V)$.

(ii) Given a submodule $W \subset V$ which determines a partial flag $0 \subset W \subset V$, we have a natural inclusion of posets

$$\mathcal{E}(W) \times \mathcal{E}(V/W) \subset \mathcal{E}(V),$$

where on the product, we take the partial order

$$(a_1, a_2) \leq (b_1, b_2) \Leftrightarrow a_1 \leq b_1 \in \mathcal{E}(W) \text{ and } a_2 \leq b_2 \in \mathcal{E}(V/W).$$

Note that $\alpha \in \mathcal{E}(V)$ lies in the sub-poset $\mathcal{E}(W) \times \mathcal{E}(V/W)$ precisely when W is one of the terms in the partial flag associated to α. Hence, if α lies in the sub-poset, so does the entire set $L(\{\alpha\})$.

(iii) With notation as above, if $\emptyset \neq S \subset SPL(V) \subset \mathcal{E}(V)$, and $L(S)$ has nonempty intersection with the image of $\mathcal{E}(W) \times \mathcal{E}(V/W) \subset \mathcal{E}(V)$, then clearly there exist nonempty subsets $S'(W) \subset SPL(W)$, $S''(W) \subset \mathbb{SPL}(V/W)$ such that

$$L(S) \cap (\mathcal{E}(W) \times \mathcal{E}(V/W)) = L(S'(W)) \times L(S''(W)) \tag{4}$$

(iv) If $\emptyset \neq S \subset SPL(V)$, then each $s \in S$ is a subset of

$$\mathcal{L}(V) = \{L \subset V | L \text{ is a free direct summand of rank 1 of } V\},$$

the set of lines in V. Let

$$T(S) = \cap_{s \in S} s = \text{ lines common to all members of } S,$$

so that $T(S) \subset \mathcal{L}(V)$. Let $M(S)$ denote the direct sum of the elements of $T(S)$, so that $M(S)$ is a free A-module of finite rank, and $0 \subset M(S) \subset V$ is a partial flag, in the sense explained earlier; further, $T(S)$ may be regarded also as an element of $SPL(M(S)) \subset \mathcal{E}(M(S))$.

(v) We now claim the following: if $\emptyset \neq S \subset SPL(V)$ and $b \in L(S)$, then there exists a unique subset $f(b) \subset T(S)$ such that if $M(b)$ is the (direct) sum of the lines in $f(b)$, then

$$b = (f(b), b') \in SPL(M(b)) \times \mathcal{E}(V/M(b)) \subset \mathcal{E}(M(b)) \times \mathcal{E}(V/M(b)) \subset \mathcal{E}(V).$$

Indeed, if

$$b = ((0 = W_0 \subset W_1 \subset \cdots \subset W_h = V), (t_1, t_2, \ldots, t_h)\}$$

where $t_i \in SPL(W_i/W_{i-1})$, then since $b \in L(S)$, we must have that $t_1 \subset s$ for all $s \in S$, which implies that $t_1 \subset T(S)$. Take $M(b) = W_1$, $t_1 = f(b) \in SPL(M(b))$.

Let $\mathcal{P}(T(S))$ be the poset of nonempty subsets of $T(S)$, with respect to inclusion. Then $b \mapsto f(b)$ gives an order-preserving map $f : L(S) \to \mathcal{P}(T(S))$.

(vi) For any $b \in L(S)$, we have

$$L(b) = L(\{f(b)\}) \times L(b') \subset \mathcal{E}(M(b)) \times \mathcal{E}(V/M(b)),$$

so that if $b_1 \in L(b)$, then $\emptyset \neq f(b_1) \subset f(b)$.

We will now complete the proof of Proposition 11. We proceed by induction on the rank of V. Suppose $S \subset SPL(V)$ is nonempty, and $L(S) \neq \emptyset$.
If $M(S) = V$, then $S = \{s\}$ for some s, and $L(S) = L(\{s\})$ is a cone, hence contractible. So assume $M(S) \neq V$.
If $T \subset T(S)$ is non-empty, and $M(T) \subset V$ the (direct) sum of the lines in T, then in the notation of (4) above, with $W = M(T)$, we have $S'(W) = \{T\}$, and so $f^{-1}(T) = \{T\} \times L(S''(W)))$ for some $S''(W) \subset SPL(V/W)$.
Now by induction, we have that $L(S''(W))$ is contractible, provided it is non-empty. Hence the non-empty fiber posets of f are contractible. If $\emptyset \neq T \subset T' \subset T(S)$, then there is a morphism of posets $f^{-1}(T) \to f^{-1}(T')$ given as follows: if $b \in f^{-1}(T)$, and

$$b = ((0 = W_0 \subset W_1 \subset \cdots \subset W_h = V), (t_1, t_2, \ldots, t_h)\}$$

where $t_i \in SPL(W_i/W_{i-1})$, then since $b \in f^{-1}(T)$, we must have $t_1 = T$, $W_1 = M(T)$. Now define $b' \in f^{-1}(T')$ using the partial flag

$$0 = W' \subset W_1 + M(T') \subset W_2 + M(T') \subset \cdots \subset W_h + M(T') = V$$

and elements $t'_i \in SPL(W_i + M(T')/W_{i-1} + M(T'))$ induced by the t_i. This is easily seen to be well-defined, and gives a morphism of posets $f^{-1}(T) \to f^{-1}(T')$. In particular, if $T \subset T' \subset T(S)$ and $f^{-1}(T)$ is non-empty, then so is $f^{-1}(T')$.
Now take any $b \in L(S)$ and put $T = f(b), T' = T(S)$ in the above to deduce that $f^{-1}(T(S)) \neq \emptyset$. By (iii) above, we see that every $f^{-1}X$ is nonempty (and therefore contractible as well) for every nonempty $X \subset T(S)$.
We see that all the fiber posets $f^{-1}(T)$ considered above are nonempty.
This makes f pre-cofibered, in the sense of Quillen (see [19], page 96), with contractible fibers. Hence by Quillen's Theorem A, f induces a homotopy equivalence on classifying spaces. But $\mathcal{P}(T(S))$ is contractible (for example, since $T(S)$ is the unique maximal element). $\square$

Remark. Proposition 5 and the remarks preceding it apply to the above Proposition. In particular, we obtain homotopy equivalences $f : \mathbb{ET}(V) \to \mathbb{FL}(V)$ so that the induced maps on homology are $GL(V)$-equivariant.

Remark. We now define the poset $\mathcal{E}(A^\infty)$ and show that $\mathbb{ET}(A^\infty) = B\mathcal{E}(A^\infty)$ is homotopy equivalent to $|\mathbb{FL}(A^\infty|$.
We have already observed that a short exact sequence of free modules of finite rank

$$0 \to V' \to V \to V'' \to 0$$

induces a natural inclusion $\mathcal{E}(V') \times \mathcal{E}(V'') \hookrightarrow \mathcal{E}(V)$ of posets. In particular, when $V'' \cong A$, this yields an inclusion $\mathcal{E}(V') \hookrightarrow \mathcal{E}(V)$.
We have ... $\subset A^n \subset A^{n+1} \subset ... \subset A^\infty$ as in the definition of $\mathbb{FL}(A^\infty)$. From the above, we obtain a direct system of posets

$$...\mathcal{E}(A^n) \hookrightarrow \mathcal{E}(A^{n+1}) \hookrightarrow ...$$

and we define $\mathcal{E}(A^\infty)$ to be the direct limit of this system of posets.

We put $P = \mathcal{E}(A^\infty)$ in lemma 10. We note that $\alpha \le \beta, \alpha \in \mathcal{E}(A^\infty), \beta \in \mathcal{E}(A^n)$ implies that $\alpha \in \mathcal{E}(A^n)$. It follows that $P = \mathcal{E}(A^\infty)$ satisfies the requirements of the lemma because each $\mathcal{E}(A^n)$ does. It is clear that $P_{min} = FL(A^\infty)$, and furthermore that $\mathcal{P}_{min} = \mathbb{FL}(A^\infty)$. This yields the homotopy equivalence of $|\mathbb{FL}(A^\infty)|$ with $\mathbb{ET}(A^\infty)$.
By Proposition 5, it follows that $\mathbb{ET}(A^\infty)//GL(A)$ and $|\mathbb{FL}(A^\infty)|//GL(A)$ are also homotopy equivalent to each other.
It has already been remarked that Corollary 7 gives the homotopy equivalence of $|\mathbb{SPL}(A^\infty|$ with $|\mathbb{FL}(A^\infty)|$. Combined with Proposition 5, this gives the homotopy equivalence of $|\mathbb{SPL}(A^\infty|//GL(A)$ with $|\mathbb{FL}(A^\infty)|//GL(A)$. The remarks preceding that proposition, combined with corollary 9, show that the action of $E(A)$ on the homology groups of $|\mathbb{SPL}(A^\infty)|$ is trivial.

3. homology of the Borel constrcuction

Let V be a free A-module of rank n. Fix $\beta \in SPL(V)$ and let $N(\beta) \subset GL(V)$ be the stabiliser of β (when $V = A^n$ and β is the standard splitting, then $N(\beta)$ is the subgroup $N_n(A)$ of the last section). That there is a $GL(V)$-equivariant $N(\beta)$-torsor on $|\mathbb{FL}(V)|$ has been observed in the previous section. In a similar manner, one may construct a $GL(V)$-equivariant $N(\beta)$-torsor on $\mathbb{ET}(V)$. This gives rise to a $N(\beta)$-torsor on $\mathbb{ET}(V)//GL(V)$. Because $BN(\beta)$ is a classifying space for such torsors, we obtain a map $\mathbb{ET}(V)//GL(V) \to BN(\beta)$, well defined up to homotopy.
On the other hand, the inclusion of β in $\mathbb{ET}(V)$ gives rise to an inclusion $BN(\beta) = \{\beta\}//N(\beta) \hookrightarrow \mathbb{ET}(V)//GL(V)$. It is clear that the composite $BN(\beta) \hookrightarrow \mathbb{ET}(V)//GL(V) \to BN(\beta)$ is homotopic to the identity. Thus $BN(\beta)$ is a homotopy retract of $\mathbb{ET}(V)//GL(V)$, but not homotopy equivalent to $\mathbb{ET}(V)//GL(V)$. Nevertheless we have the following statement:

Proposition 12. *The map $BN(\beta) \to \mathbb{ET}(V)//GL(V)$ induces an isomorphism on integral homology, provided A is as in theorem 1.*

Proof. Fix a basis for V, identifying $GL(V)$ with $GL_n(A)$. Let $\beta \in SPL(V)$ be the element naturally determined by this basis. Regarded as a vertex of $\mathbb{ET}(V)$, let $(\beta, *) \mapsto \overline{\beta}$ under the natural map

$$\pi : \mathbb{ET}(V)//GL(V) \to \mathbb{ET}(V)/GL(V)$$

from the homotopy quotient to the geometric quotient, where $* \in EGL(V)$ is the base point (corresponding to the vertex labelled by the identity element of $GL(V)$).
For any $x \in \mathbb{ET}(V)$, let $\mathcal{H}(x) \subset GL(V)$ be the isotropy group of x for the $GL(V)$-action on $\mathbb{ET}(V)$. Note that since

$$\mathbb{ET}(V)//GL(V) = (\mathbb{ET}(V) \times EGL(V))\,/GL(V),$$

the fiber $\pi^{-1}(\pi((x,*))$ may be identified with $EGL(V)/\mathcal{H}(x)$, which has the homotopy type of $B\mathcal{H}(x)$.

In particular, the fiber $\pi^{-1}(\overline{\beta})$ has the homotopy type of $BN_n(A)$, Further, the principal $N_n(A)$ bundle on $EGL(V)/\mathcal{H}(\beta)$ is naturally identified with the universal $N_n(A)$-bundle on $BN_n(A)$ – its pullback to $\{\beta\}\times EGL(V)$ is the trivial $N_n(A)$-bundle, regarded as an $N_n(A)$-equivariant principal bundle, where $N_n(A)$ acts on itself (the fiber of the trivial bundle) by translation. This means that the composite

$$\pi^{-1}(\overline{\beta}) \to \mathbb{ET}(V)/GL(V) \to BN_n(A)$$

is a homotopy equivalence, which is homotopic to the identity, if we identify $EGL(V)/\mathcal{H}(\beta)$ with $BN_n(A)$.
Thus, the lemma amounts to the assertion that $\pi^{-1}(\overline{\beta}) \to \mathbb{ET}(V)//GL(V)$ *induces an isomorphism in integral homology.*
Fix $\alpha \in FL(V)$ with $\alpha \leq \beta$ in the poset $\mathcal{E}(V)$. Let

$$P = \{\lambda \in \mathcal{E}(V) | \alpha \leq \lambda \leq \beta\}.$$

One sees easily that (i) BP is contractible, and (ii) the map $BP \to \mathbb{ET}(V)/GL(V)$ is a homoemorphism. The first assertion is obvious, since P has a maximal (as well as a minimal) element, so that BP is a cone. For the second assertion, we first note that an element $b \in P \subset \mathcal{E}(V)$ is uniquely determined by the ranks of the modules in the partial flag in V associated to b. Conversely, given any increasing sequence of numbers $n_1 < \ldots < n_h = \operatorname{rank} V$, there does exist an element of P whose partial flag module ranks are these integers. Given any element $b \in \mathcal{E}(V)$, there exists an element $g \in GL(V)$ so that $g(b) = b' \in P$; the element b' is the unique one determined by the sequence of ranks associated to b. Finally, one observes that if $b \in P$, and $g \in GL(V)$ such that $g(b) \in P$, then in fact $g(b) = b$: this is a consequence of the uniqueness of the element of P with a given sequence of ranks. These observations imply that $BP \to \mathbb{ET}(V)/GL(V)$ is bijective; it is now easy to see that it is a homeomorphism.
We may view $\mathbb{ET}(V)//GL(V)$ as the quotient of $BP \times EGL(V)$ by the equivalence relation

$$(x,y) \sim (x',y') \Leftrightarrow x = x', \text{ and } y' = g(y) \text{ for some } g \in \mathcal{H}(x). \tag{5}$$

The earlier map $\pi : \mathbb{ET}(V)//GL(V) \to \mathbb{ET}(V)/GL(V)$ may be viewed now as the map induced by the projection $BP \times EGL(V) \to BP$. We may, with this identification, also identify $\overline{\beta}$ with β.
Next, we construct a "good" fundamental system of open neighbourhoods of an arbitrary point $x \in BP$, which we need below. Such a point x lies in the relative interior of a unique simplex $\sigma(x)$ (called the *carrier* of x) corresponding to a chain $\lambda_0 < \lambda_1 < \cdots < \lambda_r$. Then one sees that the stabiliser $\mathcal{H}(x) \subset GL(V)$ is given by

$$\mathcal{H}(x) = \bigcap_{i=0}^{r} \mathcal{H}(\lambda_i),$$

since any element of $GL(V)$ which stabilizes the simplex $\sigma(x)$ must stabilize each of the vertices (for example, since the $GL(V)$ action preserves the partial order).

Let $\mathrm{star}\,(x)$ be the union of the relative interiors of all simplices in BP containing $\sigma(x)$ (this includes the relative interior of $\sigma(x)$ as well, so it contains x). It is a standard property of simplicial complexes that $\mathrm{star}\,(x)$ is an open neighbourhood of x in BP. Then if $z \in \mathrm{star}\,(x)$, clearly $\sigma(z)$ contains $\sigma(x)$, and so $\mathcal{H}(z) \subset \mathcal{H}(x)$.

Next, for such a point z, and any $y \in EGL(V)$, it makes sense to consider the path

$$t \mapsto (tz + (1-t)x, y) \in \sigma(z) \times EGL(V) \subset BP \times EGL(V)$$

(where we view the expression $tz+(1-t)x$ as a point of $\sigma(z)$, using the standard barycentric coordinates). In fact this path is contained in $\mathrm{star}\,(x) \times \{y\}$, and gives a continuous map

$$H(x) : \mathrm{star}\,(x) \times EGL(V) \times I \to \mathrm{star}\,(x) \times EGL(V)$$

which exhibits $\{x\} \times EGL(V)$ as a strong deformation retract of $\mathrm{star}\,(x) \times EGL(V)$. Further, this is compatible with the equivalence relation $\sim$ in (5) above, so that we obtain a strong deformation retraction

$$\overline{H(x)} : \pi^{-1}(\mathrm{star}\,(x)) \times I \to \pi^{-1}(\mathrm{star}\,(x)).$$

In a similar fashion, we can construct a fundamental sequence of open neighbourhoods $U_n(x)$ of x in BP, with $U_1(x) = \mathrm{star}\,(x)$, and set

$$U_n(x) = H(x)(\mathrm{star}\,(x) \times EGL(V) \times [0, 1/n)).$$

The same deformation retraction H determines, by reparametrization, a deformation retraction

$$H_n(x) : \pi^{-1}(U_n(x)) \times I \to \pi^{-1}(U_n(x))$$

of $\pi^{-1}(U_n(x)$ onto $\pi^{-1}(x)$.

Thus, if $P' = P \setminus \beta$, then

$$\pi^{-1}(\mathrm{star}\,(\beta)) = \mathbb{ET}(V)//GL(V) \setminus \pi^{-1}(BP'),$$

and from what we have just shown above, the inclusion

$$\pi^{-1}(\beta) \to \pi^{-1}(\mathrm{star}\,(\beta)) = \mathbb{ET}(V)//GL(V) \setminus \pi^{-1}(BP')$$

is a homotopy equivalence. To simplify notation, we let $X = \mathbb{ET}(V)//GL(V)$, so that we have the map $\pi : X \to BP$, and $X^0 = X \setminus \pi^{-1}(BP')$. Let $\pi^0 = \pi\,|_{X^0} : X^0 \to BP$.

We are reduced to showing, with this notation, that the inclusion of the (dense) open subset

$$X^0 \to X$$

induces an isomorphism in integral homology. Equivalently, it suffices to show that this inclusion induces an isomorphism on cohomology with arbitrary constant coefficients M. By the Leray spectral sequence, this is a consequence of showing that the maps of sheaves

$$R^i\pi_*M_X \to R^i\pi^0_*M_{X^0}$$

is an isomorphism, which is clear on stalks $x \in BP \setminus BP'$. Now consider stalks at a point $x \in BP'$. For any point $x' \in \mathrm{star}\,(x)$, note that x lies in some face of $\sigma(x')$ (the carrier of x'). We had defined a fundamental system of neighbourhoods $U_n(x)$ of x in BP; explicitly we have

$$U_n(x) = \{tx' + (1-t)x | 0 \leq t < 1/n \text{ and } x' \in \mathrm{star}\,(x)\}.$$

Here, as before, we make sense of the above expression $tx' + (1-t)x$ using barycentric coordinates in $\sigma(x')$.
Define

$$z_n(x) = \frac{1}{2n}\beta + (1 - \frac{1}{2n})x.$$

Note that $z \in BP \setminus BP' = \mathrm{star}\,(\beta)$. Further, observe that $U_n(x) \cap BP \setminus BP'$ is contractible, contains the point z, and for any $w \in U_n(x) \cap BP \setminus BP'$, contains the line segment joining z and w (this makes sense, in terms of barycentric coordinates of any simplex containing both $z_n(x)$ and w; this simplex is either the carrier of w, or the cone over it with vertex β, of which $\sigma(w)$ is a face).
This implies $\mathcal{H}(w) \subset \mathcal{H}(z_n(x)) = \mathcal{H}(x) \cap \mathcal{H}(\beta)$, for all $w \in U_n(x)$. A minor modification of the proof (indicated above) that $\pi^{-1}(x) \subset \pi^{-1}(U_n(x))$ is a strong deforamtion retract, yields the statement that

$$\pi^{-1}(z_n(x)) \to \pi^{-1}(U_n(x) \setminus BP')$$

is a strong deformation retract. Hence, the desired isomorphism on stalks follows from:

(6) $B(\mathcal{H}(x) \cap \mathcal{H}(\beta)) \to B(\mathcal{H}(x))$ induces isomorphisms in integral homology.

We now show how this statement, for the appropriate rings A, is reduced to results of [13].
First, we discuss the structure of the isotropy groups $\mathcal{H}(x)$ encountered above.
Let $\lambda \in P$, given by

$$\lambda = (F, S) = ((0 = F_0 \subset F_1 \subset \cdots \subset F_r = V), (S_1, S_2, \ldots, S_r)),$$

where we also have $\alpha \leq \lambda \leq \beta$ for our chosen elements $\alpha \in FL(V)$ and $\beta \in SPL(V)$. We may choose a basis for each of the lines in the splitting β; then $\alpha \in FL(V)$ uniquely determines an order among these basis elements, and thus a basis for the underlying free A-module V, such that the i-th submodule in the full flag α is the submodule generated by the first i elements in β. Now the stabilizer $\mathcal{H}(\alpha)$ may be viewed as the group of upper triangular matrices in $GL_n(A)$, while $\mathcal{H}(\beta)$ is the group generated by the diagonal subgroup in $GL_n(A)$ and the group of permutation matrices, identified with the permutation group S_n.
In these terms, $\mathcal{H}(\lambda)$ has the following structure. The filtration $F = (0 = F_0 \subset F_1 \subset \cdots \subset F_r = V)$ is a sub-filtration of the full flag α, and so determines a "unipotent subgroup" $U(\lambda)$ of elements fixing the elements of this partial flag,

and acting trivially on the graded quotients F_i/F_{i-1}. These are represented as matrices of the form

$$\begin{bmatrix} I_{n_1} & * & * & \cdots & * \\ 0 & I_{n_2} & * & \cdots & * \\ 0 & 0 & I_{n_3} & \cdots & * \\ & \vdots & \ddots & & \vdots \\ 0 & \cdots & & & I_{n_r} \end{bmatrix}$$

where $n_i = \mathrm{rank}\,(W_i/W_{i-1})$, I_{n_i} is the identity matrix of size n_i; these are the matrices which are strictly upper triangular with respect to a certain "ladder". Next, we may consider the group $S(\lambda) \subset S_n$ of permutation matrices, supported within the corresponding diagonal blocks, of the form

$$\begin{bmatrix} A_1 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ 0 & 0 & A_3 & \cdots & 0 \\ & \vdots & \ddots & & \vdots \\ 0 & \cdots & & & A_r \end{bmatrix}$$

where each A_j is a permutation matrix. Finally, we have the diagonal matrices $T_n(A) \subset GL_n(A)$, which are contained in $\mathcal{H}(\lambda)$ for any such λ. In fact $\mathcal{H}(\lambda) = U(\lambda)T_n(A)S(\lambda)$, where the group $T_n(A)S(\lambda)$ normalizes the subgroup $U(\lambda)$, making $\mathcal{H}(\lambda)$ a semidirect product of $U(\lambda)$ and $T_n(A)S(\lambda)$. We also have that $S(\lambda)$ normalizes $U(\lambda)T_n(A)$.

In particular, $\mathcal{H}(\alpha)$ has trivial associated permutation group $S(\alpha) = \{I_n\}$, while $\mathcal{H}(\beta)$ has trivial unipotent group $U(\beta) = \{I_n\}$ associated to it.

Now if $x \in BP$, and $\sigma(x)$ is the simplex associated to the chain $\lambda_0 < \cdots < \lambda_r$ in the poset P, then it is easy to see that $\mathcal{H}(x)$ is the semidirect product of $U(x) := U(\lambda_r)$ and $T_n(A)S(x)$, with $S(x) := S(\lambda_0)$, since as seen earlier, $\mathcal{H}(x)$ is the intersection of the $\mathcal{H}(\lambda_i)$. In other words, the "unipotent part" and the "permutation group" associated to $\mathcal{H}(x)$ are each the smallest possible ones from among the corresponding groups attached to the vertices of the carrier of x. Again we have that $S(x)$ normalizes $U(x)T_n(A)$.

We return now to the situation in (6). We see that the groups $\mathcal{H}(x) = U(x)T_n(A)S(x)$ and $\mathcal{H}(x) \cap \mathcal{H}(\beta) = T_n(A)S(x)$ both have the same associated permutation group $S(x)$, which normalizes $U(x)T_n(A)$ as well as $T_n(A)$. By comparing the spectral sequences

$$E^2_{p,q} = H_p(S(x), H_q(U(x)T_n(A), \mathbb{Z})) \Rightarrow H_{p+q}(\mathcal{H}(x), \mathbb{Z}),$$

$$E^2_{p,q} = H_p(S(x), H_q(T_n(A), \mathbb{Z})) \Rightarrow H_{p+q}(\mathcal{H}(x) \cap \mathcal{H}(\beta), \mathbb{Z})$$

we see that it thus suffices to show that the inclusion

$$T_n(A) \subset U(x)T_n(A) \tag{7}$$

induces an isomorphism on integral homology.

Now lemma 13 below finishes the proof. □

To state lemma 13 we use the following notation. Let $I = \{i_0 = 0 < i_1 < i_2 < \cdots < i_r = n\}$ be a subsequence of $\{0, 1, \ldots, n\}$, so that I determines a partial flag

$$0 \subset A^{i_1} \subset A^{i_2} \subset \cdots \subset A^{i_r} = A^n,$$

where $A^j \subset A^n$ as the submodule generated by the first j basis vectors. Let $U(I)$ be the "unipotent" subgroup of $GL_n(A)$ stabilising this flag, and acting trivially on the associated graded A-module, and let $G(I) \subset GL_n(A)$ be the subgroup generated by $U(I)$ and $T_n(A) = (A^\times)^n$, the subgroup of diagonal matrices. Then $T_n(A)$ normalises $U(I)$, and $G(I)$ is the semidirect product of $U(I)$ and $T_n(A)$.

Lemma 13. *Let A be a Nesterenko-Suslin ring. For any I as above, the homomorphism $G(I) \to G(I)/U(I) \cong T_n(A)$ induces an isomorphism on integral homology $H_*(G(I), \mathbb{Z}) \to H_*(T_n(A), \mathbb{Z})$.*

Proof. We work by induction on n, where there is nothing to prove when $n = 1$, since we must have $G(I) = T_1(A) = A^\times = GL_1(A)$. Next, if $n > 1$, and $I = \{0 < n\}$, then $U(I)$ is the trivial group, so there is nothing to prove.
Hence we may assume $n > 1$, $r \geq 2$, and thus $0 < i_1 < n$. There is then a natural homomorphism $G(I) \to G(I')$, where $I' = \{0 < i_2 - i_1 < \cdots < i_r - i_1 = n - i_1\}$, and $G(I') \subset GL_{n-i_1}(A)$. Let $n' = n - i_1$. The induced homomorphism $T_n(A) \to T_{n'}(A)$ is naturally split, with kernel $T_{i_1}(A) \subset GL_{i_1}(A) \subset GL_n(A)$.
Let

$$U_1(I) = \ker\left(U(I) \to U(I')\right) = \ker\left(G(I) \to GL_{i_1}(A) \times GL_{n'}(A)\right).$$

Then $U_1(I)$ is a normal subgroup of $G(I)$, from the last description, and

$$G(I)/U_1(I) \cong U(I') \cdot T_n(A) = T_{i_1}(A) \times G(I').$$

Now $U_1(I)$ may be identified with $M_{i_1,n'}(A)$, the additive group of matrices of size $i_1 \times n'$ over A; this matrix group has a natural action of $GL_{i_1}(A)$, and thus of the diagonal matrix group $T_{i_1}(A)$, and the resulting semidirect product of $T_{i_1}(A)$ with $U_1(A)$ is a subgroup of $G(I)$ (in fact, it is the kernel of $G(I) \to G(I')$). This matrix group $M_{i_1,n'}(A)$ is isomorphic, as $T_{i_1}(A)$-modules, to the direct sum

$$\oplus_{i=1}^{i_1} A^{n'}(i),$$

where $A^{n'}(i)$ is the free A-module of rank n', with a $T_{i_1}(A)$-action given by the the "i-th diagonal entry" character $T_{i_1}(A) \to A^\times$. Thus, the semidirect product $T_{i_1}(A)U_1(I)$ has a description as a direct product

$$T_{i_1}(A)U_1(i) \cong H \times H \times \cdots \times H = H^{n_i}$$

with $H = A^{n'} \cdot A^\times$ equal to the naturally defined semidirect product of the free A module $A^{n'}$ with $A^\times$, where $A^\times$ operates by scalar multiplication.
Proposition 1.10 and Remark 1.13 in the paper [13] of Nesterenko and Suslin implies immediately that $H \to H/A^{n'} \cong A^\times$ induces an isomorphism on integral homology.
We now use the following facts.

(i) If $H \subset K \subset G$ are groups, with H, K normal in G, and if $K \to K/H$ induces an isomorphism in integral homology, so does $G \to G/H$; this follows at once from a comparison of the two spectral sequences

$$E^2_{r,s} = H_r(G/K, H_s(K, \mathbb{Z})) \Rightarrow H_{r+s}(G, \mathbb{Z}),$$

$$E^2_{r,s} = H_r(G/K, H_s(K/H, \mathbb{Z})) \Rightarrow H_{r+s}(G/H, \mathbb{Z}).$$

(ii) If $H_i \subset G_i$ are normal subgroups, for $i = 1, \ldots, n$, such that $G_i \to G_i/H_i$ induce isomorphisms on integral homology, then for $G = \prod_{i=1}^n G_i$, $H = \prod_{i=1}^n H_i$, the map $G \to G/H$ induces an isomorphism on integral homology. This follows from the Kunneth formula.

The fact (ii) implies that $T_{i_1}(A)U_1(I) \to T_{i_1}(A)$ induces an isomorphism on integral homology. Then (i) implies that $G(I) \to T_{i_1}(A) \times G(I')$ induces an isomorphism on integral homology. By induction, we have that $G(I') \to G(I')/U(I')$ induces an isomorphism on integral homology. Hence $T_{i_1}(A) \times G(I') \to T_{i_1} \times G(I')/U(I')$ also induces an isomorphism on integral homology. Thus, we have shown that the composition $G(I) \to G(I)/U(I) = T_n(A)$ induces an isomorphism on integral homology. $\square$

4. $\mathbb{SPL}(A^\infty)^+$ and the groups $L_n(A)$

We first note that there is a small variation of Quillen's plus construction.
Let (X, x) be a pointed CW complex, (X_0, x) a contractible pointed subcomplex, G a group of homeomorphisms of X which acts transitively on the path components of X, and let H be a perfect subgroup of G, such that H stabilizes X_0.
Then $X//G$ is clearly path connected, and comes equipped with
(i) a natural map $\theta : X//G \to BG = EG/G$, induced by the projection $X \times EG \to EG$
(ii) a map $(X_0 \times EG)/H \to X//G$, induced by the H-stable contractible set $X_0 \subset X$
(iii) a homotopy equivalence $BH \to (X_0 \times EG)/H$, such that the composition $BH \to X//G \xrightarrow{\theta} BG$ is homotopic to the natural map $BH \to BG$
(iv) a natural map $(X, x) \hookrightarrow (X//G, x_0)$ determined by the base point of EG.
Note that, in particular, there is a natural inclusion $H \hookrightarrow \pi_1(X//G, x_0)$, which gives a section over $H \subset G$ of the surjection $\theta_* : \pi_1(X//G, x_0) \to \pi_1(BG, *) = G$.

Lemma 14. *In the above situation, there is a pointed CW complex (Y, y), together with a map $f : (X//G, x_0) \to (Y, y)$ such that*
(i) the natural composite map

$$H \hookrightarrow \pi_1(X//G, x_0) \xrightarrow{f} \pi_1(Y, y)$$

is trivial
(ii) if $g : (X//G, x_0) \to (Z, z)$ such that H is in the kernel of

$$\pi_1(X//G, x_0) \to \pi_1(Z, z)$$

then g factors through f, uniquely upto a pointed homotopy
(iii) f induces isomorphisms on integral homology; more generally, if L is any local system on Y, the map on homology with coefficients $H_*(X//G, f^*L) \to H_*(Y, L)$ *is an isomorphism*
(iv) $h : (X, x) \to (X', x')$ *is a pointed map of such CW complexes with G-actions, such that h is G-equivariant, then there is a map* $(Y, y) \to (Y', y')$, *making* $(X, x) \mapsto (Y, y)$ *is functorial (on the category of pointed CW complexes with suitable G actions, and equivariant maps), and f yields a natural transformation of functors.*

The pair (Y, y) is obtained by applying Quillen's plus construction to $(X//G, x_0)$ with respect to the perfect normal subgroup $\widetilde{H}$ of $\pi_1(X//G, x_0)$ which is generated by H. Part (ii) of the lemma is in fact the universal property of the plus construction. As is well-known, this may be done in a functorial way. *We sometimes write* $(Y, y) = (X//G, x_0)^+$ *to denote the above relationship.*
In what follows, the pair (G, H) is invariably $(GL_n(A), A_n)$ for $5 \leq n \leq \infty$. Here A_n is the alternating group contained in $N_n(A)$. The normal subgroups of $GL_n(A)$ generated by A_n and $E_n(A)$ coincide with each other. It follows that if we take $X = X_0$ to be a point, the Y given by the above lemma is just the "original" $BGL_n(A)^+$.
Recall that there is a natural action of $GL(A)$ on the simplicial complex $\mathbb{SPL}(A^\infty)$, and hence on its geometric realization $|\mathbb{SPL}(A^\infty)|$. We apply lemma 14 with $G = GL(A)$, $H = A_\infty$ the infinite alternating group, $X = |\mathbb{SPL}(A^\infty)|$, and $X_0 = \{x_0\}$ is the vertex of X fixed by $N(A)$ and obtain the pointed space

$$(Y(A), y) = (|\mathbb{SPL}(A^\infty)|//G), x_0)^+.$$

Taking X' to be a singleton in (iii) of the above lemma, we get a canonical map

$$\varphi : (Y(A), y) \to (BGL(A)^+, *)$$

of pointed spaces.
Let $(\mathbb{SPL}(A^\infty)^+, z)$ denote the homotopy fibre of φ. We define

$$L_n(A) = \pi_n(\mathbb{SPL}(A^\infty)^+, z) \ \forall \ n \geq 0.$$

The homotopy sequence of the fibration $\mathbb{SPL}(A^\infty)^+ \to Y(A) \to BGL(A)^+$ combined with the path-connectedness of $Y(A)$ yields:

Corollary 15. *There is an exact sequence*

$$\cdots \to K_{n+1}(A) \to L_n(A) \to \pi_n(Y(A), y) \to K_n(A) \cdots$$
$$\cdots \to L_1(A) \to \pi_1(Y(A), y) \to K_1(A) \to L_0(A) \to 0$$

where $L_0(A)$ *is regarded as a pointed set.*

Lemma 16. *The natural map* $|\mathbb{SPL}(A^\infty)| \to \mathbb{SPL}(A^\infty)^+$ *induces an isomorphism on integral homology.*

Proof. We may identify the universal covering of $BGL(A)^+$ with $BE(A)^+$, where $BE(A)^+$ is the plus construction (see lemma 14) applied to $BE(A)$ with respect to the infinite alternating group (or, what is the same thing, with respect to $E(A)$ itself). Let $\widetilde{\varphi} : \widetilde{Y} \to BE(A)^+$ be the corresponding pullback map obtained from φ.
We first note that $\mathbb{SPL}(A^\infty)^+$ is also naturally identified with the homotopy fiber of $\widetilde{\varphi}$. There is then a homotopy pullback $\widehat{\varphi} : \widehat{Y} \to BE(A)$ of $\widetilde{\varphi}$ with respect to $BE(A) \to BE(A)^+$. Thus, our map $\mathbb{SPL}(A^\infty) \to \mathbb{SPL}(A^\infty)^+$ may be viewed as the natural map on fibers associated to a map

$$\mathbb{SPL}(A^\infty)//E(A) \to \widehat{Y} \tag{8}$$

of Serre fibrations over $BE(A)$.
From a Leray-Serre spectral sequence argument, we see that since (from lemma 14) $BE(A) \to BE(A)^+$ induces a isomorphism on integral homology, so does $\widehat{Y} \to \widetilde{Y}$. Since also $\mathbb{SPL}(A^\infty)//E(A) \to \widetilde{Y}$ is a homology isomorphism (from lemma 14 again), we see that $\mathbb{SPL}(A^\infty)//E(A) \to \widehat{Y}$ induces an isomorphism on integral homology.
Now we use that the map (8) is a map between two total spaces of Serre fibrations over a common base, inducing a homology isomorphism on these total spaces. We also know that the monodromy representation of $\pi_1(BE(A)) = E(A)$ on the homology of the fibers is trivial, in both cases: for $\widehat{Y}$ this is because it is a pullback from a Serre fibration over a simply connected base, while for $\mathbb{SPL}(A^\infty)$, this is one of the key properties we have already established (see the finishing sentence of section 2). The proof is now complete modulo the remark below, which is a straightforward consequence of the Leray-Serre spectral sequence of a fibration. $\square$

Remark. Let $p : E \to B$ and $p' : E' \to B$ be fibrations with fibers F and F' respectively over the base-point $b \in B$. Let $v : E \to E'$ be a map so that $p' \circ v = p$. Assume that B is path-connected. Then $E \to E'$ is a homology isomorphism implies $F \to F'$ is a homology isomorphism under the following additional assumption:
$M \neq 0$ implies $H_0(\pi_1(B, b), M) \neq 0$ for every $\pi_1(B, b)$-subquotient M of $H_i(F), H_j(F')$ for all i, j.

5. The H-space structure

Recall that $BGL(A)^+$ has an H-space structure in a standard way, obtained from the direct sum operation on free modules of finte rank; this was constructed in [6].
The aim of this section is to prove the proposition below.

Proposition 17. *The space $Y(A)$ has an H-space structure, such that $Y(A) \to BGL(A)^+$ is homotopic to an H-map, for the standard H-space structure on $BGL(A)^+$.*

We first remark that if V is a free A-module of finite rank, then $|\mathbb{SPL}(V)|//GL(V)$ is homeomorphic to the classifying space of the following category $\mathcal{SPL}(V)$: its objects are simplices in $\mathbb{SPL}(V)$ (thus, certain finite nonempty subsets of $SPL(V)$), and morphisms $\sigma \to \tau$ are defined to be elements $g \in GL(V)$ such that $g(\sigma) \subset \tau$, that is, such that $g(\sigma)$ is a face of the simplex τ of $\mathbb{SPL}(V)$.
Let $\mathcal{A}ut(V)$ be the category with a single object $*$, with morphisms given by elements of $GL(V)$, so that the classifying space $B\mathcal{A}ut(V)$ is the standard model for $BGL(V)$. There is a functor $F_V : \mathcal{SPL}(V) \to \mathcal{A}ut(V)$, mapping every obeject σ to $*$, and mappng an arrow $\sigma \to \tau$ in $\mathcal{SPL}(V)$ to the corresponding element $g \in GLV()$. The fiber $F_V^{-1}(*)$ is the poset of simplices of $\mathbb{SPL}(V)$, whose classifying space is thus homeomorphic to $|\mathbb{SPL}(V)|$.
It is fairly straightforward to verify that $B\mathcal{SPL}(V)$ is homeomorphic to $|\mathbb{SPL}(V)|//GL(V)$ (where we have used the classifying space of the translation category of $GL(V)$ as the model for the contractible space $E(GL(V))$). One way to think of this is to consider the category $\widetilde{\mathcal{SPL}}(V)$, whose objects are pairs (σ, h) with σ a simplex of $\mathbb{SPL}(V)$, and $h \in GL(V)$, with a unique morphism $(\sigma, h) \to (\tau, g)$ precisely when $g^{-1}h(\sigma) \subset \tau$. It is clear that by considering the full subcategories of objects of the form (σ, g), where $g \in GL(V)$ is a fixed element, each of which is naturally equivalent to the poset of simplices in $\mathbb{SPL}(V)$, that the classifying space of $\widetilde{\mathcal{SPL}}(V)$ is homeomorphic to $|\mathbb{SPL}(V)| \times E(GL(V)))$. Now it is a simple matter to see (e.g., use the criterion of Quillen, given in [19], lemma 6.1, page 89) that $B\widetilde{\mathcal{SPL}}(V) \to B\mathcal{SPL}(V)$, given by $(\sigma, h) \mapsto h^{-1}(\sigma)$, is a covering space which is a principal $GL(V)$-bundle, where the deck transformations are given by the natural action of $GL(V)$ on $|\mathbb{SPL}(V)| \times E(GL(V))$.
$\mathcal{L}(V)$ denotes the collection of A-submodules $L \subset V$ so that L is free of rank one and V/L is a free module. Now we note that if V', V'' are free A-modules of finite rank, we note that there is a natural inclusion $\mathcal{L}(V') \sqcup \mathcal{L}(V'') \hookrightarrow \mathcal{L}(V' \oplus V'')$. This in turn yields a natural map

$$\varphi_{V',V''} : SPL(V') \times SPL(V'') \to SPL(V' \oplus V''),$$

given by $\varphi_{V',V''}(s, t) = s \sqcup t$.
It follows easily from the definition of $\mathbb{SPL}$ that the above map on vertices induces a simplicial map

$$\Phi_{V',V''} : \mathbb{SPL}(V') \times \mathbb{SPL}(V'') \to \mathbb{SPL}(V' \oplus V'').$$

As explained in section 0, at the level of geometric realisations, this has two descriptions. The first description may be used to show that the counterpart of lemma 8(C) is valid for $\mathbb{SPL}$, namely the homotopy class of the inclusion

$$|\mathbb{SPL}(V')| \times |\mathbb{SPL}(V'')| \to |\mathbb{SPL}(V' \oplus V'')|$$

remains unaffected by composition with the action of $g \in \mathrm{Elem}(V' \hookrightarrow V' \oplus V'')$ on $|\mathbb{SPL}(V' \oplus V'')|$.
The second description however is more useful in this context. Let us abbreviate notation and denote the (partially ordered) set of simplices of $\mathbb{SPL}(V)$ simply

by $\mathcal{S}(V)$. The desired map $\mathcal{S}(V') \times \mathcal{S}(V'') \to \mathcal{S}(V' \oplus V'')$ is given simply by $(\sigma, \tau) \mapsto \varphi_{V',V''}(\sigma \times \tau)$. The resulting map $B(\mathcal{S}(V') \times \mathcal{S}(V'')) \to B\mathcal{S}(V' \oplus V'')$ is the second description of

$$|\mathbb{SPL}(V')| \times |\mathbb{SPL}(V'')| \to |\mathbb{SPL}(V' \oplus V'')|$$

for (a) $B\mathcal{C}' \times B\mathcal{C}'' \cong B(\mathcal{C}' \times \mathcal{C}'')$ and (b) $B\mathcal{S}(V)$ is simply the barycentric subdivision of $|\mathbb{SPL}(V)|$.

This latter description also allows us to go a step further and define the functor $\mathcal{SPL}(V') \times \mathcal{SPL}(V'') \to \mathcal{SPL}(V' \oplus V'')$, given on objects by $(\sigma, \tau) \mapsto \varphi_{V',V''}(\sigma \times \tau)$ as before; on morphisms, it is given by the natural map $\mathrm{GL}\,(V') \times \mathrm{GL}\,(V'') \to GL(V' \oplus V'')$. Hence on classifying spaces, it induces a product

$$|\mathbb{SPL}(V')|//GL(V') \times |\mathbb{SPL}(V'')|//GL(V'') \to |\mathbb{SPL}(V' \oplus V'')|//GL(V' \oplus V'')|.$$

This is clearly compatible with the product

$$BGL(V') \times BGL(V'') \to BGL(V' \oplus V'')$$

under the natural maps induced by the functors $\mathcal{SPL} \to \mathcal{A}ut$ for the three free modules.

One verifies that $\mathbf{SPL}(A) = \coprod_V \mathcal{SPL}(V)$, with respect to the bifunctor

$$+ : \mathbf{SPL}(A) \times \mathbf{SPL}(A) \to \mathbf{SPL}(A)$$

induced by direct sums on free modules, and the functors $\Phi_{V',V''}$, form a symmetric monoidal category.

An equivalent category, also denoted $\mathbf{SPL}(A)$ by abuse of notation, is that whose objects are pairs (V, σ), where V is a free A-module of finite rank, and $\sigma \in \mathbb{SPL}(V)$ a simplex, and where morphisms $(V, \sigma) \to (W, \tau)$ are isomorphisms $f : V \to W$ of A-modules such that $f(\sigma)$ is a face of τ.

For the purposes of stabilization, we slightly modify the above to consider the related maps

$$\varphi_{m,n} : SPL(A^m) \times SPL(A^n) \to SPL(A^\infty)$$

given by mapping the basis vector $e_i \in A^m$ in the first factor to the basis vector $e_{2i-1} \in A^\infty$, for each $1 \le i \le m$, and the basis vector $e_j \in A^n$ in the second factor to the basis vector $e_{2j} \in A^\infty$. A pair of splittings of A^m, A^n determine one for the free module spanned by the images of the two sets of basis vectors; now one extends this to a splitting of A^∞ by adjoining the remaining basis vectors of A^∞ (that is, adjoining those vectors not in the span of the earlier images). If our first two splittings are those given by the basis vectors, which correspond to the base points in $|\mathbb{SPL}(A^m)|$ and $|\mathbb{SPL}(A^n)|$, the resulting point in $SPL(A^\infty)$ is again the base point of $|\mathbb{SPL}(A)|$.

The corresponding functors

$$\Phi_{m,n} : \mathcal{SPL}(A^m) \times \mathcal{SPL}(A^n) \to \mathcal{SPL}(A^\infty)$$

are compatible with similar functors

$$\mathcal{A}ut(A^m) \times \mathcal{A}ut(A^n) \to \mathcal{A}ut(A^\infty)$$

which, on classifying spaces, yield the diagram of product maps, preserving base points,

$$\begin{array}{ccc} |\mathbb{SPL}(A^m)//GL_n(A)| \times |\mathbb{SPL}(A^n)|//GL_n(A) & \to & |SPL(A^\infty|//GL(A) \\ \downarrow & & \downarrow \\ BGL_m(A) \times BGL_n(A) & \to & BGL(A) \end{array}$$

where the bottom arrow is the one used in [6] to define the H-space structure on $BGL(A)^+$.

As we increase m, n, the corresponding diagrams are compatible with respect to the obvious stabilization maps $|\mathbb{SPL}(A^m)| \hookrightarrow |\mathbb{SPL}(A^{m+1})|$, $|\mathbb{SPL}(A^n) \hookrightarrow |\mathbb{SPL}(A^{n+1})|$. Hence we obtain on the direct limits a diagram

$$\begin{array}{ccc} |\mathbb{SPL}(A^\infty)//GL(A)| \times |\mathbb{SPL}(A^\infty)|//GL(A) & \to & |SPL(A^\infty|//GL(A) \\ \downarrow & & \downarrow \\ BGL(A) \times BGL(A) & \to & BGL(A) \end{array}$$

From lemma 14, it follows that there is an induced diagram at the level of plus constructions

$$\begin{array}{ccc} Y(A) \times Y(A) & \to & Y(A) \\ \downarrow & & \downarrow \\ BGL(A)^+ \times BGL(A)^+ & \to & BGL(A)^+ \end{array}$$

Here we have, as remarked above, taken the homotopy equivalent model $|\mathbb{SPL}(A^\infty|//GL(A)$ for the homotopy type earlier denoted $Y(A)$. We abuse notation and use the same symbol to denote this model as well.

It is shown in [6] that the botton arrow defines an H-space structure on $BGL(A)$. We claim that, by analogous arguments, the top arrow also defines an H-space structure on $Y(A)$. Granting this, the map $Y(A) \to BGL(A)^+$ is then an H-map between path connected H-spaces, and so the homotopy fiber $Z(A)$ has the homotopy type of an H-group as well (and this was what we set out to prove here).

To show that the product $Y(A) \times Y(A) \to Y(A)$ defines an H-space structure, we need to show that left or right translation on $Y(A)$ (with respect to this product) by the base point is homotopic to the identity. This is also the main point in [6], for the case of $BGL(A)^+$. We first show:

Lemma 18. *An arbitrary inclusion $j : \{1, 2, \ldots, n\} \hookrightarrow \mathbb{N}$ determines an inclusion of A-modules $A^n \to A^\infty$, given on basis vectors by $e_i \mapsto e_{j(i)}$, which induces a map*

$$|\mathbb{SPL}(A^n)|//GL_n(A) \to Y(A)$$

which is homotopic (preserving the base point) to the map induced by standard inclusion $i_n : A^n \to A^\infty$.

Proof. We can find an automorphism g of A^∞ contained in the infinite alternating group A_∞, such that $g \circ j = i_n$, where g acts on A^∞ by permuting the basis vectors (note that the induced self-map of $|SPL(A^\infty)| \times EGL(A)$ fixes the base point). Regarding g as an element of $\pi_1(|\mathbb{SPL}(A^\infty)|//GL(A))$, this implies that the maps $(i_n)_*$ and j_*, considered as elements of the set of pointed homotopy classes of maps

$$[|\mathbb{SPL}(A^n)|//GL_n(A), |\mathbb{SPL}(A^\infty)|//GL(A)]\,,$$

are related by $g_*(j_*) = (i_n)_*$, where g_* denotes the action of the fundamental group of the target on the set of pointed homotopy classes of maps. However, g is in the kernel of the map on fundamental groups associated to the map

$$|\mathbb{SPL}(A^\infty)|//GL(A) \to (|\mathbb{SPL}(A^\infty|//GL(A))^+.$$

Hence the induced maps

$$|\mathbb{SPL}(A^n)|//GL(A) \to (|\mathbb{SPL}(A^\infty)|//GL(A))^+$$

determined by i_n and j are homotopic. □

Corollary 19. *The map $Y(A) \to Y(A)$ defined by an arbitrary injective map $\alpha : \mathbb{N} \hookrightarrow \mathbb{N}$ is homotopic, preserving the base point, to the identity.*

Proof. We first note that if for $n \geq 5$, we let $Y_n(A) = (|\mathbb{SPL}(A^n)|//GL_n(A))^+$ be the result of applying lemma 14 to $|\mathbb{SPL}(A^n)|//GL_n(A)$ for the alternating group A_n, then there are natural maps $Y_n(A) \to Y(A)$, preserving base points, and inducing an isomorphism $\varinjlim_n \pi_*(Y_n(A)) = \pi_*(Y(A))$.

We claim that if $\alpha_n : \{1, 2 \ldots, n\} \hookrightarrow \mathbb{N}$ is the inclusion induced by restricting α, then the induced map $(\alpha_n)_* : Y_n(A) \to Y(A)$ is homotopic, preserving the base points, to the natural map $Y_n(A) \to Y(A)$. This follows from lemma 18, combined with the defining universal property of the plus construction, given in lemma 14.

This implies that the map $\alpha : Y(A) \to Y(A)$ must then induce isomorphisms on homotopy groups, and hence is a homotopy equivalence, by Whitehead's theorem.

Thus, we have a map from the set of such injective maps α to the group of base-point preserving homotopy classes of self-maps of $Y(A)$. This is in fact a homomorphism of monoids, where the operation on the injective self-maps of $\mathbb{N}$ is given by composition of maps.

Now we use a trick from [6]: any homomorphism of monoids from the monoid of injective self-maps of $\mathbb{N}$ to a group is a trivial homomorphism, mapping all elements of the monoid to the identity. This is left to the reader to verify (or see [6]). □

We note that the above monoidal category $\mathbf{SPL}(A)$ can be used to give another, perhaps more insightful construction of the homotopy type $Y(A)$, analogous to Quillen's $\mathcal{S}^{-1}\mathcal{S}$ construction for $BGL(A)^+$. We sketch the argument below.

We first take $\mathbf{SPL}_0(A)$ to be the full subcategory of $\mathbf{SPL}(A)$ consisting of pairs (V, σ) where $\sigma \in SPL(V)$, i.e.,σ is a 0-simplex in $\mathbb{SPL}(V)$. This full subcategory is in fact a monoidal subcategory, which is a groupoid (all arrow are isomorphisms). Also, $\mathbf{SPL}(A)$ is a symmetric monoidal category, in that the sum operation is commutative upto coherent natural isomorphisms.

Then, using Quillen's results (see Chapter 7 in [19], particularly Theorem 7.2), one can see that $\mathbf{SPL}_0(A)^{-1}\mathbf{SPL}(A)$ is a monoidal category whose classifying space is a connected H-space, which is naturally homology equivalent to $|\mathbb{SPL}(A^\infty)|//GL(A)$. This then forces this classifying space to be homotopy equivalent to $Y(A)$, such that the H-space operations are compatible upto homotopy. This is analogous to the identification made in Theorem 7.4 in [19] of $\mathcal{S}^{-1}\mathcal{S}$ with $K_0(R) \times BGL(R)^+$ for a ring R, and appropriate $\mathcal{S}$. (We do not get the factor K_0 appearing in our situation since we work only with free modules).

6. Theorem 1 and the groups $\mathcal{H}_n(A^\times)$

Proof of Theorem 1. In view of Proposition 17, we see that $\mathbb{SPL}(A^\infty)^+$, the homotopy fiber of the H-map $Y(A) \to BGL(A)^+$, is a H-space as well. It follows that $L_0(A) = \pi_0(\mathbb{SPL}(A^\infty)^+)$ is a monoid. Furthermore, the arrow $K_1(A) \to L_0(A)$ in Corollary 15 is a monoid homomorphism. Thus this corollary produces an exact sequence of Abelian groups.
$\mathbb{SPL}(A^n)$ has a canonical base point fixed under the action of $N_n(A)$. As in sections 3 and 4, this gives a natural inclusion $BN_n(A) \to \mathbb{SPL}_n(A)//GL_n(A)$. This is a homology isomorphism by lemma 12. Taking direct limits over all $n \in \mathbb{N}$, we see that $BN(A) \to \mathbb{SPL}(A^\infty)//GL(A)$ is a homology isomorphism. Applying Quillen's plus construction with respect to the normal subgroup of $N(A)$ generated by the infinite alternating group, we obtain a space $BN(A)^+$. That $BN(A)^+$ has a canonical H-space structure follows easily by the method of the previous section. Now the map $BN(A)^+ \to Y(A)$ obtained by lemma 14(ii) is a homology isomorphism of simple path-connected CW complexes and is therefore a homotopy equivalence (see [4]) Theorem 4.37, page 371 and Theorem 4.5, page 346). This gives the isomorphism $\mathcal{H}_n(A^\times) \to L_n(A)$. The theorem now follows from corollary 15.
We now turn to the description of the groups. Let $X = B(A^\times)$. Let $X_+ = X \sqcup \{*\}$ be the pointed space with $*$ as its base-point. Let QX_+ be the direct limit of $\Omega^n\Sigma^n X_+$ where Σ denotes reduced suspension.

Proposition 20. *$\mathcal{H}_n(A^\times) \cong \pi_n(QX_+)$.*

This statement was suggested to us by Proposition 3.6 of [17].
A complete proof of the proposition was shown us by Peter May. A condensed version of what we learnt from him is given below.
Theorem 2.2, page 67 of [8] asserts that $\alpha_\infty : C_\infty X_+ \to QX_+$ is a group completion. This is proved in pages 50-59, [10]. The C_∞ here is a particular case of the construction 2.4, page 13 of [9], given for any operad. For $C_\infty(Y)$, where Y is a pointed space, the easiest definition to work with is found in May's review of [16]. It runs as follows. Let $V = \cup_{n=0}^\infty \mathbb{R}^n$. Let $C_k(Y)$ be the collection of ordered pairs (c, f) where $c \subset V$ has cardinality k and $f : c \to Y$ is any function. We identify (c, f) with (c', f') if
(i) $c' \subset c$, (ii) $f|c' = f'$, and (iii) $f(a) = *$ for all $a \in c, a \notin c'$. Here $*$ stands for the base-point of Y. Then $C_\infty Y$ is the space obtained from the disjoint union

of the $C_k(Y), k \geq 0$ by performing these identifications. This is a H-space. In our case, when $Y = X \sqcup \{*\}$, it is clear that $C_\infty Y$ is the disjoint union of all the $C_k(X)$ as a topological space. Thanks to "infinite codimension" one gets easily the homotopy equivalence of $C_k(X)$ with $X^k//S_k$ where S_k is the permutation group of $\{1, 2, .., k\}$. Now assume that X is any path-connected space equipped with a nondegenerate base-point $x \in X$. This x gives an inclusion of $X^n \hookrightarrow X^{n+1}$. Denote by X^∞ the direct limit of the X^n. Thus X^∞ is a pointed space equipped with the action of the infinite permutation group $S_\infty = \cup n S_n$. Put $Z = X^\infty//S_\infty$. As in section 4, we obtain Z^+ by the use of the infinite alternating group. As in section 5, we see that this is a H-space. It is an easy matter to check that the group completion of $\sqcup_k C_k X$ is homotopy equivalent to $\mathbb{Z} \times Z^+$. This shows that $\pi_n(QX_+) \cong \pi_n(Z^+)$ for all $n > 0$.
The proposition is the particular case: $X = B(A^\times)$.

7. Polyhedral structure of the enriched Tits building

From what has been shown so far, we see that it is of interest to determine the stable rational homology of the flag complexes $|\mathbb{FL}(A^n)|$ (or equivalently, of $|\mathbb{SPL}(A^n)|$, or $\mathbb{ET}(A^n)$). We will construct a spectral sequence that, in principle, gives an inductive procedure to do so.
But first we introduce some notation and a definition for posets.
Let P be a poset. For $p \in P$, we put $e(p) = BL(p)$ where $L(p) = \{q \in P | q \leq p\}$ and $\partial e(p) = BL'(p)$ where $L'(p) = L(p) \setminus \{p\}$. If $\partial e(p)$ is homeomorphic to a sphere for every $p \in P$, we say the poset P is *polyhedral*. We denote by $d(p)$ the dimension of $e(p)$. When P is polyhedral, the space BP gets the structure of a CW complex with $\{e(p) : p \in P\}$ as the closed cells. Its r-skeleton is BP_r where $P_r = \{p \in P : d(p) \leq r\}$. The homology of BP is then computed by the associated complex of cellular chains $Cell_\bullet(BP)$, where

$$Cell_r(BP) = \bigoplus_{\{p|\dim e(p)=r\}} H_r(e(p), \partial e(p), \mathbb{Z}).$$

Lemma 21. *$\mathcal{E}(A^n)$ is a polyhedral poset in the above sense. Its dimension is $n-1$.*

Proof. First consider the case when $p \in SPL(A^n)$ is a maximal element in $\mathcal{E}(A^n)$. Then p is an unordered collection of n lines in A^n (here, as in §2, a "line" denotes a free A-submodule of rank 1 which is a direct summand, and the set of lines in A^n is denoted by $\mathcal{L}(A^n)$). Note that the subset $p \subset \mathcal{L}(A^n)$ of cardinality n determines a poset $\widetilde{p}$, whose elements are chains $q_\bullet = \{q_1 \subset q_2 \subset \cdots \subset q_r = p\}$ of nonempty subsets, where $r_\bullet \leq q_\bullet$ if each q_i is an r_j for some j, i.e., the "filtration" $r_\bullet$ "refines" $q_\bullet$. We claim that, from the definition of the partial order on $\mathcal{E}(A^n)$, the poset $\widetilde{p}$ is naturally isomorphic to the poset $L(p)$. Indeed, an element $q \in \mathcal{E}(A^n)$ consists of a pair, consisting of a partial flag

$$0 = W_0 \subset W_1 \subset \cdots \subset W_r = A^n$$

such that W_i/W_{i-1} is free, and a sequence $t_1, \dots, t_r$ with $t_i \in SPL(W_i/W_{i-1})$. The condition that this element of $\mathcal{E}(A^n)$ lies in $L(p)$ is that each W_i is a direct sum of a subset of the lines in p, say $q_i \subset p$, giving the chain of subsets $q_1 \subset q_2 ... \subset q_r = p$; the splitting t_i is uniquely determined by the lines in $q_i \setminus q_{i-1}$.

Let $\Delta(p)$ be the $(n-1)$-simplex with p as its set of vertices. Now the chains of non-empty subsets of p correspond to simplices in the barycentric subdivison $sd\Delta(p)$, where the barycentre b corresponds to the chain $\{p\}$. Hence, from the definition of $\widetilde{p}$, it is clear that it is isomorphic to the poset whose elements are simplices in the barycentric subdivision of Δ_n with b as a vertex, with partial order given by reverse inclusion. Hence $B\widetilde{p}$ is naturally identified with the subcomplex of the second barycentric subdivision $sd^2\Delta(p)$ which is the union of all simplices containing the barycentre. This explicit description implies in particular that $BL'(p)$ is homeomorphic to S^{n-2} (with a specific triangulation).

Before proceeding to the general case, we set up the relevant notation for orientations. For a set q of cardinality r, we put $\det(q) = \wedge^r \mathbb{Z}[q]$ where $\mathbb{Z}[q]$ denotes the free Abelian group with q as basis. we observe that there is a natural isomorphism:

$$H_{n-1}(e(p), \partial e(p)) \cong H_{n-1}(\Delta(p), \partial\Delta(p)) = \det(p).$$

Now let $p \in \mathcal{E}(A^n)$ be arbitrary, corresponding to a partial flag

$$0 = W_0 \subset W_1 \subset \cdots \subset W_r = A^n.$$

and splittings $t_i \in SPL(W_i/W_{i-1})$. Then the natural map

$$\prod_{i=1}^{r} \mathcal{E}(W_i/W_{i-1}) \to \mathcal{E}(A^n)$$

is an embedding of posets, where the product has the ordering given by $(a_1, \dots, a_r) \leq (b_1, \dots, b_r)$ precisely when $a_i \leq b_i$ in $\mathcal{E}(W_i/W_{i-1})$ for each i. One sees that, by the definition of the partial order in $\mathcal{E}(A^n)$, the induced map

$$\prod_{i=1}^{r} L(t_i) \to L(p)$$

is bijective. Hence there is a homeomorphism of pairs

$$(BL(p), BL'(p)) = \prod_{i=1}^{r} (BL(t_i), BL'(t_i)),$$

and so $BL'(p) \cong S^{n-r-1}$, and $BL(p)$ is an $n-r$-cell. $\square$

We now proceed to construct the desired spectral sequence. We use the following notation: if $p \in \mathcal{E}(V)$, where V is a free A-module of finite rank, and $W_1 \subset V$ is the smallest non-zero submodule in the partial flag associated to p, define $t(p) = \operatorname{rank} W_1 - 1$. Clearly $t : \mathcal{E}(V) \to \mathbb{Z}$ is monotonic. Hence $F_r\mathcal{E}(V) = \{p \in \mathcal{E}(V) | t(p) \leq r\}$ is a sub-poset. Define

$$F_r\mathbb{ET}(V) = BF_r\mathcal{E}(V) = \cup\{e(p) | t(p) \leq r\},$$

so that

$$F_0\mathbb{ET}(V) \subset F_1\mathbb{ET}(V) \subset \cdots F_{n-1}\mathbb{ET}(V) = \mathbb{ET}(V)$$

is an increasing finite filtration of the CW complex $\mathbb{ET}(V)$ by subcomplexes. Hence there is an associated spectral sequence

$$E^1_{r,s} = H_{r+s}(F_r\mathbb{ET}(V), F_{r-1}\mathbb{ET}(V), \mathbb{Z}) \Rightarrow H_{r+s}(\mathbb{ET}(V), \mathbb{Z}).$$

Our objective now is to recognise the above E^1 terms.
It is convenient to use the complexes of cellular chains for these sub CW-complexes, which are thus sub-chain complexes of $Cell_\bullet(\mathbb{ET}(V))$. For simplicity of notation, we write $Cell_\bullet(V)$ for $Cell_\bullet(\mathbb{ET}(V))$. We have the description

$$E^1_{r,s} = H_{r+s}(\mathrm{gr}^F_r Cell_\bullet(V)).$$

We will now exhibit $\mathrm{gr}^F_r Cell_\bullet(V)$ as a direct sum of complexes. Let $W \subset V$ be a submodule such that $W, V/W$ are both free, and $\operatorname{rank} W = r+1$. Let $q \in SPL(W)$. The we have an inclusion of chain complexes

$$Cell_\bullet(e(q)) \otimes Cell_\bullet(V/W) \subset Cell_\bullet(W) \otimes Cell_\bullet(V/W) \subset F_r Cell_\bullet(V).$$

It is clear that

$$\operatorname{image} Cell_\bullet(\partial e(q)) \otimes Cell_\bullet(V/W) \subset F_{r-1}Cell_\bullet(V),$$

so that we have an induced homomorphism of complexes

$$(Cell_\bullet(e(q))/Cell_\bullet(\partial e(q))) \otimes Cell_\bullet(V/W) \to \mathrm{gr}^F_r Cell_\bullet(V).$$

Composing with the natural chain homomorphism

$$H_r(e(q), \partial e(q), \mathbb{Z})[r] \to (Cell_\bullet(e(q))/Cell_\bullet(\partial e(q)))$$

for each q, we finally obtain a chain map

$$I : \bigoplus_{(W, q \in SPL(W))} H_r(e(q), \partial e(q), \mathbb{Z})[r] \otimes Cell_\bullet(V/W) \to \mathrm{gr}^F_r Cell_\bullet(V).$$

Finally, it is fairly straightforward to verify that I is an isomorphism of complexes.
We deduce that the E^1 terms have the following description:

$$E^1_{r,s} = \bigoplus_{\substack{\operatorname{rank} W = r+1 \\ q \in SPL(W)}} \det(q) \otimes H_s(Cell_\bullet(V/W), \mathbb{Z}).$$

We define $\mathcal{L}_r(V)$ to be the collection of $q \subset \mathcal{L}(V)$ of cardinality $(r+1)$ for which (a) and (b) below hold:
(a) $\oplus\{L : L \in q\} \to V$ is injective. Its image will be denoted by $W(q)$
(b) $V/W(q)$ is free of rank $(n-r-1)$.
Summarising the above, we obtain:

Theorem 2. *There is a spectral sequence with E^1 terms*

$$E^1_{r,s} = \bigoplus_{q \in \mathcal{L}_r(V)} \det(q) \otimes H_s(\mathbb{ET}(V/W(q)), \mathbb{Z}).$$

that converges to $H_{r+s}(\mathbb{ET}(V))$. We note that $E^1_{r,s} = 0$ whenever $(r+s) \geq (n-1)$ with one exception: $(r,s) = (n-1, 0)$. Here $V \cong A^n$.

8. Compatible homotopy

It is true [3] that $i : \mathbb{ET}(W) \times \mathbb{ET}(V/W) \hookrightarrow \mathbb{ET}(V)$ has the property that $g \circ i$ is freely homotopic (not preserving base points) to i whenever $g \in \mathrm{Elem}(W \hookrightarrow V)$. There are several closed subsets of $\mathbb{ET}(A^n)$ with the property that homotopy class of the inclusion morphism into $\mathbb{ET}(A^n)$ remains unaffected by composition with the action of $g \in E_n(A)$. To prove that the union of a finite collection of such closed subsets has the same property, one would require the homotopies provided for any two members of the collection to agree on their intersection. This is the problem we are concerned with in this section.
We proceed to set up the notation for the problem.
With $q \in \mathcal{L}_r(V)$ as in theorem 2, we shall define the subspaces $U(q) \subset \mathbb{ET}(V)$ as follows. Let $W(q) = \oplus\{L | L \in q\}$. We regard q as an element of $SPL(W(q))$ and thus obtain the cell $e(q) = BL(q) \subset \mathbb{ET}(W(q))$. This gives the inclusion $\mathbb{ET}'(q) = e(q) \times \mathbb{ET}(V/W(q)) \subset \mathbb{ET}(W(q)) \times \mathbb{ET}(V/W(q)) \subset \mathbb{ET}(V)$.
We put $U(q) = \cup\{\mathbb{ET}'(t) | \emptyset \neq t \subset q\}$.

Main Question: Let $i : U(q) \hookrightarrow \mathbb{ET}(V)$ denote the inclusion. Is it true that $g \circ i$ is homotopic to i for every $g \in \mathrm{Elem}(V, q)$?
We focus on the apparently weaker question below.
Compatible Homotopy Question: Let $M \subset V$ be a submodule complementary to $W(q)$. Let $g' \in GL((W(q))$ be elementary, i.e. $g' \in \mathrm{Elem}(W(q), q)$. Define $g \in GL(V)$ by $gm = m$ for all $m \in M$ and $gw = g'w$ for all $w \in W(q)$. Is it true that $g \circ i$ is homotopic to i?
Assume that the second question has an affirmative answer in all cases. In particular, this holds when $M = 0$. Here $V = W(q)$ and $g = g'$ is an arbitrary element of $\mathrm{Elem}(V, q)$. Let t be a non-empty subset of q. Then $U(t) \subset U(q)$. We deduce that $j : U(t) \hookrightarrow \mathbb{ET}(V)$ is homotopic to $g \circ j$ for all $g \in \mathrm{Elem}(V, q)$. But $\mathrm{Elem}(V, t) = \mathrm{Elem}(V, q)$. Thus the Main Question has an affirmative answer for (q, i) replaced by (t, j), which of course, up to a change of notation, covers the general case.

Proposition 22. *The compatible homotopy question has an affirmative answer if $q \in \mathcal{L}_r(V)$ and $r \leq 2$.*

The rest of this section is devoted to the proof of this proposition. To proceed, we will require to introduce the class $\mathcal{C}$.

[3] this only requires the analogue of lemma 8(A) for the enriched Tits building. More general statements are contained in lemmas 23 and 24 .

This is our set-up. Let X be a finite set, let V_x be a finitely generated free module for each $x \in X$ and let $V = \oplus\{V_x : x \in X\}$.
Let $s = \Pi_{x\in X}s(x) \in \Pi_{x\in X}SPL(V_x)$. For each $x \in X$, we regard $s(x)$ as a subset of $\mathcal{L}(V)$ and put $Fs = \cup\{s(x)|x \in X\}$. Thus $Fs \in SPL(V)$. *The collection of maps* $f : \Pi_{x\in X}\mathbb{ET}(V_x) \to \mathbb{ET}(V)$ *with the property that* $f(\Pi_{x\in X}BL(s(x))) \subset BL(Fs)$ *for all* $s \in \Pi_{x\in X}SPL(V_x)$ *is denoted by* $\mathcal{C}$. See lemma 10 and proposition 11 and its proof for relevant notation.
Every maximal chain C of subsets of X (equivalently every total ordering of X) gives a member $i(C) \in \mathcal{C}$. For instance, if $X = \{1, 2, ..., n\}$ and C consists of the sets $\{1, 2, ..., k\}$ for $1 \leq k \leq n$, we put
$D_k = \oplus_{i=1}^k V_i$ and $E = \Pi_{i=1}^n \mathbb{ET}(D_i/D_{i-1})$, denote by
$u : \Pi_{x\in X}\mathbb{ET}(V_x) \to E$ and $v : E \to \mathbb{ET}(V)$ the natural isomorphism and natural inclusion respectively, and put $i(C) = v \circ u$.

Lemma 23. *The above space* $\mathcal{C}$ *is contractible.*

Proof. The aim is to realise $\mathcal{C}$ as the space of Λ-compatible maps for a suitable Λ and appeal to Proposition 1. Let $\Lambda(x) = \{L(S)|\emptyset \neq S \subset SPL(V_x), \emptyset \neq L(S)\}$. Proposition 11 asserts that the subspaces $\{B\lambda(x) : \lambda(x) \in \Lambda(x)\}$ give an admissible cover of $\mathbb{ET}(V_x)$.
For $\lambda = \Pi_{x\in X}\lambda(x) \in \Lambda = \Pi_{x\in X}\Lambda(x)$, we put $I(\lambda) = \Pi_{x\in X}B\lambda(x)$ and deduce that $\{I(\lambda) : \lambda \in \Lambda\}$ gives an admissible cover of $\Pi_{x\in X}\mathbb{ET}(V_x)$.
We define next a closed $J(\lambda) \subset \mathbb{ET}(V)$ for every $\lambda \in \Lambda$ with the properties:
(A): $J(\lambda) \subset J(\mu)$ whenever $\lambda \leq \mu$ and
(B): $J(\lambda)$ is contractible for every $\lambda \in \Lambda$.
For each $\lambda(x) \in \Lambda(x)$, let $U\lambda(x)$ be its set of upper bounds in $SPL(V_x)$. It follows that that $LU\lambda(x)) = \lambda(x)$. As observed before, we have
$F : \Pi_{x\in X}SPL(V_x) \to SPL(V)$. Thus given $\lambda = \Pi_x\lambda(x) \in \Lambda$,
we set $H(\lambda) = F(\Pi_{x\in X}U\lambda(x)) \subset SPL(V)$ and then put $J(\lambda) = BLH(\lambda) \subset \mathbb{ET}(V)$.
The space of Λ-compatible maps $\Pi_{x\in X}\mathbb{ET}(V_x) \to \mathbb{ET}(V)$ is seen to coincide with $\mathcal{C}$. That the $J(\lambda)$ satisfy property (A) stated above is straightforward.
The contractibility of $J(\lambda)$ for all $\lambda \in \Lambda$ is guaranteed by proposition 11 once it is checked that these sets are nonempty. But we have already noted that $\mathcal{C}$ is nonempty. Let $f \in \mathcal{C}$. Now $I(\lambda) \neq \emptyset$ and $f(I(\lambda)) \subset J(\lambda)$ implies $J(\lambda) \neq \emptyset$. Thus the $J(\lambda)$ are contractible, and as said earlier, an application of Proposition 1 completes the proof of the lemma.

$\square$

We remark that the class $\mathcal{C}$ of maps $\Pi_{i=1}^n\mathbb{ET}(W_i) \to \mathbb{ET}(\oplus_{i=1}^n W_i)$ has been defined in general.
We will continue to employ the notation: $V = \oplus\{V_x : x \in X\}$ all through this section. Let P be a partition of X. Each $p \in P$ is a subset of X and we put

$$V_p = \oplus\{V_x|x \in p\} \text{ and } \mathbb{ET}(P) = \Pi\{\mathbb{ET}(V_p)|p \in P\}.$$

When $Q \leq P$ is a partition of X (i.e. Q is finer than P), we shall define the contractible collection $\mathcal{C}(Q, P)$ of maps $f : \mathbb{ET}(Q) \to \mathbb{ET}(P)$ by demanding (a) that f is the product of maps $f(p)$

$$f(p) : \Pi\{\mathbb{ET}(V_q) : q \subset p \text{ and } q \in Q\} \to \mathbb{ET}(V_p)$$

and also (b) each $f(p)$ is in the class $\mathcal{C}$. For this one should note that $V_p = \oplus\{V_q : q \in Q \text{ and } q \subset p\}$.
We observe next that there is a distinguished collection $\mathcal{D}(Q, P) \subset \mathcal{C}(Q, P)$. To see this, recall that we had the embedding $i(C)$ for every maximal chain C of subsets of X (alternatively, for every total ordering of X). Given $Q \leq P$, denote the set of total orderings of $\{q \in Q : q \subset p\}$ by $T(p)$, for every $p \in P$. The earlier $C \mapsto i(C)$ now yields, after taking a product over $p \in P$,
$i : \Pi\{T(p) : p \in P\} \to \mathcal{C}(Q, P)$, and we denote by $\mathcal{D}(Q, P) \subset \mathcal{C}(Q, P)$ the image of i.
The lemma below is immediate from the definitions.

Lemma 24. *Given partitions $R \leq Q \leq P$ of X, if f is in $\mathcal{C}(R, Q)$ (resp. in $\mathcal{D}(R, Q)$)and g is in $\mathcal{C}(Q, P)$ (resp. in $\mathcal{D}(Q, P)$), then it follows that $g \circ f$ is in $\mathcal{C}(R, P)$ (resp. $\mathcal{D}(R, P)$).*

We will soon have to focus on the fixed points of certain unipotent $g \in GL(V)$ on $\mathbb{ET}(V)$. For instance, if $x, y \in X$ and $x \neq y$, we may consider $g = id_V + h$ where $h(V) \subset V_y$ and $h(V_z) = 0$ for all $z \in X, z \neq x$. Let C be a chain of subsets of X, so that $X \in C$. This chain C gives rise to a partition $P(C)$ of X and also $i(C) \in \mathcal{D}(P(C), \{X\})$ in a natural manner. Let $C_x = \cap\{S \in C : x \in S\}$. Then $C_x \in C$ because C is a chain. Define C_y in a similar manner. *We say the chain C is (x, y)-compatible if $C_y \subset C_x$ and $C_x \neq C_y$* . This condition on C ensures that the embedding $i(C) : \mathbb{ET}(P(C)) \to \mathbb{ET}(V)$ has its image within the fixed points of the above $g \in GL(V)$.
Now let Q be a partition of X so that $q \in Q, x \in q$ implies $y \notin q$. We shall define next the class of (x, y)-compatible $\mathcal{C}$ maps $\mathbb{ET}(Q) \to \mathbb{ET}(V)$ in the following manner. Let Λ be the set of chains C of subsets of X so that $X \in C$ and $Q \leq P(C)$ (i.e. Q is finer than the partition $P(C)$). For each $C \in \Lambda$, let $Z(C)$ be the collection of $i(C) \circ f$ where $f \in \mathcal{C}(Q, P(C))$. Finally, let $Z = \cup\{Z(C) : C \in \Lambda\}$. This set Z is defined to be the collection of (x, y)-compatible maps of class $\mathcal{C}$ from $\mathbb{ET}(Q)$ to $\mathbb{ET}(V)$. Every $z \in Z$ is a map $z : \mathbb{ET}(Q) \to \mathbb{ET}(V)$ whose image is contained in the fixed points of the above g on $\mathbb{ET}(V)$. Furthermore, in view of lemma 24, this collection of maps is contained in $\mathcal{C}(Q, \{X\})$.

Lemma 25. *Let Q be a partition of X that separates x and y. Then the collection of (x, y)-compatible class $\mathcal{C}$ maps $\mathbb{ET}(Q) \to \mathbb{ET}(V)$ is contractible.*

Proof. In view of the fact that each $\mathcal{C}(Q, P)$ is contractible, by cor 3, it follows that the space of (x, y)-compatible chains is homotopy equivalent to $B\Lambda$, where Λ is the poset of chains C in the previous paragraph. It remains to show that $B\Lambda$ is contractible.
We first consider the case where Q is the set of all singletons of X. Let $\mathcal{S}$ be the collection of subsets $S \subset X$ so that $y \in S$ and $x \notin S$. For $S \in \mathcal{S}$, let

$\mathcal{F}(S)$ be the collection of chains C of subsets of X so that $S \in C$ and $X \in C$. We see that Λ is precisely the union of $\mathcal{F}(S)$ taken over all $S \in \mathcal{S}$. Let D be a finite subset of $\mathcal{S}$. We see that the intersection of the $B\mathcal{F}(S)$, taken over $S \in D$, is nonempty if and only if D is a chain. Furthermore, when D is a chain, this intersection is clearly a cone, and therefore contractible. By cor 3, we see that $B\Lambda$ has the same homotopy type as the classifying space of the poset of chains of $\mathcal{S}$. But this is simply the barycentric subdivision of $B\mathcal{S}$. But the latter is a cone as well, with $\{y\}$ as vertex. This completes the proof that $B\Lambda$ is contractible, when Q is the finest possible partition of X.
We now come to the general case, when Q is an arbitrary partition of X that separates x, y. So we have $x', y' \in Q$ with $x \in x', y \in y'$ and $x' \neq y'$. The set Λ is identified with the collection of chains C' of subsets of Q so that
(a) $Q \in C'$, and (b) there is some $L \in C'$ so that $x' \notin L$ and $y' \in L$.
Thus the general case follows from the case considered first: one replaces (X, x, y) by (Q, x', y'). $\square$

In a similar manner, we may define, for every ordered r-tuple $(x_1, x_2, ..., x_r)$ of distinct elements of X, the set of $(x_1, x_2, ..., x_r)$-compatible chains C–we demand that for each $0 < i < r$, there is a member S of the chain so that $x_i \notin S$ and $x_{i+1} \in S$. Let Q be a partition of X that separates $x_1, x_2, ..., x_r$. Then the poset of chains C , compatible with respect to this ordered r-tuple, and for which $Q \leq i(P)$, is also contractible. One may see this through an inductive version of the proof of the above lemma. A corollary is that the collection of $(x_1, ..., x_r)$-compatible class $\mathcal{C}$ maps $\mathbb{ET}(Q) \to \mathbb{ET}(V)$ is also contractible. We skip the proof. This result is employed in the proof of Proposition 22 for $r = 2$ (which has already been verified in the above lemma), and for $r = 3$, with $\#(Q) \leq 4$. Here it is a simple verification that the poset of chains that arises as above has its classifying space homeomorphic to a point or a closed interval.

We are now ready to address the proposition. For this purpose, we assume that there is $c \in X$ so that $V_x \cong A$ for all $x \in X \setminus \{c\}$. To obtain consistency with the notation of the proposition, we set $q = X \setminus \{c\}$. The closed subset $U(q) \subset \mathbb{ET}(V)$ in the proposition is the union of $\mathbb{ET}'(t)$ taken over all $\emptyset \neq t \subset q$. For such t, we have $W(t) = V(t) = \oplus\{V_x : x \in t\}$. Recall that $\mathbb{ET}'(t)$ is the product of the cell $e(t) \subset \mathbb{ET}(V(t))$ with $\mathbb{ET}(V/V(t))$. To proceed, it will be necessary to give a contractible class of maps $D \to \mathbb{ET}(V)$ for certain closed subsets $D \subset U(q)$.
The closed subsets $D \subset U(q)$ we consider have the following shape. For each $\emptyset \neq t \subset q$, we first select a closed subset $D(t) \subset e(t)$ and then take D to be the union of the $D(t) \times \mathbb{ET}(V/V(t))$, taken over all such t. This D remains unaffected if $D(t)$ is replaced by its saturation $sD(t)$. Here $sD(t)$ is the collection of $a \in e(t)$ for which $\{a\} \times \mathbb{ET}(V/V(t))$ is contained in D.
When $\emptyset \neq t \subset q$, we denote by $p(t)$ the partition of X consisting of all the singletons contained in t, and in addition, the complement $X \setminus t$. Then there is a canonical identification $j(t) : \mathbb{ET}(p(t)) \to \mathbb{ET}(V/V(t))$.

A map $f : D \to \mathbb{ET}(V)$ is said to be in class $\mathcal{C}$ if for every $\emptyset \neq t \subset q$ and for every $a \in sD(t)$, the map $\mathbb{ET}(p(t)) \to \mathbb{ET}(V)$ given by $b \mapsto f(a, j(t)b)$ belongs to $\mathcal{C}(p(t), \{X\})$. By lemma 24, we see that it suffices to impose this condition on all $a \in D(t)$, rather than all $a \in sD(t)$.

We observe that for every $a \in e(t)$, the map $\mathbb{ET}(p(t)) \to \mathbb{ET}(V)$ given by $b \mapsto (a, j(t)b)$ belongs to $\mathcal{C}(p(t), \{X\})$. As a consequence, we see that the inclusion $D \hookrightarrow \mathbb{ET}(V)$ is of class $\mathcal{C}$.

When concerned with (x, y)-compatible maps, we will assume that $D(t) = \emptyset$ whenever t and $\{x, y\}$ are disjoint. Under this assumption, a map $f : D \to \mathbb{ET}(V)$ is said to be (x, y)-compatible of class $\mathcal{C}$ if $\mathbb{ET}(p(t)) \to \mathbb{ET}(V)$ given by $b \mapsto f(a, j(t)b)$ is a (x, y)-compatible map in $\mathcal{C}(p(t), \{X\})$ for all pairs (a, t) such that $a \in sD(t)$.

In a similar manner, we define (x, y, z)-compatible maps of class $\mathcal{C}$ as well. For this, it is necessary to assume that $D(t)$ is empty whenever the partition $p(t)$ does not separate (x, y, z), equivalently if $\{x, y, z\} \setminus t$ has at least two elements.

Lemma 26. *Assume furthermore that $D(t)$ is a simplicial subcomplex of $e(t)$. Then the space of maps $D \to \mathbb{ET}(V)$ in class $\mathcal{C}$ is contractible. The same is true of the space of such maps that are (x, y)-compatible, or (x, y, z)-compatible.*

Proof. We denote by d the cardinality of $\{t : D(t) \neq \emptyset\}$. We proceed by induction on d, beginning with $d = 0$ where the space of maps is just one point.

We choose t_0 of maximum cardinality so that $D(t_0) \neq \emptyset$. Let D' be the union of $D(t) \times \mathbb{ET}(V/V(t))$ taken over all $t \neq t_0$. Let $\mathcal{C}(D')$ and $\mathcal{C}(D)$ denote the space of class $\mathcal{C}$ maps $D' \to \mathbb{ET}(V)$ and $D \to \mathbb{ET}(V)$ respectively. By the induction hypothesis, $\mathcal{C}(D')$ is contractible. We observe that the intersection of D' and $e(t_0) \times \mathbb{ET}(V/V(t_0))$ has the form $G \times \mathbb{ET}(V/V(t_0)$ where $G \subset e(t_0)$ is a subcomplex. Furthermore, $G \cup D(t_0)$ is the saturated set $sD(t_0)$ described earlier.

For a closed subset $H \subset e(t_0)$, denote the space of $\mathcal{C}$-maps $H \times \mathbb{ET}(V/V(t_0)) \to \mathbb{ET}(V)$ by $A(H)$. Note that $A(H) = \mathrm{Maps}(H, \mathcal{C}(p(t_0), \{X\}))$. By lemma 23, the space $\mathcal{C}(p(t_0), \{X\})$ is itself contractible. It follows that $A(H)$ is contractible. In particular, both $A(G)$ and $A(sD(t_0))$ are contractible. The natural map $A(sD(t_0)) \to A(G)$ is a fibration, because the inclusion $G \hookrightarrow sD(t_0)$ is a cofibration. The fibers of $A(sD(t_0)) \to A(G)$ are thus contractible. It follows that

$$A(sD(t_0)) \times_{A(G)} \mathcal{C}(D') \to \mathcal{C}(D')$$

which is simply $\mathcal{C}(D) \to \mathcal{C}(D')$, enjoys the same properties: it is also a fibration with contractible fibers. Because $\mathcal{C}(D')$ is contractible, we deduce that $\mathcal{C}(D)$ is itself contractible. This completes the proof of the first assertion of the lemma. The remaining assertions follow in exactly the same manner by appealing to lemma 25. □

Proof of Proposition 22. Choose $x \neq y$ with $x, y \in q$. Let $g = id_V + \alpha$ where $\alpha(V) \subset V_y$ and $\alpha(V_k) = 0$ for all $k \neq x \in X$. To prove the proposition, it suffices to show that $g \circ i$ is homotopic to i where $i : U(q) \to \mathbb{ET}(V)$ is the given inclusion. *This notation x, y, α, g will remain fixed throughout the proof.*

Case 1. Here $q = \{x, y\}$. Now x, y are separated by the partitions $p(t)$ for every non-empty $t \subset q$. By the second assertion of the above lemma, there exists $f : U(q) \to \mathbb{ET}(V)$ of class $\mathcal{C}$ and (x, y)-compatible. The given inclusion $i : U(q) \to \mathbb{ET}(V)$ is also of class $\mathcal{C}$. By the first assertion of the same lemma, f is homotopic to i. Now the image of f is contained in the fixed-points of g and so we get $g \circ f = f$. It follows that $g \circ i$ is homotopic to i. This completes the proof of the proposition when $1 = r = \#(q) - 1$.

Case 2. Here $q = \{x, y, z\}$ with x, y, z all distinct.
We take Y_1 to be the union of $e(t) \times \mathbb{ET}(V/V(t))$ taken over all $t \subset q, t \neq \{z\}, t \neq \emptyset$. We put $Y_2 = \mathbb{ET}(V_z) \times \mathbb{ET}(V/V_z)$ and $Y_3 = Y_1 \cap Y_2$. We note that $U(q) = Y_1 \cup Y_2$.
The given inclusion $i : U(q) \to \mathbb{ET}(V)$ restricts to $i_k : Y_k \to \mathbb{ET}(V)$ for $k = 1, 2, 3$. The required homotopy is a path $\gamma : I \to \mathrm{Maps}(U(q), \mathbb{ET}(V))$ so that $\gamma(0) = i$ and $\gamma(1) = g \circ i$. Equivalently we require paths γ_k in $\mathrm{Maps}(Y_k, \mathbb{ET}(V))$ for $k = 1, 2$ so that
(a) $\gamma_k(0) = i_k$ and $\gamma_k(1) = g \circ i_k$ for $k = 1, 2$ and
(b) both γ_1 and γ_2 restrict to the same path in $\mathrm{Maps}(Y_3, \mathbb{ET}(V))$.
In view of the fact that $Y_3 \hookrightarrow Y_1$ is a cofibration, the weaker conditions (a') and (b') on fundamental groupoids suffice for the existence of such a γ:
(a'): $\gamma_k \in \pi_1(\mathrm{Maps}(Y_k, \mathbb{ET}(V)); i_k, g \circ i_k)$ for $k = 1, 2$
(b'): both γ_1 and γ_2 restrict to the same element of $\pi_1((\mathrm{Maps}(Y_3, \mathbb{ET}(V)); i_3, g \circ i_3)$
We have the spaces: $Z_k = \mathrm{Maps}(Y_k, \mathbb{ET}(V))$ for $k = 1, 2, 3$.These spaces come equipped with the data below:
(A)The $GL(V)$-action on $\mathbb{ET}(V)$ induces a $GL(V)$-action on Z_k
(B)The maps of class $\mathcal{C}$ give contractible subspaces $\mathcal{C}_k \subset Z_k$ for $k = 1, 2, 3$.
(C) We have $i_k \in \mathcal{C}_k$ for $k = 1, 2, 3$.
(D) The natural maps $Z_k \to Z_3$ for $k = 1, 2$ are $GL(V)$-equivariant, they take i_k to i_3 and restrict to maps $\mathcal{C}_k \to \mathcal{C}_3$.
Note that the $GL(V)$-action on Z_k turns the disjoint union:
$\mathcal{G}_k = \sqcup\{\pi_1(Z_k; i_k, hi_k) | h \in GL(V)\}$ into a group: given ordered pairs $(h_j, v_j) \in \mathcal{G}_k$, i.e. $h_j \in GL(V)$ and $v_j \in \pi_1(Z_k; i_k, h_j i_k)$ for $j = 1, 2$, we get $h_1 v_2 \in \pi_1(Z_k; h_1 i_k, h_1 h_2 i_k)$ and obtain thereby $v = (h_1 v_2).v_1 \in \pi_1(Z_k; i_k, h_1 h_2 i_k)$ and this produces the required binary operation $(h_1, v_1) * (h_2, v_2) = (h_1 h_2, v)$.
The projection $\mathcal{G}_k \to GL(V)$ is a group homomorphism. The following elementary remark will be used in an essential manner when checking condition (b'). The data (H, F, Δ) where
(i) $H \subset GL(V)$ is a subgroup,
(ii)$F \in Z_k$ is a fixed-point of H, and
(iii)$\Delta \in \pi_1(Z_k; F, i_k)$
produces the lift $H \to \mathcal{G}_k$ of the inclusion $H \hookrightarrow GL(V)$ by $h \mapsto (h, (h\Delta).\Delta^{-1})$.

Finally we observe that there are natural homomorphisms $\mathcal{G}_k \to \mathcal{G}_3$ induced by $Z_k \to Z_3$ for $k = 1, 2$.
Construction of γ_1.
The partitions $p(t)$ for $t \neq \{z\}$ separate x, y. By lemma 26, we have a (x, y)-compatible class $\mathcal{C}$-map $f : Y_1 \to \mathbb{ET}(V)$. Both i_1 and f belong to $\mathcal{C}_1$ and thus we get $\delta \in \pi_1(\mathcal{C}_1; f, i_1)$. Now f is fixed by our $g \in GL(V)$, so we also get $g\delta \in \pi_1(g\mathcal{C}_1; f, gi_1)$. The path $(g\delta).\delta^{-1}$ is the desired $\gamma_1 \in \pi_1(Z_1; i_1, gi_1)$.
Construction of γ_2.
Recall that $g = id_V + \alpha$. We choose $m : V_x \to V_z$ and $n : V_z \to V_y$ so that $nm(a) = \alpha(a)$ for all $a \in V_x$. We extend m, n by zero to nilpotent endomorphisms of V, once again denoted by $m, n : V \to V$ and put $u = id_V + n, v = id_V + m$ and note that $g = uvu^{-1}v^{-1}$.
Note that the partition $p(\{z\})$ separates both the pairs (x, z) and (z, y). We thus obtain $f', f'' \in \mathcal{C}_2$ so that f' is (x, z)-compatible and f'' is (y, z)-compatible and also $\delta' \in \pi_1(\mathcal{C}_2; f', i_2)$ and $\delta'' \in \pi_1(\mathcal{C}_2; f'', i_2)$. Noting that f', f'' are fixed by v, u respectively, we obtain
$\epsilon' = (v\delta').\delta'^{-1} \in \pi_1(Z_2; i_2, vi_2)$ and
$\epsilon'' = (u\delta'').\delta''^{-1} \in \pi_1(Z_2; i_2, ui_2)$.
Thus $v' = (v, \epsilon')$ and $u' = (u, \epsilon'')$ both belong to $\mathcal{G}_2$. We obtain γ_2 by
$u' * v' * u'^{-1} * v'^{-1} = (g, \gamma_2) \in \mathcal{G}_2$
Checking the validity of (b').
Let $\gamma_{13}, \gamma_{23} \in \pi_1(Z_3; i_3, gi_3)$ be the images of γ_1 and γ_2 respectively. We have to show that $\gamma_{13} = \gamma_{23}$.
Consider the spaces $\mathcal{H}, \mathcal{H}', \mathcal{H}''$ consisting of ordered pairs $(f_3, \delta_3), (f_3', \delta_3'), (f_3'', \delta_3'')$ respectively, where f_3, f_3', f_3'' are all in $\mathcal{C}_3$,
f_3 is (x, y)-compatible, f_3' is (x, z)-compatible, and f_3'' is (y, z)-compatible, and $\delta_3, \delta_3', \delta_3''$ are all paths in $\mathcal{C}_3$ that originate at f_3, f_3', f_3'' respectively, and they all terminate at i_3. By lemma 26, we see that the spaces $\mathcal{H}, \mathcal{H}', \mathcal{H}''$ are all contractible.
For $t = \{x, z\}, \{y, z\}, \{x, y, z\}$, the partition $p(t)$ separates (x, y, z). Note that Y_3 is contained in the union of these three $\mathbb{ET}'(t)$. By lemma 26, there is a (x, y, z)-compatible $F \in \mathcal{C}_3$. Let Δ be a path in $\mathcal{C}_3$ that originates at F and terminates at i_3. We see that $(F, \Delta) \in \mathcal{H} \cap \mathcal{H}' \cap \mathcal{H}''$.
Note that $\mathcal{H} \to \pi_1(Z_3; i_3, gi_3)$ given by $(f_3, \delta_3) \mapsto (g\delta_3).\delta_3^{-1}$ is a constant map because $\mathcal{H}$ is contractible. The (f, δ) employed in the construction of γ_1 restricts to an element of $\mathcal{H}$. Also, (F, Δ) belongs to $\mathcal{H}$. It follows that $\gamma_{13} = (g\Delta).\Delta^{-1}$.
In a similar manner, we deduce that if $\epsilon_3', \epsilon_3''$ denote the images of ϵ', ϵ'' in the fundamental groupoid of Z_3, then
$\epsilon_3' = (v\Delta).\Delta^{-1} \in \pi_1(Z_3; i_3, vi_3)$ and $\epsilon_3'' = (u\Delta).\Delta^{-1} \in \pi_1(Z_3; i_3, ui_3)$
Thus $\mathcal{G}_2 \to \mathcal{G}_3$ takes $v', u' \in \mathcal{G}_2$ to $(v, (v\Delta).\Delta^{-1}), (u, (u\Delta).\Delta^{-1}) \in \mathcal{G}_3$ respectively. It follows that their commutator $[u', v']$ maps to $(g, \gamma_{23}) \in \mathcal{G}_3$ under this homomorphism.
We apply the remark preceeding the construction of γ_1 to the subgroup H generated by u, v and F and Δ as above. We conclude that γ_{23} equals $(g\Delta).\Delta^{-1}$.

That the latter equals γ_{13} has already been shown. Thus $\gamma_{13} = \gamma_{23}$ and this completes the proof of the Proposition.

9. Low dimensional stabilisation of homology

This section contains applications of corollary 9, proposition 22 and Theorem 2 to obtain some mild information on the homology groups of $\mathbb{ET}(V)$. The notation $\mathcal{L}(V), \mathcal{L}_r(V), W(q), \det(q)$ introduced to state Theorem 2 will be freely used throughout. The spectral sequence in theorem 2 with coefficients in an Abelian group M will be denoted by $SS(V;M)$. When $V = A^n$, this is further abbreviated to $SS(n;M)$, or even to $SS(n)$ when it is clear from the context what M is.
The concept of a commutative ring with *many units* is due to Van der Kallen. An exposition of the definition and consequences of this term is given in [12]. We note that this class of rings includes semilocal rings with infinite residue fields. The three consequences of this hypothesis on A are listed as I,II,III below. These statements are followed by some elementary deductions. *Throughout this section, we will assume that our ring A has this property.*

I: $SL_n(A) = E_n(A)$.
This permits a better formulation of Lemma 8 in many instances.
Ia: Let $0 \to W \to P \to Q \to 0$ be an exact sequence of free A-modules with of ranks $a, a+b, b$. Let $d = g.c.d.(a,b)$. The group H of automorphisms of this exact sequence that induce homotheties on both W and Q may be regarded as a subgroup of $GL(P)$. This group acts trivially on the image of the embedding $i : \mathbb{ET}(W) \times \mathbb{ET}(Q) \to \mathbb{ET}(P)$. Furthermore $\{\det(g) | g \in H\}$ equals $(A^\times)^d$. Thus, if a, b are relatively prime, by lemma 8, we see that $g \circ i$ is freely homotopic to i for all $g \in GL(V)$.
We shall take $\mathrm{rank}(W) = 1$ in what follows. Here $\mathbb{ET}(W) \times \mathbb{ET}(Q)$ is canonically identified with $\mathbb{ET}(Q)$. The induced $\mathbb{ET}(Q) \to \mathbb{ET}(P)$ gives rise on homology to an arrow $H_m(\mathbb{ET}(Q)) \to H_m(\mathbb{ET}(P))$ which has a factoring:

$$H_m(\mathbb{ET}(Q)) \twoheadrightarrow H_0(PGL(Q), H_m(\mathbb{ET}(Q))) \to$$
$$\to H^0(PGL(P), H_m(\mathbb{ET}(P)) \hookrightarrow H_m(\mathbb{ET}(P)).$$

The kernel of $H_m(\mathbb{ET}(Q)) \to H_m(\mathbb{ET}(P))$ does not depend on the choice of the exact sequence. Denoting this kernel by $KH_m(Q) \subset H_m(\mathbb{ET}(Q))$ therefore gives rise to unambiguous notation. We abbreviate $H_m(\mathbb{ET}(A^n)), KH_m(A^n)$ to $H_m(n), KH_m(n)$ respectively.
Ib: In the spectral sequence $SS(n)$, we have:

(1) $H_0(PGL_{n-1}(A), H_m(n-1)) \cong H_0(PGL_n(A), E^1_{0,m})$.
(2) $E^\infty_{0,m}$ is the image of $H_m(n-1) \to H_m(n)$.
(3) $E^1_{0,m} \to E^\infty_{0,m}$ factors as follows:
$$E^1_{0,m} \twoheadrightarrow E^2_{0,m} \twoheadrightarrow H_0(PGL_n(A), E^1_{0,m}) \twoheadrightarrow E^\infty_{0,m}.$$
(4) If $H_0(PGL_n(A), E^p_{p,m+1-p}) = 0$ for all $p \geq 2$, then the given arrow $H_0(PGL_n(A), E^1_{0,m}) \to E^\infty_{0,m}$ is an isomorphism.

(5) Assume that $H_m(n-2) \to H_m(n-1)$ is surjective. Then the arrow $E^2_{0,m} \to H_0(PGL_n(A), E^1_{0,m})$ in (3) above is an isomorphism.

The factoring in part (3) above is a consequence of the factoring of $H_m(\mathbb{ET}(Q)) \to H_m(\mathbb{ET}(P))$ in part I(a).
For part (4), one notes that the composite
$E^2_{2,m-1} \to E^2_{0,m} \to H_0(PGL_n(A), E^1_{0,m})$ vanishes because $H_0(PGL_n(A), E^2_{2,m-1})$ itself vanishes. Thus we obtain a factoring: $E^2_{0,m} \to E^3_{0,m} \to H_0(PGL_n(A), E^1_{0,m})$. Proceeding inductively, we obtain the factoring:
$E^2_{0,m} \to E^\infty_{0,m} \to H_0(PGL_n(A), E^1_{0,m})$. In view of (3), we see that part (4) follows.
For part (5), it suffices to note that for every $L_0, L_1 \in \mathcal{L}(A^n)$ with $n > 1$, there is some $L_2 \in \mathcal{L}(A^n)$ with the property that both $\{L_0, L_2\}$ and $\{L_1, L_2\}$ belong to $\mathcal{L}_1(A^n)$. This fact is contained in consequence III of *many units.*

II: A is a Nesterenko-Suslin ring.
Let $r > 0, p \geq 0$. Put $N = (p+1)!$ and $n = r + p + 1$. Let $\mathcal{F}_r$ be the category with free rank A-modules of rank r as objects; the morphisms in $\mathcal{F}_r$ are A-module isomorphisms. Let F be a functor from $\mathcal{F}_r$ to the category of $\mathbb{Z}[\frac{1}{N}]$ -modules. Assume that $F(a.id_D) = id_{FD}$ for every $a \in A^\times$ and for every object D of $\mathcal{F}_r$. In other words, the natural action of $GL_r(A)$ on $F(A^r)$ factors through the action of $PGL_r(A)$.
For a free A-module V of rank n, define $Ind'F(V)$ by

$$Ind'F(V) = \oplus\{\det(q) \otimes F(V/W(q)) : q \in \mathcal{L}_p(V)\}.$$

An alternative description of $Ind'F(V)$ is as follows. Fix some $q \in \mathcal{L}_p(V)$. Let $G(q)$ be the stabiliser of q in $GL(V)$. Then $\det(q)$ and $F(V/W(q))$ are $G(q)$-modules in a natural manner. We have a natural isomorphism of $\mathbb{Z}[GL(V)]$-modules:

$$Ind'F(V) \cong \mathbb{Z}[GL(V)] \otimes_{\mathbb{Z}[G(q)]} [\det(q) \otimes_{\mathbb{Z}} F(V/W(q))].$$

IIa: If $H_i(PGL_r(A), F(A^r)) = 0$ for all $i < m$, then $H_i(PGL_n(A), Ind'F(A^n)) = 0$ for all $i < p+m$. Furthermore,

$$H_{p+m}(PGL_n(A), Ind'F(A^n)) \cong H_m(PGL_r(A), F(A^r)) \otimes Sym^p(A^\times))$$

By Shapiro's lemma, the result of [13] cited earlier, and the fact that the group homology $H_i(M, C)$ is isomorphic to $C \otimes \Lambda^i(M)$ for all commutative groups M and $\mathbb{Z}[1/i!]$-modules C given trivial M-action, IIa reduces to the statement: Let $\Sigma(q)$ denote the group of permutations of a set q of $(p+1)$ elements. Let M be an Abelian group on which $(p+1)!$ acts invertibly. Then
$H_0(\Sigma(q), \det(q) \otimes \Lambda^i(M^q))$ vanishes when $i < p$ and is isomorphic to $Sym^p(M)$ when $i = p$.

III: The standard application of the *many units* hypothesis, see ([20] for instance) is that general position is available in the precise sense given below. Let $V \cong A^n$. We denote by $K(V)$ the simplicial complex whose set of vertices

is $\mathcal{L}(V)$. A subset $S \subset \mathcal{L}(V)$ of cardinality $(r+1)$ is an r-simplex of $K(V)$ if every $T \subset S$ of cardinality $t+1 \leq n$ belongs to $\mathcal{L}_t(V)$. If $L \subset K(V)$ is a finite simplicial subcomplex, then there is some $e \in \mathcal{L}(V)$ with the property that $s \cup \{e\}$ is a $(r+1)$-simplex of $K(V)$ for every r-simplex s of L. This gives an embedding $Cone(L) \hookrightarrow K(V)$ with e as the vertex of the cone. Thus $K(V)$ is contractible. The complex of oriented chains of this simplicial complex will be denoted by $C_\bullet(V)$. Thus the reduced homologies $\widetilde{H}_i(C_\bullet(V))$ vanish for all i. The group $D(V) = Z_{n-1}C_\bullet(V) = B_{n-1}C_\bullet(V)$ comes up frequently.
IIIa:

(1) Let $p < n$. Let M be a $\mathbb{Z}[1/N]$-module where $N = (p+1)!$. Then $H_j(PGL(V), C_p \otimes M)$ vanishes for $j < p$ and is isomorphic to $Sym^p(A^\times) \otimes M$ when $j = p$.
(2) $H_j(PGL(V), C_n \otimes M)$ vanishes for all $j \geq 0$ and for all $\mathbb{Z}[1/(n+1)!]$-modules M.
(3) $H_0(PGL(V), D(V) \otimes M)$ for any Abelian group M is isomorphic to $M/2M$ if n is even, and vanishes if when n is odd
(4) $H_0(PGL(V), B_pC_\bullet(V) \otimes M) = 0$ for every $\mathbb{Z}[1/2]$-module M and for every $0 \leq p < n$.
(5) $H_1(PGL(V), Z_pC_\bullet(V) \otimes M) = 0$ for every $\mathbb{Z}[1/(p+2)!]$- module M and $1 \leq p \leq n-2$.

Note that (1) above follows from IIa when F is the constant functor M.
For (2), one observes that $PGL(V)$ acts transitively on the set of n-simplices of the simplicial complex $K(V)$. The stabiliser of an n-simplex is the permutation group Σ on $(n+1)$ letters. The claim now follows from Shapiro's lemma.
The presentation $C_{p+2}(V) \otimes M \to C_{p+1}(V) \otimes M \to B_pC_\bullet(V) \otimes M \to 0$ and the observation $H_0(PGL(V), C_{p+1}(V) \otimes M) \cong M/2M$ whenever $p < n$ suffice to take care of (3) and (4).
For assertion (5), one applies the long exact sequence of group homology to the short exact sequence:
$0 \to Z_{p+1}C_\bullet(V) \otimes M \to C_{p+1}(V) \otimes M \to Z_pC_\bullet(V) \otimes M \to 0.$
One therefore obtains the exact sequence:
$H_1(PGL(V), C_{p+1}(V) \otimes M) \to H_1(PGL(V), Z_pC_\bullet(V) \otimes M) \to H_0(PGL(V), Z_{p+1}C_\bullet(V) \otimes M)$. The end terms here vanish by (1) and (4).
IIIb:

(1) $\mathbb{ET}(V)$ is connected.
(2) $E^2_{0,0} = \mathbb{Z}, E^2_{n-1,0} = D(V), E^2_{m,0} = 0$ if $m \neq 0, n-1$ for the spectral sequence $SS(V)$.
(3) $H_1(\mathbb{ET}(V)) \cong \mathbb{Z}/2\mathbb{Z}$ if $\mathrm{rank}(V) > 2$.

Note that (1) is a consequence of (2). Part (2) is deduced by induction on $\mathrm{rank}(V) = n$. The induction hypothesis enables the identification of the $E^1_{m,0}$ terms of the spectral sequence for V (together with differentials) with the $C_m(V)$ (together with boundary operators) when $m < n$. Thus (2) follows.

For part (3), consider the spectral sequence $SS(3)$. Here $E^2_{2,0} = D(A^3)$ and $H_0(PGL_3(A), D(A^3)) = 0$ by IIIa(3). Thus the hypothesis of Ib(4) holds for SS(3). Consequently,
$H_1(3) \cong E^\infty_{1,0} \cong H_0(PGL_2(A), H_1(2)) = H_0(PGL_2(A), D(A^2)) \cong \mathbb{Z}/2\mathbb{Z}$,
the last isomorphism given by IIIa(3) once again. The isomorphism $H_1(n) \cong \mathbb{Z}/2\mathbb{Z}$ for $n > 3$ is contained in the lemma below for $N = 1$.

Lemma 27. *Let M be an Abelian group. Let $N \in \mathbb{N}$. For $0 < r < N$, we are given $m(r) \geq 0$ so that $H_r(d; M) \to H_r(d+1; M)$*
is a surjection if $d = r + m(r) + 1$, and
an isomorphism if $d > r + m(r) + 1$.
Let $m(N) = \max\{0, m(1)+1, m(2)+1, ..., m(N-1)+1\}$.
Then $H_N(d; M) \to H_N(d+1; M)$ is
(a) an isomorphism if $d > N + m(N) + 1$,
(b) is a surjection if $d = N + m(N) + 1$.
(c) The surjection in (b) above factors through an isomorphism $H_0(PGL_d(A), H_N(d)) \to H_N(d+1; M)$ if if M is a $\mathbb{Z}[1/2]$-module.

Proof. Consider the spectral sequence $SS(V; M)$ that computes the homology of $\mathbb{ET}(V)$ with coefficients in M. Here V is free of rank $N + h + 2$, where $h \geq m(N)$. We make the following claim:
*Claim:*If $E^2_{s,r} \neq 0$ and $0 < s$ and $r < N$, then $s + r \geq N + 1 + h - m(N)$. Furthermore, when equality holds, $H_0(PGL(V), E^2_{s,r} \otimes \mathbb{Z}[1/2]) = 0$.
We assume the claim and prove the lemma. We take $h = m(N)$. All the $E^2_{s,r}$ with $s + r = N$ are zero except possibly for $(s, r) = (0, N)$. Part (b) of the lemma now follows from Ib(2). We consider next the $E^2_{s,r}$ with $s + r = N + 1$ and $s \geq 2$ (or equivalently with $r < N$). It follows that $E^s_{s,r}$ is a quotient of $E^2_{s,r}$. The second assertion of the claim now show that $H_0(PGL(V), E^s_{s,r}) = 0$ if M is a $\mathbb{Z}[1/2]$-module. Part (c) of the lemma now follows from Ib(4).
We take $h > m(N)$ and prove part (a) by induction on h. The inductive hypothesis implies that $H_N(N+h; M) \to H_N(N+h+1; M)$ is surjective. By Ib(5), it follows that $H_N(N+1+h; M) \to E^2_{0,N}$ is an isomorphism. Now there are no nonzero $E^2_{s,r}$ with $s + r = N + 1$ and $s \geq 2$. Thus $E^2_{0,N} = E^\infty_{0,N}$. It follows that $H_N(N+1+h; M) \to H_N(N+2+h; M)$ is an isomorphism.
It only remains to prove the claim. We address this matter now.
For $r = 0$, both assertions of the claim are valid by IIIb(2)and IIIa(3).
So assume now that $0 < r < N$. Let $SH(r) = H_r(d; M)$ for $d = r + m(r) + 2$.
In view of our hypothesis, the chain complex
$E^1_{0,r} \leftarrow E^1_{1,r} \leftarrow ... \leftarrow E^1_{p-1,r} \leftarrow E^1_{p,r}$
for $N + h + 1 = p + r + m(r) + 1$ is identified with
$C_0(V) \otimes SH(r) \leftarrow ... \leftarrow C_{p-1}(V) \otimes SH(r) \leftarrow \oplus\{\det(q) \otimes H_r(\mathbb{ET}(V/W(q)); M) | q \in \mathcal{L}_p(V)\}$.
As in IIIb(2), it follows that $E^2_{s,r} = 0$ whenever $0 < s < p$. Furthermore, we deduce the following exact sequence for $E^2_{p,r}$:

$\oplus\{\det(q) \otimes KH_r(V/W(q);M)|q \in \mathcal{L}_p\} \to E^2_{p,r} \to Z_pC_\bullet \otimes SH(r) \to 0$. By IIIa(4), we see that $H_0(PGL(V), E^2_{p,r}) = 0$ if M is a $\mathbb{Z}[1/2]$-module.
Note that $p + r = N + h - m(r) \geq N + 1 + h - M(N)$. This completes the proof of the claim, and therefore, the proof of the lemma as well.

□

The Proposition below is an application of Proposition 22. The notation here is that of Theorem 2. We regard $B^r_{p,q}$ and $Z^r_{p,q}$ as subgroups of $E^1_{p,q}$ for all $r > 1$. The notation $KH_m(Q)$ has been introduced in Ia, the first application of *many units*.

Proposition 28. *Let rank$(V) = n$. Let M be a $\mathbb{Z}[1/2]$-module. In the spectral sequence $SS(V;M)$, we have:*

(1) $\oplus\{\det(q) \otimes KH_m(V/W(q))|q \in \mathcal{L}_1(V)\} \subset B^\infty_{1,m}$ *if $n > 1$.*
(2) $E^\infty_{1,m} = 0$ *if $n > 2$ and M is a $\mathbb{Z}[1/6]$-module.*
(3) *If, in addition, it is assumed that*
$H_{m+1}(n-2;M) \to H_{m+1}(n-1;M)$ *is surjective, then*
$\oplus\{\det(q) \otimes KH_m(V/W(q))|q \in \mathcal{L}_2(V)\} \subset B^\infty_{2,m}$.

Proof. Let $q \in \mathcal{L}_r(V)$. We have $U(q) \subset \mathbb{ET}(V)$ as in Proposition 22. The spectral sequence of Theorem 2 was constructed from an increasing filtration of subspaces of $\mathbb{ET}(V)$. Intersecting this filtration with $U(q)$ we obtain a spectral sequence that computes the homology of $U(q)$. Its terms will be denoted by $E^a_{b,c}(q)$. One notes that $E^1_{b,m}(q)$ is the direct sum of $\det(u) \otimes H_m(\mathbb{ET}(V/W(u)))$ taken over all $u \subset q$ of cardinality $(b+1)$.
We denote the terms of the spectral sequence in theorem 2 by $E^a_{b,c}(V)$. The given data also provides a homomorphism $E^a_{b,c}(q) \to E^a_{b,c}(V)$ of E^1-spectral sequences. We assume that M is a $\mathbb{Z}[1/(r+1)!]$-module.
We choose a basis $e_1, e_2, ..., e_n$ of V so that
$q = \{Ae_i : 1 \leq i \leq r+1\}$. Let $G \subset GL(V)$ be the subgroup of $g \in GL(V)$ so that
(A) $g(q) = q$, (B)$g(e_i) = e_i$ for all $i > r+1$, (C), the matrix entries of g are $0, 1, -1$ and (D) $\det(g) = 1$. Now G acts on the pair $U(q) \subset \mathbb{ET}(V)$. Thus the above homomorphism of spectral sequences is one such in the category of G-modules.We observe:
(a) G is a group of order $2(r+1)!$
(b) there are no nonzero G-invariants in $E^1_{i,m}(q)$ for $i > 0$, and consequently the same holds for all G-subquotients, in particular for $E^a_{i,m}(q)$ for all $a > 0$ as well.
Proof of part 1. Take $r = 1$. Proposition 22 implies that the image of $H_m(U(q)) \to H_m(\mathbb{ET}(V))$ has trivial G-action. In view of (b) above, this shows that $E^\infty_{1,m}(q) \to E^\infty_{1,m}(V)$ is zero. But $E^\infty_{1,m}(q) = Z^\infty_{1,m}(q) = \det(q) \otimes KH_m(V/W(q))$. It follows that $\det(q) \otimes KH_m(V/W(q)) \subset B^\infty_{1,m}(V)$. Part (1) follows.
Proof of part (2). We take $r = 2$. Here we have
$Z^\infty_{1,m}(q) = Z^2_{1,m}(q)$. Appealing to Proposition 22 and observation (b) once

again, we see that the image of the homomorphism $Z^2_{1,m}(q) \to Z^2_{1,m}(V)$ is contained in $B^\infty_{1,m}$. Part (2) therefore follows from the claim below.
Claim: $\oplus\{Z^2_{1,m}(q) | q \in \mathcal{L}_2(V)\} \to Z^2_{1,m}(V)$ is surjective.
Denote the image of $H_m(n-2;M) \to H_m(n-1;M)$ by I. A simple computation produces the exact sequences:
$0 \to \oplus\{\det(u) \otimes KH_m(V/W(u) : u \in \mathcal{L}_1(V), u \subset q\} \to Z^2_{1,m}(q) \to \det(q) \otimes I \to 0$, and
$0 \to \oplus\{\det(u) \otimes KH_m(V/W(u) : u \in \mathcal{L}_1(V)\} \to Z^2_{1,m}(V) \to Z_1 C_\bullet(V) \otimes I \to 0$.
The claim now follows from the above description of $Z^2_{1,m}(q)$ and $Z^2_{1,m}(V)$. Thus part (2) is proved.
*Proof of part (3).*We take $r = 2$ once again. The surjectivity of $H_{m+1}(n-2;M) \to H_{m+1}(n-1;M)$ implies that $E^2_{0,m+1}(q)$ has trivial G-action. By observation (b), we see that $d^2_{2,m} : E^2_{2,m}(q) \to E^2_{0,m+1}(q)$ is zero. It follows that $E^\infty_{2,m}(q) = Z^2_{2,m}(q)$ here. Proposition 22 and observation (b) once again show that the image of $Z^2_{2,m}(q) \to Z^2_{2,m}(V)$ is contained in $B^\infty_{2,m}(V)$. Because $Z^2_{2,m}(q) = \det(q) \otimes KH_m(V/W(q))$, part (3) follows.
This completes the proof of the Proposition. □

Theorem 3. *Let $H_m(n;M)$ denote $H_m(\mathbb{ET}(A^n);M)$ where M is a $\mathbb{Z}[1/6]$-module. We have:*
(1) $H_1(n;M) = 0$ for all $n > 2$,
(2) $H_0(GL_3(A), H_2(3;M)) \to H_2(n;M)$ is an isomorphism for all $n \geq 4$,
(3) $H_0(GL_4(A), H_3(4;M)) \to H_3(n;M)$ is an isomorphism for all $n \geq 5$,
(4) $H_0(GL_{2m-2}(A), H_m(2m-2;M)) \to H_m(n;M)$ is an isomorphism for all $n > 2m-2$.

Proof. Part (1) has already been proved.
Proof of part 2. For this, we study $SS(V;M)$ where $\mathrm{rank}(V) = 4$. We first note that
(i) $E^2_{3,0} = D(V)$ and therefore $H_0(PGL(V), E^2_{3,0}) = 0$.
(ii) $E^2_{1,1} = E^1_{1,1} = \oplus\{\det(q) \otimes D(V/W(q)) : q \in \mathcal{L}_1(V)\}$, and therefore $H_1(PGL(V), E^2_{1,1}) = 0$ by IIa.
(iii) $E^2_{u,v} = 0$ except when $(u,v) = (0,0), (0,2), (1,1), (3,0)$. We have $E^\infty_{1,1} = 0$ by proposition 28 and thus obtain the short exact sequence:
$0 \to E^3_{3,0} \to E^2_{3,0} \to E^2_{1,1} \to 0$.
By (i) and (ii) above, we see that $H_0(PGL(V), E^3_{3,0}) = 0$. By Ib(1,4), we see that $H_0(PGL_3(A), H_2(3;M)) \to H_2(4;M)$ is an isomorphism. In particular, $H_2(4;M)$ receives the trivial $PGL_4(A)$-action. Taking $N = 2$ and $m(1) = 0$ in lemma 27, we see that $H_2(4;M) \to H_2(n;M)$ is an isomorphism for all $n \geq 4$. This proves part (2).
Proof of part 3. We inspect $SS(V;M)$ where $V = A^5$. We note that
(1) $E^\infty_{1,2}$ and $E^\infty_{2,1}$ both vanish. This follows from proposition 28, once it is noted that $KH_1(2;M) = H_1(2;M)$.
(2) $E^2_{0,2} = H_2(4;M)$ has the trival $PGL(V)$-action.

(3) $H_i(PGL(V), E^2_{2,1}) = 0$ for all $i < 3$. This follows from IIa and IIIa(3) after observing that $H_1(2; M) \cong D(A^2) \otimes M$.
(4) From (2) and (3) we see that $d^2_{2,1} = 0$.
(5) We deduce that $E^2_{2,1} \cong E^2_{0,4}/E^3_{0,4}$ and $E^2_{1,2} = E^3_{1,2} \cong E^3_{0,4}/E^4_{0,4}$ from observations (1) and (4).
(6) $H_1(PGL(V), E^2_{1,2}) = 0$.
To see this, first note the the short exact sequence:
$0 \to P \to E^2_{1,2} \to Q \to 0$, where
$P = \oplus\{\det(q) \otimes KH_2(V/W(q) : q \in \mathcal{L}_1(V)\}$ and $Q = H_2(4; M) \otimes Z_1C_\bullet(V)$.
The vanishing of $H_1(PGL_3(A), Q)$ follows from IIIa(5). By IIa, the vanishing of $H_1(PGL_3(A), P)$ is reduced to the vanishing of $H_0(PGL_3(A), KH_2(A^3)$.
Now let I be the augmentation ideal of the group algebra $R[PGL_3(A)]$ where $R = \mathbb{Z}[1/6]$. In view of the fact that $PGL_3(A)_{ab}$ is 3-torsion, we see that $I = I^2$. It follows that for all $\mathbb{Z}[1/6]$-modules N equipped with $PGL_3(A)$-action, we have $IN = I^2N$, or equivalently, $H_0(PGL_3(A), IN) = 0$. We apply this remark to $N = H_2(3; M)$. By part (2) of the proposition, we see that $KH_2(3; M) = IN$. This proves that $H_0(PGL_3(A), KH_2(A^3)) = 0$. We have completed the proof of observation 6.
(7) $H_0(PGL(V), E^4_{4,0}) = 0$.
In view of the filtration of (5), it suffices to check that $H_1(PGL(V), E^2_{a,b}) = 0$ for $(a, b) = (1, 2)$ and $(2, 1)$ (which has been seen in observations (3) and (6)) and also that $H_0(PGL(V), E^2_{4,0}) = 0$ (and this is clear because $E^2_{4,0} = D(V)$).
(8) $H_0(PGL_4(A), H_3(4; M)) \to H_3(V; M)$ is an isomorphism.
That $H_0(PGL_4(A), H_3(4; M)) \to E^\infty_{0,3}$ is an isomorphism follows from observation (7) and Ib(4). Now $E^\infty_{a,b} = 0$ whenever $a + b = 3$ and $(a, b) \neq (3, 0)$. This proves (8).
(9) $H_3(5; M) \to H_3(n; M)$ is an isomorphism for all $n \geq 5$.
This follows from lemma 27 by taking $N = 3$ and $m(1) = m(2) = 0$. This finishes the proof of part (3).
Part (4) now follows from the same lemma and induction.

□

Remark. It can be checked that parts (1,2,4) of the above theorem are valid for $\mathbb{Z}[1/2]$-modules M. In part (3), it is true that $H_3(n; M) \cong H_3(n + 1; M)$ for $n > 4$ and also that $H_0(PGL_4(A), H_3(4; M) \to H_3(5; M)$ is a surjection.

Proposition 29. *Assume that the Compatible Homotopy Question has an affirmative answer. Then, for all $\mathbb{Z}[1/r!]$-modules M and for all $d > r + 1$, $H_0(PGL_{r+1}(A), H_r(r + 1; M)) \to H_r(d; M)$ is an isomorphism.*

Proof. For $r = 1$, this statement has been checked in IIIb(3) and lemma 27. Let $N > 1$. We assume that the above statement has been proved for all $r < N$. Let M be a $\mathbb{Z}[1/N!]$-module. In lemma 27, may may now take $m(1) = m(2) = \dots = m(N - 1) = 0$. From this lemma, we obtain:

$$H_0(PGL_{N+2}(A), H_N(N + 2; M)) \to H_N(N'; M)$$

is an isomorphism for all $N' > N+2$. So the proposition is proved once it is checked that

$$H_0(PGL_{N+1}(A), H_N(N+1; M)) \to H_N(N+2; M)$$

is an isomorphism. To prove this, we consider the spectral sequence $SS(V; M)$ where $V = A^{N+2}$. We will prove:
(i) $E^2_{a,b} = 0$ or $a = 0$ or $a+b = N$ or $(a,b) = (N+1, 0)$. Furthermore the action of $PGL(V)$ on $E^2_{0,b}$ is trivial when $b < N$.
(ii) if $a > 0$ and $b > 0$, then $H_i(PGL(V), E^2_{a,b}) = 0$ for $i = 0, 1$.
(iii) $E^\infty_{a,b} = 0$ when $a > 0$ and $b > 0$.
(iv) $E^2_{a,b} \cong E^{b+1}_{N+1,0}/E^{b+2}_{N+1,0}$ whenever $a > 0, b > 0$ and $a+b = N$.
We first observe that (iv) is true for any spectral sequence of $PGL(V)$-modules where (i),(ii) and (iii) hold. Next note that (ii) and (iv) imply that $H_0(PGL(V), E^{N+1}_{N+1,0})$ is contained in $H_0(PGL(V), E^2_{N+1,0})$. And since the latter is zero, we see that the former also vanishes.
We deduce that both arrows $H_0(PGL(V), E^1_{0,N}) \to E^\infty_{0,N} \to H_N(N+2; M)$ are isomorphisms exactly as in earlier proofs. Thus it only remains to prove (i), (ii) and (iii).
Proof of (i). This is contained in the proof of lemma 27.
Proof of (ii). $0 \to P \to E^2_{a,b} \to Q \to 0$ is exact, where
$P = \oplus\{\det(q) \otimes KH_b(V/W(q) | q \in \mathcal{L}_a(V)\}$, and $Q = Z_a C_\bullet \otimes H_b(b+2; M)$ as in the proof of the lemma 27. The required vanishing of $H_i(PGL(V), T)$ for $i = 0, 1$ holds for $T = Q$ by IIIa(5). For $T = P$ and $a > 1$, the required vanishing follows from II(a). For $T = P$ and $a = 1$, this is deduced from the vanishing of $H_0(PGL_N(A), KH_{N-1})$ (see the proof of observation (6) in the proof of theorem 3).
Proof of (iii). We follow the steps of the proof of Proposition 28. We first choose $q \in \mathcal{L}_{N-1}(V)$ and consider the inclusion $U(q) \hookrightarrow \mathbb{ET}(V)$. As in that proof we get a homomorphism of E^1 spectral sequences of G-modules with $G \subset SL(V)$ as given there. The terms of these spectral sequences are denoted by $E^a_{b,c}(q)$ and $E^a_{b,c}(V)$ respectively. From the inductive hypothesis, we deduce:
(i′) $E^2_{a,b}(q) = 0$ or $a = 0$ or $a+b = N$ or $(a,b) = (N+1, 0)$. Furthermore the action of G on $E^2_{0,b}(q)$ is trivial when $b < N$.
(ii′) if $a > 0, h > 0$, then $H_0(G, E^h_{a,b}(q)) = 0$.
These observations together imply
(iii′) $Z^\infty_{a,b}(q) = Z^2_{a,b}(q)$ when $a > 0$ and $b > 0$.
For $a > 0, b > 0$, we obtain $E^\infty_{a,b}(q) \to E^\infty_{a,b}(V)$ is zero, from the affirmative answer to the Compatible Homotopy Question. For such (a,b),
the image of $x(q) : Z^2_{a,b}(q) \to Z^2_{a,b}(V)$ is thus contained in $B^\infty_{a,b}(V)$. As in the proof of proposition 28, we see that the sum of the images of $x(q)$, taken over all $q \in \mathcal{L}_{N-1}(V)$, is all of $Z^2_{a,b}(V)$. It follows that $Z^2_{a,b}(V) = B^\infty_{a,b}(V)$ and thus $E^\infty_{a,b}(V) = 0$ whenever $a > 0, b > 0$. This proves assertion (iii) and this completes the proof of the Proposition.

□

10. a double complex

We will continue to assume that A is a commutative ring with many units. The paper [2] of Beilinson, Macpherson and Schechtman introduces a Grassmann complex, intersection and projection maps, and a torus action. The terms of the double-complex constructed below may be obtained from the quotients by the torus action of the objects of [2]. The arrows of the double-complex are signed sums of their intersection and projection maps.

$D(V), C_\bullet(V)$ etc. are as in the previous section. When $\mathrm{rank}(V) = n$, we have the resolution:

$0 \leftarrow D(V) \leftarrow C_n(V) \leftarrow C_{n+1}(V)....$

We put $\overline{C}_r(V) = H_0(PGL(V), C_r(V))$ when $r \geq n$ and define $\overline{C}_r(V)$ to be zero otherwise. We put $\overline{C}_r(A^n) = \overline{C}_r(n)$. We observe that the above resolution of $D(V)$ tensored with the rationals is a projective resolution in the category of $\mathbb{Q}[PGL(V)]$-modules. It follows that $H_i(PGL_n(A), D(A^n)) \otimes \mathbb{Q} \cong H_{n+i}(\overline{C}(n)_\bullet) \otimes \mathbb{Q}$. We denote by $\partial' : \overline{C}_r(n) \to \overline{C}_{r-1}(n)$ the boundary operator of $\overline{C}(n)_\bullet$. We will now define $\partial'' : \overline{C}_r(n) \to \overline{C}_r(n-1)$.

Let $V \cong A^n$. Let $(L_0, L_1, ..., L_r)$ be an ordered $(r+1)$-tuple in $\mathcal{L}(V)$ that gives rise to a r-simplex of $K(V)$ (see consequence III of *many units* for notation). We define $\partial_i(L_0, L_1, ..., L_r) \in C_{r-1}(V/L_i)$ by $\partial_i(L_0, L_1, ..., L_r) = (\overline{L}_0, .., \overline{L}_{i-1}, \overline{L}_{i+1}, ..., \overline{L}_r)$ where $\overline{L}_j = L_j + L_i/L_i \in \mathcal{L}(V/L_i)$ whenever $j \neq i$. Now let

$$g_r(L_0, L_1, ..., L_r) = \Sigma_{i=0}^r (-1)^i \partial(L_0, L_1, ..., L_r) \in \oplus\{C_{r-1}(V/L) : L \in \mathcal{L}(V)\}.$$

The above $g_r : C_r(V) \to \oplus\{C_{r-1}(V/L) : L \in \mathcal{L}(V)\}$ anti-commutes with the boundary operator. The functor $M \to H_0(PGL(V), M)$ takes g_r to $\partial'' : \overline{C}_r(n) \to \overline{C}_{r-1}(n-1)$. This defines ∂''.

We put $F_r(A) = \oplus\{\overline{C}_r(n) : n \geq 1\}$ and define $\partial : F_r(A) \to F_{r-1}(A)$ by $\partial = \partial' + \partial''$. The exact relation between the homology of $F_\bullet(A)$ and groups $L_n(A)$ is as yet unclear. However, we do have:

Lemma 30. $H_3(F_\bullet(A)) \otimes \mathbb{Q} \cong L_2(A) \otimes \mathbb{Q} \cong H_3(\overline{C}_\bullet(2)) \otimes \mathbb{Q}$.

We sketch a proof. In view of the H-space structure, $L_i(A) \otimes \mathbb{Q}$ is the primitive homology of $\mathbb{ET}(A^n)$ with $\mathbb{Q}$ coefficients for n large. The vanishing of $H_1(n; \mathbb{Q})$ for $n > 2$ implies that the primitive homology is all of $H_i(n; \mathbb{Q})$ for $i = 2, 3$ and n large. By theorem 3, we get $L_i(A) \otimes \mathbb{Q} \cong H_0(PGL_{i+1}(A), H_i(i+1; \mathbb{Q}))$ for $i = 2, 3$. For the computation of $H_0(PGL(V), H_2(\mathbb{ET}(V); \mathbb{Q}))$ where $V = A^3$, we recall the exact sequence obtained from $SS(V; \mathbb{Q})$:

$0 \to H_2(\mathbb{ET}(V); \mathbb{Q}) \to D(V) \otimes \mathbb{Q} \to D_2(V) = \oplus\{D(V/L) : L \in \mathcal{L}(V)\} \to 0.$

This identifies $L_2(A) \otimes \mathbb{Q}$ with the cokernel of $H_1(PGL(V), D(V)) \otimes \mathbb{Q} \to H_1(PGL(V), D_2(V)) \otimes \mathbb{Q}$. In view of IIa and the above remarks, this is readily identified with $H_3(F_\bullet(A)) \otimes \mathbb{Q}$.That gives the first isomorphism of the lemma. For the second isomorphism, what one needs is:

Claim: The arrow $H_4(\overline{C}_\bullet(3)) \to H_3(\overline{C}_\bullet(2))$ induced by ∂'' is zero.

The proof of this claim, which we address now, was already known to Spencer Bloch. Let $V = A^3$. Given an ordered 5-tuple $(L_0, ..., L_4)$ with the $L_i \in \mathcal{L}(V)$

as vertices of a 4-simplex in $K(V)$ (i.e. in general position), they belong to a conic C and the projection from the points L_i induces an isomorphism $p_i : C \to \mathbb{P}(V/L_i)$. We put $(M_0, ..., M_4) = (p_0 L_0, p_0 L_1, ..., p_0 L_4)$. Let $q_i = p_i \circ p_0^{-1}$. With the ∂_i as in the definition of g_4, we see that $\partial_i(L_0, ..., L_4) \in C_3(V/L_i)$ and $q_i \partial_i(L_0, ..., L_4) \in C_3(V/L_0)$ both give rise to the same element of $\overline{C}_3(2)$. It follows that $\partial(M_0, M_1, ..., M_4) \mapsto \partial''(L_0, ..., L_4)$ under the map $C_3(V/L_0) \to \overline{C}_3(2)$. Thus $\partial''(L_0, ..., L_4) \mapsto 0 \in H_3(\overline{C}_\bullet(2)$. This proves the claim and the lemma.

Thus we have shown that
$L_2(A) \otimes \mathbb{Q} \cong \operatorname{coker}(\overline{C}_4(A^2) \to \overline{C}_3(A^2))$.
The Bloch group tensored with $\mathbb{Q}$ is the homology of

$$\overline{C}_4(A^2) \to \overline{C}_3(A^2) \to \Lambda^2(A^\times) \otimes \mathbb{Q}.$$

Thus this discussion amounts to a proof of Suslin's theorem on the Bloch group. It remains to obtain a closed form for $L_3(A) \otimes \mathbb{Q}$ by this method.

Acknowledgements: This paper owes, most of all, to Sasha Beilinson who introduced us to the notion of a homotopy point. The formulation in terms of contractible spaces of maps occurs in many places in this manuscript. Rob de Jeu lent a patient ear while we groped towards the definition of $\mathbb{FL}(V)$. Peter May showed us a proof of Proposition 20. We thank them, Kottwitz and Gopal Prasad for interesting discussions, and also Spencer Bloch for his support throughout the project. Finally we thank the referee who brought our attention to several careless slips in the first draft.

References

[1] A. A. Beilinson, A. B. Goncharov, V. V. Schechtman, A. N. Varchenko *Aomoto dilogarithms, mixed Hodge structures and motivic cohomology of pairs of triangles on the plane.* The Grothendieck Festschrift, Vol. I, 135–172, Progr. Math., 86, Birkh?user Boston, Boston, MA, 1990

[2] A. Beilinson, R. MacPherson, V. Schechtman *Notes on motivic cohomology.* Duke Math. J. 54 (1987), no. 2, 679–710.

[3] S. Bloch, I. Kris *Mixed Tate motives.* Ann. of Math. (2) 140 (1994), no. 3, 557–605.

[4] A. Hatcher *Algebraic Topology.* Cambridge University Press, 2002.

[5] R. P. Langlands *Orbital integrals on forms of* SL(3). *I.* Amer. J. Math. 105 (1983), no. 2, 465–506.

[6] J.-L. Loday *K-théorie algébrique et représentations de groupes*, Ann. Sci. E.N.S. 9 (1976) 309-377.

[7] J. Milnor, J. Moore *On the structure of Hopf algebras*, Ann. Math. (2) 81 (1965) 211-264.

[8] J. P. May *E_∞ spaces, group completions, and permutative categories.* New developments in topology (Proc. Sympos. Algebraic Topology, Oxford, 1972), pp. 61–93. London Math. Soc. Lecture Note Ser., No. 11, Cambridge Univ. Press, London, 1974.

[9] J. P. May *The geometry of iterated loop spaces* Lecture Notes in Math, Vol. 271, Springer-Verlag.

[10] R. Cohen, T. Lada, J. P. May *The homology of iterated loop spaces.* Lecture Notes in Mathematics, Vol. 533. Springer-Verlag, Berlin-New York, 1976.

[11] J. P. May *A Concise Course in Algebraic Topology.* Chicago Lectures in Mathematics, 1999.

[12] B. Mirzaii, *Homology of* GL_n*: injectivity conjecture for* GL_4. Math. Ann. 340 (2008), no. 1, 159–184.

[13] Yu. P. Nesterenko, A. A. Suslin *Homology of the full linear group over a local ring, and Milnor's K-theory*, Math. USSR Izvestiya 34 (1990) 121-145 (translation of Russian original, in Izv. Akad. Nauk. SSSR Ser. mat 53 (1989) 121-146).

[14] D. Quillen *Higher Algebraic K-theory*, Lect. Notes in Math. No. 341, Springer-Verlag (1974).

[15] J. Rognes *A spectrum level rank filtration in algebraic K-theory* Topology 31 (1992) 813-845.

[16] G. Segal *Configuration-spaces and iterated loop-spaces.* Invent. Math. 21 (1973), 213–221.

[17] G. Segal *Categories and cohomology theories.* Topology 13 (1974), 293–312.

[18] E. Spanier *Algebraic Topology*, Springer Verlag, New York 1966.

[19] V. Srinivas *Algebraic K-theory (Second Edition)*, Progress in Math. 90, Birkhäuser (1996).

[20] A. A. Suslin *Homology of* GL_n*, characteristic classes and Milnor K-theory.* (Russian) Algebraic geometry and its applications. Trudy Mat. Inst. Steklov. 165 (1984), 188–204.

[21] A. A. Suslin K_3 *of a field, and the Bloch group.* (Russian) Translated in Proc. Steklov Inst. Math. 1991, no. 4, 217–239. Galois theory, rings, algebraic groups and their applications (Russian). Trudy Mat. Inst. Steklov. 183 (1990), 180–199, 229.

[22] A. A. Suslin *On the equivalence of K-theories.* Comm. Algebra 9 (1981), no. 15, 1559–1566.

M. V. Nori
Dept. of Mathematics
University of Chicago
5734 University Ave.
Chicago
IL-60637
USA
nori@math.uchicago.edu

V. Srinivas
School of Mathematics
Tata Institute
of Fundamental Research
Homi Bhabha Road
Colaba
Mumbai-400005
India
srinivas@math.tifr.res.in

Rationally Isotropic Quadratic Spaces Are Locally Isotropic: II

Ivan Panin and Konstantin Pimenov

Received: September 06, 2009
Revised: April 17, 2010

Abstract. The results of the present article extend the results of [Pa]. The main result of the article is Theorem 1.1 below. The proof is based on a moving lemma from [LM], a recent improvement due to O. Gabber of de Jong's alteration theorem, and the main theorem of [PR]. A purity theorem for quadratic spaces is proved as well in the same generality as Theorem 1.1, provided that R is local. It generalizes the main purity result from [OP] and it is used to prove the main result in [ChP].

1 Introduction

Let A be a commutative ring and P be a finitely generated projective A-module. An element $v \in P$ is called unimodular if the A-submodule vA of P splits off as a direct summand. If $P = A^n$ and $v = (a_1, a_2, \dots, a_n)$ then v is unimodular if and only if $a_1A + a_2A + \cdots + a_nA = A$.
Let $\frac{1}{2} \in A$. A quadratic space over A is a pair (P, α) consisting of a finitely generated projective A-module P and an A-isomorphism $\alpha : P \to P^*$ satisfying $\alpha = \alpha^*$, where $P^* = \mathrm{Hom}_R(P, R)$. Two spaces (P, α) and (Q, β) are *isomorphic* if there exists an A-isomorphism $\varphi : P \to Q$ such that $\alpha = \varphi^* \circ \beta \circ \varphi$.
Let (P, φ) be a quadratic space over A. One says that it is *isotropic* over A, if there exists a unimodular $v \in P$ with $\varphi(v) = 0$.

Theorem 1.1. *Let R be a semi-local regular integral domain containing a field. Assume that all the residue fields of R are infinite and $\frac{1}{2} \in R$. Let K be the fraction field of R and (V, φ) a quadratic space over R. If $(V, \varphi) \otimes_R K$ is isotropic over K, then (V, φ) is isotropic over R.*

This Theorem is a consequence of the following result.

Theorem 1.2. *Let k be an infinite perfect field of characteristic different from 2, B a k-smooth algebra. Let $p_1, p_2, \dots, p_n$ be prime ideals of B, $S = B - \cup_{j=1}^{n} p_j$ and $R := B_S$ be the localization of B with respect to S (note that B_S is a semi-local ring). Let K be the ring of fractions of R with respect to all non-zero divisors and (V, φ) be a quadratic space over R. If $(V, \varphi) \otimes_R K$ is isotropic over K, then (V, φ) is isotropic over R.*

For arbitrary discrete valuation rings, Theorem 1.1 holds trivially. It also holds for arbitrary regular local two-dimensional rings in which 2 is invertible, as proved by M. Ojanguren in [O].

To conclude the Introduction let us add a historical remark which might help the general reader. Let R be a regular local ring, G/R a reductive group scheme. The question whether a principal homogeneous space over R which admits a rational section actually admits a section goes back to the foundations of étale cohomology. It was raised by J.-P. Serre and A. Grothendieck (séminaire Chevalley "Anneaux de Chow"). In the geometric case, this question has essentially been solved, provided that G/R comes from a ground field k. Namely, J.-L. Colliot-Thélène and M. Ojanguren in [CT-O] deal with the case where the ground field k is infinite and perfect. There were later papers [Ra1] and [Ra2] by M.S. Raghunathan, which handled the case k infinite but not necessarily perfect. O. Gabber later announced a proof in the general case. One may then raise the question whether a similar result holds for homogeneous spaces. A specific instance is that of projective homogeneous spaces. An even more specific instance is that of smooth projective quadrics (question raised in [C-T], Montpellier 1977). This last case is handled in the present paper. Remark 3.5 deals with the semi-local case.

The key point of the proof of Theorem 1.2 is the combination of the moving lemma in [LM] and Gabber's improvement of the alteration theorem due to de Jong with the generalization of Springer's result in [PR]. Theorem 1.1 is deduced from Theorem 1.2 using D. Popescu's theorem.

2 Auxiliary results

Let k be a field. To prove Theorem 1 we need auxiliary results. We start recalling the notion of transversality as it is defined in [LM, Def.1.1.1].

Definition 2.1. *Let $f : X \to Z$, $g : Y \to Z$ be morphisms of k-smooth schemes. We say that f and g are transverse if*

1. *$Tor_q^{\mathcal{O}_Z}(\mathcal{O}_Y, \mathcal{O}_X) = 0$ for all $q > 0$.*
2. *The fibre product $X \times_Z Y$ is a k-smooth scheme.*

Lemma 2.2. *Let $f : X \to Z$ and $g : Y \to Z$ be transverse, and $pr_Y : Y \times_Z X \to Y$ and $h : T \to Y$ be transverse, then f and $g \circ h$ are transverse.*

This is just Lemma 1 from [Pa].

Since this moment and till Remark 2.6 (including that Remark) let k be an infinite perfect field of characteristic different from 2. Let U be a smooth irreducible quasi-projective variety over k and let $j : u \to U$ be a closed point of U. In particular, the field extension $k(u)/k$ is finite. It is also separable since k is perfect. Thus $u = Spec(k(u))$ is a k-smooth variety.

Lemma 2.3. *Let U be as above. Let Y be a k-smooth irreducible variety of the same dimension as U. Let $v = \{v_1, v_2, \dots, v_s\} \subset U$ be a finite set of closed points. Let $q : Y \to U$ be a projective morphism such that $q^{-1}(v) \neq \emptyset$. Assume $q : Y \to U$ and $j_v : v \hookrightarrow U$ are transverse. Then q is finite étale over an affine neighborhood of the set $v \subset U$.*

Proof. There is a $v_i \in v$ such that $q^{-1}(v_i) \neq \emptyset$. By [Pa, Lemma 2] q is finite étale over a neighborhood V_i of the point $v_i \in U$. This implies that $V_i \subset q(Y)$. It follows that $q(Y) = U$, since q is projective and U is irreducible. Whence for each $i = 1, 2, \dots, s$ one has $q^{-1}(v_i) \neq \emptyset$. By [Pa, Lemma 2] for each $m = 1, 2, \dots, s$ the morphism q is finite étale over a neighborhood V_m of the point $v_m \in U$. Since U is quasi-projective, q is finite étale over an affine neighborhood V of the set $v \subset U$.

□

Let U be as above. Let $p : \mathcal{X} \to U$ be a smooth projective k-morphism. Let $X = p^{-1}(u)$ be the fibre of p over u. Since p is smooth the $k(u)$-scheme X is smooth. Since $k(u)/k$ is separable X is smooth as a k-scheme. Thus for a morphism $f : Y \to \mathcal{X}$ of a k-smooth scheme Y it makes sense to say that f and the embedding $i : X \hookrightarrow \mathcal{X}$ are transverse. So one can state the following

Lemma 2.4. *Let $p : \mathcal{X} \to U$ be as above, let $j_v : v \hookrightarrow U$ be as in Lemma 2.3 and let $X = p^{-1}(v)$ be as above. Let Y be a k-smooth irreducible variety with $dim(Y) = dim(U)$. Let $f : Y \to \mathcal{X}$ be a projective morphism such that $f^{-1}(X) \neq \emptyset$. Suppose that f and the closed embedding $i : X \hookrightarrow \mathcal{X}$ are transverse. Then the morphism $q = p \circ f : Y \to U$ is finite étale over an affine neighborhood of the set v.*

Proof. For each $i = 1, 2, \dots, s$ the extension $k(u)/k$ is finite. Since k is perfect, the scheme v is k-smooth. The morphism $p : \mathcal{X} \to U$ is smooth. Thus the morphism j_v and the morphism p are transverse. Morphisms j_v and $q = p \circ f$ are transverse by Lemma 2.2, since j_v and f are transverse. One has $q^{-1}(v) = f^{-1}(X) \neq \emptyset$. Now Lemma 2.3 completes the proof of the Lemma.

□

For a k-smooth variety W let $CH_d(W)$ be the group of dimension d algebraic cycles modulo rational equivalence on W (see [Fu]). The next lemma is a variant of the proposition [LM, Prop. 3.3.1] for the Chow groups $Ch_d := CH_d/2CH_d$ of algebraic cycles modulo rational equivalence with $\mathbb{Z}/2\mathbb{Z}$-coefficients.

Lemma 2.5 (A moving lemma). *Suppose that k is an infinite perfect field (the characteristic of k is different from 2 as above). Let W be a k-smooth scheme* *and let $i : X \hookrightarrow W$ be a k-smooth closed subscheme. Then $Cd_d(W)$ is generated by the elements of the form $f_*([Y])$ where Y is an irreducible k-smooth variety of dimension d, $[Y] \in Cd_d(Y)$ is the fundamental class of Y, $f : Y \to W$ is a projective morphism such that f and i are transverse and $f_* : Ch_d(Y) \to Ch_d(W)$ is the push-forward.*

Proof. The group $Ch_d(W)$ is generated by cycles of the form $[Z]$, where $Z \subset W$ is a closed irreducible subvariety of dimension d. Since k is perfect of characteristic different from 2, applying a recent result due to Gabber [I, Thm. 1.3], one can find a k-smooth irreducible quasi-projective variety Z' and a proper morphism $\pi : Z' \to Z$ with k-smooth quasi-projective variety Z' and such that the degree $[k(Z') : k(Z)]$ is odd. The morphism p is necessary projective, since the k-variety Z' is quasi-projective and p is a proper morphism (see [Ha, Ch.II, Cor.4.8.e]). Write π' for the composition $Z' \to Z \hookrightarrow W$. Clearly, $\pi'_*([Z']) = [Z] \in Cd_d(W)$. The lemma is not proved yet, since π' and i are not transverse.
However to complete the proof it remains to repeat literally the proof of proposition [LM, Prop. 3.3.1]. The proof of that proposition does not use the resolution of singularities. Whence the lemma.

□

Remark 2.6. *Note that at the end of the previous proof we actually used a Chow version of [LM, Prop. 3.3.1] instead of Prop. 3.3.1 itself.*

The following theorem proved in [PR] is a generalization of a theorem of Springer. See [La, Chap.VII, Thm.2.3] for the original theorem by Springer.

Theorem 2.7. *Let R be a local Noetherian domain which has an infinite residue field of characteristic different from 2. Let $R \subset S$ be a finite R-algebra which is étale over R. Let (V, φ) be a quadratic space over R such that the space $(V, \varphi) \otimes_R S$ contains an isotropic unimodular vector. If the degree $[S : R]$ is odd then the space (V, φ) already contains a unimodular isotropic vector.*

Remark 2.8. *Theorem 2.7 is equivalent to the main result of [PR], since the R-algebra S from Theorem 2.7 one always has the form $R[T]/(F(T))$, where $F(T)$ is a separable polynomial of degree $[S : R]$ (see [AK, Chap.VI, Defn.6.11, Thm.6.12]).*

Repeating verbatim the proof of Theorem 2.7 given in [PR] we get the following result.

Theorem 2.9. *Let R be a semi-local Noetherian integral domain SUCH THAT ALL ITS residue fields ARE INFINITE of characteristic different from 2. Let $R \subset S$ be a finite R-algebra which is étale over R. Let (V, φ) be a quadratic space over R such that the space $(V, \varphi) \otimes_R S$ contains an isotropic unimodular vector. If the degree $[S : R]$ is odd then the space (V, φ) already contains a unimodular isotropic vector.*

3 Proofs of Theorems 1.2 and 1.1

Proof of Theorem 1.2. Let k be an infinite perfect field of characteristic different from 2. Let $p_1, p_2, \dots, p_n$ be prime ideals of B, $S = B - \cup_{j=1}^{n} p_j$ and $R = B_S$ be the localization of B with respect to S.
Clearly, it is sufficient to prove the theorem in the case when B is an integral domain. So, in the rest of the proof we will assume that B is an integral domain. We first reduce the proof to the localization at a set of maximal ideals. To do that we follow the arguments from [CT-O, page 101]. Clearly, there exist $f \in S$ and a quadratic space (W, ψ) over B_f such that $(W, \psi) \otimes_{B_f} B_S = (V, \varphi)$. For each index j let m_j be a maximal ideal of B containing p_j and such that $f \notin m_j$. Let $T = B - \cup_{j=1}^{n} m_j$. Now B_T is a localization of B_f and one has $B_f \subset B_T \subset B_S = R$. Replace R by B_T.
From now on and until the end of the proof of Theorem 1.2 we assume that $R = \mathcal{O}_{U,\{u_1,u_2,\dots,u_n\}}$ is the semi-local ring of a finite set of closed points $u = \{u_1, u_2, \dots, u_n\}$ on a k- smooth d-dimensional irreducible affine variety U.
Let $\mathcal{X} \subset \mathbf{P}_R(V)$ be a projective quadric given by the equation $\varphi = 0$ in the projective space $\mathbf{P}_R(V) = Proj(S^*(V^\vee))$. Let $X = p^{-1}(u)$ be the scheme-theoretic pre-image of u under the projection $p : \mathcal{X} \to Spec(R)$. Shrinking U we may assume that u is still in U and the quadratic space (V, φ) is defined over U. We still write $\mathcal{X}$ for the projective quadric in $\mathbf{P}_U(V)$ given by the equation $\varphi = 0$ and still write $p : \mathcal{X} \to U$ for the projection. Let $\eta : Spec(K) \to U$ be the generic point of U and let $\mathcal{X}_\eta$ be the generic fibre of $p : \mathcal{X} \to U$. Since the equation $\varphi = 0$ has a solution over K there exists a K-rational point y of $\mathcal{X}_\eta$. Let $Y \subset \mathcal{X}$ be its closure in $\mathcal{X}$ and let $[Y] \in Ch_d(\mathcal{X})$ be the class of Y in the Chow groups with $\mathbb{Z}/2\mathbb{Z}$-coefficients.
Since p is smooth the scheme X is $k(u)$-smooth. Since $k(u)/k$ is a finite étale algebra X is smooth as a k-scheme. By Lemma 2.5 there exist a finite family of integers $n_r \in \mathbb{Z}$ and a finite family of projective morphisms $f_r : Y_r \to \mathcal{X}$ (with k-smooth irreducible Y_r's of dimension $dim(U)$) which are transverse to the closed embedding $i : X \hookrightarrow \mathcal{X}$ and such that $\sum n_r f_{r,*}([Y_r]) = [Y]$ in $Ch_d(\mathcal{X})$. Shrinking U we may assume that for each index r one has $f_r^{-1}(X) \neq \emptyset$. By Lemma 2.4 for any index r the morphism $q_r = p \circ f_r : Y_r \to U$ is finite étale over an affine neighborhood U' of the set u. Shrinking U we may assume that $U' = U$. Let $deg : Ch_0(\mathcal{X}_\eta) \to \mathbb{Z}/2\mathbb{Z}$ be the degree map. Since $deg(y) = 1$ and $\sum n_r f_{r,*}[Y_r] = [Y] \in Ch_d(\mathcal{X})$ there exists an index r such that the degree of the finite étale morphism $q_r : Y_r \to U$ is odd. Without loss of generality we may assume that the degree of q_1 is odd. The existence of the Y_1-point $f_1 : Y_1 \to \mathcal{X}$ of $\mathcal{X}$ shows that we are under the hypotheses of Theorem 2.9. Hence shrinking U once more we see that there exists a section $s : U \to \mathcal{X}$ of the projection $\mathcal{X} \to U$. Theorem 1.2 is proven.

□

Proof of Theorem 1.1. Let R be a regular semi-local integral domain containing a field. Let k be the prime field of R. By Popescu's theorem $R = \varinjlim B_\alpha$, where

the B_α's are smooth k-algebras (see [P] or [Sw]). Let $can_\alpha : B_\alpha \to R$ be the canonical k-algebra homomorphism. We first observe that we may replace the direct system of the B_α's by a system of essentially smooth semi-local k-algebras which are integral domains. In fact, if m_j is a maximal ideal of R, we can take $p_{\alpha,j} := can_\alpha^{-1}(m_j)$, $S_\alpha := B_\alpha - \cup_{j=1}^n p_{\alpha,j}$ and replace each B_α by $(B_\alpha)_{S_\alpha}$, Note that in this case the canonical morphisms $can_\alpha : B_\alpha \to R$ take maximal ideals to maximal ones and every B_α is a regular semi-local k-algebra.
We claim that B_α is an integral domain. In fact, since B_α is a regular semi-local k-algebra it is a product $\prod_{i=1}^s B_{\alpha,i}$ of regular semi-local integral domains $B_{\alpha,i}$. The ideal $q_\alpha := can_\alpha^{-1}(0) \subset B_\alpha$ is prime and is contained in each of the maximal ideals $can_\alpha^{-1}(m_j)$ of the ring B_α. The latter ideal runs over all the maximal ideals of B_α. Thus the prime ideal q_α is contained in all maximal ideals of $B_\alpha = \prod_{i=1}^s B_{\alpha,i}$. Since q_α is prime after reordering the indices it must be of the form $q_1 \times \prod_{i=2}^s B_{\alpha,i}$. If $s \geq 2$ then the latter ideal is not contained in a maximal ideal of the form $\prod_{i=1}^{s-1} B_{\alpha,i} \times m$ for a maximal ideal m of $B_{\alpha,s}$. Whence $s = 1$ and B_α is indeed an integral domain.
There exists an index α and a quadratic space φ_α over B_α such that $\varphi_\alpha \otimes_{B_\alpha} R \cong \varphi$. For each index $\beta \geq \alpha$ we will write φ_β for the B_β-space $\varphi_\alpha \otimes_{B_\alpha} B_\beta$. Clearly, $\varphi_\beta \otimes_{B_\beta} R \cong \varphi$. The space φ_K is isotropic. Thus there exists an element $f \in R$ such that the space (V_f, φ_f) is isotropic. There exists an index $\beta \geq \alpha$ and a non-zero element $f_\beta \in B_\beta$ such that $can_\beta(f_\beta) = f$ and the space φ_β localized at f_β is isotropic over the ring $(B_\beta)_{f_\beta}$.
If $char(k) = 0$ or if $char(k) = p > 0$ and the field k is infinite perfect, then by Theorem 1.2 the space φ_β is isotropic. Whence the space φ is isotropic too.
If $char(k) = p > 0$ and the field k is finite, then choose a prime number l different from 2 and from p and take the field k_l which is the composite of all l-primary finite extensions k' of k in a fixed algebraic closure $\bar{k}$ of k. Note that for each field k'' which is between k and k_l and is finite over k the degree $[k'' : k]$ is a power of l. In particular, it is odd. Note as well that k_l is a perfect infinite field. Take the k_l-algebra $k_l \otimes_k B_\beta$. *It is a semi-local essentially k_l-smooth algebra, which is not an integral domain in general.* The element $1 \otimes f_\beta$ is not a zero divisor. In fact, k_l is a flat k-algebra and the element f is not a zero divisor in B_β.
The quadratic space $k_l \otimes_k \varphi_\beta$ localized at $1 \otimes f_\beta$ is isotropic over $(k_l \otimes_k B_\beta)_{1\otimes f_\beta} = k_l \otimes_k (B_\beta)_{f_\beta}$ and $1 \otimes f_\beta$ is not a zero divisor in $k_l \otimes_k B_\beta$. By Theorem 1.2 the space $k_l \otimes_k \varphi_\beta$ is isotropic over $k_l \otimes_k B_\beta$. Whence there exists a finite extension $k \subset k' \subset k_l$ of k such that the space $k' \otimes_k \varphi_\beta$ is isotropic over $k' \otimes_k B_\beta$. Thus the space $k' \otimes_k \varphi$ is isotropic over $k' \otimes_k R$. Now $k' \otimes_k R$ is a finite étale extension of R of odd degree. All residue fields of R are infinite. By Theorem 2.9 the space φ is isotropic over R.

□

To state the first corollary of Theorem 1.1 we need to recall the notion of unramified spaces. Let R be a Noetherian integral domain and K be its fraction field. Recall that a quadratic space (W, ψ) over K is *unramified* if for every height one prime ideal $\wp$ of R there exists a quadratic space $(V_{\mathfrak{p}}, \varphi_{\wp})$ over $R_{\wp}$ such that the spaces $(V_{\wp}, \varphi_{\wp}) \otimes_{R_{\wp}} K$ and (W, ψ) are isomorphic.

Corollary 3.1 (A purity theorem). *Let R be a regular local ring containing a field of characteristic different from 2 and such that the residue field of R is infinite. Let K be the field of fractions of R. Let (W, ψ) be a quadratic space over K which is unramified over R. Then there exists a quadratic space (V, φ) over R extending the space (W, ψ), that is the spaces $(V, \varphi) \otimes_R K$ and (W, ψ) are isomorphic.*

Proof. By the purity theorem [OP, Theorem A] there exists a quadratic space (V, φ) over R and an integer $n \geq 0$ such that $(V, \varphi) \otimes_R K \cong (W, \psi) \perp \mathbb{H}^n_K$, where $\mathbb{H}_K$ is a hyperbolic plane. If $n > 0$ then the space $(V, \varphi) \otimes_R K$ is isotropic. By Theorem 1.1 the space (V, φ) is isotropic too. Thus $(V, \varphi) \cong (V', \varphi') \perp \mathbb{H}_R$ for a quadratic space (V', φ') over R. Now Witt's Cancellation theorem over a field [La, Chap.I, Thm.4.2] shows that $(V', \varphi') \otimes_R K \cong (W, \psi) \perp \mathbb{H}^{n-1}_K$. Repeating this procedure several times we may assume that $n = 0$, which means that $(V, \varphi) \otimes_R K \cong (W, \psi)$.

□

Remark 3.2. *Corollary 3.1 is used in the proof of the main result in [ChP]. The main result in [ChP] holds now in the case of a local regular ring R containing a field provided that the residue field of R is infinite and $\frac{1}{2} \in R$.*

Corollary 3.3. *Let R be a semi-local regular integral domain containing a field. Assume that all the residue fields of R are infinite and $\frac{1}{2} \in R$. Let K be the fraction field of R. Let (V, φ) be a quadratic space over R and let $u \in R^\times$ be a unit. Suppose the equation $\varphi = u$ has a solution over K then it has a solution over R, that is there exists a vector $v \in V$ with $\varphi(v) = u$ (clearly the vector v is unimodular).*

Proof. It is very standard. However for the completeness of the exposition let us recall the arguments from [C-T, Proof of Prop.1.2]. Let $(R, -u)$ be the rank one quadratic space over R corresponding to the unit $-u$. The space $(V, \varphi)_K \perp (K, -u)$ is isotropic thus the space $(V, \varphi) \perp (R, -u)$ is isotropic by Theorem 1.1. By the lemma below there exists a vector $v \in V$ with $\varphi(v) = u$. Clearly v is unimodular.

□

Lemma 3.4. *Let (V, φ) be as above. Let $(W, \psi) = (V, \varphi) \perp (R, -u)$. The space (W, ψ) is isotropic if and only if there exists a vector $v \in V$ with $\varphi(v) = u$.*

Proof. It is standard. See [C-T, the proof of Proposition 1.2.].

□

Remark 3.5. *It would be nice to extend the result of Corollary 3.1 to the semi-local case. The difficulty is to extend the purity theorem [OP, Theorem A] to that semi-local case.*

4 Acknowledgements

The work was supported by the Presidium of the RAS Program "Fundamental Research in modern mathematics", the joint DFG-RFBR grant 09-01-91333 NNIO-a, and the RFBR-grant 10-01-00551. The Authors thanks very much J.-L. Colliot-Thélène for explaining Gabber's modification of the de Jong alteration theorem, M. Levine for his comments concerning the proof of proposition [LM, Prop. 3.3.1] and N. Durov for his stimulating interest in the present work.

References

[AK] *A. Altman, S. Kleiman.* Introduction to Grothendieck duality theory. Lect. Notes in Math. 146, Springer, 1970.

[ChP] *V. Chernousov, I. Panin.* Purity of G_2-torsors. C. R. Acad. Sci., Paris, Sér. I, Math. vol. 345, No. 6, pp. 307-312 (2007)

[C-T] *J.-L. Colliot-Thélène.* Formes quadratiques sur les anneaux semilocaux réguliers. Colloque sur les Formes Quadratiques, 2 (Montpellier, 1977). Bull. Soc. Math. France Mém. No. 59, (1979), 13–31.

[CT-O] *J.-L. Colliot-Thélène, M. Ojanguren.* Espaces principaux homogènes localement triviaux. Publ. Math. IHES No. 75, (1992), 97–122

[Fu] *W. Fulton.* Intersection theory. Springer-Verlag, 1984.

[Ha] *R. Hartshorne.* Algebraic Geometry. Graduate Texts in Math., 52, Springer-Verlag, Berlin-New York, 1977.

[I] *L. Illusie.* On Gabber's refined uniformization [Ga1], http://www.math.u-psud.fr/ illusie/ Talks at the Univ. Tokyo, Jan. 17, 22, 31, Feb. 7, 2008.

[La] *T.Y. Lam.* Introduction to quadratic forms over fields. Graduate Studies in Mathematics, vol. 67. American Mathematical Society, Providence, RI, 2005.

[LM] *M. Levine, F. Morel.* Algebraic Cobordism, Springer Monographs in Mathematics, Springer-Verlag 2007. xii+244 pp. ISBN: 978-3-540-36822-9; 3-540-36822-1

[O] *M. Ojanguren.* Unités représentées par des formes quadratiques ou par des normes réduites. Algebraic K-theory, Part II (Oberwolfach, 1980), pp. 291–299, Lecture Notes in Math., 967, Springer, Berlin-New York, 1982.

[OP] *M. Ojanguren, I. Panin.* A purity theorem for the Witt group, Ann. scient. Éc. Norm. Sup. 4ème série, 32 (1999), 71–86

[Pa] *I. Panin,* Rationally isotropic quadratic spaces are locally isotropic. Invent. math. 176, 397-403 (2009).

[P] *D. Popescu.* General Néron desingularization and approximation, Nagoya Math. Journal, 104 (1986), 85–115.

[PR] *I. Panin, U. Rehmann.* A variant of a Theorem by Springer. Algebra i Analyz, vol.19 (2007), 117-125.
`www.math.uiuc.edu/K-theory/0671/2003`

[Ra1] *M.S. Raghunathan.* Principal bundles admitting a rational section. Invent. Math. 116 (1994), no. 1-3, 409–423.

[Ra2] *M.S. Raghunathan.* Erratum: "Principal bundles admitting a rational section" [Invent. Math. 116 (1994), no. 1-3, 409–423]. Invent. Math. 121 (1995), no. 1, 223.

[Sw] *R.G. Swan.* Néron-Popescu desingularization. In Algebra and Geometry (Taipei, 1995), Lect. Algebra Geom. vol. 2 , Internat. Press, Cambridge, MA, 1998, 135 - 192.

Ivan Panin
Steklov Institute
of Mathematics
St.Petersburg
Russia
paniniv@gmail.com

Konstantin Pimenov
St.Petersburg
State University
St.Petersburg
Russia
kip302002@yahoo.com

A p-adic Regulator Map and Finiteness Results for Arithmetic Schemes

Dedicated to Andrei Suslin on his 60th birthday

Shuji Saito[1] and Kanetomo Sato[2]

Received: January 21, 2009
Revised: March 19, 2010

Abstract. A main theme of the paper is a conjecture of Bloch-Kato on the image of p-adic regulator maps for a proper smooth variety X over an algebraic number field k. The conjecture for a regulator map of particular degree and weight is related to finiteness of two arithmetic objects: One is the p-primary torsion part of the Chow group in codimension 2 of X. Another is an unramified cohomology group of X. As an application, for a regular model $\mathscr{X}$ of X over the integer ring of k, we prove an injectivity result on the torsion cycle class map of codimension 2 with values in a new p-adic cohomology of $\mathscr{X}$ introduced by the second author, which is a candidate of the conjectural étale motivic cohomology with finite coefficients of Beilinson-Lichtenbaum.

2010 Mathematics Subject Classification: Primary 14C25, 14G40; Secondary 14F30, 19F27, 11G25.
Keywords and Phrases: p-adic regulator, unramified cohomology, Chow groups, p-adic étale Tate twists

1 Introduction

Let k be an algebraic number field and let G_k be the absolute Galois group $\mathrm{Gal}(\overline{k}/k)$, where $\overline{k}$ denotes a fixed algebraic closure of k. Let X be a projective smooth variety over k and put $\overline{X} := X \otimes_k \overline{k}$. Fix a prime p and integers $r, m \geq 1$. A main theme

[1]supported by Grant-in-Aid for Scientific Research B-18340003 and S-19104001
[2]supported by JSPS Postdoctoral Fellowship for Research Abroad and EPSRC grant

of this paper is a conjecture of Bloch and Kato concerning the image of the p-adic regulator map

$$\mathrm{reg}^{r,m} \ : \ \mathrm{CH}^r(X,m) \otimes \mathbb{Q}_p \longrightarrow H^1_{\mathrm{cont}}(k, H^{2r-m-1}_{\text{ét}}(\overline{X}, \mathbb{Q}_p(r)))$$

from Bloch's higher Chow group to continuous Galois cohomology of G_k ([BK2] Conjecture 5.3). See §3 below for the definition of this map in the case $(r,m)=(2,1)$. This conjecture affirms that its image agrees with the subspace

$$H^1_g(k, H^{2r-m-1}_{\text{ét}}(\overline{X}, \mathbb{Q}_p(r))) \subset H^1_{\mathrm{cont}}(k, H^2_{\text{ét}}(\overline{X}, \mathbb{Q}_p(2)))$$

defined in loc. cit. (see §2.1 below), and plays a crucial role in the so-called Tamagawa number conjecture on special values of L-functions attached to X. In terms of Galois representations, the conjecture means that a 1-extension of continuous p-adic representations of G_k

$$0 \longrightarrow H^{2r-m-1}_{\text{ét}}(\overline{X}, \mathbb{Q}_p(r)) \longrightarrow E \longrightarrow \mathbb{Q}_p \longrightarrow 0$$

arises from a 1-extension of motives over k

$$0 \longrightarrow h^{2r-m-1}(X)(r) \longrightarrow M \longrightarrow h(\mathrm{Spec}(k)) \longrightarrow 0,$$

if and only if E is a de Rham representation of G_k. There has been only very few known results on the conjecture. In this paper we consider the following condition, which is the Bloch-Kato conjecture in the special case $(r,m)=(2,1)$:

> **H1**: *The image of the regulator map*
>
> $$\mathrm{reg} := \mathrm{reg}^{2,1} : \mathrm{CH}^2(X,1) \otimes \mathbb{Q}_p \longrightarrow H^1_{\mathrm{cont}}(k, H^2_{\text{ét}}(\overline{X}, \mathbb{Q}_p(2))).$$
>
> *agrees with* $H^1_g(k, H^2_{\text{ét}}(\overline{X}, \mathbb{Q}_p(2)))$.

We also consider a variant:

> **H1**$*$: *The image of the regulator map with* $\mathbb{Q}_p/\mathbb{Z}_p$*-coefficients*
>
> $$\mathrm{reg}_{\mathbb{Q}_p/\mathbb{Z}_p} : \mathrm{CH}^2(X,1) \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow H^1_{\mathrm{Gal}}(k, H^2_{\text{ét}}(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)))$$
>
> *agrees with* $H^1_g(k, H^2_{\text{ét}}(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)))_{\mathrm{Div}}$ (*see* §2.1 *for* $H^1_g(k,-)$). *Here for an abelian group* M, M_{Div} *denotes its maximal divisible subgroup.*

We will show that **H1** always implies **H1**$*$, which is not straight-forward. On the other hand the converse holds as well under some assumptions. See Remark 3.2.5 below for details.

Fact 1.1 *The condition* **H1** *holds in the following cases*:

(1) $H^2(X, \mathscr{O}_X) = 0$ ([CTR1], [CTR2], [Sal]).

(2) *X is the self-product of an elliptic curve over $k = \mathbb{Q}$ with square-free conductor and without complex multiplication, and $p \geq 5$* ([Md], [Fl], [LS], [La1]).

(3) *X is the elliptic modular surface of level 4 over $k = \mathbb{Q}$ and $p \geq 5$* ([La2]).

(4) *X is a Fermat quartic surface over $k = \mathbb{Q}$ or $\mathbb{Q}(\sqrt{-1})$ and $p \geq 5$* ([O]).

A main result of this paper relates the condition $\mathbf{H1}*$ to finiteness of two arithmetic objects. One is the p-primary torsion part of the Chow group $\mathrm{CH}^2(X)$ of algebraic cycles of codimension two on X modulo rational equivalence. Another is an unramified cohomology of X, which we are going to introduce in what follows.

Let $\mathfrak{o}_k$ be the integer ring of k, and put $S := \mathrm{Spec}(\mathfrak{o}_k)$. We assume the following:

ASSUMPTION 1.2 *There exists a regular scheme $\mathscr{X}$ which is proper and flat over S and whose generic fiber is X. Moreover, $\mathscr{X}$ has good or semistable reduction at each closed point of S of characteristic p.*

Let $K = k(X)$ be the function field of X. For an integer $q \geq 0$, let $\mathscr{X}^q$ be the set of all points $x \in \mathscr{X}$ of codimension q. Fix an integer $n \geq 0$. Roughly speaking, the unramified cohomology group $H^{n+1}_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(n))$ is defined as the subgroup of $H^{n+1}_{\text{ét}}(\mathrm{Spec}(K), \mathbb{Q}_p/\mathbb{Z}_p(n))$ consisting of those elements that are "unramified" along all $y \in \mathscr{X}^1$. For a precise definition, we need the p-adic étale Tate twist $\mathfrak{T}_r(n) = \mathfrak{T}_r(n)_{\mathscr{X}}$ introduced in [SH]. This object is defined in $D^b(\mathscr{X}_{\text{ét}}, \mathbb{Z}/p^r)$, the derived category of bounded complexes of étale sheaves of $\mathbb{Z}/p^r$-modules on $\mathscr{X}$, and expected to coincide with $\mathit{\Gamma}(2)^{\mathscr{X}}_{\text{ét}} \otimes^{\mathbb{L}} \mathbb{Z}/p^r$. Here $\mathit{\Gamma}(2)^{\mathscr{X}}_{\text{ét}}$ denotes the conjectural étale motivic complex of Beilinson-Lichtenbaum [Be], [Li1]. We note that the restriction of $\mathfrak{T}_r(n)$ to $\mathscr{X}[p^{-1}] := \mathscr{X} \otimes_{\mathbb{Z}} \mathbb{Z}[p^{-1}]$ is isomorphic to $\mu_{p^r}^{\otimes n}$, where μ_{p^r} denotes the étale sheaf of p^r-th roots of unity.Then $H^{n+1}_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(n))$ is defined as the kernel of the boundary map of étale cohomology groups

$$H^{n+1}_{\text{ét}}(\mathrm{Spec}(K), \mathbb{Q}_p/\mathbb{Z}_p(n)) \longrightarrow \bigoplus_{x \in \mathscr{X}^1} H^{n+2}_x(\mathrm{Spec}(\mathscr{O}_{\mathscr{X},x}), \mathfrak{T}_\infty(n)),$$

where $\mathfrak{T}_\infty(n)$ denotes $\varinjlim_{r \geq 1} \mathfrak{T}_r(n)$. There are natural isomorphisms

$$H^1_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(0)) \simeq H^1_{\text{ét}}(\mathscr{X}, \mathbb{Q}_p/\mathbb{Z}_p) \quad \text{and} \quad H^2_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(1)) \simeq \mathrm{Br}(\mathscr{X})_{p\text{-tors}},$$

where $\mathrm{Br}(\mathscr{X})$ denotes the Grothendieck-Brauer group $H^2_{\text{ét}}(\mathscr{X}, \mathbb{G}_{\mathrm{m}})$, and for an abelian group M, $M_{p\text{-tors}}$ denotes its p-primary torsion part. An intriguing question is as to whether the group $H^{n+1}_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(n))$ is finite, which is related to several significant theorems and conjectures in arithmetic geometry (see Remark 4.3.1 below). In this paper we are concerned with the case $n = 2$. A crucial role will be played by the following subgroup of $H^3_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$:

$$\begin{aligned} &H^3_{\mathrm{ur}}(K, X; \mathbb{Q}_p/\mathbb{Z}_p(2)) \\ &:= \mathrm{Im}\Big(H^3_{\text{ét}}(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \to H^3_{\text{ét}}(\mathrm{Spec}(K), \mathbb{Q}_p/\mathbb{Z}_p(2))\Big) \cap H^3_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(2)). \end{aligned}$$

It will turn out that $\mathrm{CH}^2(X)_{p\text{-tors}}$ and $H^3_{\mathrm{ur}}(K,X;\mathbb{Q}_p/\mathbb{Z}_p(2))$ are cofinitely generated over $\mathbb{Z}_p$ if $\mathrm{Coker}(\mathrm{reg}_{\mathbb{Q}_p/\mathbb{Z}_p})_{\mathrm{Div}}$ is cofinitely generated over $\mathbb{Z}_p$ (cf. Proposition 3.3.2, Lemma 5.2.3). Our main finiteness result is the following:

Theorem 1.3 *Let $\mathscr{X}$ be as in Assumption* 1.2*, and assume $p \geq 5$. Then*:

(1) $\mathbf{H1*}$ *implies that* $\mathrm{CH}^2(X)_{p\text{-tors}}$ *and* $H^3_{\mathrm{ur}}(K,X;\mathbb{Q}_p/\mathbb{Z}_p(2))$ *are finite.*

(2) *Assume that the reduced part of every closed fiber of $\mathscr{X}/S$ has simple normal crossings on $\mathscr{X}$, and that the Tate conjecture holds in codimension* 1 *for the irreducible components of those fibers* (*see the beginning of* §7 *for the precise contents of the last assumption*)*. Then the finiteness of* $\mathrm{CH}^2(X)_{p\text{-tors}}$ *and* $H^3_{\mathrm{ur}}(K,X;\mathbb{Q}_p/\mathbb{Z}_p(2))$ *implies* $\mathbf{H1*}$.

We do not need Assumption 1.2 to deduce the finiteness of $\mathrm{CH}^2(X)_{p\text{-tors}}$ from $\mathbf{H1*}$, by the alteration theorem of de Jong [dJ] (see also Remark 3.1.2 (3) below). However, we need a regular proper model $\mathscr{X}$ as above crucially in our computations on $H^3_{\mathrm{ur}}(K,X;\mathbb{Q}_p/\mathbb{Z}_p(2))$. The assertion (2) is a converse of (1) under the assumption of the Tate conjecture. We obtain the following result from Theorem 1.3 (1) (see also the proof of Theorem 1.6 in §5.1 below):

Corollary 1.4 $H^3_{\mathrm{ur}}(K,X;\mathbb{Q}_p/\mathbb{Z}_p(2))$ *is finite in the four cases in Fact* 1.1 (*under the assumption* 1.2).

We will also prove variants of Theorem 1.3 over local integer rings (see Theorems 3.1.1, 5.1.1 and 7.1.1 below). As for the finiteness of $H^3_{\mathrm{ur}}(K,\mathbb{Q}_p/\mathbb{Z}_p(2))$ over local integer rings, Spiess proved that $H^3_{\mathrm{ur}}(K,\mathbb{Q}_p/\mathbb{Z}_p(2)) = 0$, assuming that $\mathfrak{o}_k$ is an ℓ-adic local integer ring with $\ell \neq p$ and that either $H^2(X,\mathscr{O}_X) = 0$ or $\mathscr{X}$ is a product of two smooth elliptic curves over S ([Spi] §4). In [SSa], the authors extended his vanishing result to a more general situation that $\mathfrak{o}_k$ is ℓ-adic local with $\ell \neq p$ and that $\mathscr{X}$ has generalized semistable reduction. Finally we have to remark that there exists a smooth projective surface X with $p_g(X) \neq 0$ over a local field k for which the condition $\mathbf{H1*}$ does not hold and such that $\mathrm{CH}^2(X)_{\mathrm{tors}}$ is infinite [AS].

We next explain an application of the above finiteness result to a cycle class map of arithmetic schemes. Let us recall the following fact due to Colliot-Thélène, Sansuc, Soulé and Gros:

Fact 1.5 ([CTSS], [Gr]) *Let X be a proper smooth variety over a finite field of characteristic $\ell > 0$. Let p be a prime number, which may be the same as ℓ. Then the cycle class map restricted to the p-primary torsion part*

$$\mathrm{CH}^2(X)_{p\text{-tors}} \longrightarrow H^4_{\text{ét}}(X,\mathbb{Z}/p^r(2))$$

is injective for a sufficiently large $r > 0$. Here $\mathbb{Z}/p^r(2)$ denotes $\mu_{p^r}^{\otimes 2}$ if $\ell \neq p$. Otherwise $\mathbb{Z}/p^r(2)$ denotes $W_r\Omega^2_{X,\log}[-2]$ with $W_r\Omega^2_{X,\log}$ the étale subsheaf of the logarithmic part of the Hodge-Witt sheaf $W_r\Omega^2_X$ ([Bl1], [Il]).

In this paper, we study an arithmetic variant of this fact. We expect that a similar result holds for proper regular arithmetic schemes, i.e., regular schemes which are proper flat of finite type over the integer ring of a number field or a local field. To be more precise, let k, $\mathfrak{o}_k$ and X be as before and let $\mathscr{X}$ be as in Assumption 1.2. The p-adic étale Tate twist $\mathfrak{T}_r(2) = \mathfrak{T}_r(2)_{\mathscr{X}}$ mentioned before replaces $\mathbb{Z}/p^r(2)$ in Fact 1.5, and there is a cycle class map

$$\varrho_r^2 : \mathrm{CH}^2(\mathscr{X})/p^r \longrightarrow H^4_{\text{ét}}(\mathscr{X}, \mathfrak{T}_r(2)).$$

We are concerned with the induced map

$$\varrho^2_{p\text{-tors},r} : \mathrm{CH}^2(\mathscr{X})_{p\text{-tors}} \longrightarrow H^4_{\text{ét}}(\mathscr{X}, \mathfrak{T}_r(2)).$$

It is shown in [SH] that the group on the right hand side is finite. So the injectivity of this map is closely related with the finiteness of $\mathrm{CH}^2(\mathscr{X})_{p\text{-tors}}$. The second main result of this paper concerns the injectivity of this map:

THEOREM 1.6 (§5) *Assume that $H^2(X, \mathscr{O}_X) = 0$. Then $\mathrm{CH}^2(\mathscr{X})_{p\text{-tors}}$ is finite and $\varrho^2_{p\text{-tors},r}$ is injective for a sufficiently large $r > 0$.*

The finiteness of $\mathrm{CH}^2(\mathscr{X})_{p\text{-tors}}$ in this theorem is originally due to Salberger [Sal], Colliot-Thélène and Raskind [CTR1], [CTR2]. Note that there exists a projective smooth surface V over a number field with $H^2(V, \mathscr{O}_V) = 0$ for which the map

$$\mathrm{CH}^2(V)_{p\text{-tors}} \longrightarrow H^4_{\text{ét}}(V, \mu_{p^r}^{\otimes 2})$$

is not injective for some bad prime p and any $r \geq 1$ [Su] (cf. [PS]). Our result suggests that we are able to recover the injectivity of torsion cycle class maps by considering a proper regular model of V over the ring of integers in k. The fundamental ideas of Theorem 1.6 are the following. A crucial point of the proof of Fact 1.5 in [CTSS] and [Gr] is Deligne's proof of the Weil conjecture [De2]. In the arithmetic situation, the role of the Weil conjecture is replaced by the condition $\mathbf{H1}$, which implies the finiteness of $\mathrm{CH}^2(X)_{p\text{-tors}}$ and $H^3_{\text{ur}}(K, X; \mathbb{Q}_p/\mathbb{Z}_p(2))$ by Theorem 1.3 (1). The injectivity result in Theorem 1.6 is derived from the finiteness of those objects.

This paper is organized as follows. In §2, we will review some fundamental facts on Galois cohomology groups and Selmer groups which will be used frequently in this paper. In §3, we will prove the finiteness of $\mathrm{CH}^2(X)_{p\text{-tors}}$ in Theorem 1.3 (1). In §4, we will review p-adic étale Tate twists briefly and then provide some fundamental lemmas on cycle class maps and unramified cohomology groups. In §5, we will first reduce Theorem 1.6 to Theorem 1.3 (1), and then reduce the finiteness of $H^3_{\text{ur}}(K, X; \mathbb{Q}_p/\mathbb{Z}_p(2))$ in Theorem 1.3 (1) to Key Lemma 5.4.1. In §6, we will prove that key lemma, which will complete the proof of Theorem 1.3 (1). §7 will be devoted to the proof of Theorem 1.3 (2). In the appendix A, we will include an observation that the finiteness of $H^3_{\text{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$ is deduced from the Beilinson–Lichtenbaum conjectures on motivic complexes.

Acknowledgements. The second author expresses his gratitude to University of Southern California and The University of Nottingham for their great hospitality. The authors also express their gratitude to Professors Wayne Raskind, Thomas Geisser and Ivan Fesenko for valuable comments and discussions. Thanks are also due to the referee and Masanori Asakura, who read the manuscript of this paper carefully and gave them enlightening comments.

Notation

1.6. For an abelian group M and a positive integer n, ${}_nM$ and M/n denote the kernel and the cokernel of the map $M \xrightarrow{\times n} M$, respectively. See §2.3 below for other notation for abelian groups. For a field k, $\overline{k}$ denotes a fixed separable closure, and G_k denotes the absolute Galois group $\mathrm{Gal}(\overline{k}/k)$. For a discrete G_k-module M, $H^*(k, M)$ denote the Galois cohomology groups $H^*_{\mathrm{Gal}}(G_k, M)$, which are the same as the étale cohomology groups of $\mathrm{Spec}(k)$ with coefficients in the étale sheaf associated with M.

1.7. Unless indicated otherwise, all cohomology groups of schemes are taken over the étale topology. For a scheme X, an étale sheaf $\mathscr{F}$ on X (or more generally an object in the derived category of sheaves on $X_{\text{ét}}$) and a point $x \in X$, we often write $H^*_x(X, \mathscr{F})$ for $H^*_x(\mathrm{Spec}(\mathscr{O}_{X,x}), \mathscr{F})$. For a pure-dimensional scheme X and a non-negative integer q, let X^q be the set of all points on X of codimension q. For a point $x \in X$, let $\kappa(x)$ be its residue field. For an integer $n \geq 0$ and a noetherian excellent scheme X, $\mathrm{CH}_n(X)$ denotes the Chow group of algebraic cycles on X of dimension n modulo rational equivalence. If X is pure-dimensional and regular, we will often write $\mathrm{CH}^{\dim(X)-n}(X)$ for this group. For an integral scheme X of finite type over $\mathrm{Spec}(\mathbb{Q})$, $\mathrm{Spec}(\mathbb{Z})$ or $\mathrm{Spec}(\mathbb{Z}_\ell)$, we define $\mathrm{CH}^2(X, 1)$ as the cohomology group, at the middle, of the Gersten complex of Milnor K-groups

$$K_2^M(L) \longrightarrow \bigoplus_{y \in X^1} \kappa(y)^\times \longrightarrow \bigoplus_{x \in X^2} \mathbb{Z},$$

where L denotes the function field of X. As is well-known, this group coincides with a higher Chow group ([Bl3], [Le2]) by localization sequences of higher Chow groups ([Bl4], [Le1]) and the Nesterenko-Suslin theorem [NS] (cf. [To]).

1.8. In §§4–7, we will work under the following setting. Let k be an algebraic number field or its completion at a finite place. Let $\mathfrak{o}_k$ be the integer ring of k and put $S := \mathrm{Spec}(\mathfrak{o}_k)$. Let p be a prime number, and let $\mathscr{X}$ be a regular scheme which is proper flat of finite type over S and satisfies the following condition:

Assumption 1.8.1 *If p is not invertible in $\mathfrak{o}_k$, then $\mathscr{X}$ has good or semistable reduction at each closed point of S of characteristic p.*

This condition is the same as Assumption 1.2 when k is a number field.

1.9. Let k be an algebraic number field, and let $\mathscr{X} \to S = \mathrm{Spec}(\mathfrak{o}_k)$ be as in 1.8. In this situation, we will often use the following notation. For a closed point $v \in S$, let $\mathfrak{o}_v$ (resp. k_v) be the completion of $\mathfrak{o}_k$ (resp. k) at v, and let $\mathbb{F}_v$ be the residue field of k_v. We put

$$\mathscr{X}_v := \mathscr{X} \otimes_{\mathfrak{o}_k} \mathfrak{o}_v, \qquad X_v := \mathscr{X} \otimes_{\mathfrak{o}_k} k_v, \qquad Y_v := \mathscr{X} \otimes_{\mathfrak{o}_k} \mathbb{F}_v$$

and write $j_v : X_v \hookrightarrow \mathscr{X}_v$ (resp. $i_v : Y_v \hookrightarrow \mathscr{X}_v$) for the natural open (resp. closed) immersion. We put $\overline{Y_v} := Y_v \times_{\mathbb{F}_v} \overline{\mathbb{F}_v}$, and write Σ for the set of all closed point on S of characteristic p.

1.10. Let k be an ℓ-adic local field with ℓ a prime number, and let $\mathscr{X} \to S = \mathrm{Spec}(\mathfrak{o}_k)$ be as in 1.8. In this situation, we will often use the following notation. Let $\mathbb{F}$ be the residue field of k and put

$$X := \mathscr{X} \otimes_{\mathfrak{o}_k} k, \qquad Y := \mathscr{X} \otimes_{\mathfrak{o}_k} \mathbb{F}.$$

We write $j : X \hookrightarrow \mathscr{X}$ (resp. $i : Y \hookrightarrow \mathscr{X}$) for the natural open (resp. closed) immersion. Let k^{ur} be the maximal unramified extension of k, and let $\mathfrak{o}^{\mathrm{ur}}$ be its integer ring. We put

$$\mathscr{X}^{\mathrm{ur}} := \mathscr{X} \otimes_{\mathfrak{o}_k} \mathfrak{o}^{\mathrm{ur}}, \qquad X^{\mathrm{ur}} := \mathscr{X} \otimes_{\mathfrak{o}_k} k^{\mathrm{ur}}, \qquad \overline{Y} := Y \times_{\mathbb{F}} \overline{\mathbb{F}}.$$

2 PRELIMINARIES ON GALOIS COHOMOLOGY

In this section, we provide some preliminary lemmas which will be frequently used in this paper. Let k be an algebraic number field (global field) or its completion at a finite place (local field). Let $\mathfrak{o}_k$ be the integer ring of k, and put $S := \mathrm{Spec}(\mathfrak{o}_k)$. Let p be a prime number. If k is global, we often write Σ for the set of the closed points on S of characteristic p.

2.1 SELMER GROUP

Let X be a proper smooth variety over $\mathrm{Spec}(k)$, and put $\overline{X} := X \otimes_k \overline{k}$. If k is global, we fix a non-empty open subset $U_0 \subset S \setminus \Sigma$ for which there exists a proper smooth morphism $\mathscr{X}_{U_0} \to U_0$ with $\mathscr{X}_{U_0} \times_{U_0} k \simeq X$. For $v \in S^1$, let k_v and $\mathbb{F}_v$ be as in the notation 1.9. In this section we are concerned with G_k-modules

$$V := H^i(\overline{X}, \mathbb{Q}_p(n)) \quad \text{and} \quad A := H^i(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(n)).$$

For $M = V$ or A and a non-empty open subset $U \subset U_0$, let $H^*(U, M)$ denote the étale cohomology groups with coefficients in the smooth sheaf on $U_{\text{ét}}$ associated to M.

DEFINITION 2.1.1 (1) *Assume that k is local. Let $H^1_f(k, V)$ and $H^1_g(k, V)$ be as defined in* [BK2] (3.7)*. For $* \in \{f, g\}$, we define*

$$H^1_*(k, A) := \mathrm{Im}\big(H^1_*(k, V) \longrightarrow H^1(k, A)\big).$$

(2) *Assume that k is global. For $M \in \{V, A\}$ and a non-empty open subset $U \subset S$, we define the subgroup $H^1_{f,U}(k, M) \subset H^1_{\mathrm{cont}}(k, M)$ as the kernel of the natural map*

$$H^1_{\mathrm{cont}}(k, M) \longrightarrow \prod_{v \in U^1} \frac{H^1_{\mathrm{cont}}(k_v, M)}{H^1_f(k_v, M)} \times \prod_{v \in S \setminus U} \frac{H^1_{\mathrm{cont}}(k_v, M)}{H^1_g(k_v, M)}.$$

If $U \subset U_0$, we have

$$H^1_{f,U}(k, M) = \mathrm{Ker}\Big(H^1(U, M) \longrightarrow \prod\nolimits_{v \in S \setminus U} H^1_{\mathrm{cont}}(k_v, M)/H^1_g(k_v, M)\Big).$$

We define the group $H^1_g(k, M)$ and $H^1_{\mathrm{ind}}(k, M)$ as

$$H^1_g(k, M) := \varinjlim_{U \subset U_0} H^1_{f,U}(k, M), \qquad H^1_{\mathrm{ind}}(k, M) := \varinjlim_{U \subset U_0} H^1(U, M),$$

where U runs through all non-empty open subsets of U_0. These groups are independent of the choice of U_0 and $\mathscr{X}_{U_0}$ (cf. [EGA4] 8.8.2.5).

(3) *If k is local, we define $H^1_{\mathrm{ind}}(k, M)$ to be $H^1_{\mathrm{cont}}(k, M)$ for $M \in \{V, A\}$.*

Note that $H^1_{\mathrm{ind}}(k, A) = H^1(k, A)$.

2.2 p-adic Point of Motives

We provide a key lemma from p-adic Hodge theory which play crucial roles in this paper (see Corollary 2.2.3 below). Assume that k is a p-adic local field, and that there exists a regular scheme $\mathscr{X}$ which is proper flat of finite type over $S = \mathrm{Spec}(\mathfrak{o}_k)$ with $\mathscr{X} \otimes_{\mathfrak{o}_k} k \simeq X$ and which has semistable reduction. Let i and n be non-negative integers. Put

$$V^i := H^{i+1}(\overline{X}, \mathbb{Q}_p), \qquad V^i(n) := V^i \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(n),$$

and

$$H^{i+1}(\mathscr{X}, \tau_{\leq n} Rj_* \mathbb{Q}_p(n)) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_{r \geq 1} H^{i+1}(\mathscr{X}, \tau_{\leq n} Rj_* \mu_{p^r}^{\otimes n}),$$

where j denotes the natural open immersion $X \hookrightarrow \mathscr{X}$. There is a natural pull-back map

$$\alpha : H^{i+1}(\mathscr{X}, \tau_{\leq n} Rj_* \mathbb{Q}_p(n)) \longrightarrow H^{i+1}(X, \mathbb{Q}_p(n)).$$

Let $H^{i+1}(\mathscr{X}, \tau_{\leq r} Rj_* \mathbb{Q}_p(n))^0$ be the kernel of the composite map

$$\alpha' : H^{i+1}(\mathscr{X}, \tau_{\leq n} Rj_* \mathbb{Q}_p(n)) \xrightarrow{\alpha} H^{i+1}(X, \mathbb{Q}_p(n)) \longrightarrow \big(V^{i+1}(n)\big)^{G_k}.$$

For this group, there is a composite map

$$\overline{\alpha} : H^{i+1}(\mathscr{X}, \tau_{\leq n} Rj_* \mathbb{Q}_p(n))^0 \longrightarrow F^1 H^{i+1}(X, \mathbb{Q}_p(n)) \longrightarrow H^1_{\mathrm{cont}}(k, V^i(n)).$$

Here the first arrow is induced by α, the second is an edge homomorphism in a Hochschild-Serre spectral sequence

$$E_2^{u,v} := H^u_{\text{cont}}(k, V^v(n)) \Longrightarrow H^{u+v}_{\text{cont}}(X, \mathbb{Q}_p(n))(\simeq H^{u+v}(X, \mathbb{Q}_p(n))),$$

and $F^\bullet$ denotes the filtration on $H^{i+1}(X, \mathbb{Q}_p(n))$ resulting from this spectral sequence. To provide with Corollary 2.2.3 below concerning the image of $\overline{\alpha}$, we need some strong results in p-adic Hodge theory. We first recall the following comparison theorem of log syntomic complexes and p-adic vanishing cycles due to Tsuji, which extends a comparison result of Kurihara [Ku] to semistable families. Let Y be the closed fiber of $\mathscr{X} \to S$ and let $\iota : Y \hookrightarrow \mathscr{X}$ be the natural closed immersion.

THEOREM 2.2.1 ([Ts2] Theorem 5.1) *For integers n, r with $0 \leq n \leq p-2$ and $r \geq 1$, there is a canonical isomorphism*

$$\eta : s_r^{\log}(n) \xrightarrow{\sim} \iota_*\iota^*(\tau_{\leq n} Rj_*\mu_{p^r}^{\otimes n}) \quad \textit{in} \;\; D^b(\mathscr{X}_{\text{ét}}, \mathbb{Z}/p^r),$$

where $s_r^{\log}(n) = s_r^{\log}(n)_{\mathscr{X}}$ is the log syntomic complex defined by Kato [Ka2].

Put

$$H^*(\mathscr{X}, s_{\mathbb{Q}_p}^{\log}(n)) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_{r \geq 1} H^*(\mathscr{X}, s_r^{\log}(n)),$$

and define $H^{i+1}(\mathscr{X}, s_{\mathbb{Q}_p}^{\log}(n))^0$ as the kernel of the composite map

$$H^{i+1}(\mathscr{X}, s_{\mathbb{Q}_p}^{\log}(n)) \;\underset{\eta}{\xrightarrow{\sim}}\; H^{i+1}(\mathscr{X}, \tau_{\leq n} Rj_*\mathbb{Q}_p(n)) \xrightarrow{\alpha'} \left(V^{i+1}(n)\right)^{G_k},$$

where we have used the properness of $\mathscr{X}$ over S. There is an induced map

$$\overline{\eta} : H^{i+1}(\mathscr{X}, s_{\mathbb{Q}_p}^{\log}(n))^0 \;\underset{\eta}{\xrightarrow{\sim}}\; H^{i+1}(\mathscr{X}, \tau_{\leq n} Rj_*\mathbb{Q}_p(n))^0 \xrightarrow{\overline{\alpha}} H^1_{\text{cont}}(k, V^i(n)).$$

Concerning this map, we have the following fact due to Langer and Nekovář:

THEOREM 2.2.2 ([La3], [Ne2] Theorem 3.1) $\mathrm{Im}(\overline{\eta})$ *agrees with* $H^1_g(k, V^i(n))$.

As an immediate consequence of these facts, we obtain

COROLLARY 2.2.3 *Assume that $p \geq n+2$. Then* $\mathrm{Im}(\overline{\alpha}) = H^1_g(k, V^i(n))$.

REMARK 2.2.4 (1) *Theorem* 2.2.2 *is an extension of the p-adic point conjecture raised by Schneider in the good reduction case* [Sch]. *This conjecture was proved by Langer-Saito* [LS] *in a special case and by Nekovář* [Ne1] *in the general case.*

(2) *Theorem* 2.2.2 *holds unconditionally on p, if we define $H^{i+1}(\mathscr{X}, s_{\mathbb{Q}_p}^{\log}(n))$ using Tsuji's version of log syntomic complexes $\mathscr{S}_r^{\sim}(n)$ $(r \geq 1)$ in* [Ts1] §2.

2.3 Elementary Facts on $\mathbb{Z}_p$-Modules

For an abelian group M, let M_{Div} be its maximal divisible subgroup. For a torsion abelian group M, let $\mathrm{Cotor}(M)$ be the cotorsion part M/M_{Div}.

Definition 2.3.1 *Let M be a $\mathbb{Z}_p$-module.*

(1) *We say that M is cofinitely generated over $\mathbb{Z}_p$ (or simply, cofinitely generated), if its Pontryagin dual $\mathrm{Hom}_{\mathbb{Z}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p)$ is a finitely generated $\mathbb{Z}_p$-module.*

(2) *We say that M is cofinitely generated up to a finite-exponent group, if M_{Div} is cofinitely generated and $\mathrm{Cotor}(M)$ has a finite exponent.*

(3) *We say that M is divisible up to a finite-exponent group, if $\mathrm{Cotor}(M)$ has a finite exponent.*

Lemma 2.3.2 *Let $0 \to L \to M \to N \to 0$ be a short exact sequence of $\mathbb{Z}_p$-modules.*

(1) *Assume that L, M and N are cofinitely generated. Then there is a positive integer r_0 such that for any $r \geq r_0$ we have an exact sequence of finite abelian p-groups*

$$0 \to {}_{p^r}L \to {}_{p^r}M \to {}_{p^r}N \to \mathrm{Cotor}(L) \to \mathrm{Cotor}(M) \to \mathrm{Cotor}(N) \to 0.$$

Consequently, taking the projective limit of this exact sequence with respect to $r \geq r_0$ there is an exact sequence of finitely generated $\mathbb{Z}_p$-modules

$$0 \to T_p(L) \to T_p(M) \to T_p(N) \to \mathrm{Cotor}(L) \to \mathrm{Cotor}(M) \to \mathrm{Cotor}(N) \to 0,$$

where for an abelian group A, $T_p(A)$ denotes its p-adic Tate module.

(2) *Assume that L is cofinitely generated up to a finite-exponent group. Assume further that M is divisible, and that N is cofinitely generated and divisible. Then L and M are cofinitely generated.*

(3) *Assume that L is divisible up to a finite-exponent group. Then for a divisible subgroup $D \subset N$ and its inverse image $D' \subset M$, the induced map $(D')_{\mathrm{Div}} \to D$ is surjective. In particular, the natural map $M_{\mathrm{Div}} \to N_{\mathrm{Div}}$ is surjective.*

(4) *If $L_{\mathrm{Div}} = N_{\mathrm{Div}} = 0$, then we have $M_{\mathrm{Div}} = 0$.*

Proof. (1) There is a commutative diagram with exact rows

$$\begin{CD}
0 @>>> L @>>> M @>>> N @>>> 0 \\
@. @V{\times p^r}VV @V{\times p^r}VV @V{\times p^r}VV @. \\
0 @>>> L @>>> M @>>> N @>>> 0.
\end{CD}$$

One obtains the assertion by applying the snake lemma to this diagram, noting $\mathrm{Cotor}(A) \simeq A/p^r$ for a cofinitely generated $\mathbb{Z}_p$-module A and a sufficiently large $r \geq 1$.

(2) Our task is to show that $\mathrm{Cotor}(L)$ is finite. By a similar argument as for (1), there is an exact sequence for a sufficiently large $r \geq 1$

$$0 \longrightarrow {}_{p^r}L \longrightarrow {}_{p^r}M \longrightarrow {}_{p^r}N \longrightarrow \mathrm{Cotor}(L) \longrightarrow 0,$$

where we have used the assumptions on L and M. Hence the finiteness of $\mathrm{Cotor}(L)$ follows from the assumption that N is cofinitely generated.

(3) We have only to show the case $D = N_{\mathrm{Div}}$. For a $\mathbb{Z}_p$-module A, we have

$$A_{\mathrm{Div}} = \mathrm{Im}\left(\mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p, A) \to A\right)$$

by [J1] Lemma (4.3.a). Since $\mathrm{Ext}^1_{\mathbb{Z}_p}(\mathbb{Q}_p, L) = 0$ by the assumption on L, the following natural map is surjective:

$$\mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p, M) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p, N).$$

By these facts, the natural map $M_{\mathrm{Div}} \to N_{\mathrm{Div}}$ is surjective.

(4) For a $\mathbb{Z}_p$-module A, we have

$$A_{\mathrm{Div}} = 0 \iff \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p, A) = 0$$

by [J1] Remark (4.7). The assertion follows from this fact and the exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p, L) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p, M) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p, N).$$

This completes the proof of the lemma. □

2.4 Divisible Part of $H^1(k, A)$

Let the notation be as in §2.1. We prove here the following general lemma, which will be used frequently in §§3–7:

Lemma 2.4.1 *Under the notation in Definition* 2.1.1 *we have*

$$\mathrm{Im}\big(H^1_{\mathrm{ind}}(k, V) \to H^1(k, A)\big) = H^1(k, A)_{\mathrm{Div}},$$
$$\mathrm{Im}\big(H^1_g(k, V) \to H^1(k, A)\big) = H^1_g(k, A)_{\mathrm{Div}}.$$

Proof. The assertion is clear if k is local. Assume that k is global. Without loss of generality we may assume that A is divisible. We prove only the second equality and omit the first one (see Remark 2.4.9 (2) below). Let $U_0 \subset S$ be as in §2.1. We have

$$\mathrm{Im}\big(H^1_{f,U}(k, V) \to H^1(U, A)\big) = H^1_{f,U}(k, A)_{\mathrm{Div}} \tag{2.4.2}$$

for non-empty open $U \subset U_0$. This follows from a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & H^1_{f,U}(k,V) \arrow[r] \arrow[d] & H^1(U,V) \arrow[r] \arrow[d, "\alpha"'] & \prod_{v\in S\setminus U} H^1_{\mathrm{cont}}(k_v,V)/H^1_g(k_v,V) \arrow[d, "\beta"'] \\
0 \arrow[r] & H^1_{f,U}(k,A) \arrow[r] & H^1(U,A) \arrow[r] & \prod_{v\in S\setminus U} H^1(k_v,A)/H^1_g(k_v,A)
\end{tikzcd}$$

and the facts that $\mathrm{Coker}(\alpha)$ is finite and that $\mathrm{Ker}(\beta)$ is finitely generated over $\mathbb{Z}_p$. By (2.4.2), the second equality of the lemma is reduced to the following assertion:

$$\varinjlim_{U\subset U_0} \big(H^1_{f,U}(k,A)_{\mathrm{Div}}\big) = \left(\varinjlim_{U\subset U_0} H^1_{f,U}(k,A))\right)_{\mathrm{Div}}. \tag{2.4.3}$$

To show this equality, we will prove the following sublemma:

Sublemma 2.4.4 *For an open subset $U \subset U_0$, put*

$$C_U := \mathrm{Coker}\big(H^1_{f,U_0}(k,A) \to H^1_{f,U}(k,A)\big).$$

Then there exists a non-empty open subset $U_1 \subset U_0$ such that the quotient C_U/C_{U_1} is divisible for any open subset $U \subset U_1$.

We first finish our proof of (2.4.3) admitting this sublemma. Let $U_1 \subset U_0$ be a non-empty open subset as in Sublemma 2.4.4. Noting that $H^1_{f,U}(k,A)$ is cofinitely generated, there is an exact sequence of finite groups

$$\mathrm{Cotor}\big(H^1_{f,U_1}(k,A)\big) \longrightarrow \mathrm{Cotor}\big(H^1_{f,U}(k,A)\big) \longrightarrow \mathrm{Cotor}(C_U/C_{U_1}) \longrightarrow 0$$

for open $U \subset U_1$ by Lemma 2.3.2 (1). By this exact sequence and Sublemma 2.4.4, the natural map $\mathrm{Cotor}(H^1_{f,U_1}(k,A)) \to \mathrm{Cotor}(H^1_{f,U}(k,A))$ is surjective for any open $U \subset U_1$, which implies that the inductive limit

$$\varinjlim_{U\subset U_0} \mathrm{Cotor}(H^1_{f,U}(k,A))$$

is a finite group. The equality (2.4.3) follows easily from this.

Proof of Sublemma 2.4.4. We need the following general fact:

Sublemma 2.4.5 *Let $\underline{N} = \{N_\lambda\}_{\lambda\in\Lambda}$ be an inductive system of cofinitely generated $\mathbb{Z}_p$-modules indexed by a filtered set Λ such that* $\mathrm{Coker}(N_\lambda \to N_{\lambda'})$ *is divisible for any two $\lambda, \lambda' \in \Lambda$ with $\lambda' \geq \lambda$. Let L be a cofinitely generated $\mathbb{Z}_p$-module and $\{f_\lambda : N_\lambda \to L\}_{\lambda\in\Lambda}$ be $\mathbb{Z}_p$-homomorphisms compatible with the transition maps of $\underline{N}$. Then there exists $\lambda_0 \in \Lambda$ such that* $\mathrm{Coker}\big(\mathrm{Ker}(f_{\lambda_0}) \to \mathrm{Ker}(f_\lambda)\big)$ *is divisible for any $\lambda \geq \lambda_0$.*

Proof of Sublemma 2.4.5. Let $f_\infty : N_\infty \to L$ be the limit of f_λ. The assumption on $\underline{N}$ implies that for any two $\lambda, \lambda' \in \Lambda$ with $\lambda' \geq \lambda$, the quotient $\mathrm{Im}(f_{\lambda'})/\mathrm{Im}(f_\lambda)$ is divisible, so that

$$\mathrm{Cotor}(\mathrm{Im}(f_\lambda)) \to \mathrm{Cotor}(\mathrm{Im}(f_{\lambda'})) \text{ is surjective.} \tag{2.4.6}$$

By the equality $\mathrm{Im}(f_\infty) = \varinjlim_{\lambda\in\Lambda} \mathrm{Im}(f_\lambda)$, there is a short exact sequence

$$0 \longrightarrow \varinjlim_{\lambda\in\Lambda} \big(\mathrm{Im}(f_\lambda)_{\mathrm{Div}}\big) \longrightarrow \mathrm{Im}(f_\infty) \longrightarrow \varinjlim_{\lambda\in\Lambda} \mathrm{Cotor}(\mathrm{Im}(f_\lambda)) \longrightarrow 0,$$

and the last term is finite by the fact (2.4.6) and the assumption that L is cofinitely generated. Hence we get

$$\varinjlim_{\lambda\in\Lambda} \big(\mathrm{Im}(f_\lambda)_{\mathrm{Div}}\big) = \mathrm{Im}(f_\infty)_{\mathrm{Div}}.$$

Since $\mathrm{Im}(f_\infty)_{\mathrm{Div}}$ has finite corank, there exists an element $\lambda_0 \in \Lambda$ such that $\mathrm{Im}(f_\lambda)_{\mathrm{Div}} = \mathrm{Im}(f_\infty)_{\mathrm{Div}}$ for any $\lambda \geq \lambda_0$. This fact and (2.4.6) imply the equality

$$\mathrm{Im}(f_\lambda) = \mathrm{Im}(f_{\lambda_0}) \quad \text{for any } \lambda \geq \lambda_0. \tag{2.4.7}$$

Now let $\lambda \in \Lambda$ satisfy $\lambda \geq \lambda_0$. Applying the snake lemma to the commutative diagram

$$\begin{tikzcd}
 & N_{\lambda_0} \arrow[r] \arrow[d, "f_{\lambda_0}"'] & N_\lambda \arrow[r] \arrow[d, "f_\lambda"'] & N_\lambda/N_{\lambda_0} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L \arrow[r, equal] & L \arrow[r] & 0, &
\end{tikzcd}$$

we get an exact sequence

$$\mathrm{Ker}(f_{\lambda_0}) \longrightarrow \mathrm{Ker}(f_{\lambda_0}) \longrightarrow N_\lambda/N_{\lambda_0} \xrightarrow{0} \mathrm{Coker}(f_{\lambda_0}) \xrightarrow{\sim} \mathrm{Coker}(f_\lambda),$$

which proves Sublemma 2.4.5, beucase N_λ/N_{λ_0} is divisible by assumption. □

We now turn to the proof of Sublemma 2.4.4. For non-empty open $U \subset U_0$, there is a commutative diagram with exact rows

$$\begin{tikzcd}
 & H^1(U_0, A) \arrow[r] \arrow[d, "r_{U_0}"'] & H^1(U, A) \arrow[r] \arrow[d, "r_U"'] & \bigoplus_{v\in U_0\setminus U} A(-1)^{G_{\mathbb{F}_v}} \arrow[r, "\beta_U"] \arrow[d, "\alpha_U"'] & H^2(U_0, A) \\
0 \arrow[r] & \bigoplus_{v\in S\setminus U_0} H^1_{/g}(k_v, A) \arrow[r] & \bigoplus_{v\in S\setminus U} H^1_{/g}(k_v, A) \arrow[r] & \bigoplus_{v\in U_0\setminus U} H^1_{/g}(k_v, A), &
\end{tikzcd}$$

where we put

$$H^1_{/g}(k_v, A) := H^1(k_v, A)/H^1_g(k_v, A)$$

for simplicity. The upper row is obtained from a localization exact sequence of étale cohomology and the isomorphism

$$H^2_v(U_0, A) \simeq H^1(k_v, A)/H^1(\mathbb{F}_v, A) \simeq A(-1)^{G_{\mathbb{F}_v}} \quad \text{for } v \in U_0 \setminus U,$$

where we have used the fact that the action of G_k on A is unramified at $v \in U_0$. The map α_U is obtained from the facts that $H^1_g(k_v, A) = H^1(k_v, A)_{\mathrm{Div}}$ if $v \notin \Sigma$ and that $H^1(\mathbb{F}_v, A)$ is divisible (recall that A is assumed to be divisible). It gives

$$\mathrm{Ker}(\alpha_U) = \bigoplus_{v \in U_0 \setminus U} \big(A(-1)^{G_{\mathbb{F}_v}}\big)_{\mathrm{Div}}. \tag{2.4.8}$$

Now let ϕ_U be the composite map

$$\phi_U : \mathrm{Ker}(\alpha_U) \hookrightarrow \bigoplus_{v \in U_0 \setminus U} A(-1)^{G_{\mathbb{F}_v}} \xrightarrow{\beta_U} H^2(U_0, A),$$

and let

$$\psi_U : \mathrm{Ker}(\phi_U) \longrightarrow \mathrm{Coker}(r_{U_0})$$

be the map induced by the above diagram. Note that

$$C_U \simeq \mathrm{Ker}(\psi_U), \text{ since } H^1_{f,U}(k, A) = \mathrm{Ker}(r_U).$$

By (2.4.8), the inductive system $\{\mathrm{Ker}(\alpha_U)\}_{U \subset U_0}$ and the maps $\{\phi_U\}_{U \subset U_0}$ satisfy the assumptions in Sublemma 2.4.5. Hence there exists a non-empty open subset $U' \subset U_0$ such that $\mathrm{Ker}(\phi_U)/\mathrm{Ker}(\phi_{U'})$ is divisible for any open $U \subset U'$. Then applying Sublemma 2.4.5 again to the inductive system $\{\mathrm{Ker}(\phi_U)\}_{U \subset U'}$ and the maps $\{\psi_U\}_{U \subset U'}$, we conclude that there exists a non-empty open subset $U_1 \subset U'$ such that the quotient

$$\mathrm{Ker}(\psi_U)/\mathrm{Ker}(\psi_{U_1}) = C_U/C_{U_1}$$

is divisible for any open subset $U \subset U_1$. This completes the proof of Sublemma 2.4.4 and Lemma 2.4.1. □

Remark 2.4.9 (1) *By the argument after Sublemma* 2.4.4, $\mathrm{Cotor}(H^1_g(k, A))$ *is finite if A is divisible.*

(2) *One obtains the first equality in Lemma* 2.4.1 *by replacing the local terms* $H^1_{/g}(k_v, A)$ *in the above diagram with* $\mathrm{Cotor}(H^1(k_v, A))$.

2.5 Cotorsion Part of $H^1(k, A)$

Assume that k is global, and let the notation be as in §2.1. We investigate here the boundary map

$$\delta_{U_0} : H^1(k, A) \longrightarrow \bigoplus_{v \in (U_0)^1} A(-1)^{G_{\mathbb{F}_v}}$$

arising from a localization exact sequence of étale cohomology and the purity for discrete valuation rings. Concerning this map, we prove the following standard lemma, which will be used in our proof of Theorem 1.3:

LEMMA 2.5.1 (1) *The map*

$$\delta_{U_0,\mathrm{Div}} : H^1(k,A)_{\mathrm{Div}} \longrightarrow \bigoplus_{v\in(U_0)^1} \big(A(-1)^{G_{\mathbb{F}_v}}\big)_{\mathrm{Div}}$$

induced by δ_{U_0} has cofinitely generated cokernel.

(2) *The map*

$$\delta_{U_0,\mathrm{Cotor}} : \mathrm{Cotor}(H^1(k,A)) \longrightarrow \bigoplus_{v\in(U_0)^1} \mathrm{Cotor}\big(A(-1)^{G_{\mathbb{F}_v}}\big)$$

induced by δ_{U_0} has finite kernel and cofinitely generated cokernel.

We have nothing to say about the finiteness of the cokernel of these maps.

Proof. For a non-empty open $U \subset U_0$, there is a commutative diagram of cofinitely generated $\mathbb{Z}_p$-modules

$$\begin{tikzcd} & H^1(U,A)_{\mathrm{Div}} \arrow[r, "\gamma_U"] \arrow[d, hook] & \bigoplus_{v\in U_0\setminus U} \big((A(-1)^{G_{\mathbb{F}_v}}\big)_{\mathrm{Div}} \arrow[d, hook] & \\ H^1(U_0,A) \arrow[r] & H^1(U,A) \arrow[r, "\alpha_U"] & \bigoplus_{v\in U_0\setminus U} A(-1)^{G_{\mathbb{F}_v}} \arrow[r, "\beta_U"] & H^2(U_0,A), \end{tikzcd}$$

where the lower row is obtained from a localization exact sequence of étale cohomology and the purity for discrete valuation rings, and γ_U is induced by α_U. Let

$$f_U : \mathrm{Cotor}(H^1(U,A)) \longrightarrow \bigoplus_{v\in U_0\setminus U} \mathrm{Cotor}\big(A(-1)^{G_{\mathbb{F}_v}}\big)$$

be the map induced by α_U. By a diagram chase, we obtain an exact sequence

$$\mathrm{Ker}(f_U) \longrightarrow \mathrm{Coker}(\gamma_U) \longrightarrow \mathrm{Coker}(\alpha_U) \longrightarrow \mathrm{Coker}(f_U) \longrightarrow 0.$$

Taking the inductive limit with respect to all non-empty open subsets $U \subset U_0$, we obtain an exact sequence

$$\mathrm{Ker}(\delta_{U_0,\mathrm{Cotor}}) \to \mathrm{Coker}(\delta_{U_0,\mathrm{Div}}) \to \varinjlim_{U\subset U_0} \mathrm{Coker}(\alpha_U) \to \mathrm{Coker}(\delta_{U_0,\mathrm{Cotor}}) \to 0,$$

where we have used Lemma 2.4.1 to obtain the equalities

$$\mathrm{Ker}(\delta_{U_0,\mathrm{Cotor}}) = \varinjlim_{U\subset U_0} \mathrm{Ker}(f_U) \quad \text{and} \quad \mathrm{Coker}(\delta_{U_0,\mathrm{Div}}) = \varinjlim_{U\subset U_0} \mathrm{Coker}(\gamma_U).$$

Since $\varinjlim_{U\subset U_0} \mathrm{Coker}(\alpha_U)$ is a subgroup of $H^2(U_0,A)$, it is cofinitely generated. Hence the assertions in Lemma 2.5.1 are reduced to showing that $\mathrm{Ker}(\delta_{U_0,\mathrm{Cotor}})$ is finite. We prove this finiteness assertion. The lower row of the above diagram yields exact sequences

$$\mathrm{Cotor}(H^1(U_0,A)) \longrightarrow \mathrm{Cotor}(H^1(U,A)) \longrightarrow \mathrm{Cotor}(\mathrm{Im}(\alpha_U)) \longrightarrow 0, \qquad (2.5.2)$$

$$T_p(\mathrm{Im}(\beta_U)) \longrightarrow \mathrm{Cotor}(\mathrm{Im}(\alpha_U)) \longrightarrow \bigoplus_{v\in U_0\setminus U} \mathrm{Cotor}\big(A(-1)^{G_{\mathbb{F}_v}}\big), \qquad (2.5.3)$$

where the second exact sequence arises from the short exact sequence

$$0 \longrightarrow \mathrm{Im}(\alpha_U) \longrightarrow \bigoplus_{v\in U_0\setminus U} A(-1)^{G_{\mathbb{F}_v}} \longrightarrow \mathrm{Im}(\beta_U) \longrightarrow 0$$

(cf. Lemma 2.3.2 (1)). Taking the inductive limit of (2.5.2) with respect to all non-empty open $U \subset U_0$, we obtain the finiteness of the kernel of the map

$$\mathrm{Cotor}(H^1(k,A)) \longrightarrow \varinjlim_{U\subset U_0} \mathrm{Cotor}(\mathrm{Im}(\alpha_U)).$$

Taking the inductive limit of (2.5.3) with respect to all non-empty open $U \subset U_0$, we see that the kernel of the map

$$\varinjlim_{U\subset U_0} \mathrm{Cotor}(\mathrm{Im}(\alpha_U)) \longrightarrow \bigoplus_{v\in (U_0)^1} \mathrm{Cotor}\big(A(-1)^{G_{\mathbb{F}_v}}\big),$$

is finite, because we have

$$\varinjlim_{U\subset U_0} T_p(\mathrm{Im}(\beta_U)) \subset T_p(H^2(U_0,A))$$

and the group on the right hand side is a finitely generated $\mathbb{Z}_p$-module. Thus $\mathrm{Ker}(\delta_{U_0,\mathrm{Cotor}})$ is finite and we obtain Lemma 2.5.1. □

2.6 Local-Global Principle

Let the notation be as in §2.1. If k is local, then the Galois cohomological dimension $\mathrm{cd}(k)$ is 2 (cf. [Se] II.4.3). In the case that k is global, we have $\mathrm{cd}(k) = 2$ either if $p \geq 3$ or if k is totally imaginary. Otherwise, $H^q(k,A)$ is finite 2-torsion for $q \geq 3$ (cf. loc. cit. II.4.4 Proposition 13, II.6.3 Theorem B). As for the second Galois cohomology groups, the following local-global principle due to Jannsen [J2] plays a fundamental role in this paper (see also loc. cit. §7 Corollary 7):

Theorem 2.6.1 ([J2] §4 Theorem 4) *Assume that k is global and that $i \neq 2(n-1)$. Let P be the set of all places of k. Then the map*

$$H^2(k, H^i(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(n))) \longrightarrow \bigoplus_{v\in P} H^2(k_v, H^i(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(n)))$$

has finite kernel and cokernel, and the map

$$H^2(k, H^i(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(n))_{\mathrm{Div}}) \longrightarrow \bigoplus_{v\in P} H^2(k_v, H^i(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(n))_{\mathrm{Div}})$$

is bijective.

We apply these facts to the filtration $F^\bullet$ on $H^*(X, \mathbb{Q}_p/\mathbb{Z}_p(n))$ resulting from the Hochschild-Serre spectral sequence

$$E_2^{u,v} = H^u(k, H^v(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(n))) \Longrightarrow H^{u+v}(X, \mathbb{Q}_p/\mathbb{Z}_p(n)). \tag{2.6.2}$$

COROLLARY 2.6.3 *Assume that k is global and that $i \neq 2n$. Then*:

(1) *$F^2H^i(X, \mathbb{Q}_p/\mathbb{Z}_p(n))$ is cofinitely generated up to a finite-exponent group.*

(2) *For $v \in P$, put $X_v := X \otimes_k k_v$. Then the natural maps*

$$F^2H^i(X, \mathbb{Q}_p/\mathbb{Z}_p(n)) \longrightarrow \bigoplus_{v \in P} F^2H^i(X_v, \mathbb{Q}_p/\mathbb{Z}_p(n)),$$
$$F^2H^i(X, \mathbb{Q}_p/\mathbb{Z}_p(n))_{\mathrm{Div}} \longrightarrow \bigoplus_{v \in P} F^2H^i(X_v, \mathbb{Q}_p/\mathbb{Z}_p(n))_{\mathrm{Div}}$$

have finite kernel and cokernel (*and the second map is surjective*).

Proof. Let $\mathfrak{o}_k$ be the integer ring of k, and put $S := \mathrm{Spec}(\mathfrak{o}_k)$. Note that the set of all finite places of k agrees with S^1.

(1) The group $H^2(k_v, H^{i-2}(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(n))_{\mathrm{Div}})$ is divisible and cofinitely generated for any $v \in S^1$, and it is zero if $p \nmid v$ and X has good reduction at v, by the local Poitou-Tate duality [Se] II.5.2 Théorème 2 and Deligne's proof of the Weil conjecture [De2] (see [Sat2] Lemma 2.4 for details). The assertion follows from this fact and Theorem 2.6.1.

(2) We prove the assertion only for the first map. The assertion for the second map is similar and left to the reader. For simplicity, we assume that

($\sharp$) $p \geq 3$ or k is totally imaginary.

Otherwise one can check the assertion by repeating the same arguments as below in the category of abelian groups modulo finite abelian groups. By ($\sharp$), we have $\mathrm{cd}_p(k) = 2$ and there is a commutative diagram

$$\begin{tikzcd}
H^2(k, H^{i-2}(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(n))) \arrow[r] \arrow[d, two heads] & \bigoplus_{v \in S^1} H^2(k_v, H^{i-2}(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(n))) \arrow[d, two heads] \\
F^2H^i(X, \mathbb{Q}_p/\mathbb{Z}_p(n)) \arrow[r] & \bigoplus_{v \in S^1} F^2H^i(X_v, \mathbb{Q}_p/\mathbb{Z}_p(n)),
\end{tikzcd}$$

where the vertical arrows are edge homomorphisms of Hochschild-Serre spectral sequences and these arrows are surjective. Since

$$H^2(k_v, H^{i-2}(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(n))) = 0 \quad \text{for archimedean places } v$$

by ($\sharp$), the top horizontal arrow has finite kernel and cokernel by Theorem 2.6.1. Hence it is enough to show that the right vertical arrow has finite kernel. For any $v \in S^1$, the

v-component of this map has finite kernel by Deligne's criterion [De1] (see also [Sat2] Remark 1.2). If v is prime to p and X has good reduction at v, then the v-component is injective. Indeed, there is an exact sequence resulting from a Hochschild-Serre spectral sequence and the fact that $\mathrm{cd}(k_v) = 2$:

$$H^{i-1}(X_v, \mathbb{Q}_p/\mathbb{Z}_p(n)) \xrightarrow{d} H^{i-1}(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(n)))^{G_{k_v}} \\ \to H^2(k_v, H^{i-2}(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(n))) \to F^2H^i(X_v, \mathbb{Q}_p/\mathbb{Z}_p(n)).$$

The edge homomorphism d is surjective by the commutative square

$$\begin{tikzcd} H^{i-1}(Y_v, \mathbb{Q}_p/\mathbb{Z}_p(n)) \arrow[r, two heads] \arrow[d] & H^{i-1}(\overline{Y_v}, \mathbb{Q}_p/\mathbb{Z}_p(n)))^{G_{\mathbb{F}_v}} \arrow[d, "\wr"] \\ H^{i-1}(X_v, \mathbb{Q}_p/\mathbb{Z}_p(n)) \arrow[r, "d"] & H^{i-1}(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(n)))^{G_{k_v}}. \end{tikzcd}$$

Here Y_v denotes the reduction of X at v and $\overline{Y_v}$ denotes $Y_v \otimes_{\mathbb{F}_v} \overline{\mathbb{F}_v}$. The left (resp. right) vertical arrow arises from the proper base-change theorem (resp. proper smooth base-change theorem), and the top horizontal arrow is surjective by the fact that $\mathrm{cd}(\mathbb{F}_v) = 1$. Thus we obtain the assertion. □

3 Finiteness of Torsion in a Chow Group

Let k, S, p and Σ be as in the beginning of §2, and let X be a proper smooth geometrically integral variety over $\mathrm{Spec}(k)$. We introduce the following technical condition:

> **H0**: *The group* $H^3_{\text{ét}}(\overline{X}, \mathbb{Q}_p(2))^{G_k}$ *is trivial.*

If k is global, **H0** always holds by Deligne's proof of the Weil conjecture [De2]. When k is local, **H0** holds if $\dim(X) = 2$ or if X has good reduction (cf. [CTR2] §6); it is in general a consequence of the monodromy-weight conjecture.

3.1 Finiteness of $\mathrm{CH}^2(X)_{p\text{-tors}}$

The purpose of this section is to show the following result, which is a generalization of a result of Langer [La4] Proposition 3 and implies the finiteness assertion on $\mathrm{CH}^2(X)_{p\text{-tors}}$ in Theorem 1.3 (1):

Theorem 3.1.1 *Assume* **H0**, **H1∗** *and either* $p \geq 5$ *or the equality*

$$H^1_g(k, H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)))_{\mathrm{Div}} = H^1(k, H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)))_{\mathrm{Div}}. \qquad (*_g)$$

Then $\mathrm{CH}^2(X)_{p\text{-tors}}$ *is finite.*

Remark 3.1.2 (1) $(*_g)$ *holds if* $H^2(X, \mathscr{O}_X) = 0$ *or if* k *is* ℓ*-adic local with* $\ell \neq p$*.*

(2) *Crucial facts to this theorem are Lemmas* 3.2.2*,* 3.3.5 *and* 3.5.2 *below. The short exact sequence in Lemma* 3.2.2 *is an important consequence of the Merkur'ev-Suslin theorem* [MS].

(3) *In Theorem* 3.1.1*, we do not need to assume that* X *has good or semistable reduction at any prime of* k *dividing* p (*cf.* 1.8.1)*, because we do not need this assumption in Lemma* 3.5.2 *by the alteration theorem of de Jong* [dJ].

3.2 Regulator Map

We recall here the definition of the regulator maps

$$\mathrm{reg}_{\Lambda} : \mathrm{CH}^2(X,1) \otimes \Lambda \longrightarrow H^1_{\mathrm{ind}}(k, H^2(\overline{X}, \Lambda(2))) \tag{3.2.1}$$

with $\Lambda = \mathbb{Q}_p$ or $\mathbb{Q}_p/\mathbb{Z}_p$, assuming **H0**. The general framework on étale Chern class maps and regulator maps is due to Soulé [So1], [So2]. We include here a more elementary construction of reg_{Λ}, which will be useful in this paper. Let $K := k(X)$ be the function field of X. Take an open subset $U_0 \subset S \setminus \Sigma = S[p^{-1}]$ and a smooth proper scheme $\mathscr{X}_{U_0}$ over U_0 satisfying $\mathscr{X}_{U_0} \times_{U_0} \mathrm{Spec}(k) \simeq X$. For an open subset $U \subset U_0$, put $\mathscr{X}_U := \mathscr{X}_{U_0} \times_{U_0} U$ and define

$$N^1H^3(\mathscr{X}_U, \mu_{p^r}^{\otimes 2}) := \mathrm{Ker}\big(H^3(\mathscr{X}_U, \mu_{p^r}^{\otimes 2}) \to H^3(K, \mu_{p^r}^{\otimes 2})\big).$$

Lemma 3.2.2 *For an open subset* $U \subset U_0$*, there is an exact sequence*

$$0 \longrightarrow \mathrm{CH}^2(\mathscr{X}_U, 1)/p^r \longrightarrow N^1H^3(\mathscr{X}_U, \mu_{p^r}^{\otimes 2}) \longrightarrow {}_{p^r}\mathrm{CH}^2(\mathscr{X}_U) \longrightarrow 0$$

See §1.7 *for the definition of* $\mathrm{CH}^2(\mathscr{X}_U, 1)$.

Proof. The following argument is due to Bloch [Bl], Lecture 5. We recall it for the convenience of the reader. There is a localization spectral sequence

$$E_1^{u,v} = \bigoplus_{x \in (\mathscr{X}_U)^u} H_x^{u+v}(\mathscr{X}_U, \mu_{p^r}^{\otimes 2}) \Longrightarrow H^{u+v}(\mathscr{X}_U, \mu_{p^r}^{\otimes 2}). \tag{3.2.3}$$

By the relative smooth purity, there is an isomorphism

$$E_1^{u,v} \simeq \bigoplus_{x \in (\mathscr{X}_U)^u} H^{v-u}(x, \mu_{p^r}^{\otimes 2-u}), \tag{3.2.4}$$

which implies that $N^1H^3(\mathscr{X}_U, \mu_{p^r}^{\otimes 2})$ is isomorphic to the cohomology of the Bloch-Ogus complex

$$H^2(K, \mu_{p^r}^{\otimes 2}) \longrightarrow \bigoplus_{y \in (\mathscr{X}_U)^1} H^1(y, \mu_{p^r}) \longrightarrow \bigoplus_{x \in (\mathscr{X}_U)^2} \mathbb{Z}/p^r.$$

By Hilbert's theorem 90 and the Merkur'ev-Suslin theorem [MS], this complex is isomorphic to the Gersten complex

$$K_2^M(K)/p^r \longrightarrow \bigoplus_{y \in (\mathscr{X}_U)^1} \kappa(y)^\times/p^r \longrightarrow \bigoplus_{x \in (\mathscr{X}_U)^2} \mathbb{Z}/p^r.$$

On the other hand, there is an exact sequence obtained by a diagram chase

$$0 \longrightarrow \mathrm{CH}^2(\mathscr{X}_U, 1) \otimes \mathbb{Z}/p^r \longrightarrow \mathrm{CH}^2(\mathscr{X}_U, 1; \mathbb{Z}/p^r) \longrightarrow {}_{p^r}\mathrm{CH}^2(\mathscr{X}_U) \longrightarrow 0.$$

Here $\mathrm{CH}^2(\mathscr{X}_U, 1; \mathbb{Z}/p^r)$ denotes the cohomology of the above Gersten complex and it is isomorphic to $N^1H^3(\mathscr{X}_U, \mu_{p^r}^{\otimes 2})$. Thus we obtain the lemma. □

Put

$$M^q := H^q(\overline{X}, \Lambda(2)) \quad \text{with} \quad \Lambda \in \{\mathbb{Q}_p, \mathbb{Q}_p/\mathbb{Z}_p\}.$$

For an open subset $U \subset U_0$ let $H^*(U, M^q)$ be the étale cohomology with coefficients in the smooth sheaf associated with M^q. There is a Leray spectral sequence

$$E_2^{u,v} = H^u(U, M^v) \Longrightarrow H^{u+v}(\mathscr{X}_U, \Lambda(2)).$$

By Lemma 3.2.2, there is a natural map

$$\mathrm{CH}^2(\mathscr{X}_U, 1) \otimes \Lambda \longrightarrow H^3(\mathscr{X}_U, \Lambda(2)).$$

Noting that $E_2^{0,3}$ is zero or finite by $\mathbf{H0}$, we define the map

$$\mathrm{reg}_{\mathscr{X}_U, \Lambda} : \mathrm{CH}^2(\mathscr{X}_U, 1) \otimes \Lambda \longrightarrow H^1(U, M^2)$$

as the composite of the above map with an edge homomorphism of the Leray spectral sequence. Finally we define reg_Λ in (3.2.1) by passing to the limit over all non-empty open $U \subset U_0$. Our construction of reg_Λ does not depend on the choice of U_0 or $\mathscr{X}_{U_0}$.

Remark 3.2.5 *By Lemma* 2.4.1, $\mathbf{H1}$ *always implies* $\mathbf{H1*}$. *If k is local,* $\mathbf{H1*}$ *conversely implies* $\mathbf{H1}$. *If k is global, one can check that* $\mathbf{H1*}$ *implies* $\mathbf{H1}$, *assuming that the group* $\mathrm{Ker}(\mathrm{CH}^2(\mathscr{X}_{U_0}) \to \mathrm{CH}^2(X))$ *is finitely generated up to torsion and that the Tate conjecture for divisors holds for almost all closed fibers of $\mathscr{X}_{U_0}/U_0$.*

3.3 Proof of Theorem 3.1.1

We start the proof of Theorem 3.1.1, which will be completed in §3.5 below. By Lemma 3.2.2, there is an exact sequence

$$0 \longrightarrow \mathrm{CH}^2(X, 1) \otimes \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\phi} N^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \longrightarrow \mathrm{CH}^2(X)_{p\text{-tors}} \longrightarrow 0, \tag{3.3.1}$$

where we put

$$N^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) := \mathrm{Ker}(H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \to H^3(K, \mathbb{Q}_p/\mathbb{Z}_p(2))).$$

In view of (3.3.1), Theorem 3.1.1 is reduced to the following two propositions:

PROPOSITION 3.3.2 (1) *If k is local, then* $\mathrm{CH}^2(X)_{p\text{-tors}}$ *is cofinitely generated over* $\mathbb{Z}_p$.

(2) *Assume that k is global, and that* $\mathrm{Coker}\big(\mathrm{reg}_{\mathbb{Q}_p/\mathbb{Z}_p}\big)_{\mathrm{Div}}$ *is cofinitely generated over* $\mathbb{Z}_p$. *Then* $\mathrm{CH}^2(X)_{p\text{-tors}}$ *is cofinitely generated over* $\mathbb{Z}_p$.

PROPOSITION 3.3.3 *Assume* $\mathbf{H0}$, $\mathbf{H1}*$ *and either* $p \geq 5$ *or* $(*_g)$. *Then we have*

$$\mathrm{Im}(\phi) = N^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))_{\mathrm{Div}}.$$

We will prove Proposition 3.3.2 in §3.4 below, and Proposition 3.3.3 in §3.5 below.

REMARK 3.3.4 (1) *If k is local, then* $H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))$ *is cofinitely generated. Hence Proposition* 3.3.2 (1) *immediately follows from the exact sequence* (3.3.1).

(2) *When k is global, then* $H^1(k, A)_{\mathrm{Div}}/H^1_g(k, A)_{\mathrm{Div}}$ *with* $A := H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))$ *is cofinitely generated by Lemma* 2.4.1. *Hence* $\mathbf{H1}*$ *implies the second assumption of Proposition* 3.3.2 (2).

Let $F^\bullet$ be the filtration on $H^*(X, \mathbb{Q}_p/\mathbb{Z}_p(2))$ resulting from the Hochschild-Serre spectral sequence (2.6.2). The following fact due to Salberger will play key roles in our proof of the above two propositions:

LEMMA 3.3.5 ([Sal] Main Lemma 3.9) *The following group has a finite exponent*:

$$N^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \cap F^2H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)).$$

3.4 PROOF OF PROPOSITION 3.3.2

For (1), see Remark 3.3.4 (1). We assume that k is global, and prove (2). Put

$$H^3 := H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \quad \text{and} \quad \Gamma := \phi(\mathrm{CH}^2(X,1) \otimes \mathbb{Q}_p/\mathbb{Z}_p) \subset H^3$$

(cf. (3.3.1)). Let $F^\bullet$ be the filtration on H^3 resulting from the spectral sequence (2.6.2), and put $N^1H^3 := N^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))$. We have $\Gamma \subset (F^1H^3)_{\mathrm{Div}} = (H^3)_{\mathrm{Div}}$ by $\mathbf{H0}$, and there is a filtration on H^3

$$0 \subset \Gamma + (F^2H^3)_{\mathrm{Div}} \subset (F^1H^3)_{\mathrm{Div}} \subset H^3.$$

By (3.3.1), the inclusion $N^1H^3 \subset H^3$ induces an inclusion $\mathrm{CH}^2(X)_{p\text{-tors}} \subset H^3/\Gamma$. We show that the image of this inclusion is cofinitely generated, using the above filtration on H^3. It suffices to show the following lemma:

LEMMA 3.4.1 (1) *The kernel of* $\mathrm{CH}^2(X)_{p\text{-tors}} \to H^3/(\Gamma + (F^2H^3)_{\mathrm{Div}})$ *is finite.*

(2) *The image of* $\mathrm{CH}^2(X)_{p\text{-tors}} \to H^3/(F^1H^3)_{\mathrm{Div}}$ *is finite.*

(3) *The second assumption of Proposition* 3.3.2 (2) *implies that the group*

$$M := (F^1H^3)_{\mathrm{Div}}/(\Gamma + (F^2H^3)_{\mathrm{Div}})$$

is cofinitely generated.

Proof. (1) There is an exact sequence

$$0 \longrightarrow \frac{N^1H^3 \cap (F^2H^3)_{\mathrm{Div}}}{\Gamma \cap (F^2H^3)_{\mathrm{Div}}} \longrightarrow \mathrm{CH}^2(X)_{p\text{-tors}} \longrightarrow \frac{H^3}{\Gamma + (F^2H^3)_{\mathrm{Div}}}.$$

Hence (1) follows from Lemma 3.3.5 and Corollary 2.6.3 (1).

(2) Let U_0 and $\mathscr{X}_{U_0} \to U_0$ be as in §3.2. For non-empty open $U \subset U_0$, there is a commutative diagram up to a sign

$$\begin{tikzcd} N^1H^3(\mathscr{X}_U, \mathbb{Q}_p/\mathbb{Z}_p(2)) \arrow[r] \arrow[d, hook] & \mathrm{CH}^2(\mathscr{X}_U) \otimes \mathbb{Z}_p \arrow[d, "\varrho"] \\ H^3(\mathscr{X}_U, \mathbb{Q}_p/\mathbb{Z}_p(2)) \arrow[r] & H^4(\mathscr{X}_U, \mathbb{Z}_p(2)) \end{tikzcd}$$

by the same argument as for [CTSS], §1, Proposition 1. Here the top arrow is the composite of $N^1H^3(\mathscr{X}_U, \mathbb{Q}_p/\mathbb{Z}_p(2)) \to \mathrm{CH}^2(\mathscr{X}_U)_{p\text{-tors}}$ (cf. Lemma 3.2.2) with the natural inclusion. The bottom arrow is a Bockstein map and the right vertical arrow is the cycle class map of $\mathscr{X}_U$. Taking the inductive limit with respect to all non-empty $U \subset U_0$, we see that the left square of the following diagram commutes (up to a sign):

$$\begin{tikzcd} N^1H^3 \arrow[r] \arrow[d, hook] & \mathrm{CH}^2(X) \otimes \mathbb{Z}_p \arrow[d, "\varrho_{\mathrm{ind}}"] \arrow[dr, "\varrho_{\mathrm{cont}}"] & \\ H^3 \arrow[r] & H^4_{\mathrm{ind}}(X, \mathbb{Z}_p(2)) \arrow[r] & H^4_{\mathrm{cont}}(X, \mathbb{Z}_p(2)), \end{tikzcd}$$

where $H^*_{\mathrm{cont}}(X, \mathbb{Z}_p(2))$ denotes the continuous étale cohomology [J1] and the bottom right arrow is by definition the inductive limit, with respect to $U \subset U_0$, of the natural restriction map

$$H^4(\mathscr{X}_U, \mathbb{Z}_p(2)) = H^4_{\mathrm{cont}}(\mathscr{X}_U, \mathbb{Z}_p(2)) \longrightarrow H^4_{\mathrm{cont}}(X, \mathbb{Z}_p(2)).$$

The right triangle of the diagram commutes by the definition of cycle classes in loc. cit. Theorem (3.23). This diagram and the exact sequence (3.3.1) yield a commutative diagram (up to a sign)

$$\begin{tikzcd} \mathrm{CH}^2(X)_{p\text{-tors}} \arrow[r] \arrow[d] & \mathrm{CH}^2(X) \otimes \mathbb{Z}_p \arrow[d, "\varrho_{\mathrm{cont}}"] \\ H^3/(F^1H^3)_{\mathrm{Div}} \arrow[r, hook] & H^4_{\mathrm{cont}}(X, \mathbb{Z}_p(2)), \end{tikzcd}$$

where the bottom arrow is injective by **H0** and loc. cit. Theorem (5.14). Now the assertion follows from that fact that $\mathrm{Im}(\varrho_{\mathrm{cont}})$ is finitely generated over $\mathbb{Z}_p$ ([Sa] Theorem (4-4)).

(3) Put

$$N := (F^1H^3)_{\mathrm{Div}}/\{\Gamma + (F^2H^3 \cap (F^1H^3)_{\mathrm{Div}})\} = \mathrm{Coker}\big(\mathrm{reg}_{\mathbb{Q}_p/\mathbb{Z}_p}\big)_{\mathrm{Div}},$$

which is cofinitely generated by assumption and fits into an exact sequence

$$(F^2H^3 \cap (F^1H^3)_{\mathrm{Div}})/(F^2H^3)_{\mathrm{Div}} \longrightarrow M \longrightarrow N \longrightarrow 0.$$

The first group in this sequence has a finite exponent by Corollary 2.6.3 (1), N is divisible and cofinitely generated, and M is divisible. Hence M is cofinitely generated by Lemma 2.3.2 (2). This completes the proof of Lemma 3.4.1 and Proposition 3.3.2. □

3.5 Proof of Proposition 3.3.3

We put

$$NF^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) := N^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \cap F^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)).$$

Note that $N^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))_{\mathrm{Div}} = NF^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))_{\mathrm{Div}}$ by **H0**. There is an edge homomorphism of the spectral sequence (2.6.2)

$$\psi : F^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \longrightarrow H^1(k, H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))). \qquad (3.5.1)$$

The composite of ϕ in (3.3.1) and ψ agrees with $\mathrm{reg}_{\mathbb{Q}_p/\mathbb{Z}_p}$. Hence by Lemma 3.3.5, the assertion of Proposition 3.3.3 is reduced to the following lemma, which generalizes [LS] Lemma (5.7) and extends [La1] Lemma (3.3):

Lemma 3.5.2 *Assume either $p \geq 5$ or $(*_g)$ (but we do not assume* **H1∗**). *Then we have*

$$\psi(NF^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))_{\mathrm{Div}}) \subset H^1_g(k, H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))).$$

We start the proof of this lemma. The assertion is obvious under the assumption $(*_g)$. Hence we are done if k is ℓ-adic local with $\ell \neq p$ (cf. Remark 3.1.2 (1)). It remains to deal with the following two cases:

(1) k is p-adic local with $p \geq 5$.

(2) k is global and $p \geq 5$.

Put $A := H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))$ for simplicity. We first reduce the case (2) to the case (1). Suppose that k is global. Then there is a commutative diagram

$$\begin{tikzcd}
NF^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))_{\mathrm{Div}} \arrow[r] \arrow[d] & H^1(k, A) \arrow[d] \\
\prod_{v \in S^1} NF^1H^3(X_v, \mathbb{Q}_p/\mathbb{Z}_p(2))_{\mathrm{Div}} \arrow[r] & \prod_{v \in S^1} H^1(k_v, A),
\end{tikzcd}$$

where the vertical arrows are natural restriction maps. By this diagram and the definition of $H^1_g(k,A)$, the case (2) is reduced to the case (1).

We prove the case (1). We first reduce the problem to the case where X has semistable reduction. By the alteration theorem of de Jong [dJ], there exists a proper flat generically finite morphism $X' \to X$ such that X' is projective smooth over k and has a proper flat regular model over the integral closure $\mathfrak{o}'$ of $\mathfrak{o}_k$ in $\Gamma(X', \mathscr{O}_{X'})$ with semistable reduction. Put

$$L := \mathrm{Frac}(\mathfrak{o}') \quad \text{and} \quad A' := H^2(X' \otimes_L \overline{k}, \mathbb{Q}_p/\mathbb{Z}_p(2)).$$

Then there is a commutative diagram whose vertical arrows are natural restriction maps

$$\begin{CD}
NF^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))_{\mathrm{Div}} @>>> H^1(k,A) @>>> H^1(k,A)/H^1_g(k,A) \\
@VVV @VVV @VVV \\
NF^1H^3(X', \mathbb{Q}_p/\mathbb{Z}_p(2))_{\mathrm{Div}} @>>> H^1(L,A') @>>> H^1(L,A')/H^1_g(L,A').
\end{CD}$$

Our task is to show that the composite of the upper row is zero. Because X' and X are proper smooth varieties over k, the restriction map $r : A \to A'$ has a quasi-section $s : A' \to A$ with $s \circ r = d \cdot \mathrm{id}_A$, where d denotes the extension degree of the function field of $X' \otimes_L \overline{k}$ over that of $\overline{X}$. Hence by the functoriality of $H^1_g(k,A)$ in A, the right vertical arrow in the above diagram has finite kernel, and the problem is reduced to showing that the composite of the lower row is zero. Thus we are reduced to the case that X has a proper flat regular model $\mathscr{X}$ over $S = \mathrm{Spec}(\mathfrak{o}_k)$ with semistable reduction. We prove this case in what follows.

Let $j : X \hookrightarrow \mathscr{X}$ be the natural open immersion. There is a natural injective map

$$\alpha_r : H^3(\mathscr{X}, \tau_{\leq 2} Rj_*\mu_{p^r}^{\otimes 2}) \hookrightarrow H^3(X, \mu_{p^r}^{\otimes 2})$$

induced by the natural morphism $\tau_{\leq 2} Rj_*\mu_{p^r}^{\otimes 2} \to Rj_*\mu_{p^r}^{\otimes 2}$. By Corollary 2.2.3, it suffices to show the following two lemmas (see also Remark 3.5.6 below):

Lemma 3.5.3 *$N^1H^3(X, \mu_{p^r}^{\otimes 2}) \subset \mathrm{Im}(\alpha_r)$ for any $r \geq 1$.*

Lemma 3.5.4 *Put*

$$H^3(\mathscr{X}, \tau_{\leq 2} Rj_*\mathbb{Q}_p/\mathbb{Z}_p(2)) := \varinjlim_{r \geq 1} H^3(\mathscr{X}, \tau_{\leq 2} Rj_*\mu_{p^r}^{\otimes 2}),$$

and define $H^3(\mathscr{X}, \tau_{\leq 2} Rj_*\mathbb{Q}_p/\mathbb{Z}_p(2))^0$ as the kernel of the natural map

$$H^3(\mathscr{X}, \tau_{\leq 2} Rj_*\mathbb{Q}_p/\mathbb{Z}_p(2)) \to H^3(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)).$$

Then the canonical map

$$H^3(\mathscr{X}, \tau_{\leq 2} Rj_*\mathbb{Q}_p(2))^0 \longrightarrow H^3(\mathscr{X}, \tau_{\leq 2} Rj_*\mathbb{Q}_p/\mathbb{Z}_p(2))^0$$

has finite cokernel, where $H^3(\mathscr{X}, \tau_{\leq 2} Rj_*\mathbb{Q}_p(2))^0$ is as we defined in §2.2.

To prove Lemma 3.5.3, we need the following fact due to Hagihara, whose latter vanishing will be used later in §6:

LEMMA 3.5.5 ([SH] A.2.4, A.2.6) *Let n, r and c be integers with $n \geq 0$ and $r, c \geq 1$. Then for any $q \leq n + c$ and any closed subscheme $Z \subset Y$ with* $\mathrm{codim}_{\mathscr{X}}(Z) \geq c$, *we have*

$$H^q_Z(\mathscr{X}, \tau_{\leq n} Rj_* \mu_{p^r}^{\otimes n}) = 0 = H^{q+1}_Z(\mathscr{X}, \tau_{\geq n+1} Rj_* \mu_{p^r}^{\otimes n}).$$

Proof of Lemma 3.5.3. We compute the local-global spectral sequence

$$E_1^{u,v} = \bigoplus_{x \in \mathscr{X}^a} H_x^{u+v}(\mathscr{X}, \tau_{\leq 2} Rj_* \mu_{p^r}^{\otimes 2}) \Longrightarrow H^{u+v}(\mathscr{X}, \tau_{\leq 2} Rj_* \mu_{p^r}^{\otimes 2}).$$

By the first part of Lemma 3.5.5 and the smooth purity for points on X, we have

$$E_1^{u,v} = \begin{cases} H^v(K, \mu_{p^r}^{\otimes 2}) & (\text{if } u = 0) \\ \bigoplus_{x \in X^u} H^{v-u}(x, \mu_{p^r}^{\otimes 2-u}) & (\text{if } v \leq 2). \end{cases}$$

Repeating the same computation as in the proof of Lemma 3.2.2, we obtain

$$N^1 H^3(X, \mu_{p^r}^{\otimes 2}) \simeq E_2^{1,2} = E_\infty^{1,2} \hookrightarrow H^3(\mathscr{X}, \tau_{\leq 2} Rj_* \mu_{p^r}^{\otimes 2}),$$

which implies Lemma 3.5.3. □

REMARK 3.5.6 *Lemma* 3.5.3 *extends a result of Langer-Saito* ([LS] *Lemma* (5.4)) *to regular semistable families and removes the assumption in* [La1] *Lemma* (3.1) *concerning Gersten's conjecture for algebraic K-groups. Therefore the same assumption in loc. cit. Theorem* A *has been removed as well.*

Proof of Lemma 3.5.4. By the Bloch-Kato-Hyodo theorem on the structure of p-adic vanishing cycles ([BK1], [Hy]), there is a distinguished triangle of the following form in $D^b(\mathscr{X}_{\text{ét}})$ (cf. [SH], (4.3.3)):

$$\tau_{\leq 2} Rj_* \mu_{p^r}^{\otimes 2} \longrightarrow \tau_{\leq 2} Rj_* \mu_{p^{r+s}}^{\otimes 2} \longrightarrow \tau_{\leq 2} Rj_* \mu_{p^s}^{\otimes 2} \longrightarrow (\tau_{\leq 2} Rj_* \mu_{p^r}^{\otimes 2})[1]$$

Taking étale cohomology groups, we obtain a long exact sequence

$$\begin{aligned} \cdots \to H^q(\mathscr{X}, \tau_{\leq 2} Rj_* \mu_{p^r}^{\otimes 2}) \to H^q(\mathscr{X}, \tau_{\leq 2} Rj_* \mu_{p^{r+s}}^{\otimes 2}) \to H^q(\mathscr{X}, \tau_{\leq 2} Rj_* \mu_{p^s}^{\otimes 2}) \\ \to H^{q+1}(\mathscr{X}, \tau_{\leq 2} Rj_* \mu_{p^r}^{\otimes 2}) \to \cdots . \qquad (3.5.7) \end{aligned}$$

We claim that $H^q(\mathscr{X}, \tau_{\leq 2} Rj_* \mu_{p^r}^{\otimes 2})$ is finite for any q and r. Indeed, the claim is reduced to the case $r = 1$ by the exactness of (3.5.7) and this case follows from the Bloch-Kato-Hyodo theorem mentioned above and the properness of $\mathscr{X}$ over S. Hence taking the projective limit of (3.5.7) with respect to r and then taking the inductive limit with respect to s we obtain a long exact sequence

$$\begin{aligned} \cdots \to H^q(\mathscr{X}, \tau_{\leq 2} Rj_* \mathbb{Z}_p(2)) \to H^q(\mathscr{X}, \tau_{\leq 2} Rj_* \mathbb{Q}_p(2)) \to H^q(\mathscr{X}, \tau_{\leq 2} Rj_* \mathbb{Q}_p/\mathbb{Z}_p(2)) \\ \to H^{q+1}(\mathscr{X}, \tau_{\leq 2} Rj_* \mathbb{Z}_p(2)) \to \cdots , \end{aligned}$$

where $H^q(\mathscr{X}, \tau_{\leq 2} Rj_* \mathbb{Z}_p(2))$ is finitely generated over $\mathbb{Z}_p$ for any q. The assertion in the lemma easily follows from this exact sequence and a similar long exact sequence of étale cohomology groups of $\overline{X}$. The details are straight-forward and left to the reader. □

This completes the proof of Lemma 3.5.2, Proposition 3.3.3 and Theorem 3.1.1.

4 Cycle Class Map and Unramified Cohomology

Let $k, S, p, \mathscr{X}$ and K be as in the notation 1.8. In particular, we always assume that $\mathscr{X}$ satisfies 1.8.1. In this section we give a brief review of p-adic étale Tate twists and provide some preliminary results on cycle class maps. The main result of this section is Corollary 4.4.3 below.

4.1 p-adic Étale Tate Twist

Let n and r be positive integers. We recall here the fundamental properties $(\mathbf{S1})$–$(\mathbf{S7})$ listed below of the object $\mathfrak{T}_r(n) = \mathfrak{T}_r(n)_{\mathscr{X}} \in D^b(\mathscr{X}_{\text{ét}}, \mathbb{Z}/p^r)$ introduced by the second author [SH]. The properties $(\mathbf{S1})$, $(\mathbf{S2})$, $(\mathbf{S3})$ and $(\mathbf{S4})$ characterizes $\mathfrak{T}_r(n)$ uniquely up to a unique isomorphism in $D^b(\mathscr{X}_{\text{ét}}, \mathbb{Z}/p^r)$.

$(\mathbf{S1})$ *There is an isomorphism* $t : \mathfrak{T}_r(n)|_V \simeq \mu_{p^r}^{\otimes n}$ *on* $V := \mathscr{X}[p^{-1}]$.

$(\mathbf{S2})$ $\mathfrak{T}_r(n)$ *is concentrated in* $[0, n]$.

$(\mathbf{S3})$ *Let* $Z \subset \mathscr{X}$ *be a locally closed regular subscheme of pure codimension* c *with* $\mathrm{ch}(Z) = p$. *Let* $i : Z \to \mathscr{X}$ *be the natural immersion. Then there is a canonical Gysin isomorphism*

$$\mathrm{Gys}_i^n : W_r\Omega_{Z,\log}^{n-c}[-n-c] \xrightarrow{\sim} \tau_{\leq n+c} Ri^! \mathfrak{T}_r(n) \quad \text{in } D^b(Z_{\text{ét}}, \mathbb{Z}/p^r),$$

where $W_r\Omega_{Z,\log}^q$ *denotes the étale subsheaf of the logarithmic part of the Hodge-Witt sheaf* $W_r\Omega_Z^q$ ([Bl1], [Il]).

$(\mathbf{S4})$ *For* $x \in \mathscr{X}$ *and* $q \in \mathbb{Z}_{\geq 0}$, *we define* $\mathbb{Z}/p^r(q) \in D^b(x_{\text{ét}}, \mathbb{Z}/p^r)$ *as*

$$\mathbb{Z}/p^r(q) := \begin{cases} \mu_{p^r}^{\otimes q} & (\textit{if } \mathrm{ch}(x) \neq p) \\ W_r\Omega_{x,\log}^q[-q] & (\textit{if } \mathrm{ch}(x) = p). \end{cases}$$

Then for $y, x \in \mathscr{X}$ *with* $c := \mathrm{codim}(x) = \mathrm{codim}(y) + 1$, *there is a commutative diagram*

$$\begin{array}{ccc} H^{n-c+1}(y, \mathbb{Z}/p^r(n-c+1)) & \xrightarrow{-\partial^{\mathrm{val}}} & H^{n-c}(x, \mathbb{Z}/p^r(n-c)) \\ \downarrow{\scriptstyle \mathrm{Gys}_{i_y}^n} & & \downarrow{\scriptstyle \mathrm{Gys}_{i_x}^n} \\ H_y^{n+c-1}(\mathscr{X}, \mathfrak{T}_r(n)) & \xrightarrow{\delta^{\mathrm{loc}}} & H_x^{n+c}(\mathscr{X}, \mathfrak{T}_r(n)). \end{array}$$

Here for $z \in \mathscr{X}$, $\mathrm{Gys}^n_{i_z}$ is induced by the Gysin map in ($\mathbf{S3}$) (*resp. the absolute purity* [RZ], [Th], [FG]) *if* $\mathrm{ch}(z) = p$ (*resp.* $\mathrm{ch}(z) \neq p$). *The arrow δ^{loc} denotes the boundary map of a localization exact sequence and ∂^{val} denotes the boundary map of Galois cohomology groups due to Kato* [KCT] §1.

($\mathbf{S5}$) *Let Y be the union of the fibers of $\mathscr{X}/S$ of characteristic p. We define the étale sheaf $\nu^{n-1}_{Y,r}$ on Y as*

$$\nu^{n-1}_{Y,r} := \mathrm{Ker}\Big(\partial^{\mathrm{val}} : \bigoplus\nolimits_{y \in Y^0} i_{y*} W_r\Omega^{n-1}_{y,\log} \longrightarrow \bigoplus\nolimits_{x \in Y^1} i_{x*} W_r\Omega^{n-2}_{x,\log}\Big),$$

where for $y \in Y$, i_y denotes the canonical map $y \hookrightarrow Y$. Let i and j be as follows:

$$V = \mathscr{X}[p^{-1}] \xhookrightarrow{\ j\ } \mathscr{X} \xhookleftarrow{\ i\ } Y.$$

Then there is a distinguished triangle in $D^b(\mathscr{X}_{\mathrm{\acute{e}t}}, \mathbb{Z}/p^r)$

$$i_*\nu^{n-1}_{Y,r}[-n-1] \xrightarrow{\ g\ } \mathfrak{T}_r(n) \xrightarrow{\ t'\ } \tau_{\leq n} Rj_*\mu_{p^r}^{\otimes n} \xrightarrow{\ \sigma\ } i_*\nu^{n-1}_{Y,r}[-n],$$

where t' is induced by the isomorphism t in ($\mathbf{S1}$) *and the acyclicity property* ($\mathbf{S2}$). *The arrow g arises from the Gysin morphisms in* ($\mathbf{S3}$), *σ is induced by the boundary maps of Galois cohomology groups* (*cf.* ($\mathbf{S4}$)).

($\mathbf{S6}$) *There is a canonical distinguished triangle of the following form in $D^b(\mathscr{X}_{\mathrm{\acute{e}t}})$*:

$$\mathfrak{T}_{r+s}(n) \longrightarrow \mathfrak{T}_s(n) \xrightarrow{\ \delta_{s,r}\ } \mathfrak{T}_r(n)[1] \xrightarrow{\ \underline{p^s}\ } \mathfrak{T}_{r+s}(n)[1].$$

($\mathbf{S7}$) *$H^i(\mathscr{X}, \mathfrak{T}_r(n))$ is finite for any r and i* (*by the properness of $\mathscr{X}$*).

When k is p-adic local with $p \geq n+2$ and $\mathscr{X}$ is smooth over S, then $i^*\mathfrak{T}_r(n)$ is isomorphic to the syntomic complex $\mathscr{S}_r(n)$ of Kato [Ka1], which is the derived image of a syntomic sheaf of Fontaine-Messing [FM]. This fact follows from a result of Kurihara [Ku] and ($\mathbf{S5}$). Therefore our object $\mathfrak{T}_r(n)$ extends the syntomic complexes to the global situation. Note also that $i^*\mathfrak{T}_r(n)$ is not the log syntomic complex $s^{\log}_r(n)$ unless $n > \dim(\mathscr{X})$, because the latter object is isomorphic to $\tau_{\leq n} i^*Rj_*\mu_{p^r}^{\otimes n}$ by Theorem 2.2.1.

REMARK 4.1.1 *The above properties of $\mathfrak{T}_r(n)$ deeply rely on the computation on the étale sheaf of p-adic vanishing cycles due to Bloch-Kato* [BK1] *and Hyodo* [Hy].

LEMMA 4.1.2 *Put*

$$H^q(\mathscr{X}, \mathfrak{T}_{\mathbb{Z}_p}(n)) := \varprojlim_{r \geq 1} H^q(\mathscr{X}, \mathfrak{T}_r(n)), \quad H^q(\mathscr{X}, \mathfrak{T}_\infty(n)) := \varinjlim_{r \geq 1} H^q(\mathscr{X}, \mathfrak{T}_r(n))$$

and $H^q(\mathscr{X}, \mathfrak{T}_{\mathbb{Q}_p}(n)) := H^q(\mathscr{X}, \mathfrak{T}_{\mathbb{Z}_p}(n)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$*. Then there is a long exact sequence of* $\mathbb{Z}_p$*-modules*

$$\begin{aligned} \cdots \longrightarrow\ & H^q(\mathscr{X}, \mathfrak{T}_{\mathbb{Z}_p}(n)) \longrightarrow H^q(\mathscr{X}, \mathfrak{T}_{\mathbb{Q}_p}(n)) \longrightarrow H^q(\mathscr{X}, \mathfrak{T}_{\infty}(n)) \\ & \longrightarrow H^{q+1}(\mathscr{X}, \mathfrak{T}_{\mathbb{Z}_p}(n)) \longrightarrow \cdots, \end{aligned}$$

where $H^q(\mathscr{X}, \mathfrak{T}_{\mathbb{Z}_p}(n))$ *is finitely generated over* $\mathbb{Z}_p$*,* $H^q(\mathscr{X}, \mathfrak{T}_{\infty}(n))$ *is cofinitely generated over* $\mathbb{Z}_p$*, and* $H^q(\mathscr{X}, \mathfrak{T}_{\mathbb{Q}_p}(n))$ *is finite-dimensional over* $\mathbb{Q}_p$*.*

Proof. The assertions immediately follow from (**S6**) and (**S7**). The details are straight-forward and left to the reader. □

4.2 Cycle Class Map

Let us review the definition of the cycle map

$$\varrho_r^n : \mathrm{CH}^n(\mathscr{X})/p^r \longrightarrow H^{2n}(\mathscr{X}, \mathfrak{T}_r(n)).$$

Consider the local-global spectral sequence

$$E_1^{u,v} = \bigoplus_{x \in \mathscr{X}^u} H_x^{u+v}(\mathscr{X}, \mathfrak{T}_r(n)) \Longrightarrow H^{u+v}(\mathscr{X}, \mathfrak{T}_r(n)). \tag{4.2.1}$$

By (**S3**) and the absolute cohomological purity [FG] (cf. [RZ], [Th]), we have

$$E_1^{u,v} \simeq \bigoplus_{x \in \mathscr{X}^u} H^{v-u}(x, \mathbb{Z}/p^r(n-u)) \qquad \text{for } v \leq n. \tag{4.2.2}$$

This implies that there is an edge homomorphism $E_2^{n,n} \to H^{2n}(\mathscr{X}, \mathfrak{T}_r(n))$ with

$$\begin{aligned} E_2^{n,n} &\simeq \mathrm{Coker}\Big(\partial^{\mathrm{val}} : \bigoplus\nolimits_{y \in \mathscr{X}^{n-1}} H^1(y, \mathbb{Z}/p^r(1)) \longrightarrow \bigoplus\nolimits_{x \in \mathscr{X}^n} H^0(x, \mathbb{Z}/p^r)\Big) \\ &= \mathrm{CH}^n(\mathscr{X})/p^r, \end{aligned}$$

where ∂^{val} is as in (**S4**). We define ϱ_r^n as the composite map

$$\varrho_r^n : \mathrm{CH}^n(\mathscr{X})/p^r \simeq E_2^{n,n} \longrightarrow H^{2n}(\mathscr{X}, \mathfrak{T}_r(n)).$$

In what follows, we restrict our attention to the case $n = 2$.

Lemma 4.2.3 *Let* $Z \subset \mathscr{X}$ *be a closed subscheme of pure codimension* 1*, and let* K *be the function field of* $\mathscr{X}$*. Put*

$$\begin{aligned} N^1 H^i(\mathscr{X}, \mathfrak{T}_r(2)) &:= \mathrm{Ker}\big(H^i(\mathscr{X}, \mathfrak{T}_r(2)) \to H^i(K, \mu_{p^r}^{\otimes 2})\big), \\ N^2 H_Z^i(\mathscr{X}, \mathfrak{T}_r(2)) &:= \mathrm{Ker}\Big(H_Z^i(\mathscr{X}, \mathfrak{T}_r(2)) \to \bigoplus\nolimits_{z \in Z^0} H_z^i(\mathscr{X}, \mathfrak{T}_r(2))\Big). \end{aligned}$$

(1) $N^1H^3(\mathscr{X}, \mathfrak{T}_r(2))$ *is isomorphic to the cohomology of the Gersten complex modulo* p^r

$$K_2^M(K)/p^r \longrightarrow \bigoplus_{y \in \mathscr{X}^1} \kappa(y)^\times/p^r \longrightarrow \bigoplus_{x \in \mathscr{X}^2} \mathbb{Z}/p^r,$$

and there is an exact sequence

$$0 \longrightarrow \mathrm{CH}^2(\mathscr{X}, 1)/p^r \longrightarrow N^1H^3(\mathscr{X}, \mathfrak{T}_r(2)) \longrightarrow {}_{p^r}\mathrm{CH}^2(\mathscr{X}) \longrightarrow 0.$$

See §1.7 *for the definition of* $\mathrm{CH}^2(\mathscr{X}, 1)$.

(2) *There are isomorphisms*

$$\begin{aligned} H^3_Z(\mathscr{X}, \mathfrak{T}_r(2)) &\simeq \mathrm{Ker}\Big(\partial^{\mathrm{val}} : \bigoplus\nolimits_{z \in Z^0} \kappa(z)^\times/p^r \to \bigoplus\nolimits_{x \in Z^1} \mathbb{Z}/p^r\Big), \\ N^2H^4_Z(\mathscr{X}, \mathfrak{T}_r(2)) &\simeq \mathrm{Coker}\Big(\partial^{\mathrm{val}} : \bigoplus\nolimits_{z \in Z^0} \kappa(y)^\times/p^r \to \bigoplus\nolimits_{x \in Z^1} \mathbb{Z}/p^r\Big) \\ &= \mathrm{CH}_{d-2}(Z)/p^r, \end{aligned}$$

where d *denotes the Krull dimension of* $\mathscr{X}$.

Proof. (1) follows from a similar argument as for the proof of Lemma 3.2.2, using the spectral sequence

$$E_1^{u,v} = \bigoplus_{x \in \mathscr{X}^u} H_x^{u+v}(\mathscr{X}, \mathfrak{T}_r(2)) \Longrightarrow H^{u+v}(\mathscr{X}, \mathfrak{T}_r(2)) \quad \text{((4.2.1) with } n = 2) \tag{4.2.4}$$

and the purity isomorphism

$$E_1^{u,v} \simeq \bigoplus_{x \in \mathscr{X}^u} H^{v-u}(x, \mathbb{Z}/p^r(2-u)) \qquad \text{for } v \le 2 \qquad \text{((4.2.1) with } n = 2). \tag{4.2.5}$$

More precisely, since $E_1^{u,v} = 0$ for (u, v) with $u > v$ and $v \le 2$, we have $N^1H^3(\mathscr{X}, \mathfrak{T}_r(2)) \simeq E_2^{1,2}$, which is isomorphic to the cohomology of the Gersten complex in the assertion by Hilbert's theorem 90, the Merkur'ev-Suslin theorem [MS] and $(\mathbf{S4})$. One can prove (2) in the same way as for (1), using the spectral sequence

$$E_1^{u,v} = \bigoplus_{x \in \mathscr{X}^u \cap Z} H_x^{u+v}(\mathscr{X}, \mathfrak{T}_r(2)) \Longrightarrow H_Z^{u+v}(\mathscr{X}, \mathfrak{T}_r(2))$$

and the purity isomorphism

$$E_1^{u,v} \simeq \bigoplus_{x \in Z^{u-1}} H^{v-u}(x, \mathbb{Z}/p^r(2-u)) \qquad \text{for } v \le 2$$

instead of (4.2.4) and (4.2.5). The details are straight-forward and left to the reader. □

Corollary 4.2.6 *${}_{p^r}\mathrm{CH}^2(\mathscr{X})$ is finite for any $r \geq 1$, and $\mathrm{CH}^2(\mathscr{X})_{p\text{-tors}}$ is cofinitely generated.*

Proof. The finiteness of ${}_{p^r}\mathrm{CH}^2(\mathscr{X})$ follows from the exact sequence in Lemma 4.2.3 (1) and (**S7**) in §4.1. The second assertion follows from Lemma 4.1.2 and the facts that $\mathrm{CH}^2(\mathscr{X})_{p\text{-tors}}$ is a subquotient of $H^3(\mathscr{X}, \mathfrak{T}_\infty(2))$. $\square$

4.3 Unramified Cohomology

Let K be the function field of $\mathscr{X}$. We define the unramified cohomology groups $H^{n+1}_{\mathrm{ur}}(K, \mathbb{Z}/p^r(n))$ and $H^{n+1}_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(n))$ as follows:

$$H^{n+1}_{\mathrm{ur}}(K, \mathbb{Z}/p^r(n)) := \mathrm{Ker}\left(H^{n+1}(K, \mu_{p^r}^{\otimes n}) \to \bigoplus\nolimits_{y \in \mathscr{X}^1} H^{n+2}_y(\mathscr{X}, \mathfrak{T}_r(n))\right),$$
$$H^{n+1}_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(n)) := \varinjlim_{r \geq 1} H^{n+1}_{\mathrm{ur}}(K, \mathbb{Z}/p^r(n)).$$

We mention some remarks on these groups:

Remark 4.3.1 (1) *For $n = 0$, we have*

$$H^1_{\mathrm{ur}}(K, \mathbb{Z}/p^r(0)) = H^1(\mathscr{X}, \mathbb{Z}/p^r) \quad \textit{and} \quad H^1_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(0)) = H^1(\mathscr{X}, \mathbb{Q}_p/\mathbb{Z}_p).$$

If k is global, then $H^1_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(0))$ is finite by a theorem of Katz-Lang [KL].

(2) *For $n = 1$, we have*

$$H^2_{\mathrm{ur}}(K, \mathbb{Z}/p^r(1)) = {}_{p^r}\mathrm{Br}(\mathscr{X}) \quad \textit{and} \quad H^2_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(1)) = \mathrm{Br}(\mathscr{X})_{p\text{-tors}}.$$

If k is global, the finiteness of $H^2_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(1))$ is equivalent to the finiteness of the Tate-Shafarevich group of the Picard variety of X (cf. [G] III, [Ta1]).

(3) *For $n = d := \dim(\mathscr{X})$, $H^{d+1}_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(d))$ agrees with a group considered by Kato* [KCT]*, who conjectures that*

$$H^{d+1}_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(d)) = 0 \quad \textit{if } p \neq 2 \textit{ or } k \textit{ has no embedding into } \mathbb{R}.$$

His conjecture is a generalization, to higher-dimensional proper arithmetic schemes, of the corresponding classical fact on the Brauer groups of local and global integer rings. The $d = 2$ case is proved in [KCT] *and the $d = 3$ case is proved in* [JS].

We restrict our atttention to the case $n = 2$ in what follows. The following standard proposition relates $H^3_{\mathrm{ur}}(K, \mathbb{Z}/p^r(2))$ with the cycle class map ϱ^2_r, which will be useful later.

PROPOSITION 4.3.2 *For a positive integer r, there is an exact sequence*

$$\begin{aligned} 0 \longrightarrow N^1H^3(\mathscr{X}, \mathfrak{T}_r(2)) &\longrightarrow H^3(\mathscr{X}, \mathfrak{T}_r(2)) \longrightarrow H^3_{\mathrm{ur}}(K, \mathbb{Z}/p^r(2)) \\ \longrightarrow \quad \mathrm{CH}^2(\mathscr{X})/p^r &\xrightarrow{\varrho^2_r} H^4(\mathscr{X}, \mathfrak{T}_r(2)) . \end{aligned}$$

Consequently, taking the inductive limit on $r \geq 1$, we get an exact sequence

$$\begin{aligned} 0 \longrightarrow \ N^1H^3(\mathscr{X}, \mathfrak{T}_\infty(2)) &\longrightarrow H^3(\mathscr{X}, \mathfrak{T}_\infty(2)) \longrightarrow H^3_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(2)) \\ \longrightarrow \mathrm{CH}^2(\mathscr{X}) \otimes \mathbb{Q}_p/\mathbb{Z}_p &\xrightarrow{\varrho^2_{\mathbb{Q}_p/\mathbb{Z}_p}} H^4(\mathscr{X}, \mathfrak{T}_\infty(2)) . \end{aligned} \tag{4.3.3}$$

Proof. Consider the spectral sequence (4.2.4). Since $E_1^{u,v} = 0$ for (u, v) with $u > v$ and $v \leq 2$ by (4.2.5), there is an exact sequnece

$$0 \to E_2^{1,2} \to H^3(\mathscr{X}, \mathfrak{T}_r(2)) \to E_2^{0,3} \to E_2^{2,2} \to H^4(\mathscr{X}, \mathfrak{T}_r(2)).$$

One obtains the assertion by rewriting these E_2-terms by similar arguments as for the proof of Lemma 4.2.3 (1). $\square$

REMARK 4.3.4 *Because the groups $H^*(\mathscr{X}, \mathfrak{T}_\infty(2))$ are cofinitely generated by Lemma* 4.1.2*, the sequence* (4.3.3) *implies that $H^3_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$ is cofinitely generated if and only if $\mathrm{CH}^2(\mathscr{X}) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ is cofinitely generated.*

We next prove that $H^3_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$ is related with the torsion part of the cokernel of a cycle class map, assuming its finiteness. This result will not be used in the rest of this paper, but shows an arithmetic meaning of $H^3_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$. See also Appendix B below for a zeta value formula for threefolds over finite fields using unramified cohomology.

PROPOSITION 4.3.5 *Assume that $H^3_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$ is finite. Then the order of*

$$\mathrm{Coker}\big(\varrho^2_{\mathbb{Z}_p} : \mathrm{CH}^2(\mathscr{X}) \otimes \mathbb{Z}_p \longrightarrow H^4(\mathscr{X}, \mathfrak{T}_{\mathbb{Z}_p}(2))\big)_{p\text{-tors}}$$

agrees with that of $H^3_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$.

Proof. Note that $H^4(\mathscr{X}, \mathfrak{T}_{\mathbb{Z}_p}(2))$ is finitely generated over $\mathbb{Z}_p$ by Lemma 4.1.2, so that $\mathrm{Coker}(\varrho^2_{\mathbb{Z}_p})_{p\text{-tors}}$ is finite. Consider the following commutative diagram with exact rows (cf. Lemma 4.1.2):

$$\begin{array}{ccccccccc} 0 \to & \mathrm{CH}^2(\mathscr{X})_{p\text{-tors}} & \to & \mathrm{CH}^2(\mathscr{X}) \otimes \mathbb{Z}_p & \xrightarrow{b} & \mathrm{CH}^2(\mathscr{X}) \otimes \mathbb{Q}_p & \to & \mathrm{CH}^2(\mathscr{X}) \otimes \mathbb{Q}_p/\mathbb{Z}_p & \to 0 \\ & {\scriptstyle a}\downarrow & & {\scriptstyle \varrho^2_{\mathbb{Z}_p}}\downarrow & & {\scriptstyle \varrho^2_{\mathbb{Q}_p}}\downarrow & & {\scriptstyle \varrho^2_{\mathbb{Q}_p/\mathbb{Z}_p}}\downarrow & \\ 0 \to & \mathrm{Cotor}(H^3(\mathscr{X}, \mathfrak{T}_\infty(2))) & \to & H^4(\mathscr{X}, \mathfrak{T}_{\mathbb{Z}_p}(2)) & \xrightarrow{c} & H^4(\mathscr{X}, \mathfrak{T}_{\mathbb{Q}_p}(2)) & \to & H^4(\mathscr{X}, \mathfrak{T}_\infty(2)), & \end{array}$$

where the arrow a denotes the map obtained from the short exact sequence in Lemma 4.2.3 (1) and the arrows b and c are natural maps. See Lemma 4.4.2 below for the commutativity of the left square. By the finiteness of $H^3_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$, we see that

$$\mathrm{Coker}(a) \simeq \mathrm{gr}^0_N H^3(\mathscr{X}, \mathfrak{T}_\infty(2)) := H^3(\mathscr{X}, \mathfrak{T}_\infty(2))/N^1H^3(\mathscr{X}, \mathfrak{T}_\infty(2))$$

(cf. Lemma 4.2.3 (1)) and that the natural map $\mathrm{Ker}(\varrho^2_{\mathbb{Q}_p}) \to \mathrm{Ker}(\varrho^2_{\mathbb{Q}_p/\mathbb{Z}_p})$ is zero (cf. (4.3.3)). The latter conclusion further implies that the kernel of the induced map $\mathrm{Im}(b) \to \mathrm{Im}(c)$ is divisible. Noting these facts, we obtain a short exact sequence

$$0 \longrightarrow \mathrm{gr}^0_N H^3(\mathscr{X}, \mathfrak{T}_\infty(2)) \longrightarrow \mathrm{Coker}(\varrho^2_{\mathbb{Z}_p})_{p\text{-tors}} \longrightarrow \mathrm{Ker}(\varrho^2_{\mathbb{Q}_p/\mathbb{Z}_p}) \longrightarrow 0$$

by a diagram chase on the above diagram. Comparing this sequence with (4.3.3), we obtain the assertion. □

4.4 Torsion Cycle Class Map of Codimension Two

We define $H^3_{\mathrm{ur}}(K, X; \mathbb{Q}_p/\mathbb{Z}_p(2))$ as the following subgroup of $H^3_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$:

$$\mathrm{Im}\big(H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \to H^3(K, \mathbb{Q}_p/\mathbb{Z}_p(2))\big) \cap H^3_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(2)).$$

In this subsection we relate the finiteness of this group with the injectivity of torsion cycle class maps of codimension two (see Corollary 4.4.3 below), which will be used in the proof of Theorem 1.6. We start with the following proposition.

Proposition 4.4.1 *Assume that the quotient*

$$\mathrm{gr}^0_N H^3(\mathscr{X}, \mathfrak{T}_\infty(2)) := H^3(\mathscr{X}, \mathfrak{T}_\infty(2))/N^1 H^3(\mathscr{X}, \mathfrak{T}_\infty(2))$$

is finite. Then there exists a positive integer r_0 such that the kernel of the map

$$\varrho^2_{p\text{-tors},r} : \mathrm{CH}^2(\mathscr{X})_{p\text{-tors}} \longrightarrow H^4(\mathscr{X}, \mathfrak{T}_r(2))$$

agrees with $(\mathrm{CH}^2(\mathscr{X})_{p\text{-tors}})_{\mathrm{Div}}$ for any $r \geq r_0$.

We need the following lemma to prove this proposition (cf. [CTSS] Proposition 1):

Lemma 4.4.2 *For integers $r, s > 0$, there is a commutative diagram up to a sign*

$$\begin{tikzcd} N^1H^3(\mathscr{X}, \mathfrak{T}_s(2)) \arrow[r, "\alpha_s"] \arrow[d, hook] & {}_{p^s}\mathrm{CH}^2(\mathscr{X}) \arrow[d, "\varrho^2_{s,r}"] \\ H^3(\mathscr{X}, \mathfrak{T}_s(2)) \arrow[r, "\delta_{s,r}"] & H^4(\mathscr{X}, \mathfrak{T}_r(2)), \end{tikzcd}$$

where α_s denotes the boundary map in the short exact sequence of Lemma 4.2.3 (1) *and $\varrho^2_{s,r}$ denotes the cycle class map ϱ^2_r restricted to ${}_{p^s}\mathrm{CH}^2(\mathscr{X})$. The arrow $\delta_{s,r}$ is the connecting morphism of the distinguished triangle in* (**S6**):

$$\mathfrak{T}_{r+s}(2) \longrightarrow \mathfrak{T}_s(2) \xrightarrow{\delta_{s,r}} \mathfrak{T}_r(2)[1] \xrightarrow{p^s} \mathfrak{T}_{r+s}(2)[1].$$

Proof of Lemma 4.4.2. Note that $N^1H^3(\mathscr{X}, \mathfrak{T}_s(2))$ is generated by the image of $H^3_Z(\mathscr{X}, \mathfrak{T}_s(2))$ for closed subsets $Z \subset \mathscr{X}$ of pure codimension 1. We fix such a

Z, and endow it with the reduced subscheme structure. There is a diagram which commutes obviously

$$\begin{tikzcd} H^3_Z(\mathscr{X},\mathfrak{T}_s(2)) \arrow[r, "\delta_{s,r}"] \arrow[d, "\text{canonical}"'] & H^4_Z(\mathscr{X},\mathfrak{T}_r(2)) \arrow[d, "\text{canonical}"] \\ H^3(\mathscr{X},\mathfrak{T}_s(2)) \arrow[r, "\delta_{s,r}"] & H^4(\mathscr{X},\mathfrak{T}_r(2)). \end{tikzcd}$$

We show that the image of the upper $\delta_{s,r}$ lies in the subgroup

$$\begin{aligned} N^2H^4_Z(\mathscr{X},\mathfrak{T}_r(2)) &= \mathrm{Ker}\Big(H^4_Z(\mathscr{X},\mathfrak{T}_r(2)) \to \bigoplus\nolimits_{z\in Z^0} H^4_z(\mathscr{X},\mathfrak{T}_r(2))\Big) \\ &\simeq \mathrm{Coker}\Big(\partial^{\mathrm{val}} : \bigoplus\nolimits_{z\in Z^0} \kappa(z)^\times/p^r \to \bigoplus\nolimits_{x\in Z^1} \mathbb{Z}/p^r\Big) \\ &= \mathrm{CH}_{d-2}(Z)/p^r \qquad\qquad (d := \dim(\mathscr{X})) \end{aligned}$$

(cf. Lemma 4.2.3). Indeed, there is a commutative diagram with exact bottom row

$$\begin{tikzcd} & H^3_Z(\mathscr{X},\mathfrak{T}_s(2)) \arrow[r, "\delta_{s,r}"] \arrow[d] & H^4_Z(\mathscr{X},\mathfrak{T}_r(2)) \arrow[d] \\ \bigoplus_{z\in Z^0} H^3_z(\mathscr{X},\mathfrak{T}_{r+s}(2)) \arrow[r, two heads] & \bigoplus_{z\in Z^0} H^3_z(\mathscr{X},\mathfrak{T}_s(2)) \arrow[r, "\delta_{s,r}"] & \bigoplus_{z\in Z^0} H^4_z(\mathscr{X},\mathfrak{T}_r(2)), \end{tikzcd}$$

whose bottom left arrow is surjective by the purity in ($\mathbf{S3}$) and Hilbert's theorem 90:

$$H^3_z(\mathscr{X},\mathfrak{T}_t(2)) \simeq H^1(z,\mathbb{Z}/p^t(1)) \simeq \kappa(z)^\times/p^t \quad \text{for} \quad t = r+s,\ s.$$

Hence the lower $\delta_{s,r}$ is the zero map and the image of the upper $\delta_{s,r}$ is contained in $N^2H^4_Z(\mathscr{X},\mathfrak{T}_r(2))$. Now the composite map

$$H^3_Z(\mathscr{X},\mathfrak{T}_s(2)) \xrightarrow{\delta_{s,r}} N^2H^4_Z(\mathscr{X},\mathfrak{T}_r(2)) \simeq \mathrm{CH}_{d-2}(Z)/p^r \longrightarrow \mathrm{CH}^2(\mathscr{X})/p^r$$

agrees, up to a sign, with the composite map

$$H^3_Z(\mathscr{X},\mathfrak{T}_s(2)) \to N^1H^3(\mathscr{X},\mathfrak{T}_s(2)) \xrightarrow{\alpha_s} {}_{p^s}\mathrm{CH}^2(\mathscr{X}) \hookrightarrow \mathrm{CH}^2(\mathscr{X}) \twoheadrightarrow \mathrm{CH}^2(\mathscr{X})/p^r$$

by ($\mathbf{S4}$) and computations on boudary maps (see [CTSS] Proof of Proposition 1, Step 6). We obtain the commutativity in question from these facts, because the cycle class map ϱ^2_r for cycles on Z is given by the composite map

$$\mathrm{CH}_{d-2}(Z)/p^r \simeq N^2H^4_Z(\mathscr{X},\mathfrak{T}_r(2)) \hookrightarrow H^4_Z(\mathscr{X},\mathfrak{T}_r(2)) \to H^4(\mathscr{X},\mathfrak{T}_r(2))$$

by definition. □

Proof of Proposition 4.4.1. The following argument is essentially the same as the proof of [CTSS] Corollaire 3. Taking the inductive limit of the diagram in Lemma

4.4.2 with respect to $s \geq 1$, we obtain a diagram whose square commutes up to a sign and whose bottom row is exact

$$\begin{tikzcd} & N^1H^3(\mathscr{X},\mathfrak{T}_\infty(2)) \arrow[r, "\alpha_\infty"] \arrow[d, hook] & \mathrm{CH}^2(\mathscr{X})_{p\text{-tors}} \arrow[d, "\varrho^2_{p\text{-tors},r}"] \\ H^3(\mathscr{X},\mathfrak{T}_\infty(2)) \arrow[r, "\times p^r"] & H^3(\mathscr{X},\mathfrak{T}_\infty(2)) \arrow[r, "\delta_{\infty,r}"] & H^4(\mathscr{X},\mathfrak{T}_r(2)). \end{tikzcd}$$

Since $\mathrm{Ker}(\alpha_\infty)$ is divisible by Lemma 4.2.3 (1), this diagram induces the following commutative diagram up to a sign:

$$\begin{tikzcd} & \mathrm{Cotor}(N^1H^3(\mathscr{X},\mathfrak{T}_\infty(2))) \arrow[r, "\overline{\alpha_\infty}", "\sim"'] \arrow[d, hook] & \mathrm{Cotor}(\mathrm{CH}^2(\mathscr{X})_{p\text{-tors}}) \arrow[d, "\overline{\varrho^2_{p\text{-tors},r}}"] \\ \mathrm{Cotor}(H^3(\mathscr{X},\mathfrak{T}_\infty(2))) \arrow[r, "\times p^r"] & \mathrm{Cotor}(H^3(\mathscr{X},\mathfrak{T}_\infty(2))) \arrow[r, "\overline{\delta_{\infty,r}}"] & H^4(\mathscr{X},\mathfrak{T}_r(2)), \end{tikzcd}$$

where the bottom row remains exact, and the injectivity of the central vertical arrow follows from the finiteness of $\mathrm{gr}^0_N H^3(\mathscr{X},\mathfrak{T}_\infty(2))$. Because $\mathrm{Cotor}(H^3(\mathscr{X},\mathfrak{T}_\infty(2)))$ is finite by (**S7**) and Lemma 4.1.2, the map $\overline{\delta_{\infty,r}}$ is injective for any r for which p^r annihilates $\mathrm{Cotor}(H^3(\mathscr{X},\mathfrak{T}_\infty(2)))$. Thus we obtain Proposition 4.4.1. □

Corollary 4.4.3 *If $H^3_{\mathrm{ur}}(K,X;\mathbb{Q}_p/\mathbb{Z}_p(2))$ is finite, then there is a positive integer r_0 such that* $\mathrm{Ker}(\varrho^2_{p\text{-tors},r}) = (\mathrm{CH}^2(\mathscr{X})_{p\text{-tors}})_{\mathrm{Div}}$ *for any $r \geq r_0$.*

Proof. Since $\mathrm{gr}^0_N H^3(\mathscr{X},\mathfrak{T}_\infty(2))$ is a subgroup of $H^3_{\mathrm{ur}}(K,X;\mathbb{Q}_p/\mathbb{Z}_p(2))$ (cf. (4.3.3)), the assumption implies that $\mathrm{gr}^0_N H^3(\mathscr{X},\mathfrak{T}_\infty(2))$ is finite. Hence the assertion follows from Proposition 4.4.1. □

Remark 4.4.4 *If k is ℓ-adic local with $\ell \neq p$, then we have $\mathfrak{T}_\infty(2) = \mathbb{Q}_p/\mathbb{Z}_p(2)$ by definition and*

$$H^3(\mathscr{X},\mathfrak{T}_\infty(2)) = H^3(\mathscr{X},\mathbb{Q}_p/\mathbb{Z}_p(2)) \simeq H^3(Y,\mathbb{Q}_p/\mathbb{Z}_p(2))$$

by the proper base-change theorem, where Y denotes the closed fiber of $\mathscr{X}/S$. The last group is finite by Deligne's proof of the Weil conjecture [De2]. *Hence $\varrho^2_{p\text{-tors},r}$ for $\mathscr{X}$ is injective for a sufficiently large $r \geq 1$ by Proposition* 4.4.1. *On the other hand, if k is global or p-adic local, then $H^3(\mathscr{X},\mathfrak{T}_\infty(2))$ is not in general finite. Therefore we need to consider the finiteness of the group $H^3_{\mathrm{ur}}(K,X;\mathbb{Q}_p/\mathbb{Z}_p(2))$ to investigate the injectivity of $\varrho^2_{p\text{-tors},r}$.*

5 Finiteness of an Unramified Cohomology Group

Let $k, S, p, \mathscr{X}$ and K be as in the notation 1.8. We always assume 1.8.1 throughout this and the next section.

5.1 Finiteness of $H^3_{\mathrm{ur}}(K, X; \mathbb{Q}_p/\mathbb{Z}_p(2))$

In this and the next section, we prove the following result, which implies the finiteness of $H^3_{\mathrm{ur}}(K, X; \mathbb{Q}_p/\mathbb{Z}_p(2))$ in Theorem 1.3 (1). See the beginning of §3 for **H0**.

Theorem 5.1.1 *Assume* **H0**, **H1**$*$ *and either* $p \geq 5$ *or the equality*

$$H^1_g(k, H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)))_{\mathrm{Div}} = H^1(k, H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)))_{\mathrm{Div}}. \qquad (*_g)$$

Then $H^3_{\mathrm{ur}}(K, X; \mathbb{Q}_p/\mathbb{Z}_p(2))$ *is finite.*

In this section we reduce Theorem 5.1.1 to Key Lemma 5.4.1 stated in §5.4 below. We will prove the key lemma in §6. We first prove Theorem 1.6 admitting Theorem 5.1.1.

Proof of Theorem 1.6. The assumption $H^2(X, \mathscr{O}_X) = 0$ implies **H1**$*$ and $(*_g)$ (cf. Fact 1.1, Remark 3.2.5, Remark 3.1.2 (1)). Hence $H^3_{\mathrm{ur}}(K, X; \mathbb{Q}_p/\mathbb{Z}_p(2))$ is finite by Theorem 5.1.1. By Corollary 4.4.3, there is a positive integer r_0 such that

$$\mathrm{Ker}(\varrho^2_{p\text{-tors},r}) = (\mathrm{CH}^2(\mathscr{X})_{p\text{-tors}})_{\mathrm{Div}} \quad \text{for any } r \geq r_0.$$

Thus it remains to check that $\mathrm{CH}^2(\mathscr{X})_{p\text{-tors}}$ is finite, which follows from the finiteness of $\mathrm{CH}^2(X)_{p\text{-tors}}$ (cf. Theorem 3.1.1) and [CTR2] Lemma 3.3. This completes the proof. □

5.2 Proof of Theorem 5.1.1, Step 1

We reduce Theorem 5.1.1 to Proposition 5.2.2 below. Let $N^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))$ (resp. $\mathrm{gr}^0_N H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))$) be the kernel (resp. the image) of the natural map

$$H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \longrightarrow H^3(K, \mathbb{Q}_p/\mathbb{Z}_p(2)).$$

In view of Lemma 4.2.3, there is a commutative diagram with exact rows

$$\begin{array}{ccccc} N^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) & \hookrightarrow & H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) & \to & \mathrm{gr}^0_N H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \\ {\scriptstyle \delta_1}\downarrow & & {\scriptstyle \delta_2}\downarrow & & {\scriptstyle \overline{\mathfrak{d}}}\downarrow \\ \bigoplus_{v \in S^1} N^2H^4_{Y_v}(\mathscr{X}, \mathfrak{T}_\infty(2)) & \hookrightarrow & \bigoplus_{v \in S^1} H^4_{Y_v}(\mathscr{X}, \mathfrak{T}_\infty(2)) & \to & \bigoplus_{v \in S^1} \bigoplus_{y \in Y_v^0} H^4_y(\mathscr{X}, \mathfrak{T}_\infty(2)), \end{array} \qquad (5.2.1)$$

where the arrows δ_2 and $\overline{\mathfrak{d}}$ arise from boundary maps of localization exact sequences and δ_1 is induced by the right square. Note that we have

$$\mathrm{Ker}(\overline{\mathfrak{d}}) = H^3_{\mathrm{ur}}(K, X; \mathbb{Q}_p/\mathbb{Z}_p(2)).$$

Proposition 5.2.2 *Assume* **H0**, **H1**$*$ *and either* $p \geq 5$ *or* $(*_g)$*. Then we have*

$$\mathrm{Ker}(\overline{\mathfrak{d}})_{\mathrm{Div}} = 0.$$

The proof of this proposition will be started in §5.3 below and finished in the next section. We first finish the proof of Theorem 5.1.1, admitting Proposition 5.2.2. It suffices to show the following lemma (see also Remark 3.3.4 (2)):

Lemma 5.2.3 (1) *If k is local, then* $\mathrm{Ker}(\overline{\mathfrak{d}})$ *is cofinitely generated over* $\mathbb{Z}_p$.

(2) *Assume that k is global, and that* $\mathrm{Coker}(\mathrm{reg}_{\mathbb{Q}_p/\mathbb{Z}_p})_{\mathrm{Div}}$ *is cofinitely generated over $\mathbb{Z}_p$, where* $\mathrm{reg}_{\mathbb{Q}_p/\mathbb{Z}_p}$ *denotes the regulator map* (3.2.1). *Then* $\mathrm{Ker}(\overline{\mathfrak{d}})$ *is cofinitely generated over* $\mathbb{Z}_p$.

Proof. (1) is obvious, because $H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))$ is cofinitely generated. We prove (2). We use the notation fixed in 1.9. By Lemma 4.1.2, $H^3(\mathscr{X}, \mathfrak{T}_\infty(2))$ is cofinitely generated. Hence it suffices to show $\mathrm{Coker}(\delta_1)$ is cofinitely generated, where δ_1 is as in (5.2.1). There is a commutative diagram

$$\begin{array}{ccc} \mathrm{CH}^2(X,1)\otimes\mathbb{Q}_p/\mathbb{Z}_p & \longrightarrow & N^1H^3(X,\mathbb{Q}_p/\mathbb{Z}_p(2)) \\ \Big\downarrow{\scriptstyle\partial} & & \Big\downarrow{\scriptstyle\delta_1} \\ \bigoplus_{v\in S^1}\mathrm{CH}_{d-2}(Y_v)\otimes\mathbb{Q}_p/\mathbb{Z}_p & \xrightarrow{\sim} & \bigoplus_{v\in S^1} N^2H^4_{Y_v}(\mathscr{X},\mathfrak{T}_\infty(2)), \end{array}$$

where the bottom isomorphism follows from Lemma 4.2.3 (2) and ∂ is the boundary map of the localization sequence of higher Chow groups. See (3.3.1) for the top arrow. Since $N^2H^4_{Y_v}(\mathscr{X}, \mathfrak{T}_\infty(2))$ is cofinitely generated for any $v \in S^1$, it suffices to show that for a sufficiently small non-empty open subset $U \subset S$, the cokernel of the boundary map

$$\partial_U : \mathrm{CH}^2(X,1)\otimes\mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \bigoplus_{v\in(U)^1}\mathrm{CH}_{d-2}(Y_v)\otimes\mathbb{Q}_p/\mathbb{Z}_p$$

is cofinitely generated. Note that $\mathrm{CH}_{d-2}(Y_v) = \mathrm{CH}^1(Y_v)$ if Y_v is smooth. Now let U be a non-empty open subset of $S \setminus \Sigma$ for which $\mathscr{X} \times_S U \to U$ is smooth. Put $A := H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))$, viewed as a smooth sheaf on $U_{\text{ét}}$. There is a commutative diagram up to a sign

$$\begin{array}{ccc} \mathrm{CH}^2(X,1)\otimes\mathbb{Q}_p/\mathbb{Z}_p & \xrightarrow{\mathrm{reg}_{\mathbb{Q}_p/\mathbb{Z}_p}} & H^1(k,A) \\ \Big\downarrow{\scriptstyle\partial_U} & & \Big\downarrow{\scriptstyle\delta_U} \\ \bigoplus_{v\in U^1}\mathrm{CH}^1(Y_v)\otimes\mathbb{Q}_p/\mathbb{Z}_p & \xrightarrow{\tau_U} & \bigoplus_{v\in U^1} A(-1)^{G_{\mathbb{F}_v}}. \end{array}$$

See §2.5 for δ_U. The bottom arrow τ_U is defined as the composite map

$$\mathrm{CH}^1(Y_v)\otimes\mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow H^2(Y_v,\mathbb{Q}_p/\mathbb{Z}_p(1)) \xrightarrow{\epsilon} H^2(\overline{Y_v},\mathbb{Q}_p/\mathbb{Z}_p(1))^{G_{\mathbb{F}_v}} = A(-1)^{G_{\mathbb{F}_v}},$$

where the first injective map is the cycle class map for divisors on Y_v. Note that $\mathrm{Coker}(\partial_U)$ is divisible and that $\mathrm{Ker}(\tau_U)$ has a finite exponent by the isomorphisms

$$\mathrm{Ker}(\epsilon) \simeq H^1(\mathbb{F}_v, H^1(\overline{Y_v}, \mathbb{Q}_p/\mathbb{Z}_p(1))) \simeq H^1(\mathbb{F}_v, \mathrm{Cotor}(H^1(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(1))))$$

for $v \in U^1$, where the first isomorphism follows from the Hochschild-Serre spectral sequence for Y_v, and the second follows from Deligne's proof of the Weil conjecture [De2] and the proper smooth base-change theorem $H^1(\overline{Y_v}, \mathbb{Q}_p/\mathbb{Z}_p(1)) \simeq H^1(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(1))$. Hence to prove that $\mathrm{Coker}(\partial_U)$ is cofinitely generated, it suffices to show that the map

$$\partial' := \tau_U \circ \partial_U : \mathrm{CH}^2(X,1) \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \bigoplus_{v \in U^1} \big(A(-1)^{G_{\mathbb{F}_v}}\big)_{\mathrm{Div}}$$

has cofinitely generated cokernel (cf. Lemma 2.3.2 (2)). Finally $\mathrm{Coker}(\mathrm{reg}_{\mathbb{Q}_p/\mathbb{Z}_p})_{\mathrm{Div}}$ is cofinitely generated by assumption, which implies that ∂' has cofinitely generated cokernel by Lemma 2.5.1 (1). Thus we obtain Lemma 5.2.3. □

5.3 Proof of Theorem 5.1.1, Step 2

We construct a key commutative diagram (5.3.3) below and prove Lemma 5.3.5, which play key roles in our proof of Proposition 5.2.2. We need some preliminaries. We suppose that k is global until the end of Lemma 5.3.1. Let $\Sigma \subset S$ be the set of the closed points on S of characteristic p. For non-empty open $U \subset S$, put

$$\mathscr{X}_U := \mathscr{X} \times_S U \quad \text{and} \quad \mathscr{X}_U[p^{-1}] := \mathscr{X}_U \times_S (S \setminus \Sigma).$$

Let $j_U : \mathscr{X}_U[p^{-1}] \to \mathscr{X}_U$ be the natural open immersion. There is a natural injective map

$$\alpha_{U,r} : H^3(\mathscr{X}_U, \tau_{\leq 2} Rj_{U*}\mu_{p^r}^{\otimes 2}) \hookrightarrow H^3(\mathscr{X}_U[p^{-1}], \mu_{p^r}^{\otimes 2})$$

induced by the canonical morphism $\tau_{\leq 2} Rj_{U*}\mu_{p^r}^{\otimes 2} \to Rj_{U*}\mu_{p^r}^{\otimes 2}$.

Lemma 5.3.1 *We have $N^1H^3(\mathscr{X}_U[p^{-1}], \mu_{p^r}^{\otimes 2}) \subset \mathrm{Im}(\alpha_{U,r})$.*

Proof. We compute the local-global spectral sequence

$$E_1^{u,v} = \bigoplus_{x \in (\mathscr{X}_U)^u} H_x^{u+v}(\mathscr{X}_U, \tau_{\leq 2} Rj_{U*}\mu_{p^r}^{\otimes 2}) \Longrightarrow H^{u+v}(\mathscr{X}_U, \tau_{\leq 2} Rj_{U*}\mu_{p^r}^{\otimes 2}).$$

By the absolute cohomological purity [FG] and Lemma 3.5.5 (1), we have

$$E_1^{u,v} \simeq \begin{cases} H^v(K, \mu_{p^r}^{\otimes 2}) & (\text{if } u = 0) \\ \bigoplus_{x \in (\mathscr{X}_U[p^{-1}])^u} H^{v-u}(x, \mu_{p^r}^{\otimes 2-u}) & (\text{if } v \leq 2). \end{cases}$$

Repeating the same computation as in the proof of Lemma 3.2.2, we obtain

$$N^1H^3(\mathscr{X}_U[p^{-1}], \mu_{p^r}^{\otimes 2}) \simeq E_2^{1,2} = E_\infty^{1,2} \hookrightarrow H^3(\mathscr{X}_U, \tau_{\leq 2} Rj_{U*}\mu_{p^r}^{\otimes 2}),$$

which completes the proof of Lemma 5.3.1. □

Now we suppose that k is either local or global, and put

$$\mathscr{W} := \begin{cases} H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) & \text{(if } k \text{ is } \ell\text{-adic local with } \ell \neq p) \\ H^3(\mathscr{X}, \tau_{\leq 2} Rj_*\mathbb{Q}_p/\mathbb{Z}_p(2)) & \text{(if } k \text{ is } p\text{-adic local)} \\ \varinjlim_{\Sigma \subset U \subset S} H^3(\mathscr{X}_U, \tau_{\leq 2} Rj_{U*}\mathbb{Q}_p/\mathbb{Z}_p(2)) & \text{(if } k \text{ is global),} \end{cases} \tag{5.3.2}$$

where j in the second case denotes the natural open immersion $X \hookrightarrow \mathscr{X}$, and the limit in the last case is taken over all non-empty open subsets $U \subset S$ which contain Σ. By Lemma 3.5.3 and Lemma 5.3.1, there are inclusions

$$N^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \subset \mathscr{W} \subset H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))$$

and a commutative diagram

$$\begin{tikzcd} NF^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))_{\mathrm{Div}} \arrow[r, hook] \arrow[dr, "\nu"'] & (\mathscr{W}^0)_{\mathrm{Div}} \arrow[d, "\omega"] \\ & H^1(k, H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))). \end{tikzcd} \tag{5.3.3}$$

Here $NF^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))$ is as we defined in §3.5, and we put

$$\mathscr{W}^0 := \mathrm{Ker}\big(\mathscr{W} \longrightarrow H^3(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))\big). \tag{5.3.4}$$

The arrows ω and ν are induced by the edge homomorphism (3.5.1). We show here the following lemma, which extends Lemma 3.5.2 under Assumption 1.8.1:

Lemma 5.3.5 *Assume either $p \geq 5$ or $(*_g)$. Then we have*

$$\mathrm{Im}(\omega) \subset H^1_g(k, H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))).$$

Remark 5.3.6 *We will prove the equality* $\mathrm{Im}(\omega) = H^1_g(k, H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)))_{\mathrm{Div}}$ *under the same assumptions, later in Lemma* 7.2.2.

The following corollary of Lemma 5.3.5 will be used later in §5.4:

Corollary 5.3.7 *Assume* **H0**, **H1∗** *and either $p \geq 5$ or $(*_g)$. Then we have*

$$\mathrm{Im}(\nu) = \mathrm{Im}(\omega) = H^1_g(k, H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)))_{\mathrm{Div}}.$$

Proof of Lemma 5.3.5. The assertion under the second condition is rather obvious. In particular, we are done if k is ℓ-adic local with $\ell \neq p$ (cf. Remark 3.1.2 (1)). If k is p-adic local with $p \geq 5$, the assertion follows from Corollary 2.2.3 and Lemma 3.5.4. Before proving the global case, we show the following sublemma:

SUBLEMMA 5.3.8 *Let k be an ℓ-adic local field with $\ell \neq p$. Let $\mathscr{X}$ be a proper smooth scheme over $S := \mathrm{Spec}(\mathfrak{o}_k)$. Put $A := H^i(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(n))$ and*

$$H^{i+1}_{\mathrm{ur}}(X, \mathbb{Q}_p/\mathbb{Z}_p(n)) := \mathrm{Im}\big(H^{i+1}(\mathscr{X}, \mathbb{Q}_p/\mathbb{Z}_p(n)) \to H^{i+1}(X, \mathbb{Q}_p/\mathbb{Z}_p(n))\big).$$

Then we have

$$H^1_f(k, A) \subset \mathrm{Im}\big(F^1H^{i+1}(X, \mathbb{Q}_p/\mathbb{Z}_p(n)) \cap H^{i+1}_{\mathrm{ur}}(X, \mathbb{Q}_p/\mathbb{Z}_p(n)) \to H^1(k, A)\big)$$

and the quotient is annihilated by $\#(A/A_{\mathrm{Div}})$, where $F^\bullet$ denotes the filtration induced by the Hochschild-Serre spectral sequence (2.6.2)*.*

Proof. Put $\Lambda := \mathbb{Q}_p/\mathbb{Z}_p$, and let $\mathbb{F}$ be the residue field of k. By the proper smooth base-change theorem, G_k acts on A through the quotient $G_{\mathbb{F}}$. It suffices to show the following two claims:

(i) *We have*

$$\mathrm{Im}\big(F^1H^{i+1}(X, \Lambda(n)) \cap H^{i+1}_{\mathrm{ur}}(X, \Lambda(n)) \to H^1(k, A)\big) = H^1(\mathbb{F}, A),$$

where $H^1(\mathbb{F}, A)$ is regarded as a subgroup of $H^1(k, A)$ by an inflation map.

(ii) *We have*

$$H^1_f(k, A) \subset H^1(\mathbb{F}, A)$$

and the quotient is annihilated by $\#(A/A_{\mathrm{Div}})$.

We show these claims. Let Y be the closed fiber of $\mathscr{X}/S$, and consider a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & H^1(\mathbb{F}, H^i(\overline{Y}, \Lambda(n))) \arrow[r] \arrow[d, hook, "\sigma_1"'] & H^{i+1}(Y, \Lambda(n)) \arrow[r] \arrow[d, "\sigma_2"'] & H^{i+1}(\overline{Y}, \Lambda(n)))^{G_{\mathbb{F}}} \arrow[d, "\sigma_3", "\wr"'] \\
0 \arrow[r] & H^1(k, A) \arrow[r] & H^{i+1}(X, \Lambda(n))/F^2 \arrow[r] & H^{i+1}(\overline{X}, \Lambda(n)))^{G_k},
\end{tikzcd}$$

where the exactness of the upper (resp. lower) row follows from the fact that $\mathrm{cd}(G_{\mathbb{F}}) = 1$ (resp. $\mathrm{cd}(G_k) = 2$). The arrows σ_1 and σ_3 are induced by the isomorphism

$$H^*(\overline{Y}, \Lambda(n)) \simeq H^*(\overline{X}, \Lambda(n)) \quad \text{(proper smooth base-change theorem).}$$

The arrow σ_2 is induced by

$$\sigma_2' : H^{i+1}(Y, \Lambda(n)) \simeq H^{i+1}(\mathscr{X}, \Lambda(n)) \longrightarrow H^{i+1}(X, \Lambda(n)).$$

Since $\mathrm{Im}(\sigma_2') = H^{i+1}_{\mathrm{ur}}(X, \Lambda(n))$ by definition, the claim (i) follows from the above diagram. The second assertion immediately follows from the fact that $H^1_f(k, A) = \mathrm{Im}(H^1(\mathbb{F}, A)_{\mathrm{Div}} \to H^1(k, A))$. This completes the proof of Sublemma 5.3.8. □

We prove Lemma 5.3.5 in the case that k is global with $p \geq 5$. Let $\mathscr{W}$ and $\mathscr{W}^0$ be as in (5.3.2) and (5.3.4), respectively, and put

$$A := H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)).$$

Note that $(\mathscr{W}^0)_{\mathrm{Div}} = \mathscr{W}_{\mathrm{Div}}$ by **H0**. By a similar argument as for Lemma 2.4.1, we have

$$\mathscr{W}_{\mathrm{Div}} = \varinjlim_{\Sigma \subset U \subset S} H^3(\mathscr{X}_U, \tau_{\leq 2} Rj_{U*}\mathbb{Q}_p/\mathbb{Z}_p(2))_{\mathrm{Div}}.$$

Here the limit is taken over all non-empty open subsets $U \subset S$ which contain Σ, and j_U denotes the natural open immersion $\mathscr{X}_U[p^{-1}] \hookrightarrow \mathscr{X}_U$. By this equality and the definition of $H^1_g(k, A)$ (cf. Definition 2.1.1), it suffices to show the following sublemma:

Sublemma 5.3.9 *Let U be an open subset of S containing Σ, and fix an open subset U' of $U \setminus \Sigma$ for which $\mathscr{X}_{U'} \to U'$ is smooth (and proper). Put $\mathscr{W}_U := H^3(\mathscr{X}_U, \tau_{\leq 2} Rj_{U*}\mathbb{Q}_p/\mathbb{Z}_p(2))$. Then for any $x \in (\mathscr{W}_U)_{\mathrm{Div}}$, its diagonal image*

$$\overline{x} = (\overline{x}_v)_{v \in S^1} \in \prod_{v \in (U')^1} \frac{H^1(k_v, A)}{H^1_f(k_v, A)} \times \prod_{v \in S \setminus U'} \frac{H^1(k_v, A)}{H^1_g(k_v, A)}$$

is zero.

Proof. Since $(\mathscr{W}_U)_{\mathrm{Div}}$ is divisible, it suffices to show that $\overline{x}$ is killed by a positive integer independent of x. By Lemma 5.3.8, $\overline{x}_v$ with $v \in (U')^1$ is killed by $\#(A/A_{\mathrm{Div}})$. Next we compute $\overline{x}_v$ with $v \in \Sigma$. Let $\mathscr{X}_v$ and $j_v : X_v \hookrightarrow \mathscr{X}_v$ be as in 1.9, and put

$$\mathscr{W}_v := H^3(\mathscr{X}_v, \tau_{\leq 2} Rj_{v*}\mathbb{Q}_p/\mathbb{Z}_p(2)).$$

By **H0** over k, we have

$$\mathrm{Im}\big((\mathscr{W}_U)_{\mathrm{Div}} \to \mathscr{W}_v\big) \subset \mathrm{Ker}\big(\mathscr{W}_v \to H^3(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)\big)_{\mathrm{Div}}.$$

Hence Corollary 2.2.3 and Lemma 3.5.4 imply that $\overline{x}_v = 0$ for $v \in \Sigma$. Finally, because the product of the other components

$$\prod_{v \in S \setminus (U' \cup \Sigma)} \frac{H^1(k_v, A)}{H^1_g(k_v, A)}$$

is a finite group, we see that all local components of $\overline{x}$ are annihilated by a positive integer independent of x. Thus we obtain the sublemma and Lemma 5.3.5. $\square$

5.4 Proof of Theorem 5.1.1, Step 3

We reduce Proposition 5.2.2 to Key Lemma 5.4.1 below. We replace the conditions in Proposition 5.2.2 with another condition

> **N1**: *We have* $\mathrm{Im}(\omega) = \mathrm{Im}(\nu)$ *in* (5.3.3)*, and* $\mathrm{Coker}\big(\mathrm{reg}_{\mathbb{Q}_p/\mathbb{Z}_p}\big)_{\mathrm{Div}}$ *is cofinitely generated over* $\mathbb{Z}_p$*. Here* $\mathrm{reg}_{\mathbb{Q}_p/\mathbb{Z}_p}$ *denotes the regulator map* (3.2.1)*.*

Indeed, assuming **H0**, **H1**$*$ and either $p \geq 5$ or $(*_g)$, we see that **N1** holds by Corollary 5.3.7 and the fact that the quotient $H(k,A)_{\mathrm{Div}}/H^1_g(k,A)_{\mathrm{Div}}$, with $A = H^2(\overline{X},\mathbb{Q}_p/\mathbb{Z}_p(2))$, is cofinitely generated over $\mathbb{Z}_p$ for (cf. Lemma 2.4.1). Thus Proposition 5.2.2 is reduced to the following:

Key Lemma 5.4.1 *Assume* **H0** *and* **N1***. Then we have* $\mathrm{Ker}(\overline{\mathfrak{d}})_{\mathrm{Div}} = 0$.

This lemma will be proved in the next section.

6 Proof of the Key Lemma

The notation remains as in the previous section. We always assume 1.8.1 throughout this section. The aim of this section is to prove Key Lemma 5.4.1.

6.1 Proof of Key Lemma 5.4.1

Let

$$\mathfrak{d} : H^3(X,\mathbb{Q}_p/\mathbb{Z}_p(2)) \longrightarrow \bigoplus_{v\in S^1}\ \bigoplus_{y\in(Y_v)^0} H^4_y(\mathscr{X},\mathfrak{T}_\infty(2))$$

be the map induced by $\overline{\mathfrak{d}}$ in (5.2.1). Put

$$\Theta := H^3(X,\mathbb{Q}_p/\mathbb{Z}_p(2))\,\big/\big(N^1H^3(X,\mathbb{Q}_p/\mathbb{Z}_p(2))_{\mathrm{Div}}\big)$$

and let $\widetilde{\Theta} \subset \Theta$ be the image of $\mathrm{Ker}(\mathfrak{d})$. Note that we have

$$\widetilde{\Theta} = \mathrm{Ker}\Big(\Theta \to \mathrm{gr}^0_N H^3(X,\mathbb{Q}_p/\mathbb{Z}_p(2)) \xrightarrow{\overline{\mathfrak{d}}} \bigoplus\nolimits_{v\in S^1}\ \bigoplus\nolimits_{y\in(Y_v)^0} H^4_y(\mathscr{X},\mathfrak{T}_\infty(2))\Big)$$

and a short exact sequence

$$0 \longrightarrow \mathrm{Cotor}(N^1H^3(X,\mathbb{Q}_p/\mathbb{Z}_p(2))) \longrightarrow \widetilde{\Theta} \longrightarrow \mathrm{Ker}(\overline{\mathfrak{d}}) \longrightarrow 0.$$

If k is global, the assumption of Proposition 3.3.2 (2) is satisfied by the condition **N1**. Hence $\mathrm{Cotor}(N^1H^3(X,\mathbb{Q}_p/\mathbb{Z}_p(2)))$ is finite in both cases k is local and global (cf. Proposition 3.3.2, (3.3.1)). By the above short exact sequence and Lemma 2.3.2 (3), our task is to show

$$\widetilde{\Theta}_{\mathrm{Div}} = 0, \quad \text{assuming } \mathbf{H0} \text{ and } \mathbf{N1}.$$

Let $F^\bullet$ be the filtration on $H^3(X,\mathbb{Q}_p/\mathbb{Z}_p(2))$ resulting from the Hochschild-Serre spectral sequence (2.6.2). We define the filtration $F^\bullet$ on Θ as that induced by $F^\bullet H^3(X,\mathbb{Q}_p/\mathbb{Z}_p(2))$, and define the filtration $F^\bullet\widetilde{\Theta} \subset \widetilde{\Theta}$ as the pull-back of $F^\bullet\Theta$. Since **H0** implies the finiteness of $\mathrm{gr}^0_F\widetilde{\Theta}$, it suffices to show

$$(F^1\widetilde{\Theta})_{\mathrm{Div}} = 0, \quad \text{assuming } \mathbf{N1}. \tag{6.1.1}$$

The following lemma will play key roles:

Lemma 6.1.2 *Suppose that k is local. Then the following composite map has finite kernel*:

$$\mathfrak{d}_2 : H^2(k, H^1(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)) \longrightarrow H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \xrightarrow{\mathfrak{d}} \bigoplus_{y \in Y^0} H^4_y(\mathscr{X}, \mathfrak{T}_\infty(2)).$$

Here the first map is obtained by the Hochschild-Serre spectral sequence (2.6.2) *and the fact that* $\mathrm{cd}(k) = 2$ (*cf.* §2.6). *Consequently, the group* $F^2H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \cap$ $\mathrm{Ker}(\mathfrak{d})$ *is finite.*

Admitting this lemma, we will prove (6.1.1) in §§6.2–6.3. We will prove Lemma 6.1.2 in §6.4.

6.2 Proof of (6.1.1) in the Local Case

We prove (6.1.1) assuming that k is local and that Lemma 6.1.2 holds. Let $\mathbb{F}$ be the residue field of k. By Lemma 6.1.2, $F^2\widetilde{\Theta}$ is finite. We prove that $\mathrm{Im}(F^1\widetilde{\Theta} \to \mathrm{gr}^1_F\Theta)$ is finite, which is exactly the finiteness of $\mathrm{gr}^1_F\widetilde{\Theta}$ and implies (6.1.1). Let $\mathscr{W}$ and $\mathscr{W}^0$ be as in (5.3.2) and (5.3.4), respectively. **N1** implies

$$\mathrm{gr}^1_F\Theta \simeq F^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \big/ \big((\mathscr{W}^0)_{\mathrm{Div}} + F^2H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))\big). \qquad (6.2.1)$$

If $p \neq \mathrm{ch}(\mathbb{F})$, then the group on the right hand side is clearly finite. If $p = \mathrm{ch}(\mathbb{F})$, then Lemma 6.2.2 below implies that the image of $F^1\widetilde{\Theta} \to \mathrm{gr}^1_F\Theta$ is a subquotient of $\mathrm{Cotor}(\mathscr{W}^0)$, which is finite by the proof of Lemma 3.5.4. Thus we are reduced to

Lemma 6.2.2 *If* $p = \mathrm{ch}(\mathbb{F})$*, then* $\mathrm{Ker}(\mathfrak{d}) \subset \mathscr{W}$.

We do not need to assume **H0** or **N1** in this lemma.

Proof. Let the notation be as in 1.10. Note that $\mathscr{W} = H^3(\mathscr{X}, \tau_{\leq 2}Rj_*\mathbb{Q}_p/\mathbb{Z}_p(2))$ by definition. There is a commutative diagram with distinguished rows in $D^b(\mathscr{X}, \mathbb{Z}/p^r)$

$$\begin{array}{ccccccc}
\mathfrak{T}_r(2) & \longrightarrow & Rj_*\mu_{p^r}^{\otimes 2} & \longrightarrow & Ri_*Ri^!\mathfrak{T}_r(2)[1] & \longrightarrow & \mathfrak{T}_r(2)[1] \\
\downarrow{\scriptstyle t} & & \| & & \downarrow{\scriptstyle Ri_*Ri^!(t)[1]} & & \downarrow{\scriptstyle t[1]} \\
\tau_{\leq 2}Rj_*\mu_{p^r}^{\otimes 2} & \longrightarrow & Rj_*\mu_{p^r}^{\otimes 2} & \longrightarrow & Ri_*Ri^!(\tau_{\leq 2}Rj_*\mu_{p^r}^{\otimes 2})[1] & \longrightarrow & (\tau_{\leq 2}Rj_*\mu_{p^r}^{\otimes 2})[1],
\end{array}$$

where t is as in (**S5**) in §4.1. The central square of this diagram gives rise to the left square of the following commutative diagram (whose rows are not exact):

$$\begin{array}{ccccc}
H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) & \longrightarrow & H^4_Y(\mathscr{X}, \mathfrak{T}_\infty(2)) & \longrightarrow & \bigoplus_{y \in Y^0} H^4_y(\mathscr{X}, \mathfrak{T}_\infty(2)) \\
\| & & \downarrow & & \downarrow \\
H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) & \xrightarrow{\epsilon_1} & H^0(Y, i^*R^3j_*\mathbb{Q}_p/\mathbb{Z}_p(2)) & \xrightarrow{\epsilon_2} & \bigoplus_{y \in Y^0} H^0(y, i^*R^3j_*\mathbb{Q}_p/\mathbb{Z}_p(2)).
\end{array} \qquad (6.2.3)$$

Here the middle and the right vertical arrow is obtained from the composite morphism

$$Ri_*Ri^!\mathfrak{T}_r(2)[1] \xrightarrow{Ri_*Ri^!(t)[1]} Ri_*Ri^!\tau_{\leq 2}Rj_*\mu_{p^r}^{\otimes 2}[1] \xleftarrow{\sim} \tau_{\geq 3}Rj_*\mu_{p^r}^{\otimes 2},$$

and the right square commutes by the functoriality of restriction maps. The composite of the upper row is $\mathfrak{d}$. We have $\mathrm{Ker}(\epsilon_1) = \mathscr{W}$ obviously, and ϵ_2 is injective by the second assertion of Lemma 3.5.5. Hence we have $\mathrm{Ker}(\mathfrak{d}) \subset \mathrm{Ker}(\epsilon_2 \circ \epsilon_1) = \mathrm{Ker}(\epsilon_1) = \mathscr{W}$. □

6.3 Proof of (6.1.1) in the Global Case

We prove (6.1.1) assuming that k is global and that Lemma 6.1.2 holds. Let $\mathscr{W}$ and $\mathscr{W}^0$ be as in (5.3.2) and (5.3.4), respectively. **N1** implies

$$\mathrm{gr}^1_F\Theta \simeq F^1H^3(X,\mathbb{Q}_p/\mathbb{Z}_p(2))\big/\big((\mathscr{W}^0)_{\mathrm{Div}} + F^2H^3(X,\mathbb{Q}_p/\mathbb{Z}_p(2))\big). \tag{6.3.1}$$

We first prove the following lemma, where we do not assume **H0** or **N1**:

Lemma 6.3.2 $\mathrm{Ker}(\mathfrak{d}) \subset \mathscr{W}$.

Proof. We use the notation in 1.9. By the same argument as for the proof of Lemma 6.2.2, we obtain a commutative diagram analogous to (6.2.3)

$$\begin{array}{ccccc} H^3(X,\mathbb{Q}_p/\mathbb{Z}_p(2)) & \longrightarrow & \bigoplus_{v\in S^1} H^4_{Y_v}(\mathscr{X},\mathfrak{T}_\infty(2)) & \longrightarrow & \bigoplus_{v\in S^1}\bigoplus_{y\in(Y_v)^0} H^4_y(\mathscr{X},\mathfrak{T}_\infty(2)) \\ \| & & \downarrow & & \downarrow \\ H^3(X,\mathbb{Q}_p/\mathbb{Z}_p(2)) & \xrightarrow{\epsilon_1} & \bigoplus_{v\in\Sigma} H^0(Y_v, i_v^*R^3j_{v*}\mathbb{Q}_p/\mathbb{Z}_p(2)) & \xrightarrow{\epsilon_2} & \bigoplus_{v\in\Sigma}\bigoplus_{y\in(Y_v)^0} H^0(y, i_v^*R^3j_{v*}\mathbb{Q}_p/\mathbb{Z}_p(2)) \end{array}$$

The composite of the upper row is $\mathfrak{d}$. The assertion follows from the facts that $\mathrm{Ker}(\epsilon_1) = \mathscr{W}$ and that ϵ_2 is injective (cf. Lemma 3.5.5). □

We prove (6.1.1). By Lemma 2.3.2 (4), it suffcies to show that

$$(F^2\widetilde{\Theta})_{\mathrm{Div}} = 0 = (\mathrm{gr}^1_F\widetilde{\Theta})_{\mathrm{Div}}.$$

Since $F^2H^3(X,\mathbb{Q}_p/\mathbb{Z}_p(2))\cap\mathrm{Ker}(\mathfrak{d})$ has a finite exponent (Corollary 2.6.3 (2), Lemma 6.1.2), we have $(F^2\widetilde{\Theta})_{\mathrm{Div}} = 0$. We show $(\mathrm{gr}^1_F\widetilde{\Theta})_{\mathrm{Div}} = 0$. By (6.3.1) and Lemma 6.3.2, we have

$$\mathrm{gr}^1_F\widetilde{\Theta} \subset \Xi := \mathscr{W}^0/((\mathscr{W}^0)_{\mathrm{Div}} + Z) \quad \text{with } Z := \mathscr{W}^0 \cap F^2H^3(X,\mathbb{Q}_p/\mathbb{Z}_p(2)).$$

By Corollary 2.6.3 (1), $\mathrm{Cotor}(Z)$ has a finite exponent, which implies

$$(\mathrm{gr}^1_F\widetilde{\Theta})_{\mathrm{Div}} \subset \Xi_{\mathrm{Div}} = \mathrm{Cotor}(\mathscr{W}^0)_{\mathrm{Div}} = 0$$

(cf. Lemma 2.3.2 (3)). Thus we obtain (6.1.1).

6.4 Proof of Lemma 6.1.2

The case that k is p-adic local follows from [Sat1] Theorem 3.1, Lemma 3.2 (1) (cf. [Ts3]). More precisely, $\mathscr{X}$ is assumed in [Sat1] §3 to have strict semistable reduction, but one can remove the strictness assumption easily. The details are left to the reader.

We prove Lemma 6.1.2 assuming that k is ℓ-adic local with $\ell \neq p$. Note that in this case $\mathscr{X}/S$ may not have semistable reduction. If $\mathscr{X}/S$ has strict semistable reduction, then the assertion is proved in [Sat1] Theorem 2.1. We prove the general case. Put

$$\Lambda := \mathbb{Q}_p/\mathbb{Z}_p$$

for simplicity. By the alteration theorem of de Jong [dJ], we take a proper generically finite morphism $f : \mathscr{X}' \to X$ such that $\mathscr{X}'$ has strict semistable reduction over the normalization $S' = \mathrm{Spec}(\mathfrak{o}_{k'})$ of S in $\mathscr{X}'$. sNote that $\eth_2$ is the composite of a composite map

$$\eth_3 : H^2(k, H^1(\overline{X}, \Lambda(2)) \longrightarrow H^3(X, \Lambda(2)) \xrightarrow{\delta^{\mathrm{loc}}} H^4_Y(\mathscr{X}, \Lambda(2))$$

with a pull-back map

$$H^4_Y(\mathscr{X}, \Lambda(2)) \longrightarrow \bigoplus_{y \in Y^0} H^4_y(\mathscr{X}, \Lambda(2)). \tag{6.4.1}$$

Here the arrow δ^{loc} is the boundary map of a localization exact sequence. There is a commutative diagram

$$\begin{tikzcd}
H^2(k, H^1(\overline{X}, \Lambda(2))) \arrow[r, "\eth_3"] \arrow[d, "f^*"'] & H^4_Y(\mathscr{X}, \Lambda(2)) \arrow[d, "f^*"] \\
H^2(k', H^1(\overline{X}', \Lambda(2))) \arrow[r, "\eth_3'"] & H^4_{Y'}(\mathscr{X}', \Lambda(2)),
\end{tikzcd}$$

where $\overline{X}' := \mathscr{X}' \otimes_{\mathfrak{o}_{k'}} \overline{k}$ and Y' denotes the closed fiber of $\mathscr{X}'/S'$. We have already shown that $\mathrm{Ker}(\eth_3')$ is finite, and a standard norm argument shows that the left vertical arrow has finite kernel. Thus $\mathrm{Ker}(\eth_3)$ is finite as well. It remains to show

Lemma 6.4.2 $\mathrm{Im}(\eth_3) \cap N^2H^4_Y(\mathscr{X}, \Lambda(2))$ *is finite, where* $N^2H^4_Y(\mathscr{X}, \Lambda(2))$ *denotes the kernel of the map* (6.4.1)*.*

Proof. First we note that

$$\mathrm{Im}(\eth_3) \subset \mathrm{Im}\big(H^1(\mathbb{F}, H^3_{\overline{Y}}(\mathscr{X}^{\mathrm{ur}}, \Lambda(2))) \to H^4_Y(\mathscr{X}, \Lambda(2))\big).$$

Indeed, this follows from the fact that $\eth_3$ factors as follows:

$$\begin{aligned}
&H^2(k, H^1(\overline{X}, \Lambda(2))) \simeq H^1(\mathbb{F}, H^1(k^{\mathrm{ur}}, H^1(\overline{X}, \Lambda(2)))) \\
&\longrightarrow H^1(\mathbb{F}, H^2(X^{\mathrm{ur}}, \Lambda(2))) \longrightarrow H^1(\mathbb{F}, H^3_{\overline{Y}}(\mathscr{X}^{\mathrm{ur}}, \Lambda(2))) \longrightarrow H^4_Y(\mathscr{X}, \Lambda(2)).
\end{aligned}$$

Hence it suffices to show the finiteness of the kernel of the composite map

$$\upsilon : H^1(\mathbb{F}, H^3_{\overline{Y}}(\mathscr{X}^{\mathrm{ur}}, \Lambda(2))) \longrightarrow H^4_Y(\mathscr{X}, \Lambda(2)) \longrightarrow \bigoplus_{y \in Y^0} H^4_y(\mathscr{X}, \Lambda(2)).$$

There is a commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & \mathrm{CH}_{d-2}(Y) \otimes \Lambda \arrow[r, "\iota"] \arrow[d, hook] & \mathrm{CH}_{d-2}(\overline{Y}) \otimes \Lambda \arrow[d, hook] \\
H^1(\mathbb{F}, H^3_{\overline{Y}}(\mathscr{X}^{\mathrm{ur}}, \Lambda(2))) \arrow[r, hook] \arrow[dr, "\upsilon"'] & H^4_Y(\mathscr{X}, \Lambda(2)) \arrow[r] \arrow[d] & H^4_{\overline{Y}}(\mathscr{X}^{\mathrm{ur}}, \Lambda(2)) \arrow[d] \\
 & \bigoplus_{y \in Y^0} H^4_y(\mathscr{X}, \Lambda(2)) \arrow[r] & \bigoplus_{\eta \in (\overline{Y})^0} H^4_\eta(\mathscr{X}^{\mathrm{ur}}, \Lambda(2)),
\end{tikzcd}$$

whose columns arise from the isomorphisms

$$N^2 H^4_Y(\mathscr{X}, \Lambda(2)) \simeq \mathrm{CH}_{d-2}(Y) \otimes \Lambda, \quad N^2 H^4_{\overline{Y}}(\mathscr{X}^{\mathrm{ur}}, \Lambda(2)) \simeq \mathrm{CH}_{d-2}(\overline{Y}) \otimes \Lambda$$

with $d := \dim(\mathscr{X})$ (see Lemma 4.2.3 (2), noting that $\mathfrak{T}_\infty(2) = \Lambda(2)$ in this situation). The middle exact row arises from the Hochschild-Serre spectral sequence for the covering $\mathscr{X}^{\mathrm{ur}} \to \mathscr{X}$. A diagram chase shows that $\mathrm{Ker}(\upsilon) \simeq \mathrm{Ker}(\iota)$, and we are reduced to showing the finiteness of $\mathrm{Ker}(\iota)$. Because the natural restriction map

$$\mathrm{CH}_{d-2}(Y)/\mathrm{CH}_{d-2}(Y)_{\mathrm{tors}} \to \mathrm{CH}_{d-2}(\overline{Y})/\mathrm{CH}_{d-2}(\overline{Y})_{\mathrm{tors}}$$

is injective by the standard norm argument, the finiteness of $\mathrm{Ker}(\iota)$ follows from the following general lemma:

LEMMA 6.4.3 *Let e be a positive integer and let Z be a scheme which is separated of finite type over $F := \overline{\mathbb{F}}$ with $\dim(Z) \le e$. Then the group $\mathrm{CH}_{e-1}(Z)/\mathrm{CH}_{e-1}(Z)_{\mathrm{tors}}$ is a finitely generated abelian group.*

Proof of Lemma 6.4.3. Obviously we may suppose that Z is reduced. We first reduced the problem to the case where Z is proper. Take a dense open immersion $Z \hookrightarrow Z'$ with $\overline{Z}'$ is proper. Writing Z'' for $Z' \setminus Z$, there is an exact sequence

$$\mathrm{CH}_{e-1}(Z'') \longrightarrow \mathrm{CH}_{e-1}(Z') \longrightarrow \mathrm{CH}_{e-1}(Z) \longrightarrow 0,$$

where $\mathrm{CH}_{e-1}(Z'')$ is finitely generated free abelian group because $\dim(Z'') \le e-1$. Let $f : \widetilde{Z} \to Z$ be the normalization. Since f is birational and finite, one easily sees that the cokernel of $f_* : \mathrm{CH}_{e-1}(\widetilde{Z}) \to \mathrm{CH}_{e-1}(Z)$ is finite. Thus we may assume Z is a proper normal variety of dimension e over F. Since F is algebraically closed, Z has an F-rational point. Now the theory of Picard functor (cf. [Mu] §5) implies the functorial isomorphisms $\mathrm{CH}_{e-1}(Z) \simeq \mathrm{Pic}_{Z/F}(F)$, where $\mathrm{Pic}_{Z/F}$ denotes the Picard functor for Z/F. This functor is representable by a group scheme and fits into the exact sequence of group schemes

$$0 \longrightarrow \mathrm{Pic}^{\tau}_{Z/F} \longrightarrow \mathrm{Pic}_{Z/F} \longrightarrow \mathrm{NS}_{Z/F} \longrightarrow 0,$$

where $\mathrm{Pic}^{\tau}_{Z/F}$ is quasi-projective over F and the reduce part of $\mathrm{NS}_{Z/F}$ is associated with a finitely generated abelian group. Since F is the algebraic closure of a finite field, the group $\mathrm{Pic}^{\tau}_{Z/F}(F)$ is torsion. Lemma 6.4.3 follows immediately from these facts. □

This completes the proof of Lemma 6.4.2, Lemma 6.1.2 and the key lemma 5.4.1. □

7 Cokernel of the Regulator Map

Let $k, S, p, \mathscr{X}$ and K be as in the notation 1.8. We always assume 1.8.1 throughout this section. For a proper smooth geometrically integral variety Z over a finite field $\mathbb{F}$. we say that *the Tate conjecture holds in codimension* 1 *for* Z, if the étale cycle class map

$$\mathrm{CH}^1(Z) \otimes \mathbb{Q}_\ell \longrightarrow H^2(Z \otimes_{\mathbb{F}} \overline{\mathbb{F}}, \mathbb{Q}_\ell(1))^{G_{\mathbb{F}}}$$

is surjective for a prime number $\ell \neq \mathrm{ch}(\mathbb{F})$ ([Ta1], [Ta2]). By [Mi1] Theorems 4.1 and 6.1, this condition is independent of $\ell \neq \mathrm{ch}(\mathbb{F})$ and equivalent to that the Grothendieck-Brauer group $\mathrm{Br}(Z) = H^2(Z, \mathbb{G}_{\mathrm{m}})$ is finite.

7.1 Statement of the Result

Let $\mathscr{C}$ be the category of $\mathbb{Z}_p$-modules modulo the Serre subcategory consisting of p-primary torsion abelian groups of finite-exponent. In this section, we prove the following result, which implies Theorem 1.3 (2) (see the beginning of §3 for **H0**):

Theorem 7.1.1 *Assume* **H0** *and either* $p \geq 5$ *or the equality*

$$H^1_g(k, H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)))_{\mathrm{Div}} = H^1(k, H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)))_{\mathrm{Div}}. \qquad (*_g)$$

Assume further the following conditions:

- **T**: *The reduced part of every closed fiber of $\mathscr{X}/S$ has simple normal crossings on $\mathscr{X}$, and the Tate conjecture holds in codimension* 1 *for the irreducible components of those fibers.*
- **F**: $\mathrm{Cotor}(\mathrm{CH}^2(X)_{p\text{-tors}})$ *has a finite exponent.*

Then there exists a short exact sequence in $\mathscr{C}$

$$0 \longrightarrow \mathrm{CH}^2(X)_{p\text{-tors}} \longrightarrow \frac{H^1_g(k, H^2_{\text{ét}}(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)))}{\mathrm{Im}(\mathrm{reg}_{\mathbb{Q}_p/\mathbb{Z}_p})} \longrightarrow H^3_{\mathrm{ur}}(K, X; \mathbb{Q}_p/\mathbb{Z}_p(2)).$$

Moreover the image of the last map contains $H^3_{\mathrm{ur}}(K, X; \mathbb{Q}_p/\mathbb{Z}_p(2))_{\mathrm{Div}}$.

This result is a generalization of Theorems 3.1.1 and 5.1.1 under the assumption **T**. The formulation of Theorem 7.1.1 in this final version was much inspired by discussions with Masanori Asakura.

7.2 PROOF OF THEOREM 7.1.1

Let

$$\mathfrak{d} : H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \longrightarrow \bigoplus_{v \in S} \bigoplus_{y \in (Y_v)^0} H^4_y(\mathscr{X}, \mathfrak{T}_\infty(2)).$$

be the map induced by $\overline{\mathfrak{d}}$ in (5.2.1). Let $\mathscr{W}$ be as in (5.3.2) and let $\mathscr{W}^0$ be as in (5.3.4). We need the following two lemmas:

LEMMA 7.2.1 *Assume that* $\mathbf{T}$ *holds. Then we have*

$$\mathscr{W}_{\mathrm{Div}} \subset \mathrm{Ker}(\mathfrak{d}) + F^2H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)).$$

LEMMA 7.2.2 *Assume either* $p \geq 5$ *or* $(*_g)$. *Then we have*

$$\mathrm{Im}(\omega) = H^1_g(k, H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)))_{\mathrm{Div}},$$

where ω *is as in* (5.3.3).

Lemma 7.2.1 will be proved in §§7.4–7.5 below, and Lemma 7.2.2 will be proved in §7.3 below. We prove Theorem 7.1.1 in this subsection, admitting these lemmas. Let

$$H^3 := H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)), \quad N^1H^3 \subset H^3 \quad \text{and} \quad F^iH^3 \subset H^3$$

be as in §3.4, and put

$$NF^1H^3 := N^1H^3 \cap F^1H^3 \quad \text{and} \quad A := H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)).$$

We see that

$$\mathrm{Cotor}(N^1H^3) \text{ has a finite exponent} \tag{7.2.3}$$

by the exact sequence (3.3.1) and the assumption $\mathbf{F}$, and moreover that

$$\mathrm{Cotor}(NF^1H^3) \text{ has a finite exponent} \tag{7.2.4}$$

by $\mathbf{H0}$.

We work in $\mathscr{C}$ for a while to simplify the arguments. By $\mathbf{H0}$, (3.3.1) and Lemma 3.3.5, there is a short exact sequence in $\mathscr{C}$

$$0 \longrightarrow \mathrm{CH}^2(X)_{p\text{-tors}} \longrightarrow H^1(k, A)/\mathrm{Im}(\mathrm{reg}_{\mathbb{Q}_p/\mathbb{Z}_p}) \longrightarrow H^1(k, A)/NF \longrightarrow 0,$$

where NF denotes the image of NF^1H^3 in $H^1(k, A)$. By the assumption '$p \geq 5$ or $(*_g)$' and Lemma 3.5.2, the composite map $(NF^1H^3)_{\mathrm{Div}} \to F^1H^3 \to H^1(k, A)$ factors through $H^1_g(k, A)$. Therefore we obtain a short exact sequence in $\mathscr{C}$

$$0 \longrightarrow \mathrm{CH}^2(X)_{p\text{-tors}} \longrightarrow H^1_g(k, A)/\mathrm{Im}(\mathrm{reg}_{\mathbb{Q}_p/\mathbb{Z}_p}) \longrightarrow H^1_g(k, A)/NF_{\mathrm{Div}} \longrightarrow 0,$$

whose exactness follows from (7.2.4) and the fact that the natural map $(NF^1H^3)_{\mathrm{Div}} \to NF_{\mathrm{Div}}$ is surjective (cf. Corollary 2.6.3 (1) and Lemma 2.3.2 (3)). It remains to construct an injective map

$$\alpha : H^1_g(k, A)/NF_{\mathrm{Div}} \hookrightarrow H^3_{\mathrm{ur}}(K, X; \mathbb{Q}_p/\mathbb{Z}_p(2)) \quad \text{in} \quad \mathscr{C}$$

whose image contains $H^3_{\mathrm{ur}}(K,X;\mathbb{Q}_p/\mathbb{Z}_p(2))_{\mathrm{Div}}$.

We work in the usual category $\mathbb{Z}_p$-modules in what follows. There are $\mathbb{Z}_p$-homomorphisms

$$\begin{tikzcd} H^3_{\mathrm{ur}}(K,X;\mathbb{Q}_p/\mathbb{Z}_p(2)) \arrow[r, hook, "\iota"] & H^3/N^1H^3 \arrow[d, "\pi"', "\text{(finite-exponent kernel)}"] \\ & H^3/(N^1H^3+F^2H^3), \end{tikzcd} \tag{7.2.5}$$

where ι is induced by the definition of $H^3_{\mathrm{ur}}(K,X;\mathbb{Q}_p/\mathbb{Z}_p(2))$ and $\mathrm{Ker}(\pi)$ has a finite exponent by Lemma 3.3.5 (i.e., π is an isomorphism in $\mathscr{C}$). On the other hand, there is a diagram of $\mathbb{Z}_p$-submodules of H^3

$$\begin{tikzcd} (N^1H^3)_{\mathrm{Div}} \arrow[r, hook] \arrow[d, hook] & N^1H^3 \cap \mathscr{W}_{\mathrm{Div}} \arrow[r, hook] \arrow[dl, hook] & \mathscr{W}_{\mathrm{Div}} \arrow[r, hook, "7.2.1"] \arrow[d, hook] & \mathrm{Ker}(\mathfrak{d}) + F^2H^3 \\ N^1H^3 \arrow[r, hook] & \mathrm{Ker}(\mathfrak{d}) \arrow[r, hook, "\text{6.2.2 and 6.3.2}"] & \mathscr{W}, & \end{tikzcd}$$

where the inclusion $\mathrm{Ker}(\mathfrak{d}) \subset \mathscr{W}$ is obvious when k is ℓ-adic local with $\ell \neq p$. Since $N^1H^3 \cap \mathscr{W}_{\mathrm{Div}}$ is divisible up to a finite-exponent group by (7.2.3), we have

$$\mathscr{W}_{\mathrm{Div}}/(N^1H^3 \cap \mathscr{W}_{\mathrm{Div}}) \simeq (\mathscr{W}/N^1H^3)_{\mathrm{Div}} \tag{7.2.6}$$

by Lemma 2.3.2 (3). Now for a subgroup $M \subset H^3$, let $\overline{M}$ be its image into $H^3/(N^1H^3+F^2H^3)$. Then we have $\overline{\mathscr{W}_{\mathrm{Div}}} \subset \overline{\mathrm{Ker}(\mathfrak{d})} \subset \overline{\mathscr{W}}$ by the above diagram, and moreover

$$\overline{\mathscr{W}_{\mathrm{Div}}} = \left(\overline{\mathrm{Ker}(\mathfrak{d})}\right)_{\mathrm{Div}} = (\overline{\mathscr{W}})_{\mathrm{Div}} \tag{7.2.7}$$

by (7.2.6) and the fact that $\mathrm{Ker}(\pi)$ has a finite exponent (cf. Lemma 2.3.2 (3)). Noting that $\mathrm{Coker}(H^1_g(k,A))$ has a finite exponent (Remark 2.4.9 (1)), we define the desired map α in $\mathscr{C}$ as that induced by the following diagram:

$$\begin{tikzcd} H^1_g(k,A)/NF_{\mathrm{Div}} \arrow[rr, dotted, "\alpha"] & & H^3_{\mathrm{ur}}(K,X;\mathbb{Q}_p/\mathbb{Z}_p(2)) \arrow[d, "\pi\circ\iota"] \\ H^1_g(k,A)_{\mathrm{Div}}/NF_{\mathrm{Div}} \arrow[u, hook] \arrow[r, two heads, "\tau"] & (\overline{\mathscr{W}})_{\mathrm{Div}} \arrow[r, hook] & \overline{\mathrm{Ker}(\mathfrak{d})}, \end{tikzcd}$$

where the right vertical map is surjective by the definition of $H^3_{\mathrm{ur}}(K,X;\mathbb{Q}_p/\mathbb{Z}_p(2))$ and has finite-exponent kernel by the diagram (7.2.5). The arrow τ is obtained from Lemma 7.2.2, which is surjective by **H0** and has finite-exponent kernel by (7.2.4). By the last fact, α is injective in $\mathscr{C}$. Finally $\mathrm{Im}(\alpha)$ contains $H^3_{\mathrm{ur}}(K,X;\mathbb{Q}_p/\mathbb{Z}_p(2))_{\mathrm{Div}}$ by the surjectivity of τ and (7.2.7). Thus we obtain Theorem 7.1.1 assuming Lemmas 7.2.1 and 7.2.2.

7.3 PROOF OF LEMMA 7.2.2

If k is local, then the assertion follows from Corollary 2.2.3 and Lemma 3.5.4. We show the inclusion $\mathrm{Im}(\omega) \supset H^1_g(k, H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)))_{\mathrm{Div}}$, assuming that k is global (the inclusion in the other direction has been proved in Lemma 5.3.5). Put $A := H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))$. By Lemma 2.3.2 (3), it is enough to show the following:

(i) *The image of the composite map*

$$\mathscr{W}^0 \hookrightarrow F^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \xrightarrow{\psi} H^1(k, A)$$

contains $H^1_g(k, A)_{\mathrm{Div}}$, where the arrow ψ is as in (3.5.1)*.*

(ii) *The kernel of this composite map is cofinitely generated up to a finite-exponent group.*

(ii) follows from Corollary 2.6.3 (1). We prove (i) in what follows. We use the notation fixed in 1.9. Let $U \subset S$ be a non-empty open subset which contains Σ and for which $\mathscr{X}_U \to U$ is smooth outside of Σ. Let $j_U : \mathscr{X}_U[p^{-1}] \to \mathscr{X}_U$ be the natural open immersion. Put $U' := U \setminus \Sigma$ and $\Lambda := \mathbb{Q}_p/\mathbb{Z}_p$. For $v \in \Sigma$, put

$$M_v := F^1H^3(X_v, \Lambda(2))/(H^3(\mathscr{X}_v, \tau_{\leq 2}R(j_v)_*\Lambda(2))^0)_{\mathrm{Div}},$$

where the superscript 0 means the subgroup of elements which vanishes in

$$H^3(\overline{X}, \Lambda(2)) \simeq H^3(X_v \otimes_{k_v} \overline{k_v}, \Lambda(2)).$$

We construct a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(r_U) \arrow[r] \arrow[d, "c_U"] & F^1H^3(\mathscr{X}_U[p^{-1}], \Lambda(2)) \arrow[r, "r_U"] \arrow[d, "\psi_U"] & \bigoplus_{v\in\Sigma} M_v \arrow[d, "b_\Sigma"] \\
0 \arrow[r] & \mathrm{Ker}(a_U) \arrow[r] & H^1(U, A) \arrow[r, "a_U"] & \bigoplus_{v\in\Sigma} H^1_{/g}(k_v, A),
\end{tikzcd}$$

where F^1 on $H^3(\mathscr{X}_U[p^{-1}], \Lambda(2))$ means the filtration resulting from the Hochschild-Serre spectral sequence (3.2.3) for $\mathscr{X}_U[p^{-1}]$, and ψ_U is an edge homomorphism of that spectral sequence. The arrows r_U and a_U are natural pull-back maps, and we put

$$H^1_{/g}(k_v, A) := H^1(k_v, A)/H^1_g(k_v, A).$$

The existence of b_Σ follows from the local case of Lemma 7.2.2, and c_U denotes the map induced by the right square. Note that $\mathrm{Ker}(a_U)$ contains $H^1_{f,U}(k, A)$. Now let

$$c : \mathscr{W}^\dagger := \varinjlim_{\Sigma\subset U\subset S} \mathrm{Ker}(r_U) \longrightarrow \varinjlim_{\Sigma\subset U\subset S} \mathrm{Ker}(a_U)$$

be the inductive limit of c_U, where U runs through all non-empty open subsets of S which contains Σ and for which $\mathscr{X}_U \to U$ is smooth outside of Σ. Because the group on the right hand side contains $H^1_g(k, A)$, it remains to show that

(iii) $\mathrm{Coker}(c)$ *has a finite exponent.*

(iv) $\mathscr{W}^\dagger$ *is contained in* $\mathscr{W}^0$.

(iv) is rather straight-forward and left to the reader. We prove (iii). For $U \subset S$ as above, applying the snake lemma to the above diagram, we see that the kernel of the natural map

$$\mathrm{Coker}(c_U) \longrightarrow \mathrm{Coker}(\psi_U)$$

is a subquotient of $\mathrm{Ker}(b_\Sigma)$. By the local case of Lemma 7.2.2, we have

$$\mathrm{Ker}(b_\Sigma) \simeq \bigoplus_{v \in \Sigma} \mathrm{Im}(F^2H^3(X_v, \Lambda(2)) \to M_v)$$

and the group on the right hand side is finite by Lemma 7.5.1 below. On the other hand $\mathrm{Coker}(\psi_U)$ is zero if $p \geq 3$, and killed by 2 if $p = 2$. Hence passing to the limit, we see that $\mathrm{Coker}(c)$ has a finite exponent. This completes the proof of Lemma 7.2.2.

7.4 Proof of Lemma 7.2.1, Step 1

We start the proof of Lemma 7.2.1. Our task is to show the inclusion

$$\mathfrak{d}(\mathscr{W}_{\mathrm{Div}}) \subset \mathfrak{d}(F^2H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))). \tag{7.4.1}$$

If k is global, then the assertion is reduced to the local case, because the natural map

$$F^2H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \longrightarrow \bigoplus_{v \in S^1} F^2H^3(X_v, \mathbb{Q}_p/\mathbb{Z}_p(2))$$

has finite cokernel by Corollary 2.6.3 (2).

Assume now that k is local. In this subsection, we treat the case that k is ℓ-adic local with $\ell \neq p$. We use the notation fixed in 1.10. Recall that Y has simple normal crossings on $\mathscr{X}$ by the assumption $\mathbf{T}$. Note that $\mathfrak{d}$ factors as

$$H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \longrightarrow H^4_Y(\mathscr{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)) \xrightarrow{\iota} \bigoplus_{y \in Y^0} H^4_y(\mathscr{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)),$$

and that $\mathrm{Im}(\mathfrak{d}) \subset \mathrm{Im}(\iota)$. There is a short exact sequence

$$\begin{aligned} 0 \to H^1(\mathbb{F}, H^3_{\overline{Y}}(\mathscr{X}^{\mathrm{ur}}, \mathbb{Q}_p/\mathbb{Z}_p(2))) &\to H^4_Y(\mathscr{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)) \\ &\to H^4_{\overline{Y}}(\mathscr{X}^{\mathrm{ur}}, \mathbb{Q}_p/\mathbb{Z}_p(2))^{G_{\mathbb{F}}} \to 0 \end{aligned}$$

arising from a Hochschild-Serre spectral sequence. We have $\mathrm{Ker}(\iota) \simeq \mathrm{CH}_{d-2}(Y) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ with $d := \dim(\mathscr{X})$ by Lemma 4.2.3 (2). Hence to show the inclusion (7.4.1), it suffices to prove

Proposition 7.4.2 (1) *Assume that* $\mathbf{T}$ *holds. Then the composite map*

$$\mathrm{CH}_{d-2}(Y) \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow H^4_Y(\mathscr{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)) \longrightarrow H^4_{\overline{Y}}(\mathscr{X}^{\mathrm{ur}}, \mathbb{Q}_p/\mathbb{Z}_p(2))^{G_{\mathbb{F}}} \tag{7.4.3}$$

is an isomorphism up to finite groups. Consequently, we have

$$\mathrm{Im}(\iota) \simeq H^1(\mathbb{F}, H^3_{\overline{Y}}(\mathscr{X}^{\mathrm{ur}}, \mathbb{Q}_p/\mathbb{Z}_p(2)))$$

up to finite groups.

(2) *The image of the composite map*

$$H^2(k, H^1(X, \mathbb{Q}_p/\mathbb{Z}_p(2))) \longrightarrow H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \longrightarrow H^4_Y(\mathscr{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))$$

contains $H^1(\mathbb{F}, H^3_{\overline{Y}}(\mathscr{X}^{\mathrm{ur}}, \mathbb{Q}_p/\mathbb{Z}_p(2)))_{\mathrm{Div}}$.

We first show the following lemma:

LEMMA 7.4.4 (1) *Consider the Mayer-Vietoris spectral sequence obtained from the absolute purity* (*cf.* [RZ], [Th], [FG])

$$E_1^{u,v} = H^{2u+v-2}(\overline{Y}^{(-u+1)}, \mathbb{Q}_p/\mathbb{Z}_p(u+1)) \Longrightarrow H^{u+v}_{\overline{Y}}(\mathscr{X}^{\mathrm{ur}}, \mathbb{Q}_p/\mathbb{Z}_p(2)), \tag{7.4.5}$$

where $\overline{Y}^{(q)}$ *denotes the disjoint union of q-fold intersections of distinct irreducible components of the reduced part of* $\overline{Y}$. *Then there are isomorphisms up to finite groups*

$$H^1(\mathbb{F}, H^3_{\overline{Y}}(\mathscr{X}^{\mathrm{ur}}, \mathbb{Q}_p/\mathbb{Z}_p(2))) \simeq H^1(\mathbb{F}, E_2^{-1,4}),$$
$$H^4_{\overline{Y}}(\mathscr{X}^{\mathrm{ur}}, \mathbb{Q}_p/\mathbb{Z}_p(2))^{G_{\mathbb{F}}} \simeq (E_2^{0,4})^{G_{\mathbb{F}}}.$$

(2) *As a $G_{\mathbb{F}}$-module,* $H^0(k^{\mathrm{ur}}, H^2(\overline{X}, \mathbb{Q}_p))$ *has weight* ≤ 2.

Proof of Lemma 7.4.4. (1) Since $E_1^{u,v} = 0$ for any (u,v) with $u > 0$ or $2u + v < 2$, there is a short exact sequence

$$0 \longrightarrow E_2^{0,3} \longrightarrow H^3_{\overline{Y}}(\mathscr{X}^{\mathrm{ur}}, \mathbb{Q}_p/\mathbb{Z}_p(2))) \longrightarrow E_2^{-1,4} \longrightarrow 0, \tag{7.4.6}$$

and the edge homomorphism

$$E_2^{0,4} \hookrightarrow H^4_{\overline{Y}}(\mathscr{X}^{\mathrm{ur}}, \mathbb{Q}_p/\mathbb{Z}_p(2))), \tag{7.4.7}$$

where we have $E_2^{-1,4} = \mathrm{Ker}(d_1^{-1,4})$ and $E_2^{0,4} = \mathrm{Coker}(d_1^{-1,4})$ and $d_1^{-1,4}$ is the Gysin map $H^0(\overline{Y}^{(2)}, \mathbb{Q}_p/\mathbb{Z}_p) \to H^2(\overline{Y}^{(1)}, \mathbb{Q}_p/\mathbb{Z}_p(1)))$. Note that $E_1^{u,v}$ is pure of weight $v-4$ by Deligne's proof of the Weil conjecture [De2], so that $H^i(\mathbb{F}, E_\infty^{u,v})$ $(i = 0, 1)$ is finite unless $v = 4$. The assertions immediately follow from these facts.

(2) By the alteration theorem of de Jong [dJ], we may assume that $\mathscr{X}$ is projective and has semistable reduction over S. If X is a surface, then the assertion is proved in [RZ]. Otherwise, take a closed immersion $\mathscr{X} \hookrightarrow \mathbb{P}^N_S =: \mathbb{P}$. By [JS] Proposition 4.3 (b), there exists a hyperplane $H \subset \mathbb{P}$ which is flat over S and for which $\mathscr{Z} := \mathscr{X} \times_{\mathbb{P}} H$ is regular with semistable reduction over S. The restriction map

$H^2(\overline{X},\mathbb{Q}_p) \to H^2(\overline{Z},\mathbb{Q}_p)$ ($\overline{Z} := \mathscr{X} \times_{\mathfrak{o}_k} \overline{k}$) is injective by the weak and hard Lefschetz theorems. Hence the claim is reduced to the case of surfaces. This completes the proof of the lemma. □

Proof of Proposition 7.4.2. (1) Note that the composite map (7.4.3) in question has finite kernel by Lemma 6.4.3 and the arguments in the proof of Lemma 6.4.2. We prove that (7.4.3) has finite cokernel, assuming $\mathbf{T}$. By the Kummer theory, there is a short exact sequence

$$0 \longrightarrow \mathrm{Pic}(\overline{Y}^{(1)}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow H^2(\overline{Y}^{(1)},\mathbb{Q}_p/\mathbb{Z}_p(1)) \longrightarrow \mathrm{Br}(\overline{Y}^{(1)})_{p\text{-tors}} \longrightarrow 0 \tag{7.4.8}$$

and the differential map $d_1^{-1,4}$ of the spectral sequence (7.4.5) factors through the Gysin map

$$H^0(\overline{Y}^{(2)},\mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow \mathrm{Pic}(\overline{Y}^{(1)}) \otimes \mathbb{Q}_p/\mathbb{Z}_p,$$

whose cokernel is $\mathrm{CH}_{d-2}(\overline{Y}) \otimes \mathbb{Q}_p/\mathbb{Z}_p$. Hence in view of the computations in the proof of Lemma 7.4.4 (1), the Gysin map

$$\mathrm{CH}_{d-2}(\overline{Y}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to H^4_{\overline{Y}}(\mathscr{X}^{\mathrm{ur}},\mathbb{Q}_p/\mathbb{Z}_p(2))$$

(cf. Lemma 4.2.3 (2)) factors through the map (7.4.7) and we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{CH}_{d-2}(Y) \otimes \mathbb{Q}_p/\mathbb{Z}_p \arrow[r, "(7.4.3)"] \arrow[d] & H^4_{\overline{Y}}(\mathscr{X}^{\mathrm{ur}},\mathbb{Q}_p/\mathbb{Z}_p(2))^{G_{\mathbb{F}}} \\
(\mathrm{CH}_{d-2}(\overline{Y}) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^{G_{\mathbb{F}}} \arrow[r] & (E_2^{0,4})^{G_{\mathbb{F}}}, \arrow[u, "(7.4.7)"']
\end{tikzcd}$$

where the left vertical arrow has finite cokernel (and kernel) by Lemma 6.4.3 and a standard norm argument. The right vertical arrow has finite cokernel (and is injective) by Lemma 7.4.4 (1). Thus it suffices to show that the bottom horizontal arrow has finite cokernel. By the exact sequence (7.4.8), there is a short exact sequence

$$0 \longrightarrow \mathrm{CH}_{d-2}(\overline{Y}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow E_2^{0,4} \longrightarrow \mathrm{Br}(\overline{Y}^{(1)})_{p\text{-tors}} \longrightarrow 0.$$

Our task is to show that $\big(\mathrm{Br}(\overline{Y}^{(1)})_{p\text{-tors}}\big)^{G_{\mathbb{F}}}$ is finite, which follows from the assumption $\mathbf{T}$ and the finiteness of the kernel of the natural map

$$H^1(G_{\mathbb{F}},\mathrm{Pic}(\overline{Y}^{(1)}) \otimes \mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow H^1(G_{\mathbb{F}},H^2(\overline{Y}^{(1)},\mathbb{Q}_p/\mathbb{Z}_p(1)))$$

(cf. Lemma 7.6.2 in §7.6 below). Thus we obtain the assertion.

(2) Since $\mathrm{cd}(k^{\mathrm{ur}}) = 1$, there is a short exact sequence

$$0 \to H^1(k^{\mathrm{ur}},H^1(\overline{X},\mathbb{Q}_p/\mathbb{Z}_p(2))) \to H^2(X^{\mathrm{ur}},\mathbb{Q}_p/\mathbb{Z}_p(2)) \to H^2(\overline{X},\mathbb{Q}_p/\mathbb{Z}_p(2))^{G_{k^{\mathrm{ur}}}} \to 0$$

arising from a Hochschild-Serre spectral sequence. By Lemma 7.4.4 the last group has weight ≤ -2, and we have isomorphisms up to finite groups

$$\begin{aligned} H^2(k, H^1(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))) &\simeq H^1(\mathbb{F}, H^1(k^{\mathrm{ur}}, H^1(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)))) \\ &\simeq H^1(\mathbb{F}, H^2(X^{\mathrm{ur}}, \mathbb{Q}_p/\mathbb{Z}_p(2))). \end{aligned} \tag{7.4.9}$$

Now we plug the short exact sequence (7.4.6) into the localization exact sequence

$$H^2(X^{\mathrm{ur}}, \mathbb{Q}_p/\mathbb{Z}_p(2)) \longrightarrow H^3_{\overline{Y}}(\mathscr{X}^{\mathrm{ur}}, \mathbb{Q}_p/\mathbb{Z}_p(2))) \xrightarrow{\alpha} H^3(\mathscr{X}^{\mathrm{ur}}, \mathbb{Q}_p/\mathbb{Z}_p(2)).$$

Note that $H^3(\mathscr{X}^{\mathrm{ur}}, \mathbb{Q}_p/\mathbb{Z}_p(2)) \simeq H^3(\overline{Y}, \mathbb{Q}_p/\mathbb{Z}_p(2))$, so that it has weight ≤ -1 (cf. [De2]). Let $E_2^{u,v}$ be as in (7.4.5). Since $E_2^{-1,4}$ is pure of weight 0, the induced map

$$E_2^{-1,4} \longrightarrow H^3(\mathscr{X}^{\mathrm{ur}}, \mathbb{Q}_p/\mathbb{Z}_p(2)))/\alpha(E_2^{0,3})$$

has finite image. Hence the composite map

$$H^2(X^{\mathrm{ur}}, \mathbb{Q}_p/\mathbb{Z}_p(2)) \longrightarrow H^3_{\overline{Y}}(\mathscr{X}^{\mathrm{ur}}, \mathbb{Q}_p/\mathbb{Z}_p(2))) \longrightarrow E_2^{-1,4}$$

has finite cokernel, and the following map has finite cokernel as well:

$$H^1(\mathbb{F}, H^2(X^{\mathrm{ur}}, \mathbb{Q}_p/\mathbb{Z}_p(2))) \longrightarrow H^1(\mathbb{F}, E_2^{-1,4}).$$

Now Proposition 7.4.2 (2) follows from this fact together with (7.4.9) and the first isomorphism in Lemma 7.4.4 (1). $\square$

REMARK 7.4.10 *Let J be the set of the irreducible components of $Y^{(2)}$ and put*

$$\varDelta := \mathrm{Ker}\big(g' : \mathbb{Z}^J \to \mathrm{NS}(Y^{(1)})\big) \quad \textit{with} \quad \mathrm{NS}(Y^{(1)}) := \bigoplus_{y \in Y^0} \mathrm{NS}(Y_y),$$

where for $y \in Y^0$, Y_y denotes the closure $\overline{\{y\}} \subset Y$ and $\mathrm{NS}(Y_y)$ denotes its Néron-Severi group. The arrow g' arises from the Gysin map $\mathbb{Z}^J \to \mathrm{Pic}(Y^{(1)})$. One can easily show, assuming $\mathbf{T}$ and using Lemma 7.6.2 in §7.6 below, that the corank of $H^1(\mathbb{F}, E_2^{-1,4})$ over $\mathbb{Z}_p$ is equal to the rank of $\varDelta$ over $\mathbb{Z}$. Hence Proposition 7.4.2 (2) implies the inequality

$$\dim_{\mathbb{Q}_p}(H^2(k, H^1(\overline{X}, \mathbb{Q}_p(2))) \geq \dim_{\mathbb{Q}}(\varDelta \otimes \mathbb{Q}), \tag{7.4.11}$$

which will be used in the next subsection.

7.5 PROOF OF LEMMA 7.2.1, STEP 2

We prove Lemma 7.2.1, assuming that k is p-adic local (see 1.10 for notation). We first show the following lemma:

LEMMA 7.5.1 *We have*

$$F^2H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \subset H^3(\mathscr{X}, \tau_{\leq 2}Rj_*\mathbb{Q}_p/\mathbb{Z}_p(2)) \ (= \mathscr{W}).$$

Proof of Lemma 7.5.1. There is a distinguished triangle in $D^b(\mathscr{X}_{\text{ét}}, \mathbb{Z}/p^r)$ by §4.1 $(\mathbf{S5})$

$$i_*\nu^1_{Y,r}[-3] \xrightarrow{\ g\ } \mathfrak{T}_r(2) \xrightarrow{\ t'\ } \tau_{\leq 2}Rj_*\mu_{p^r}^{\otimes 2} \xrightarrow{\ \sigma\ } i_*\nu^1_{Y,r}[-2]. \tag{7.5.2}$$

Applying $Ri^!$ to this triangle, we obtain a distinguished triangle in $D^b(Y_{\text{ét}}, \mathbb{Z}/p^r)$

$$\nu^1_{Y,r}[-3] \xrightarrow{\mathrm{Gys}^2_i} Ri^!\mathfrak{T}_r(2) \xrightarrow{Ri^!(t)} i^*(\tau_{\geq 3}Rj_*\mu_{p^r}^{\otimes 2})[-1] \longrightarrow \nu^1_{Y,r}[-2], \tag{7.5.3}$$

where $\mathrm{Gys}^2_i := Ri^!(g)$ and we have used the natural isomorphism

$$i^*(\tau_{\geq 3}Rj_*\mu_{p^r}^{\otimes 2})[-1] \simeq Ri^!(\tau_{\leq 2}Rj_*\mu_{p^r}^{\otimes 2}).$$

Now let us recall the commutative diagram (6.2.3):

$$\begin{array}{ccccc}
H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) & \longrightarrow & H^4_Y(\mathscr{X}, \mathfrak{T}_\infty(2)) & \longrightarrow & \bigoplus_{y\in Y^0} H^4_y(\mathscr{X}, \mathfrak{T}_\infty(2)) \\
\| & & \downarrow & & \downarrow{\scriptstyle\lambda} \\
H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) & \xrightarrow{\epsilon_1} & H^0(Y, i^*R^3j_*\mathbb{Q}_p/\mathbb{Z}_p(2)) & \xrightarrow{\epsilon_2} & \bigoplus_{y\in Y^0} H^0(y, i^*R^3j_*\mathbb{Q}_p/\mathbb{Z}_p(2)),
\end{array}$$

where the middle and the right vertical arrows are induced by $Ri^!(t)$ in (7.5.3). By the proof of Lemma 6.2.2, we have $H^3(\mathscr{X}, \tau_{\leq 2}Rj_*\mathbb{Q}_p/\mathbb{Z}_p(2)) = \mathrm{Ker}(\epsilon_2\epsilon_1)$. Hence it suffices to show the image of the composite map

$$\mathfrak{d}_2 : H^2(k, H^1(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)) \longrightarrow H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \xrightarrow{\ \mathfrak{d}\ } \bigoplus_{y\in Y^0} H^4_y(\mathscr{X}, \mathfrak{T}_\infty(2))$$

is contained in $\mathrm{Ker}(\lambda)$. By the distinguished triangle (7.5.3), $\mathrm{Ker}(\lambda)$ agrees with the image of the Gysin map

$$\mathrm{Gys} := \bigoplus_{y\in Y^0} \mathrm{Gys}^2_{i_y} : \bigoplus_{y\in Y^0} H^1(y, W_\infty\Omega^1_{y,\log}) \hookrightarrow \bigoplus_{y\in Y^0} H^4_y(\mathscr{X}, \mathfrak{T}_\infty(2)).$$

On the other hand, $\mathfrak{d}_2$ factors into the following maps up to a sign, by the commutativity of the central square in [SH] (4.4.2):

$$\begin{aligned}
&H^2(k, H^1(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))) \simeq H^1(\mathbb{F}, H^1(k^{\mathrm{ur}}, H^1(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)))) \\
&\longrightarrow H^1(\mathbb{F}, H^2(X^{\mathrm{ur}}, \mathbb{Q}_p/\mathbb{Z}_p(2))) \xrightarrow{\ \sigma\ } H^1(\mathbb{F}, H^0(\overline{Y}, \nu^1_{\overline{Y},\infty})) \\
&\longrightarrow H^1\Big(\mathbb{F}, \textstyle\bigoplus_{\eta\in\overline{Y}^0} H^0(\eta, W_\infty\Omega^1_{\eta,\log})\Big) \longrightarrow \textstyle\bigoplus_{y\in Y^0} H^1(y, W_\infty\Omega^1_{y,\log}) \\
&\xrightarrow{\mathrm{Gys}} \textstyle\bigoplus_{y\in Y^0} H^4_y(\mathscr{X}, \mathfrak{T}_\infty(2)),
\end{aligned}$$

where $\nu^1_{\overline{Y},\infty} := \varinjlim_r \nu^1_{\overline{Y},r}$. Thus we obtain the assertion. □

We start the proof of Lemma 7.2.1, i.e., the inclusion (7.4.1), assuming that k is p-adic local. The triangle (7.5.2) gives rise to the upper exact row of the following commutative diagram with exact rows (cf. [SH] (4.4.2)):

$$\begin{tikzcd} H^3(\mathscr{X}, \mathfrak{T}_r(2)) \arrow[r] \arrow[d, equal] & H^3(\mathscr{X}, \tau_{\leq 2} Rj_* \mu_{p^r}^{\otimes 2}) \arrow[r, "-\sigma"] \arrow[d] & H^1(Y, \nu_{Y,r}^1) \arrow[d, "\mathrm{Gys}_i^2"] \\ H^3(\mathscr{X}, \mathfrak{T}_r(2)) \arrow[r] & H^3(X, \mu_{p^r}^{\otimes 2}) \arrow[r] & H^4_Y(\mathscr{X}, \mathfrak{T}_r(2)), \end{tikzcd}$$

where σ (resp. Gys_i^2) is as in (7.5.2) (resp. (7.5.3)). Hence the map $\mathfrak{d}$ restricted to $\mathscr{W} = H^3(\mathscr{X}, \tau_{\leq 2} Rj_* \mathbb{Q}_p/\mathbb{Z}_p(2))$ factors as

$$\mathscr{W} \longrightarrow H^1(Y, \nu_{Y,\infty}^1) \longrightarrow H^4_Y(\mathscr{X}, \mathfrak{T}_\infty(2)) \longrightarrow \bigoplus_{y \in Y^0} H^4_y(\mathscr{X}, \mathfrak{T}_\infty(2)),$$

where $\nu_{Y,\infty}^1 := \varinjlim_r \nu_{Y,r}^1$. By Lemmas 7.5.1 and 6.1.2, it suffices to show that the corank of

$$\mathrm{Im}\Big(H^1(Y, \nu_{Y,\infty}^1) \to \bigoplus\nolimits_{y \in Y^0} H^4_y(\mathscr{X}, \mathfrak{T}_\infty(2))\Big) \tag{7.5.4}$$

is not greater than $\dim_{\mathbb{Q}_p} H^2(k, H^1(\overline{X}, \mathbb{Q}_p(2)))$. We pursue an analogy to the case $p \neq \mathrm{ch}(\mathbb{F})$ by replacing $H^4_Y(\mathscr{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))$ with $H^1(Y, \nu_{Y,\infty}^1)$. There is an exact sequence

$$0 \longrightarrow H^1(\mathbb{F}, H^0(\overline{Y}, \nu_{\overline{Y},\infty}^1)) \longrightarrow H^1(Y, \nu_{Y,\infty}^1) \longrightarrow H^1(\overline{Y}, \nu_{\overline{Y},\infty}^1)^{G_{\mathbb{F}}} \longrightarrow 0$$

arising from a Hochschild-Serre spectral sequence. By [Sat3] Corollary 2.2.7, there is a Mayer-Vietoris spectral sequence

$$E_1^{a,b} = H^{a+b}(\overline{Y}^{(1-a)}, W_\infty \Omega^{1+a}_{\overline{Y}^{(1-a)},\log}) \Longrightarrow H^{a+b}(\overline{Y}, \nu_{\overline{Y},\infty}^1).$$

Note that $E_1^{a,b}$ is of weight $b-1$ so that $H^i(\mathbb{F}, E_1^{a,b})$ is finite unless $b = 1$. Thus we obtain isomorphisms up to finite groups

$$H^1(\mathbb{F}, H^0(\overline{Y}, \nu_{\overline{Y},\infty}^1)) \simeq H^1(\mathbb{F}, E_2^{-1,1}), \tag{7.5.5}$$

$$H^1(\overline{Y}, \nu_{\overline{Y},\infty}^1)^{G_{\mathbb{F}}} \simeq (E_2^{0,1})^{G_{\mathbb{F}}} \tag{7.5.6}$$

with $E_2^{-1,1} = \mathrm{Ker}(d_1^{-1,1})$ and $E_2^{0,1} = \mathrm{Coker}(d_1^{-1,1})$, where $d_1^{-1,1}$ is the Gysin map

$$H^0(\overline{Y}^{(2)}, \mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow H^1(\overline{Y}^{(1)}, W_\infty \Omega^1_{\overline{Y}^{(1)},\log}).$$

There is an exact sequence of $G_{\mathbb{F}}$-modules (cf. (7.6.1) below)

$$0 \longrightarrow \mathrm{Pic}(\overline{Y}^{(1)}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow H^1(\overline{Y}^{(1)}, W_\infty \Omega^1_{\overline{Y}^{(1)},\log}) \longrightarrow \mathrm{Br}(\overline{Y}^{(1)})_{p\text{-tors}} \longrightarrow 0.$$

Hence we see that the group (7.5.4) coincides with the image of $H^1(\mathbb{F}, H^0(\overline{Y}, \nu^1_{Y,\infty}))$ up to finite groups by the condition $\mathbf{T}$, the same computation as for Proposition 7.4.2 (1) and [CTSS] p. 782 Théorème 3. Now we are reduced to showing

$$\begin{aligned}\dim_{\mathbb{Q}_p} H^2(k, H^1(\overline{X}, \mathbb{Q}_p(2))) &\geq \mathrm{corank}\big(H^1(\mathbb{F}, H^0(\overline{Y}, \nu^1_{\overline{Y},\infty}))\big) \\ &= \mathrm{corank}\big(H^1(\mathbb{F}, E_2^{-1,1})\big),\end{aligned}$$

where the last equality follows from (7.5.5). As is seen in Remark 7.4.10, the right hand side is equal to $\dim_{\mathbb{Q}}(\varDelta \otimes \mathbb{Q})$ under the condition $\mathbf{T}$. On the other hand, by [J2] Corollary 7, the left hand side does not change when one replaces p with another prime p'. Thus the desired inequality follows from (7.4.11). This completes the proof of Lemma 7.2.1 and Theorem 7.1.1. □

7.6 Appendix to Section 7

Let Z be a proper smooth variety over a finite field $\mathbb{F}$. For a positive integer m, we define the object $\mathbb{Z}/m\mathbb{Z}(1) \in D^b(Z_{\text{ét}}, \mathbb{Z}/m\mathbb{Z})$ as

$$\mathbb{Z}/m\mathbb{Z}(1) := \mu_{m'} \oplus (W_r\Omega^1_{Z,\log}[-1])$$

where we factorized m as $m' \cdot p^r$ with $(p, m') = 1$. There is a distinguished triangle of Kummer theory for $\mathbb{G}_{\mathrm{m}} := \mathbb{G}_{\mathrm{m},Z}$ in $D^b(Z_{\text{ét}})$

$$\mathbb{Z}/m\mathbb{Z}(1) \longrightarrow \mathbb{G}_{\mathrm{m}} \xrightarrow{\times m} \mathbb{G}_{\mathrm{m}} \longrightarrow \mathbb{Z}/m\mathbb{Z}(1)[1].$$

So there is a short exact sequence of $G_{\mathbb{F}}$-modules

$$0 \longrightarrow \mathrm{Pic}(\overline{Z})/m \longrightarrow H^2(\overline{Z}, \mathbb{Z}/m\mathbb{Z}(1)) \longrightarrow {}_m\mathrm{Br}(\overline{Z}) \longrightarrow 0,$$

where $\overline{Z} := Z \otimes_{\mathbb{F}} \overline{\mathbb{F}}$. Taking the inductive limit with respect to $m \geq 1$, we obtain a short exact sequence of $G_{\mathbb{F}}$-modules

$$0 \longrightarrow \mathrm{Pic}(\overline{Z}) \otimes \mathbb{Q}/\mathbb{Z} \xrightarrow{\alpha} H^2(\overline{Z}, \mathbb{Q}/\mathbb{Z}(1)) \longrightarrow \mathrm{Br}(\overline{Z}) \longrightarrow 0. \tag{7.6.1}$$

Concerning the arrow α, we prove the following lemma, which has been used in this section.

Lemma 7.6.2 *The map $H^1(\mathbb{F}, \mathrm{Pic}(\overline{Z}) \otimes \mathbb{Q}/\mathbb{Z}) \to H^1(\mathbb{F}, H^2(\overline{Z}, \mathbb{Q}/\mathbb{Z}(1))$ induced by α has finite kernel.*

Proof. Note that $\mathrm{Pic}(\overline{Z}) \otimes \mathbb{Q}/\mathbb{Z} \simeq (\mathrm{NS}(\overline{Z})/\mathrm{NS}(\overline{Z})_{\mathrm{tors}}) \otimes \mathbb{Q}/\mathbb{Z}$. By a theorem of Matsusaka [Ma] Theorem 4, the group $\mathrm{Div}(\overline{Z})/\mathrm{Div}(\overline{Z})_{\mathrm{num}}$ is isomorphic to $\mathrm{NS}(\overline{Z})/\mathrm{NS}(\overline{Z})_{\mathrm{tors}}$, where $\mathrm{Div}(\overline{Z})$ denotes the group of Weil divisors on $\overline{Z}$, $\mathrm{Div}(Z)_{\mathrm{num}}$ denotes the subgroup of Weil divisors numerically equivalent to zero. By this fact and the fact that $\mathrm{NS}(\overline{Z})$ is finitely generated, there exists a finite family $\{C_i\}_{i \in I}$ of proper smooth curves over $\mathbb{F}$ which are finite over Z and for which the kernel of the natural map $\mathrm{NS}(\overline{Z}) \to \bigoplus_{i \in I} \mathrm{NS}(\overline{C}_i)$ with $\overline{C}_i := C_i \otimes_{\mathbb{F}} \overline{\mathbb{F}}$ is torsion. Now consider a commutative diagram

$$\begin{CD} H^1(\mathbb{F}, \mathrm{NS}(\overline{Z}) \otimes \mathbb{Q}/\mathbb{Z}) @>>> H^1(\mathbb{F}, H^2(\overline{Z}, \mathbb{Q}/\mathbb{Z}(1))), \\ @VVV @VVV \\ \bigoplus_{i \in I} H^1(\mathbb{F}, \mathrm{NS}(\overline{C}_i) \otimes \mathbb{Q}/\mathbb{Z}) @>\sim>> \bigoplus_{i \in I} H^1(\mathbb{F}, H^2(\overline{C}_i, \mathbb{Q}/\mathbb{Z}(1))). \end{CD}$$

By a standard norm argument, one can easily show that the left vertical map has finite kernel. The bottom horizontal arrow is bijective, because $\mathrm{Br}(\overline{C}_i) = 0$ for any $i \in I$ by Tsen's theorem (cf. [Se] II.3.3). Hence the top horizontal arrow has finite kernel and we obtain the lemma. $\square$

A Relation with Conjectures of Beilinson and Lichtenbaum

Let $k, p, S, \mathscr{X}$ and K be as in the notation 1.8. We always assume 1.8.1 in what follows. The Zariski site Z_{Zar} on a scheme Z always means $(\text{ét}/Z)_{\mathrm{Zar}}$, and $Z_{\text{ét}}$ means the usual small étale site. The main result of this appendix is Proposition A.1.3 below.

A.1 Motivic Complex and Conjectures

Let $\mathbb{Z}(2)_{\mathrm{Zar}} = \mathbb{Z}(2)^{\mathscr{X}}_{\mathrm{Zar}}$ be the motivic complex on $\mathscr{X}_{\mathrm{Zar}}$ defined by using Bloch's cycle complex, and let $\mathbb{Z}(2)_{\text{ét}}$ be its étale sheafification, which are, by works of Levine ([Le1], [Le2]), considered as strong candidates for motivic complexes of Beilinson-Lichtenbaum ([Be], [Li1]) in Zariski and étale topology, respectively (see also [Li2], [Li3]). We put

$$H^*_{\mathrm{Zar}}(\mathscr{X}, \mathbb{Z}(2)) := H^*_{\mathrm{Zar}}(\mathscr{X}, \mathbb{Z}(2)_{\mathrm{Zar}}), \qquad H^*_{\text{ét}}(\mathscr{X}, \mathbb{Z}(2)) := H^*_{\text{ét}}(\mathscr{X}, \mathbb{Z}(2)_{\text{ét}}).$$

In this appendix, we observe that the finiteness of $H^3_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$ is deduced from the following conjectures on motivic complexes:

Conjecture A.1.1 *Let $\epsilon : \mathscr{X}_{\text{ét}} \to \mathscr{X}_{\mathrm{Zar}}$ be the natural continuous map of sites. Then*:

(1) (*Beilinson-Lichtenbaum conjecture*). *We have*

$$\mathbb{Z}(2)_{\mathrm{Zar}} \xrightarrow{\sim} \tau_{\leq 2} R\epsilon_* \mathbb{Z}(2)_{\text{ét}} \quad \text{in } D(\mathscr{X}_{\mathrm{Zar}}).$$

(2) (*Hilbert's theorem* 90). *We have $R^3\epsilon_*\mathbb{Z}(2)_{\text{ét}} = 0$.*

(3) (*Kummer theory on $\mathscr{X}[p^{-1}]_{\text{ét}}$*). *We have $(\mathbb{Z}(2)_{\text{ét}})|_{\mathscr{X}[p^{-1}]} \otimes^{\mathbb{L}} \mathbb{Z}/p^r \simeq \mu_{p^r}^{\otimes 2}$.*

This conjecture holds if $\mathscr{X}$ is smooth over S by a result of Geisser [Ge1] Theorem 1.2 and the Merkur'ev-Suslin theorem [MS] (see also [GL2] Remark 5.9).

Conjecture A.1.2 *Let γ^2 be the canonical map*

$$\gamma^2 : \mathrm{CH}^2(\mathscr{X}) = H^4_{\mathrm{Zar}}(\mathscr{X}, \mathbb{Z}(2)) \longrightarrow H^4_{\text{ét}}(\mathscr{X}, \mathbb{Z}(2)).$$

Then the p-primary torsion part of $\mathrm{Coker}(\gamma^2)$ *is finite.*

This conjecture is based on Lichtenbaum's conjecture [Li1] that $H^4_{\text{ét}}(\mathscr{X}, \mathbb{Z}(2))$ is a finitely generated abelian group (by the properness of $\mathscr{X}/S$). The aim of this appendix is to prove the following:

Proposition A.1.3 *If Conjectures* A.1.1 *and* A.1.2 *hold, then* $H^3_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$ *is finite.*

This proposition is reduced to the following lemma:

Lemma A.1.4 (1) *If Conjecture* A.1.1 *holds, then for $r \geq 1$ there is an exact sequence*

$$0 \longrightarrow \mathrm{Coker}(\alpha_r) \longrightarrow H^3_{\mathrm{ur}}(K, \mathbb{Z}/p^r(2)) \longrightarrow \mathrm{Ker}(\varrho^2_r) \longrightarrow 0,$$

where α_r denotes the map

$$\alpha_r : {}_{p^r}\mathrm{CH}^2(\mathscr{X}) \longrightarrow {}_{p^r}H^4_{\text{ét}}(\mathscr{X}, \mathbb{Z}(2))$$

induced by γ^2, and ϱ^2_r denotes the cycle class map

$$\varrho^2_r : \mathrm{CH}^2(\mathscr{X})/p^r \longrightarrow H^4_{\text{ét}}(\mathscr{X}, \mathfrak{T}_r(2)).$$

(2) *If Conjectures* A.1.1 *and* A.1.2 *hold, then* $\mathrm{Coker}(\alpha_{\mathbb{Q}_p/\mathbb{Z}_p})$ *and* $\mathrm{Ker}(\varrho^2_{\mathbb{Q}_p/\mathbb{Z}_p})$ *are finite, where $\alpha_{\mathbb{Q}_p/\mathbb{Z}_p} := \varinjlim_{r \geq 1} \alpha_r$ and $\varrho^2_{\mathbb{Q}_p/\mathbb{Z}_p} := \varinjlim_{r \geq 1} \varrho^2_r$.*

To prove this lemma, we need the following sublemma, which is a variant of Geisser's arguments in [Ge1] §6 and provides a Kummer theory on the whole $\mathscr{X}$ extending Conjecture A.1.1 (3):

Sublemma A.1.5 *Put $\mathbb{Z}/p^r(2)_{\text{ét}} := \mathbb{Z}(2)_{\text{ét}} \otimes^{\mathbb{L}} \mathbb{Z}/p^r$. If Conjecture* A.1.1 *holds, then there is a unique isomorphism*

$$\mathbb{Z}/p^r(2)_{\text{ét}} \xrightarrow{\sim} \mathfrak{T}_r(2) \quad \textit{in } D(\mathscr{X}_{\text{ét}}, \mathbb{Z}/p^r)$$

that extends the isomorphism in Conjecture A.1.1 (3).

We prove Sublemma A.1.5 in §A.2 below and Lemma A.1.4 in §A.3 below.

A.2 Proof of Sublemma A.1.5

By Conjecture A.1.1 (3), we have only to consider the case where p is not invertible on S. Let us note that

$(*)\quad \mathbb{Z}/p^r(2)_{\text{ét}}$ is concentrated in degrees ≤ 2

by Conjecture A.1.1 (1) and (2). Let V, Y, i and j be as follows:

$$V := \mathscr{X}[p^{-1}] \xhookrightarrow{\ j\ } \mathscr{X} \xhookleftarrow{\ i\ } Y,$$

where Y denotes the union of the fibers of $\mathscr{X}/S$ of characteristic p. In étale topology, we define $Ri^!$ and Rj_* for unbounded complexes by the method of Spaltenstein [Spa]. We will prove

$$\tau_{\leq 3} Ri^! \mathbb{Z}/p^r(2)_{\text{ét}} \simeq \nu^1_{Y,r}[-3] \quad \text{in } D(Y_{\text{ét}}, \mathbb{Z}/p^r), \tag{A.2.1}$$

using $(*)$ (see $(\mathbf{S5})$ in §4.1 for $\nu^1_{Y,r}$). We first prove Sublemma A.1.5 admitting this isomorphism. Since $(\mathbb{Z}(2)_{\text{ét}})|_V \otimes^{\mathbb{L}} \mathbb{Z}/p^r \simeq \mu_{p^r}^{\otimes 2}$ by Conjecture A.1.1 (3), we obtain a distinguished triangle from (A.2.1) and $(*)$

$$i_*\nu^1_{Y,r}[-3] \longrightarrow \mathbb{Z}/p^r(2)_{\text{ét}} \longrightarrow \tau_{\leq 2} Rj_*\mu_{p^r}^{\otimes 2} \longrightarrow i_*\nu^1_{Y,r}[-2].$$

Hence comparing this distinguished triangle with that of $(\mathbf{S5})$ in §4.1, we obtain the desired isomorphism in the sublemma, whose uniqueness follows from [SH] Lemmas 1.1 and 1.2 (1).

In what follows, we prove (A.2.1). Put $\mathscr{K} := \mathbb{Z}(2)_{\text{Zar}} \otimes^{\mathbb{L}} \mathbb{Z}/p^r$ and $\mathscr{L} := \mathbb{Z}/p^r(2)_{\text{ét}}$ for simplicity. Let $\epsilon : \mathscr{X}_{\text{ét}} \to \mathscr{X}_{\text{Zar}}$ be as in Conjecture A.1.1. In Zariski topology, we define $Ri^!_{\text{Zar}}$ and $Rj_{\text{Zar}*}$ for unbounded complexes in the usual way by the finiteness of cohomological dimension. Because $\mathscr{L} = \epsilon^*\mathscr{K}$ is concentrated in degrees ≤ 2 by $(*)$, there is a commutative diagram with distinguished rows in $D^b(\mathscr{X}_{\text{ét}}, \mathbb{Z}/p^r)$

$$\begin{tikzcd}
\epsilon^*\mathscr{K} \arrow[r] \arrow[d, equal] & \tau_{\leq 2}\epsilon^* Rj_{\text{Zar}*}j^*_{\text{Zar}}\mathscr{K} \arrow[r] \arrow[d, "\alpha"'] & (\tau_{\leq 3}\epsilon^* i_{\text{Zar}*}Ri^!_{\text{Zar}}\mathscr{K})[1] \arrow[r] \arrow[d, "\beta"'] & \epsilon^*\mathscr{K}[1] \arrow[d, equal] \\
\mathscr{L} \arrow[r] & \tau_{\leq 2}Rj_{\text{ét}*}j^*_{\text{ét}}\mathscr{L} \arrow[r] & (\tau_{\leq 3} i_{\text{ét}*}Ri^!_{\text{ét}}\mathscr{L})[1] \arrow[r] & \mathscr{L}[1],
\end{tikzcd}$$

where the upper (resp. lower) row is obtained from the localization triangle in the Zariski (resp. étale) topology and the arrows α and β are canonical base-change morphisms. Since α is an isomorphism ([MS], [SV], [GL2]), β is an isomorphism as well. Hence (A.2.1) is reduced to showing

$$\tau_{\leq 3} Ri^!_{\text{Zar}}\mathscr{K} \simeq \epsilon_{Y*}\nu^1_{Y,r}[-3] \quad \text{in} \in D(Y_{\text{Zar}}, \mathbb{Z}/p^r), \tag{A.2.2}$$

where $\epsilon_Y : Y_{\text{ét}} \to Y_{\text{Zar}}$ denotes the natural continuous map of sites and we have used the base-change isomorphism $\epsilon^* i_{\text{Zar}*} = i_{\text{ét}*}\epsilon^*_Y$ ([Ge1] Proposition 2.2 (a)). Finally we show (A.2.2). Consider the local-global spectral sequence in the Zariski topology

$$E_1^{u,v} = \bigoplus_{x \in \mathscr{X}^u \cap Y} R^{u+v} i_{x*}(Ri^!_x Ri^!_{\text{Zar}}\mathscr{K}) \Longrightarrow R^{u+v} i^!_{\text{Zar}}\mathscr{K},$$

where for $x \in Y$, i_x denotes the natural map $x \to Y$. We have

$$E_1^{u,v} \simeq \begin{cases} \bigoplus_{x \in \mathscr{X}^u \cap Y} i_{x*}\epsilon_{x*} W_r\Omega^{2-u}_{x,\log} & \text{(if } v = 2\text{)} \\ 0 & \text{(otherwise)} \end{cases}$$

by the localization sequence of higher Chow groups [Le1] and results of Geisser-Levine ([GL1] Proposition 3.1, Theorem 7.1), where for $x \in Y$, ϵ_x denotes the natural continuous map $x_{\text{ét}} \to x_{\text{Zar}}$ of sites. By this description of E_1-terms and the compatibility of boundary maps ([GL2] Lemma 3.2, see also [Sz] Appendix), we obtain (A.2.2). This completes the proof of Sublemma A.1.5.

A.3 Proof of Lemma A.1.4

(1) By Sublemma A.1.5, there is an exact sequence

$$0 \longrightarrow H^3_{\text{ét}}(\mathscr{X}, \mathbb{Z}(2))/p^r \longrightarrow H^3_{\text{ét}}(\mathscr{X}, \mathfrak{T}_r(2)) \longrightarrow {}_{p^r}H^4_{\text{ét}}(\mathscr{X}, \mathbb{Z}(2)) \longrightarrow 0.$$

By Conjecture A.1.1 (1) and (2), we have

$$H^3_{\text{ét}}(\mathscr{X}, \mathbb{Z}(2)) \simeq H^3_{\text{Zar}}(\mathscr{X}, \mathbb{Z}(2)) \simeq \mathrm{CH}^2(\mathscr{X}, 1).$$

Thus we get an exact sequence

$$0 \longrightarrow \mathrm{CH}^2(\mathscr{X}, 1)/p^r \longrightarrow H^3_{\text{ét}}(\mathscr{X}, \mathfrak{T}_r(2)) \longrightarrow {}_{p^r}H^4_{\text{ét}}(\mathscr{X}, \mathbb{Z}(2)) \longrightarrow 0.$$

On the other hand, there is an exact sequence

$$0 \to N^1H^3_{\text{ét}}(\mathscr{X}, \mathfrak{T}_r(2)) \to H^3_{\text{ét}}(\mathscr{X}, \mathfrak{T}_r(2)) \to H^3_{\text{ur}}(K, \mathbb{Z}/p^r(2)) \to \mathrm{Ker}(\varrho^2_r) \to 0$$

by Proposition 4.3.2 (see Lemma 4.2.3 for N^1). In view of these facts and the short exact sequence in Lemma 4.2.3 (1), we get the desired exact sequence.

(2) By Conjecture A.1.1 (1) and (2), the map γ^2 in Conjecture A.1.2 is injective. Hence we get an exact sequence

$$0 \longrightarrow \mathrm{Coker}(\alpha_r) \longrightarrow {}_{p^r}\mathrm{Coker}(\gamma^2) \longrightarrow \mathrm{CH}^2(\mathscr{X})/p^r \xrightarrow{\gamma^2/p^r} H^4_{\text{ét}}(\mathscr{X}, \mathbb{Z}(2))/p^r.$$

Noting that the composite of γ^2/p^r and the injective map

$$H^4_{\text{ét}}(\mathscr{X}, \mathbb{Z}(2))/p^r \hookrightarrow H^4_{\text{ét}}(\mathscr{X}, \mathfrak{T}_r(2))$$

obtained from Sublemma A.1.5 coincides with ϱ^2_r, we get a short exact sequence

$$0 \longrightarrow \mathrm{Coker}(\alpha_r) \longrightarrow {}_{p^r}\mathrm{Coker}(\gamma^2) \longrightarrow \mathrm{Ker}(\varrho^2_r) \longrightarrow 0,$$

which implies the finiteness of $\mathrm{Coker}(\alpha_{\mathbb{Q}_p/\mathbb{Z}_p})$ and $\mathrm{Ker}(\varrho^2_{\mathbb{Q}_p/\mathbb{Z}_p})$ under Conjecture A.1.2. This completes the proof of Lemma A.1.4 and Proposition A.1.3. □

B Zeta Value of Threefolds over Finite Fields

In this appendix B, all cohomology groups of schemes are taken over the étale topology. Let X be a projective smooth geometrically integral threefold over a finite field $\mathbb{F}_q$, and let K be the function field of X (the case of fourfolds is treated in a recent paper of Kohmoto [Ko]). We define the unramified cohomology $H^{n+1}_{\mathrm{ur}}(K, \mathbb{Q}/\mathbb{Z}(n))$ in the same way as in 1.8. We show that the groups

$$H^2_{\mathrm{ur}}(K, \mathbb{Q}/\mathbb{Z}(1)) = \mathrm{Br}(X) \quad \text{and} \quad H^3_{\mathrm{ur}}(K, \mathbb{Q}/\mathbb{Z}(2))$$

are related with the value of the Hasse-Weil zeta function $\zeta(X, s)$ at $s = 2$:

$$\zeta^*(X, 2) := \lim_{s \to 2} \zeta(X, s)(1 - q^{2-s})^{-\varrho_2}, \qquad \text{where } \varrho_2 := \mathrm{ord}_{s=2}\, \zeta(X, s).$$

Let

$$\theta \ : \ \mathrm{CH}^2(X) \longrightarrow \mathrm{Hom}(\mathrm{CH}^1(X), \mathbb{Z})$$

be the map induced by the intersection pairing and the degree map

$$\mathrm{CH}^2(X) \times \mathrm{CH}^1(X) \longrightarrow \mathrm{CH}^3(X) = \mathrm{CH}_0(X) \xrightarrow{\deg} \mathbb{Z}.$$

The map θ has finite cokernel by a theorem of Matsusaka [Ma] Theorem 4. We define

$$\mathscr{R} := |\mathrm{Coker}(\theta)|.$$

We prove the following formula (compare with the formula in [Ge2]):

Theorem B.1 *Assume that* $\mathrm{Br}(X)$ *and* $H^3_{\mathrm{ur}}(K, \mathbb{Q}/\mathbb{Z}(2))$ *are finite. Then* $\zeta^*(X, 2)$ *equals the following rational number up to a sign*:

$$q^{\chi(X, \mathscr{O}_X, 2)} \cdot \frac{|H^3_{\mathrm{ur}}(K, \mathbb{Q}/\mathbb{Z}(2))|}{|\mathrm{Br}(X)| \cdot \mathscr{R}} \cdot \prod_{i=0}^{3} |\mathrm{CH}^2(X, i)_{\mathrm{tors}}|^{(-1)^i} \cdot \prod_{i=0}^{1} |\mathrm{CH}^1(X, i)_{\mathrm{tors}}|^{(-1)^i},$$

where $\mathrm{CH}^2(X, i)$ *and* $\mathrm{CH}^1(X, i)$ *denote Bloch's higher Chow groups* [Bl3] *and* $\chi(X, \mathscr{O}_X, 2)$ *denotes the following integer*:

$$\chi(X, \mathscr{O}_X, 2) := \sum_{i,j} (-1)^{i+j}\, (2 - i)\, \dim_{\mathbb{F}_q} H^j(X, \Omega^i_X) \quad (0 \le i \le 2,\ \ 0 \le j \le 3).$$

This theorem follows from a theorem of Milne ([Mi2] Theorem 0.1) and Proposition B.2 below. For integers $i, n \ge 0$, we define

$$H^i(X, \widehat{\mathbb{Z}}(n)) := \prod_{\text{all } \ell} H^i(X, \mathbb{Z}_\ell(n)),$$

where ℓ runs through all prime numbers, and $H^i(X, \mathbb{Z}_p(n))$ ($p := \mathrm{ch}(\mathbb{F}_q)$) is defined as

$$H^i(X, \mathbb{Z}_p(n)) := \varprojlim_{r \ge 1} H^{i-n}(X, W_r\Omega^n_{X,\log}).$$

Proposition B.2 (1) *We have*

$$H^i(X, \widehat{\mathbb{Z}}(2)) \simeq \begin{cases} \mathrm{CH}^2(X, 4-i)_{\mathrm{tors}} & (i = 0, 1, 2, 3) \\ (\mathrm{CH}^1(X, i-6)_{\mathrm{tors}})^* & (i = 6, 7), \end{cases} \tag{B.3}$$

where for an abelian group M, we put

$$M^* := \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z}).$$

Furthermore, $\mathrm{CH}^1(X, j)_{\mathrm{tors}}$ *and* $\mathrm{CH}^2(X, j)_{\mathrm{tors}}$ *are finite for any $j \geq 0$, and we have*

$$\mathrm{CH}^1(X, j)_{\mathrm{tors}} = 0 \ \textit{for} \ j \geq 2 \quad \textit{and} \quad \mathrm{CH}^2(X, j)_{\mathrm{tors}} = 0 \ \textit{for} \ j \geq 4.$$

(2) *Assume that* $\mathrm{Br}(X)$ *is finite. Then we have*

$$H^5(X, \widehat{\mathbb{Z}}(2))_{\mathrm{tors}} \simeq \mathrm{Br}(X)^*,$$

and the cycle class map

$$\mathrm{CH}^2(X) \otimes \mathbb{Z}_\ell \longrightarrow H^4(X, \mathbb{Z}_\ell(2))$$

has finite cokernel for any prime number ℓ.

(3) *Assume that* $\mathrm{Br}(X)$ *and* $H^3_{\mathrm{ur}}(K, \mathbb{Q}/\mathbb{Z}(2))$ *are finite. Then the following map given by the cup product with the canonical element $1 \in \widehat{\mathbb{Z}} \simeq H^1(\mathbb{F}_q, \widehat{\mathbb{Z}})$ has finite kernel and cokernel*:

$$\epsilon^4 : H^4(X, \widehat{\mathbb{Z}}(2)) \longrightarrow H^5(X, \widehat{\mathbb{Z}}(2)),$$

and we have the following equality of rational numbers:

$$\frac{|\mathrm{Ker}(\epsilon^4)|}{|\mathrm{Coker}(\epsilon^4)|} = \frac{|H^3_{\mathrm{ur}}(K, \mathbb{Q}/\mathbb{Z}(2))| \cdot |\mathrm{CH}^2(X)_{\mathrm{tors}}|}{|\mathrm{Br}(X)| \cdot \mathscr{R}}.$$

Proof of Proposition B.2. (1) By standard arguments on limits, there is a long exact sequence

$$\cdots \longrightarrow H^i(X, \widehat{\mathbb{Z}}(2)) \longrightarrow H^i(X, \widehat{\mathbb{Z}}(2)) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow H^i(X, \mathbb{Q}/\mathbb{Z}(2)) \\ \longrightarrow H^{i+1}(X, \widehat{\mathbb{Z}}(2)) \longrightarrow \cdots .$$

By [CTSS] p. 780 Théorème 2, p. 782 Théorème 3, we see that

$$H^i(X, \widehat{\mathbb{Z}}(2)) \text{ and } H^i(X, \mathbb{Q}/\mathbb{Z}(2)) \text{ are finite for } i \neq 4, 5.$$

Hence we have

$$H^i(X, \widehat{\mathbb{Z}}(2)) \simeq H^{i-1}(X, \mathbb{Q}/\mathbb{Z}(2)) \quad \text{for } i \neq 4, 5, 6.$$

On the other hand, there is an exact sequence

$$0 \longrightarrow \mathrm{CH}^2(X, 5-i) \otimes \mathbb{Q}/\mathbb{Z} \longrightarrow H^{i-1}(X, \mathbb{Q}/\mathbb{Z}(2)) \longrightarrow \mathrm{CH}^2(X, 4-i)_{\mathrm{tors}} \longrightarrow 0$$

for $i \leq 3$ ([MS], [SV], [GL1], [GL2]), where $\mathrm{CH}^2(X, 5-i) \otimes \mathbb{Q}/\mathbb{Z}$ must be zero because it is divisible and finite. Thus we get the isomorphism (B.3) for $i \leq 3$, the finiteness of $\mathrm{CH}^2(X, j)_{\mathrm{tors}}$ for $j \geq 1$ and the vanishing of $\mathrm{CH}^2(X, j)_{\mathrm{tors}}$ for $j \geq 4$. The finiteness of $\mathrm{CH}^2(X, 0)_{\mathrm{tors}} = \mathrm{CH}^2(X)_{\mathrm{tors}}$ (cf. [CTSS] p. 780 Théorème 1) follows from the exact sequence

$$0 \longrightarrow \mathrm{CH}^2(X, 1) \otimes \mathbb{Q}/\mathbb{Z} \longrightarrow N^1H^3(X, \mathbb{Q}/\mathbb{Z}(2)) \longrightarrow \mathrm{CH}^2(X)_{\mathrm{tors}} \longrightarrow 0$$

(cf. Lemma 3.2.2), where we put

$$N^1H^3(X, \mathbb{Q}/\mathbb{Z}(2)) := \mathrm{Ker}(H^3(X, \mathbb{Q}/\mathbb{Z}(2)) \to H^3(K, \mathbb{Q}/\mathbb{Z}(2))).$$

As for the case $i = 6, 7$ of (B.3), we have

$$H^i(X, \widehat{\mathbb{Z}}(2))^* \simeq H^{7-i}(X, \mathbb{Q}/\mathbb{Z}(1))$$

by a theorem of Milne [Mi2] Theorem 1.14 (a). It remains to show

$$\mathrm{CH}^1(X, j)_{\mathrm{tors}} \simeq H^{1-j}(X, \mathbb{Q}/\mathbb{Z}(1)) \quad \text{for } j \geq 0,$$

which can be checked by similar arguments as before.

(2) We have $H^5(X, \widehat{\mathbb{Z}}(2))^* \simeq H^2(X, \mathbb{Q}/\mathbb{Z}(1))$ and an exact sequence

$$0 \longrightarrow \mathrm{CH}^1(X) \otimes \mathbb{Q}/\mathbb{Z} \longrightarrow H^2(X, \mathbb{Q}/\mathbb{Z}(1)) \longrightarrow \mathrm{Br}(X) \longrightarrow 0. \tag{B.4}$$

Hence we have $(H^5(X, \widehat{\mathbb{Z}}(2))_{\mathrm{tors}})^* \simeq \mathrm{Br}(X)$, assuming $\mathrm{Br}(X)$ is finite. To show the second assertion for $\ell \neq \mathrm{ch}(\mathbb{F}_q)$, it is enough to show that the cycle class map

$$\mathrm{CH}^2(X) \otimes \mathbb{Q}_\ell \longrightarrow H^4(\overline{X}, \mathbb{Q}_\ell(2))^{\Gamma}$$

is surjective, where $\Gamma := \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$. The assumption on $\mathrm{Br}(X)$ implies the bijectivity of the cycle class map

$$\mathrm{CH}^1(X) \otimes \mathbb{Q}_\ell \xrightarrow{\sim} H^2(\overline{X}, \mathbb{Q}_\ell(1))^{\Gamma}$$

by [Ta2] Proposition (4.3) (see also [Mi1] Theorem 4.1), and the assertion follow from [Ta2] Proposition (5.1). As for the case $\ell = \mathrm{ch}(\mathbb{F}_q)$, one can easily pursue an analogy using crystalline cohomology, whose details are left to the reader.

(3) The finiteness assumption on $\mathrm{Br}(X)$ implies the condition $\boldsymbol{SS}(X, 1, \ell)$ in [Mi2] for all prime numbers ℓ by loc. cit. Proposition 0.3. Hence $\boldsymbol{SS}(X, 2, \ell)$ holds by the Poincaré duality, and ϵ^4 has finite kernel and cokernel by loc. cit. Theorem 0.1.

To show the equality assertion, we put

$$\widehat{\mathrm{CH}}^2(X) := \varprojlim_{n\geq 1} \mathrm{CH}^2(X)/n,$$

and consider the following commutative square (cf. [Mi3] Lemma 5.4):

$$\begin{tikzcd}
\widehat{\mathrm{CH}}^2(X) \arrow[r, "\Theta"] \arrow[d, "\alpha"'] & \mathrm{Hom}(\mathrm{CH}^1(X), \widehat{\mathbb{Z}}) \\
H^4(X, \widehat{\mathbb{Z}}(2)) \arrow[r, "\epsilon^4"] & H^5(X, \widehat{\mathbb{Z}}(2)), \arrow[u, "\beta"']
\end{tikzcd}$$

where the top arrow Θ denotes the map induced by θ. The arrow α denotes the cycle class map of codimension 2, and β denotes the Pontryagin dual of the cycle class map with $\mathbb{Q}/\mathbb{Z}$-coefficients in (B.4). The arrow α is injective (cf. (4.3.3)) and we have

$$|\mathrm{Coker}(\alpha)| = |H^3_{\mathrm{ur}}(K, \mathbb{Q}/\mathbb{Z}(2))|$$

by the finiteness assumption on $H^3_{\mathrm{ur}}(K, \mathbb{Q}/\mathbb{Z}(2))$ and (2) (cf. Proposition 4.3.5). The arrow β is surjective and we have

$$\mathrm{Ker}(\beta) = H^5(X, \widehat{\mathbb{Z}}(2))_{\mathrm{tors}} \overset{(2)}{\simeq} \mathrm{Br}(X)^*,$$

by Milne's lemma ([Mi3] Lemma 5.3) and the isomorphism $\mathrm{CH}^1(X) \otimes \widehat{\mathbb{Z}} \simeq H^2(X, \widehat{\mathbb{Z}}(1))$ (cf. [Ta2] Proposition (4.3)), where we have used again the finiteness assumption on $\mathrm{Br}(X)$. Therefore in view of the finiteness of $\mathrm{Ker}(\epsilon^4)$, the map Θ has finite kernel and we obtain

$$\mathrm{Ker}(\Theta) = \widehat{\mathrm{CH}}^2(X)_{\mathrm{tors}} = \mathrm{CH}^2(X)_{\mathrm{tors}},$$

where we have used the finiteness of $\mathrm{CH}^2(X)_{\mathrm{tors}}$ in (1). Finally the assertion follows from the following equality concerning the above diagram:

$$\frac{|\mathrm{Ker}(\Theta)|}{|\mathrm{Coker}(\Theta)|} = \frac{|\mathrm{Ker}(\alpha)|}{|\mathrm{Coker}(\alpha)|} \cdot \frac{|\mathrm{Ker}(\epsilon^4)|}{|\mathrm{Coker}(\epsilon^4)|} \cdot \frac{|\mathrm{Ker}(\beta)|}{|\mathrm{Coker}(\beta)|}$$

This completes the proof of Proposition B.2 and Theorem B.1. □

References

[AS] Asakura, M., Saito, S.: Surfaces over a p-adic field with infinite torsion in the Chow group of 0-cycles. Algebra Number Theory **1**, 163–181 (2008)

[Be] Beilinson, A. A.: Height pairings between algebraic cycles. In: Manin, Yu. I. (ed.) *K-theory, Arithmetic and Geometry*, (Lecture Notes in Math. 1289), pp. 1–27, Berlin, Springer, 1987

[Bl1] Bloch, S.: Algebraic K-theory and crystalline cohomology. Inst. Hautes Études Sci. Publ. Math. **47**, 187–268 (1977)

[Bl2] Bloch, S.: *Lectures on Algebraic Cycles*. (Duke Univ. Math. Series 4), Durham, Duke Univ. Press, 1980

[Bl3] Bloch, S.: Algebraic cycles and higher K-theory. Adv. Math. **61**, 267–304 (1986)

[Bl4] Bloch, S.: The moving lemma for higher Chow groups. J. Algebraic Geom. **3**, 537–568 (1994)

[BK1] Bloch, S., Kato, K.: p-adic étale cohomology. Inst. Hautes Études Sci. Publ. Math. **63**, 107–152 (1986)

[BK2] Bloch, S., Kato, K.: L-functions and Tamagawa numbers of motives. In: Cartier, P., Illusie, L., Katz, N. M., Laumon, G., Manin, Yu. I., Ribet, K. A. (eds.) *The Grothendieck Festscherift I*, (Progr. Math. 86), pp. 333–400, Boston, Birkhäuser, 1990

[CTR1] Colliot-Thélène, J.-L., Raskind, W.: K_2-cohomology and the second Chow group. Math. Ann. **270**, 165–199 (1985)

[CTR2] Colliot-Thélène, J.-L., Raskind, W.: Groupe de Chow de codimension deux des variété sur un corps de numbres: Un théorème de finitude pour la torsion, Invent. Math. **105**, 221–245 (1991)

[CTSS] Colliot-Thélène, J.-L., Sansuc, J.-J., Soulé, C.: Torsion dans le groupe de Chow de codimension deux. Duke Math. J. **50**, 763–801 (1983)

[dJ] de Jong, A. J.: Smoothness, semi-stability and alterations. Inst. Hautes Études Sci. Publ. Math. **83**, 51–93 (1996)

[De1] Deligne, P.: Théorème de Lefschetz difficile et critères de dégéméresence de suites spectrales. Inst. Hautes Études Sci. Publ. Math. **35**, 107–126 (1968)

[De2] Deligne, P.: La conjecture de Weil II. Inst. Hautes Études Sci. Publ. Math. **52**, 313–428 (1981)

[Fl] Flach, M.: A finiteness theorem for the symmetric square of an elliptic curve. Invent. Math. **109**, 307–327 (1992)

[FM] Fontaine, J.-M., Messing, W.: p-adic periods and p-adic étale cohomology. In: Ribet, K. A. (ed.) *Current Trends in Arithmetical Algebraic Geometry*, (Contemp. Math. 67), pp. 179–207, Providence, Amer. Math. Soc., 1987

[FG] Fujiwara, K.: A proof of the absolute purity conjecture (after Gabber). In: Usui, S., Green, M., Illusie, L., Kato, K., Looijenga, E., Mukai, S., Saito, S. (eds.) *Algebraic Geometry, Azumino, 2001*, (Adv. Stud. in Pure Math. 36), pp. 153–184, Tokyo, Math. Soc. Japan, 2002

[Ge1] Geisser, T.: Motivic cohomology over Dedekind rings. Math. Z. **248**, 773–794 (2004)

[Ge2] Geisser, T.: Weil-étale cohomology over finite fields. Math. Ann. **330**, 665-692 (2004)

[GL1] Geisser, T., Levine, M.: The p-part of K-theory of fields in characteristic p. Invent. Math. **139**, 459–494 (2000)

[GL2] Geisser, T., Levine, M.: The Bloch-Kato conjecture and a theorem of Suslin-Voevodsky. J. Reine Angew. Math. **530**, 55–103 (2001)

[Gr] Gros, M.: Sur la partie p-primaire du groupe de Chow de codimension deux. Comm. Algebra **13**, 2407–2420 (1985)

[G] Grothendieck, A.: Le groupe de Brauer. In: *Dix Exposés sur la Cohomologie des Schémas*, pp. 46–188, Amsterdam, North-Holland, 1968

[Hy] Hyodo, O.: A note on p-adic etale cohomology in the semi-stable reduction case. Invent. Math. **91** 543–557 (1988)

[Il] Illusie, L.: Complexe de de Rham-Witt et cohomologie cristalline. Ann. Sci. École Norm. Sup. (4) **12**, 501–661 (1979)

[J1] Jannsen, U.: Continuous étale cohomology. Math. Ann. **280**, 207–245 (1987)

[J2] Jannsen, U.: On the ℓ-adic cohomology of varieties over number fields and its Galois cohomology. In: Ihara, Y., Ribet, K. A., Serre, J.-P. (eds.) *Galois Group over* $\mathbb{Q}$, pp. 315–360, Berlin, Springer, 1989

[JS] Jannsen, U., Saito, S.: Kato homology of arithmetic schemes and higher class field theory over local fields. Documenta Math. *Extra Volume*: *Kazuya Kato's Fiftieth Birthday*, 479–538 (2003)

[JSS] Jannsen, U., Saito, S., Sato, K.: Étale duality for constructible sheaves on arithmetic schemes. preprint 2009

`http://arxiv.org/abs/0910.3759`

[Ka1] Kato, K.: On p-adic vanishing cycles (Application of ideas of Fontaine-Messing). In: *Algebraic Geometry, Sendai, 1985*, (Adv. Stud. in Pure Math. 10), pp. 207–251, Tokyo, Kinokuniya, 1987

[Ka2] Kato, K.: Semi-stable reduction and p-adic étale cohomology. In: *Périodes p-adiques, Séminaire de Bures, 1988*, (Astérisuque 223), pp. 269–293, Marseille, Soc. Math. France, 1994

[KCT] Kato, K.: A Hasse principle for two-dimensional global fields. (with an appendix by Colliot-Thélène, J.-L.), J. Reine Angew. Math. **366**, 142–183 (1986)

[KL] Katz, N., Lang, S.: Finiteness theorem for higher geometric class field theory. Enseign. Math. **27**, 285–319 (1981)

[Ko] Kohmoto, D.: A generalization of the Artin-Tate formula for fourfolds. preprint 2010

[Ku] Kurihara, M.: A note on p-adic étale cohomology. Proc. Japan Acad. Ser. A **63**, 275–278 (1987)

[La1] Langer, A.: Selmer groups and torsion zero cycles on the self-product of a semistable elliptic curve. Doc. Math. **2**, 47–59 (1997)

[La2] Langer, A.: 0-cycles on the elliptic modular surface of level 4. Tohoku Math. J. **50**, 315–360 (1998)

[La3] Langer, A.: Local points of motives in semistable reduction. Compositio Math. **116**, 189–217 (1999)

[La4] Langer, A.: Finiteness of torsion in the codimension-two Chow group: an axiomatic approach. In: Gordon, B. B., Lewis, J. D., Müller-Stach, S., Saito, S., Yui, N. (eds.) *The Arithmetic and Geometry of Algebraic Cycles, Banff, 1988*, (NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. 548), pp. 277–284, Dordrecht, Kluwer, 2000

[LS] Langer A., Saito, S.: Torsion zero-cycles on the self-product of a modular elliptic curve. Duke Math. J. **85**, 315–357 (1996)

[Le1] Levine, M.: Techniques of localization in the theory of algebraic cycles. J. Algebraic Geom. **10**, 299-363 (2001)

[Le2] Levine, M.: K-theory and motivic cohomology of schemes. preprint, 1999

[Li1] Lichtenbaum, S.: Values of zeta functions at non-negative integers. In: Jager, H. (ed.) *Number Theory, Noordwijkerhout, 1983*, (Lecture Notes in Math. 1068), pp. 127-138, Berlin, Springer, 1984

[Li2] Lichtenbaum, S.: The construction of weight-two arithmetic cohomology. Invent. Math. **88**, 183–215 (1987)

[Li3] Lichtenbaum, S.: New results on weight-two motivic cohomology. In: Cartier, P., Illusie, L., Katz, N. M., Laumon, G., Manin, Y., Ribet, K. A. (eds.) *The Grothendieck Festschrift III*, (Progr. Math. 88), pp. 35–55, Boston, Birkhäuser, 1990

[Ma] Matsusaka, T.: The criteria for algebraic equivalence and the torsion group. Amer. J. Math. **79**, 53–66 (1957)

[MS] Merkur'ev, A. S., Suslin, A. A.: K-cohomology of Severi-Brauer varieties and the norm residue homomorphism. Math. USSR Izv. **21**, 307–341 (1983)

[Md] Mildenhall, S.: Cycles in a product of elliptic curves, and a group analogous to the class group. Duke Math. J. **67**, 387–406 (1992)

[Mi1] Milne, J. S.: On a conjecture of Artin and Tate. Ann. of Math. **102**, 517–533 (1975)

[Mi2] Milne, J. S.: Values of zeta functions of varieties over finite fields. Amer. J. Math. **108**, 297–360 (1986)

[Mi3] Milne, J. S.: Motivic cohomology and values of zeta functions. Compositio Math. **68**, 59–102 (1988)

[Mu] Mumford, D.: *Geometric Invariant Theory*. (Ergebnisse der Math. 34), Berlin, Springer, 1965

[Ne1] Nekovář, J.: Syntomic cohomology and p-adic regulators. preprint, 1997

[Ne2] Nekovář, J.: p-adic Abel-Jacobi maps and p-adic height pairings. In: Gordon, B. B., Lewis, J. D., Müller-Stach, S., Saito, S., Yui, N. (eds.) *The Arithmetic and Geometry of Algebraic Cycles, Banff, 1998*, (CRM Proc. Lecture Notes 24), pp. 367–379, Providence, Amer. Math. Soc., 2000

[NS] Nesterenko, Yu. P., Suslin, A. A.: Homology of the general linear group over a local ring, and Milnor's K-theory. Math. USSR Izv. **34**, 121–145 (1990)

[O] Otsubo, N.: Selmer groups and zero-cycles on the Fermat quartic surface. J. Reine Angew. Math. **525**, 113–146 (2000)

[PS] Parimala, R., Suresh, V.: Zero-cycles on quadric fibrations: Finiteness theorems and the cycle map. Invent. Math. **122**, 83–117 (1995)

[RZ] Rapoport, M., Zink, T.: Über die lokale Zetafunktion von Shimuravarietäten. Monodromiefiltration und verschwindende Zyklen in ungleicher Charakteristik. Invent. Math. **68**, 21–101 (1982)

[Sa] Saito, S.: On the cycle map for torsion algebraic cycles of codimension two. Invent. Math. **106**, 443–460 (1991)

[SSa] Saito, S., Sato, K.: Finiteness theorem on zero-cycles over p-adic fields. to appear in Ann. of Math.

`http://arxiv.org/abs/math.AG/0605165`

[SSu] Saito, S., Sujatha, R.: A finiteness theorem for cohomology of surfaces over p-adic fields and an application to Witt groups. In: Jacob, B., Rosenberg, A. (eds.) *Algebraic K-Theory and Algebraic Geometry*: *Connections with quadratic forms and division algebras, Santa Barbara, 1992*, (Proc. of Sympos. Pure Math. 58, Part 2), pp. 403–416, Providence, Amer. Math. Soc., 1995

[Sal] Salberger, P.: Torsion cycles of codimension two and ℓ-adic realizations of motivic cohomology. In: David, S. (ed.) *Séminaire de Thórie des Nombres 1991/92*, (Progr. Math. 116), pp. 247–277, Boston, Birkhäuser, 1993

[Sat1] Sato, K.: Injectivity of the torsion cycle map of codimension two of varieties over p-adic fields with semi-stable reduction. J. Reine Angew. Math. **501**, 221–235 (1998)

[Sat2] Sato, K.: Abel-Jacobi mappings and finiteness of motivic cohomology groups. Duke Math. J. **104**, 75–112 (2000)

[Sat3] Sato, K.: Logarithmic Hodge-Witt sheaves on normal crossing varieties. Math. Z. **257**, 707–743 (2007)

[SH] Sato, K.: p-adic étale Tate twists and arithmetic duality. (with an appendix by Hagihara, K.), Ann. Sci. École Norm. Sup. (4) **40**, 519–588 (2007)

[Sch] Schneider, P.: p-adic point of motives. In: Jannsen, U. (ed.) *Motives*, (Proc. Symp. Pure Math. 55-II), pp. 225–249, Providence, Amer. Math. Soc., 1994

[Se] Serre, J.-P.: *Cohomologie Galoisienne.* 5^e éd., (Lecture Notes in Math. 5), Berlin, Springer, 1992

[So1] Soulé, C.: Operations on étale K-theory. Applications. In: Dennis, R. K. (ed.) *Algebraic K-theory, Oberwolfach, 1980, Part I.* (Lecture Notes in Math. 966), pp. 271–303 Berlin, Springer, 1982

[So2] Soulé, C.: Opérations en K-théorie algébrique. Canad. J. Math. **37**, 488–550 (1985)

[Spa] Spaltenstein, N.: Resolutions of unbounded complexes. Compositio Math. **65**, 121–154 (1988)

[Spi] Spiess, M.: On indecomposable elements of K_1 of a product of elliptic curves. K-Theory **17**, 363–383 (1999)

[Su] Suresh, V.: Zero cycles on conic fibrations and a conjecture of Bloch. K-Theory **10**, 597–610 (1996)

[SV] Suslin, A. A., Voevodsky, V.: Bloch-Kato conjecture and motivic cohomology with finite coefficients. In: Gordon, B. B., Lewis, J. D., Müller-Stach, S., Saito, S., Yui, N. (eds.) *The Arithmetic and Geometry of Algebraic Cycles, Banff, 1998*, (NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. 548), pp. 117–189, Dordrecht, Kluwer, 2000

[Sz] Szamuely, T.: Sur la théorie des corps de classes pour les variétés sur les corps p-adiques. J. Reine Angew. Math. **525**, 183–212 (2000)

[Ta1] Tate, J.: On the conjecture of Birch and Swinnerton-Dyer and a geometric analog. Séminaire Bourbaki 1965/66, Exposé 306, Benjamin, New York, 1966

[Ta2] Tate, J.: Conjectures on algebraic cycles in ℓ-adic cohomology. In: Jannsen, U. (ed.) *Motives*, (Proc. Symp. Pure Math. 55-I), pp. 71–83, Providence, Amer. Math. Soc., 1994

[Th] Thomason, R. W.: Absolute cohomological purity. Bull. Soc. Math. France **112**, 397–406 (1984)

[To] Totaro, B.: Milnor K-theory is the most simplest part of algebraic K-theory. K-Theory **6**, 177-189 (1992)

[Ts1] Tsuji, T.: p-adic étale cohomology and crystalline cohomology in the semi-stable reduction case. Invent. Math. **137**, 233–411 (1999)

[Ts2] Tsuji, T.: On p-adic nearby cycles of log smooth families. Bull. Soc. Math. France **128**, 529–575 (2000)

[Ts3] Tsuji, T.: On the maximal unramified quotients of p-adic étale cohomology groups and logarithmic Hodge-Witt sheaves. Documenta Math. *Extra Volume*: *Kazuya Kato's Fiftieth Birthday*, 833–890 (2003)

[EGA4] Grothendieck, A., Dieudonné, J.: Etude locale des schémas et des morphismes de schémas. Inst. Hautes Études Sci. Publ. Math. **20** (1964), **24** (1965), **28** (1966), **32** (1967)

[SGA4] Grothendieck, A., Artin, M., Verdier, J.-L., with Deligne, P., Saint-Donat, B.: *Théorie des Topos et Cohomologie Étale des Schémas*. (Lecture Notes in Math. 269, 270, 305), Berlin, Springer, 1972–73

Shuji Saito
Department of Mathematics
University of Tokyo
Komaba, Meguro-ku
Tokyo 153-8914, JAPAN
sshuji@ms.u-tokyo.ac.jp

Kanetomo Sato
Graduate School of Mathematics, Nagoya University
Furo-cho, Chikusa-ku
Nagoya 464-8602 JAPAN
kanetomo@math.nagoya-u.ac.jp

Twists of Drinfeld–Stuhler Modular Varieties

Michael Spiess

Received: October 26, 2009
Revised: May 6, 2010

Abstract. Let $\mathcal{A}$ be a maximal (or more generally a hereditary) order in a central simple algebra over a global field F of positive characteristic. We show that certain modular scheme of $\mathcal{A}$-elliptic sheaves– for different $\mathcal{A}$ – are twists of each other and deduce that the *uniformization at* ∞ and the *Cherednik-Drinfeld uniformization* for these varieties are equivalent.

2010 Mathematics Subject Classification: Primary 11G09; Secondary 11G18, 14G22
Keywords and Phrases: Drinfeld modular varieties, rigid analytic uniformization

Dedicated to A.A.Suslin on his 60th birthday

1 Introduction

In [Dr1], Drinfeld has introduced the analogues of Shimura varieties for GL_d over a global field F of positive characteristic. Following a suggestion of U. Stuhler the corresponding varieties for an inner form of GL_d, i.e. the group of invertible elements A^* of a central simple algebra A of dimension d^2 over F, have been introduced by Laumon, Rapoport and Stuhler in [LRS]. For $d = 2$ these are the analogues of Shimura curves. In this paper we show that some of these varieties (for different A) are twists of each other.
Let us recall the latter in the simplest case (i.e. over $\mathbb{Q}$ and by neglecting level structure). Let D be an indefinite quaternion algebra over $\mathbb{Q}$ and $\mathcal{D}$ a maximal order in D. The Shimura curve S_D is the (coarse) moduli space corresponding to the moduli problem

$$(S \to \operatorname{Spec}\mathbb{Z}) \mapsto \text{abelian surfaces over } S \text{ with } \mathcal{D}\text{-action}.$$

By fixing an isomorphism $D \otimes \mathbb{R} \cong M_2(\mathbb{R})$ the group of units $\mathcal{D}^*$ acts on the symmetric space $\mathcal{H}_\infty := \mathbb{P}^1_{/\mathbb{R}} - \mathbb{P}^1(\mathbb{R})$ (the upper and lower half plane) through linear transformations. The curve $S_D \otimes_{\mathbb{Q}} \mathbb{R}$ admits the following concrete description

$$S_D \otimes_{\mathbb{Q}} \mathbb{R} = \mathcal{D}^* \backslash \mathcal{H}_\infty. \tag{1}$$

If p is a prime number which is ramified in D then there is a similar explicite description over $\mathbb{Q}_p$. For that let $\overline{D}$ be the definite quaternion algebra over $\mathbb{Q}$ given by the local data $\overline{D} \otimes \mathbb{Q}_\ell \cong D \otimes \mathbb{Q}_\ell$ for all prime numbers ℓ different from p and $\overline{D} \otimes \mathbb{Q}_p \cong M_2(\mathbb{Q}_p)$. Let $\overline{\mathcal{D}}$ denote a maximal $\mathbb{Z}[\frac{1}{p}]$-order in $\overline{D}$ and denote by $\mathbb{Q}_p^{\mathrm{nr}}$ the quotient field of the ring of Witt vectors of $W(\overline{\mathbb{F}}_p)$. The Theorem of Cherednik-Drinfeld asserts that

$$S_D \otimes_{\mathbb{Q}} \mathbb{Q}_p = \overline{\mathcal{D}}^* \backslash (\mathcal{H}_p \otimes_{\mathbb{Q}_p} \mathbb{Q}_p^{\mathrm{nr}}). \tag{2}$$

(see [Ce], [Dr2] or [BC]). Here $\overline{\mathcal{D}}^*$ acts on $\mathbb{Q}_p^{\mathrm{nr}}$ via $\gamma \mapsto \mathrm{Frob}_p^{-\operatorname{ord}_p(\mathrm{Nrd}(\gamma))}$ and on the p-adic upper half plane $\mathcal{H}_p := \mathbb{P}^1_{/\mathbb{Q}_p} - \mathbb{P}^1(\mathbb{Q}_p)$ via linear transformations.
Now let F be a global field of positive characteristic, i.e. F is the function field of a smooth proper curve X over a finite field $\mathbb{F}_q$. The analogues of Shimura curves over F are the moduli spaces of $\mathcal{A}$-elliptic sheaves as introduced in [LRS]. In this paper we generalize this notion slightly by making systematically use of hereditary orders. Let $\infty \in X$ be a fixed closed point. For simplicity we assume in the introduction that $\deg(\infty) = 1$. Let A be a central simple F-algebra of dimension d^2 and let $\mathcal{A}$ be a locally principal hereditary $\mathcal{O}_X$-order in A. The condition *locally principal* means that the radical $\mathrm{Rad}(\mathcal{A}_x)$ of $\mathcal{A}_x := \mathcal{A} \otimes_{\mathcal{O}_X} \widehat{\mathcal{O}}_{X,x}$ is a principal ideal for every closed point $x \in X$. There exists a positive integer $e = e_x(\mathcal{A})$ such that $\mathrm{Rad}(\mathcal{A}_x)^e$ is the ideal $\mathcal{A}_x \varpi_x$ generated by a uniformizer ϖ_x of X at x. The number $e_x(\mathcal{A})$ divides d for al l x and is equal to 1 for almost all x. We assume in the following that $e_\infty(\mathcal{A}) = d$. If A is unramified at ∞ then this amounts to require that $\mathcal{A}_\infty$ is isomorphic to the subring of matrices in $M_d(\widehat{\mathcal{O}}_{X,\infty})$ which are upper triangular modulo ϖ_∞. Roughly, an $\mathcal{A}$-elliptic sheaf with pole ∞ is a locally free $\mathcal{A}$-module of rank 1 together with a meromorphic $\mathcal{A}$-linear Frobenius having a simple pole at ∞ and a simple zero. The precise definition is as follows.
An $\mathcal{A}$-elliptic sheaf over an $\mathbb{F}_q$-scheme S is a pair $E = (\mathcal{E}, t)$ consisting of a locally free right $\mathcal{A} \boxtimes \mathcal{O}_S$-module of rank 1 and an injective homomorphism of $\mathcal{A} \boxtimes \mathcal{O}_S$-modules

$$t : (\mathrm{id}_X \times \mathrm{Frob}_S)^*(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{A}(-\frac{1}{d}\infty)) \longrightarrow \mathcal{E}$$

such that the cokernel of t is supported on the graph $\Gamma_z \subseteq X \times_{\operatorname{Spec} \mathbb{F}_q} S$ of a morphism $z : S \to X$ (called the zero) and is – when considered as a sheaf on $S \cong \Gamma_z$ – a locally free $\mathcal{O}_S$-module of rank d.
Here $\mathcal{A}(-\frac{1}{d}\infty)$ denotes the two-sided ideal in $\mathcal{A}$ given by $\mathcal{A}(-\frac{1}{d}\infty)_x = \mathcal{A}_x$ for all $x \neq \infty$ and $\mathcal{A}(-\frac{1}{d}\infty)_\infty = \mathrm{Rad}(\mathcal{A}_\infty)$. This definition differs, but, as will be

proved in the appendix, is equivalent to the one given in [LRS][1]. Unlike in loc. cit. we do not require the zero z to be disjoint from the pole ∞ nor from the closed points which are ramified in A. Also we allow ∞ to be ramified in A. For an arbitrary effective divisor I on X there is the notion of a level-I-structure on E. We will show (Theorem 4.11) that the moduli stack of $\mathcal{A}$-elliptic sheaves with level-I-structure $\mathcal{E}\ell\ell_{\mathcal{A},I}^{\infty}$ is a Deligne- Mumford stack which is locally of finite type and of relative dime nsion $d-1$ over $X-I$. If $I \neq 0$, it is a smooth and quasiprojective scheme over $X'-I$ where X' denotes the complement of set of closed points $x \in X$ with $e_x(\mathcal{A}) > 1$.

Let B be another central simple F-algebra of dimension d^2 and assume that there exists a closed point $\mathfrak{p} \in X - \{\infty\}$ such that the local invariants of B are given by $\mathrm{inv}_{\infty}(B) = \mathrm{inv}_{\infty}(A) + \frac{1}{d}$, $\mathrm{inv}_{\mathfrak{p}}(B) = \mathrm{inv}_{\mathfrak{p}}(A) - \frac{1}{d}$ and $\mathrm{inv}_x(B) = \mathrm{inv}_x(A)$ for all $x \neq \infty, \mathfrak{p}$. Let $\mathcal{B}$ be a locally principal hereditary $\mathcal{O}_X$-order in B with $e_x(\mathcal{B}) = e_x(\mathcal{A})$ for all x. Our main result is that the moduli stack $\mathcal{E}\ell\ell_{\mathcal{B},I}^{\mathfrak{p}}$ is a twist of $\mathcal{E}\ell\ell_{\mathcal{A},I}^{\infty}$. To state this more precisely we assume for simplicity that $\deg(\mathfrak{p}) = 1$ and $I = 0$ (see 4.24 and 4.25 for the general statement). We have

$$\mathcal{E}\ell\ell_{\mathcal{B}}^{\mathfrak{p}} \quad \cong \quad (\mathcal{E}\ell\ell_{\mathcal{A}}^{\infty} \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q)/ < w_{\mathfrak{p}} \otimes \mathrm{Frob}_q > . \tag{3}$$

Here $w_{\mathfrak{p}}$ is a certain modular automorphism of $\mathcal{E}\ell\ell_{\mathcal{A}}^{\infty}$ (in the case $d=2$ it is the analogue of the Atkin-Lehner involution at p for a modular or a Shimura curve).

We explain briefly our strategy for proving (3). We consider invertible $\mathcal{A}$-$\mathcal{B}$-bimodules $\mathcal{L}$ together with a meromorphic Frobenius Φ having a simple zero at ∞ and simple pole at $\mathfrak{p}$. More precisely, for an $\mathbb{F}_q$-scheme S, we consider pairs $L = (\mathcal{L}, \Phi)$ where $\mathcal{L}$ is an invertible $\mathcal{A} \boxtimes \mathcal{O}_S$-$\mathcal{B} \boxtimes \mathcal{O}_S$-bimodule and Φ is an isomorphism of bimodules

$$\Phi : (\mathrm{id}_X \times \mathrm{Frob}_S)^*(\mathcal{L} \otimes_{\mathcal{A}} \mathcal{A}(-\frac{1}{d}\mathfrak{p})) \longrightarrow \mathcal{L} \otimes_{\mathcal{A}} \mathcal{A}(-\frac{1}{d}\infty).$$

These will be called *invertible Frobenius bimodules of slope* $D = \frac{1}{d}\infty - \frac{1}{d}\mathfrak{p}$ and their moduli space will be denoted by $\mathrm{SE}_{\mathcal{A},\mathcal{B}}^{D}$. We will show in section 4.4 that it is a torsor over $\mathrm{Spec}\,\mathbb{F}_q$ of the finite group of modular automorphisms of $\mathcal{E}\ell\ell_{\mathcal{A}}^{\infty}$ and compute it explicitely (it is instructive to view $\mathrm{SE}_{\mathcal{A},\mathcal{B}}^{D}$ as an analogue of the moduli space of supersingular elliptic curves with a fixed ring of endomorphisms). In section 4.5 we construct a canonical tensor product $\mathcal{E}\ell\ell_{\mathcal{A}}^{\infty} \times \mathrm{SE}_{\mathcal{A},\mathcal{B}}^{D} \to \mathcal{E}\ell\ell_{\mathcal{B}}^{\mathfrak{p}}, (E, L) \mapsto E \otimes_{\mathcal{A}} L$. The isomorphism (3) is then a simple consequence.

From the global result (3) we deduce that the uniformization at ∞ and the analogue of the Cherednik-Drinfeld uniformization for the moduli spaces $\mathcal{E}\ell\ell_{\mathcal{A},I}^{\infty}$ are equivalent. In fact an analogue of the uniformization result (1) for $\mathcal{E}\ell\ell_{\mathcal{A},I}^{\infty}$ has been proved by Blum and Stuhler in [BS] (in case where the level I is prime to ∞). On the other hand Hausberger has shown in [Hau] (again under the

[1]In [LRS], the authors work with hereditary orders $\mathcal{A}$ with $\mathcal{A}_{\infty} \cong M_d(\widehat{\mathcal{O}}_{X,\infty})$ and parabolic structures at ∞ on $\mathcal{E}$ instead.

assumption that ∞ does not divide the level I) that there is also an analogue of the Cherednik-Drinfeld theorem.
We describe briefly the contents of each section. In section 2 and 3 we discuss hereditary orders in central simple algebras over local fields and global function fields. We show in particular that any hereditary order is Morita equivalent to a (locally) principal hereditary order. This is the reason why it suffices to consider $\mathcal{A}$-elliptic sheaves for locally principal $\mathcal{A}$. In section 4 we introduce $\mathcal{A}$-elliptic sheaves and study their moduli spaces and sections 2.6 and 4.4 are devoted to invertible Frobenius bimodules. In section 3.3 we introduce the notion of a *special* $\mathcal{A}$*-module*. If $E = (\mathcal{E}, t)$ is an $\mathcal{A}$-elliptic sheaf then $\mathrm{Coker}(t)$ is special. The stack $\mathrm{Coh}_{\mathcal{A},\mathrm{sp}}$ of special $\mathcal{A}$-modules plays a key role in the study of the bad fibers of the characteristic morphism $char : \mathcal{E}\ell\ell^{\infty}_{\mathcal{A},I} \to X$ in section 4.3. In fact $\mathrm{Coh}_{\mathcal{A},\mathrm{sp}}$ is an Artin stack and *char* admits a canonical factorization $\mathcal{E}\ell\ell^{\infty}_{\mathcal{A},I} \to \mathrm{Coh}_{\mathcal{A},\mathrm{sp}} \to X$. We shall show that the first map is smooth and the second semistable. In section 4.5 we construct the tensor product of an $\mathcal{A}$-elliptic sheaf (with level-I-structure) and an invertible Frobenius bimodules (with level-I-structure) and prove our main result (Theorems 4.24 and 4.25). Finally, in section 4.6 we discuss the application to uniformization of $\mathcal{E}\ell\ell^{\infty}_{\mathcal{A},I}$ by Drinfeld's symmetric spaces and its coverings.
Acknowledgements. I thank E. Lau and T. Zink for helpful conversations. Part of this work has been done during a stay at the Max-Planck-Institute in Bonn in the Winter 2004/05, so I am grateful for the hospitality.

Notation As an orientation for the reader we collect here a few basic notations which are used in the entire work. However most notations listed below will be introduced again somewhere in this work.
For a scheme S we let $|S|$ be the set of closed points of S. The category of S-schemes is denoted by Sch/S. If $S = \mathrm{Spec}\, k$ for a field k then we also write Sch/k.
The algebraic closure of a field k is denote by $\overline{k}$. If k is finite then $k_n \subset \overline{k}$ denotes the extension of degree n of k.
In chapters 3, 4 and in 5.2, X denotes a smooth proper curve over some base field k. In chapter 3, k is an arbitrary perfect field of cohomological dimension 1, whereas in chapter 3 k is the finite field $\mathbb{F}_q$. The function field of X is denoted by F. For $Y, Z \in \mathrm{Sch}/k$ we write $X \times Y$ for their product over k.
For a closed point $x \in X$ we denote by $k(x)$ its residue field and by $\deg(x)$ the degree $[k(x) : k]$. If S is a $k(x)$-scheme, then x_S will denote the morphism $S \to \mathrm{Spec}\, k(x) \hookrightarrow X$. If $S = \mathrm{Spec}\, k'$ is a field then we also write $x_{k'}$ instead of $x_{\mathrm{Spec}\, k'}$.
For a non-zero effective divisor I on X, we denote the corresponding closed subscheme of X by I as well. If $\mathcal{M}$ is a sheaf of $\mathcal{O}_X$-modules then we use $\mathcal{M}_I$ for $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_I$.
In chapter 4, for $S \in \mathrm{Sch}/\mathbb{F}_q$ we denote by Frob_S its Frobenius endomorphism (over $\mathbb{F}_q$). In the case where $S = \mathrm{Spec}\, k'$ for some algebraic extension field k' of $\mathbb{F}_q$ we also sometimes write Frob_q for $\mathrm{Frob}_{\mathrm{Spec}\, k'}$ and Frobenius in the Galois

group $G(k'/\mathbb{F}_q)$. If $S \in \mathrm{Sch}/\mathbb{F}_q$ and $\mathcal{E}$ is a sheaf of $\mathcal{O}_{X\times S}$-modules then ${}^\tau\mathcal{E}$ denotes the sheaf $(\mathrm{id}_X \times \mathrm{Frob}_S)^*(\mathcal{E})$.
We denote by $\mathbb{A}$ the Adele ring of F and for a finite set of closed points T of X we let $\mathbb{A}^T$ denote the Adele ring outside of T.

Contents

2 Local theory of hereditary orders

2.1 Basic definitions

Let X be a scheme and $\mathcal{A}$ a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. We denote by $\mathrm{Mod}_{\mathcal{A}}$ the category of sheaves of right $\mathcal{A}$-modules. Let $\mathcal{B}$ be another quasi-coherent $\mathcal{O}_X$-algebra. An $\mathcal{A}$-$\mathcal{B}$-bimodule $\mathcal{I}$ is an $\mathcal{O}_X$-module with a left $\mathcal{A}$- and right $\mathcal{B}$-action which are compatible with the $\mathcal{O}_X$-action.
$\mathcal{A}$ and $\mathcal{B}$ are said to be *(Morita) equivalent* (notation: $\mathcal{A} \simeq \mathcal{B}$) if there exists a quasi-coherent $\mathcal{A}$-$\mathcal{B}$-bimodule $\mathcal{I}$ and a quasi-coherent $\mathcal{B}$-$\mathcal{A}$-bimodule $\mathcal{J}$ such that the following equivalent conditions hold:

(i) There exists bimodule isomorphisms

$$\mathcal{I} \otimes_{\mathcal{B}} \mathcal{J} \longrightarrow \mathcal{A}, \qquad \mathcal{J} \otimes_{\mathcal{A}} \mathcal{I} \longrightarrow \mathcal{B}.$$

(ii) The functors

$$\cdot \otimes_{\mathcal{A}} \mathcal{I} : \mathrm{Mod}_{\mathcal{A}} \longrightarrow \mathrm{Mod}_{\mathcal{A}}, \qquad \cdot \otimes_{\mathcal{B}} \mathcal{J} : \mathrm{Mod}_{\mathcal{B}} \longrightarrow \mathrm{Mod}_{\mathcal{A}}$$

are equivalences of categories and mutually quasi-invers.
In this case $\mathcal{I}$ and $\mathcal{J}$ are called *invertible* bimodules and $\mathcal{J}$ is called the *inverse* of $\mathcal{I}$. The group of isomorphism classes of invertible $\mathcal{A}$-$\mathcal{A}$-bimodules will be denoted by $\mathrm{Pic}(\mathcal{A})$.
Now assume that X is a Dedekind scheme that is a one-dimensional connected regular noetherian scheme with function field K, i.e. $\mathrm{Spec}\, K \to X$ is the generic point. Let A be a central simple algebra over K. An $\mathcal{O}_X$*-order* in A is a sheaf of $\mathcal{O}_X$-algebras $\mathcal{A}$ with generic fiber A which is coherent and locally free as an $\mathcal{O}_X$-module. If $\mathcal{B}$ is an $\mathcal{O}_X$-order in another central simple K-algebra then it is easy to see that an invertible $\mathcal{A}$-$\mathcal{B}$-bimodule is a coherent and locally free $\mathcal{O}_X$-module.
The $\mathcal{O}_X$-order $\mathcal{A}$ in A is called *maximal* if for any open affine $U = \mathrm{Spec}\, R \subseteq X$ the set of sections $\Gamma(U, \mathcal{A})$ is a maximal R-order in A. $\mathcal{A}$ is called *hereditary* if its sections $\Gamma(U, \mathcal{A})$ over any open affine $U = \mathrm{Spec}\, R \subseteq X$ is a hereditary R-order in A that is any left ideal in $\Gamma(U, \mathcal{A})$ is projective (equivalently any right ideal is projective; compare ([Re], (10.7)). Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of finite rank which has a left or right $\mathcal{A}$-action compatible with the $\mathcal{O}_X$-action. Then the set of sections of $\mathcal{E}$ over any affine open $U = \mathrm{Spec}\, R \subseteq X$ are a projective $\Gamma(U, \mathcal{A})$-module.
If X is affine, i.e. the spectrum of a Dedekind ring $\mathcal{O}$ we usually identify $\mathcal{A}$ with its sections $\Gamma(X, \mathcal{A})$. An $\mathcal{O}$*-lattice* is a finitely generated torsionfree (hence projective) $\mathcal{O}$-module. A (left or right) $\mathcal{A}$*-lattice* is a (left or right) $\mathcal{A}$-module which is an $\mathcal{O}$-lattice. By ([Re], (10.7)) $\mathcal{A}$ is hereditary if and only if every (left or right) $\mathcal{A}$-lattice is projective.

2.2 Structure theory

Let $\mathcal{O}$ be a henselian discrete valuation ring with maximal ideal $\mathfrak{p}$ and residue field $k = \mathcal{O}/\mathfrak{p}$. Let $\varpi \in \mathfrak{p}$ be a fixed prime element. We will recall the structure theory of hereditary $\mathcal{O}$-orders in central simple K-algebras (a reference for what follows is [Re], section 39). Since we are only interested in applications to the case where $\mathcal{O}$ is the henselisation or completion of a local ring in a global field we will assume for simplicity that k is perfect and of cohomological dimension ≤ 1.
Let $\mathcal{A}$ be a hereditary $\mathcal{O}$-order in a central simple K-algebra A of dimension n^2. Its Jacobson radical will be denoted by $\mathfrak{P} = \mathfrak{P}_{\mathcal{A}}$. By ([Re], 39.1 and exercise 6 on p. 365) $\mathfrak{P}$ is an invertible two-sided ideal and any other two-sided invertible fractional ideal is an integral power of $\mathfrak{P}$. Let B be a central

simple K-algebra equivalent to A and $\mathcal{B}$ be a maximal order in B. We denote its radical by $\mathfrak{M} = \mathfrak{P}_{\mathcal{B}}$. Let I be an invertible A-B-bimodule. Its inverse is $J\colon = \mathrm{Hom}_K(I, K)$. Let $\mathcal{I}$ be a $\mathcal{A}$-$\mathcal{B}$-stable lattice in I, i.e. $a\mathcal{I}, \mathcal{I}b \subseteq \mathcal{I}$ for all $a \in \mathcal{A}, b \in \mathcal{B}$. Such a lattice exists. In fact if $\mathcal{L} \subseteq I$ is any $\mathcal{O}$-lattice then the $\mathcal{O}$-module generated by the set $\{axb \mid a \in \mathcal{A}, x \in \mathcal{L}, b \in \mathcal{B}\}$ is a $\mathcal{A}$-$\mathcal{B}$-stable lattice. There exists a positive integer t – called the *type* of $\mathcal{A}$ – such that $\mathfrak{P}^t\mathcal{I} = \mathcal{I}\mathfrak{M}$ (see [Re], 39.18 (i)). It is also equal to the number of isomorphism classes of indecomposable left (or right) $\mathcal{A}$-lattices. If $\mathcal{M}$ is an indecomposable left $\mathcal{A}$-lattice then $\{\mathfrak{P}^i\mathcal{M} \mid i = 0, 1, \ldots, t-1\}$ is a full set of representatives of the set of indecomposable left $\mathcal{A}$-lattices. For $i \in \mathbb{Z}$ we set $\mathcal{I}_i\colon = \mathfrak{P}^{-i}\mathcal{I}$ and $\mathcal{J}_i\colon = \mathrm{Hom}_{\mathcal{O}}(\mathcal{I}_{-i}, \mathcal{O})$. The sequences $\{\mathcal{I}_i \mid i \in \mathbb{Z}\}$ and $\{\mathcal{J}_i \mid i \in \mathbb{Z}\}$ satisfy the following conditions:

(i) $\mathfrak{P}\mathcal{I}_i = \mathcal{I}_{i-1},\ \mathcal{I}_i\mathfrak{M} = \mathcal{I}_{i-t},\ \mathcal{J}_i\mathfrak{P} = \mathcal{J}_{i-1},\ \mathfrak{M}\mathcal{J}_i = \mathcal{J}_{i-t}$ for all $i \in \mathbb{Z}$.

(ii) Let $\mathcal{A}_i\colon = \{x \in A \mid x\mathcal{I}_i \subseteq \mathcal{I}_i\} = \{x \in A \mid \mathcal{J}_{-i}x \subseteq \mathcal{J}_{-i}\}$. Then $\mathcal{A}_1, \ldots, \mathcal{A}_t$ are the different maximal orders containing $\mathcal{A}$ and we have $\mathcal{A} = \mathcal{A}_1 \cap \ldots \cap \mathcal{A}_t$ (note that $\mathcal{A}_i = \mathcal{A}_j$ if $i \equiv j \mod t$). The lattice $\mathcal{I}_i$ is an invertible $\mathcal{A}_i$-$\mathcal{B}$-bimodule with inverse $\mathcal{J}_{-i}$. Note that $\mathcal{A}_i = \mathcal{A}_j$ if $i \equiv j \mod t$.

(iii) Let $\overline{\mathcal{A}}\colon = \mathcal{A}/\mathfrak{P}, \overline{\mathcal{B}}\colon = \mathcal{B}/\mathfrak{M}$ and let

$$\overline{\mathcal{A}}^{(i)}\colon = \mathrm{Im}(\overline{\mathcal{A}} \to \mathrm{End}_{\overline{\mathcal{B}}}(\mathcal{I}_i/\mathcal{I}_{i-1})) \cong \mathrm{Im}(\overline{\mathcal{A}} \to \mathrm{End}_{\overline{\mathcal{B}}}(\mathcal{J}_{-i}/\mathcal{J}_{-i-1}))$$

for $i = 1, \ldots, t$. Then, considered as a $\overline{\mathcal{A}}^{(i)}$-$\overline{\mathcal{B}}$-bimodule, $\mathcal{I}_i/\mathcal{I}_{i-1}$ is invertible with inverse $\mathcal{J}_{-i}/\mathcal{J}_{-i-1}$. We have

$$\overline{\mathcal{A}} \cong \overline{\mathcal{A}}^{(1)} \times \ldots \times \overline{\mathcal{A}}^{(t)}$$

and $\overline{\mathcal{A}}^{(i)} \cong M_{n_i}(k')$ for $i = 1, \ldots, t$. Here k' is the center of $\overline{\mathcal{B}}$ and $n_i = \mathrm{rank}_{\overline{\mathcal{B}}}(\mathcal{I}_i/\mathcal{I}_{i-1})$. The numbers $(n_1, \ldots, n_t)$ are called the *invariants* of $\mathcal{A}$. They are well-defined up to cyclic permutation.

Definition 2.1. *The positive integer $e = e(\mathcal{A})$ with $\mathfrak{P}^e = \varpi\mathcal{A}$ will be called the index of $\mathcal{A}$.*

We will see below (Lemma 2.4) that $e(\mathcal{A})$ does not change under finite étale base change. If d is the order of $[A]$ in $Br(F)$ (hence $d = [k' : k]$) and t is the type of $\mathcal{A}$ then $e = dt$.
Recall ([BF], p. 216) that $\mathcal{A}$ is said to be *principal* if every two-sided invertible ideal of $\mathcal{A}$ is a principal ideal or equivalently if there exists $\Pi \in \mathfrak{P}$ with $\mathcal{A}\Pi = \Pi\mathcal{A} = \mathfrak{P}$. For example $\mathcal{A}$ is principal if it is a maximal $\mathcal{O}$-order in A or if $e(\mathcal{A}) = n$. This is a consequence of the following characterization of principal orders.

Lemma 2.2. *Let $\mathcal{A}$ be a hereditary order in a central simple K-algebra A of dimension n^2. The following conditions are equivalent.*

(i) $\mathcal{A}$ is a principal order.
(ii) If $(n_1, \ldots, n_t)$ are the invariants of $\mathcal{A}$ then $n_1 = \ldots = n_t$.
(iii) Let $\mathcal{M}_1, \ldots, \mathcal{M}_t$ be a full set of representatives of the isomorphism classes of indecomposable right $\mathcal{A}$-lattices. Then there exists an integer $f \in \mathbb{N}$ such that

$$\mathcal{A} \cong (\mathcal{M}_1 \oplus \ldots \oplus \mathcal{M}_t)^f$$

as right $\mathcal{A}$-modules. In this case we have $f = n_1 = \ldots = n_t$ and $n = ef$.

Proof. (i) $\Leftrightarrow$ (ii) see ([BF], Theorem 1.3.2, p. 217).
(i) $\Leftrightarrow$ (iii) Since $\mathcal{A}$ is principal if and only if $\mathcal{A} \cong \mathfrak{P}$ as right $\mathcal{A}$-modules this follows from the fact that the map $[\mathcal{M}] \mapsto [\mathcal{M}\mathfrak{P}]$ is a cyclic permutation of the set isomorphism classes of indecomposable right $\mathcal{A}$-lattices ([Re], 39.23).
For the last assertion note that if $\mathcal{A}$ is principal then on the one hand

$$\dim_k(\mathcal{A}/\mathfrak{P}) = \sum_{j=1}^{t} \dim_k(\overline{\mathcal{A}}^{(j)}) = \sum_{j=1}^{t} dn_j^2 = tdn_i^2 = en_i^2$$

for $i \in \{1, \ldots, t\}$. On the other hand since $\mathcal{M}_j/\mathcal{M}_j\mathfrak{P}$ is an irreducible $\overline{\mathcal{A}}^{(j)}$-module we have

$$\dim_k(\mathcal{A}/\mathfrak{P}) = f \sum_{j=1}^{t} \dim_k(\mathcal{M}_j/\mathcal{M}_j\mathfrak{P}) = ftdn_i = fen_i$$

Therefore we get $f = n_i$. Finally because of

$$n^2 = \dim_k(\mathcal{A}/\varpi\mathcal{A}) = \sum_{i=0}^{e-1} \dim_k(\mathfrak{P}^i/\mathfrak{P}^{i+1}) = e \dim_k(\mathcal{A}/\mathfrak{P})$$

we obtain $ef = n$. $\square$

Suppose that $\mathcal{A}$ is principal. We denote the subgroup of A^* of elements $x \in A^*$ with $x\mathcal{A} = \mathcal{A}x$ by $N(\mathcal{A})$. For $x \in N(\mathcal{A})$ there exists a unique $m \in \mathbb{Z}$ with $x\mathcal{A} = \mathfrak{P}^m$ and we set $v_{\mathcal{A}}(x) = \frac{m}{e}$. We have a commuative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{O}^* & \longrightarrow & K^* & \xrightarrow{v_K} & \mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathcal{A}^* & \longrightarrow & N(\mathcal{A}) & \xrightarrow{v_{\mathcal{A}}} & \frac{1}{e}\mathbb{Z} & \longrightarrow & 0 \end{array}$$

where v_K denoted the normalized valuation of K and the vertical maps are the natural inclusions.
Next we consider the special case where $A = \mathrm{End}_K(V)$ for a finite-dimensional K-vector space V (i.e. A is split). A *lattice chain* in V is a sequence of $\mathcal{O}$-lattices $\mathcal{L}_\star = \{\mathcal{L}_i \mid i \in \mathbb{Z}\}$ such that

(i) $\mathcal{L}_i \subseteq \mathcal{L}_{i+1}$ for all $i \in \mathbb{Z}$.

(ii) There exists a positive integer e, the *period* of $\mathcal{L}_\star$, such that $\mathcal{L}_{i-e} = \varpi\mathcal{L}_i$ for all $i \in \mathbb{Z}$.

The ring

$$\mathcal{A} = \mathrm{End}(\mathcal{L}_\star) : = \{f \in A \mid f(\mathcal{L}_i) \subseteq \mathcal{L}_i \ \forall i \in \mathbb{Z}\} \tag{4}$$

is a hereditary $\mathcal{O}$-order in A of index (= type) e with invariants $n_i = \dim_k(L_i/L_{i-1})$. We have:

$$\mathfrak{P}_{\mathcal{A}}^{-m} = \mathrm{End}^m(\mathcal{L}_\star) : = \{f \in A \mid f(\mathcal{L}_i) \subseteq \mathcal{L}_{i+m} \ \forall i \in \mathbb{Z}\}. \tag{5}$$

Any hereditary $\mathcal{O}$-order in A is of the form (4) for some lattice chain.

2.3 Étale base change

We keep the notation and assumption of the last section. Let A be a central simple algebra and $\mathcal{A}$ an $\mathcal{O}$-order in A with radical $\mathfrak{P}$.

Lemma 2.3. *The following conditions are equivalent:*
(i) $\mathcal{A}$ is hereditary.
(ii) There exists a two-sided invertible ideal $\mathfrak{M}$ in $\mathcal{A}$ such that $\mathcal{A}/\mathfrak{M}$ is semisimple and $\mathfrak{M}^e = \varpi\mathcal{A}$ for some $e \geq 1$.
Moreover if $\mathfrak{M}$ is as in (ii) then $\mathfrak{M} = \mathfrak{P}$.

Proof. (i) $\Rightarrow$ (ii) follows from ([Re], (39.18) (iii)) (for $\mathfrak{M} = \mathfrak{P}$).
(ii) $\Rightarrow$ (i) In view of ([Re], (39.1)) it suffices to show that $\mathfrak{M} = \mathfrak{P}$. The inclusion $\mathfrak{M} \supseteq \mathfrak{P}$ is a consequence of the assumption that $\mathcal{A}/\mathfrak{M}$ is semisimple. The converse inclusion follows from ([Re], exercise 1). □

Lemma 2.4. *Let K'/K be a finite unramified extension and $\mathcal{O}'$ the integral closure of $\mathcal{O}$ in K'. Then $\mathcal{A}$ is hereditary (resp. principal) if only if $\mathcal{A} \otimes_{\mathcal{O}} \mathcal{O}'$ is hereditary (resp. principal). In this case $\mathfrak{P} \otimes_{\mathcal{O}} \mathcal{O}'$ is the radical of the latter.*

Proof. We will prove only the statement for hereditary orders and leave the case of principal orders to the reader. If $\mathcal{A}$ is hereditary then $\mathfrak{M} : = \mathfrak{P} \otimes_{\mathcal{O}} \mathcal{O}'$ satisfies the condition (ii) of Lemma 2.3. Hence $\mathcal{A} \otimes_{\mathcal{O}} \mathcal{O}'$ is hereditary.
To prove the converse let $\mathcal{P}$ be a left $\mathcal{A}$-lattice. We have to show that

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{P}, \cdot) : \mathrm{Mod}_{\mathcal{A}} \to \mathrm{Mod}_{\mathcal{O}}$$

is an exact functor or – since $\mathcal{O}'$ is a faithfully flat $\mathcal{O}$-algebra – that

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{P}, \cdot) \otimes_{\mathcal{O}} \mathcal{O}' \cong \mathrm{Hom}_{\mathcal{A} \otimes_{\mathcal{O}} \mathcal{O}'}(\mathcal{P} \otimes_{\mathcal{O}} \mathcal{O}', \cdot \otimes_{\mathcal{O}} \mathcal{O}')$$

is exact. However the assumption implies that $\mathcal{P} \otimes_{\mathcal{O}} \mathcal{O}'$ is a projective $\mathcal{A} \otimes_{\mathcal{O}} \mathcal{O}'$-module. □

2.4 Morita equivalence

Let A be a central simple algebra and $\mathcal{A}$ a hereditary $\mathcal{O}$-order in A with radical $\mathfrak{P}$. If $\mathcal{A}'$ is another $\mathcal{O}$-order in A containing $\mathcal{A}$ then $\mathcal{A}'$ is hereditary as well and $\mathfrak{P}_{\mathcal{A}'}\subseteq\mathfrak{P}$.

Lemma 2.5. *Let $\mathcal{A}_1, \ldots, \mathcal{A}_s$ be a collection of $\mathcal{O}$-orders in A containing $\mathcal{A}$ with radicals $\mathfrak{P}_1, \ldots, \mathfrak{P}_s$. If $\mathcal{A}_1 \cap \ldots \cap \mathcal{A}_s = \mathcal{A}$ then $\mathfrak{P}_1 + \ldots + \mathfrak{P}_s = \mathfrak{P}$.*

Proof. Clearly $\mathfrak{P}_1 + \ldots + \mathfrak{P}_s \subseteq \mathfrak{P}$. By Lemma 2.4 to prove equality we may pass to a finite unramified extension K'/K. Hence we can assume $A = \mathrm{End}_K(V)$ for some finite-dimensional K vector space V and that there exists a lattice chain $\mathcal{L}_\star = \{\mathcal{L}_i \mid i \in \mathbb{Z}\}$ in V with period $e = e(\mathcal{A})$ such that

$$\mathcal{A} = \{f \in A \mid f(\mathcal{L}_i)\subseteq\mathcal{L}_i \ \forall i \in \mathbb{Z}\}, \quad \mathfrak{P} = \{f \in A \mid f(\mathcal{L}_i)\subseteq\mathcal{L}_{i-1} \ \forall i \in \mathbb{Z}\}.$$

Clearly it is enough to consider the case where $s = e$ and $\mathcal{A}_i = \{f \in A \mid f(\mathcal{L}_i)\subseteq\mathcal{L}_i\}$, $i = 1, \ldots, e$ are the different maximal orders containing $\mathcal{A}$. We proceed by induction on e so we can assume that $e > 1$ and that the radical

$$\mathfrak{P}' = \{f \in A \mid f(\mathcal{L}_i)\subseteq\mathcal{L}_{i-1} \ \forall\, i \not\equiv 0, 1 \bmod e \ \text{ and } f(\mathcal{L}_i)\subseteq\mathcal{L}_{i-2} \ \forall\, i \equiv 1 \bmod e\},$$

of $\mathcal{B}\colon = \mathcal{A}_1 \cap \ldots \cap \mathcal{A}_{e-1}$ is $= \mathfrak{P}_1 + \ldots + \mathfrak{P}_{e-1}$.
Let $f \in \mathfrak{P}$. Consider the diagram of k-vector spaces

$$\begin{tikzcd} \mathcal{L}_1/\mathcal{L}_0 \arrow[r, "\overline{f}"] \arrow[d, hook] & \mathcal{L}_0/\mathcal{L}_{-1} \\ \mathcal{L}_e/\mathcal{L}_0 \arrow[r, dashed, "\overline{g}"] & \mathcal{L}_0/\mathcal{L}_{-e} \arrow[u, two heads] \end{tikzcd}$$

where the vertical maps are induced by $\mathcal{L}_1 \hookrightarrow \mathcal{L}_e$ and $id : \mathcal{L}_0 \to \mathcal{L}_0$ respectively and the upper horizontal map by f. There exists a dotted arrow $\overline{g}$ making the diagram commutative. Let $g \in \mathrm{Hom}_{\mathcal{O}}(\mathcal{L}_e, \mathcal{L}_0) = \mathfrak{P}_e$ be a "lift" of $\overline{g}$. Then $g(x) \equiv f(x) \mod \mathcal{L}_{-1}$ for every $x \in \mathcal{L}_1$. Therefore $(f - g)(\mathcal{L}_1)\subseteq\mathcal{L}_{-1}$ and $(f - g)(\mathcal{L}_i)\subseteq f(\mathcal{L}_i) + g(\mathcal{L}_e)\subseteq\mathcal{L}_{i-1} + \mathcal{L}_0 = \mathcal{L}_{i-1}$ for $i = 2, \ldots, e - 1$ and consequently $f - g \in \mathfrak{P}'$. This proves $\mathfrak{P}\subseteq\mathfrak{P}' + \mathfrak{P}_e = \mathfrak{P}_1 + \ldots + \mathfrak{P}_{e-1} + \mathfrak{P}_e$. $\square$

Corollary 2.6. *Let $\mathcal{A}$ be a hereditary $\mathcal{O}$-order of type t with radical $\mathfrak{P}$ in the central simple K-algebra A and let $\mathcal{A}_1, \ldots, \mathcal{A}_t$ denote the different maximal orders containing $\mathcal{A}$. Then*

$$\mathcal{A}_1 + \ldots + \mathcal{A}_t = \mathfrak{P}^{-t+1}$$

is a two-sided invertible ideal.

Proof. By ([Re], section 39, exercise 10) and Lemma 2.5 above we have

$$\mathfrak{P} = \mathfrak{P}_1 + \ldots + \mathfrak{P}_t = \mathfrak{P}^t \mathcal{A}_1 + \ldots + \mathfrak{P}^t \mathcal{A}_t = \mathfrak{P}^t(\mathcal{A}_1 + \ldots + \mathcal{A}_t)$$

hence $\mathcal{A}_1 + \ldots + \mathcal{A}_t = \mathfrak{P}^{-t+1}$. □

Let B be another central simple K-algebra which is equivalent to A and let $\mathcal{B}$ a maximal order in B with radical $\mathfrak{M}$. Let I be an invertible A-B-bimodule, $J\colon = \mathrm{Hom}_K(I, K)$ and let $\{\mathcal{I}_i \mid i \in \mathbb{Z}\}$ and $\{\mathcal{J}_i \mid i \in \mathbb{Z}\}$ be as in section 2.2.

Lemma 2.7. *Consider $\mathcal{I}_i \otimes_{\mathcal{B}} \mathcal{J}_j$ (resp. $\mathcal{J}_i \otimes_{\mathcal{A}} \mathcal{I}_j$) as a submodule of $(\mathcal{I}_i \otimes_{\mathcal{B}} \mathcal{J}_j) \otimes_{\mathcal{O}} K = I \otimes_B J$ (resp. $J \otimes_A I$).*
(a) $\sum_{i+j=-t+1} \mathcal{I}_i \otimes_{\mathcal{B}} \mathcal{J}_j \cong \mathcal{A}$ as an $\mathcal{A}$-$\mathcal{A}$-bimodule.
(b) $\mathcal{J}_i \otimes_{\mathcal{A}} \mathcal{I}_j$ is an invertible bimodule. If $i + j = 0$ then $\mathcal{J}_i \otimes_{\mathcal{A}} \mathcal{I}_j \cong \mathcal{B}$ (as $\mathcal{B}$-$\mathcal{B}$-bimodule). We have

$$\mathcal{J}_{i+1} \otimes_{\mathcal{A}} \mathcal{I}_j = \mathcal{J}_i \otimes_{\mathcal{A}} \mathcal{I}_{j+1} = \begin{cases} \mathfrak{M}^{-1}(\mathcal{J}_i \otimes_{\mathcal{A}} \mathcal{I}_j) & \text{if } i+j \equiv 0 \mod r; \\ \mathcal{J}_i \otimes_{\mathcal{A}} \mathcal{I}_j & \text{if } i+j \not\equiv 0 \mod r \end{cases}$$

Proof. (a) Under the identification $I \otimes_B J = \mathrm{Hom}_K(J, K) \otimes_B J = \mathrm{Hom}_B(J, J)$ the submodule $\mathcal{I}_i \otimes_{\mathcal{B}} \mathcal{J}_j$ corresponds to $\mathrm{Hom}_{\mathcal{B}}(\mathcal{J}_{-i}, \mathcal{J}_j)$. Hence if we fix an A-A-bimodule isomorphism $I \otimes_B J \cong A$ so that $\mathrm{Hom}_{\mathcal{B}}(\mathcal{J}_0, \mathcal{J}_0)$ is mapped to $\mathcal{A}_0$ then for arbitrary $i, j \in \mathbb{Z}$ with $i + j = 0$ the module $\mathrm{Hom}_{\mathcal{B}}(\mathcal{J}_{-i}, \mathcal{J}_j)$ is mapped to $\mathcal{A}_i$. It follows $\sum_{i+j=0} \mathcal{I}_i \otimes_{\mathcal{B}} \mathcal{J}_j \cong \mathcal{A}_1 + \ldots + \mathcal{A}_r$ hence together with Lemma 2.5 the assertion.
(b) The proof of the first two statements is similar and will be left to the reader. For the last statement note that

$$\mathrm{Coker}(\mathcal{J}_i \otimes_{\mathcal{A}} \mathcal{I}_j \to \mathcal{J}_{i+1} \otimes_{\mathcal{A}} \mathcal{I}_j) \cong \mathcal{J}_{i+1}/\mathcal{J}_i \otimes_{\mathcal{A}} \mathcal{I}_j \cong \mathcal{J}_{i+1}/\mathcal{J}_i \otimes_{\overline{\mathcal{A}}} \mathcal{I}_j/\mathcal{I}_{j-1}$$

By (iii) above we have

$$\mathcal{J}_{i+1}/\mathcal{J}_i \otimes_{\overline{\mathcal{A}}} \mathcal{I}_j/\mathcal{I}_{j-1} \cong \mathcal{J}_{i+1}/\mathcal{J}_i \otimes_{\overline{\mathcal{A}}^{(j)}} \mathcal{I}_j/\mathcal{I}_{j-1} \cong \overline{\mathcal{B}}$$

if $i + j \equiv 0 \mod t$ and $\mathcal{J}_{i+1}/\mathcal{J}_i \otimes_{\overline{\mathcal{A}}} \mathcal{I}_j/\mathcal{I}_{j-1} = 0$ if $i + j \not\equiv 0 \mod t$. □

Corollary 2.8. *The assignment*

$$\mathcal{M} \mapsto \{\mathcal{M} \otimes_{\mathcal{A}} \mathcal{I}_i \mid i \in \mathbb{Z}\}$$

defines an equivalence between the category of right $\mathcal{A}$-lattices and the category of increasing chains $\{\mathcal{M}_i \mid i \in \mathbb{Z}\}$ of right $\mathcal{B}$-lattices such that $\mathcal{M}_i\mathfrak{M} = \mathcal{M}_{i-t}$ for all $i \in \mathbb{Z}$. A quasi-inverse is given by

$$\{\mathcal{M}_i \mid i \in \mathbb{Z}\} \mapsto \sum_{i+j=-t+1} \mathcal{M}_i \otimes_{\mathcal{B}} \mathcal{J}_j.$$

Here the sum is taken inside of $(\bigcup_{i \in \mathbb{Z}} \mathcal{M}_i) \otimes_B J$.

Proposition 2.9. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be hereditary $\mathcal{O}$-orders in central simple K-algebras A_1 and A_2. The following conditions are equivalent:*

(i) $\mathcal{A}_1$ and $\mathcal{A}_2$ are Morita equivalent.

(ii) A_1 and A_2 are equivalent and $\mathcal{A}_1$ and $\mathcal{A}_2$ have the same index.

Proof. We will show only that (ii) implies (i). The proof of the converse is easier and will be left to the reader. Suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ have the same period. Let D be a central division algebra over K equivalent to A_1 and A_2 and $\mathcal{D}$ be the maximal $\mathcal{O}$-order in D. For $\nu = 1, 2$ we fix increasing sequences of $\mathcal{A}_\nu$-$\mathcal{D}$- and $\mathcal{D}$-$\mathcal{A}_\nu$-bimodules $\{\mathcal{I}_i^{(\nu)} \mid i \in \mathbb{Z}\}$ and $\{\mathcal{I}_i^{(\nu)} \mid i \in \mathbb{Z}\}$ as in 2.2. Put $I^{(\nu)} = \bigcup_{i\in\mathbb{Z}} \mathcal{I}_i^{(\nu)}$ and $J^{(\nu)} = \bigcup_{i\in\mathbb{Z}} \mathcal{J}_i^{(\nu)}$. The assumption implies that $\mathcal{X} \colon = \sum_{i+j=-t+1} \mathcal{I}_i^{(1)} \otimes_{\mathcal{D}} \mathcal{J}_j^{(2)}$ is an $\mathcal{A}_1$-$\mathcal{A}_2$-lattice (the summation takes place in $I^{(1)} \otimes_D J^{(2)}$) and $\mathcal{Y} \colon = \sum_{i+j=-t+1} \mathcal{I}_i^{(2)} \otimes_{\mathcal{D}} \mathcal{J}_j^{(1)}$ a $\mathcal{A}_2$-$\mathcal{A}_1$-lattice. By Corollary 2.8 above the assignment $\mathcal{M} \mapsto \mathcal{M} \otimes_{\mathcal{A}_1} \; cX$ defines an equivalence between the category of right $\mathcal{A}_1$-lattices and the category of right $\mathcal{A}_2$-lattices. A quasi-inverse is given by $\mathcal{N} \mapsto \mathcal{N} \otimes_{\mathcal{A}_2} \mathcal{Y}$. This implies that $\mathcal{A}_1$ and $\mathcal{A}_2$ are Morita equivalent. In fact using Lemma 2.7 it is easy to see that the $\mathcal{X} \otimes_{\mathcal{A}_2} \mathcal{Y} \cong \mathcal{A}_1$ and $\mathcal{Y} \otimes_{\mathcal{A}_1} \mathcal{X} \cong \mathcal{A}_2$. $\square$

Recall that a right $\mathcal{A}$-lattice $\mathcal{M}$ is called *stably free* if there exists integers $r \geq 1, s \geq 0$ such that $\mathcal{M}^r \cong \mathcal{A}^s$.

Lemma 2.10. *Let $\mathcal{A}$ is a principal $\mathcal{O}$-order of index e in a central simple K-algebra of dimension n^2. Let $\mathcal{M}_1, \ldots, \mathcal{M}_t$ be representatives of isomorphism classes of indecomposable right $\mathcal{A}$-lattices. For a right $\mathcal{A}$-lattice $\mathcal{M} \neq 0$ the following conditions are equivalent.*

(i) $\mathcal{M}$ is stably free.

(ii) $\mathcal{M} \cong (\mathcal{M}_1 \oplus \ldots \oplus \mathcal{M}_t)^r$ for some positive integer r.

(iii) $\mathcal{D} \colon = \mathrm{End}_{\mathcal{A}}(\mathcal{M})$ is a principal $\mathcal{O}$-order of index e in a central simple K-algebra D.

Moreover in this case $\mathcal{A}$ and $\mathcal{D}$ are Morita equivalent and $\mathcal{M}$ is an invertible $\mathcal{D}$-$\mathcal{A}$-bimodule. If $\mathrm{rank}_{\mathcal{O}} \mathcal{M} = rne$ then $\dim_K(D) = (er)^2$.

Proof. The equivalence of (i) and (ii) follows immediately from Lemma 2.2.

(ii) $\Leftrightarrow$ (iii) By Lemma 2.4 we may pass to a finite unramified extension K'/K. Therefore we can assume that $A = \mathrm{End}_K(V)$ for an n-dimensional K-vector space V and $\mathcal{A} = \mathrm{End}(\mathcal{L}_\star)$ for a lattice chain $\mathcal{L}_\star$ with period e in V. There exists $r_1, \ldots, r_e \geq 0$ with

$$\mathcal{M} \cong \mathcal{L}_1^{r_1} \oplus \ldots \oplus \mathcal{L}_e^{r_1}$$

Since $\mathrm{Hom}_{\mathcal{A}}(\mathcal{L}_i, \mathcal{L}_j) \cong \mathfrak{p}^{\mu}$ with $i - j \leq \mu e < i - j + e$ we have

$$\mathrm{End}_{\mathcal{A}}(\mathcal{M}) \cong \begin{pmatrix} M_{r_1,r_1}(\mathcal{O}) & M_{r_1,r_2}(\mathcal{O}) & \ldots & M_{r_1,r_e}(\mathcal{O}) \\ M_{r_2,r_1}(\mathfrak{p}) & M_{r_2,r_2}(\mathcal{O}) & \ldots & M_{r_2,r_e}(\mathcal{O}) \\ \vdots & \vdots & \ddots & \vdots \\ M_{r_e,r_1}(\mathfrak{p}) & M_{r_e,r_2}(\mathfrak{p}) & \ldots & M_{r_e,r_e}(\mathcal{O}) \end{pmatrix}$$

By ([Re], 39.14) the order on the right is a hereditary order in $M_m(K)$ where $m = \sum_{i=1}^{e} r_i$. Its index is $= e$ if and only if $r_i \geq 1$ for all $i \in \{1, \ldots, e\}$ and in this case the invariants are $(r_1, \ldots, r_e)$. The equivalence of (ii) and (iii) follows. The proof of the last assertion will be left to the reader. □

Corollary 2.11. *Let $\mathcal{A}$ be as in 2.10 and let $\mathcal{M}$ be a stably free $\mathcal{A}$-module. We have:*
(a) $\mathrm{rank}_{\mathcal{O}}\, \mathcal{M}$ *is a multiple of en.*
(b) $\mathcal{M}$ is free if and only if $\mathrm{rank}_{\mathcal{O}}\, \mathcal{M}$ *is a multiple of n^2. In particular if $e = n$ then $\mathcal{M}$ is free.*

Proof. If $A \cong M_m(D)$ where D is the central division algebra equivalent to A then $\mathrm{rank}_{\mathcal{O}}\, \mathcal{M}_i = md^2$ with $d^2 = \dim_K(D)$. Hence if $\mathcal{M} \cong (\mathcal{M}_1 \oplus \ldots \oplus \mathcal{M}_t)^r$ for $r \in \mathbb{N}$ then $\mathrm{rank}_{\mathcal{O}}\, \mathcal{M} = rtmd^2 = ren$. The second assertion is obvious.
□

Corollary 2.12. *Let $\mathcal{A}$ and $\mathcal{B}$ be a principal orders in central simple K-algebras A and B and assume that* $\dim_K(A) = \dim_K(B) = n^2$ *and* $e(\mathcal{A}) = e(\mathcal{B}) = e$. *Let $\mathcal{I}$ be an $\mathcal{A}$-$\mathcal{B}$-bimodule. The following conditions are equivalent:*
(i) $\mathcal{I}$ is an invertible $\mathcal{A}$-$\mathcal{B}$-bimodule.
(ii) $\mathcal{I}$ is a free left $\mathcal{A}$-module of rank 1.
(iii) $\mathcal{I}$ is a free right $\mathcal{B}$-module of rank 1.

Proof. (i) ⇒ (iii) We show first that $\mathcal{I}$ is a lattice. Let $\mathcal{J}$ be an inverse of $\mathcal{I}$ and ${}_{\mathcal{B}-\mathrm{tor}}\mathcal{J}$ its $\mathcal{B}$-torsion ($\mathcal{A}$-)submodule. Since ${}_{\mathcal{B}-\mathrm{tor}}\mathcal{J} \otimes_{\mathcal{A}} \mathcal{I} \hookrightarrow \mathcal{J} \otimes_{\mathcal{A}} \mathcal{I} \cong \mathcal{B}$ we have ${}_{\mathcal{B}-\mathrm{tor}}\mathcal{J} \otimes_{\mathcal{A}} \mathcal{I} = 0$ and therefore ${}_{\mathcal{B}-\mathrm{tor}}\mathcal{J} = {}_{\mathcal{B}-\mathrm{tor}}\mathcal{J} \otimes_{\mathcal{A}} \mathcal{I} \otimes_{\mathcal{B}} \mathcal{J} = 0$. For $m \in \mathcal{J}, m \neq 0$ we get $\mathcal{B}m \cong \mathcal{B}$ as left $\mathcal{B}$-module and therefore

$$\mathcal{I} \cong \mathcal{I} \otimes_{\mathcal{B}} \mathcal{B}m \hookrightarrow \mathcal{I} \otimes_{\mathcal{B}} \mathcal{J} \cong \mathcal{A}.$$

Hence $\mathcal{I}$ is a lattice. Let $\mathcal{D}\colon = \mathrm{End}_{\mathcal{B}}(\mathcal{I}) \supseteq \mathcal{A}$. Thus $\mathcal{I}$ is a $\mathcal{D}$-$\mathcal{B}$-bimodule and so $\mathcal{I} \otimes_{\mathcal{B}} \mathcal{J} \cong \mathcal{A}$ is a $\mathcal{D}$-$\mathcal{A}$-bimodule. But $\mathrm{End}_{\mathcal{A}}(\mathcal{A}_{\mathcal{A}}) = \mathcal{A}$ and therefore $\mathcal{D} = \mathcal{A}$. By 2.10 and 2.11, $\mathcal{I}$ is a free $\mathcal{B}$-module of rank 1.
(iii) ⇒ (i) By 2.10, $\mathcal{D}$ is a principal order of index e in a central simple K-algebra D of dimension d^2 and $\mathcal{I}$ is an invertible $\mathcal{D}$-$\mathcal{B}$-bimodule. Since $\mathcal{D} \supseteq \mathcal{A}$ this implies $\mathcal{D} = \mathcal{A}$. □

Corollary 2.13. *Let $\mathcal{A}$ be a hereditary $\mathcal{O}$-order in central simple K-algebra A of index e. Then there exists a principal $\mathcal{O}$-order $\mathcal{D}$ in a central simple K-algebra which is Morita equivalent to $\mathcal{A}$. In fact that $\mathcal{D}$ can be chosen such that* $\operatorname{rank}_{\mathcal{O}}(\mathcal{D}) = e^2$.

Proof. Let $\mathcal{A}'$ be a principal $\mathcal{O}$-order of index e in $A' :\, = M_e(A)$ (since e^2 divides $\dim_K(A')$ and e is a multiple of the order of $[A'] = [A]$ in $\operatorname{Br}(F)$ such an order clearly exists). By Proposition 2.9 $\mathcal{A}'$ is Morita equivalent to $\mathcal{A}$. The second assertion follows immediately from 3.11. □

2.5 Maximal tori

Let A be a central simple K-algebra of dimension n^2 and $\mathcal{A}$ a hereditary $\mathcal{O}$-order in A with radical $\mathfrak{P}$. In this section we consider commutative étale $\mathcal{O}$-subalgebras of $\mathcal{A}$. Note that a commutative finite flat $\mathcal{O}$-algebra $\mathcal{T}$ is étale if and only if $\operatorname{Rad}(\mathcal{T}) = \varpi\mathcal{T}$.

Lemma 2.14. *Let $\mathcal{T}$ be a commutative étale $\mathcal{O}$-subalgebra of $\mathcal{A}$. Then we have* $\operatorname{Rad}(\mathcal{T}) = \mathcal{T} \cap \mathfrak{P}$.

Proof. Since $\mathcal{T}$ is a direct product of local $\mathcal{O}$-algebras $\mathcal{T} = \prod \mathcal{T}_i$ and $\operatorname{Rad}(\mathcal{T}) = \prod \operatorname{Rad}(\mathcal{T}_i)$ it suffices to prove the assertion for each factor. Thus we may assume that $\mathcal{T}$ is a local ring. Hence $\operatorname{Rad}(\mathcal{T})$ is the maximal ideal of $\mathcal{T}$ which implies $\mathcal{T} \cap \mathfrak{P} \subseteq \operatorname{Rad}(\mathcal{T})$. On the other hand, by the assumption, we have $\operatorname{Rad}(\mathcal{T}) = \varpi\mathcal{T}$ hence $\operatorname{Rad}(\mathcal{T}) \subseteq \mathcal{T} \cap \mathfrak{P}$. □

A commutative étale $\mathcal{O}$-subalgebra $\mathcal{T}$ of $\mathcal{A}$ is called *maximal torus* if $\operatorname{rank}_{\mathcal{O}} \mathcal{T} = n$. It follows immediately from the structure theory for hereditary $\mathcal{O}$-orders in central simple K-algebras ([Re], 39.14) that there exists a maximal torus in $\mathcal{A}$. We have the following characterization of maximal tori:

Lemma 2.15. *Let $\mathcal{T}$ be a commutative étale $\mathcal{O}$-subalgebra of $\mathcal{A}$. The following conditions are equivalent.*
(i) $\mathcal{T}$ is a maximal torus.
(ii) $\mathcal{T}$ is a maximal commutative étale $\mathcal{O}$-subalgebras of $\mathcal{A}$.
(iii) $\mathcal{T} = Z_{\mathcal{A}}(\mathcal{T}) = \{x \in \mathcal{A} \mid xt = tx \quad \forall t \in \mathcal{T}\}$.
(iv) $\mathcal{T}/\operatorname{Rad}(\mathcal{T})$ is a maximal commutative separable k-subalgebra of $\mathcal{A}/\mathfrak{P}$.

Proof. The simple proof of the equivalence of the first three conditions will be left to the reader.
(iii) $\Leftrightarrow$ (iv) By 2.14 above we have $\operatorname{Rad}(\mathcal{T}) = \mathcal{T} \cap \mathfrak{P} = \varpi\mathcal{T}$. Thus it follows from Lemma 5.1 of the appendix that (iv) holds if and only if $\operatorname{rank}_{\mathcal{O}} \mathcal{T} = \dim_k(\mathcal{T}/\varpi\mathcal{T}) = n$. □

Lemma 2.16. *(a) If $k = \mathbb{F}_q$ and $\mathcal{A}$ is a maximal order in A then $\mathcal{A}$ admits a maximal torus isomorphic to $\mathcal{O}_n$, the ring of integers of the unramified extension of degree n of K.*
(b) Let $\mathcal{O}'$ be a finite étale local $\mathcal{O}$-algebra and $\mathcal{T}$ be a maximal torus in $\mathcal{A}$. Then $\mathcal{T} \otimes_{\mathcal{O}} \mathcal{O}'$ is a maximal torus in $\mathcal{A} \otimes_{\mathcal{O}} \mathcal{O}'$.
(c) For any two maximal tori $\mathcal{T}, \mathcal{T}'$ of $\mathcal{A}$ there exists a finite étale local $\mathcal{O}$-algebra $\mathcal{O}'$ such that $\mathcal{T} \otimes_{\mathcal{O}} \mathcal{O}'$ and $\mathcal{T}' \otimes_{\mathcal{O}} \mathcal{O}'$ are conjugated (by some $a \in (\mathcal{A} \otimes_{\mathcal{O}} \mathcal{O}')^*$).

Proof. (a) and (b) are obvious.
To prove (c) we may pass to a finite unramified extension of K if necessary so that $A = \operatorname{End}_K(V)$ and $\mathcal{A} = \operatorname{End}(\mathcal{L}_\star)$ where V is a finite-dimensional K-vector space and $\mathcal{L}_\star$ is a lattice chain in V. We may also assume that $\mathcal{T} \cong \mathcal{O}^n \cong \mathcal{T}'$ where $n = \dim(V)$. Let e be the period of $\mathcal{L}_\star$ and let $\overline{\mathcal{L}}_i\colon = \mathcal{L}_i/\mathcal{L}_{i-1}$. Consider the $\overline{\mathcal{A}}\colon = \mathcal{A}/\mathfrak{P}$-module $\overline{\mathcal{L}}\colon = \bigoplus_{i=1}^e \overline{\mathcal{L}}_i$. As a $\overline{\mathcal{T}}\colon = \mathcal{T}/\operatorname{Rad}(\mathcal{T})$- and $\overline{\mathcal{T}}'\colon = \mathcal{T}'/\operatorname{Rad}(\mathcal{T}')$-module it is free of rank 1 (by Lemma 5.3 of the appendix). Hence there exists an isomorphism $\overline{\Theta} : \overline{\mathcal{T}} \to \overline{\mathcal{T}}'$ such that $\overline{\Theta}(\bar{t})x = \bar{t}x$ for all $\bar{t} \in \overline{\mathcal{T}}, x \in \overline{\mathcal{L}}$. We choose a lifting Θ of $\overline{\Theta}$ i.e. an isomorphism of $\mathcal{O}$-algebras $\Theta : \mathcal{T} \to \mathcal{T}'$ which reduces to $\overline{\Theta}$ modulo ϖ. Then for any $i \in \mathbb{Z}$ we h ave

$$\Theta(t)x = tx \qquad \text{for all } t \in \mathcal{T}, x \in \overline{\mathcal{L}}_i \tag{6}$$

Since $\mathcal{L}_0$ is a free $\mathcal{T}$- and $\mathcal{T}'$-module of rank 1 there exists $f \in \operatorname{Aut}_{\mathcal{O}}(\mathcal{L}_0) \subseteq A^*$ such that $f(tx) = \Theta(t)f(x)$ for all $t \in \mathcal{T}, x \in \mathcal{L}_0$. Hence $\Theta(t) = ftf^{-1}$ for all $t \in \mathcal{T}$ and therefore $\mathcal{T}' = f\mathcal{T}f^{-1}$. We claim that $f \in \mathcal{A}^*$, i.e. $f(\mathcal{L}_i) = \mathcal{L}_i$ for all $i \in \mathbb{Z}$. For that it is enough to see that $f(\mathcal{L}_i) \subseteq \mathcal{L}_i$ for all $i = 1, 2, \ldots e$ and in fact for $i = 1$ (by induction). Note that $f(\mathcal{L}_1) \subseteq f(\mathcal{L}_e) = \varpi^{-1} f(\mathcal{L}_0) = \mathcal{L}_e$. Choose $i \in \{1, 2, \ldots, e\}$ minimal with $f(\mathcal{L}_1) \subseteq \mathcal{L}_i$ and assume that $i \geq 2$. Then f induces a nontrivial $\mathcal{T}$-linear homomorphism $\bar{f} : \overline{\mathcal{L}}_1 \longrightarrow \overline{\mathcal{L}}_i$ such that

$$\bar{f}(tx) = \Theta(t)\bar{f}(x) = t\bar{f}(x) \qquad \text{for all } t \in \mathcal{T}, x \in \overline{\mathcal{L}}_i.$$

On the other hand since $\overline{\mathcal{L}}$ is a free $\overline{\mathcal{T}}$-module of rank 1 we have $\operatorname{Hom}_{\mathcal{T}}(\overline{\mathcal{L}}_1, \overline{\mathcal{L}}_i) = 0$, a contradiction. This proves $f \in \mathcal{A}^*$. □

We need the following two simple Lemmas in section 3.3.

Lemma 2.17. *Suppose that $\mathcal{A}$ is principal and let $\mathcal{T}$ be a maximal torus in $\mathcal{A}$. Let $\mathcal{M}$ be a $\mathcal{A}$-lattice and put $\overline{\mathcal{T}}\colon = \mathcal{T}/\operatorname{Rad}(\mathcal{T})$. The following conditions are equivalent.*
(i) $\mathcal{M}$ is stably free.
(ii) $\mathcal{M}/\mathfrak{P}\mathcal{M}$ is a free $\overline{\mathcal{T}}$-module.

The proof will be left to the reader.

Lemma 2.18. *Assume that $\mathcal{A}$ is principal and let $\mathcal{T}$ be a maximal torus in $\mathcal{A}$. Let $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{N} \to 0$ be a short exact sequence of $\mathcal{A}$-modules and*

assume that $\mathcal{M}$ is a stably free $\mathcal{A}$-lattice and $\varpi\mathcal{N} = 0$. The following conditions are equivalent.
(i) $\mathcal{M}'$ is stably free.
(ii) $\mathcal{N}$ is a free $\overline{\mathcal{T}}$-module.

Proof. By using the exact sequence

$$0 \longrightarrow \mathrm{Ker}(\mathcal{N} \otimes_{\mathcal{A}} \mathfrak{P} \to \mathcal{N}) \to \mathcal{M}'/\mathfrak{P}\mathcal{M}' \longrightarrow \mathcal{M}/\mathfrak{P}\mathcal{M} \longrightarrow \mathcal{N}/\mathfrak{P}\mathcal{N} \longrightarrow 0$$

we see that

$$\begin{aligned}[\mathcal{M}'/\mathfrak{P}\mathcal{M}'] &= [\mathcal{M}/\mathfrak{P}\mathcal{M}] + [\mathrm{Ker}(\mathcal{N} \otimes_{\mathcal{A}} \mathfrak{P} \to \mathcal{N})] - [\mathcal{N}/\mathfrak{P}\mathcal{N}] \\ &= [\mathcal{M}/\mathfrak{P}\mathcal{M}] + [\mathcal{N} \otimes_{\mathcal{A}} \mathfrak{P}] - [\mathcal{N}]\end{aligned}$$

in the Grothendieck group $K_0(\overline{\mathcal{T}})$. Note that $[\mathcal{N}] = [\mathcal{N} \otimes_{\mathcal{A}} \mathfrak{P}]$ if and only if $\mathcal{N}$ is a free $\overline{\mathcal{T}}$-module. Hence (ii) is equivalent to the equality $[\mathcal{M}'/\mathfrak{P}\mathcal{M}'] = [\mathcal{M}/\mathfrak{P}\mathcal{M}]$ in $K_0(\overline{\mathcal{T}})$. The assertion follows from 2.17. □

2.6 Local theory of invertible Frobenius bimodules

Let $\mathcal{O}$ be a henselian discrete valuation ring with quotient field K, maximal ideal $(\varpi) = \mathfrak{p}$ and residue field $k = \mathcal{O}/\mathfrak{p}$. We assume that k is finite of characteristic p. Let v_K be the normalized valuation of K. We denote by inv the canonical isomorphism $\mathrm{Br}(K) \to \mathbb{Q}/\mathbb{Z}$ of class field theory. Let $\mathcal{O}'$ be a finite étale local $\mathcal{O}$-algebra with quotient field K'. By $\sigma \in \mathrm{G}(K'/K)$ we denote Frobenius isomorphism (i.e. $\sigma(x) \equiv x^{\sharp(k)} \mod \mathfrak{p}$). For an $\mathcal{O}'$-module M we write ${}^{\sigma}M$ for $M \otimes_{\mathcal{O}',\sigma} \mathcal{O}'$ (or equivalently ${}^{\sigma}M = M$ with the new $\mathcal{O}'$-action $x \cdot m = \sigma(x)m$).
Let A be a central simple K-algebra of dimension d^2 and $\mathcal{A}$ a principal $\mathcal{O}$-order in A with radical $\mathfrak{P}$ and index $e = e(\mathcal{A})$ (note that we have $e\,\mathrm{inv}(A) = 0$). Let $\mathcal{M}$ be a free right $\mathcal{A}_{\mathcal{O}'}$-module of rank 1 together with an isomorphism of $\mathcal{A}_{\mathcal{O}'}$-modules

$$\phi : {}^{\sigma}\mathcal{M}\mathfrak{P}^m \longrightarrow \mathcal{M}$$

for some $m \in \mathbb{Z}$. We set

$$\mathcal{B} \colon = \mathrm{End}_{\mathcal{A}_{\mathcal{O}'}}(\mathcal{M}, \phi) = \{f \in \mathrm{End}_{\mathcal{A}_{\mathcal{O}'}}(\mathcal{M}) \mid \phi \circ f = {}^{\sigma}f \circ \phi\}.$$

Lemma 2.19. *The $\mathcal{O}$-algebra $\mathcal{B}$ is a principal order of index e in the central simple K-algebra $B \colon = \mathcal{B}_K$ of dimension d^2. We have*

$$\mathrm{inv}(B) = \mathrm{inv}(A) + \frac{m}{e} \mod \mathbb{Z} \tag{7}$$

Proof. Let $\phi_{K'} \colon = \phi \otimes_{\mathcal{O}'} \mathrm{id}_{K'} : {}^{\sigma}(M_{K'}) \to M_{K'}$. By Lemma 2.10 the $\mathcal{O}'$-algebra $\mathcal{B}' \colon = \mathrm{End}_{\mathcal{A}_{\mathcal{O}'}}(\mathcal{M})$ is a principal $\mathcal{O}'$-order of index e in

$B' :=\operatorname{End}_{A_{K'}}(\mathcal{M}_{K'})$. Define a σ-linear isomorphism $\psi : B' \to B'$ by $\psi(f) := \phi_{K'}^{-1} \circ f \circ \phi_{K'}$. We have

$$\mathcal{B} = \{b \in \mathcal{B}' \mid \psi(b) = b\} \quad \text{and} \quad \mathcal{B}' \cong \mathcal{B}_{\mathcal{O}'}.$$

Together with Lemma 2.4 this implies the first statement. The proof of the second assertion will be left to the reader. □

Conversely suppose that we have given a second central simple K-algebra B of dimension d^2 and a principal $\mathcal{O}$-order $\mathcal{B}$ in B of index e. We also assume that $[K' : K]$ is a multiple of the order of $[B \otimes A^{\mathrm{opp}}]$ in $\mathrm{Br}(K)$. Let m be any integer such that (7) holds.

Lemma 2.20. *There exists an invertible $\mathcal{B}_{\mathcal{O}'}$-$\mathcal{A}_{\mathcal{O}'}$-bimodule $\mathcal{M}$ and an isomorphism of bimodules*

$$\phi : {}^{\sigma}\mathcal{M}\mathfrak{P}^m \longrightarrow \mathcal{M}.$$

Proof. By Proposition 2.9 the principal orders $\mathcal{B}_{\mathcal{O}'}$ and $\mathcal{A}_{\mathcal{O}'}$ are Morita equivalent. Let $\mathcal{M}$ be an invertible $\mathcal{B}_{\mathcal{O}'}$-$\mathcal{A}_{\mathcal{O}'}$-bimodule. Then ${}^{\sigma}\mathcal{M}$ is invertible as well. Hence there exists an isomorphism $\phi' : {}^{\sigma}\mathcal{M}\mathfrak{P}^{m'} \to \mathcal{M}$ for some $m' \in \mathbb{Z}$. By 2.19 we have

$$\operatorname{inv}(B) = \operatorname{inv}(A) + \frac{m'}{e} \mod \mathbb{Z}$$

and therefore $m \equiv m' \mod e$. Put $\phi := \varpi^{\frac{m'-m}{e}}\phi'$. □

For the rest of this section we assume that $\mathcal{O}$ is an $\mathbb{F}_q$-algebra ($q = p^r$ for some $r \in \mathbb{Z}$) and let k' be an (possibly infinite) algebraic extension of k whose degree (over k) is a multiple of e. Let $\mathcal{O}' := \mathcal{O} \otimes_{\mathbb{F}_q} k'$ and $\sigma := \mathrm{id}_{\mathcal{O}} \otimes \mathrm{Frob}_q \in \mathrm{G}(\mathcal{O}'/\mathcal{O})$. For $\rho \in \operatorname{Hom}_{\mathbb{F}_q}(k, k') \cong \operatorname{Hom}_{k'}(k \otimes_{\mathbb{F}_q} k', k')$ we denote the kernel of $\mathcal{O}' \to k \otimes_{\mathbb{F}_q} k' \to k'$ by $\mathfrak{p}'_\rho$ and we set $\mathcal{O}'_\rho := \mathcal{O}'_{\mathfrak{p}'_\rho}$. Then $\mathcal{O}'_\rho$ is a (pro-)finite (pro-)étale local $\mathcal{O}$-algebra whose degree is a multiple of e and $\mathcal{O}' \cong \bigoplus_\rho \mathcal{O}'_\rho$. Similarly

$$\mathcal{A}_{\mathcal{O}'} = \bigoplus_\rho \mathcal{A}'_\rho \qquad \text{with } \mathcal{A}'_\rho = \mathcal{A}_{\mathcal{O}'_\rho}$$

and $\mathfrak{P}_{\mathcal{O}'} = \operatorname{Rad}(\mathcal{A}_{\mathcal{O}'})$ is equal to the product $\prod_\rho \mathfrak{P}'_\rho$ where $\mathfrak{P}'_\rho$ denotes the maximal invertible two-sided ideal $\operatorname{Ker}(\mathcal{A}_{\mathcal{O}'} \to \mathcal{A}'_\rho/\operatorname{Rad}(\mathcal{A}'_\rho))$ of $\mathcal{A}_{\mathcal{O}'}$. For the distinguished element $\iota := \mathrm{incl} : k \hookrightarrow k'$ in $\operatorname{Hom}_{\mathbb{F}_q}(k, k')$ we put $\mathfrak{p}' = \mathfrak{p}'_\rho$ and $\mathfrak{P}' := \mathfrak{P}'_\iota$. Let $\mathcal{M}$ be a free right $\mathcal{A}_{\mathcal{O}'}$-module of rank 1. For $m \in \mathbb{Z}$ the $\mathcal{A}_{\mathcal{O}'}$-module ${}^{\sigma}(\mathcal{M}(\mathfrak{P}')^m)$ is also free of rank 1. Hence there exists an isomorphism

$$\phi : {}^{\sigma}(\mathcal{M}(\mathfrak{P}')^m) \longrightarrow \mathcal{M}.$$

If we set

$$\mathcal{B} := \operatorname{End}_{\mathcal{A}_{\mathcal{O}'}}(\mathcal{M}, \phi) = \{f \in \operatorname{End}_{\mathcal{A}_{\mathcal{O}'}}(\mathcal{M}) \mid \phi \circ f = \circ\, \phi\}.$$

then one can deduce easily from Lemma 2.19 that $\mathcal{B}$ is a principal $\mathcal{O}$-order of index e in the central simple K-algebra $B = \mathcal{B}_K$ and that equation (7) holds. Conversely given such a principal $\mathcal{O}$-order $\mathcal{B}$ of index e and $m \in \mathbb{Z}$ such that (7) holds there exists a pair $(\mathcal{M}, \phi)$ as above with $\mathcal{B} = \mathrm{End}_{\mathcal{A}_{\mathcal{O}'}}(\mathcal{M}, \phi)$. To see this let $\mathcal{M}$ be any invertible $\mathcal{B}_{\mathcal{O}'}$-$\mathcal{A}_{\mathcal{O}'}$-bimodule. Since ${}^{\sigma}\mathcal{M}$ is invertible as well we have

$$ {}^{\sigma}(\mathcal{M}\prod_{\rho}(\mathfrak{P}'_{\rho})^{m_{\rho}}) \cong \mathcal{M} \tag{8} $$

for certain $m_{\rho} \in \mathbb{Z}$. Since ${}^{\sigma}(\mathfrak{P}'_{\rho}) \cong \mathfrak{P}'_{\mathrm{Frob}_q \circ \rho}$ we may assume – after replacing $\mathcal{M}$ by $\mathcal{M}\mathfrak{A}$ for a suitable invertible two-sided $\mathcal{A}_{\mathcal{O}'}$-ideal $\mathfrak{A}$ – that $m_{\rho} = 0$ for all $\rho \in \mathrm{Hom}_{\mathbb{F}_q}(k, k')$ except $\rho = \iota$. As in the proof of Lemma 2.20 we deduce

$$ \mathrm{inv}(B) = \mathrm{inv}(A) + \frac{m_{\iota}}{e} \mod \mathbb{Z} $$

hence $m_{\iota} \cong m \mod e$ and therefore $(\mathfrak{P}')^{m_{\iota}} \cong (\mathfrak{P}')^{m}$. Hence there also exists an isomorphism ${}^{\sigma}(\mathcal{M}(\mathfrak{P}')^{m}) \cong \mathcal{M}$.

Definition 2.21. *A pair $(\mathcal{M}, \phi)$ consisting of an invertible $\mathcal{B}_{\mathcal{O}'}$-$\mathcal{A}_{\mathcal{O}'}$-bimodule $\mathcal{M}$ and an isomorphism $\phi : {}^{\sigma}(\mathcal{M}(\mathfrak{P}')^{m}) \to \mathcal{M}$ is called an invertible ϕ-$\mathcal{A}$-$\mathcal{B}$-bimodule of slope $-\frac{m}{e}$ over $\mathcal{O}'$.*

We have seen that an invertible $\mathcal{B}_{\mathcal{O}'}$-$\mathcal{A}_{\mathcal{O}'}$-bimodule of a given slope $r \in \mathbb{Q}$ exists and only if $r = \mathrm{inv}(A) - \mathrm{inv}(B) \mod \mathbb{Z}$. It is also easy to see that any two invertible $\mathcal{B}_{\mathcal{O}'}$-$\mathcal{A}_{\mathcal{O}'}$-bimodules of the same slope differ (up to isomorphism) by a fractional $\mathcal{A}$-ideal. This implies that if k'' is an algebraic extension of k' and $\mathcal{O}'' = \mathcal{O} \otimes_{\mathbb{F}_q} k''$ then any ϕ-$\mathcal{A}$-$\mathcal{B}$-bimodule over $\mathcal{O}''$ is obtained by base change from an ϕ-$\mathcal{A}$-$\mathcal{B}$-bimodule over $\mathcal{O}'$.

Remark 2.22. Assume that $[k' : k] = e$ and let $n = [k' : \mathbb{F}_q]$. Let $(\mathcal{M}, \phi)$ be an invertible $\mathcal{B}_{\mathcal{O}'}$-$\mathcal{A}_{\mathcal{O}'}$-bimodule of slope $-\frac{m}{e}$. For $r \in \mathbb{Z}/n\mathbb{Z}$ we put $\mathfrak{P}'_r = {}^{\sigma^r}\mathfrak{P}'$. We have $\prod_{r \in \mathbb{Z}/n\mathbb{Z}} \mathfrak{P}'_r = \mathfrak{P}^e_{\mathcal{O}'} = \mathfrak{p}\mathcal{A}_{\mathcal{O}'}$. For each two-sided invertible ideal $\mathfrak{A}'$ of $\mathcal{A}_{\mathcal{O}'}$ and $r \in \mathbb{Z}/n\mathbb{Z}$, the map ϕ induces isomorphisms $({}^{\sigma^r}\mathcal{M})\mathfrak{A}' \to ({}^{\sigma^{r-1}}\mathcal{M})\mathfrak{A}'\mathfrak{P}'^{\,m}_r$ which will be also denoted by ϕ. Consider the map

$$ \phi^n : \mathcal{M} = ({}^{\sigma^n}\mathcal{M}) \xrightarrow{\phi} ({}^{\sigma^{n-1}}\mathcal{M})\mathfrak{P}'^{\,m}_n \xrightarrow{\phi} \ldots \longrightarrow \mathcal{M} \prod_{r \in \mathbb{Z}/n\mathbb{Z}} \mathfrak{P}'^{\,m}_r = \mathcal{M}\mathfrak{p}^m \tag{9} $$

Since (9) is $\mathcal{B}_{\mathcal{O}'}$-$\mathcal{A}_{\mathcal{O}'}$-bilinear and commutes with ϕ there exists an element $x \in K$ with $v_K(x) = m$ such that (9) is given by multiplication with x. This fact will be used later when we discuss level structure at the pole of $\mathcal{A}$-elliptic sheaves.

3 Global theory of hereditary orders

In this section we study hereditary orders in a central simple algebras over a function field of one variable (though most results hold also for number fields).

We shall show that two hereditary orders are Morita equivalent if their generic fibers are equivalent and all their local indices are the same. Furthermore any such hereditary order is Morita equivalent to a locally principal one. We will then study the Picard group of a locally principal order $\mathcal{A}$ and introduce the notion of $\mathcal{A}$-degree of a locally free $\mathcal{A}$-module of finite rank. In the final part we will introduce the notion of a special $\mathcal{A}$-module.
In this chapter k denotes a fixed perfect field of cohomological dimension ≤ 1 and X a smooth projective geometrically connected curve over k with function field F. For $x \in |X|$ we denote by $\mathcal{O}_x$ the completion of $\mathcal{O}_{X,x}$ and by F_x the quotient field of $\mathcal{O}_x$. The maximal ideal of $\mathcal{O}_x$ will be denoted by $\mathfrak{p}_x$. If $\mathcal{V}$ is a coherent $\mathcal{O}_X$-module then we set $\mathcal{V}_x = \mathcal{V}\otimes_{\mathcal{O}_X}\mathcal{O}_x$ and if V is a finite-dimensional F-vector space we put $V_x = V \otimes_F F_x$.

3.1 MORITA EQUIVALENCE.

Let V be a finite-dimensional F-vector space. The set of locally free coherent $\mathcal{O}_X$-modules $\mathcal{V}$ with generic fiber $\mathcal{V}_\eta = V$ is in one-to-one correspondence with the set of $\mathcal{O}_x$-lattices $\mathcal{V}_x$ in V_x for all $x \in |X|$ such that there exists an F-basis B of V with $\mathcal{V}_x = \sum_{b\in B}\mathcal{O}_x b$ for almost all x. Consequently if $U \subseteq X$ is an open subscheme then there is a one-to-one correspondence between coherent and locally free $\mathcal{O}_X$-modules $\mathcal{V}$ and coherent and locally free $\mathcal{O}_U$-module $\mathcal{V}_U$ and together with an $\mathcal{O}_x$-lattice $\mathcal{V}_x$ in $\mathcal{V}_U \otimes F_x$ for all $x \in X - U$.
Let A be a central simple F-algebra and $\mathcal{A}$ a hereditary $\mathcal{O}_X$-order in A. We put $e_x(\mathcal{A})\colon = e(\mathcal{A}_x)$. There are only finitely many points $x \in |X|$ with $e_x(\mathcal{A}) > 1$. Define the divisor $\underline{\mathrm{Disc}}(\mathcal{A})$ as $\underline{\mathrm{Disc}}(\mathcal{A})\colon = \sum_{x\in|X|}(e_x(\mathcal{A})-1)x$. If k is finite and $x \in |X|$ then $\mathrm{inv}_x(A)$ denotes the image of the class of A_x under the canonical isomorphism of class field theory $\mathrm{Br}(F_v) \to \mathbb{Q}/\mathbb{Z}$.

PROPOSITION 3.1. *Let A_1, A_2 be central simple algebras over F and let $\mathcal{A}_1$ and $\mathcal{A}_2$ be hereditary $\mathcal{O}_X$-orders in A_1 and A_2 respectively. The following conditions are equivalent.*
(i) $\mathcal{A}_1$ and $\mathcal{A}_2$ are equivalent.
(ii) A_1 and A_2 are equivalent and $(\mathcal{A}_1)_x$ and $(\mathcal{A}_2)_x$ are equivalent for all $x \in |X|$.
(iii) A_1 and A_2 are equivalent and $\underline{\mathrm{Disc}}(\mathcal{A}_1) = \underline{\mathrm{Disc}}(\mathcal{A}_2)$.
Moreover if k is a finite field then the above conditions are also equivalent to:
(iv) $\mathrm{inv}_x(A_1) = \mathrm{inv}_x(A_2)$ for all $x \in |X|$ and $\underline{\mathrm{Disc}}(\mathcal{A}_1) = \underline{\mathrm{Disc}}(\mathcal{A}_2)$.

Proof. (i) $\Rightarrow$ (ii) is clear. (ii) $\Leftrightarrow$ (iii) follows from Proposition 2.9 and (iii) $\Leftrightarrow$ (iv) from the Theorem of Brauer–Hasse–Noether. It remains to show that (ii) implies (i). Let U be an affine open subscheme of X contained in the complement of $\underline{\mathrm{Disc}}(\mathcal{A}_1) = \underline{\mathrm{Disc}}(\mathcal{A}_2)$ in X. By ([Re], 21.7) $\mathcal{A}_1|_U$ and $\mathcal{A}_2|_U$ are Morita equivalent. Let $\mathcal{I}_U$ be an invertible $\mathcal{A}_1|_U$-$\mathcal{A}_2|_U$-bimodule and let $\mathcal{I}_x$ be an invertible $(\mathcal{A}_1)_x$-$(\mathcal{A}_2)_x$-bimodule for each $x \in X - U$. Since there is only one invertible $(A_1)_x$-$(A_2)_x$-bimodule up to isomorphism we may assume that $\mathcal{I}_x \otimes F_x = \mathcal{I}_U \otimes F_x$ i.e. that $\mathcal{I}_x$ is a lattice in $\mathcal{I}_U \otimes F_x$. It is easy to see that the locally free $\mathcal{O}_X$-module $\mathcal{I}$ corresponding to $\mathcal{I}_U$ and the $\mathcal{I}_x$, $x \in X - U$ is then an invertible $\mathcal{A}_1$-$\mathcal{A}_2$-bimodule. $\square$

A *locally principal $\mathcal{O}_X$-order* $\mathcal{A}$ is a hereditary $\mathcal{O}_X$-order in a central simple F-algebra A such that $\mathcal{A}_x$ is principal for all $x \in |X|$. The rank of $\mathcal{A}$ is its rank as an $\mathcal{O}_X$-module, hence $= \dim_F(A)$. If $\mathcal{A}$ is a hereditary $\mathcal{O}_X$-order in A then it is locally principal if for example $\mathcal{A}_x$ is either maximal or $e_x(\mathcal{A}) = d$ for all $x \in |\underline{\mathrm{Disc}}(\mathcal{A})|$.
Suppose that $\mathcal{A}$ is a locally principal $\mathcal{O}_X$-order of rank d^2. We define two positive integers $e(\mathcal{A}), \delta(\mathcal{A})$ by

$$e(\mathcal{A}) : = \mathrm{lcm}\{e_x(\mathcal{A}) \mid x \in |X|\} \tag{10}$$

$$\delta(\mathcal{A}) : = \mathrm{lcm}\{ \text{ numerator of } \tfrac{e_x(\mathcal{A})}{\deg(x)} \mid x \in |X|\}$$

According to Lemma 2.2 we have $\delta(\mathcal{A}) \mid e(\mathcal{A}) \mid d$. If $\mathcal{A}$ is locally principal then one can easily see that

$$\deg(\mathcal{A}) = -\frac{d^2}{2} \sum_{x \in |X|} (1 - \frac{1}{e_x(\mathcal{A})}) \deg(x).$$

In particular if $\mathcal{B}$ is a second locally principal $\mathcal{O}_X$-order of rank d^2 with $\underline{\mathrm{Disc}}(\mathcal{A}) = \underline{\mathrm{Disc}}(\mathcal{B})$ then

$$\deg(\mathcal{A}) = \deg(\mathcal{B}). \tag{11}$$

Corollary 3.2. *Let $\mathcal{A}$ be a hereditary $\mathcal{O}_X$-order in a central simple F-algebra A. Then there exists a locally principal $\mathcal{O}_X$-order $\mathcal{D}$ which is Morita equivalent to $\mathcal{A}$. In fact $\mathcal{D}$ can be chosen such that* $\mathrm{rank}_{\mathcal{O}_X}(\mathcal{D}) = e(\mathcal{A})^2$.

Proof. That $\mathcal{A}$ is equivalent to a locally principal $\mathcal{O}_X$-order follows easily from the corresponding local statement 2.13. In fact if $B : = M_e(A)$ then for all $x \in |\underline{\mathrm{Disc}}(\mathcal{A})|$ we can pick a principal $\mathcal{O}_x$-order $\mathcal{B}_x$ in B_x equivalent to $\mathcal{A}_x$. If $U : = X - |\underline{\mathrm{Disc}}(\mathcal{A})|$ and $\mathcal{B}_U$ is a maximal $\mathcal{O}_U$-order in B then there exists a uniquely determined hereditary $\mathcal{O}_X$-order $\mathcal{B}$ in B with $\mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{O}_x = \mathcal{B}_x$ for all $x \in |\underline{\mathrm{Disc}}(\mathcal{A})|$ and $\mathcal{B}|_U = \mathcal{B}_U$. The order $\mathcal{B}$ is locally principal and equivalent to $\mathcal{A}$ by 3.1.
Thus to prove the second statement we may assume that $\mathcal{A}$ is locally principal. Let $\mathcal{I}$ be a locally stably free $\mathcal{A}$-module which is of rank de as an $\mathcal{O}_X$-module. By Lemma 2.10 and 3.1 above it follows that $\mathcal{D} : = \underline{\mathrm{End}}_{\mathcal{A}}(\mathcal{I})$ is a locally principal $\mathcal{O}_X$-order in $\mathrm{End}_A(\mathcal{I}_\eta)$. Moreover $\mathcal{D}$ is equivalent to $\mathcal{A}$ and $\mathrm{rank}_{\mathcal{O}_X}(\mathcal{D}) = e(\mathcal{A})^2$. $\square$

3.2 Locally free $\mathcal{A}$-modules

The Picard group of a locally principal order. In this section $\mathcal{A}$ denotes a locally principal $\mathcal{O}_X$-order of rank d^2. We are going to compute the

Picard group of $\mathcal{A}$. We define

$$\mathrm{Div}(\mathcal{A})\colon =\{\sum_{x\in|X|} n_x x\in \mathrm{Div}(X)\otimes\mathbb{Q}\mid\ e_x(\mathcal{A})n_x\in\mathbb{Z}\ \forall\, x\in|X|\}.$$

Note that $\deg(\mathrm{Div}(\mathcal{A}))=\frac{1}{\delta(\mathcal{A})}\mathbb{Z}$. For a divisor $D=\sum_{x\in|X|} n_x x\in \mathrm{Div}(\mathcal{A}))$ we denote by $\mathcal{A}(D)$ the invertible $\mathcal{A}$-$\mathcal{A}$-bimodule given by $\mathcal{A}(D)|_{X-|D|}=\mathcal{A}|_{X-|D|}$ and $\mathcal{A}(D)_x=\mathfrak{P}_{\mathcal{A}_x}^{-n_x e_x(\mathcal{A})}$ for all $x\in|X|$. If $D\in\mathrm{Div}(X)$ then $\mathcal{A}(D)=\mathcal{A}\otimes_{\mathcal{O}_X}\mathcal{O}_X(D)$.

Proposition 3.3. *The sequence*

$$0\longrightarrow F^*/k^*\xrightarrow{\mathrm{div}}\mathrm{Div}(\mathcal{A})\xrightarrow{D\mapsto\mathcal{A}(\mathbf{D})}\mathrm{Pic}(\mathcal{A})\longrightarrow 0$$

is exact.

Proof. This follows from ([Re], 40.9). □

We also need to consider the group of isomorphism classes of invertible $\mathcal{A}$-$\mathcal{A}$-bimodules with level structure and give a description of it as an idele class group. Let $I=\sum_x n_x x$ be an effective divisor on X. The corresponding finite closed subscheme of X will be also denoted by I. A *level-I-structure* on an invertible $\mathcal{A}$-$\mathcal{A}$-bimodule $\mathcal{L}$ is an isomorphism $\beta:\mathcal{A}_I\to\mathcal{L}_I$ of right $\mathcal{A}_I$-modules. We denote by $\mathrm{Pic}_I(\mathcal{A})$ the set of isomorphism classes of invertible $\mathcal{A}$-$\mathcal{A}$-bimodules with level-I-structure. If $(\mathcal{L}_1,\beta_1),(\mathcal{L}_2,\beta_2)$ are invertible $\mathcal{A}$-$\mathcal{A}$-bimodules with level-I-structures we define the level-I-structure $\beta_1\beta_2$ on $\mathcal{L}_1\otimes_{\mathcal{A}}\mathcal{L}_2$ as the composite

$$\beta_1\beta_2:\mathcal{A}_I\xrightarrow{\ \beta_2\ }(\mathcal{L}_2)_I=\mathcal{A}_I\otimes_{\mathcal{A}_I}(\mathcal{L}_2)_I\xrightarrow{\beta_1\otimes\mathrm{id}}(\mathcal{L}_1\otimes_{\mathcal{A}}\mathcal{L}_2)_I \tag{12}$$

thus defining a group structure on $\mathrm{Pic}_I(\mathcal{A})$. Note that unlike $\mathrm{Pic}(\mathcal{A})$, $\mathrm{Pic}_I(\mathcal{A})$ is in general not abelian. In fact we have a short exact sequence

$$0\longrightarrow\Gamma(I,\mathcal{A}_I)^*/k^*\longrightarrow\mathrm{Pic}_I(\mathcal{A})\longrightarrow\mathrm{Pic}(\mathcal{A})\longrightarrow 0 \tag{13}$$

where the first map is given by $a\in\Gamma(I,\mathcal{A}_I)^*\mapsto(\mathcal{A},l_a:\mathcal{A}_I\xrightarrow{a\cdot}\mathcal{A}_I)$.
Let $U_I(\mathcal{A})\colon=\mathrm{Ker}(\prod_{x\in|X|}\mathcal{A}_x^*\to\prod_{x\in|X|}(\mathcal{A}_x/\mathfrak{p}_x^{n_x}\mathcal{A}_x)^*=\Gamma(I,\mathcal{A}_I)^*)$ and let

$$\mathcal{C}_I(\mathcal{A})\colon=(\textstyle\prod'_{x\in|X|}N(\mathcal{A}_x))/U_I(\mathcal{A})F^*$$

where $\prod'_{x\in|X|}N(\mathcal{A}_x))$ denotes the restricted direct product of the groups $\{N(\mathcal{A}_x))\}_{x\in|X|}$ with respect to $\{\mathcal{A}_x^*\}_{x\in|X|}$. Given $a=\{a_x\}_x\in\prod'_{x\in|X|}N(\mathcal{A}_x)$ we put $\mathrm{div}(a)=\sum_{x\in|X|}v_{\mathcal{A}_x}(a_x)x$. Left multiplication by a induces a level-I-structure $\beta_a:\mathcal{A}_I\to\mathcal{A}(\mathrm{div}(a))_I$.

Corollary 3.4. *The assignement $a\mapsto(\mathcal{A}(\mathrm{div}(a)),\beta_a)$ induces an isomorphism $\mathcal{C}_I(\mathcal{A})\cong\mathrm{Pic}_I(\mathcal{A})$.*

Relative divisors and invertible bimodules. Let S be a k-scheme and let $\pi : X \times S \to S$ be the projection. We need to define the bimodule $\mathcal{A}(D)$ also for elements of a certain group of relative divisors $\mathrm{Div}(\mathcal{A} \boxtimes \mathcal{O}_S)$. For the latter we use the following ad hoc definition. Assume first that S is of finite type over k. Let $\mathcal{S}$ be the collection of all connected components of $x \times S$ where x runs through all closed points of X. Thus if $S' \in \mathcal{S}$ there exists a unique closed point $x\colon = \pi(S')$ with $S' \subseteq x \times S$. We set

$$\mathrm{Div}(\mathcal{A} \boxtimes \mathcal{O}_S)\colon = \bigoplus_{S' \in \mathcal{S}} \frac{1}{e_{\pi(S')}(\mathcal{A})} \mathbb{Z}.$$

Let R be the integral closure of k in $\Gamma(S, \mathcal{O}_S)$. Note that for $x \in |X|$ the set of open and closed subschemes of $x \times S$ corresponds to the set of idempotents in $k(x) \otimes_k \Gamma(S, \mathcal{O}_S)$. If $f : S_1 \to S_2$ is a morphism of k-schemes there is an obvious notion of a pull-back $f^* : \mathrm{Div}(\mathcal{A} \boxtimes \mathcal{O}_{S_2}) \to \mathrm{Div}(\mathcal{A} \boxtimes \mathcal{O}_{S_1})$. For an arbitrary k-scheme we define $\mathrm{Div}(\mathcal{A} \boxtimes \mathcal{O}_S)$ as the direct limit of $\mathrm{Div}(\mathcal{A} \boxtimes \mathcal{O}_{S'})$ over the category of pairs (S', g) consisting of a k-scheme S' of finite type and a morphism $g : S \to S'$ in Sch/k.
Let $S \in \mathrm{Sch}/k$. A k-morphism $x_S : S \to X$ which factors as $S \to \mathrm{Spec}\, k(x) \to X$ for some $x \in |X|$ yields an element – denoted by x_S as well – of the group $\mathrm{Div}(\mathcal{A} \boxtimes \mathcal{O}_S)$. For that we can assume that S is of finite type. Since the graph $\Gamma_{x_S} = (x_S, \mathrm{id}_S) : S \longrightarrow X \times S$ is an open and closed subscheme of $x \times S$ it is a disjoint union of connected components and we define $x_S \in \mathrm{Div}(\mathcal{A} \boxtimes \mathcal{O}_S)$ to be the sum of these components.
There exists a unique homomorphism

$$\mathrm{Div}(\mathcal{A} \boxtimes \mathcal{O}_S) \to \mathrm{Pic}(\mathcal{A} \boxtimes \mathcal{O}_S), \quad D \mapsto (\mathcal{A} \boxtimes \mathcal{O}_S)(D) \tag{14}$$

compatible with pull-backs which agrees with the previously defined map in case $S = \mathrm{Spec}\, k'$ for a finite extension k'/k. It suffices to define (14) for $\frac{1}{e_x(\mathcal{A})} D$, where D is a connected component of $x \times S$ for some $x \in |X|$. It is also enough to consider the case where S is connected and of finite type over k. Let R be the integral closure of k in $\Gamma(S, \mathcal{O}_S)$. Then $\mathrm{Spec}\, R$ is connected and finite over $\mathrm{Spec}\, k$, i.e. R is an artinian finite local k-algebra. Let k' denotes the residue field of R. Since k is perfect the canonical projection $R \to k'$ has a unique section. Therefore the structural morphism $S \to \mathrm{Spec}\, k$ factors as $S \to \mathrm{Spec}\, k' \to \mathrm{Spec}\, k$. Thus by replacing k, X and $\mathcal{A}$ by k' and $X_{k'}$ and $\mathcal{A} \boxtimes k'$ respectively we can assume that the residue field of R is k. However, in this case, $x \times S$ is connected for all $x \in |X|$, hence $D = x \times S$ with $x = \pi(D)$. S o we are forced to define $(\mathcal{A} \boxtimes \mathcal{O}_S)(\frac{1}{e_x(\mathcal{A})} D)\colon = \pi^*(\mathcal{A}(\frac{1}{e_x(\mathcal{A})} x))$.

$\mathcal{A}$-rank and $\mathcal{A}$-degree. Let $f : S \to X$ be a morphism. For $\mathcal{E}$ in ${}_{f^*(\mathcal{A})}\mathrm{Mod}$ and $\mathcal{F}$ in $\mathrm{Mod}_{f^*(\mathcal{A})}$ we put $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F}\colon = \mathcal{E} \otimes_{f^*(\mathcal{A})} \mathcal{F}$. If $D = \sum_{x \in |X|} n_x x \in \mathrm{Div}(\mathcal{A})$ we set $\mathcal{E}(D)\colon = \mathcal{E} \otimes_{\mathcal{A}} f^*(\mathcal{A}(D))$ and $\mathcal{F}(D)\colon = f^*(\mathcal{A}(D) \boxtimes \mathcal{O}_S) \otimes_{\mathcal{A}} \mathcal{F}$. Let S be a k-scheme. We denote by ${}_{\mathcal{A}}\mathrm{Vect}(S)$ (resp. $\mathrm{Vect}_{\mathcal{A}}(S)$) the category coherent and locally free left (resp. right) $\mathcal{A} \boxtimes \mathcal{O}_S$-modules. For $\mathcal{F}$ in ${}_{\mathcal{A}}\mathrm{Vect}(S)$ or

$\mathrm{Vect}_{\mathcal{A}}(S)$ let $\mathrm{rank}_{\mathcal{A}} \mathcal{F}$ be the locally constant function $s \mapsto \mathrm{rank}_{\mathcal{A}\boxtimes k(s)}(\mathcal{F}|_{X\times s})$ on S (hence $\mathrm{rank}_{\mathcal{A}} \mathcal{F}$ can be viewed as an element of $\mathbb{Z}^{\pi_0(S)}$). For a positive integer r we denote by ${}_{\mathcal{A}}\mathrm{Vect}^{\mathrm{r}}(S)$ (resp. $\mathrm{Vect}^{\mathrm{r}}_{\mathcal{A}}(S)$) the subcategory of $\mathcal{F} \in {}_{\mathcal{A}}\mathrm{Vect}(S)$ (resp. $\mathcal{F} \in \mathrm{Vect}_{\mathcal{A}}(S)$) with $\mathrm{rank}_{\mathcal{A}} \mathcal{F} = r$.
Let $\mathcal{F}$ be a locally free $\mathcal{A}\boxtimes\mathcal{O}_S$-module of rank r. Define $\det_{\mathcal{A}} \mathcal{F}$ as the image of the isomorphism class of $\mathcal{F}$ (viewed as an element of $H^1(X \times S, \mathrm{GL}_r(\mathcal{A} \boxtimes \mathcal{O}_S))$ under the map

$$H^1_{\mathrm{zar}}(X \times S, \mathrm{GL}_r(\mathcal{A} \boxtimes \mathcal{O}_S) \longrightarrow H^1_{\mathrm{zar}}(X \times S, \mathcal{O}^*) = \mathrm{Pic}(X \times S)$$

induced by the reduced norm $\mathrm{Nrd} : M_r(A) \to F$. We obtain a locally constant function

$$\deg_{\mathcal{A}}(\mathcal{F}) : S \to \frac{1}{d}\mathbb{Z}, s \mapsto \deg((\det_{\mathcal{A}} \mathcal{F})|_{X\times s})$$

It is easy to see that

$$\deg_{\mathcal{A}}(\mathcal{F}) = \frac{1}{d^2}(\deg(\mathcal{F}) - \mathrm{rank}_{\mathcal{A}}(\mathcal{F}) \deg(\mathcal{A})).$$

In particular since $\deg(\mathcal{A}(D)) = \deg(\mathcal{A}) + d^2 \deg(D)$ we have

$$\deg_{\mathcal{A}}(\mathcal{A}(D)) = \deg(D)$$

for $D \in \mathrm{Div}(\mathcal{A})$.

Lemma 3.5. *(a) Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of coherent and locally free $\mathcal{A} \boxtimes \mathcal{O}_S$-modules. Then*

$$\deg_{\mathcal{A}}(\mathcal{F}_2) = \deg_{\mathcal{A}}(\mathcal{F}_1) + \deg_{\mathcal{A}}(\mathcal{F}_3).$$

(b) Let $\mathcal{E}$ be an object of $\mathrm{Vect}^{\mathrm{r}}_{\mathcal{A}}(S)$ and $\mathcal{F}$ be an object of ${}_{\mathcal{A}}\mathrm{Vect}^{\mathrm{s}}(S)$. Then

$$\frac{1}{d^2}(\deg(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F}) - rs \deg(\mathcal{A})) = r \deg_{\mathcal{A}}(\mathcal{F}) + s \deg_{\mathcal{A}}(\mathcal{E}).$$

(c) Let $\mathcal{B}$ be a second locally principal $\mathcal{O}_X$-order of rank d^2 equivalent to $\mathcal{A}$. Let $\mathcal{E}$ be an object of $\mathrm{Vect}^{\mathrm{r}}_{\mathcal{A}}(S)$ and let $\mathcal{I}$ be an invertible $\mathcal{A}$-$\mathcal{B}$-bimodule. Then

$$\deg_{\mathcal{B}}(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{I}) = \deg_{\mathcal{A}}(\mathcal{E}) + r \deg_{\mathcal{A}}(\mathcal{I}).$$

(d) Let $\mathcal{E}$ be an object of ${}_{\mathcal{A}}\mathrm{Vect}^{\mathrm{r}}(S)$ and $D \in \mathrm{Div}(\mathcal{A})$. Then

$$\deg_{\mathcal{A}}(\mathcal{E}(D)) = \deg_{\mathcal{A}}(\mathcal{E}) + r \deg(D)$$

Proof. (a) is obvious, (c) follows from (b) and (11) and (d) is a special case of (c). Note that by 2.12 the bimodule $\mathcal{L}$ in (c) is a locally-free left $\mathcal{A}$- and right $\mathcal{B}$-module of rank 1.

For (b) it is enough to consider the case when S is a connected k-scheme of finite type and therefore – by chosing a fixed closed point $s \in S$ and taking the base change $\operatorname{Spec} k(s) \to \operatorname{Spec} k$ – to consider the case $S = \operatorname{Spec} k$. If

$$\begin{tikzcd}[row sep=small] & \mathcal{E} \\ \mathcal{E}'' \arrow[ur] \arrow[dr] & \\ & \mathcal{E}' \end{tikzcd} \tag{15}$$

is a diagram of locally free $\mathcal{A} \boxtimes \mathcal{O}_S$-modules of the same rank r and injective $\mathcal{A} \boxtimes \mathcal{O}_S$-linear homomorphisms then it is easy to see that (b) holds for $\mathcal{E}$ if and only if it holds for $\mathcal{E}'$. Since $\mathcal{E}|_U \cong \mathcal{A}^r|_U$ for some non-empty open subscheme $U \subseteq X$ there exists a diagram (15) with $\mathcal{E}' = \mathcal{A}^r$. The assertion follows. $\square$

It follows from 3.3 or 3.5 (b) that $\deg_{\mathcal{A}} : \operatorname{Pic}(\mathcal{A}) \to \mathbb{Q}$ is a homomorphism. We denote its kernel by $\operatorname{Pic}_0(\mathcal{A})$. Also if $I \in \operatorname{Div}(X)$ we let $\operatorname{Pic}_{I,0}(\mathcal{A})$ be the subgroup of $(\mathcal{L}, \beta) \in \operatorname{Pic}_I(\mathcal{A})$ with $\deg_{\mathcal{A}}(\mathcal{L}) = 0$. The image $\deg_{\mathcal{A}}(\operatorname{Pic}(\mathcal{A}))$ is equal to $\frac{1}{\delta(\mathcal{A})}\mathbb{Z}$.

Remark 3.6. Let $\mathcal{A}, \mathcal{B}$ be locally principal $\mathcal{O}_X$-order of rank d^2 and suppose that $\mathcal{A}$ and $\mathcal{B}$ are equivalent. The set of isomorphism classes of invertible $\mathcal{A}$-$\mathcal{B}$-bimodule has a simple transitive left $\operatorname{Pic}(\mathcal{A})$-action. Hence for any two invertible $\mathcal{A}$-$\mathcal{B}$-bimodule $\mathcal{I}, \mathcal{J}$ the degrees $\deg_{\mathcal{A}}(\mathcal{J})$ and $\deg_{\mathcal{A}}(\mathcal{I})$ differ by a multiple of $\frac{1}{\delta(\mathcal{A})}$. Call $\mathcal{A}$ and $\mathcal{B}$ strongly Morita equivalent if there exists an invertible $\mathcal{A}$-$\mathcal{B}$-bimodule $\mathcal{I}$ with $\deg_{\mathcal{A}}(\mathcal{I}) = 0$. It is easy to see that a given equivalence class of locally principal $\mathcal{O}_X$-orders of rank d^2 decomposes into $\frac{e}{\delta}$ strong equivalence classes (where e and δ are defined in (10)).

3.3 Special $\mathcal{A}$-modules

Let $\mathcal{A}$ be a locally principal $\mathcal{O}_X$-order of rank d^2. If $g : U \to X$ is an étale morphism then a maximal torus in $\mathcal{A}_U : = g^*(\mathcal{A})$ is a maximal commutative étale $\mathcal{O}_U$-subalgebra of $\mathcal{A}_U$.

Definition 3.7. *A right $\mathcal{A} \boxtimes \mathcal{O}_S$-module $\mathcal{K}$ is called special of rank r if the following holds:*

(i) $\mathcal{K}$ is coherent as an $\mathcal{O}_{X\times S}$-module and the map $\operatorname{Supp}(\mathcal{K}) \hookrightarrow X \times S \to S$ is an isomorphism. Hence $\operatorname{Supp}(\mathcal{K})$ is the image of the graph of a morphism $N = N(\mathcal{K}) : S \to X$ and $\mathcal{K}$ is the direct image of a $N^*(\mathcal{A})$-module – also denoted by $\mathcal{K}$ – by the graph $\Gamma_N = (N, \mathrm{id}_S) : S \to X \times S$.

(ii) Consider $\mathcal{K}$ as a sheaf on S as in (i). For any étale morphism $g : U \to X$ and maximal torus $\mathcal{T}$ of $\mathcal{A}_U$, $(g_S)^*(\mathcal{K})$ is a locally free $(N_U)^*(\mathcal{T})$-module of rank r. Here g_S (resp. N_U) denote the base change of g (resp. N) with respect to N (resp. g).

We denote by $\mathrm{Coh}^r_{\mathcal{A},\mathrm{sp}}$ *the stack over* k *such that for each* $S \in \mathrm{Sch}/k$, $\mathrm{Coh}^r_{\mathcal{A},\mathrm{sp}}(S)$ *is the groupoid of special* $\mathcal{A}\boxtimes\mathcal{O}_S$*-modules of rank* r. *The morphism* $\mathcal{K}\mapsto N(\mathcal{K})$ *will be denoted by* $N:\mathrm{Coh}^r_{\mathcal{A},\mathrm{sp}}\to X$.

Remarks 3.8. (a) By Lemma 2.16 it suffices to check condition (ii) for a fixed étale covering $\{U_i\to U\}$ and maximal tori $\mathcal{T}_i$ of $\mathcal{A}_{U_i}$.
(b) Let $\mathcal{K}$ be as in 3.7 satisfying (i) and assume that $N(\mathcal{K}):S\to X$ factors through $X-|\,\underline{\mathrm{Disc}}(\mathcal{A})|$. Then $\mathcal{K}$ is special of rank r if and only if $\mathcal{K}$ is a locally free of rank rd as an $\mathcal{O}_S$-module.
(c) Let $\mathcal{A}'$ be another locally principal $\mathcal{O}_X$-order of rank d^2 equivalent to $\mathcal{A}$ and let $\mathcal{I}$ be an invertible $\mathcal{A}$-$\mathcal{A}'$-bimodule. Tensoring with $\mathcal{I}$ maps $\mathrm{Coh}^r_{\mathcal{A},\mathrm{sp}}$ isomorphically to $\mathrm{Coh}^r_{\mathcal{A}',\mathrm{sp}}$. This follows easily from the fact that, locally on X, $\mathcal{A}$ and $\mathcal{A}'$ are isomorphic. More generally if $\mathcal{A}$ are equivalent on some open subscheme $U\subseteq X$ and $\mathcal{I}$ is a $\mathcal{A}$-$\mathcal{A}'$-bimodule which is invertible on U then tensoring with $\mathcal{I}$ yields an isomorphism $\cdot\otimes_{\mathcal{A}}\mathcal{I}:\mathrm{Coh}^r_{\mathcal{A},\mathrm{sp}}\times_X U\to\mathrm{Coh}^r_{\mathcal{A}',\mathrm{sp}}\times_X U$.

Except in the appendix, we need to consider only the case $r=1$. In the following we investigate the geometric properties of $\mathrm{Coh}_{\mathcal{A},\mathrm{sp}}:=\mathrm{Coh}^r_{\mathcal{A},\mathrm{sp}}$. Recall that a morphism $f:Y\to X$ is said to be semistable if its generic fiber is smooth and for any $y\in Y$ there exists an étale neighbourhood Y' of y, an open affine neighbourhood $\mathrm{Spec}\,R$ of $x=f(y)$ and a smooth X-morphism $Y'\xrightarrow{g}\mathrm{Spec}\,R[T_1,\dots,T_r]/(T_1\cdots T_r-\varpi)$ for some $r\geq 1$, where ϖ is a local parameter at x. Equivalently, Y is a smooth k-scheme, the generic fiber Y_η is smooth over F and the closed fiber Y_x is a reduced divisor with normal crossings for all $x\in|X|$. Therefore if f is semistable it is flat.
We have the following simple Lemma whose proof will be left to the reader:

Lemma 3.9. *Let* $Y_1\xrightarrow{f}Y_2\xrightarrow{g}X$ *be morphism of schemes such that* f *is smooth and surjective. Then* g *is semistable if and only if* $g\circ f$ *is semistable.*

Let $\mathcal{Y}$ be an algebraic stack over k. We will call a morphism $f:\mathcal{Y}\to X$ semistable if there exists a scheme Y and a presentation $P:Y\to\mathcal{Y}$ (i.e. P is smooth and surjective) such that $f\circ P:Y\to X$ is semistable. It follows from 3.9 that if this holds then any presentation $P':Y'\to\mathcal{Y}$ (with Y' a scheme) has this property. In particular if $\mathcal{Y}$ is a scheme the two notions of semistability agree.
Our aim in this section is to prove the following result.

Proposition 3.10. $\mathrm{Coh}_{\mathcal{A},\mathrm{sp}}$ *is an algebraic stack over* $\mathbb{F}_q$. *The morphism* $N:\mathrm{Coh}_{\mathcal{A},\mathrm{sp}}\to X$ *is semistable of relative dimension* -1. *Its restriction to the open subset* $X-\underline{\mathrm{Disc}}(\mathcal{A})$ *is smooth. Consequently* $\mathrm{Coh}_{\mathcal{A},\mathrm{sp}}$ *is locally of finite type and smooth over* $\mathbb{F}_q$.

Proof. The last assertion follows from ([Lau], 3.2.1). Since the assertion is étale local on X we may assume that $X=\mathrm{Spec}\,R$ is affine with R a principal ideal domain, $|\,\underline{\mathrm{Disc}}(\mathcal{A})|=\{\mathfrak{p}\}$ and the generic fiber of $\mathcal{A}$ is $\cong M_d(F)$. By 3.2 we may also assume that $e_{\mathfrak{p}}(\mathcal{A})=d$. Let ϖ be a generator of $\mathfrak{p}$. Then

$\Gamma(\operatorname{Spec} R, \mathcal{A})$ is isomorphic to the R-subalgebra of $M_d(R)$ of matrices which are upper triangular modulo $\mathfrak{p}$. Hence $\Gamma(\operatorname{Spec} R, \mathcal{A})$ can be identified with the R-algebra $R^d\{\Pi\}$ defined by the relations

$$\Pi(x_1, \ldots, x_d) = (x_2, \ldots, x_d, x_1)\Pi, \qquad \Pi^d = \varpi.$$

Let $\mathrm{Coh}^{\square}_{\mathcal{A},\mathrm{sp}}(S)$ denote the groupoid of pairs $(\mathcal{K}, \alpha)$ where $\mathcal{K} \in \mathrm{Coh}_{\mathcal{A},\mathrm{sp}}(S)$ and $\alpha : \mathcal{O}_S^n \to N^*(\mathcal{K})$ is an isomorphism. The action of Π on $\mathcal{K}$ yields – by transport of structure via α – a map $\mathcal{O}_S^n \to \mathcal{O}_S^n$ of the form $(x_1, \ldots, x_d) \mapsto (x_2 a_1, \ldots, x_d a_{d-1}, x_1 a_d)$ for some $(a_1, \ldots, a_d) \in \Gamma(S, \mathcal{O}_S)$ such that $a_1 \cdots a_d = N^*(\varpi)$. Thus the assignement $(\mathcal{K}, \alpha) \mapsto (N, a_1, \ldots, a_d)$ defines an isomorphism

$$\mathrm{Coh}^{\square}_{\mathcal{A},\mathrm{sp}} \cong \operatorname{Spec} R[T_1, \ldots, T_d]/(T_1 \cdots T_d - \varpi)$$

Finally the forgetful morphism $\mathrm{Coh}^{\square}_{\mathcal{A},\mathrm{sp}} \to \mathrm{Coh}_{\mathcal{A},\mathrm{sp}}$ is a presentation. In fact it induces an isomorphism $\mathbb{G}_m^d \backslash \mathrm{Coh}^{\square}_{\mathcal{A},\mathrm{sp}} \cong \mathrm{Coh}_{\mathcal{A},\mathrm{sp}}$. Here the $\mathbb{G}_m^d$ action on $\mathrm{Coh}^{\square}_{\mathcal{A},\mathrm{sp}}$ is defined by the natural $\mathbb{G}_m^d(S)$-action on the set of isomorphisms $\alpha : \mathcal{O}_S^n \to N^*(\mathcal{K})$. $\square$

We finish this section with the following criterion for an $\mathcal{A} \boxtimes \mathcal{O}_S$-module $\mathcal{E}$ to be a locally free.

Lemma 3.11. *Let $U \subseteq X$ be a non-empty open subscheme such that $\mathcal{E}|_{U \times S}$ is a locally free $\mathcal{A}_U \boxtimes \mathcal{O}_S$-module. The following conditions are equivalent.*
(i) $\mathcal{E}$ is a locally free $\mathcal{A} \boxtimes \mathcal{O}_S$-module of rank r.
(ii) For $x \in |X - U|$ and any pair of k-morphism $g : S' \to S$ and $x_{S'} : S' \to \operatorname{Spec} k(x) \to X$ the quotient $g^*(\mathcal{E})/g^*(\mathcal{E})(-\frac{1}{e_x(\mathcal{A})} x_{S'})$ is a special $\mathcal{A}$-module of rank r.
(iii) For $x \in |\underline{\mathrm{Disc}}(\mathcal{A})| - U$ and any pair of k-morphism $g : S' \to S$ and $x_{S'} : S' \to \operatorname{Spec} k(x) \to X$ the quotient $g^*(\mathcal{E})/g^*(\mathcal{E})(-\frac{1}{e_x(\mathcal{A})} x_{S'})$ is a special $\mathcal{A}$-module of rank r.

Proof. That (i) implies (ii) follows from Lemma 2.17 and the equivalence of (ii) and (iii) from 3.8 (b) above.
(ii) $\Rightarrow$ (i) We may assume that S is affine, hence that S and of finite type over k. For $y \in |X \times S|$ we have to show that $\mathcal{E} \otimes \mathcal{O}_{(X \times S),y}$ is a free $(\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{O}_{(X \times S),y}$-module. It follows from ([Laf], I.2, lemme 4) that we may even replace S by the image s of $y \to X \times S \to S$. It follows from ([Laf], I.2, lemme 4). Thus it is enough to prove (i) if $S = \operatorname{Spec} \overline{k}$ is the algebraic closure of k. However in this case the assertion follows from 2.11 and 2.17. $\square$

We have the following generalization of ([Lau], Lemma 1.2.6).

Lemma 3.12. *Let $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{K} \to 0$ be a short exact sequence of right $\mathcal{A} \boxtimes \mathcal{O}_S$-modules. We assume $\mathcal{K}$ is coherent as an $\mathcal{O}_{X \times S}$-module, the map*

$\mathrm{Supp}(\mathcal{K}) \hookrightarrow X \times S \to S$ is an isomorphism and $\mathcal{K}$ is as an $\mathcal{O}_S$-module locally free of rank rd. We also assume that $\mathcal{E}$ is a locally free $\mathcal{A} \boxtimes \mathcal{O}_S$-module of rank r. Then the following conditions are equivalent.
(i) $\mathcal{E}'$ is a locally free $\mathcal{A} \boxtimes \mathcal{O}_S$-module of rank r.
(ii) $\mathcal{K}$ is special of rank r.

Proof. Again by using Lafforgues Lemma ([Laf], I.2.4) (applied to $\mathcal{A}$ and maximal tori in $\mathcal{A}$) it suffices to consider the case where k is algebraically closed and $S = \mathrm{Spec}\, k$. The assertion follows then from Lemma 2.18. □

4 The moduli space of $\mathcal{A}$-elliptic sheaves

4.1 $\mathcal{A}$-elliptic sheaves

In this chapter X denotes a smooth projective geometrically connected curve over the finite field $\mathbb{F}_q$ of characteristic p, F the function field of X. We also fix a closed point $\infty \in |X|$. Let $\mathcal{A}$ be a locally principal $\mathcal{O}_X$-order of rank d^2 and let A be its generic fiber. We make the following

Assumption 4.1. *$e_\infty(\mathcal{A}) = d$.*

Definition 4.2. *Let S be an $\mathbb{F}_q$-scheme. An $\mathcal{A}$-elliptic sheaf over S with pole ∞ is a triple $E = (\mathcal{E}, \infty_S, t)$, where $\mathcal{E}$ is a locally free right $\mathcal{A} \boxtimes \mathcal{O}_S$-module of rank 1, where $\infty_S : S \to X$ is an $\mathbb{F}_q$-morphism with $\infty_S(S) = \{\infty\}$ and where*

$$t : {}^\tau(\mathcal{E}(-\frac{1}{d}\infty_S)) \longrightarrow \mathcal{E}$$

is an injective $\mathcal{A} \boxtimes \mathcal{O}_S$-linear homomorphism such that the following condition holds:
(*) The map $\mathrm{Supp}(\mathrm{Coker}(t)) \hookrightarrow X \times S \to S$ is an isomorphism. Considered as a sheaf on S, $\mathcal{K}$ is a locally free $\mathcal{O}_S$-module of rank d.
Hence $\mathrm{Supp}(\mathrm{Coker}(t))$ is the image of the graph of a $\mathbb{F}_q$-morphism $\iota_0 : S \to X$ called the zero (or characteristic) of E.
We denote by $\mathcal{E}\ell\ell_{\mathcal{A}}^\infty$ the stack over $\mathbb{F}_q$ such that for each $S \in \mathrm{Sch}/k$, $\mathcal{E}\ell\ell_{\mathcal{A}}^\infty(S)$ is the category whose objects are $\mathcal{A}$-elliptic sheaves over S and whose morphisms are isomorphisms between $\mathcal{A}$-elliptic sheaves.

For $n \in \frac{1}{d}\mathbb{Z}$ we define $\mathcal{E}\ell\ell_{\mathcal{A},n}^\infty$ to be the open and closed substack of $\mathcal{A}$-elliptic sheaves $E = (\mathcal{E}, \infty_S, t)$ with fixed degree $\deg_{\mathcal{A}}(\mathcal{E}) = n$. The functor which maps an $\mathcal{A}$-elliptic sheaf over $E = (\mathcal{E}, \infty_S, t)$ over S to its zero $\iota_0 : S \to X$ defines a morphism *char* : $\mathcal{E}\ell\ell_{\mathcal{A}}^\infty \to X$ (called the *characteristic morphism*). Similarly $E = (\mathcal{E}, \infty_S, t) \mapsto \infty_S$ defines a morphism *pole* : $\mathcal{E}\ell\ell_{\mathcal{A}}^\infty \to \mathrm{Spec}\, k(\infty)$. By Lemma 3.12, $\mathrm{Coker}(t)$ is a special $\mathcal{A}$-module of rank 1. This fact allows us to compare the above condition (*) with the *condition spéciale* in ([Hau], section 3) (see also 5.11 (b) below). It follows that the characteristic morphism factors as

$$char : \mathcal{E}\ell\ell_{\mathcal{A}}^\infty \longrightarrow \mathrm{Coh}_{\mathcal{A},sp} \xrightarrow{N} X \tag{16}$$

We will see in the proof of Theorem 4.11 below that the first arrow is smooth.

Remarks 4.3. (a) The concept of an $\mathcal{A}$-elliptic sheaf is due to Laumon, Rapoport and Stuhler ([LRS], section 2). The definition given above is different but, as will be explained in the appendix, equivalent to the one given in ([LRS], section 2). In fact our Definition 4.2 is slightly more general. Their notion corresponds to an $\mathcal{A}$-elliptic sheaf where (i) A is a division algebra which is unramified at ∞, (ii) $\mathcal{A}|_{X-\{\infty\}}$ is a maximal order in A and (iii) the zero ι_0 is disjoint from $|\,\underline{\mathrm{Disc}}(\mathcal{A})|$, i.e. ι_0 factors through $X - |\,\underline{\mathrm{Disc}}(\mathcal{A})| \hookrightarrow X$ (the latter condition was weakened in [BS] and [Hau] to require only that ι_0 factors through $(X - |\,\underline{\mathrm{Disc}}(\mathcal{A})|) \cup \{\infty\} \cup \{x \in |X| \mid \mathrm{inv}_x(A) = \frac{1}{d}\}$).
(b) Let $\mathcal{A}$ be the subsheaf of $M_d(\mathcal{O}_X)$ of matrices which are upper triangular modulo ∞. In this case $\mathcal{E}\ell\ell_{\mathcal{A}}^{\infty}$ is isomorphic to the stack $\mathcal{E}\ell\ell_X^{(d)}$ of elliptic sheaves of rank d (hence above $X - \{\infty\}$ it is isomorphic to the stack of Drinfeld modules of rank d; compare ([BS], section 3)). In fact by Proposition 5.10 of the appendix we have $\mathcal{E}\ell\ell_{\mathcal{A}}^{\infty} \cong \mathcal{P}\mathcal{E}\ell\ell_{M_d(\mathcal{O}_X)}^{\infty}$ and the latter is isomorphic to the stack of $\mathcal{E}\ell\ell_X^{(d)}$ by Morita equivalence.
(c) If A is a division algebra then $\mathcal{A}$-elliptic sheaves are special cases of *right $\mathcal{A}$-shtukas* of rank 1 ([Laf], 1.1). Recall that an $\mathcal{A}$-shtuka of rank 1 is a diagram

$$\begin{tikzcd}[column sep=small] \mathcal{E} \arrow[dr, "j"] & \\ & \mathcal{E}' \\ {}^{\tau}\mathcal{E} \arrow[ur, "t"] & \end{tikzcd}$$

where $\mathcal{E}, \mathcal{E}'$ are locally free right $\mathcal{A} \boxtimes \mathcal{O}_S$-modules of rank 1 and where j and t are injective $\mathcal{A} \boxtimes \mathcal{O}_S$-linear homomorphism such that the cokernels of j and t and of the dual morphisms $j^\vee$ and $t^\vee$ satisfy condition (*) above (actually, it follows from Lemma 3.12 (compare also the proof of 4.14 (b) below) that it is enough to require that the cokernels of j and t satisfies (*)). Hence we have $\mathrm{Coker}(j), \mathrm{Coker}(t) \in \mathrm{Coh}_{\mathcal{A},\mathrm{sp}}(S)$. In fact if $E = (\mathcal{E}, \infty_S, t) \in \mathcal{E}\ell\ell_{\mathcal{A}}^{\infty}(S)$ is an $\mathcal{A}$-elliptic sheaf with zero $\iota_0 : S \to X$ then the diagram

$$\begin{tikzcd}[column sep=small] \mathcal{E}(-\frac{1}{d}\infty_S) \arrow[dr, "j"] & \\ & \mathcal{E} \\ {}^{\tau}(\mathcal{E}(-\frac{1}{d}\infty_S)) \arrow[ur, "t"] & \end{tikzcd} \tag{17}$$

is an $\mathcal{A}$-shtuka with pole ∞_S and zero ι_0. Therefore we have a 2-cartesian square

$$\begin{tikzcd} \mathcal{E}\ell\ell_{\mathcal{A}}^{\infty} \arrow[r] \arrow[d, "pole"] & \mathrm{Sht}_{\mathcal{A}}^1 \arrow[d] \\ \mathrm{Spec}\, k(\infty) \arrow[r] & \mathrm{Coh}_{\mathcal{A},sp} \end{tikzcd} \tag{18}$$

Here the second vertical arrow is given by mapping an $\mathcal{A}$-shtuka $(\mathcal{E}, \mathcal{E}', j, t)$ to $\mathrm{Coker}(j)$. The lower horizontal arrow is defined by $\mathcal{A}/\mathcal{A}(-\frac{1}{d}\infty_{\mathrm{Spec}\, k(\infty)}) \in \mathrm{Coh}_{\mathcal{A},sp}(\mathrm{Spec}\, k(\infty))$. It is easy to see that it is representable and a closed immersion. Hence the morphism $\mathcal{E}\ell\ell_{\mathcal{A}}^{\infty} \to \mathrm{Sht}_{\mathcal{A}}^{1}$ given by (17) is a closed immersion.
(d) One could consider $\mathcal{A}$-elliptic sheaves more generally for a hereditary $\mathcal{O}_X$-order $\mathcal{A}$. However since any hereditary $\mathcal{O}_X$-order is Morita equivalent to a locally principal $\mathcal{O}_X$-order we do not obtain new moduli spaces in this way.
(e) If we consider $\mathcal{E}\ell\ell_{\mathcal{A}}^{\infty}$ as a $k(\infty)$- rather than a $\mathbb{F}_q$-stack we can (and will) drop ∞_S from the definition. More precisely for $S \in \mathrm{Sch}/k(\infty)$ the objects of $\mathcal{E}\ell\ell_{\mathcal{A}}^{\infty}(S)$ are just pairs $E = (\mathcal{E}, t)$ such that $(\mathcal{E}, \infty_S, t)$ is an $\mathcal{A}$-elliptic sheaf as in 4.2 where ∞_S is composite $S \to \mathrm{Spec}\, k(\infty) \hookrightarrow X$.
(f) Define an automorphism of stacks $\theta : \mathcal{E}\ell\ell_{\mathcal{A}}^{\infty} \to \mathcal{E}\ell\ell_{\mathcal{A}}^{\infty}$ by

$$\theta(\mathcal{E}, \infty_S, t) = (\mathcal{E}(\frac{1}{d}{}^{\tau}\infty_S), {}^{\tau}\infty_S, t(\frac{1}{d}{}^{\tau}\infty_S)) \tag{19}$$

where ${}^{\tau}\infty_S = \infty_S \circ \mathrm{Frob}_S$. We have $\theta(\mathcal{E}\ell\ell_{\mathcal{A},n}^{\infty}) = \mathcal{E}\ell\ell_{\mathcal{A},n+\frac{1}{d}}^{\infty}$ for all $n \in \frac{1}{d}\mathbb{Z}$ and $\theta^{\deg(\infty)}(E) = E \otimes_{\mathcal{A}} \mathcal{A}(\frac{1}{d}\infty)$ for all $\mathcal{A}$-elliptic sheaves E.
(g) Let $\mathcal{A}'$ be a locally principal $\mathcal{O}_X$-order which is Morita equivalent to $\mathcal{A}$ and let $\mathcal{L}$ be an invertible $\mathcal{A}$-$\mathcal{A}'$-bimodule. Then

$$E = (\mathcal{E}, \infty_S, t) \mapsto E \otimes_{\mathcal{A}} \mathcal{L} :\,= (\mathcal{E} \otimes_{\mathcal{A}} \mathcal{L}, \infty_S, t \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{L}})$$

defines an isomorphism between $\mathcal{E}\ell\ell_{\mathcal{A}}^{\infty}$ and $\mathcal{E}\ell\ell_{\mathcal{A}'}^{\infty}$. If $m = \deg_{\mathcal{A}}(\mathcal{L})$ then it maps the substack $\mathcal{E}\ell\ell_{\mathcal{A},n}^{\infty}$ isomorphically onto the substack $\mathcal{E}\ell\ell_{\mathcal{A},m+n}^{\infty}$. In particular $E \mapsto E \otimes_{\mathcal{A}} \mathcal{L}$ defines an action of the abelian group $\mathrm{Pic}(\mathcal{A})$ on $\mathcal{E}\ell\ell_{\mathcal{A}}^{\infty}$.

We define $\mathrm{Pic}(\mathcal{A})[\theta]$ to be the group generated by its subgroup $\mathrm{Pic}(\mathcal{A})$ and the element θ which satisfies the relations $\theta^{\deg(\infty)} = \mathcal{A}(\frac{1}{d}\infty)$ and $\theta\mathcal{L} = \mathcal{L}\theta$ for all $\mathcal{L} \in \mathrm{Pic}(\mathcal{A})$. Thus $\mathrm{Pic}(\mathcal{A})[\theta]$ acts on $\mathcal{E}\ell\ell_{\mathcal{A}}^{\infty}$. The group $\mathrm{Pic}(\mathcal{A})[\theta]$ is an extension of $\mathbb{Z}/\deg(\infty)\mathbb{Z} \cong \mathrm{G}(k(\infty)/\mathbb{F}_q)$ by $\mathrm{Pic}(\mathcal{A})$. The map $\deg_{\mathcal{A}} : \mathrm{Pic}(\mathcal{A}) \to \frac{1}{d}\mathbb{Z}$ extends to a homomorphism $\deg_{\mathcal{A}} : \mathrm{Pic}(\mathcal{A})[\theta] \to \frac{1}{d}\mathbb{Z}$ by defining $\deg_{\mathcal{A}}(\theta) = \frac{1}{d}$.

Definition 4.4. *The group of modular automorphisms* $\mathcal{W}(\mathcal{A}, \infty)$ *is defined as the kernel of* $\deg_{\mathcal{A}} : \mathrm{Pic}(\mathcal{A})[\theta] \to \frac{1}{d}\mathbb{Z}$.

$\mathcal{W}(\mathcal{A}, \infty)$ stabilizes the substack $\mathcal{E}\ell\ell_{\mathcal{A},n}^{\infty}$ for all $n \in \frac{1}{d}\mathbb{Z}$. There exists a canonical homomorphism

$$\mathcal{W}(\mathcal{A}, \infty) \to \mathrm{G}(k(\infty)/\mathbb{F}_q) \tag{20}$$

so that *pole* : $\mathcal{E}\ell\ell_{\mathcal{A}}^{\infty} \to \mathrm{Spec}\, k(\infty)$ is $\mathcal{W}(\mathcal{A}, \infty)$-equivariant. The kernel of (20) is $\mathrm{Pic}_0(\mathcal{A})$ and the image is of order $\frac{\delta(\mathcal{A})\deg(\infty)}{d}$ (thus (20) is surjective if and only if $\deg_{\mathcal{A}} : \mathrm{Pic}(\mathcal{A}) \to \frac{1}{d}\mathbb{Z}$ is surjective).

4.2 Level structure

We reformulate now the notion of a level structure on an $\mathcal{A}$-elliptic sheaf given in ([LRS], 2.7 and 8.4) in our framework. Let $I = \sum_x n_x x$ be an effective divisor on X. We recall first from ([LRS], 2.7; see also [Dr3]) the notion of a level-I-structure when ∞ does not divide I, i.e. $n_\infty = 0$.

Definition 4.5. *Suppose that $\infty \notin |I|$. Let $E = (\mathcal{E}, \infty_S, t)$ be an $\mathcal{A}$-elliptic sheaf over an $\mathbb{F}_q$-scheme S with zero $\iota_0 : S \to X$ disjoint from I i.e. $\iota_0(S) \cap I = \emptyset$. A level-$I$-structure on E is an $\mathcal{A}_I \boxtimes \mathcal{O}_S$-linear isomorphism*

$$\alpha : \mathcal{A}_I \boxtimes \mathcal{O}_S \longrightarrow \mathcal{E}|_{I\times S} \otimes_{\mathcal{A}} \mathcal{A}(\frac{1}{d}\infty)$$

compatible with t, i.e. the diagram

$$\begin{tikzcd} {}^\tau\mathcal{E}|_{I\times S} \arrow[rr, "t|_{I\times S}"] & & \mathcal{E}|_{I\times S} \\ & \mathcal{A}_I \boxtimes \mathcal{O}_S \arrow[ul, bend left, "{}^\tau\alpha"] \arrow[ur, bend right, "\alpha"'] & \end{tikzcd}$$

commutes.

We denote by $\mathcal{E}\ell\ell^\infty_{\mathcal{A},I}$ the stack of $\mathcal{A}$-elliptic sheaves with level I-structure and for $n \in \frac{1}{d}\mathbb{Z}$ by $\mathcal{E}\ell\ell^\infty_{\mathcal{A},I,n}$ the open and closed substack of $\mathcal{A}$-elliptic sheaves with level I-structure with fixed degree $\deg_{\mathcal{A}} = n$. Again we obtain morphisms $char : \mathcal{E}\ell\ell^\infty_{\mathcal{A},I} \to X - I$ and $pole : \mathcal{E}\ell\ell^\infty_{\mathcal{A},I} \to \operatorname{Spec} k(\infty)$. The automorphism (19) of Remark 4.3 (f) extends canonically to an automorphism $\theta : \mathcal{E}\ell\ell^\infty_{\mathcal{A},I} \to \mathcal{E}\ell\ell^\infty_{\mathcal{A},I}$. The right action of $\operatorname{Pic}(\mathcal{A})$ on $\mathcal{E}\ell\ell^\infty_{\mathcal{A}}$ lifts to a right action of $\operatorname{Pic}_I(\mathcal{A})$ on $\mathcal{E}\ell\ell^\infty_{\mathcal{A},I}$ as follows. If $(\mathcal{L}, \beta)$ is an invertible $\mathcal{A}$-$\mathcal{A}$-bimodule with level-I-structure and (E, α) an $\mathcal{A}$-elliptic sheaf with level-I-structure (E, α) over S then we define $(E, \alpha) \otimes (\mathcal{L}, \beta) := (E \otimes \mathcal{L}, \alpha \bullet \beta)$ with

$$\alpha \bullet \beta : \mathcal{A}_I \boxtimes \mathcal{O}_S \stackrel{\beta\boxtimes \mathrm{id}}{\longrightarrow} \mathcal{L}_I \boxtimes \mathcal{O}_S = (\mathcal{A}_I \boxtimes \mathcal{O}_S) \otimes_{\mathcal{A}} \mathcal{L} \boxtimes \mathcal{O}_S \stackrel{\alpha\boxtimes \mathrm{id}}{\longrightarrow} (\mathcal{E} \otimes_{\mathcal{A}} \mathcal{L})|_{I\times S}. \quad (21)$$

As before we have $\theta^{\deg(\infty)}(E, \alpha) = (E, \alpha) \otimes (\mathcal{A}(\frac{1}{d}\infty), \mathrm{id})$.
Suppose now that $|I| = \{\infty\}$, i.e. $I = n\infty$ with $n > 0$. Let $k(\infty)_d$ be a fixed extension of degree d of $k(\infty)$. According to section 2.6 there exists a pair $(\mathcal{M}_\infty, \phi_\infty)$ consisting of a free right $\mathcal{A}_\infty \otimes_{\mathbb{F}_q} k(\infty)_d$-module $\mathcal{M}_\infty$ of rank 1 and an isomorphism

$$\phi_\infty : {}^\sigma(\mathcal{M}_\infty \mathfrak{P}) \longrightarrow \mathcal{M}_\infty$$

where $\mathfrak{P}$ denotes the maximal invertible two-sided ideal of $\mathcal{A}_\infty \otimes_{\mathbb{F}_q} k(\infty)_d$ corresponding to the inclusion $k(\infty) \hookrightarrow k(\infty)_d$. Let $\mathcal{M}_I$ denote the sheaf of $\mathcal{A}_I \otimes_{\mathbb{F}_q} k(\infty)_d$-modules associated to the $\mathcal{M}_\infty/\mathcal{M}_\infty \mathfrak{p}^n_\infty$. The map ϕ_∞ induces an isomorphism

$$\phi_I : {}^\tau(\mathcal{M}_I(-\frac{1}{d}\infty_d)) \longrightarrow \mathcal{M}_I$$

where ∞_d denotes the morphism $\infty_{k(\infty)_d} : \operatorname{Spec} k(\infty)_d \to \operatorname{Spec} k(\infty) \hookrightarrow X$.

Definition 4.6. *Let $E = (\mathcal{E}, \infty_S, t)$ be an $\mathcal{A}$-elliptic sheaf over an $\mathbb{F}_q$-scheme S with zero $\iota_0 : S \to X$ disjoint from I.*
(a) Suppose that $I = n\infty$ with $n > 0$. Let $E = (\mathcal{E}, \infty_S, t)$ be an $\mathcal{A}$-elliptic sheaf over an $\mathbb{F}_q$-scheme S with zero $\iota_0 : S \to X$ disjoint from I. A level-I-structure on E consist of a pair (λ, α) where $\lambda : S \to \operatorname{Spec} k(\infty)_d$ is an $\mathbb{F}_q$-morphism of schemes which lifts the pole ∞_S and where α is an $\mathcal{A}_{I_\infty} \boxtimes \mathcal{O}_S$-linear isomorphism

$$\alpha : (\mathrm{id}_I \times \lambda)^*(\mathcal{M}_I) \longrightarrow \mathcal{E}|_{I \times S}$$

such that the diagram

$$\begin{tikzcd}
({}^\tau(\mathcal{E}(-\frac{1}{d}\infty_S))|_{I\times S} \arrow[rr, "t|_{I\times S}"] & & \mathcal{E}|_{I\times S} \\
(\mathrm{id}_I \times\lambda)^*({}^\tau(\mathcal{M}_I(-\frac{1}{d}\infty_d))) \arrow[u, "{}^\tau(\alpha(-\frac{1}{d}\infty_S))"] \arrow[rr, "(\mathrm{id}_I\times\lambda)^*(\phi_I)"] & & (\mathrm{id}_I\times\lambda)^*(\mathcal{M}_I) \arrow[u, "\alpha"']
\end{tikzcd}$$

commutes.
(b) Suppose that I is an arbitrary effective divisor on X with $\infty \in |I|$ and write $I = n\infty + I^\infty = I_\infty + I^\infty$ with $n > 0$ such that ∞ does not divide I^∞. A level-I-structure on E is a triple $(\alpha_f, \lambda, \alpha_\infty)$ consisting of a level-I^∞-structure α_f and a level-I_∞-structure (λ, α_∞).

Let I be an effective divisor on X with $\infty \in |I|$. Again we define $\mathcal{E}\ell\ell^\infty_{\mathcal{A},I}$ as the stack of $\mathcal{A}$-elliptic sheaves with level-I-structure $(\mathcal{E}, t, \alpha_f, \lambda, \alpha_\infty)$ and denote for $n \in \frac{1}{d}\mathbb{Z}$ by $\mathcal{E}\ell\ell^\infty_{\mathcal{A},I,n}$ the substack where $\deg_{\mathcal{A}}(\mathcal{E}) = n$. There are canonical morphisms

$$\mathit{char} : \mathcal{E}\ell\ell^\infty_{\mathcal{A},I} \to X - I, \qquad \mathit{pole} : \mathcal{E}\ell\ell^\infty_{\mathcal{A},I} \to \operatorname{Spec} k(\infty)_d$$

(the latter is given by $(E, \alpha_f, \lambda, \alpha_\infty) \mapsto \lambda$; it lifts the morphism $\mathit{pole} : \mathcal{E}\ell\ell^\infty_{\mathcal{A}} \to \operatorname{Spec} k(\infty)$).

Modular automorphisms. Next we are going to extend the definition of the automorphisms (19) and define a natural right action of a certain idele class group on $\mathcal{E}\ell\ell^\infty_{\mathcal{A},I}$ (thus lifting the action of $\mathrm{Pic}_I(\mathcal{A})$ when $\infty \notin |I|$). Define $\theta : \mathcal{E}\ell\ell^\infty_{\mathcal{A},I} \to \mathcal{E}\ell\ell^\infty_{\mathcal{A},I}$ by

$$\theta(\mathcal{E}, \infty_S, t, \alpha_f, \lambda, \alpha_\infty) = (\mathcal{E}(\frac{1}{d}{}^\tau\infty_S), {}^\tau\infty_S, t(\frac{1}{d}{}^\tau\infty_S), \alpha_f, {}^\tau\lambda, \alpha^\sharp_\infty) \tag{22}$$

where $\alpha^\sharp_\infty$ is the composite

$$(\mathrm{id}_I \times {}^\tau\lambda)^*(\mathcal{M}_I) \xrightarrow{\phi_I(\frac{1}{d}{}^\tau\infty_d)} (\mathrm{id}_I \times \lambda)^*(\mathcal{M}_I(\frac{1}{d}{}^\tau\infty_d)) \xrightarrow{\alpha_\infty(\frac{1}{d}{}^\tau\infty_S)} \mathcal{E}(\frac{1}{d}{}^\tau\infty_S)|_{I\times S}.$$

Write $I = n\infty + I^\infty = I_\infty + I^\infty$ with $n > 0$ and $\infty \notin |I^\infty|$. Let $\mathcal{D}_\infty$ be a principal order in a central F-algebra D_∞ of dimension d^2 such that

$$e(\mathcal{D}_\infty) = e(\mathcal{A}_\infty) \quad \text{and} \quad \mathrm{inv}(D_\infty) = \mathrm{inv}(A_\infty) + \frac{1}{d}.$$

We have seen in section 2.6 that

$$\mathcal{D}_\infty \cong \mathrm{End}_{\mathcal{A}_\infty \otimes_{\mathbb{F}_q} k(\infty)_d}(\mathcal{M}_\infty, \phi_\infty).$$

We choose an isomorphism (thus making $(\mathcal{M}_\infty, \phi_\infty)$ into an invertible ϕ-$\mathcal{D}_\infty$-$\mathcal{A}_\infty$-bimodule of slope $-\frac{1}{d}$). Let

$$U_I(\mathcal{A}^\infty \times \mathcal{D}_\infty) \colon = \mathrm{Ker}(\prod_{x \in |X| - \{\infty\}} \mathcal{A}_x^* \times \mathcal{D}_\infty^* \to \Gamma(I^\infty, \mathcal{A}_{I^\infty})^* \times (\mathcal{D}_\infty / \mathfrak{p}_\infty^n \mathcal{D}_\infty)^*)$$

and define

$$\mathcal{C}_I(\mathcal{A}^\infty \times \mathcal{D}_\infty) \colon = ({\prod}'_{x \in |X| - \{\infty\}} N(\mathcal{A}_x)) \times N(\mathcal{D}_\infty) / U_I(\mathcal{A}^\infty \times \mathcal{D}_\infty) F^*$$

For $a = (a_f, a_\infty) = (\{a_x\}_{x \neq \infty}, a_\infty) \in ({\prod}'_{x \in |X| - \{\infty\}} N(\mathcal{A}_x)) \times N(\mathcal{D}_\infty)$ let $\mathrm{div}(a) = \sum_{x \in |X| - \{\infty\}} v_{\mathcal{A}_x}(a_x) x + v_{\mathcal{D}_\infty}(a_\infty) \infty \in \mathrm{Div}(\mathcal{A})$. Let $(E, \alpha_f, \lambda, \alpha_\infty) \in \mathcal{E}\ell\ell^\infty_{\mathcal{A},I}$. Left multiplication by a_f on ${\prod}'_{x \in |X| - \{\infty\}} N(\mathcal{A}_x)$ induces a level-I^∞-structure $\alpha_f \cdot a_f$ on $\mathcal{E}(\mathrm{div}(a))$. Similarly left multiplication by a_∞ on $\mathcal{M}_\infty$ yields a level-I_∞-structure $\alpha_\infty \cdot a_\infty$ on $E \otimes \mathcal{A}(\mathrm{div}(a))$. One easily verifies that

$$(E, \alpha_f, \lambda, \alpha_\infty) \cdot a \colon = (E \otimes \mathcal{A}(\mathrm{div}(a)), \alpha_f \cdot a_f, \lambda, \alpha_\infty \cdot a_\infty)$$

yields a right $({\prod}'_{x \in |X| - \{\infty\}} N(\mathcal{A}_x)) \times N(\mathcal{D}_\infty)$-action on $\mathcal{E}\ell\ell^\infty_{\mathcal{A},I}$ and that it factors through $\mathcal{C}_I(\mathcal{A}^\infty \times \mathcal{D}_\infty)$.
The canonical projection $\mathcal{C}_I(\mathcal{A}^\infty \times \mathcal{D}_\infty) \to \mathcal{C}_{I^\infty}(\mathcal{A})$ (given on the ∞-factor by $N(\mathcal{D}_\infty) \xrightarrow{v_{\mathcal{D}_\infty}} \frac{1}{d}\mathbb{Z} \cong N(\mathcal{A}_\infty)/\mathcal{A}_\infty^*$) followed by the isomorphism $\mathcal{C}_{I^\infty}(\mathcal{A}) \to \mathrm{Pic}_{I^\infty}(\mathcal{A})$ from 3.4 yields also a $\mathcal{C}_I(\mathcal{A}^\infty \times \mathcal{D}_\infty)$-action on $\mathcal{E}\ell\ell^\infty_{\mathcal{A},I^\infty}$ and one checks that the forgetful morphism of stacks $\mathcal{E}\ell\ell^\infty_{\mathcal{A},I} \longrightarrow \mathcal{E}\ell\ell^\infty_{\mathcal{A},I^\infty}$ commutes with the $\mathcal{C}_I(\mathcal{A}^\infty \times \mathcal{D}_\infty)$-actions.
By Remark 2.22, there exists a prime element $\varpi_\infty \in \mathcal{O}_\infty$ such that the class $\xi \in \mathcal{C}_I(\mathcal{A}^\infty \times \mathcal{D}_\infty)$ of the idele $(\{1\}_{x \neq \infty}, \varpi_\infty)$ satisfies $\theta^{d \deg(\infty)}(E) = E \cdot \xi$ for all $E \in \mathcal{E}\ell\ell^\infty_{\mathcal{A},I}(S)$.
If ∞ does not divide the level I we define the group $\mathrm{Pic}_I(\mathcal{A})[\theta]$ similar to $\mathrm{Pic}(\mathcal{A})[\theta]$ in the last section. $\mathrm{Pic}_I(\mathcal{A})[\theta]$ contains $\mathrm{Pic}_I(\mathcal{A})$ as a subgroup and the element θ lies in the center and satisfies the relation $\theta^{\deg(\infty)} = (\mathcal{A}(\frac{1}{d}\infty), \mathrm{id})$. Let $\deg_\mathcal{A} : \mathrm{Pic}_I(\mathcal{A})[\theta] \to \frac{1}{d}\mathbb{Z}$ be given by $(\mathcal{L}, \beta) \mapsto \deg_\mathcal{A}(\mathcal{L})$ on $\mathrm{Pic}_I(\mathcal{A})$ and $\deg_\mathcal{A}(\theta) = \frac{1}{d}$.
Assume that ∞ divides I and write $I = n\infty + I^\infty = I_\infty + I^\infty$ as above. Define $\mathcal{C}_I(\mathcal{A}^\infty \times \mathcal{D}_\infty)[\theta]$ as the group generated by $\mathcal{C}_I(\mathcal{A}^\infty \times \mathcal{D}_\infty)$ and a central element θ satisfying the relation $\theta^{d \deg(\infty)} = \xi$. The homomorphism $\mathcal{C}_I(\mathcal{A}^\infty \times \mathcal{D}_\infty) \to \mathcal{C}_{I^\infty}(\mathcal{A}) \cong \mathrm{Pic}_{I^\infty}(\mathcal{A}) \xrightarrow{\deg_\mathcal{A}} \frac{1}{d}\mathbb{Z}$ extends to a homomorphism $\deg_\mathcal{A} : \mathcal{C}_I(\mathcal{A}^\infty \times \mathcal{D}_\infty)[\theta] \to \frac{1}{d}\mathbb{Z}$ by setting $\deg_\mathcal{A}(\theta) = \frac{1}{d}\mathbb{Z}$.

Definition 4.7. *Let I be an effective divisor on X. The group of modular automorphisms $\mathcal{W}(\mathcal{A}, I, \infty)$ of $\mathcal{E}\ell\ell^\infty_{\mathcal{A},I}$ is defined as follows:*

$$\mathcal{W}(\mathcal{A}, I, \infty) \colon = \begin{cases} \mathrm{Ker}(\deg_{\mathcal{A}} : \mathrm{Pic}_I(\mathcal{A})[\theta] \to \frac{1}{d}\mathbb{Z}) & \text{if } \infty \notin |I|, \\ \mathrm{Ker}(\deg_{\mathcal{A}} : \mathcal{C}_I(\mathcal{A}^\infty \times \mathcal{D}_\infty)[\theta] \to \frac{1}{d}\mathbb{Z}) & \text{if } \infty \in |I|. \end{cases}$$

Remarks 4.8. (a) If $\infty \notin |I|$ (resp. $\infty \in |I|$) there exists a canonical homomorphism $\mathcal{W}(\mathcal{A}, I, \infty) \to \mathrm{G}(k(\infty)/\mathbb{F}_q)$ (resp. $\mathcal{W}(\mathcal{A}, I, \infty) \to \mathrm{G}(k(\infty)_d/\mathbb{F}_q)$) with kernel $\mathrm{Pic}_{I,0}(\mathcal{A})$ (resp. $\mathcal{C}_I(\mathcal{A}^\infty \times \mathcal{D}_\infty)_0$) such that $\mathcal{E}\ell\ell^\infty_{\mathcal{A},I} \to \mathrm{Spec}\, k(\infty)$ (resp. $\mathcal{E}\ell\ell^\infty_{\mathcal{A},I} \to \mathrm{Spec}\, k(\infty)_d$) is $\mathcal{W}(\mathcal{A}, I, \infty)$-equivariant.
(b) Let $I < J$ be effective divisors on X there exists a canonical projection $\mathcal{W}(\mathcal{A}, J, \infty) \to \mathcal{W}(\mathcal{A}, I, \infty)$ such that the forgetful morphism $\mathcal{E}\ell\ell^\infty_{\mathcal{A},J} \longrightarrow \mathcal{E}\ell\ell^\infty_{\mathcal{A},I}$ is $\mathcal{W}(\mathcal{A}, J, \infty)$-equivariant.
(c) The map $x \mapsto x\theta^{-d\deg_{\mathcal{A}}(x)}$ induces an isomorphism

$$\mathcal{C}_I(\mathcal{A}^\infty \times \mathcal{D}_\infty)/\xi_\infty^{\mathbb{Z}} \quad \cong \quad \mathcal{W}(\mathcal{A}, I, \infty).$$

This fact will be used in section 4.6.

We have (compare ([LRS], 8.10) and ([Laf], I.3.5))

Lemma 4.9. *Let $I < J$ be effective divisors on X. Over $X - J$ the forgetful morphism*

$$\mathcal{E}\ell\ell^\infty_{\mathcal{A},J} \longrightarrow \mathcal{E}\ell\ell^\infty_{\mathcal{A},I}$$

is representable and is a finite, étale Galois covering. Its Galois group is $\cong$ $\mathrm{Ker}(\mathcal{W}(\mathcal{A}, J, \infty) \to \mathcal{W}(\mathcal{A}, I, \infty))$. If $\infty \notin |J| - |I|$ it is $\cong \mathrm{Ker}(\mathcal{A}_J^* \to \mathcal{A}_I^*)$.

Corollary 4.10. *Let $\mathcal{A}'$ be a locally principal $\mathcal{O}_X$-suborder of $\mathcal{A}$ with the same generic fiber A and denote by $\iota : Y \hookrightarrow X$ the reduced closed subscheme with $|Y| = \{x \in |X| \mid e_x(\mathcal{A}') > e_x(\mathcal{A}')\}$. Note that $\infty \notin Y$. Let I be an effective divisors disjoint from Y and put $J \colon = I + Y$. Then over $X - J$ the forgetful morphism factors canonically as*

$$\mathcal{E}\ell\ell^\infty_{\mathcal{A},J} \longrightarrow \mathcal{E}\ell\ell^\infty_{\mathcal{A}',I} \longrightarrow \mathcal{E}\ell\ell^\infty_{\mathcal{A},I}. \tag{23}$$

Both maps are representable and finite and étale. Moreover the first arrow is Galois.

Proof. Let $\mathcal{P} \colon = \mathrm{Im}(\mathcal{A}'|_Y \to \mathcal{A}|_Y)$. Then the diagram

$$\begin{array}{ccc} \mathcal{A}' & \longrightarrow & \iota_*(\mathcal{P}) \\ \big\downarrow & & \big\downarrow \\ \mathcal{A} & \longrightarrow & \iota_*(\mathcal{A}_Y) \end{array}$$

is cartesian. Here we view J as a closed subscheme of X and denote by $\iota :$ $J \to X$ the inclusion. For $E = (\mathcal{E}, t, \infty_S, \alpha)$ in $\mathcal{E}\ell\ell^\infty_{\mathcal{A},J}(S)$ we decompose α into a level-I-structure α_I and a level-Y-structure α_Y. Define $\mathcal{E}'$ by the cartesian square

$$\begin{tikzcd} \mathcal{E}' \arrow[rr] \arrow[d] & & \iota_*(\mathcal{P}) \arrow[d] \\ \mathcal{E} \arrow[r] & \iota_*(\mathcal{E}|_{Y\times S}) \arrow[r, "{\alpha_Y}^{-1}"] & \iota_*(\mathcal{A}_Y) \end{tikzcd}$$

Then $\mathcal{E}'$ is a locally free $\mathcal{A}'\boxtimes\mathcal{O}_S$-module of rank 1. The first morphism in (23) is induced by $\mathcal{E}\mapsto\mathcal{E}'$ whereas the second by $\mathcal{E}'\mapsto\mathcal{E}'\otimes_{\mathcal{A}'}\mathcal{A}$. The proof of the remaining assertions is left to the reader. $\square$

4.3 The coarse moduli scheme

Our aim now is to prove the following theorem.

Theorem 4.11. *(a) $\mathcal{E}\ell\ell_{\mathcal{A},I}^\infty$ is a Deligne-Mumford stack over $\mathbb{F}_q$. It is locally of finite type over X. The morphism char : $\mathcal{E}\ell\ell_{\mathcal{A},I}^\infty\to X-I$ is semistable of relative dimension $d-1$.*
(b) The open and closed substack $\mathcal{E}\ell\ell_{\mathcal{A},I,0}^\infty$ of $\mathcal{E}\ell\ell_{\mathcal{A},I}^\infty$ is of finite type over $X' := X-(\underline{\mathrm{Disc}}(\mathcal{A})\cup I)$. It admits a coarse moduli scheme which will be denote by $\mathrm{Ell}_{\mathcal{A},I}^\infty$. The structural map $\mathcal{E}\ell\ell_{\mathcal{A},I,0}^\infty|_{X'}\to\mathrm{Ell}_{\mathcal{A},I}^\infty$ is an isomorphism if $I\neq 0$.
(c) The morphism char : $\mathrm{Ell}_{\mathcal{A},I}^\infty\to X'$ is quasiprojective and smooth of relative dimension $d-1$. In particular $\mathrm{Ell}_{\mathcal{A},I}^\infty$ is a smooth, quasiprojective $\mathbb{F}_q$-scheme.

Remark 4.12. This is known if A is a division algebra or $A=M_d(F)$ and if we restrict $\mathcal{E}\ell\ell_{\mathcal{A},I}^\infty$ to the open subset X' ([LRS], Theorem 4.1 and [Dr1]). In fact if we assume that A is a division algebra and let S denote the subset of $\underline{\mathrm{Disc}}(\mathcal{A})$ consisting of all points $\mathfrak{p}\in\underline{\mathrm{Disc}}(\mathcal{A})-\{\infty\}$ with $\mathrm{inv}_{\mathfrak{p}}A=1$ and of ∞ if $\mathrm{inv}_\infty A=0$ then $\mathcal{E}\ell\ell_{\mathcal{A},I,0}^\infty$ admits a coarse moduli scheme $\mathrm{Ell}_{\mathcal{A},I}^\infty$ over $X\cup S$ which is projective and semistable of relative dimension $d-1$ (at the pole ∞ this is proved in [BS]; at $\mathfrak{p}\in S-\{\infty\}$ it is proved in certain cases by [Hau] and can be deduce in general from the first case using the main result of section 4.5).

The proof of 4.11 consists essentially of two parts. In the first part one shows that $\mathcal{E}\ell\ell_{\mathcal{A}}^\infty\to X$ is a Deligne-Mumford stack and semistable. In the second part one proves that for $I\neq 0$, $\mathcal{E}\ell\ell_{\mathcal{A},I,n}^\infty$ is a quasiprojective scheme over X' by showing that for a large m the map $\mathcal{E}\ell\ell_{\mathcal{A},mI,n}^{\infty,\mathrm{stab}}\to\mathcal{E}\ell\ell_{\mathcal{A},I,n}^\infty$ is surjective. Here $\mathcal{E}\ell\ell_{\mathcal{A},I}^{\infty,\mathrm{stab}}$ denotes the substack of $\mathcal{A}$-elliptic sheaves whose underlying vector bundle is *I-stable*. It is a consequence of a theorem of Seshadri that $\mathcal{E}\ell\ell_{\mathcal{A},I}^{\infty,\mathrm{stab}}$ is a quasiprojective scheme. For the surjectivity one can follow the arguments in ([LRS], section 5) so we will omit the proof.
The proof of the first part is also mainly a reproduction of the corresponding arguments in ([LRS], section 4; compare also ([Laf] Chapitre I), [La] and ([Lau],

1.3 and 1.4)) so we will be rather brief and elaborate only on those steps were essential modification have to be made. We follow ([Lau], 1.2) and work with the factorization (16), i.e. we consider $\mathcal{E}\ell\ell_{\mathcal{A},I}^{\infty}$ mostly over $\mathrm{Coh}_{\mathcal{A},\mathrm{sp}}$ rather than over X. Let $\mathrm{Inj}_{\mathcal{A},\mathrm{sp}}$ be the stack over $k(\infty)$ such that for each $S \in \mathrm{Sch}\,/k(\infty)$, $\mathrm{Inj}_{\mathcal{A},\mathrm{sp}}(S)$ is the groupoid of injective morphisms $j : \mathcal{E}' \to \mathcal{E}$ locally free right $\mathcal{A} \boxtimes \mathcal{O}_S$-modules of rank 1 with $\mathrm{Coker}(j) \in \mathrm{Coh}_{\mathcal{A},\mathrm{sp}}(S)$.

Lemma 4.13. *(a) The two morphism*

$$\mathrm{Inj}_{\mathcal{A},\mathrm{sp}} \longrightarrow \mathrm{Vect}^1_{\mathcal{A},0} \times \mathrm{Coh}_{\mathcal{A},\mathrm{sp}}$$

given by $(j : \mathcal{E}' \to \mathcal{E}) \mapsto (\mathcal{E}, \mathrm{Coker}(j))$ *and* $(j : \mathcal{E}' \to \mathcal{E}) \mapsto (\mathcal{E}', \mathrm{Coker}(j))$ *are representable and quasiaffine of finite type and smooth of relative dimension d. Consequently* $\mathrm{Inj}_{\mathcal{A},\mathrm{sp}}$ *is algebraic, smooth and of finite type over* $\mathbb{F}_q$.
(b) The two morphism

$$\mathrm{Inj}_{\mathcal{A},\mathrm{sp}} \longrightarrow \mathrm{Vect}_X^{d^2}$$

given by $\mathcal{E}$ *and* $\mathcal{E}'$ *are representable and quasiprojective and in particular of finite type.*

The proof of (a) for the first morphism is literally the same as ([Lau], 1.3.2). The statement for second morphism can be deduce from that for the first as in ([Lau], 1.3.2). We need to remark only that for a short exact sequence $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{K} \to 0$ of right $\mathcal{A} \otimes \mathcal{O}_S$-modules with $\mathcal{E}', \mathcal{E} \in \mathrm{Vect}^1_{\mathcal{A}}(S)$ and $\mathcal{K} \in \mathrm{Coh}_{\mathcal{A},\mathrm{sp}}(S)$ the third term of the dual sequence of $\mathcal{A}^{\mathrm{opp}} \otimes \mathcal{O}_S$-modules $0 \to \mathcal{E}^{\vee} \to \mathcal{E}'^{\vee} \to \underline{\mathrm{Ext}}^1_{\mathcal{A}\otimes\mathcal{O}_S}(\mathcal{K}, \mathcal{A} \boxtimes \mathcal{O}_S) \to 0$ lies, by Lemma 3.12, in $\mathrm{Coh}_{\mathcal{A}^{\mathrm{opp}},\mathrm{sp}}(S)$. (b) follows from ([Laf], I.2.2 and I.2.8). □

Consider now the following obvious diagram of stacks

$$\begin{array}{ccc}
\mathcal{E}\ell\ell_{\mathcal{A}}^{\infty} & \longrightarrow & \mathrm{Vect}^1_{\mathcal{A}} \otimes_{\mathbb{F}_q} k(\infty) \\
\downarrow & & \downarrow \\
\mathrm{Inj}_{\mathcal{A},\mathrm{sp}} & \longrightarrow & (\mathrm{Vect}^1_{\mathcal{A}} \times \mathrm{Vect}^1_{\mathcal{A}}) \otimes_{\mathbb{F}_q} k(\infty) \\
\downarrow & & \\
\mathrm{Coh}_{\mathcal{A},\mathrm{sp}} & &
\end{array} \tag{24}$$

where the right vertical arrow in the (2-cartesian) square is the graph of the endomorphism $\mathrm{Frob} \circ \theta^{-1} : \mathrm{Vect}^1_{\mathcal{A}} \otimes_{\mathbb{F}_q} k(\infty) \to \mathrm{Vect}^1_{\mathcal{A}} \otimes_{\mathbb{F}_q} k(\infty)$ (for the definition of θ^{-1} compare 4.3 (f); if $\deg(\infty) = 1$ it is given by $\mathcal{E} \mapsto \mathcal{E}(-\frac{1}{d}\infty)$). By ([Laf], I.2.5) the stack $\mathrm{Vect}^1_{\mathcal{A}}$ is algebraic, locally of finite type and smooth over $\mathbb{F}_q$. Together with Proposition 3.10, Lemma 4.9 and Lemma 4.13 above the same argument as in ([LRS], section 4; see also ([Laf], I.2.5) and ([Lau], I.3.5)) imply part (a) of

Lemma 4.14. *(a) Let I be an effective divisor on X. The morphism $\mathcal{E}\ell\ell^{\infty}_{\mathcal{A},I} \to \mathrm{Coh}_{\mathcal{A},\mathrm{sp}}$ is algebraic, locally of finite type and smooth of relative dimension d. The morphism $char : \mathcal{E}\ell\ell^{\infty}_{\mathcal{A},I} \to X$ is semistable of relative dimension $d-1$.*
(b) $\mathcal{E}\ell\ell^{\infty}_{\mathcal{A}}$ is a Deligne-Mumford stack, locally of finite type and smooth over $\mathbb{F}_q$. Moreover if $I \neq 0$ then $\mathcal{E}\ell\ell^{\infty}_{\mathcal{A},I}|_{X'}$ is isomorphic to an algebraic space.

Proof of (b). Everything is clear if we replace "Deligne-Mumford" by "algebraic". To prove that $\mathcal{E}\ell\ell^{\infty}_{\mathcal{A}}$ is indeed a Deligne-Mumford stack we use ([LM], 8.1). If we replace the lower vertical map in (24) by $\mathrm{Inj}_{\mathcal{A},\mathrm{sp}} \to \mathrm{Spec}\, k(\infty)$ then, by Lemma 4.13 and ([La], Lemma on p. 60), the diagonal morphism $\mathcal{E}\ell\ell^{\infty}_{\mathcal{A}} \to \mathcal{E}\ell\ell^{\infty}_{\mathcal{A}} \times_{\mathbb{F}_q} \mathcal{E}\ell\ell^{\infty}_{\mathcal{A}}$ is unramified. Note that for $E \in \mathcal{E}\ell\ell^{\infty}_{\mathcal{A},I}(S)$ with zero $S \to X'$ we have $\mathrm{Aut}(E) = {\mathbb{F}_q}^*$ if $I = 0$ or $\mathrm{Aut}(E) = 1$ otherwise. Hence the last assertion follows from ([LM], 8.1.1). $\square$

Remarks 4.15. (a) Note that by 4.3 (f) we could have defined $\mathrm{Ell}^{\infty}_{\mathcal{A}}$ also as the coarse moduli scheme of the quotient $\mathcal{E}\ell\ell^{\infty}_{\mathcal{A}}/\theta^{\mathbb{Z}}$ or of $\mathcal{E}\ell\ell^{\infty}_{\mathcal{A},n}$ for any $n \in \frac{1}{d}\mathbb{Z}$.
(b) Let $I \hookrightarrow X$ be a reduced closed subscheme with $\infty \notin I$ and let $\mathcal{A}$ be the subsheaf of $M_2(\mathcal{O}_X)$ of matrices which are upper triangular modulo I. Then by using 4.10 and 4.3 (b) it is easy to see that the $\mathrm{Ell}^{\infty}_{\mathcal{A}}$ is isomorphic to the (open) Drinfeld modular curve $Y_0(I) = Y_0^{\infty}(I)$.
(c) If A is a central division algebra which is unramified at ∞ and $\mathcal{A}|_{X-\{\infty\}}$ is a maximal order in A then $char : \mathrm{Ell}^{\infty}_{\mathcal{A}} \to X$ is proper (see [LRS], Theorem 6.1 and [Hau], 6.4). In the general case this is not true anymore even if A is a division algebra. In fact if $d = 2$ and A is ramified only at ∞ and at $\mathfrak{p} \in |X|$ and if $\mathcal{A}$ is a maximal $\mathcal{O}_X$-order in A then we will show in section 4.5 that $\mathrm{Ell}^{\mathfrak{p}}_{\mathcal{A}}$ is a twist of the affine curve $Y_0^{\infty}(\mathfrak{p}) \to X$.

4.4 Invertible Frobenius bimodules

We consider now two locally principal $\mathcal{O}_X$-orders $\mathcal{A}$ and $\mathcal{B}$, both of rank d^2 with $\underline{\mathrm{Disc}}(\mathcal{A}) = \underline{\mathrm{Disc}}(\mathcal{B})$ and assume that $e(\mathcal{A}) = d = e(\mathcal{B})$. We denote by A and B the generic fibers of $\mathcal{A}$ and $\mathcal{B}$ respectively. Let $D = \sum_{x\in|X|} m_x x \in \mathrm{Div}(\mathcal{A})$ be a divisor such that $\sum_{x\in|X|} m_x = 0$. We consider the following moduli problem associated to $\mathcal{A}, \mathcal{B}, D$.

Definition 4.16. *Let S be an $\mathbb{F}_q$-scheme. An invertible Frobenius $\mathcal{A}$-$\mathcal{B}$-bimodule (or Φ-$\mathcal{A}$-$\mathcal{B}$-bimodule for short) over S of slope D is a tuple $L = (\mathcal{L}, (x_S)_{x\in|D|}, \Phi)$ where $\mathcal{L}$ is an invertible $\mathcal{A}\boxtimes\mathcal{O}_S$-$\mathcal{B}\boxtimes\mathcal{O}_S$-bimodule which is locally free of rank 1 as a left $\mathcal{A}\boxtimes\mathcal{O}_S$- and right $\mathcal{B}\boxtimes\mathcal{O}_S$-module, where for $x \in |D|$, $x_S : S \to X$ is a morphism in $\mathrm{Sch}/\mathbb{F}_q$ which factors through $x \to X$ and where Φ is a bimodule isomorphism*

$$\Phi : {}^{\tau}(\mathcal{L}(D_S)) \longrightarrow \mathcal{L}.$$

with $D_S\colon = \sum_{x\in|D|} m_x x_S$. The morphisms x_S are called the poles of L.

Note we have $\deg_{\mathcal{A}}(\mathcal{L}) = \deg_{\mathcal{B}}(\mathcal{L})$. Note also that $(\mathcal{A} \boxtimes \mathcal{O}_S)(D_S) \otimes_{\mathcal{A}} \mathcal{L} = \mathcal{L} \otimes_{\mathcal{B}} (\mathcal{B} \boxtimes \mathcal{O}_S)(D_S)$. Thus the notion $\mathcal{L}(D_S)$ is unambiguous.
Obviously the concept of invertible Φ-$\mathcal{A}$-$\mathcal{B}$-bimodules of slope D defines a stack which we denote by $\mathcal{SE}^D_{\mathcal{A},\mathcal{B}}$. It is equipped with canonical morphisms $\mathcal{SE}^D_{\mathcal{A},\mathcal{B}} \to \operatorname{Spec} k(x)$ for all $x \in |D|$. For $n \in \frac{1}{d}\mathbb{Z}$ let $\mathcal{SE}^D_{\mathcal{A},\mathcal{B},n}$ be the substack of $(\mathcal{L}, (x_S)_{x\in T}, \Phi) \in \mathcal{SE}^D_{\mathcal{A},\mathcal{B}}(S)$ with $\deg_{\mathcal{A}}(\mathcal{L}) = n$. There is a canonical left $\operatorname{Pic}(\mathcal{A})$- and right $\operatorname{Pic}(\mathcal{B})$-action on $\mathcal{SE}^D_{\mathcal{A},\mathcal{B}}$ compatible with $\deg_{\mathcal{A}}$ (in fact the left and right action are the same if we identify the two groups under the canonical isomorphism $\operatorname{Pic}(\mathcal{A}) \cong \operatorname{Div}(\mathcal{A})/F^* \cong \operatorname{Pic}(\mathcal{B})$).
For $x \in |D|$ we define an automorphism $\theta_x : \mathcal{SE}^D_{\mathcal{A},\mathcal{B}} \to \mathcal{SE}^D_{\mathcal{A},\mathcal{B}}$ by

$$\theta_x(\mathcal{L}, (x'_S)_{x'\in|D|}, \Phi) = (\mathcal{L}(-m_x{}^\tau x_S), {}^\tau x_S, (x'_S)_{x'\in|D|, x'\neq x}, \Phi(-m_x{}^\tau x)). \quad (25)$$

The automorphisms θ_x for different $x \in |D|$ commute with each other and with the $\operatorname{Pic}(\mathcal{A})$- and $\operatorname{Pic}(\mathcal{B})$-action. We have $\theta_x(\mathcal{SE}^D_{\mathcal{A},\mathcal{B},n}) = \mathcal{SE}^D_{\mathcal{A},\mathcal{B},n-m_x}$ for all $n \in \frac{1}{d}\mathbb{Z}$ and $\theta_x^{\deg(x)}(L) = \mathcal{A}(-m_x x) \otimes L = L \otimes \mathcal{B}(-m_x x)$ for all $L \in \mathcal{SE}^D_{\mathcal{A},\mathcal{B}}(S)$. Moreover if $|D| = \{x_1, \ldots, x_m\}$ and if we put $\Theta_D\colon = \theta_{x_1} \circ \ldots \circ \theta_{x_m}$ then $\Theta_D(L) = \operatorname{Frob}^*_S(L)$ for all $S \in \operatorname{Sch}/\mathbb{F}_q$ and $L \in \mathcal{SE}^D_{\mathcal{A},\mathcal{B}}(S)$. Hence $\Theta_D = \operatorname{Frob}_{\mathcal{SE}^D_{\mathcal{A},\mathcal{B}}}$.

Level structure. Let $L = (\mathcal{L}, (x_S)_{x\in|D|}, \Phi) \in \mathcal{SE}^D_{\mathcal{A},\mathcal{B}}(S)$. We view $\mathcal{L}$ as a right $\mathcal{B}$-module only and proceed as in section 4.2. Let I be an effective divisor on X. Assume first that $|I| \cap |D| = \emptyset$. Then a *level-I-structure on* L is an isomorphism of right $\mathcal{B}_I \boxtimes \mathcal{O}_S$-modules $\beta : \mathcal{B}_I \boxtimes \mathcal{O}_S \longrightarrow \mathcal{L}|_{I\times S}$ such that the diagram

$$\begin{tikzcd}
{}^\tau\mathcal{L}|_{I\times S} \arrow[rr, "\Phi|_{I\times S}"] & & \mathcal{L}|_{I\times S} \\
& \mathcal{A}_I \boxtimes \mathcal{O}_S \arrow[ul, bend left, "{}^\tau\beta"] \arrow[ur, bend right, "\alpha"'] &
\end{tikzcd}$$

commutes.
Next assume that $I = nx$ with $n > 0$ for some $x \in |D|$. Put $e = e_x(\mathcal{A}), m = m_x$ and let $k(x)_e$ be an extension of degree e of $k(x)$. If $m > 0$ we denote by $M = (\mathcal{M}_x, \phi_x)$ a fixed invertible ϕ-$\mathcal{A}_x$-$\mathcal{B}_x$-bimodule of slope $-m$ over $\mathcal{O}_x \otimes_{\mathbb{F}_q} k(x)_e$. In case $m < 0$, $M = (\mathcal{M}_x, \phi_x)$ denotes a ϕ-$\mathcal{B}_x$-$\mathcal{A}_x$-bimodule of slope m over $\mathcal{O}_x \otimes_{\mathbb{F}_q} k(x)_e$. Thus if $m > 0$ (resp. $m < 0$) then ϕ_x is an isomorphism

$$\phi_x : {}^\sigma(\mathcal{M}_x \mathfrak{P}^{me}) \longrightarrow \mathcal{M}_x \quad (\text{resp. } \phi_x : {}^\sigma(\mathfrak{P}^{-me}\mathcal{M}_x) \longrightarrow \mathcal{M}_x)$$

where $\mathfrak{P}$ denotes the maximal ideal of $\mathcal{B}_x \otimes_{\mathbb{F}_q} k(x)_e$ corresponding to the inclusion $k(x) \hookrightarrow k(x)_e$. As in 4.2 the pair $(\mathcal{M}_x, \phi_x)$ induces a pair $(\mathcal{M}_I, \phi_I)$ consisting of $\mathcal{A}_I \otimes_{\mathbb{F}_q} k(x)_e$-$\mathcal{B}_I \otimes_{\mathbb{F}_q} k(x)_e$-bimodule and an isomorphism $\phi_I : {}^\tau(\mathcal{M}_I(mx_e)) \longrightarrow \mathcal{M}_I$ where x_e is the map $\operatorname{Spec} k(x)_e \to x \hookrightarrow X$. A *level-$I$-structure on* L consists of a pair (μ, β) where $\mu : S \to \operatorname{Spec} k(x)_e$ is an

$\mathbb{F}_q$-morphism which lifts x_S and an isomorphism of right $\mathcal{B}_I \boxtimes \mathcal{O}_S$-modules β. If $m < 0$, then β is a map

$$\beta : (\mathrm{id}_I \times \mu)^*(\mathcal{M}_I) \longrightarrow \mathcal{L}|_{I\times S} \tag{26}$$

such that

$$\begin{tikzcd}
{}^{\tau}(\mathcal{L}(D))|_{I\times S} \arrow[r] & \mathcal{L}|_{I\times S} \\
(\mathrm{id}_I \times \mu)^*({}^{\tau}(\mathcal{M}_I(mx_e)) \arrow[u] \arrow[r] & (\mathrm{id}_I \times \mu)^*(\mathcal{M}_I) \arrow[u]
\end{tikzcd}$$

commutes. If $m > 0$ then

$$\beta : \mathcal{B}_I \boxtimes \mathcal{O}_S \longrightarrow (\mathrm{id}_I \times \mu)^*(\mathcal{M}_I) \otimes_{\mathcal{A}} \mathcal{L}|_{I\times S} \tag{27}$$

and

$$\begin{tikzcd}
{}^{\tau}((\mathrm{id}_I \times \mu)^*(\mathcal{M}_I) \otimes_{\mathcal{A}} \mathcal{L}|_{I\times S}) \arrow[rr, "\phi_I \otimes \Phi|_{I\times S}"] & & (\mathrm{id}_I \times \mu)^*(\mathcal{M}_I) \otimes_{\mathcal{A}} \mathcal{L}|_{I\times S} \\
& \mathcal{B}_I \boxtimes \mathcal{O}_S \arrow[ul, bend left, "{}^{\tau}\beta"] \arrow[ur, bend right, "\beta"'] &
\end{tikzcd}$$

should commute.

For an arbitrary effective divisor I on X we write $I = I_0 + \sum_{x\in |I|\cap|D|} n_x x$ $= I_0 + \sum_{x\in |I|\cap|D|} I_x$ with $|I_0| \cap |D| = \emptyset$ and $n_x > 0$ for $x \in |I| \cap |D|$. Then a level-I-structure on L is a tuple $(\beta_0, (\mu_x, \beta_x)_{x\in|I|\cap|D|})$ consisting of a level-I_0-structure β_0 and level-I_x-structures (μ_x, β_x) for all $x \in |I| \cap |D|$. This yields stacks $\mathcal{SE}^D_{\mathcal{A},\mathcal{B},I}, \mathcal{SE}^D_{\mathcal{A},\mathcal{B},I,n}$ equipped with forgetful morphisms

$$\mathcal{SE}^D_{\mathcal{A},\mathcal{B},I} \to \mathcal{SE}^D_{\mathcal{A},\mathcal{B}}, \qquad \mathcal{SE}^D_{\mathcal{A},\mathcal{B},I} \to \operatorname{Spec} k(x)_{e_x(\mathcal{A})} \quad \text{for all } x \in |I| \cap |D|$$

(the latter lifts the morphism $\mathcal{SE}^D_{\mathcal{A},\mathcal{B}} \to \operatorname{Spec} k(x)$).

Modular automorphisms. Let $T\colon = \{x \in |D| \mid m_x > 0\}$. If $|I| \cap T = \emptyset$ then there is a canonical left $\mathrm{Pic}_I(\mathcal{A})$-action on $\mathcal{SE}^D_{\mathcal{A},\mathcal{B},I}$ lifting the $\mathrm{Pic}(\mathcal{A})$-action on $\mathcal{SE}^D_{\mathcal{A},\mathcal{B}}$. We want to extend this to a natural left action of an idele class group $\mathcal{C}_I(\mathcal{A}^T \times \mathcal{B}_T)$ on $\mathcal{SE}^D_{\mathcal{A},\mathcal{B},I}$ for arbitrary I (similarly to the right action of $\mathcal{C}_I(\mathcal{A}^\infty \times \mathcal{D}_\infty)$-action on $\mathcal{E}\ell\ell^\infty_{\mathcal{A},I^\infty}$ defined in 4.2). Write $I = I^T + I_T$ with $|I^T| \cap T = \emptyset$ and $|I_T| \subseteq T$. Put

$$U_I(\mathcal{A}^T \times \mathcal{B}_T)\colon = \mathrm{Ker}(\prod_{x\in|X|-T} \mathcal{A}^*_x \times \prod_{x\in T} \mathcal{B}^*_x \to \Gamma(I^T, \mathcal{A}_{I^T})^* \times \Gamma(I_T, \mathcal{B}_{I_T})^*)$$

and define

$$\mathcal{C}_I(\mathcal{A}^T \times \mathcal{B}_T)\colon = {\prod}'_{x\in|X|-T} N(\mathcal{A}_x) \times \prod_{x\in T} N(\mathcal{B}_x)/U_I(\mathcal{A}^T \times \mathcal{B}_T)F^*.$$

There is a canonical epimorphism $\mathcal{C}_I(\mathcal{A}^T \times \mathcal{B}_T) \longrightarrow \mathrm{Pic}(\mathcal{A})$ given on the class $[g]$ represented by $g = (\{a_x\}_{x\in|X|-T}, \{b_x\}_{x\in T}) \in \prod'_{x\in|X|-T} N(\mathcal{A}_x)) \times \prod_{x\in T} N(\mathcal{B}_x)$ by $\mathcal{A}(\mathrm{div}(g))$ where

$$\mathrm{div}(g) = \sum_{x\in|X|-T} v_{\mathcal{A}_x}(a_x)x \ + \sum_{x\in T} v_{\mathcal{B}_x}(a_x)x$$

The kernel of the composition $\mathcal{C}_I(\mathcal{A}^T \times \mathcal{B}_T) \to \mathrm{Pic}(\mathcal{A}) \xrightarrow{\deg_{\mathcal{A}}} \mathbb{Q}$ will be denoted by $\mathcal{C}_I(\mathcal{A}^T \times \mathcal{B}_T)_0$.
Let $g = (a^T, b_T) = (\{a_x\}_{x\notin T}, \{b_x\}_{x\in T}) \in \prod'_{x\in|X|-T} N(\mathcal{A}_x)) \times \prod_{x\in T} N(\mathcal{B}_x)$ and $L = (L, \beta_0, (\mu_x, \beta_x)_{x\in|I|\cap|D|}) \in \mathcal{SE}^D_{\mathcal{A},\mathcal{B},I}(S)$. Left multiplication by a^T on the target of β_0 and β_x for $x \in |I^T|$ (respectively by b_T on the target of β_x for $x \in |I_T|$) yields a level-I-structure on $\mathcal{A}(\mathrm{div}(g)) \otimes L$. This defines a left action of $\prod'_{x\in|X|-T} N(\mathcal{A}_x)) \times \prod_{x\in T} N(\mathcal{B}_x)$ on $\mathcal{SE}^D_{\mathcal{A},\mathcal{B},I}$ which factors through $\mathcal{C}_I(\mathcal{A}^T \times \mathcal{B}_T)$.

4.17. Similar to (22), for $x \in |D|$ there exists a canonical lift of (25) to an automorphism $\theta_x : \mathcal{SE}^D_{\mathcal{A},\mathcal{B},I} \to \mathcal{SE}^D_{\mathcal{A},\mathcal{B},I}$ having the following properties:

(i) The following diagram commutes

$$\begin{array}{ccc} \mathcal{SE}^D_{\mathcal{A},\mathcal{B},I} & \xrightarrow{\theta_x} & \mathcal{SE}^D_{\mathcal{A},\mathcal{B},I} \\ \downarrow & & \downarrow \\ \mathrm{Spec}\, k(x)_* & \xrightarrow{\mathrm{Frob}_q} & \mathrm{Spec}\, k(x)_* \end{array}$$

where $* = 1$ or $* = e_x(\mathcal{A})$ depending on whether $x \notin |I|$ or $x \in |I|$.

(ii) For $n \in \frac{1}{d}\mathbb{Z}$ we have $\theta_x(\mathcal{SE}^D_{\mathcal{A},\mathcal{B},I,n}) = \mathcal{SE}^D_{\mathcal{A},\mathcal{B},I,n-m_x}$.

(iii) The automorphisms θ_x for different $x \in |D|$ commute with each other and with the $\mathcal{C}_I(\mathcal{A}^T \times \mathcal{B}_T)$-action.

(iv) For $x \in |D|$ there exists $\xi_x \in \mathcal{C}_I(\mathcal{A}^T \times \mathcal{B}_T)$ such that $\theta_x^{\deg(x)}(L) = \xi_x L$ (resp. $\theta_x^{e_x(\mathcal{A})\deg(x)}(L) = \xi_x L$) for all $L \in \mathcal{SE}^D_{\mathcal{A},\mathcal{B},I}(S)$.

(v) If $|D| = \{x_1, \ldots, x_m\}$ put $\Theta_D\colon = \theta_{x_1} \circ \ldots \circ \theta_{x_m}$. Then $\Theta_D = \mathrm{Frob}_{\mathcal{SE}^D_{\mathcal{A},\mathcal{B},I}}$.

Let $\mathcal{G}$ be the group of automorphism of $\mathcal{SE}^D_{\mathcal{A},\mathcal{B},I}$ generated by $\mathcal{C}_I(\mathcal{A}^T \times \mathcal{B}_T)$ and the set $\{\theta_x \mid x \in |D|\}$. For $g \in \mathcal{G}$ the degree $m \in \frac{1}{d}\mathbb{Z}$ of g is defined by $g\mathcal{SE}^D_{\mathcal{A},\mathcal{B},I,n} = \mathcal{SE}^D_{\mathcal{A},\mathcal{B},I,n+m}$ for all $n \in \frac{1}{d}\mathbb{Z}$. Let $\mathcal{G}_0$ be the subgroup of elements of degree 0. Since the degree of θ_x, $x \in |D|$ is $-m_x$ we have $\Theta_D \in \mathcal{G}_0$.

Definition 4.18. *Suppose that $\mathcal{SE}^D_{\mathcal{A},\mathcal{B},I,0} \neq \emptyset$. We define $\mathcal{W}(\mathcal{A},\mathcal{B},I,D)$ to be the group of automorphisms of $\mathcal{SE}^D_{\mathcal{A},\mathcal{B},I,0}$ of the form $g|_{\mathcal{SE}^D_{\mathcal{A},\mathcal{B},I,0}}$ for $g \in \mathcal{G}_0$.*

Remark 4.19. Assume that $\mathcal{SE}^D_{\mathcal{A},\mathcal{B},I,0} \neq \emptyset$. For all $x \in |D|$ there exists canonical homomorphisms $\mathcal{W}(\mathcal{A},\mathcal{B},I,D) \to \mathrm{G}(k(x)_*/\mathbb{F}_q)$ where $* = \emptyset$ or $* = e_x(\mathcal{A})$ depending on whether $x \notin |I|$ or $x \in |I|$. It is surjective since Θ_D is mapped to $\mathrm{Frob}_{k(x)_*}$ by property (v) above. The kernel of the homomorphism

$$\mathcal{W}(\mathcal{A},\mathcal{B},I,D) \longrightarrow \prod_{x\in|D|-|I|} \mathrm{G}(k(x)/\mathbb{F}_q) \times \prod_{x\in|D|\cap|I|} \mathrm{G}(k(x)_{e_x(\mathcal{A})})/\mathbb{F}_q) \tag{28}$$

is $\mathcal{C}_I(\mathcal{A}^T \times \mathcal{B}_T)_0$.
It is easy to see that (28) is surjective provided that $\delta(\mathcal{A}) = d$ (i.e. $\deg_{\mathcal{A}}: \mathrm{Pic}(\mathcal{A}) \to \frac{1}{d}\mathbb{Z}$ is surjective). Under this condition $\mathcal{W}(\mathcal{A},\mathcal{B},I,D)$ can be defined in a similar way as 4.7, i.e. as the subgroup of $\deg_{\mathcal{A}} = 0$ elements in the abstract group $\mathcal{C}_I(\mathcal{A}^T \times \mathcal{B}_T)[\theta_x, x \in |D|]$ generated by $\mathcal{C}_I(\mathcal{A}^T \times \mathcal{B}_T)$ and a set of central element $\{\theta_x \mid x \in |D|\}$ with $\deg_{\mathcal{A}}(\theta_x) = -m_x$ and such that the relations (iv) above hold.

Tensor product and Inverse. There is also a notion of a tensor product of invertible Frobenius bimodules and of an inverse. These constructions are needed in the proof of Proposition 4.20 below. Let $\mathcal{C}$ be a third locally principal $\mathcal{O}_X$-order of rank d^2 with $\underline{\mathrm{Disc}}(\mathcal{C}) = \underline{\mathrm{Disc}}(\mathcal{A})$. Let $D_1 = \sum_{x\in|X|} m_x^{(1)} x, D_2 = \sum_{x\in|X|} m_x^{(2)} x \in \mathrm{Div}(\mathcal{A})$ with $\sum_{x\in|X|} m_x^{(i)} = 0$ for $i = 1,2$. Let $Y = \mathrm{Spec}\,\mathbb{F}_q$ if $|D_1| \cap |D_2| = \emptyset$ or $Y = \mathrm{Spec}(\bigotimes_{x\in|D_1|\cap|D_2|} k(x))$ otherwise. We view $\mathcal{SE}^{D_1}_{\mathcal{A},\mathcal{B}}$ and $\mathcal{SE}^{D_2}_{\mathcal{B},\mathcal{C}}$ as stacks over Y. Let $S \in \mathrm{Sch}/Y$ and let $L = (\mathcal{L}, (x_S)_{x\in|D|}, \Phi) \in \mathcal{SE}^{D_1}_{\mathcal{A},\mathcal{B}}(S)$, $M = (\mathcal{M}, (x_S)_{x\in|D_2|}, \Psi) \in \mathcal{SE}^{D_2}_{\mathcal{B},\mathcal{C}}(S)$ (hence for $x \in |D_1| \cap |D_2|$, the morphisms x_S for L and M agree and are equal to the canonical morphism $S \to \mathrm{Spec}\,k(x) \to X$). Define

$$L \otimes M = (\mathcal{L} \otimes \mathcal{M}, (x_S)_{x\in|D_1+D_2|}, \Phi \otimes_{\mathcal{B}} \Psi) \in \mathcal{SE}^{D_1+D_2}_{\mathcal{A},\mathcal{C}}(S).$$

Thus we get a morphism of stacks

$$\otimes : \mathcal{SE}^{D_1}_{\mathcal{A},\mathcal{B}} \times_Y \mathcal{SE}^{D_2}_{\mathcal{B},\mathcal{C}} \longrightarrow \mathcal{SE}^{D_1+D_2}_{\mathcal{A},\mathcal{C}} \tag{29}$$

which is compatible with degrees.
The inverse L^{-1} of $L = (\mathcal{L}, (x_S)_{x\in|D|}, \Phi) \in \mathcal{SE}^D_{\mathcal{A},\mathcal{B}}(S)$ is defined as

$$L^{-1} = (\mathcal{L}^\vee, (x_S)_{x\in|D|}, (\Phi^\vee)^{-1}) \in \mathcal{SE}^{-D}_{\mathcal{B},\mathcal{A}}(S). \tag{30}$$

We leave it to the reader to extend the Definition (29) and (30) to invertible Frobenius bimodules with level-I-structure (see also the next section where the tensor product of an $\mathcal{A}$-elliptic sheaf with level-I-structure with a Frobenius bimodule with level-I-structure is defined).

Moduli spaces. Let $D = \sum_{x\in|X|} m_x x \in \mathrm{Div}(\mathcal{A})$, $D \neq 0$ be such that $\sum_{x\in|X|} m_x = 0$ and let $I \in \mathrm{Div}(X)$ with $I \geq 0$. Our aim is to prove the following result.

Proposition 4.20. *(a)* $\mathcal{SE}^D_{\mathcal{A},\mathcal{B},I} \neq \emptyset$ *if and only if*

$$\sum_{x\in|X|} \mathrm{inv}_x(B)x = (\sum_{x\in|X|} \mathrm{inv}_x(A)x) + D \mod \mathrm{Div}(X). \tag{31}$$

(b) $\mathcal{SE}^D_{\mathcal{A},\mathcal{B},I}$ *is a Deligne-Mumford stack which is étale over* $\mathbb{F}_q$*. The open and closed substack* $\mathcal{SE}^D_{\mathcal{A},\mathcal{B},I,n}$ *is finite over* $\mathbb{F}_q$ *for all* $n \in \frac{1}{d}\mathbb{Z}$*.*
(c) $\mathcal{SE}^D_{\mathcal{A},\mathcal{B},I,0}$ *admits a coarse moduli space* $\mathrm{SE}^D_{\mathcal{A},\mathcal{B},I}$*. The structural morphism* $\mathcal{SE}^D_{\mathcal{A},\mathcal{B},I,0} \to \mathrm{SE}^D_{\mathcal{A},\mathcal{B},I}$ *is an isomorphims if* $I \neq 0$*.*
(d) Suppose that $\mathcal{SE}^D_{\mathcal{A},\mathcal{B},I,0} \neq \emptyset$*. Then* $\mathrm{SE}^D_{\mathcal{A},\mathcal{B},I} \to \mathrm{Spec}\,\mathbb{F}_q$ *is a* $\mathcal{W}(\mathcal{A},\mathcal{B},I,D)$*-torsor. In particular* $\mathrm{SE}^D_{\mathcal{A},\mathcal{B},I}$ *is a finite, étale* $\mathbb{F}_q$*-scheme.*

We begin with the proof of (a). Since $\mathcal{SE}^D_{\mathcal{A},\mathcal{B},I}$ is locally of finite presentation it suffices to show that $\mathcal{SE}^D_{\mathcal{A},\mathcal{B},I}(\mathrm{Spec}\,\overline{\mathbb{F}}_q) \neq \emptyset$ if and only if (32) holds. We write $\overline{X}$, $\overline{\mathcal{A}}$ etc. for $X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$, $\mathcal{A} \otimes \overline{\mathbb{F}}_q$ etc. Let $\sigma\colon = \mathrm{id}_X \otimes \mathrm{Frob}_q : \overline{X} \to \overline{X}$ and let $\pi : \overline{X} \to X$ be the projection. Define $\mathrm{div}(\pi) : \mathrm{Div}(\overline{X}) \otimes \mathbb{Q} \to \mathrm{Div}(X) \otimes \mathbb{Q}$ by $\mathrm{div}(\pi)(\sum_i n_i\bar{x}_i) = \sum_i n_i\pi(\bar{x}_i)$ (Note that $\deg(\mathrm{div}(\pi)(D)) \neq \deg(D)$ in general). Part (a) of Proposition 4.20 follows from the following slighty more general result.

Lemma 4.21. *Let* $\overline{D} \in \mathrm{Div}_0(\overline{\mathcal{A}})$*. The following conditions are equivalent:*
(i) There exists an invertible $\overline{\mathcal{A}}$*-*$\overline{\mathcal{B}}$*-bimodule* $\mathcal{L}$ *such that* ${}^{\tau}(\mathcal{L}(\overline{D})) \cong \mathcal{L}$*.*
(ii) We have

$$\sum_{x\in|X|} \mathrm{inv}_x(B)x = (\sum_{x\in|X|} \mathrm{inv}_x(A)x) + \mathrm{div}(\pi)(\overline{D}) \mod \mathrm{Div}(X). \tag{32}$$

Proof. That (i) implies (ii) can be easily deduced from the corresponding local result. To show the converse we consider first the special case $\mathrm{div}(\pi)(\overline{D}) \in \mathrm{Div}(X)$, i.e. $\mathcal{A} \simeq \mathcal{B}$. Then $\overline{D}$ can be written as a sum of divisors of the form $\pi^*(D_1), D_1 \in \mathrm{Div}_0(X)$ and of the form $\frac{1}{e_x(\mathcal{A})}(x - \sigma(x))$ for $x \in |X|$. Hence we can assume that either $\overline{D} = \frac{1}{e_x(\mathcal{A})}(x - \sigma(x))$ or $\overline{D} = pr^*(D_1)$. In the first case the assertion is obvious. In the second case it follows from the fact that the homomorphism of abelian varieties

$$\mathrm{id} - \mathrm{Frob} : \mathrm{Jac}_X \to \mathrm{Jac}_X$$

is an isogeny hence faithfully flat.
Returning to the general case note that by 3.1 at least $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$ are Morita equivalent. Let $\mathcal{L}$ be an arbitrary invertible $\overline{\mathcal{A}}$-$\overline{\mathcal{B}}$-bimodule. Then ${}^{\tau}\mathcal{L}$ is also invertible hence ${}^{\tau}(\mathcal{L}(\overline{D}')) \cong \mathcal{L}$ for some $\overline{D}' \in \mathrm{Div}_0(\overline{\mathcal{A}})$. It follows that the congruence (32) holds with $\overline{D}'$ instead of $\overline{D}$ as well and therefore $\mathrm{div}(\pi)(\overline{D} - \overline{D}') \in \mathrm{Div}(X)$. Hence by what we have shown above we may alter $\mathcal{L}$ by some element of $\mathrm{Pic}(\overline{\mathcal{A}})$ so that ${}^{\tau}(\mathcal{L}(\overline{D})) \cong \mathcal{L}$. □

To prove the other assertions of 4.20 we first note that for a connected $S \in \mathrm{Sch}/\mathbb{F}_q$ and $L \in \mathcal{SE}^D_{\mathcal{A},\mathcal{B},I,n}(S)$ the group of automorphisms $\mathrm{Aut}(L)$ of L is $= {\mathbb{F}_q}^*$ if $I = 0$ or $= 1$ otherwise. Hence for $I > 0$ the presheaf $\mathrm{SE}^D_{\mathcal{A},\mathcal{B},I}$ defined by

$$\mathrm{SE}^D_{\mathcal{A},\mathcal{B},I,n}(S) \colon = \text{isomorphism classes of objects of } \mathcal{SE}^D_{\mathcal{A},\mathcal{B},I,0}(S)$$

is a *fppf* sheaf and the canonical morphism $\mathcal{SE}^D_{\mathcal{A},\mathcal{B},I,n} \longrightarrow \mathrm{SE}^D_{\mathcal{A},\mathcal{B},I,n}$ is an isomorphism. We put $\mathrm{SE}^D_{\mathcal{A},\mathcal{B},I} = \mathrm{SE}^D_{\mathcal{A},\mathcal{B},I,0}$. For $I > 0$, 4.20 (c), (d) follows from:

Lemma 4.22. *Suppose that $I \neq 0$ and $\mathcal{SE}^D_{\mathcal{A},\mathcal{B},I,0} \neq \emptyset$. Then $\mathrm{SE}^D_{\mathcal{A},\mathcal{B},I}$ is a $\mathcal{W}(\mathcal{A},\mathcal{B},I,D)$-torsor.*

Proof. Assume first that $D = 0$, $\mathcal{A} = \mathcal{B}$. It follows from ([Laf], I.3, Théorème 2) that the map

$$(f : S \to \operatorname{Spec}\mathbb{F}_q) \mapsto f^* : \mathrm{Pic}_I(\mathcal{A}) \to \mathcal{SE}^0_{\mathcal{A},\mathcal{A},I}(S)$$

yields an isomorphism between $\mathcal{SE}^0_{\mathcal{A},\mathcal{A},I}$ and the trivial $\mathrm{Pic}_I(\mathcal{A})$-torsor over $\mathbb{F}_q$. In particular $\mathrm{SE}^0_{\mathcal{A},\mathcal{A},I}$ is isomorphic to the trivial $\mathrm{Pic}_{I,0}(\mathcal{A})$-torsor.
Now let $D \neq 0$. To simplify the notation we assume $|D| \cap |I| = \emptyset$ so that $\mathcal{C}_I(\mathcal{A}^T \times \mathcal{B}_T)_0 \cong \mathrm{Pic}_{I,0}(\mathcal{A})$ (the proof in the general case is analogous). Let $S \in \mathrm{Sch}/\mathbb{F}_q$ be connected and let $L_1, L_2 \in \mathrm{SE}^D_{\mathcal{A},\mathcal{B},I}(S)$. If L_1 and L_2 have the same poles then $\xi = L_2 \otimes L_1^{-1} \in \mathrm{SE}^0_{\mathcal{A},\mathcal{A},I}(S) \cong \mathrm{Pic}_{I,0}(\mathcal{A})$ by the remark above, hence $\xi L_1 = L_2$. In general there exists suitable $r_x \in \mathbb{Z}$ such that L_2 and $(\prod_{x\in|D|} \theta_x^{r_x})(L_1)$ have the same poles, hence $\xi(\prod_{x\in|D|} \theta_x^{r_x})(L_1) = L_2$ for some $\xi \in \mathrm{Pic}_I(\mathcal{A})$. Thus $wL_1 = L_2$ for $w = \xi(\prod_{x\in|D|} \theta_x^{r_x}) \in \mathcal{G}_0$.
Let $w \in \mathcal{W}(\mathcal{A},\mathcal{B},I,D)$, $L \in \mathrm{SE}^D_{\mathcal{A},\mathcal{B},I}(S)$ such that $wL = L$. Write $w = \xi\prod_{x\in|D|}\theta_x^{r_x}$ with $\xi \in \mathrm{Pic}_I(\mathcal{A})$ and $r_x \in \mathbb{Z}$. By 4.17 (ii), for $x \in |D|$ and the pole x_S of $wL = L$ we have $x_S \circ \mathrm{Frob}_S^{r_x} = x_S$, hence $\deg(x) \mid r_x$. By 4.17 (iv) it follows that $w \in \mathrm{Pic}_{I,0}(\mathcal{A})$. However $wL = L$ implies that w corresponds to $(wL) \otimes L^{-1} = L \otimes L^{-1} \in \mathrm{SE}^0_{\mathcal{A},\mathcal{A},I,0}(S)$ under the canonical bijection $\mathrm{Pic}_I(\mathcal{A}) \cong \mathrm{SE}^0_{\mathcal{A},\mathcal{A},I,0}(S)$, i.e. $w = 1$. This proves that for a connected $S \in \mathrm{Sch}/\mathbb{F}_q$, $\mathrm{SE}^D_{\mathcal{A},\mathcal{B},I}(S)$ is either empty or $\mathcal{W}(\mathcal{A},\mathcal{B},I,D)$ acts simply transitively on it.
To finish the proof we have to show that $\mathcal{SE}^D_{\mathcal{A},\mathcal{B},I,0} \neq \emptyset$ implies that $\mathrm{SE}^D_{\mathcal{A},\mathcal{B},I}(\operatorname{Spec}\overline{\mathbb{F}}_q) \neq \emptyset$. This is a consequence of the fact that $\mathcal{SE}^D_{\mathcal{A},\mathcal{B},I,0}$ is locally of finite presentation. □

Similarly one shows that $\mathrm{SE}^D_{\mathcal{A},\mathcal{B},I,n}$ is a $\mathrm{Im}(\mathcal{G}_0 \to \mathrm{Aut}(\mathcal{SE}^D_{\mathcal{A},\mathcal{B},I,n}))$-torsor for all $n \in \frac{1}{d}\mathbb{Z}$. In particular each $\mathrm{SE}^D_{\mathcal{A},\mathcal{B},I,n}$ is a finite étale $\mathbb{F}_q$-scheme. This proves (b) for $I \neq 0$.
It remains to consider the case $I = 0$. Choose an auxiliary level $J \in \mathrm{Div}(X)$ with $J > 0$ and $|D| \cap |J| = \emptyset$. A similar argument as in 4.3 shows that

$$\mathcal{SE}^D_{\mathcal{A},\mathcal{B},n} \quad \cong \quad \Gamma(J,\mathcal{A}_J)^* \backslash \mathrm{SE}^D_{\mathcal{A},\mathcal{B},J,n}$$

Hence $\mathcal{SE}^D_{\mathcal{A},\mathcal{B}}$ is a Deligne-Mumford stack. Moreover as in 4.3 one shows that the quotient $\mathrm{SE}^D_{\mathcal{A},\mathcal{B}} \colon = (\Gamma(J,\mathcal{A}_J)^*/\mathbb{F}_q{}^*)\backslash \mathrm{SE}^D_{\mathcal{A},\mathcal{B},J}$ is a coarse moduli scheme of $\mathcal{SE}^D_{\mathcal{A},\mathcal{B},0}$ and that $\mathcal{SE}^D_{\mathcal{A},\mathcal{B},0} \cong \mathbb{F}_q{}^*\backslash \mathrm{SE}^D_{\mathcal{A},\mathcal{B}}$. Finally since $\mathcal{W}(\mathcal{A},\mathcal{B},D) \cong \mathcal{W}(\mathcal{A},\mathcal{B},J,D)/(\Gamma(J,\mathcal{A}_J)^*/\mathbb{F}_q{}^*)$ it follows from Lemma 4.22 that $\mathrm{SE}^D_{\mathcal{A},\mathcal{B}}$ is a $\mathcal{W}(\mathcal{A},\mathcal{B},D)$-torsor over $\mathbb{F}_q$. This completes the proof of 4.20.

Remarks 4.23. (a) Let $D = \sum_{x\in|X|} m_x x \in \mathrm{Div}(\mathcal{A})$, $D \neq 0$ be such that $\sum_{x\in|X|} m_x = 0$. Condition (31) is not sufficient for $\mathcal{SE}^D_{\mathcal{A},\mathcal{B},I,0} \neq \emptyset$ (compare Remark 3.6). However if additionally we have $\sum_{x\in|X|}\mathbb{Z}m_x = \frac{1}{d}\mathbb{Z}$ then by taking suitable products of θ_x's we obtain elements g in the center of G of arbitrary degree $m \in \frac{1}{d}\mathbb{Z}$. Thus $\mathcal{SE}^D_{\mathcal{A},\mathcal{B},I,m} \neq \emptyset$ implies $\mathcal{SE}^D_{\mathcal{A},\mathcal{B},I,0} \neq \emptyset$. We also see that an automorphism $g \in \mathcal{G}$ of degree zero is uniquely determined by its restriction to $\mathcal{SE}^D_{\mathcal{A},\mathcal{B},I,0}$, i.e. we have $\mathcal{W}(\mathcal{A},\mathcal{B},I,D) = \mathcal{G}_0$.
(b) Suppose that $\mathcal{SE}^D_{\mathcal{A},\mathcal{B},I,0} \neq \emptyset$. One can describe the $\mathcal{W}(\mathcal{A},\mathcal{B},I,D)$-torsor $\mathrm{SE}^D_{\mathcal{A},\mathcal{B},I}/\mathbb{F}_q$ explicitely as follows. For $L \in \mathcal{SE}^D_{\mathcal{A},\mathcal{B},I,0}(\overline{\mathbb{F}}_q)$ let

$$\psi_L : \mathcal{W}(\mathcal{A},\mathcal{B},I,D) \times \operatorname{Spec}\overline{\mathbb{F}}_q = \coprod_{w\in\mathcal{W}(\mathcal{A},\mathcal{B},I,D)} \operatorname{Spec}\overline{\mathbb{F}}_q \to \mathrm{SE}^D_{\mathcal{A},\mathcal{B},I}$$

be given on the w-component by the morphism corresponding to wL. By 4.17 (v) the diagram

$$\begin{array}{ccc}
\mathcal{W}(\mathcal{A},\mathcal{B},I,D) \times \operatorname{Spec}\overline{\mathbb{F}}_q & \xrightarrow{\psi_L\times\mathrm{id}} & \mathrm{SE}^D_{\mathcal{A},\mathcal{B},I} \times \operatorname{Spec}\overline{\mathbb{F}}_q \\
\Big\downarrow{\scriptstyle \Theta_D^{-1}\times\mathrm{Frob}_q} & & \Big\downarrow{\scriptstyle \mathrm{id}\times\mathrm{Frob}_q} \\
\mathcal{W}(\mathcal{A},\mathcal{B},I,D) \times \operatorname{Spec}\overline{\mathbb{F}}_q & \xrightarrow{\psi_L\times\mathrm{id}} & \mathrm{SE}^D_{\mathcal{A},\mathcal{B},I} \times \operatorname{Spec}\overline{\mathbb{F}}_q
\end{array}$$

commutes. Thus ψ_L induces an isomorphism

$$\mathrm{SE}^D_{\mathcal{A},\mathcal{B},I} \cong (\mathcal{W}(\mathcal{A},\mathcal{B},I,D) \times \operatorname{Spec}\overline{\mathbb{F}}_q)/ < \Theta_D^{-1} \times \mathrm{Frob}_q >$$

4.5 Twists of moduli spaces of $\mathcal{A}$-elliptic sheaves

In this section $\mathcal{A}$ denotes a locally principal $\mathcal{O}_X$-order of rank d^2 with generic fiber A such that $e_\infty(\mathcal{A}) = d$. We also assume that there is a second closed point $\mathfrak{p} \neq \infty$ such that $e_{\mathfrak{p}}(\mathcal{A}) = d$ and we put $D \colon = \frac{1}{d}\infty - \frac{1}{d}\mathfrak{p}$. Let $\mathcal{B}$ be a locally principal $\mathcal{O}_X$-order of rank d^2 with $\underline{\mathrm{Disc}}(\mathcal{B}) = \underline{\mathrm{Disc}}(\mathcal{A})$ and such that for the generic fiber B of $\mathcal{B}$ we have

$$\sum_{x\in|X|} \mathrm{inv}_x(B)x = (\sum_{x\in|X|} \mathrm{inv}_x(A)x) + D \mod \mathrm{Div}(X).$$

In order to show that the moduli spaces $\mathrm{Ell}^\infty_{\mathcal{A},I}$ and $\mathrm{Ell}^{\mathfrak{p}}_{\mathcal{B},I}$ are twists of each other we are going to define a canonical tensor product $\mathcal{E}\ell\ell^\infty_{\mathcal{A},I} \times \mathcal{SE}^D_{\mathcal{A},\mathcal{B},I} \to \mathcal{E}\ell\ell^{\mathfrak{p}}_{\mathcal{B},I}$.

We introduce more notation. Recall that the notion of level structure at ∞ for objects of $\mathcal{E}\ell\ell^{\infty}_{\mathcal{A},I}$ and $\mathcal{SE}^{D}_{\mathcal{A},\mathcal{B},I}$ and at $\mathfrak{p}$ for objects of $\mathcal{E}\ell\ell^{\mathfrak{p}}_{\mathcal{B},I}$ and $\mathcal{SE}^{D}_{\mathcal{A},\mathcal{B},I}$ depend on the choice of certain local Frobenius bimodules. In order to define the tensor product (33) below these choices have to be compatibly matched. For ∞ let $M = (\mathcal{M}_\infty, \phi_\infty)$ be an invertible ϕ-$\mathcal{B}_\infty$-$\mathcal{A}_\infty$-bimodule of slope $-\frac{1}{d}$. For $\mathfrak{p}$ we choose an invertible ϕ-$\mathcal{A}_\mathfrak{p}$-$\mathcal{B}_\mathfrak{p}$-bimodule $N = (\mathcal{N}_\mathfrak{p}, \phi_\mathfrak{p})$ also of slope $-\frac{1}{d}$. We use M to define level structure at ∞ and N to define level structure at $\mathfrak{p}$. By Remark 2.22 there exists prime elements $\varpi_\infty \in \mathcal{O}_\infty$ and $\varpi_\mathfrak{p} \in \mathcal{O}_\mathfrak{p}$ such that

$$\phi_\infty^{d\deg(\infty)} = \varpi_\infty, \quad \phi_\mathfrak{p}^{d\deg(\mathfrak{p})} = \varpi_\mathfrak{p}.$$

Now fix a level $I \in \mathrm{Div}(X), I \geq 0$. We put

$$k(\infty)_\star = \begin{cases} k(\infty) & \text{if } \infty \notin |I|, \\ k(\infty)_d & \text{if } \infty \in |I|, \end{cases} \qquad k(\mathfrak{p})_\sharp = \begin{cases} k(\mathfrak{p}) & \text{if } \mathfrak{p} \notin |I|, \\ k(\mathfrak{p})_d & \text{if } \mathfrak{p} \in |I|. \end{cases}$$

There exists a canonical map $\mathcal{W}(\mathcal{A}, I, \infty) \to \mathcal{W}(\mathcal{A}, \mathcal{B}, I, D)$ induced by $C_I(\mathcal{A}^\infty \times \mathcal{B}_\infty)[\theta] \to C_I(\mathcal{A}^\infty \times \mathcal{B}_\infty)[\theta_\infty, \theta_\mathfrak{p}]$ given by $\theta \mapsto \theta_\infty^{-1}$. Using the diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & C_I(\mathcal{A}^\infty \times \mathcal{B}_\infty)_0 \arrow[r] \arrow[d] & \mathcal{W}(\mathcal{A}, I, \infty) \arrow[r] \arrow[d] & \mathrm{G}(k(\infty)_\star/\mathbb{F}_q) \arrow[d] \\
0 \arrow[r] & C_I(\mathcal{A}^\infty \times \mathcal{B}_\infty)_0 \arrow[r] & \mathcal{W}(\mathcal{A}, \mathcal{B}, I, D) \arrow[r] & \mathrm{G}(k(\infty)_\star/\mathbb{F}_q) \times \mathrm{G}(k(\mathfrak{p})_\sharp/\mathbb{F}_q)
\end{tikzcd}$$

it is easy to see that $\mathcal{W}(\mathcal{A}, I, \infty) \to \mathcal{W}(\mathcal{A}, \mathcal{B}, I, D)$ is injective and is equal to the kernel of the canonical projection $\mathcal{W}(\mathcal{A}, \mathcal{B}, I, D) \to \mathrm{G}(k(\mathfrak{p})_\sharp/\mathbb{F}_q)$ (compare 4.19). Recall that $C_I(\mathcal{A}^\infty \times \mathcal{B}_\infty)_0 \cong \mathrm{Pic}_{I,0}(\mathcal{A})$ if ∞ does not divide I. In the following we will consider $\mathcal{W}(\mathcal{A}, I, \infty)$ as a subgroup of $\mathcal{W}(\mathcal{A}, \mathcal{B}, I, D)$. From Proposition 4.20 we deduce that $\mathrm{SE}^{D}_{\mathcal{A},\mathcal{B},I}$ is a $\mathcal{W}(\mathcal{A}, I, \infty)$-torsor over $\mathrm{Spec}\, k(\mathfrak{p})_\sharp$. By 4.17 (iv) there exist $\xi_\infty, \xi_\mathfrak{p} \in C_I(\mathcal{A}^\infty \times \mathcal{B}_\infty)_0$ such that

$$\theta_\infty^{[k(\infty)_\star:\mathbb{F}_q]} = \xi_\infty, \qquad \theta_\mathfrak{p}^{[k(\mathfrak{p})_\sharp:\mathbb{F}_q]} = \xi_\mathfrak{p}$$

ξ_∞ and $\xi_\mathfrak{p}$ are given as follows. Let $\Pi_\infty \in \mathcal{B}_\infty$ (resp. $\Pi_\mathfrak{p} \in \mathcal{A}_\mathfrak{p}$) be a generator of the radical of $\mathcal{B}_\infty$ (resp. of $\mathcal{A}_\mathfrak{p}$). If $\infty \notin |I|$ (resp. $\infty \in |I|$) then ξ_∞ denotes the class in $C_I(\mathcal{A}^\infty \times \mathcal{B}_\infty)$ of the idele $(\{1\}_{x\neq\infty}, \Pi_\infty^{-1})$ (resp. $(\{1\}_{x\neq\infty}, \varpi_\infty^{-1})$) in $C_I(\mathcal{A}^\infty \times \mathcal{B}_\infty)$. If $\mathfrak{p} \notin |I|$ (resp. $\mathfrak{p} \in |I|$) then $\xi_\mathfrak{p}$ denotes the class in $C_I(\mathcal{A}^\infty \times \mathcal{B}_\infty)$ of the idele $(\{1\}_{x\neq\mathfrak{p}}, \Pi_\mathfrak{p})$ (resp. $(\{1\}_{x\neq\mathfrak{p}}, \varpi_\mathfrak{p})$) in $C_I(\mathcal{A}^\infty \times \mathcal{B}_\infty)$. The tensor product

$$\otimes : \mathcal{E}\ell\ell^{\infty}_{\mathcal{A},I} \otimes_{k(\infty)_\star} \mathcal{SE}^{D}_{\mathcal{A},\mathcal{B},I} \longrightarrow \mathcal{E}\ell\ell^{\mathfrak{p}}_{\mathcal{B},I}, (E, L) \mapsto E \otimes L \tag{33}$$

is a morphism of $\mathrm{Spec}\, k(\mathfrak{p})_\star$-stacks having the following properties:

(i) The morphism (33) is compatible with $\deg_\mathcal{A}$ and $\deg_\mathcal{B}$, i.e. for $m, n \in \frac{1}{d}\mathbb{Z}$ it induces a morphism

$$\mathcal{E}\ell\ell^{\infty}_{\mathcal{A},I,m} \otimes_{k(\infty)_\star} \mathcal{SE}^{D}_{\mathcal{A},\mathcal{B},I,n} \longrightarrow \mathcal{E}\ell\ell^{\mathfrak{p}}_{\mathcal{B},I,m+n}.$$

(ii) The morphism of stacks

$$\mathcal{E}\ell\ell^{\infty}_{\mathcal{A},I} \otimes_{k(\infty)_\star} \mathcal{SE}^D_{\mathcal{A},\mathcal{B},I} \longrightarrow \mathcal{E}\ell\ell^{\mathfrak{p}}_{\mathcal{B},I} \otimes_{k(\mathfrak{p})_\sharp} \mathcal{SE}^D_{\mathcal{A},\mathcal{B},I}, (E,L) \mapsto (E \otimes L, L)$$

is an isomorphism with quasi-inverse

$$\mathcal{E}\ell\ell^{\mathfrak{p}}_{\mathcal{B},I} \otimes_{k(\mathfrak{p})_\sharp} \mathcal{SE}^D_{\mathcal{A},\mathcal{B},I} \longrightarrow \mathcal{E}\ell\ell^{\infty}_{\mathcal{A},I} \otimes_{k(\infty)_\star} \mathcal{SE}^D_{\mathcal{A},\mathcal{B},I}, (E,L) \mapsto (E \otimes L^{-1}, L).$$

(iii) The following diagram commutes

$$\begin{tikzcd}
\mathcal{E}\ell\ell^{\infty}_{\mathcal{A},I} \otimes_{k(\infty)_\star} \mathcal{SE}^D_{\mathcal{A},\mathcal{B},I} \arrow[d, "\theta\otimes\theta_\infty"] \arrow[dr, "\otimes"] & \\
& \mathcal{E}\ell\ell^{\infty}_{\mathcal{B},I} \\
\mathcal{E}\ell\ell^{\infty}_{\mathcal{A},I} \otimes_{k(\infty)_\star} \mathcal{SE}^D_{\mathcal{A},\mathcal{B},I} \arrow[ur, "\otimes"] &
\end{tikzcd}$$

(iv) For $\xi \in \mathcal{C}_I(\mathcal{A}^\infty \times \mathcal{B}_\infty)$ the following diagram commutes

$$\begin{tikzcd}
\mathcal{E}\ell\ell^{\infty}_{\mathcal{A},I} \otimes_{k(\infty)_\star} \mathcal{SE}^D_{\mathcal{A},\mathcal{B},I} \arrow[d, "\xi\otimes\xi^{-1}"] \arrow[dr, "\otimes"] & \\
& \mathcal{E}\ell\ell^{\infty}_{\mathcal{B},I} \\
\mathcal{E}\ell\ell^{\infty}_{\mathcal{A},I} \otimes_{k(\infty)_\star} \mathcal{SE}^D_{\mathcal{A},\mathcal{B},I} \arrow[ur, "\otimes"] &
\end{tikzcd}$$

.

To define (33) let $S \in \mathrm{Sch}/\mathbb{F}_q$ and let $\infty_S : S \to X$, $\mathfrak{p}_S : S \to X$ be morphisms in $\mathrm{Sch}/\mathbb{F}_q$ which factor through $\infty \to X$ and $\mathfrak{p} \to X$ respectively. Let $E = (\mathcal{E}, \infty_S, t)$ be an $\mathcal{A}$-elliptic sheaf over S with zero $z : S \to X$ and let $L = (\mathcal{L}, \Phi)$ be an invertible $\mathcal{A}$-$\mathcal{B}$-bimodule of slope $\frac{1}{d}(\infty_S - \mathfrak{p}_S)$. Define

$$E \otimes L \colon = (\mathcal{E} \otimes_{\mathcal{A}} \mathcal{L}, \mathfrak{p}_S, t \otimes_{\mathcal{A}} \Phi).$$

Note that $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{L}(-\frac{1}{d}\mathfrak{p}_S) = \mathcal{E}(-\frac{1}{d}\infty_S) \otimes_{\mathcal{A}} \mathcal{L}(\frac{1}{d}(\infty_S - \mathfrak{p}_S))$. One easily checks that $t \otimes_{\mathcal{A}} \Phi$ is an injective $\mathcal{B} \boxtimes \mathcal{O}_S$-linear homomorphism with $\mathrm{Coker}(t \otimes_{\mathcal{A}} \Phi) \cong \mathrm{Coker}(t) \otimes_{\mathcal{A}} \mathcal{L}$. It follows from 3.8 (c) that $E \otimes L$ is a $\mathcal{B}$-elliptic sheaf with pole $\mathfrak{p}$ and zero z. Thus we have defined (33) if $I = 0$.
When considering additionally level-I-structure, it is enough to treat separately the three cases $\infty, \mathfrak{p} \notin |I|$, $|I| = \{\infty\}$ and $|I| = \{\mathfrak{p}\}$. In the first case if E carries a level-I-structure α and L a level-I-structure β then one defines a level-I-structure $\alpha \bullet \beta$ on $E \otimes L$ as in (21).

Suppose now $I = n\infty$, $n > 0$ and let $E = (\mathcal{E}, \infty_S, t)$, $L = (\mathcal{L}, \Phi, \infty_S, \mathfrak{p}_S)$ be as above. Let (α, λ), (μ, β) be level-I-structures on E and L respectively such that $\lambda = \mu : S \to \operatorname{Spec} k(\infty)_d$ lifts ∞_S. Thus

$$\alpha : (\mathrm{id}_I \times \lambda)^*(\mathcal{M}_I) \xrightarrow{\cong} \mathcal{E}|_{I\times S}, \quad \beta : \mathcal{B}_I \boxtimes \mathcal{O}_S \xrightarrow{\cong} (\mathrm{id}_I \times \mu)^*(\mathcal{M}_I) \otimes_{\mathcal{A}} \mathcal{L}|_{I\times S}.$$

Let $\alpha \bullet \beta$ be the composition

$$\alpha \bullet \beta : \mathcal{B}_I \boxtimes \mathcal{O}_S \xrightarrow{\beta} (\mathrm{id}_I \times \mu)^*(\mathcal{M}_I) \otimes_{\mathcal{A}} \mathcal{L}|_{I\times S} \xrightarrow{\alpha\otimes \mathrm{id}} (\mathcal{E} \otimes_{\mathcal{A}} \mathcal{L})|_{I\times S}$$

Finally let $I = n\mathfrak{p}$, $n > 0$ and let α and (μ, β) be level-I-structures on E and L. In this case level-I-structures on E and L are given by

$$\alpha : \mathcal{A}_I \boxtimes \mathcal{O}_S \longrightarrow \mathcal{E}|_{I\times S}, \quad \beta : (\mathrm{id}_I \times \mu)^*(\mathcal{N}_I) \longrightarrow \mathcal{L}|_{I\times S}$$

where $\mu : S \to \operatorname{Spec} k(\mathfrak{p})_d$ is a lift of $\mathfrak{p}_S$. We set

$$\alpha \bullet \beta : (\mathrm{id}_I \times \mu)^*(\mathcal{N}_I) \xrightarrow{\beta} \mathcal{L}|_{I\times S} = \mathcal{A}_I \otimes_{\mathcal{A}} \mathcal{L}|_{I\times S} \xrightarrow{\alpha\otimes \mathrm{id}} (\mathcal{E} \otimes_{\mathcal{A}} \mathcal{L})|_{I\times S}.$$

In both cases one easily checks that $\alpha \bullet \beta$ defines a level-I-structure on $E \otimes L$. Thus we have defined (33). The straight forward but tedious verification of the properties (i)–(iv) will be left to the reader.

Recall that $\mathrm{Ell}^{\infty}_{\mathcal{A},I}$, $\mathrm{Ell}^{\mathfrak{p}}_{\mathcal{B},I}$ and $\mathrm{SE}^{D}_{\mathcal{A},\mathcal{B},I}$ denote the coarse moduli spaces of $\mathcal{E}\ell\ell^{\infty}_{\mathcal{A},I,0}$, $\mathcal{E}\ell\ell^{\mathfrak{p}}_{\mathcal{B},I,0}$ and $\mathcal{SE}^{D}_{\mathcal{A},\mathcal{B},I,0}$ respectively (these are fine moduli spaces if $I \neq 0$). By (i)–(iv), (33) induces an $\mathcal{W}(\mathcal{A}, I, \infty)$-equivariant isomorphism of $\mathbb{F}_q$-schemes

$$\mathrm{Ell}^{\infty}_{\mathcal{A},I} \otimes_{k(\infty)_\star} \mathrm{SE}^{D}_{\mathcal{A},\mathcal{B},I} \longrightarrow \mathrm{Ell}^{\mathfrak{p}}_{\mathcal{B},I} \otimes_{k(\mathfrak{p})_\sharp} \mathrm{SE}^{D}_{\mathcal{A},\mathcal{B},I}\,. \tag{34}$$

Here the (free) action of the finite group $\mathcal{W}(\mathcal{A}, I, \infty)$ on the right is given by $\mathrm{id} \otimes \xi$, $\xi \in \mathcal{W}(\mathcal{A}, I, \infty) \hookrightarrow \mathcal{W}(\mathcal{A}, \mathcal{B}, I, D)$ whereas on the left it is given by $\xi^{-1} \otimes \xi$. Consequently by passing to quotients under the action and using the fact that $\mathrm{SE}^{D}_{\mathcal{A},\mathcal{B},I}/k(\mathfrak{p})_\sharp$ is a $\mathcal{W}(\mathcal{A}, I, \infty)$-torsor we obtain:

THEOREM 4.24. *The isomorphism (34) induces an isomorphism of $k(\mathfrak{p})_\sharp$-schemes*

$$(\mathrm{Ell}^{\infty}_{\mathcal{A},I} \otimes_{k(\infty)_\star} \mathrm{SE}^{D}_{\mathcal{A},\mathcal{B},I})/\mathcal{W}(\mathcal{A}, I, \infty) \quad \cong \quad \mathrm{Ell}^{\mathfrak{p}}_{\mathcal{B},I}$$

We shall give now another formulation of this result. Note that

$$\Theta_D^{[k(\mathfrak{p})_\sharp:\mathbb{F}_q]} = \theta_\infty^{[k(\mathfrak{p})_\sharp:\mathbb{F}_q]}\xi_{\mathfrak{p}} = \mathrm{Frob}_{\mathrm{SE}^{D}_{\mathcal{A},\mathcal{B},I}/k(\mathfrak{p})_\sharp}$$

In particular $\Theta_D^{-[k(\mathfrak{p})_\sharp:\mathbb{F}_q]}$ lies in $\mathcal{W}(\mathcal{A}, I, \infty)$ and is equal to $= \theta^{[k(\mathfrak{p})_\sharp:\mathbb{F}_q]}\xi_{\mathfrak{p}}^{-1}$. The fact that (34) is in particular $\Theta_D^{[k(\mathfrak{p})_\sharp:\mathbb{F}_q]}$-equivariant implies that the following diagram commutes

$$\begin{tikzcd}
\mathrm{Ell}^{\infty}_{\mathcal{A},I} \otimes_{k(\infty)_\star} \mathrm{SE}^{D}_{\mathcal{A},\mathcal{B},I} \arrow[r, "(34)"] \arrow[d, "\theta^{[k(\mathfrak{p})_\sharp:\mathbb{F}_q]}\xi_{\mathfrak{p}}^{-1}\otimes \mathrm{Frob}_{\mathrm{SE}/k(\mathfrak{p})_\sharp}"] & \mathrm{Ell}^{\mathfrak{p}}_{\mathcal{B},I} \otimes_{k(\mathfrak{p})_\sharp} \mathrm{SE}^{D}_{\mathcal{A},\mathcal{B},I} \arrow[d, "\mathrm{id}\otimes \mathrm{Frob}_{\mathrm{SE}/k(\mathfrak{p})_\sharp}"] \\
\mathrm{Ell}^{\infty}_{\mathcal{A},I} \otimes_{k(\infty)_\star} \mathrm{SE}^{D}_{\mathcal{A},\mathcal{B},I} \arrow[r, "(34)"] & \mathrm{Ell}^{\mathfrak{p}}_{\mathcal{B},I} \otimes_{k(\mathfrak{p})_\sharp} \mathrm{SE}^{D}_{\mathcal{A},\mathcal{B},I}
\end{tikzcd}$$

Fix $L \in \mathrm{SE}^D_{\mathcal{A},\mathcal{B},I}(\overline{\mathbb{F}}_q)$. Its poles correspond to $\mathbb{F}_q$-embeddings $\lambda : k(\infty)_\star \to \overline{\mathbb{F}}_q$, $\mu : k(\mathfrak{p})_\sharp \to \overline{\mathbb{F}}_q$. By taking base change of the above diagram with respect to the morphism $\mathrm{Spec}\,\overline{\mathbb{F}}_q \to \mathrm{SE}^D_{\mathcal{A},\mathcal{B},I}$ corresponding to L we obtain:

Theorem 4.25. *Let $m = [k(\mathfrak{p})_\sharp : \mathbb{F}_q]$. Thus $m = \deg(\mathfrak{p})$ if $\mathfrak{p} \notin |I|$ and $m = d\deg(\mathfrak{p})$ otherwise. The isomorphism $\cdot \otimes L : \mathrm{Ell}^\infty_{\mathcal{A},I} \otimes_{k(\infty)_\star,\lambda} \overline{\mathbb{F}}_q \to \mathrm{Ell}^{\mathfrak{p}}_{\mathcal{B},I} \otimes_{k(\mathfrak{p})_\sharp,\mu} \overline{\mathbb{F}}_q$ induces an isomorphism of $k(\mathfrak{p})_\sharp$-schemes*

$$(\mathrm{Ell}^\infty_{\mathcal{A},I} \otimes_{k(\infty)_\star,\lambda} \overline{\mathbb{F}}_q)/ < \theta^m \xi_{\mathfrak{p}}^{-1} \otimes \mathrm{Frob}_q^m > \quad \cong \quad \mathrm{Ell}^{\mathfrak{p}}_{\mathcal{B},I}\,.$$

Remark 4.26. A pair $(\lambda,\mu) \in \mathrm{Hom}_{\mathbb{F}_q}(k(\infty)_\star, \overline{\mathbb{F}}_q) \times \mathrm{Hom}_{\mathbb{F}_q}(k(\mathfrak{p})_\sharp, \overline{\mathbb{F}}_q)$ will be called *admissible for* $(\mathcal{A},\mathcal{B},I)$ if there exists $L \in \mathrm{SE}^D_{\mathcal{A},\mathcal{B},I}(\overline{\mathbb{F}}_q)$ with poles λ and μ. The surjectivity of the homomorphism $\mathcal{W}(\mathcal{A},\mathcal{B},I,D) \to \mathrm{G}(k(\infty)_\star/\mathbb{F}_q)$ implies that for all λ there exists a μ such that (λ,μ) is admissible.

4.6 Application to uniformization

Let $\mathcal{A}$ be locally principal $\mathcal{O}_X$-order of rank d^2 with generic fiber A such that $e_\infty(\mathcal{A}) = d$. Let $I \in \mathrm{Div}(X)$ denote an effective divisor. For a closed point $x \in |X| - |I|$ we denote by $\widehat{\mathrm{Ell}}^\infty_{\mathcal{A},I}/\,\mathrm{Spf}(\mathcal{O}_x)$ the formal completion of $\mathrm{Ell}^\infty_{\mathcal{A},I}$ along the fiber at x of the characteristic morphism $\mathrm{Ell}^\infty_{\mathcal{A},I} \to X - I$. Also for an arbitrary $x \in |X|$ we let $\mathrm{Ell}^{\infty,\mathrm{an}}_{\mathcal{A},I}/F_x$ denote the rigid analytic space associated to $\mathrm{Ell}^\infty_{\mathcal{A},I} \times_X \mathrm{Spec}\,F_x$. There exists two types of uniformization of $\mathrm{Ell}^\infty_{\mathcal{A},I}$, i.e. explicite descriptions of $\widehat{\mathrm{Ell}}^\infty_{\mathcal{A},I}/\,\mathrm{Spf}(\mathcal{O}_\infty)$ and $\mathrm{Ell}^{\infty,\mathrm{an}}_{\mathcal{A},I}/F_x$ as (finite unions of) certain quotients of Drinfeld's symmetric spaces and its coverings. These are called *uniformization at the pole* and *Cherednik-Drinfeld uniformization*. The first concerns the point $x = \infty$ (under the assumption $\mathrm{inv}_\infty A = 0$) whereas the second the points $\mathfrak{p} \in |X| - \{\infty\}$ with $\mathrm{inv}_{\mathfrak{p}} A = \frac{1}{d}$. By using Theorem 4.25 we show that the two types of uniformization are equivalent (see Proposition 4.28 below).

In order to introduce the quotients of symmetric spaces appearing in the uniformization results below we have to introduce more notation. Fix a closed point $x \in X$. We denote by $\widehat{\mathcal{O}}^{\mathrm{nr}}_x$ the completion of the strict henselisation of $\mathcal{O}_x$ and by $\widehat{F}^{\mathrm{nr}}_x$ its function field. For each positive integer m we denote by $F_{x,m}$ the unramified extension of degree m of F_x in $\widehat{F}^{\mathrm{nr}}_x$ and let $\mathcal{O}_{x,m}$ be its ring of integers. Note that the projection $\mathcal{O}_{x,m} \to k(x)_m$ has a canonical section, i.e. $k(x)_m \subseteq F_{x,m}$. Similarly $\overline{k(x)} \subset \widehat{\mathcal{O}}^{\mathrm{nr}}_x$. Denote by D_x the central division algebra over F_x with invariant $\frac{1}{d}$ and let $\mathcal{D}_x$ be the maximal order in D_x. We also fix a uniformizer $\varpi_x \in \mathcal{O}_x$ and an element $\Pi_x \in \mathcal{D}_x$ with $\Pi_x^d = \varpi_x$. Let σ denote the automorphism on $\mathcal{O}_{x,m}$ and $\widehat{\mathcal{O}}^{\mathrm{nr}}_x$ which induces the Frob_q on the residue fields.

Let Ω^d_x be Drinfeld's $(d-1)$-dimensional symmetric space over F_x and $\widehat{\Omega}^d_x/\,\mathrm{Spf}(\mathcal{O}_x)$ its canonical formal model (see e.g. [Ge]). The rigid analytic variety Ω^d_x parametrizes certain formal groups. The formal scheme $\widehat{\Omega}^d_x$ is equipped with a canonical $\mathrm{GL}_d(F_x)$-action.

We define an action of $\mathrm{GL}_d(F_x)$ on $\widehat{\Omega}_x^d \widehat{\otimes}_{\mathcal{O}_x} \mathcal{O}_{x,m} = \widehat{\Omega}_x^d \otimes_{k(x)} k(x)_m$ and $\widehat{\Omega}_x^d \widehat{\otimes}_{\mathcal{O}_x} \widehat{\mathcal{O}}_x^{\mathrm{nr}} = \widehat{\Omega}_x^d \otimes_{k(x)} \overline{k(x)}$ by letting $g \in \mathrm{GL}_d(F_x)$ act canonically on $\widehat{\Omega}_x^d$ and by $\sigma^{-v_x(\det(g))}$ on $\mathcal{O}_{x,m}$ and $\mathcal{O}_x$ respectively. There exists a tower $\ldots \Sigma_{n+1,x}^d \to \Sigma_{n,x}^d \to \ldots \to \Sigma_{1,x}^d \to \Sigma_{0,x}^d = \Omega_x^d \otimes_{F_x} F_{x,d}$ of finite étale Galois coverings ([Ge], IV.1). Each $\Sigma_{n,x}^d$ carries a $\mathrm{GL}_d(F_x)/\varpi_x^{\mathbb{Z}}$- and $D_x^*/\varpi_x^{\mathbb{Z}}$-action and the covering maps $\Sigma_{n+1,x}^d \to \Sigma_{n,x}^d$ are equivariant. Finally for $n \geq 0$ we equip $\Sigma_{n,x}^d \otimes_{F_{x,d}} \widehat{F}_x^{\mathrm{nr}} = \Sigma_{n,x}^d \otimes_{k(x)_d} \overline{k(x)}$ with a $\mathrm{GL}_d(F_x)$- and $\mathcal{D}_x$-action by letting $g \in \mathrm{GL}_d(F_x)$ (or inD_x) act canonically on the first factor and by $\sigma^{-v_x(\mathrm{Nrd}(g))}$ on the second factor.

Rigid analytic Drinfeld-Stuhler varieties. Suppose that $\mathrm{inv}_\infty A = 0$ and fix an isomorphism $A_\infty \cong M_d(F_\infty)$. We write $I = n\infty + I^\infty$ with $\infty \notin |I^\infty|$. Assume first that $n = 0$. We define

$$\widehat{\mathrm{Sh}}_{\mathcal{A},I}^{\infty} : = A^* \backslash \left(A^*(\mathbb{A}^\infty)/U_I(\mathcal{A}^\infty) \times \widehat{\Omega}_\infty^d \right).$$

This is formal scheme over $\mathrm{Spf}(\mathcal{O}_\infty)$.
Next assume $\infty \in |I|$ and write $I = n\infty + I^\infty$ with $\infty \notin |I^\infty|$. Then we define

$$\mathrm{Sh}_{\mathcal{A},I}^{\infty} : = A^* \backslash \left(A^*(\mathbb{A}^\infty)/U_{I^\infty}(\mathcal{A}^\infty) \times \Sigma_{n,\infty}^d \right).$$

This is rigid analytic space over F_∞.
There exists a canonical right action of the group $\mathcal{C}_I(\mathcal{A}^\infty \times \mathcal{D}_\infty)$ on $\widehat{\mathrm{Sh}}_{\mathcal{A},I}^{\infty}$ and $\mathrm{Sh}_{\mathcal{A},I}^{\infty}$ which is defined as follows. Let $a = (\{a_x\}_{x\neq\infty}, d_\infty) \in (\prod'_{x\in|X|-\{\infty\}} N(\mathcal{A}_x)) \times N(\mathcal{D}_\infty)$ and assume first $n = 0$. Then the right action of the class $[a] \in \mathcal{C}_I(\mathcal{A}^\infty \times \mathcal{D}_\infty)$ of a on $\widehat{\mathrm{Sh}}_{\mathcal{A},I}^{\infty}$ is given by right multiplication by $\{a_x\}_{x\neq\infty}$ on $A^*(\mathbb{A}^\infty)/U_I(\mathcal{A}^\infty)$. Now assume that $n > 0$. Then $[a]$ acts on $\mathrm{Sh}_{\mathcal{A},I}^{\infty}$ by right multiplication of $\{a_x\}_{x\neq\infty}$ on $A^*(\mathbb{A}^\infty)/U_{I^\infty}(\mathcal{A}^\infty)$ and letting d_∞^{-1} act on $\Sigma_{n,\infty}^d$.
There are canonical morphism

$$pole : \widehat{\mathrm{Sh}}_{\mathcal{A},I}^{\infty} \to \mathrm{Spec}\, k(\infty) \qquad \text{if } n = 0, \tag{35}$$

$$pole : \mathrm{Sh}_{\mathcal{A},I}^{\infty} \to \mathrm{Spec}\, k(\infty)_d \qquad \text{if } n > 0. \tag{36}$$

which we are going to defined now. Denote by $l = l_\infty : A^*(\mathbb{A}^\infty) \to \mathbb{Z}$ the composite

$$l_\infty : A^*(\mathbb{A}^\infty) \xrightarrow{\mathrm{Nrd}} F^*(\mathbb{A}^\infty) \xrightarrow{\mathrm{div}} \bigoplus_{x\neq\infty} \mathbb{Z}x \xrightarrow{\deg} \mathbb{Z}.$$

Note that for $a \in A^* \subset A^*(\mathbb{A}^\infty)$ we have $l_\infty(a) = -\deg(\infty) v_\infty(\mathrm{Nrd}(a))$. First assume $n = 0$. Let

$$A^*(\mathbb{A}^\infty)/U_I(\mathcal{A}^\infty) \times \widehat{\Omega}_\infty^d \longrightarrow \mathrm{Spec}\, k(\infty) \tag{37}$$

be given on the component $\eta U_I(\mathcal{A}^\infty) \times \widehat{\Omega}^d_\infty$ by

$$\widehat{\Omega}^d_\infty \longrightarrow \operatorname{Spec} k(\infty) \xrightarrow{\mathrm{Frob}_q^{-l(\eta)}} \operatorname{Spec} k(\infty). \tag{38}$$

Clearly, (38) factors through $\widehat{\mathrm{Sh}}^\infty_{\mathcal{A},I}$, hence it induces (35).
Now suppose $n > 0$. Since $k(\infty)_d \subseteq F_{\infty,d}$, we get a map $\Sigma^d_{n,\infty} \to \operatorname{Spec} F_{\infty,d} \to \operatorname{Spec} k(\infty)_d$. Note that for $g \in \mathrm{GL}_d(F_\infty)$ the diagram

$$\begin{array}{ccc} \Sigma^d_{n,\infty} & \xrightarrow{g\cdot} & \Sigma^d_{n,\infty} \\ \big\downarrow & & \big\downarrow \\ \operatorname{Spec} k(\infty)_d & \xrightarrow{\mathrm{Frob}_q^{-v_\infty(\det(g))}} & \operatorname{Spec} k(\infty)_d \end{array} \tag{39}$$

commutes. We define

$$A^*(\mathbb{A}^\infty)/U_I(\mathcal{A}^\infty) \times \Sigma^d_{n,\infty} \to \operatorname{Spec} k(\infty)_d \tag{40}$$

on the component corresponding to $\eta U_I(\mathcal{A}^\infty) \in A^*(\mathbb{A}^\infty)/U_I(\mathcal{A}^\infty)$ by

$$\Sigma^d_{n,\infty} \longrightarrow \operatorname{Spec} k(\infty)_d \xrightarrow{\mathrm{Frob}_q^{-l(\eta)}} \operatorname{Spec} k(\infty)_d.$$

The commutativity of (39) implies that (40) factors through $\mathrm{Sh}^\infty_{\mathcal{A},I}$, i.e. it yields the map (36).

Cherednik-Drinfeld varieties. Let $\widetilde{\xi} = \{\widetilde{\xi}_x\}_{x\neq\infty} \in (\prod'_{x\in|X|-\{\infty\}} N(\mathcal{A}_x))$ and let $\xi \in \mathcal{C}_I(\mathcal{A}^\infty \times \mathcal{D}_\infty)$ be the idele class represented by $(\{\widetilde{\xi}_x\}_{x\neq\infty}, 1) \in (\prod'_{x\in|X|-\{\infty\}} N(\mathcal{A}_x)) \times N(\mathcal{D}_\infty)$. We assume that ξ is a central element in $\mathcal{C}_I(\mathcal{A}^\infty \times \mathcal{D}_\infty)$ and that $m = -d\deg_{\mathcal{A}}(\xi) = -l_\infty(\widetilde{\xi}) \neq 0$. We define

$$\begin{aligned} \widehat{\mathrm{Sh}}^\xi_{\mathcal{A},I,\infty} &:= A^*\backslash \left(A^*(\mathbb{A}^\infty)/U_I(\mathcal{A}^\infty)\widetilde{\xi}^{\mathbb{Z}} \times \widehat{\Omega}^d_\infty \otimes_{k(\infty)} \overline{k(\infty)}\right) && \text{if } n = 0, \\ \mathrm{Sh}^\xi_{\mathcal{A},I,\infty} &:= A^*\backslash \left(A^*(\mathbb{A}^\infty)/U_{I^\infty}(\mathcal{A}^\infty)\widetilde{\xi}^{\mathbb{Z}} \times \Sigma^d_{n,\infty} \otimes_{k(\infty)_d} \overline{k(\infty)}\right) && \text{if } n > 0. \end{aligned}$$

As above one defines a right action of $\mathcal{C}_I(\mathcal{A}^\infty \times \mathcal{D}_\infty)$ on $\widehat{\mathrm{Sh}}^\xi_{\mathcal{A},I,\infty}$ and $\mathrm{Sh}^\xi_{\mathcal{A},I,\infty}$ by letting $a = (\{a_x\}_{x\neq\infty}, d_\infty) \in (\prod'_{x\in|X|-\{\infty\}} N(\mathcal{A}_x)) \times N(\mathcal{D}_\infty)$ act by right multiplication by $\{a_x\}_{x\neq\infty}$ on $A^*(\mathbb{A}^\infty)/U_I(\mathcal{A}^\infty)$ and letting d_∞^{-1} act on $\widehat{\Omega}^d_\infty \widehat{\otimes}_{\mathcal{O}_\infty} \widehat{\mathcal{O}}^{\mathrm{nr}}_\infty$ (if $n = 0$) and $\Sigma^d_{n,\infty} \otimes_{F_{\infty,d}} \widehat{F}^{\mathrm{nr}}_\infty$ (if $n > 0$). Note that ξ acts trivially.
Let $k(\xi)$ denote the fixed field of Frob_q^m in $\overline{k(\infty)}$. There are canonical morphisms

$$\widehat{\mathrm{Sh}}^\xi_{\mathcal{A},I,\infty} \to \operatorname{Spec} k(\xi) \qquad \text{if } n = 0, \tag{41}$$

$$\mathrm{Sh}^\xi_{\mathcal{A},I,\infty} \to \operatorname{Spec} k(\xi) \qquad \text{if } n > 0. \tag{42}$$

Their definition is similar to the definition of (35) and (35). For example (41) is induced by the maps

$$\eta(U_I(\mathcal{A}^\infty)\widetilde{\xi}^{\mathbb{Z}}) \times \widehat{\Omega}^d_\infty \otimes_{k(\infty)} \overline{k(\infty)} \longrightarrow \operatorname{Spec} k(\xi) \xrightarrow{\mathrm{Frob}_q^{-l(\eta)}} \operatorname{Spec} k(\xi).$$

The rigid analytic varieties $\mathrm{Sh}^\infty_{\mathcal{A},I}$ and $\mathrm{Sh}^\xi_{\mathcal{A},I,\infty}$ are twists of each other. More precisely we have the following result.

LEMMA 4.27. *(a) There exists a canonical isomorphism of formal schemes over* $\mathrm{Spf}(\mathcal{O}_\infty)$ *(resp. rigid analytic varieties over F_∞)*

$$\widehat{\mathrm{Sh}}^\infty_{\mathcal{A},I} \otimes_{k(\infty)} \overline{k(\infty)} / < \xi \otimes \mathrm{Frob}_q^m > \;\cong\; \widehat{\mathrm{Sh}}^\xi_{\mathcal{A},I,\infty} \quad \textit{for } n=0, \textit{ resp.} \tag{43}$$

$$\mathrm{Sh}^\infty_{\mathcal{A},I} \otimes_{k(\infty)_d} \overline{k(\infty)} / < \xi \otimes \mathrm{Frob}_q^m > \;\cong\; \mathrm{Sh}^\xi_{\mathcal{A},I,\infty} \quad \textit{for } n>0. \tag{44}$$

Here $\widehat{\mathrm{Sh}}^\infty_{\mathcal{A},I} \otimes_{k(\infty)} \overline{k(\infty)}$ *(resp.* $\mathrm{Sh}^\infty_{\mathcal{A},I} \otimes_{k(\infty)_d} \overline{k(\infty)_d}$*) denotes the base change to* $\overline{k(\infty)}$ *of the morphism (35) (resp. (36)).*
(b) Let $\xi_\infty \in \mathcal{C}_I(\mathcal{A}^\infty \times \mathcal{D}_\infty)$ *be the class of the idele* $(\{1\}_{x\neq\infty}, \Pi_\infty^{-1})$ *(if $n=0$) resp. of* $(\{1\}_{x\neq\infty}, \varpi_\infty^{-1})$ *(if $n>0$). Then we have*

$$\widehat{\mathrm{Sh}}^\xi_{\mathcal{A},I,\infty} \otimes_{k(\xi)} \overline{k(\xi)} / < \xi_\infty \otimes \mathrm{Frob}_q^{\deg(\infty)} > \cong \widehat{\mathrm{Sh}}^\infty_{\mathcal{A},I} \quad \textit{for } n=0, \textit{ resp.} \tag{45}$$

$$\mathrm{Sh}^\xi_{\mathcal{A},I,\infty} \otimes_{k(\xi)} \overline{k(\xi)} / < \xi_\infty \otimes \mathrm{Frob}_q^{d\deg(\infty)} > \cong \mathrm{Sh}^\infty_{\mathcal{A},I} \quad \textit{for } n>0. \tag{46}$$

Proof. We prove only the existence of (43). The other cases are similar and will be left to the reader. For $\eta U_I(\mathcal{A}^\infty) \in A^*(\mathbb{A}^\infty)/U_I(\mathcal{A}^\infty)$ we denote the base change of the map (38) to $\overline{k(\infty)}$ by $(\eta U_I(\mathcal{A}^\infty) \times \widehat{\Omega}^d_\infty) \otimes_{k(\infty)} \overline{k(\infty)}$. Let

$$(\eta U_I(\mathcal{A}^\infty) \times \widehat{\Omega}^d_\infty) \otimes_{k(\infty)} \overline{k(\infty)} \xrightarrow{\mathrm{id}\otimes \mathrm{Frob}_q^{l(\eta)}} \widehat{\Omega}^d_\infty \otimes_{k(\infty)} \overline{k(\infty)} \tag{47}$$

and let

$$\left(A^*(\mathbb{A}^\infty)/U_I(\mathcal{A}^\infty) \times \widehat{\Omega}^d_\infty\right) \otimes_{k(\infty)} \overline{k(\infty)} \to A^*(\mathbb{A}^\infty)/U_I(\mathcal{A}^\infty) \times (\widehat{\Omega}^d_\infty \otimes_{k(\infty)} \overline{k(\infty)}) \tag{48}$$

be made up of all the morphisms (45). One easily checks that it is A^*-equivariant and that the following diagram commutes:

$$\begin{array}{ccc}
(A^*(\mathbb{A}^\infty)/U_I(\mathcal{A}^\infty) \times \widehat{\Omega}^d_\infty) \otimes \overline{k(\infty)} & \xrightarrow{(48)} & A^*(\mathbb{A}^\infty)/U_I(\mathcal{A}^\infty) \times (\widehat{\Omega}^d_\infty \otimes \overline{k(\infty)}) \\
\Big\downarrow{\scriptstyle \cdot\xi\otimes \mathrm{Frob}_q^m} & & \Big\downarrow{\scriptstyle \cdot\widetilde{\xi}\otimes \mathrm{id}} \\
(A^*(\mathbb{A}^\infty)/U_I(\mathcal{A}^\infty) \times \widehat{\Omega}^d_\infty) \otimes \overline{k(\infty)} & \xrightarrow{(48)} & A^*(\mathbb{A}^\infty)/U_I(\mathcal{A}^\infty) \times (\widehat{\Omega}^d_\infty \otimes \overline{k(\infty)})
\end{array}$$

Hence (48) induces the isomorphism (43). □

Note that, since ξ_∞ acts trivially on $\widehat{\mathrm{Sh}}^\infty_{\mathcal{A},I}$ (resp. $\mathrm{Sh}^\infty_{\mathcal{A},I}$), the $\mathcal{C}_I(\mathcal{A}^\infty\times\mathcal{D}_\infty)$-action on $\widehat{\mathrm{Sh}}^\infty_{\mathcal{A},I}$ (resp. $\mathrm{Sh}^\infty_{\mathcal{A},I}$) induces a right $\mathcal{W}(\mathcal{A},I,\infty)$-action (by Remark 4.8 (c)). In terms of the latter, Lemma 4.27 (a) can be reformulated as follows:

$$\widehat{\mathrm{Sh}}^\infty_{\mathcal{A},I}\otimes_{k(\infty)}\overline{k(\infty)}/<\xi\theta^m\otimes\mathrm{Frob}_q^m> \quad\cong\quad \widehat{\mathrm{Sh}}^\xi_{\mathcal{A},I,\infty} \qquad \text{for } n=0, \text{ resp.} \tag{49}$$

$$\mathrm{Sh}^\infty_{\mathcal{A},I}\otimes_{k(\infty)_d}\overline{k(\infty)}/<\xi\theta^m\otimes\mathrm{Frob}_q^m> \quad\cong\quad \mathrm{Sh}^\xi_{\mathcal{A},I,\infty} \qquad \text{for } n>0. \tag{50}$$

Uniformisation at the pole. Suppose that $\mathrm{inv}_\infty A=0$ and assume first that ∞ does not divide the level I. Then there exists an isomorphism of formal schemes over $\mathrm{Spf}(\mathcal{O}_\infty)$

$$\widehat{\mathrm{Ell}}^\infty_{\mathcal{A},I}/\,\mathrm{Spf}(\mathcal{O}_\infty) \quad\cong\quad \widehat{\mathrm{Sh}}^\infty_{\mathcal{A},I} \tag{51}$$

which is compatible with the $\mathcal{W}(\mathcal{A},I,\infty)$-action and the morphisms *pole*.
Now assume $\infty\in|I|$. Then it is expected

$$\mathrm{Ell}^{\infty,\mathrm{an}}_{\mathcal{A},I}\,/\,\mathrm{Spec}\,F_\infty \quad\cong\quad \mathrm{Sh}^\infty_{\mathcal{A},I}\,. \tag{52}$$

Again, (52) should be compatible with the $\mathcal{W}(\mathcal{A},I,\infty)$-action and the morphisms *pole*.
We say that $\mathrm{Ell}^\infty_{\mathcal{A},I}$ admits *uniformization at the pole* if (51) (resp. (52)) holds. Suppose that $\infty\notin|I|$. (51) has been proved in ([BS], 4.4) if A is a division algebra or $A=M_d(F)$. As in loc. cit. the general case can be easily deduced from ([St], Corollary, p. 531 and Theorem 1, p. 538) or ([Ge], III.3.1.1). If $\infty\in|I|$ then (52) in known in the case of Drinfeld modular varieties (i.e. $A=M_d(F)$) the uniformization (52) is proved in [Dr4].

Cherednik-Drinfeld uniformization: Let $\mathfrak{p}\in|X|-\{\infty\}$ and assume that $\mathrm{inv}_\mathfrak{p} A=\frac{1}{d}$. Let $\mathcal{B}$ be a locally principal $\mathcal{O}_X$-order of rank d^2 with $\underline{\mathrm{Disc}}(\mathcal{B})=\underline{\mathrm{Disc}}(\mathcal{A})$ and such that the local invariants of the generic fiber B of $\mathcal{B}$ are given by $\mathrm{inv}_\infty(B)=\mathrm{inv}_\infty(A)+\frac{1}{d}$, $\mathrm{inv}_\mathfrak{p}(B)=0$ and $\mathrm{inv}_x(B)=\mathrm{inv}_x(A)$ for all $x\in|X|-\{\infty,\mathfrak{p}\}$. We fix an isomorphism $B_\mathfrak{p}\cong M_d(F_\mathfrak{p})$ and isomorphisms $\mathcal{B}_x\cong\mathcal{A}_x$ for all $x\in|X|-\{\infty,\mathfrak{p}\}$. Using the latter we can identify the groups $\mathcal{C}_I(\mathcal{A}^\infty\times\mathcal{B}_\infty)$ and $C_I(\mathcal{B}^\mathfrak{p}\times\mathcal{A}_\mathfrak{p})$. Since ξ_∞ acts trivially on $\widehat{\mathrm{Sh}}^{\xi_\infty}_{\mathcal{B},I,\mathfrak{p}}$ (resp. $\mathrm{Sh}^{\xi_\infty}_{\mathcal{B},I,\mathfrak{p}}$) we obtain a right $\mathcal{W}(\mathcal{A},I,\infty)\cong C_I(\mathcal{B}^\mathfrak{p}\times\mathcal{A}_\mathfrak{p})/\xi_\infty^\mathbb{Z}$-action on $\widehat{\mathrm{Sh}}^{\xi_\infty}_{\mathcal{B},I,\mathfrak{p}}$ (resp. $\mathrm{Sh}^{\xi_\infty}_{\mathcal{B},I,\mathfrak{p}}$). We also fix an isomorphism $k(\infty)_\star\cong k(\xi_\infty)\subset\overline{k(\mathfrak{p})}$ such that the pair $(k(\infty)_\star\cong k(\xi_\infty)\hookrightarrow\overline{k(\mathfrak{p})},k(\mathfrak{p})_\sharp\hookrightarrow\overline{k(\mathfrak{p})})$ is admissible in the sense of Remark 4.26 ($k(\infty)_\star$ and $k(\mathfrak{p})_\sharp$ are defined as in the last section) and define

$$\textit{pole}:\widehat{\mathrm{Sh}}^{\xi_\infty}_{\mathcal{A},I,\infty}\xrightarrow{(41)}\mathrm{Spec}\,k(\xi_\infty)\cong\mathrm{Spec}\,k(\infty)_\star \qquad \text{if } n=0,$$

$$\textit{pole}:\mathrm{Sh}^{\xi_\infty}_{\mathcal{A},I,\infty}\xrightarrow{(42)}\mathrm{Spec}\,k(\xi_\infty)\cong\mathrm{Spec}\,k(\infty)_\star \qquad \text{if } n>0.$$

Assume that $\mathfrak{p} \notin |I|$ (resp. $\mathfrak{p} \in |I|$). Then we expect that there is a canonical isomorphism of formal schemes over $\mathrm{Spf}(\mathcal{O}_\mathfrak{p})$ (resp. of rigid analytic spaces over $F_\mathfrak{p}$)

$$\widehat{\mathrm{Ell}}^{\infty}_{\mathcal{A},I}/\,\mathrm{Spf}(\mathcal{O}_\mathfrak{p}) \quad\cong\quad \widehat{\mathrm{Sh}}^{\xi_\infty}_{\mathcal{B},I,\mathfrak{p}} \qquad\qquad \text{if } \mathfrak{p} \notin |I|, \tag{53}$$

$$\mathrm{Ell}^{\infty,\mathrm{an}}_{\mathcal{A},I}/F_\mathfrak{p} \quad\cong\quad \mathrm{Sh}^{\xi_\infty}_{\mathcal{B},I,\infty} \qquad\qquad \text{if } \mathfrak{p} \in |I| \tag{54}$$

compatible with $\mathcal{W}(\mathcal{A}, I, \infty)$-action and the morphisms *pole*.
We say that $\mathrm{Ell}^\infty_{\mathcal{A},I}$ admits *Cherednik-Drinfeld uniformization* if (53) (resp. (54)) holds. Both (53) and (54) are proved in ([Hau], 8.1 and 8.3) in the case $\deg(\infty) = 1$, $\mathrm{inv}_\infty A = 0$ and $\infty \notin |I|$. Under these assumptions the formal scheme $\widehat{\mathrm{Sh}}^{\xi_\infty}_{\mathcal{B},I,\mathfrak{p}}$ and the rigid analytic variety $\mathrm{Sh}^{\xi_\infty}_{\mathcal{B},I,\infty}$ have the following simpler description

$$\widehat{\mathrm{Sh}}^{\xi_\infty}_{\mathcal{B},I,\mathfrak{p}} = B^*\backslash \left(B^*(\mathbb{A}^{\mathfrak{p},\infty})/U_I(\mathcal{B}^{\infty,\mathfrak{p}}) \times \widehat{\Omega}^d_\mathfrak{p} \otimes_{k(\mathfrak{p})} \overline{k(\mathfrak{p})}\right) \qquad \text{if } n = 0,$$

$$\mathrm{Sh}^{\xi_\infty}_{\mathcal{B},I,\mathfrak{p}} = B^*\backslash \left(B^*(\mathbb{A}^{\mathfrak{p},\infty})/U_I(\mathcal{B}^{\infty,\mathfrak{p}}) \times \Sigma^d_{n,\mathfrak{p}} \otimes_{k(\mathfrak{p})_d} \overline{k(\mathfrak{p})}\right) \qquad \text{if } n > 0,$$

where n denotes now the exact multiple of $\mathfrak{p}$ occuring in I.
By combining Theorem 4.25, (49), (50) and Lemma 4.27 (b) we obtain:

Proposition 4.28. *Let $\mathfrak{p} \in |X| - \{\infty\}$ and let $\mathcal{A}$ and $\mathcal{B}$ be locally principal $\mathcal{O}_X$-orders of rank d^2 with $\underline{\mathrm{Disc}}(\mathcal{B}) = \underline{\mathrm{Disc}}(\mathcal{A})$ such that the local invariants of the generic fibers A and B are given by $\mathrm{inv}_\infty(A) = 0, \mathrm{inv}_\infty(B) = \frac{1}{d}$, $\mathrm{inv}_\mathfrak{p}(A) = \frac{1}{d}, \mathrm{inv}_\mathfrak{p}(B) = 0$ and $\mathrm{inv}_x(B) = \mathrm{inv}_x(A)$ for all $x \in |X| - \{\infty, \mathfrak{p}\}$. The following conditions are equivalent:*
(i) $\mathrm{Ell}^\infty_{\mathcal{A},I}$ admits uniformization at the pole.
(ii) $\mathrm{Ell}^\mathfrak{p}_{\mathcal{B},I}$ admits Cherednik-Drinfeld uniformization.

By applying 4.28 to the results of [BS] and [Hau] we obtain further cases where $\mathrm{Ell}^\infty_{\mathcal{A},I}$ admits uniformization at the pole or Cherednik-Drinfeld uniformization. For example if $\mathrm{inv}_\infty(A) = 0$, $\infty \in |I|$ and if there exists a point $\mathfrak{p} \in |X| - \{\infty\}$ such that $\mathrm{inv}_\mathfrak{p}(A) = \frac{1}{d}$ and $\deg(\mathfrak{p}) = \frac{1}{d}$ then $\mathrm{Ell}^\infty_{\mathcal{A},I}$ admits uniformization at the pole. Conversely Cherednik-Drinfeld uniformization for $\mathrm{Ell}^\infty_{\mathcal{A},I}$ holds whenever if $\mathfrak{p}$ does not divide the level.

5 Appendix

5.1 Commutative subalgebras in semisimple algebras

Let k be a perfect field and A a finite-dimensional semisimple k-algebra. We collect a few facts about maximal separable and commutative subalgebras of A for which we could not find any references.
Let Z denote the center of A. By Wedderburns Theorem we have $Z \cong k_1 \times \ldots \times k_r$ for some finite separable extensions k_i/k. For a finite Z-module M, $\mathrm{rank}_Z M$ denotes the (not necessarily constant) rank of the corresponding locally free $\mathcal{O}_{\mathrm{Spec}\, Z}$-module.

Lemma 5.1. *Let T be a commutative separable k-subalgebra of A. The following conditions are equivalent.*
(i) $T = Z_A(T) = \{x \in A \mid tx = xt \quad \forall t \in T\}$.
(ii) T is a maximal commutative separable k-subalgebra of A.
(iii) $T \supseteq Z$ *and* $(\mathrm{rank}_Z T)^2 = \mathrm{rank}_Z A$.
Moreover if $A = \mathrm{End}_{k_1}(V_1) \times \ldots \times \mathrm{End}_{k_r}(V_r)$ *where V_i a finite-dimensional k_i-vector space for $i = 1, \ldots, r$, then (i) – (iii) are equivalent to*
(iv) $V_1 \oplus \ldots \oplus V_r$ is a free T-module of rank 1.

A commutative separable k-subalgebra T of A satisfying the equivalent conditions (i) – (iii) above will be called a *maximal torus* of A.

Lemma 5.2. *Let T_1, T_2 be two maximal tori of A. Then there exists a finite extension k'/k such that $T_1 \otimes_k k'$ and $T_2 \otimes_k k'$ are conjugated in $A \otimes_k k'$.*

A finite A-module M is called a generator of Mod_A if the functor

$$\mathrm{Hom}_A(M, \cdot) : \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_k$$

is faithfull. M is called a minimal generator if $\dim_k(M)$ is minimal. Assume now that A is split, i.e. $A = \mathrm{End}_{k_1}(V_1) \times \ldots \times \mathrm{End}_{k_r}(V_r)$ as in condition (iv) of Lemma 5.1 and let T be a maximal torus in A. We have

Lemma 5.3. *Let M be a finite A-module. The following conditions are equivalent.*
(i) M is a minimal generator.
(ii) $M \cong V_1 \oplus \ldots \oplus V_r$
(iii) M is a free T-module of rank 1.

5.2 $\mathcal{A}$-elliptic sheaves according to Laumon-Rapoport-Stuhler

The aim of this section is to show that under suitable assumptions on $\mathcal{A}$ the moduli stack $\mathcal{E}\ell\ell_{\mathcal{A}}^{\infty}$ defined in section 4.1 is isomorphic to the stack defined in ([LRS], 2.4).
Firstly, we establish an equivalence between certain parabolic vector bundles and locally free modules of a hereditary algebra. We use the following notations and assumptions. Let k be a perfect field of cohomological dimension ≤ 1 and let X be a smooth connected curve over k and F is the function field of X. We also fix a closed point $\infty \in X$. To simplify the notation we assume that $\deg(\infty) = 1$ (see Remark 5.11 below for the case $\deg(\infty) > 1$).
Let $\mathcal{A}'$ be a locally principal $\mathcal{O}_X$-order of rank d^2 with generic fiber A'. We assume that $e_\infty(\mathcal{A}') = 1$, i.e. $\mathcal{A}'_\infty \cong M_d(\mathcal{O}_\infty)$. To begin with we introduce the notion of a parabolic $\mathcal{A}'$-modules and parabolic vector bundles with $\mathcal{A}'$-action (compare [Yo]). A *filtered object* in a category $\mathcal{C}$ is a functor $C_\star : \mathbb{Z} \to \mathcal{C}$. Morphisms of filtered objects are natural transformations. Here we regard the ordered set $\mathbb{Z}$ as a category in the usual way. The set of objects is $\mathbb{Z}$ and for $i, j \in \mathbb{Z}$ we have

$$\sharp(\mathrm{Mor}(i,j)) = \begin{cases} 1 & \text{if } i \leq j \\ 0 & \text{otherwise.} \end{cases}$$

For $i \in \mathbb{Z}$ the morphism $C_i \to C_{i+1}$ will be denoted by $j_i = j_i^C$. For a filtered object $C_\star$ in $\mathcal{C}$ and $n \in \mathbb{Z}$ the shifted filtered object $C[n]_\star$ is defined as the composite $\mathbb{Z} \xrightarrow{+n} \mathbb{Z} \to \mathcal{C}$. A morphism $\phi : C_\star \to D_\star$ of filtered objects induces a morphism $\phi[n] : C[n]_\star \to D[n]_\star$.
Recall that for $S \in \mathrm{Sch}/k$ we have set ${}_{\mathcal{A}'}\mathrm{Mod}(S) := {}_{\mathcal{A}' \boxtimes \mathcal{O}_S}\mathrm{Mod}$ (resp. $\mathrm{Mod}_{\mathcal{A}'}(S) := \mathrm{Mod}_{\mathcal{A}' \boxtimes \mathcal{O}_S}$).

DEFINITION 5.4. *Let S be a k-scheme.*
(a) For $e \in \mathbb{Z}$ with $e \geq 1$ let $\mathrm{PMod}_{\mathcal{A}',e}(S)$ denote the category of pairs $(\mathcal{F}_\star, \psi_\star)$ consisting of a filtered $\mathrm{Mod}_{\mathcal{A}'}(S)$-object $\mathcal{F}_\star$ and an isomorphism $\psi_\star : \mathcal{F}[e]_\star \to \mathcal{F}_\star(\infty) := \mathcal{F}_\star \otimes_{\mathcal{O}_{X \times S}} (\mathcal{O}_X(\infty) \boxtimes \mathcal{O}_S)$ such that the restriction of $j_i : \mathcal{F} \to \mathcal{F}_{i+1}$ to $X - \{\infty\} \times S$ is an isomorphism and such that the following diagram commutes

$$\begin{tikzcd}
 & \mathcal{F}_{i+e} \arrow[d, "\psi_i"] \\
\mathcal{F}_i \arrow[ur, "j_{i+e-1}\circ\ldots\circ j_i"] \arrow[r, "\mathrm{id}\otimes\iota"'] & \mathcal{F}_i(\infty)
\end{tikzcd} \tag{55}$$

where $\iota : \mathcal{O}_{X \times S} \hookrightarrow \mathcal{O}_X(\infty) \boxtimes \mathcal{O}_S$ is the inclusion. Morphisms in $\mathrm{PMod}_{\mathcal{A}',e}(S)$ are morphisms of filtered objects compatible with the isomorphisms ψ.
(b) Let $\mathrm{PCoh}^r_{\mathcal{A}',sp,e}(S)$ denote the groupoid of $(\mathcal{K}_\star, \psi_\star)$ in $\mathrm{PMod}_{\mathcal{A}',e}(S)$ such that $\mathcal{K}_i \in \mathrm{Coh}^r_{\mathcal{A}',sp}(S)$ and $N(\mathcal{K}_\star) := N(\mathcal{K}_i) = N(\mathcal{K}_{i+1})$ for all $i \in \mathbb{Z}$.
(c) For $e, r \in \mathbb{Z}$ with $e, r \geq 1$ and $e \mid rd$. We denote by $\mathrm{PVect}^r_{\mathcal{A}',e}(S)$ the full subcategory of $(\mathcal{F}_\star, \psi_{\mathcal{F}_\star})$ in $\mathrm{PMod}_{\mathcal{A}',e}(S)$ such that $\mathcal{F}_i \in \mathrm{Vect}^r_{\mathcal{A}'}(S)$ for all $i \in \mathbb{Z}$ and such that $\mathrm{Coker}(j_\star : \mathcal{F}_\star \to \mathcal{F}[1]_\star) \in \mathrm{PCoh}^s_{\mathcal{A}',sp,e}(S)$ with $s = \frac{rd}{e}$.
Similarly one defines ${}_{\mathcal{A}'}\mathrm{PMod}_e(S)$ and ${}_{\mathcal{A}'}\mathrm{PVect}^r_e(S)$ using left $\mathcal{A}' \boxtimes \mathcal{O}_S$-modules.

Note that for $(\mathcal{F}_\star, \psi_\star)$ in $\mathrm{PMod}_{\mathcal{A}',e}(S)$ with $\mathcal{F}_i \in \mathrm{Vect}^r_{\mathcal{A}'}(S)$ for all $i \in \mathbb{Z}$, the commutativity of diagram (55) implies that $j_i : \mathcal{F}_i \to \mathcal{F}_{i+1}$ is injective and $\mathrm{Coker}(j_i)$ is a sheaf on $\infty \times S$. For $\mathcal{A}' = \mathcal{O}_X$ we write Mod_X, $\mathrm{PMod}_{X,e}$, Vect_X etc. for $\mathrm{Mod}_{\mathcal{O}_X}$, $\mathrm{PMod}_{\mathcal{O}_X,e}(S)$, $\mathrm{Vect}_{\mathcal{O}_X}$ etc.
Let $\mathcal{E}_\star \in \mathrm{PMod}_{\mathcal{A}',e}(S)$ and $\mathcal{F}_\star \in {}_{\mathcal{A}'}\mathrm{PMod}_e(S)$. We are going to define now a tensor product $(\mathcal{E}_\star \otimes_{\mathcal{A}'} \mathcal{F}_\star)_\star$. For $i \in \mathbb{Z}$ we set

$$T_i(\mathcal{E}_\star, \mathcal{F}_\star) := \bigoplus_{\lambda+\mu=i, \lambda,\mu \in \mathbb{Z}} \mathcal{E}_\lambda \otimes_{\mathcal{A}'} \mathcal{F}_\mu.$$

For $i \in \mathbb{Z}$ with we define homomorphisms

$$\alpha_i : T_i(\mathcal{E}_\star, \mathcal{F}_\star) \longrightarrow T_{i+1}(\mathcal{E}_\star, \mathcal{F}_\star), \quad \beta_i : T_i(\mathcal{E}_\star, \mathcal{F}_\star) \longrightarrow T_{i+1}(\mathcal{E}_\star, \mathcal{F}_\star)$$

as the direct sums of the inclusions $j_\lambda \otimes \mathrm{id} : \mathcal{E}_\lambda \otimes_{\mathcal{A}'} \mathcal{F}_\mu \to \mathcal{E}_{\lambda+1} \otimes_{\mathcal{A}'} \mathcal{F}_\mu$ (resp. $\mathrm{id} \otimes j_\mu : \mathcal{E}_\lambda \otimes_{\mathcal{A}'} \mathcal{F}_\mu \to \mathcal{E}_\lambda \otimes_{\mathcal{A}'} \mathcal{F}_{\mu+1}$). Also let

$$\gamma_i : T_i(\mathcal{E}_\star, \mathcal{F}_\star) \longrightarrow T_i(\mathcal{E}_\star, \mathcal{F}_\star)$$

be the isomorphism given on the summand $\mathcal{E}_\lambda \otimes_{\mathcal{A}'} \mathcal{F}_\mu$ by

$$\mathcal{E}_\lambda \otimes_{\mathcal{A}'} \mathcal{F}_\mu \cong \mathcal{E}_\lambda(\infty) \otimes_{\mathcal{A}'} \mathcal{F}_\mu(-\infty) \stackrel{\psi^{-1}\otimes\psi^{-1}}{\longrightarrow} \mathcal{E}_{\lambda+e} \otimes_{\mathcal{A}'} \mathcal{F}_{\mu-e}.$$

Finally let

$$\delta_i : T_{i-1}(\mathcal{E}_\star, \mathcal{F}_\star) \oplus T_i(\mathcal{E}_\star, \mathcal{F}_\star) \longrightarrow T_i(\mathcal{E}_\star, \mathcal{F}_\star)$$

be given on the summand $T_{i-1}(\mathcal{E}_\star, \mathcal{F}_\star)$ by $\alpha_{i-1} - \beta_{i-1}$ and by $id - \gamma_i$ on $T_i(\mathcal{E}_\star, \mathcal{F}_\star)$. We define

$$\sum_{\lambda+\mu=i,\lambda,\mu\in\mathbb{Z}} \mathcal{E}_\lambda \otimes_{\mathcal{A}'} \mathcal{F}_\mu = \mathrm{Coker}(\delta_i) \tag{56}$$

There are canonical morphisms

$$\sum_{\lambda+\mu=i-1,\lambda,\mu\in\mathbb{Z}} \mathcal{E}_\lambda \otimes_{\mathcal{A}'} \mathcal{F}_\mu \longrightarrow \sum_{\lambda+\mu=i,\lambda,\mu\in\mathbb{Z}} \mathcal{E}_\lambda \otimes_{\mathcal{A}'} \mathcal{F}_\mu. \tag{57}$$

The isomorphisms

$$\mathcal{E}_{\lambda+d} \otimes_{\mathcal{A}'} \mathcal{F}_\mu \stackrel{\psi\otimes\mathrm{id}}{\longrightarrow} (\mathcal{E}_\lambda \otimes_{\mathcal{A}'} \mathcal{F}_\mu)(\infty), \qquad \mathcal{E}_\lambda \otimes_{\mathcal{A}'} \mathcal{F}_{\mu+d} \stackrel{\mathrm{id}\otimes\psi}{\longrightarrow} (\mathcal{E}_\lambda \otimes_{\mathcal{A}'} \mathcal{F}_\mu)(\infty)$$

induces an isomorphism

$$\sum_{\lambda+\mu=i+d,\lambda,\mu\in\mathbb{Z}} \mathcal{E}_\lambda \otimes_{\mathcal{A}'} \mathcal{F}_\mu \longrightarrow (\sum_{\lambda+\mu=i,\lambda,\mu\in\mathbb{Z}} \mathcal{E}_\lambda \otimes_{\mathcal{A}'} \mathcal{F}_\mu)(\infty). \tag{58}$$

Definition 5.5. *The tensor product $(\mathcal{E}_\star \otimes_{\mathcal{A}'} \mathcal{F}_\star)_\star \in \mathrm{PMod}_{X,e}(S)$ is defined as the collection of $\mathcal{O}_{X\times S}$-modules*

$$(\mathcal{E}_\star \otimes_{\mathcal{A}'} \mathcal{F}_\star)_i = \sum_{\lambda+\mu=i,\lambda,\mu\in\mathbb{Z}} \mathcal{E}_\lambda \otimes_{\mathcal{A}'} \mathcal{F}_\mu$$

(for $i \in \mathbb{Z}$) together with the maps (57) and (58).

Lemma 5.6. *Let $\mathcal{E}_\star \in \mathrm{PVect}^r_{\mathcal{A}',e}(S)$ and $\mathcal{F}_\star \in {}_{\mathcal{A}'}\mathrm{PVect}^r_e(S)$. Then $(\mathcal{E}_\star \otimes_{\mathcal{A}'} \mathcal{F}_\star)_\star$ lies in $\mathrm{PVect}^{rd^2}_{X,e}(S)$. In particular $\sum_{\lambda+\mu=i,\lambda,\mu\in\mathbb{Z}} \mathcal{E}_\lambda \otimes_{\mathcal{A}'} \mathcal{F}_\mu$ is a locally free $\mathcal{O}_{X\times S}$-module of rank rd^2 for all $i \in \mathbb{Z}$.*

Proof. For $\mathcal{A}' = \mathcal{O}_X$ this follows immediately from the fact that a parabolic $\mathcal{O}_{X\times S}$-module is a parabolic vector bundle if and only if it is parabolically flat ([Yo], Proposition 3.1). The general case can be deduce from this special case by Morita equivalence. More precisely since the question is local we can replace X by an étale neighbourhood of ∞ and therefore can assume that $\mathcal{A}' \cong M_d(\mathcal{O}_X)$. Let $\mathcal{I}$ be an invertible $\mathcal{A}'$-$\mathcal{O}_X$-bimodule and $\mathcal{J}$ its inverse. Since $(\mathcal{E}_\star \otimes_{\mathcal{A}'} \mathcal{F}_\star)_\star = ((\mathcal{E}_\star \otimes_{\mathcal{A}'} \mathcal{I}) \otimes_{\mathcal{O}_X} (\mathcal{J} \otimes_{\mathcal{A}'} \mathcal{F}_\star))_\star$ the assertion follows from the case $\mathcal{A}' = \mathcal{O}_X$. $\square$

Let $\mathcal{A}$ be another locally principal $\mathcal{O}_X$-order of rank d^2 and assume that $e_\infty(\mathcal{A}) = d$ and $\mathcal{A}|_U$ and $\mathcal{A}'|_U$ are Morita equivalent where $U = X - \{\infty\}$. There exists an increasing families of $\mathcal{A}$-$\mathcal{A}'$-bimodules $\{\mathcal{I}_i \mid i \in \mathbb{Z}\}$ and of $\mathcal{A}'$-$\mathcal{A}$-bimodules $\{\mathcal{J}_i \mid i \in \mathbb{Z}\}$ such that $(\mathcal{I}_i)|_U = (\mathcal{I}_{i+1})_U =: \mathcal{I}_U$ and $(\mathcal{J}_i)|_U = (\mathcal{J}_{i+1})_U =: \mathcal{J}_U$ for all $i \in \mathbb{Z}$, $\mathcal{I}_U$ is an invertible $\mathcal{A}_U$-$\mathcal{A}'_U$-bimodule with inverse $\mathcal{J}_U$ and such that $\{(\mathcal{I}_i)_\infty \mid i \in \mathbb{Z}\}$ and $\{(\mathcal{J}_i)_\infty \mid i \in \mathbb{Z}\}$ are as in 2.2. It follows from Corollary 2.12 that $\mathcal{I}_i$ and $\mathcal{J}_j$ are locally free $\mathcal{A}'$-modules of rank 1. Also we have

$$\mathcal{A}(\frac{1}{d}\infty) \otimes_{\mathcal{A}} \mathcal{I}_i = \mathcal{I}_{i+1}, \quad \mathcal{J}_i \otimes_{\mathcal{A}} \mathcal{A}(\frac{1}{d}\infty) = \mathcal{J}_{i+1}$$

for all $i \in \mathbb{Z}$.

PROPOSITION 5.7. *Put* $\mathrm{Vect}_{\mathcal{A}} = \mathrm{Vect}^1_{\mathcal{A}}$ *and* $\mathrm{PVect}_{\mathcal{A}'} = \mathrm{PVect}^1_{\mathcal{A}',d}$. *The morphisms*

$$\cdot \otimes_{\mathcal{A}} \mathcal{I}_\star : \mathrm{Vect}_{\mathcal{A}} \longrightarrow \mathrm{PVect}_{\mathcal{A}'}, \quad (\cdot \otimes_{\mathcal{A}'} \mathcal{J}_\star)_{d-1} : \mathrm{PVect}_{\mathcal{A}'} \longrightarrow \mathrm{Vect}_{\mathcal{A}} \tag{59}$$

given by $\mathcal{E} \mapsto \mathcal{E} \otimes_{\mathcal{A}} \mathcal{I}_\star : = \{\mathcal{F} \otimes_{\mathcal{A}} \mathcal{I}_i \mid i \in \mathbb{Z}\}$ *and* $\mathcal{E}_\star \mapsto \sum_{\lambda+\mu=d-1} \mathcal{E}_\lambda \otimes_{\mathcal{A}'} \mathcal{J}_\mu$ *are mutually inverse isomorphisms of stacks. Define* $\theta : \mathrm{Vect}_{\mathcal{A}} \to \mathrm{Vect}_{\mathcal{A}}$ *and* $\theta' : \mathrm{PVect}_{\mathcal{A}'} \to \mathrm{PVect}_{\mathcal{A}'}$ *by* $\theta(\mathcal{E}) = \mathcal{E}(\frac{1}{d}\infty)$ *and* $\theta'(\mathcal{E}_\star, \psi_\star) = (\mathcal{E}[1]_\star, \psi[1]_\star)$. *Then the diagrams*

$$\begin{array}{ccc} \mathrm{Vect}_{\mathcal{A}} & \xrightarrow{\otimes_{\mathcal{A}}\mathcal{I}_\star} & \mathrm{PVect}_{\mathcal{A}'} \\ \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta'} \\ \mathrm{Vect}_{\mathcal{A}} & \xrightarrow{\otimes_{\mathcal{A}}\mathcal{I}_\star} & \mathrm{PVect}_{\mathcal{A}'} \end{array} \qquad \begin{array}{ccc} \mathrm{PVect}_{\mathcal{A}'} & \xrightarrow{(\cdot\otimes_{\mathcal{A}'}\mathcal{J}_\star)_{d-1}} & \mathrm{Vect}_{\mathcal{A}} \\ \downarrow{\scriptstyle\theta'} & & \downarrow{\scriptstyle\theta} \\ \mathrm{PVect}_{\mathcal{A}'} & \xrightarrow{(\cdot\otimes_{\mathcal{A}'}\mathcal{J}_\star)_{d-1}} & \mathrm{Vect}_{\mathcal{A}} \end{array} \tag{60}$$

are 2-commutative.

Proof. In view of 5.6 we only have to show that the second morphism is well-defined. By 5.6 and 3.11 we have to prove that for $\mathcal{E}_\star \in \mathrm{PVect}_{\mathcal{A}'}(S)$ the quotient

$$(\sum_{\lambda+\mu=0,\lambda,\mu\in\mathbb{Z}} \mathcal{E}_\lambda \otimes_{\mathcal{A}'} \mathcal{J}_\mu)/(\sum_{\lambda+\mu=-1,\lambda,\mu\in\mathbb{Z}} \mathcal{E}_\lambda \otimes_{\mathcal{A}'} \mathcal{J}_\mu) \cong \sum_{\lambda+\mu=0,\lambda,\mu\in\mathbb{Z}} \overline{\mathcal{E}}_\lambda \otimes_{\mathcal{A}'} \mathcal{J}_\mu$$

is a special $\mathcal{A}$-module on $S \cong \infty \times S$ where $\overline{\mathcal{E}}_\star : = \mathrm{Coker}(\mathcal{E}_\star[-1] \hookrightarrow \mathcal{E}_\star) \in_{\mathcal{A}'} \mathrm{PMod}_d(S)$. However this follows from:

LEMMA 5.8. *The assignement* $\mathcal{K}_\star \mapsto \sum_{\lambda+\mu=d-1} \mathcal{K}_\lambda \otimes_{\mathcal{A}'} \mathcal{J}_\mu$ *defines a morphism*

$$(\cdot \otimes_{\mathcal{A}'} \mathcal{J}_\star)_{d-1} : \mathrm{PCoh}^1_{\mathcal{A}',sp,d} \longrightarrow \mathrm{Coh}^1_{\mathcal{A},\mathrm{sp}}.$$

Proof. By Lafforgue's Lemma ([Laf], I.2.4) (applied to a maximal tori in $\mathcal{A}$) it suffices to consider the case where $S = \mathrm{Spec}\, k$ and k is an algebraically closed field. If $N(\mathcal{K}_\star) \neq \infty$ then the assertion follows from Remarks 3.8 (b), (c). Now assume $N(\mathcal{K}_\star) = \infty$. Let $\mathcal{O} = \mathcal{O}_\infty$, $K = F_\infty$. Since the question is local with respect to the étale topology we can replace X by $\operatorname{Spec} \mathcal{O}$ where $\mathcal{O} = \mathcal{O}_\infty$. Then $\mathcal{A}' \cong M_d(\mathcal{O})$ and $\mathcal{A} \cong \operatorname{End}(\mathcal{L}_\star)$ for a lattice chain $\mathcal{L}_\star$ of period d in K^d. Morita equivalence allows us to replace $\mathcal{A}'$ by $\mathcal{O}$, i.e. we can assume that $\mathcal{A}' = \mathcal{O}$. Then $\mathcal{M}_i \colon = \Gamma(\operatorname{Spec} \mathcal{O}, \mathcal{K}_i)$ is a onedimensional k-vector space for all $i \in \mathbb{Z}$ and we have to show that

$$\sum_{i+j=0} \mathcal{M}_i \otimes_{\mathcal{O}} \mathcal{J}_j \tag{61}$$

is a free $\overline{\mathcal{T}} = \mathcal{T} \otimes k$-module of rank 1 where $\mathcal{T} \cong \mathcal{O}^d$ is any maximal torus in $\mathcal{A}$. If $1 = e_1 + \ldots + e_d$ is a decomposition of $1 \in \mathcal{T}$ into primitive idempotents we obtain a corresponding decompositon of (61) into

$$\sum_{i+j=0} \mathcal{M}_i \otimes_{\mathcal{O}} \mathcal{J}_j^{(\nu)}, \qquad \nu = 1, \ldots, d$$

where $\mathcal{J}_j^{(\nu)} \colon = \mathcal{J}_j e_\nu$. Since $\mathcal{J}_j$ is free of rank 1 as a $\mathcal{T}$-module, ${\mathcal{J}_\star}^{(\nu)}$ is a *shifted parabolic line bundle* (compare [Yo]) for each $\nu \in \{1, \ldots, d\}$. Therefore $(\mathcal{M}_\star \otimes_{\mathcal{O}} {\mathcal{J}_\star}^{(\nu)})_\star \cong \mathcal{M}_\star[m]$ for some $m \in \mathbb{Z}$. Consequently

$$\sum_{i+j=0} \mathcal{M}_i \otimes_{\mathcal{O}} \mathcal{J}_j^{(\nu)} \cong \mathcal{M}_m$$

is a onedimensional k-vector space. It follows that (61) is a free $\mathcal{T} \otimes_{\mathcal{O}} R$-module of rank 1. $\square$

Now assume that $k = \mathbb{F}_q$ and that $\mathcal{A}'$ is a maximal $\mathcal{O}_X$-order in a central division algebra A' of dimension d^2 with $A'_\infty \cong M_d(F_\infty)$. Let us recall the definition of an $\mathcal{A}'$-elliptic sheaf given in ([LRS], 2.2) and ([BS], 4.4.1) (here we do not require $\deg(\infty) = 1$).

Definition 5.9. *Let $S \in \operatorname{Sch}/\mathbb{F}_q$. An $\mathcal{A}'$-elliptic sheaf $E' = (\mathcal{E}_i, j_i, t_i)_{i \in \mathbb{Z}}$ with pole ∞ in the sense of [LRS] consists of a commutative diagram*

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & \mathcal{E}_{i-1} & \xrightarrow{j_{i-1}} & \mathcal{E}_i & \xrightarrow{j_i} & \mathcal{E}_{i+1} & \longrightarrow & \ldots \\
 & & \uparrow{\scriptstyle t_{i-2}} & & \uparrow{\scriptstyle t_{i-1}} & & \uparrow{\scriptstyle t_i} & & \\
\ldots & \longrightarrow & {}^\tau\mathcal{E}_{i-2} & \xrightarrow{j_{i-1}} & {}^\tau\mathcal{E}_{i-1} & \xrightarrow{j_{i-1}} & {}^\tau\mathcal{E}_i & \longrightarrow & \ldots
\end{array}$$

where $\mathcal{E}_i$ are locally free $\mathcal{O}_{X \times S}$-modules of rank d^2 additionally equipped with a right action of $\mathcal{A}'$ compatible with the $\mathcal{O}_X$-action. The maps are injective $\mathcal{A}' \boxtimes \mathcal{O}_S$-linear homomorphisms.
Furthermore the following conditions should hold:

(i) (Periodicity) $\mathcal{E}_{i+e\deg(\infty)} = \mathcal{E}_i(\infty) \colon = \mathcal{E}_i \otimes_{\mathcal{O}_{X\times S}} (\mathcal{O}(\infty) \boxtimes \mathcal{O}_S)$ *where the canonical embedding of* $\mathcal{E}_i$ *on the right side corresponds on the left to the compostion* $\mathcal{E}_i \stackrel{j}{\hookrightarrow} \dots \stackrel{j}{\hookrightarrow} \mathcal{E}_{i+d' deg(\infty)}$.
(ii) The quotient sheaf $\mathcal{E}_i/j_{i-1}(\mathcal{E}_{i-1})$ *is a locally free sheaf of rank d on the graph of a morphism* $\iota_{\infty,i} : S \to X$.
(iii) There exists a morphism $z : S \to X - |\underline{\mathrm{Disc}}(\mathcal{A}')|$ *– called the zero or characteristic of* E' *– such that for all* $i \in \mathbb{Z}$, $\mathrm{Coker}(t_i)$ *is supported on the graph of a morphism* z *and is a direct image of a locally free* $\mathcal{O}_S$*-module of rank* d *by* $\Gamma_z = (z, \mathrm{id}_S) : S \longrightarrow X \times S$.

We first remark that condition (iii) implies that $\mathcal{E}_i$ is actually a locally free $\mathcal{A}' \boxtimes \mathcal{O}_S$-module. This follows from ([Laf], I.4, proposition 7) or can be deduced from Lemma 3.11 together with ([LRS], 2.6). Secondly condition (i) implies that $\iota_{\infty,i}(S) = \{\infty\}$ and we have

$$\iota_{\infty,i} \circ \mathrm{Frob}_S = \iota_{\infty,i+1}$$

for all $i \in \mathbb{Z}$. For that consider the two filtrations of $\mathcal{E}_{i+1}/t_{i-1}({}^\tau\mathcal{E}_{i-1})$

$$0 \subseteq \mathcal{E}_i/t_{i-1}({}^\tau\mathcal{E}_{i-1}) \subseteq \mathcal{E}_{i+1}/t_{i-1}({}^\tau\mathcal{E}_{i-1}),$$
$$0 \subseteq t_i({}^\tau\mathcal{E}_i)/t_{i-1}({}^\tau\mathcal{E}_{i-1}) \subseteq \mathcal{E}_{i+1}/t_{i-1}({}^\tau\mathcal{E}_{i-1}).$$

The first shows that the support of $\mathcal{E}_{i+1}/t_{i-1}({}^\tau\mathcal{E}_{i-1})$ is $\Gamma_z + \Gamma_{\iota_{\infty,i+1}}$ and the second that it is $\Gamma_z + \Gamma_{\iota_{\infty,i}\circ \mathrm{Frob}_S}$.
Suppose again that $\deg(\infty) = 1$. Hence the stack $\mathcal{PE}\ell\ell^\infty_{\mathcal{A}'}(S)$ of $\mathcal{A}'$-elliptic sheaves as defined in 5.9 is isomorphic to the stack of triples $E' = (\mathcal{E}_\star, t_\star)$ where $\mathcal{E}_\star = (\mathcal{E}_\star, \psi_\star) \in \mathrm{PVect}_{\mathcal{A}'}(S)$ and $t_\star : {}^\tau\mathcal{E}[-1]_\star \to \mathcal{E}_\star$ is a morphism in $\mathrm{PVect}_{\mathcal{A}'}(S)$ such that (iii) above holds for $\mathrm{Coker}(t_\star)$.
We show that the isomorphisms (59) yield isomorphisms between $\mathcal{PE}\ell\ell^\infty_{\mathcal{A}'}$ and $\mathcal{E}\ell\ell^\infty_{\mathcal{A}}|_{X-|\underline{\mathrm{Disc}}(\mathcal{A}')|} = \mathcal{E}\ell\ell^\infty_{\mathcal{A}} \times_X (X - |\underline{\mathrm{Disc}}(\mathcal{A}')|)$. Define

$$\cdot \otimes_{\mathcal{A}} \mathcal{I}_\star : \mathcal{E}\ell\ell^\infty_{\mathcal{A}}|_{X-|\underline{\mathrm{Disc}}(\mathcal{A}')|}(S) \longrightarrow \mathcal{PE}\ell\ell^\infty_{\mathcal{A}'}(S)$$

by $(\mathcal{E}, t) \mapsto (\mathcal{E} \otimes_{\mathcal{A}} \mathcal{I}_\star, t \otimes_{\mathcal{A}} \mathcal{I}_\star)$. The commutativity of the first diagram (60) shows that $t \otimes_{\mathcal{A}} \mathcal{I}_\star$ is a map ${}^\tau\mathcal{E} \otimes_{\mathcal{A}} \mathcal{I}_\star[-1] \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{I}_\star$. That E' has property (iii) above follows from Remark 3.8 (c). Conversely, we define

$$(\cdot \otimes_{\mathcal{A}'} \mathcal{J}_\star)_{d-1} : \mathcal{PE}\ell\ell^\infty_{\mathcal{A}'}(S) \longrightarrow \mathcal{E}\ell\ell^\infty_{\mathcal{A}}|_{X-|\underline{\mathrm{Disc}}(\mathcal{A}')|}(S)$$

by $(\mathcal{E}_\star, t_\star) \mapsto ((\mathcal{E}_\star \otimes_{\mathcal{A}'} \mathcal{J}_\star)_{d-1}, (t_\star \otimes_{\mathcal{A}'} \mathcal{J}_\star)_{d-1})$. Again the commutativity of the second diagram of (60) implies that $(t_\star \otimes_{\mathcal{A}'} \mathcal{J}_\star)_{d-1}$ is a morphism ${}^\tau(\mathcal{E}(-\frac{1}{d}\infty_S)) \to \mathcal{E}$ where $\mathcal{E} = (\mathcal{E}_\star \otimes_{\mathcal{A}'} \mathcal{J}_\star)_{d-1}$. Finally condition (*) of Definition 4.2 follows Lemma 5.8. We deduce from 5.7:

PROPOSITION 5.10. *Let S be a k-scheme. The morphisms*

$$\cdot \otimes_{\mathcal{A}} \mathcal{I}_\star : \mathcal{E}\ell\ell^\infty_{\mathcal{A}}|_{X-|\underline{\mathrm{Disc}}(\mathcal{A}')|} \longrightarrow \mathcal{PE}\ell\ell^\infty_{\mathcal{A}'}, \tag{62}$$
$$(\cdot \otimes_{\mathcal{A}'} \mathcal{J}_\star)_{d-1} : \mathcal{PE}\ell\ell^\infty_{\mathcal{A}'} \longrightarrow \mathcal{E}\ell\ell^\infty_{\mathcal{A}}|_{X-|\underline{\mathrm{Disc}}(\mathcal{A}')|} \tag{63}$$

are mutually inverse isomorphisms.

REMARKS 5.11. (a) In order to extend 5.10 to the case $\deg(\infty) > 1$ we have to modify Definition 5.4 (b) as follows. For $S \in \mathrm{Sch}/k$ let $\mathrm{PVect}_{\mathcal{A}'}(S)$ denote the category of triples $(\mathcal{F}_\star, \psi_{\mathcal{F}_\star}, \infty_S)$ where $\infty_S : S \to X$ is a k-morphism which factors through $\infty \to X$ and $(\mathcal{F}_\star, \psi_{\mathcal{F}_\star})$ is an element of $\mathrm{PMod}_{\mathcal{A}', d\deg(\infty)}(S)$ such that $\mathcal{F}_i \in \mathrm{Vect}^1_{\mathcal{A}'}(S)$ for all $i \in \mathbb{Z}$ and such that the sheaf $\mathrm{Coker}(j_i)$ is a locally free sheaf of rank d on the graph of $\infty_S \circ \mathrm{Frob}^i_S : S \to X$. To define isomorphisms similar to (59) we consider increasing families of $\mathcal{A} \boxtimes k(\infty)$-$\mathcal{A}' \boxtimes k(\infty)$-bimodules $\{\mathcal{I}_i \mid i \in \mathbb{Z}\}$ and $\mathcal{A}' \boxtimes k(\infty)$-$\mathcal{A} \boxtimes k(\infty)$-bimodules $\{\mathcal{J}_i \mid i \in \mathbb{Z}\}$ with the following properties:

(i) $\mathcal{A}(\frac{1}{d}\infty_i) \otimes_{\mathcal{A}} \mathcal{I}_i = \mathcal{I}_{i+1}, \quad \mathcal{J}_i \otimes_{\mathcal{A}} \mathcal{A}(\frac{1}{d}\infty_i) = \mathcal{J}_{i+1}$ for all $i \in \mathbb{Z}$. Here ∞_0 denotes the canonical morphism $\mathrm{Spec}\, k(\infty) \to X$ and $\infty_i :\, = \infty_0 \circ \mathrm{Frob}^i :$ $\mathrm{Spec}\, k(\infty) \to X$.

(i) $\mathcal{I}_U = (\mathcal{I}_i)|_{U \times_{\mathbb{F}_q} k(\infty)}$ is an invertible $\mathcal{A}_U \boxtimes k(\infty)$-$\mathcal{A}'_U \boxtimes k(\infty)$-bimodule with inverse $\mathcal{J}_U = (\mathcal{J}_i)|_{U \times_{\mathbb{F}_q} k(\infty)}$.

(iii) For all $i \in \mathbb{Z}$, $\mathcal{I}_i$ and $\mathcal{J}_j$ are locally free $\mathcal{A}' \boxtimes k(\infty)$-modules of rank 1.

As in 5.7 one defines isomorphisms

$$\begin{aligned} \cdot \otimes_{\mathcal{A}} \mathcal{I}_\star &: \mathrm{Vect}_{\mathcal{A}} \times_{\mathbb{F}_q} k(\infty) \longrightarrow \mathrm{PVect}_{\mathcal{A}'}, \\ (\cdot \otimes_{\mathcal{A}'} \mathcal{J}_\star)_{d-1} &: \mathrm{PVect}_{\mathcal{A}'} \longrightarrow \mathrm{Vect}_{\mathcal{A}} \times_{\mathbb{F}_q} k(\infty) \end{aligned}$$

which then yield the isomorphisms (62), (63) above.
(b) Let $\mathfrak{p}$ be a closed point of X such that $\mathrm{inv}_{\mathfrak{p}}(A') = \frac{1}{d}$. In [Hau], Hausberger constructed a flat proper model of $\mathcal{E}\ell\ell^{\infty}_{\mathcal{A}'}$ over $(X - |\,\underline{\mathrm{Disc}}(\mathcal{A}')|) \cup \{\mathfrak{p}\}$ by extending the definition of the moduli problem 5.9 of Laumon-Rapoport-Stuhler to characteristic $\mathfrak{p}$. By using ([Hau], 2.16) it is easy to see using that his *condition spéciale* ([Hau], section 3) corresponds to our condition (*).

REFERENCES

[BC] J.-F. Boutot and H. Carayol, *Uniformisation p-adic des courbes de Shimura: les théorèmes de Cerednik et Drinfeld*. In: Courbes modulaires et courbes de Shimura, Astérisque 196–197 (1991).

[BF] C. J. Bushnell and A. Fröhlich, *Nonabelian congruence Gauss sums and p-adic simple algebras*. Proc. London Math. Soc. (3) 50 (1985), 207–264.

[BS] A. Blum and U. Stuhler, *Drinfeld modules and elliptic sheaves*. In: Vector bundles on curves—new directions (Cetraro, 1995), Lect. Notes in Math. 1649, Springer Verlag 1997, 110–193.

[Ce] I. V. Cherednik, *Uniformization of algebraic curves by discrete arithmetic subgroups of* $\mathrm{PGL}_2(k_w)$ *with compact quotients*, Math. USSR Sbornik 29 (1976), 55–78.

[DM] P. Deligne and D. Mumford, *The irreducibility of the space of curves of given genus*, Publ. Math. IHES 36 (1969), 79–105.

[Dr1] V. G. Drinfel'd, *Elliptic modules*, Math. USSR Sbornik 23 (1974), 561–592.

[Dr2] V. G. Drinfel'd, *Coverings of p-adic symmetric regions.* Funct. Anal. and its Appl. 10 (1976), 107–115.

[Dr3] V. G. Drinfel'd, *Elliptic modules. II.* Mat. Sb. (N.S.) 102 (1977), 182–194, 325.

[Dr4] V. G. Drinfel'd, Letter to H. Carayol from 12.1.1980.

[Ge] A. Genestier, *Espaces symétriques de Drinfeld*, Astérisque 234 (1996).

[Hau] T. Hausberger, *Uniformisation des variétés de Laumon-Rapoport-Stuhler et conjecture de Drinfeld-Carayol*,

[Ja] G.J. Janusz, *Tensor products of orders*, J. London Math. Soc. (2) 20 (1979), 186–192.

[Laf] L. Lafforgue, *Chtoucas de Drinfeld et conjecture de Ramanujan-Petersson*, Astérisque 243 (1997).

[Lau] E. Lau, *On generalized $\mathcal{D}$-Shtukas*, Bonner Mathematische Schriften Nr. 369 (2004).

[La] G. Laumon, *Drinfeld Shtukas.* In: Vector bundles on curves—new directions (Cetraro, 1995), Lect. Notes in Math. 1649, Springer Verlag 1997, 50–109.

[LM] G. Laumon and L. Moret-Bailly, *Champs algébriques.* Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. 39. Springer Verlag 2000.

[LRS] G. Laumon, M. Rapoport and U. Stuhler, *$\mathcal{D}$-elliptic sheaves and the Langlands correspondence*, Invent. Math. 113 (1993), 217–338.

[Re] I. Reiner, *Maximal orders*, London Mathematical Society Monographs, No. 5, Academic Press 1975.

[St] U. Stuhler, *p-adic homogeneous spaces and moduli problems.* Math. Z. 192 (1986), 491–540.

[Yo] K. Yokogawa, *Infinitesimal deformation of parabolic Higgs sheaves.* Internat. J. Math. 6 (1995), 125–148.

Michael Spieß
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
D-33501 Bielefeld
Germany

Artin-Hasse Functions and their Invertions in Local Fields

Vitaly Valtman, Sergey Vostokov[1]

Received: September 24, 2009
Revised: February 5, 2010

Abstract. The extension of Artin-Hasse functions and the inverse maps on formal modules over local fields is presented.

2010 Mathematics Subject Classification: 11S31
Keywords and Phrases: Artin-Hasse functions, formal groups

Андрею Суслину, прекрасному человеку и ученому, на 60-летие

1 Introduction

The classical Artin-Hasse function was defined in 1928 in [1] in order to find Ergängungsätze for the global reciprocity law in the cyclotomic field $\mathbb{Q}(\zeta)$, where ζ is a primitive root of unity of order p^n. The Artin-Hasse functions turned out to be a convenient tool for describing the arithmetic of the multiplicative group of a local field. Using them, I.R. Shafarevich constructed in [2] a canonical basis, which gives the decomposition of elements up to p^nth powers. When in the early 1960s Lubin and Tate constructed formal groups with complex multiplication in a local field to define the reciprocity relation explicitly, the role of Artin-Hasse functions became clear. They turned out to be the isomorphism between the canonical formal group with logarithm $x + \frac{x^p}{p} + \frac{x^{p^2}}{p^2} + \ldots$ and the multiplicative formal group (see [3]).
S.V. Vostokov in [4] generalized Artin-Hasse functions in the multiplicative case using the Frobenius operator Δ, defined on the ring of Laurent series over the ring of Witt vectors, to the function E_Δ:

$$E_\Delta(f(x)) = \exp\left(1 + \frac{\Delta}{p} + \frac{\Delta^2}{p^2} + \ldots\right)(f(x)),$$

[1]The work is supported by RFBR (grant no. 08-01-00777a); the second author also acknowledges support from SFB 478.

where Δ acts on x as raising to the power p, and on the coefficients of the ring of Witt vectors as the usual Frobenius. The map E_Δ induces a homomorphism from the additive group of the ring of power series to the multiplicative group. In addition, [4] contains the definitions of the inverse operator function l_Δ:

$$l_\Delta(f(x)) = \left(1 - \frac{\Delta}{p}\right) \log f(x), \qquad f(x) \equiv 1 \pmod{x},$$

which induces the inverse homomorphism.

To find explicit formulas for the Hilbert pairing on the Lubin-Tate formal groups, the work [5] generalized Artin-Hasse functions $E_\Delta(f(x))$ and their inverse functions $l_\Delta(f(x))$ to the Lubin-Tate formal groups. Using them, explicit formulas for the Hilbert pairing for formal Lubin-Tate modules were obtained in [4], [5]. They played a key role in the construction of the arithmetic of formal modules (see [5]).

The Artin-Hasse functions for Honda formal groups were defined in the work [6]. In the construction of generalised Artin-Hasse functions one uses the existence of a classification of this type of formal groups. Until 2002 such a classification was known in two cases only:

1. Lubin-Tate formal groups, i.e. formal groups of minimal height defined over the ring of integers of a local field isomorphic to the endomorphism ring of a formal group;

2. Honda formal groups, i.e. formal groups over the ring of Witt vectors of a finite field.

In 2002 M. Bondarko and S. Vostokov [6] obtained an explicit classification of formal groups in arbitrary local fields.

To study the arithmetic of the formal modules in an arbitrary local field and to derive explicit formulas for the generalized Hilbert pairing on these modules we should extend Artin-Hasse functions and the inverse maps on these formal modules. This paper contains a solution of this problem.

2 Notation

Suppose that $K/\mathbb{Q}$ is a local field with ramification index e and ring of integers O_K. By N we will denote its inertia subfield, and by O the ring of integers of N; σ will denote the Frobenius map on N.

Define a linear operator $\Delta : O[[x]] \to O[[x]]$ which acts as follows: $\Delta(\sum a_i x^i) = \sum \sigma(a_i) x^{pi}$.

Consider the ring $W = O[[t]]$. We can extend the action of σ on $O[[t]]$ by defining $\sigma(t) = t^p$. Then we can define Δ on $W[[x]]$ just as on $O[[x]]$. Let $\psi : W \to O_K$ be a homomorphism satisfying $\psi(t) = \pi$ and $\psi|_O = id_O$. Define a map $\phi_0 : O_K \to W$ by the formula: $\phi_0(\sum\limits_{i=0}^{e-1} w_i \pi^i) = \sum\limits_{i=0}^{e-1} w_i t^i$, where $w_i \in O$.

$V = N((t))$ will denote the field of fractions of W. We can easily extend ψ on it. We can also extend Δ on $V[[x]]$.
Denote by R_O the set of Teichmüller representatives in the ring O.
Let F_1 and F_2 be isomorphic formal groups over O_K with logarithms λ_1 and λ_2 respectively. Suppose that F_1 is a p-typical formal group. We will denote the isomorphism between them by $f(x) = \lambda_1^{-1}(\lambda_2(x)) \in O_K[[x]]$. Since F_1 is p-typical, we get $\lambda_1(x) = \Lambda_1(\Delta)x$, where $\Lambda_1(\Delta) \in O_K[[\Delta]]\Delta$.
We use the following convention: sometimes we write λ or Λ or F without an index if the index is equal to 1.

Lemma 1. *There exists a formal group $\overline{F} \in W[[x, y]]$ with p-typical logarithm $\overline{\lambda} \in xV[[x]]$ such that the following two statements are true:*

1. $\psi(\overline{\lambda}) = \lambda$.
2. $\psi(\overline{F}) = F$.

Proof. Let $G(R)$ be a universal formal group. Then the homomorphism between G and F can be factored through W. This is what we wanted. □

From now on we fix some $\overline{F}$ and $\overline{\lambda}$ from the lemma above. Since $\overline{\lambda}$ is p-typical, we can write it in the form $\overline{\lambda}(x) = \overline{\Lambda}(\Delta)x$, where $\overline{\Lambda}(\Delta) \in V[[\Delta]]\Delta$.

3 Definition of $\overline{l}$

Define the function $\overline{l} : xW[[x]] \to xV[[x]]$ by formula

$$\overline{l}(g(x)) = \overline{\Lambda}^{-1}(\Delta)\overline{\lambda}(g(x)). \tag{1}$$

This is an analog of the inverted Artin-Hasse function.

Theorem 1. $\overline{l}(xW[[x]]) \subset xW[[x]]$. *Moreover, $\overline{l}$ is a bijection from $xW[[x]]$ onto $xW[[x]]$.*

Proof. We begin by proving several lemmas.

Lemma 2. *Let $f_1, f_2 \in xW[[x]]$ be such that $\overline{l}(f_1), \overline{l}(f_2) \in W[[x]]$. Then $\overline{l}(f_1 +_{\overline{F}} f_2) = \overline{l}(f_1) + \overline{l}(f_2) \in xW[[x]]$.*

Proof. $\overline{l}(f_1 +_{\overline{F}} f_2) = \overline{\Lambda}^{-1}(\Delta)(\lambda(f_1 +_{\overline{F}} f_2)) = \overline{\Lambda}^{-1}(\Delta)(\lambda(f_1) + \lambda(f_2)) = \overline{l}(f_1) + \overline{l}(f_2)$. □

Lemma 3. *Suppose that $f \in xW[[x]]$ is such that $\overline{l}(f) \in xW[[x]]$ and $a \in W$ is such that $a = \theta t^n$, where $\theta \in R_O$. Suppose that $m \in N$ and $g(x) = f(ax^m)$. Then $\overline{l}(g(x)) = \overline{l}(f(x))(ax^m) \in xW[[x]]$.*

Proof. Denote $y = ax^m$. We know that $\sigma(\theta) = \theta^p$ and $\sigma(t) = t^p$, so $\Delta(y) = y^p$. So Δ acts on $f(x)$ just as on $f(y)$. This implies the statement of the lemma. □

By definition $\overline{\lambda}(x) = \overline{\Lambda}(\Delta)x$, so $\overline{l}(x) = \overline{\Lambda}(\Delta)^{-1}\overline{\lambda}(x) = \overline{\Lambda}(\Delta)^{-1}\overline{\Lambda}(\Delta)x = x$. So $\overline{l}(x) = x \in W[[x]]$. From the previous lemma we get $\overline{l}(x^n) = x^n \in W[[x]]$.
Then, $\overline{l}([p^m]x^n) = \overline{l}(x^n +_{\overline{F}} x^n +_{\overline{F}} + \cdots +_{\overline{F}} x^n) = p^m\overline{l}(x^n) = p^m x^n$.
Now, suppose that $f(x) = a_k x^k + a_{k+1}x^{k+1} + \ldots$ is an element of $x^k W[[x]]$, where $k > 0$. Then a_k (as any element of W) can be written in the form

$$a_k = \sum_{i=0}^{\infty} p^i \sum_{j=0}^{\infty} t^j r_{i,j}, \tag{2}$$

where $r_{i,j} \in R_O$. Define g_k by the formula:

$$g_k(x) = \overline{\sum_{i=0}^{\infty}}[p^i]\overline{\sum_{j=0}^{\infty}} t^j r_{i,j} x^k, \tag{3}$$

where overlined sums are taken using the $\overline{F}$ formal group law.

Lemma 4. *$g_k(x)$ is a convergent series and $g_k(x) \in x^k W[[x]]$.*

Proof. The second part of the lemma follows from the definition, so we will prove only the first part.
Denote $U_l = \overline{\sum_{i=0}^{l}}[p^i]\overline{\sum_{j=0}^{\infty}} t^j r_{i,j} x^k$. We will increase l and look at the coefficient at x^t. Consider the cooefficients of x^1, x^2 $\ldots x^l$ of $\overline{\lambda}^{-1}$ and $\overline{\lambda}$. Denote by Q the maximal degree of p in the denominators of these coefficients. Then $U_l = \overline{\lambda}(\sum_{i=0}^{l} p^i \sum_{j=0}^{\infty} \overline{\lambda}^{-1}(t^j r_{i,j} x^k))$. If we add a term with $l \geq 2Q$, the coefficient will increase by a multiple of p^{l-2Q}. Therefore, the coefficient will converge. $\square$

We now prove Theorem 1.
We have $f_{k+1} = f -_{\overline{F}} g_k \in x^{k+1}W[[x]]$, since we have constructed g_k so that it starts with the same term as f. Then we can construct $g_{k+1} \in x^{k+1}W[[x]]$ and so on. So any $f \in xW[[x]]$ can be written as $f(x) = \overline{\sum_{k=1}^{\infty}} g_k$. (As above, the line over the sum symbol denotes that we use $\overline{F}$ group law.)
Then

$$\overline{l}(f(x)) = \overline{l}\left(\overline{\sum_{k=1}^{\infty}} g_k(x)\right) = \sum_{k=1}^{\infty} \overline{l}(g_k(x)) = \sum_{k=1}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} p^j \overline{l}(r_{i,j,k} t^i x^k) =$$
$$= \sum_{k=1}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} p^j r_{i,j,k} t^i x^k \in xW[[x]]. \tag{4}$$

This proves the first part of the theorem. The bijectivity of $\overline{l}$ follows from formulas (3) and (4): $\overline{l}$ is a bijection between $xW[[x]]$ and all $\{(r_{i,j,k})\}$. $\square$

Now we can introduce the function $\overline{E} : xW[[x]] \to xW[[x]]$ as $\overline{l}^{-1}$. Immediately from the definition we get the formula

$$\overline{E}(f(x)) = \overline{\lambda}^{-1}(\overline{\Lambda}(\Delta)f(x)). \tag{5}$$

Theorem 2. *$\overline{l}$ is an isomorphism $xW[[x]]_{\overline{F}}$ into $xW[[x]]_+$ and $\overline{E}$ is its inverse.*

Proof. From Lemma (2) we get that $\overline{l}$ is a homomorphism. And from Theorem (1) it is a bijection. □

4 Definition of E

Suppose that $\phi : O_K \to W$ is any map such that the following statements hold:

1. $\phi\psi = id_{O_K}$.
2. $\phi(x + y) = \phi(x) + \phi(y)$.
3. $\phi|_{O_N} = id_{O_N}$.

Such a map exists, because we can take $\phi = \phi_0$. Then we can extend ϕ on $W[[x]]$ by taking $\phi(ax^n) = \phi(a)\phi(x^n)$.
Now define the map $E : xO_K[[x]] \to xO_K[[x]]$ by the following formula:

$$E(f(x)) = \psi(\overline{E}(\phi(x))). \tag{6}$$

Note that it is possible that we will get different E, if we take different ϕ.

Lemma 5. *Suppose that $f(x) \in xO_K[[x]]$, and $f(x) - a_k x^k \in x^{k+1}O_K[[x]]$. Then $E(f(x)) - a_k x^k \in x^{k+1}O_K[[x]]$.*

Proof. Denote $\phi(f(x))$ by $h(x)$. Then $h(x) - \phi(a_k)x^k \in x^{k+1}W[[x]]$. Now, from formula (4) we derive $\overline{E}(h(x)) - \psi(a_k)x^k \in x^{k+1}W[[x]]$. Therefore, $\psi(\overline{E}(h(x)))$ starts with $\psi(\phi(a_k))x^k = a_k x^k$. □

Theorem 3. *E is an isomorphism $xO_K[[x]]_+$ into $xO_K[[x]]_F$.*

Proof. 1. E is a homomorphism: $E(f_1 + f_2) = \psi(\overline{E}(\phi(f_1 + f_2))) = \psi(\overline{E}(\phi(f_1) + \phi(f_2))) = \psi(\overline{E}(\phi(f_1)) +_{\overline{F}} \overline{E}(\phi(f_2))) = \psi(\overline{E}(\phi(f_1))) +_F \psi(\overline{E}(\phi(f_2))) = E(f_1) +_F E(f_2)$.

2. Injectivity is obvious from Lemma 5.

3. It remains to prove that E is a surjection. Let $f(x) \in x^k O_K[[x]]$. Suppose that it starts with $a_k x^k$. Then we can take $g_k(x) = a_k x^k$: $E(g_k(x))$ then also starts with $a_k x^k$. Then denote $f_{k+1}(x) = f(x) -_F g_k(x)$. It starts with $b_{k+1}x^{k+1}$, so we can construct g_{k+1} and so on. Then put $g(x) = \sum g_k(x)$, so $E(g(x)) = \overline{\sum} g_k(x) = f(x)$. □

Moreover, we can define an isomorphism $xO_K[[x]]_+$ into $xO_K[[x]]_{F_2}$ by the formula

$$E(\hat{g}(x)) = f(E(g(x))) \tag{7}$$

References

1. E. Artin, H. Hasse, Die beiden Ergänzungssatz zum Reziprozitätsgesetz der l^n-ten Potenzreste im Körper der l^n-ten Einheitswurzeln, Hamb. Abh. 6 (1928), 146–162.

2. I.R. Shafarevich, On p-extentions, Rec. Math. [Mat. Sbornik] N.S., 20(62):2 (1947), 351–363.

3. J. Lubin, J. Tate, Formal complex multiplication in local fields, Ann. Math. 81 (1965), 380–387.

4. S.V. Vostokov, Explicit form of the law of reciprocity, Math USSR Izv., 1979, 43:3, 557–588.

5. S.V. Vostokov, Norm pairing in formal modules, Math USSR Izv., 1980, 15(1), 25–51.

6. S.V. Vostokov, M.V. Bondarko, An explicit classification of formal groups over local fields, Tr. Mat. Inst. Steklova, 241, Nauka, Moscow, 2003, 43–67.

7. O.V. Demchenko, Formal Honda groups: the arithmetic of the group of points, Algebra i analiz, 12:1 (2000), 115–133.

Vitaly Valtman,
Faculty of Mathematics
and Mechanics
St. Petersburg State University
St. Petersburg, Russia
valtmanva@gmail.com

Sergey Vostokov,
Faculty of Mathematics
and Mechanics
St. Petersburg State University
St. Petersburg, Russia
sergei.vostokov@gmail.com

Rationality of Integral Cycles

Alexander Vishik

Received: January 31, 2010
Revised: March 5, 2010

Abstract. In this article we provide the sufficient condition for a Chow group element to be defined over the ground field. This is an integral version of the result known for $\mathbb{Z}/2$-coefficients. We also show that modulo 2 and degree r cohomological invariants of algebraic varieties can not affect rationality of cycles of codimensions up to $2^{r-1}-2$.

2010 Mathematics Subject Classification: 11E04, 14C15, 14C25, 19D45, 55S25
Keywords and Phrases: Quadrics, Chow groups, algebraic cobordism, Landweber-Novikov operations, Symmetric operations

1 Introduction

In many situations it is important to know, if the element of the Chow group of some variety which exists over algebraic closure is actually defined over k. In particular, this question arises while studying various discrete invariants of quadrics. An effective tool here is the method introduced in [9] which, in particular, gives that for an element of codimension m with $\mathbb{Z}/2$-coefficients, it is sufficient to check that it is defined over the function field of a sufficiently large quadric (of dimension $> 2m$). This approach was successfully applied to various questions from quadratic form theory. In particular, to the construction of fields with the new values of the u-invariant (see [11]). The above core result can be extended in various directions. One of them due to K.Zainoulline (see [13]) establishes similar statement for the quadric substituted by a norm-variety of Rost for the pure symbol in $K_n^M(k)/p$ (p-prime) and for Chow groups with $\mathbb{Z}/p$-coefficients. This is a generalisation of the case of a Pfister quadric, which is a norm-variety for the pure symbol modulo 2.
But all the mentioned results are dealing with Chow groups with torsion coefficients. In the current article I would like to address similar question for

integral Chow groups. The results obtained are similar to the $\mathbb{Z}/2$-case, but one requires an additional condition on Q (aside from its size) saying that Q has a projective line defined over the generic point of Q. Although, the "generic quadric" does not have this property, such quadrics are quite widespread. If one imposes stronger conditions on a quadric, one can show that the map

$$\mathrm{CH}^m(Y) \to \mathrm{CH}^m(Y_{k(Q)})$$

is surjective. In particular, this happens for a Pfister quadric, and $m < 2^{r-1}-1$. This latter result can be used to show that (modulo 2) and degree r cohomological invariants of algebraic varieties do not effect rationality of Chow group elements of codimension up to $2^{r-1}-2$.
The main tool we use is "Symmetric Operations" in Algebraic Cobordism (see [10]). These are "formal halves" of the "negative parts" of Steenrod operations ($mod\,2$) there. If one does not care about 2-torsion effects, one can use more simple Landweber-Novikov operations instead. But the symmetric operations provide the only (known) way to get "clean" results on rationality.
Acknowledgements: I would like to thank the referee for the very useful suggestions. The support of the EPSRC Responsive Mode grant EP/G032556/1 is gratefully acknowledged.

2 Symmetric operations

For any field k of characteristic 0, M.Levine and F.Morel have defined the Algebraic Cobordism theory Ω^*, which is the universal generalised oriented cohomology theory on the category Sm/k of smooth quasi-projective varieties over k (see [3, Theorem 1.2.6]), which means that for any other such theory A^* there is unique map of theories $\Omega^* \to A^*$. For a given smooth quasi-projective X, the ring $\Omega^*(X)$ is additively generated by the classes $[v : V \to X]$ with V-smooth and v-projective, modulo some relations. The value of Ω^* on $\mathrm{Spec}(k)$ coincides with $MU^{2*}(pt) = \mathbb{L}$ - the Lazard ring - see [3, Theorem 1.2.7]. Since Chow groups form a generalised oriented cohomology theory, one has a canonical map $pr : \Omega^* \to \mathrm{CH}^*$ (given by $[v : V \to X] \mapsto v_*(1_V) \in \mathrm{CH}_{\dim(V)}(X)$) which is surjective, and moreover, $\mathrm{CH}^*(X) = \Omega^*(X)/\mathbb{L}_{>0} \cdot \Omega^*(X)$ - see [3, Theorem 1.2.19]. Thus, one can reconstruct Chow groups if the Algebraic Cobordism is known. On Ω^* we have the action of Landweber-Novikov operations - [3, Example 4.1.25]. Such operations can be parametrised by the polynomials $g \in \mathbb{L}[\sigma_1, \sigma_2, \ldots]$, with

$$S^g_{L.-N.}([v : V \to X]) := v_*(g(c_1, c_2, \ldots) \cdot 1_V) \in \Omega^*(X),$$

where $c_i = c_i(-T_V + v^*T_X)$ is the i-th Chern class of the virtual normal bundle of v. If one does not mind moding out the 2-torsion, then all the results of the next section can be obtained using the Landweber-Novikov operations only. But to obtain precise statements one needs more subtle "Symmetric operations". These operations were introduced in [8] and [10]. It is convenient

to parametrise them by $q(s) \in \mathbb{L}[[s]]$, where $\Phi^{s^r} : \Omega^d(X) \to \Omega^{2d+r}(X)$. The operation $\Phi^{q(s)}$ is constructed as follows. For a smooth morphism $W \to U$, let $\widetilde{\square}(W/U)$ denotes the blow-up of $W \times_U W$ at the diagonal W. For a smooth variety W denote: $\widetilde{\square}(W) := \widetilde{\square}(W/\operatorname{Spec}(k))$. Denote as $\widetilde{C}^2(W)$ and $\widetilde{C}^2(W/U)$ the quotient variety of $\widetilde{\square}(W)$, respectively, $\widetilde{\square}(W/U)$ by the natural $\mathbb{Z}/2$-action. These are smooth varieties. Notice that they have natural line bundle $\mathcal{L}$, which lifted to $\widetilde{\square}$ becomes $\mathcal{O}(1)$ - see [4], or [10, p.492]. Let $\tilde{\rho} := c_1(\mathcal{L}) \in \Omega^1(\widetilde{C}^2)$.
If $[v] \in \Omega^d(X)$ is represented by $v : V \to X$, then v can be decomposed as $V \xrightarrow{g} W \xrightarrow{f} X$, where g is a regular embedding, and f is smooth projective. One gets natural morphisms:

$$\widetilde{C}^2(V) \overset{\alpha}{\hookrightarrow} \widetilde{C}^2(W) \overset{\beta}{\hookleftarrow} \widetilde{C}^2(W/X) \xrightarrow{\gamma} X.$$

Now, $\Phi^{q(s)}([v]) := \gamma_* \beta^* \alpha_*(q(\tilde{\rho}))$. Denote as $\phi^{q(s)}([v])$ the composition $pr \circ \Phi^{q(s)}([v])$. As was proven in [10, Theorem 2.24], $\Phi^{q(s)}$ gives a well-defined operation $\Omega^*(X) \to \Omega^*(X)$. I should note that in [10] and [9] we use slightly different parametrisation for symmetric operations. To stress this difference I used the different name for the uniformiser. In [10] the parameter is t and its relation to our s is given by: $t = [-1]_\Omega(s)$, where $[-1]_\Omega(s) \in \mathbb{L}[[s]]$ is the inverse in terms of the universal formal group law. The difference basically amounts to signs, and with the new choice the formulae are just a little bit nicer.
It was proven in [8] that the Chow-trace of Φ is the half of the Chow-trace of certain Landweber-Novikov operation.

Proposition 2.1 ([8, Propositions 3.8, 3.9], [10, Proposition 3.14]) *For* $[v] \in \Omega^d(X)$,

(1) $2\phi^{s^r}([v]) = pr(-S^{r+d}_{L-N}([v]))$, *for* $r > 0$;

(2) $2\phi^{s^0}([v]) = pr([v]^2 - S^d_{L-N}([v]))$,

where we denote $S^{\sigma_r}_{L-N}$ *as* S^r_{L-N}.

The additive properties of ϕ are given by the following:

Proposition 2.2 ([10, Proposition 2.8])

(1) *Operation* ϕ^{s^r} *is additive for* $r > 0$;

(2) $\phi^{s^0}(x + y) = \phi^{s^0}(x) + \phi^{s^0}(y) + pr(x \cdot y)$.

Let $[v] \in \Omega^*(X)$ be some cobordism class, and $[u] \in \mathbb{L}$ be the class of a smooth projective variety U over k of positive dimension. We will use the notation $\eta_2(U)$ for the (minus) Rost invariant $\frac{deg(c_{\dim(U)}(-T_U))}{2} \in \mathbb{Z}$ (see [4]).

Proposition 2.3 ([10, Proposition 3.15]) *In the above notations, let* $r = (\operatorname{codim}(v) - 2\dim(u))$. *Then, for any* $i \geqslant \max(r; 0)$,

$$\phi^{s^{i-r}}([v] \cdot [u]) = -\eta_2(U) \cdot (pr \circ S^i_{L-N})([v]).$$

The following proposition describes the behaviour of Φ with respect to pull-backs and regular push-forwards. For $q(s) = \sum_{i\geqslant 0} q_i s^i \in CH^*(X)[[s]]$ let us define $\phi^{q(s)} := \sum_{i\geqslant 0} q_i \phi^{s^i}$. For a vector bundle $\mathcal{V}$ denote $c(\mathcal{V})(s) := \prod_i (s+\lambda_i)$, where $\lambda_i \in CH^1$ are the *roots* of $\mathcal{V}$. This is the usual total Chern class of $\mathcal{V}$.

Proposition 2.4 ([10, Propositions 3.1, 3.4]) *Let $f : Y \to X$ be some morphism of smooth quasi-projective varieties, and $q(s) \in CH^*(X)[[s]]$. Then*

(1) $f^*\phi^{q(s)}([v]) = \phi^{f^*q(s)}(f^*[v])$;

(2) *If f is a regular embedding, then $\phi^{q(s)}(f_*([w])) = f_*(\phi^{f^*q(s)\cdot c(\mathcal{N}_f)(s)}([w]))$, where $\mathcal{N}_f$ is the normal bundle of the embedding.*

And, consequently, for f - a regular embedding:

(3) $\phi^{q(s)}(f_*([1_Y])\cdot[v]) = \phi^{q(s)\cdot f_*(c(\mathcal{N}_f)(s))}([v])$

3 Rationality of cycles over function fields of quadrics

Let k be a field of characteristic 0, Y be a smooth quasi-projective variety, and Q be a smooth projective quadric defined over k. For $\overline{y} \in \mathrm{CH}^m(Y_{\overline{k}})$ we say that $\overline{y}$ is *k-rational*, if it belongs to the image of the natural restriction map:

$$\mathrm{CH}^m(Y) \to \mathrm{CH}^m(Y_{\overline{k}}).$$

The following result shows that rationality of $\overline{y}$ can be checked over the function field of Q provided Q is sufficiently large and a little bit "special".

Theorem 3.1 *In the above notations, suppose that $m < \dim(Q)/2$, and $\mathrm{i}_1(Q) > 1$. Then*

$$\overline{y} \text{ is defined over } k \;\Leftrightarrow\; \overline{y}|_{\overline{k(Q)}} \text{ is defined over } k(Q).$$

With stronger conditions on Q we can prove more subtle result.

Proposition 3.2 *Let Q be smooth projective quadric with $\mathrm{i}_1(Q) > m < \dim(Q)/2$. Then the map $\mathrm{CH}^m(Y) \twoheadrightarrow \mathrm{CH}^m(Y_{k(Q)})$ is surjective, for all smooth quasi-projective Y.*

Applying it to the case of a Pfister quadric, we get:

Corollary 3.3 *Let Q_α be an r-fold Pfister quadric. Then the map*

$$\mathrm{CH}^m(Y) \twoheadrightarrow \mathrm{CH}^m(Y_{k(Q_\alpha)})$$

is surjective, for all smooth quasi-projective Y, and all $m < 2^{r-1} - 1$.

Proof: In this case, $\mathrm{i}_1(Q_\alpha) = 2^{r-1}$, and $\dim(Q_\alpha)/2 = 2^{r-1} - 1$ (actually, the case of a Pfister quadric is the only one where the second inequality in Proposition 3.2 is needed). $\square$

This immediately implies:

THEOREM 3.4 *Let k be any field of characteristic 0, and $r \in \mathbb{N}$, then there exists a field extension F/k such that:*

- $\mathrm{K}_r^M(F)/2 = 0$*;*
- *The map $\mathrm{CH}^m(Y) \to \mathrm{CH}^m(Y_F)$ is surjective, for all $m < 2^{r-1} - 1$, for all smooth quasi-projective Y defined over k.*

Proof: Take F - the standard *Merkurjev's tower of fields*, that is, $F := \varinjlim F_i$, where $F_{i+1} = F_i'$, and for any field G, the extension G' of G is defined as $\varinjlim G(\times_{i\in I} Q_i)$, where the latter limit is taken over all finite sets I of r-fold Pfister quadrics defined over G. Then $\mathrm{K}_r^M(F)/2 = 0$, and it is sufficient to prove the respective property for the map $\mathrm{CH}^m(Y_E) \to \mathrm{CH}^m(Y_{E(Q_\alpha)})$, where Q_α is an r-fold Pfister quadric. It remains to apply Corollary 3.3. $\square$

This result shows that $(\mathit{mod}\, 2)$ and degree $\geqslant r$ cohomological invariants of smooth algebraic varieties could not affect rationality of cycles of codimension up to $2^{r-1} - 2$. As the example of a Pfister quadric itself shows, this boundary is sharp.

REMARK: The above Corollary can also be proven by other means. Namely, the computations of M.Rost ([6, Theorem 5], see also [2, Theorem 8.1], or [10, Theorem 4.1]) show:

PROPOSITION 3.5 (M.Rost) *Let Q_α be r-fold Pfister quadric over the field k. Then, for any field extension F/k, the map*

$$\mathrm{CH}^n(Q_\alpha) \twoheadrightarrow \mathrm{CH}^n(Q_\alpha|_F)$$

is surjective, for any $n < \dim(Q_\alpha)/2 = 2^{r-1} - 1$.

It is remarkable that the respective map

$$\Omega^n(Q_\alpha) \hookrightarrow \Omega^n(Q_\alpha|_F)$$

on algebraic cobordism groups is instead injective, for any n - see [10, Theorem 4.1].
Combined with the following general result Proposition 3.5 gives our Corollary 3.3.

Proposition 3.6 (R.Elman, N.Karpenko, A. Merkurjev, [1, Lemma 88.5]) *Let X, Y be smooth varieties over k, such that, for any field extension F/k, and for any $n \leqslant m$, the map:*

$$\mathrm{CH}^n(X) \twoheadrightarrow \mathrm{CH}^n(X_F)$$

is surjective. Then, the map:

$$\mathrm{CH}^m(Y) \twoheadrightarrow \mathrm{CH}^m(Y_{k(X)})$$

is surjective as well.

Here I should point out, that although one gets a different "elementary" proof of Corollary 3.3, it involves a quite non-trivial ingredient - the Rost computation of the Chow groups of Pfister quadrics. The Pfister quadrics and also few small-dimensional quadrics are the only ones for which such computation is known. In particular, to prove the whole Proposition 3.2 using this method one needs an analogue of Proposition 3.5.

Both Theorem 3.1 and Proposition 3.2 are consequences of the following statement.

Proposition 3.7 *Let Q be a smooth projective quadric of dimension $> 2m$ with $\mathfrak{i}_1(Q) > 1$, E/k be field extension such that $i_W(q_E) > m$, and Y be a smooth quasi-projective k-variety. Then, for any $y \in \mathrm{CH}^m(Y_{k(Q)})$ there exists $x \in \mathrm{CH}^m(Y)$ such that $x_{E(Q)} = y_{E(Q)}$.*

Proof: Consider $y \in \mathrm{CH}^m(Y_{k(Q)})$. Using the surjections

$$\Omega^m(Y \times Q) \overset{pr}{\twoheadrightarrow} \mathrm{CH}^m(Y \times Q) \twoheadrightarrow \mathrm{CH}^m(Y_{k(Q)}),$$

we can lift y to some element $v \in \Omega^m(Y \times Q)$.
Since $i_W(Q_E) > m$, $q_E = (\perp_{i=0}^m \mathbb{H}) \perp q'$, for some quadratic form q' defined over E. Consider the cobordism motive of our quadric Q_E (see [5], or [12]) By [7, Proposition 2] and [12, Corollary 2.8], we have that $M^{\Omega}(Q_E) = (\oplus_{i=0}^m \mathbf{L}(i)[2i]) \oplus M'$, where

$$M' = M^{\Omega}(Q')(m+1)[2m+2] \oplus (\oplus_{i=0}^m \mathbf{L}(\dim(Q) - i)[2\dim(Q) - 2i]),$$

where $\mathbf{L}(j)[2j]$ is the cobordism Tate-motive (see [12]). Moreover, we can always choose the generator of $\mathbb{L} \cong \Omega^*(\mathbf{L}(i)[2i]) \subset \Omega^*(Q_E)$ to be h^i. Let us denote the passage $k \to E$ as $\bar{\ }$. Then, our element $v \in \Omega^m(Y \times Q)$ restricted to E can be presented as $\overline{v} = \sum_{i=0}^m \overline{v}^i \cdot h^i + \overline{v}'$, where $\overline{v}^i \in \Omega^{m-i}(Y_E)$, and $\overline{v}' \in \Omega^m(M^{\Omega}(Y) \otimes M')$. Applying the composition

$$\Omega^m(Y \times Q_E) \to \mathrm{CH}^m(Y \times Q_E) \to \mathrm{CH}^m(Y_{E(Q)})$$

to $\overline{v}$ we get $pr(\overline{v}^0)_{E(Q)}$. On the other hand, from commutativity of the diagram

$$\begin{array}{ccccc}
\Omega^m(Y \times Q_E) & \longrightarrow & \mathrm{CH}^m(Y \times Q_E) & \longrightarrow & \mathrm{CH}^m(Y_{E(Q)}) \\
\uparrow & & \uparrow & & \uparrow \\
\Omega^m(Y \times Q) & \longrightarrow & \mathrm{CH}^m(Y \times Q) & \longrightarrow & \mathrm{CH}^m(Y_{k(Q)})
\end{array}$$

it must coincide with $y_{E(Q)}$. Thus, it is sufficient to show that $pr(\overline{v}^0) \in \mathrm{CH}^m(Y_E)$ is defined over k.

Lemma 3.8 *In the above situation, let $e : P \hookrightarrow Q$ be a linear embedding of smooth quadrics, with $\dim(P) = m$. Let $\rho : M^\Omega(Q) \to M^\Omega(Q)$ be cobordism-motivic endomorphism of Q. Let $v \in \Omega^m(Y \times Q)$. Then*

$$(id \times e)^*(id \times \rho)^*(\overline{v}) = \sum_{i=0}^{m} \left(\sum_{j=0}^{i} \alpha_{i,j} \cdot \overline{v}^j \right) \cdot h^i,$$

where $\alpha_{i,j} \in \mathbb{L}_{i-j}$. Moreover, $\alpha_{i,i} \in \mathbb{Z}$ is visible on the level of Chow groups: $\rho_{CH}(h^i) = \alpha_{i,i} \cdot h^i$.

Proof: By dimensional reasons, any map from $M^\Omega(P)$ to M' is zero. Thus, $(id\times e)^*(id\times\rho)^*(\overline{v}) = \sum_{j=0}^{m} \overline{v}^j \cdot (e\circ\rho)^*(h^j)$. Clearly, $\rho^*(h^j) = \sum_{i\geqslant j} \alpha_{i,j} \cdot h^i + \beta'_j$, where $\alpha_{i,j} \in \mathbb{L}_{i-j}$, and $\beta'_j \in \Omega^j(M')$. Again, by the same reasons, $e^*(\beta'_j) = 0$, and we get the first statement. Projecting to CH^* we get the description of $\alpha_{i,i}$. □

Lemma 3.9 *In the situation of Proposition 3.7, let $e : P \hookrightarrow Q$ be a linear embedding of smooth quadrics, where $\dim(P) = m$, and $v \in \Omega^m(Y \times Q)$. Then there exist $w, z \in \Omega^m(Y \times P)$ such that: $\overline{w} = \sum_{i=0}^{m} \overline{w}^i \cdot h^i$, $\overline{z} = \sum_{i=0}^{m} \overline{z}^i \cdot h^i$, and*

1) $\overline{w}^i = \overline{v}^i$;

2) $\overline{z}^i = \sum_{j=0}^{i} \alpha_{i,j} \cdot \overline{v}^j$, *where* $\alpha_{i,j} \in \mathbb{L}_{i-j}$;

3) $\alpha_{0,0} = 1$, *and* $\alpha_{i,i} = 0$, *for all odd* i.

Proof: Since $\mathrm{i}_1(Q) > 1$, the (Chow) motive of Q is decomposable, and if N is indecomposable direct summand containing $\mathbf{Z}$ (when restricted to $\overline{k}$), then N does not contain $\mathbf{Z}(i)[2i]$, for any odd i, by the result of R.Elman, N.Karpenko, A.Merkurjev - [1, Proposition 83.2] (here $\mathbf{Z}(j)[2j]$ is the Tate-motive in the Chow-motivic category - see [5], or [12]).
Let $\rho_{CH} \in \mathrm{End}_{Chow(k)}(M^{CH}(Q))$ be the projector corresponding to N. Then $\rho^*_{CH}(1) = 1$, and $\rho^*_{CH}(h^i) = 0$, for any odd i. Using the surjective map $pr : \Omega^* \twoheadrightarrow \mathrm{CH}^*$, we can lift ρ_{CH} to a cobordism-motivic morphism $\rho : M^\Omega(Q) \to M^\Omega(Q)$. Take $z := (id \times (e \circ \rho))^*(v)$, and $w := (id \times e)^*(v)$. □

In the above notations, let $\overline{u} = \sum_{i=0}^{m} \alpha_{i,i}\overline{v}^i \cdot h^i \in \Omega^m(Y \times P_E)$. Then:

Lemma 3.10 *For any $0 \leqslant k \leqslant [m/2]$, the element $pr(\Phi^{s^{m-2k}}(\pi_*(h^k \cdot (\overline{z} - \overline{u}))))$ is a linear combination of $pr(S^j_{L-N}(\overline{v}^j))$ with even coefficients.*

Proof: We have: $\pi_*(h^k \cdot (\overline{z} - \overline{u})) = \sum_{i=1}^{m} \sum_{0 \leqslant j < i} [P_{m-k-i}] \cdot \alpha_{i,j} \cdot \overline{v}^j$, where $[P_l] \in \mathbb{L}$ is the class of an l-dimensional quadric. Thus, by Proposition 2.3,

$$pr(\Phi^{s^{m-2k}}(\pi_*(h^k \cdot (\overline{z} - \overline{u})))) = -\sum_{i=1}^{m} \sum_{0 \leqslant j < i} \eta([P_{m-k-i}] \cdot \alpha_{i,j}) \cdot pr(S^j_{L-N}(\overline{v}^j)).$$

But $2 \cdot \eta$ is multiplicative, P_{m-k-i} is a quadric (possibly, zero-dimensional), and $\dim(\alpha_{i,j}) > 0$. Thus, all the coefficients are even. □

Lemma 3.11 *If $w \in \Omega^m(Y \times P)$ decomposes over E as $\overline{w} = \sum_{i=0}^{m} (\overline{w}^i \cdot h^i)$, then any linear combination of $pr(S^i_{L-N}(\overline{w}^i))$ with even coefficients is defined over k.*

Proof: Consider the projection $Y \times P \xrightarrow{\pi} Y$. It is sufficient to observe that, for all $0 \leqslant i \leqslant m$, the elements $pr(S^i_{L-N}\pi_*(h^{m-i} \cdot \overline{w})) = 2pr(S^i_{L-N}(\overline{w}^i)) + 2\sum_{0 \leqslant j < i} \eta(P_{i-j}) pr(S^j_{L-N}(\overline{w}^j))$ are defined over k. □

Denote as $\eta(x)$ the power series $\sum_{i \geqslant 0} \eta(P_i) \cdot x^i$, where $\eta(P_l) = \frac{deg(c_l(-T_{P_l}))}{2}$ is the (minus) *Rost invariant* of an l-dimensional quadric P_l.

Proposition 3.12 *Let Y be smooth quasi-projective variety, P - smooth projective quadric of dimension m, and $z \in \Omega^m(Y \times P)$ such element that $\overline{z} = \sum_{i=0}^{m} \overline{z}^i \cdot h^i$, where $\overline{z}^i \in \Omega^{m-i}(Y_E)$. Then, for any polynomial $f \in \mathbb{Z}[x]$ of degree $\leqslant [m/2]$, the linear combination*

$$\sum_{j=0}^{m} g_{m-j} \cdot pr(S^j_{L-N}(\overline{z}^j)),$$

is defined over k, where $g(x) = \sum_l g_l \cdot x^l$ is "the degree $\leqslant m$ part" of the product $f(x) \cdot \eta(x)$.

Proof: Let $f \in \mathbb{Z}[x]$ be some polynomial of degree $\leqslant [m/2]$, and f_i be its coefficients. Consider the element $y := pr(\sum_i f_i \cdot \Phi^{s^{m-2i}}(\pi_*(h^i \cdot z)))$. Then $\overline{y} = \sum_i f_i \sum_j pr(\Phi^{s^{m-2i}}([P_{m-i-j}] \cdot \overline{z}^j))$, where $[P_l]$ is the class of quadric of dimension l in $\mathbb{L}$. By Proposition 2.3, this expression is equal to

$$-\sum_i \sum_j f_i \cdot \eta(P_{m-i-j}) \cdot pr(S^j_{L-N}(\overline{z}^j)) = -\sum_{j=0}^{m} g_{m-j} \cdot pr(S^j_{L-N}(\overline{z}^j)),$$

where the polynomial $g(x) = \sum_l g_l \cdot x^l$ is the degree $\leqslant m$ part of the product $f(x) \cdot \eta(x)$, where $\eta(x) = \sum_{r \geqslant 0} \eta(P_r) \cdot x^r$. $\square$

It follows from Proposition 3.12, and Lemmas 3.9, 3.10 and 3.11 that, for any $f \in \mathbb{Z}[x]$ of degree $\leqslant [m/2]$, the element

$$\sum_{j=0}^{m} g_{m-j} \cdot pr(S^j_{L-N}(\overline{u}^j)) = \sum_{j=0}^{m} g_{m-j}\alpha_{j,j} \cdot pr(S^j_{L-N}(\overline{v}^j))$$

is defined over k. Now it is sufficient to find a polynomial $f \in \mathbb{Z}[x]$ of degree $\leqslant [m/2]$ such that the polynomial $g(x) := (f(x) \cdot \eta(x))_{\leqslant m}$ has an odd coefficient at x^m and even coefficients at all the smaller monomials of the same pairity. We can pass to $\mathbb{Z}/2$-coefficients, where we have:

Lemma 3.13 *There exists such polynomial $f \in \mathbb{Z}/2[x]$ of degree $\leqslant [m/2]$ that $(f(x) \cdot \eta(x))_{\leqslant m}$ $(mod 2) = x^m +$ terms of parity $(m-1)$.*

Proof: Recall that $\eta_l = \eta(P_l) = (-1)^l \frac{(2l)!}{l!(l+1)!}$, and $\eta(x)$ $(mod 2) = \sum_{i \geqslant 0} x^{2^i - 1} = \gamma^{-1}$, where $\gamma = 1 + \sum_{i \geqslant 0} x^{2^i}$. We will use some facts about these power series obtained in [11].
In the case $m = 2n+1$ - odd, consider $f(x) := (\gamma^m)_{\leqslant n}$ ($= a_m$ in the notations of [11]). Since $\gamma^m = (\gamma^m)_{\leqslant n} + (\gamma^m)_{>n}$, we have that

$$((\gamma^m)_{\leqslant n} \cdot \gamma^{-1})_{\leqslant n} = \gamma^{2n}_{\leqslant n}$$

contains only terms of even degree. But by [11, Corollary 3.10 and (1)],

$$((\gamma^m)_{\leqslant n} \cdot \gamma^{-1})_{\leqslant 2n+1} = ((\gamma^m)_{\leqslant n} \cdot \gamma^{-1})_{\leqslant n} + x^{2n} + x^{2n+1}.$$

Hence, $(f \cdot \gamma^{-1})_{<m}$ consists of terms of even degree, and $(f \cdot \gamma^{-1})_m = x^m$.
In the case $m = 2n$ - even, it remains to take $f(x) := a_{m-1} \cdot x$. $\square$

Remark: Actually, the above polynomial f is unique. Moreover, it is exactly the polynomial δ appearing in the proof of [11, Proposition 3.5], where a completely different selection criterion was used!

Proposition 3.7 is proven.

$\square$

References

[1] R.Elman, N.Karpenko, A.Merkurjev, *The Algebraic and Geometric Theory of Quadratic Forms*, AMS Colloquium Publications, 56, 2008, 435pp.

[2] N.Karpenko, A.Merkurjev, *Rost projectors and Steenrod operations*, Documenta Math., 7 (2002), 481-493.

[3] M.Levine, F.Morel, *Algebraic cobordism*, Springer Monographs in Mathematics, Springer-Verlag, 2007.

[4] A.Merkurjev, *Rost degree formula*, Preprint, 2000, 1-19.

[5] A.Nenashev, K.Zainoulline, *Oriented Cohomology and motivic decompositions of relative cellular spaces*, J.Pure Appl. Algebra 205 (2006), no.2, 323-340.

[6] M.Rost, *Some new results on the Chowgroups of quadrics*, Preprint, 1990, 1-5. (http://www.math.uni-bielefeld.de/ rost/data/chowqudr.pdf)

[7] M.Rost, *The motive of a Pfister form*, Preprint, 1998, 1-13. (http://www.math.uni-bielefeld.de/ rost/motive.html)

[8] A.Vishik, *Symmetric operations* (in Russian), Trudy Mat. Inst. Steklova 246 (2004), Algebr. Geom. Metody, Svyazi i Prilozh., 92-105. English transl.: Proc. of the Steklov Institute of Math. 246 (2004), 79-92.

[9] A.Vishik, *Generic points of quadrics and Chow groups*, Manuscripta Math 122 (2007), No.3, 365-374.

[10] A.Vishik, *Symmetric operations in Algebraic Cobordism*, Adv. Math 213 (2007), 489-552.

[11] A.Vishik, *Fields of u-invariant* 2^r+1, *Algebra, Arithmetic and Geometry - Manin Festschrift*, Birkhauser, 2010, in press.

[12] A.Vishik, N.Yagita, *Algebraic cobordisms of a Pfister quadric*, J. of London Math. Soc., 76 (2007), n.2, 586-604.

[13] K. Zainoulline, *Special correspondences and Chow traces of Landweber-Novikov operations*, J. Reine und Angew. Math. 628 (2009), 195-204.

Alexander Vishik
School of Mathematical Sciences
University of Nottingham
University Park
Nottingham
NG7 2RD, United Kingdom
alexander.vishik@nottingham.ac.uk

Cancellation Theorem

Vladimir Voevodsky

Received: December 8, 2008
Revised: May 15, 2010

Abstract. In this paper we give a direct proof of the fact that for any schemes of finite type X, Y over a Noetherian scheme S the natural map of presheaves with transfers

$$\underline{Hom}(\mathbf{Z}_{tr}(X), \mathbf{Z}_{tr}(Y)) \to \underline{Hom}(\mathbf{Z}_{tr}(X) \otimes_{tr} \mathbf{G}_m, \mathbf{Z}_{tr}(Y) \otimes_{tr} \mathbf{G}_m)$$

is a (weak) $\mathbf{A}^1$-homotopy equivalence. As a corollary we deduce that the Tate motive is quasi-invertible in the triangulated categories of motives over perfect fields.

2010 Mathematics Subject Classification: 14F42, 19E15

Contents

1 Introduction

Let $SmCor(k)$ be the category of finite correspondences between smooth schemes over a field k. Denote by $\mathbf{G}_m$ the scheme $\mathbf{A}^1 - \{0\}$. One defines the sheaf with transfers S_t^1 by the condition that $\mathbf{Z}_{tr}(\mathbf{G}_m) = S_t^1 \oplus \mathbf{Z}$ where $\mathbf{Z}$ is split off by the projection to the point and the point 1. For any scheme Y consider the sheaf with transfers $F_Y = \underline{Hom}(S_t^1, S_t^1 \otimes \mathbf{Z}_{tr}(Y))$ which maps a smooth scheme X to $Hom(S_t^1 \otimes \mathbf{Z}_{tr}(X), S_t^1 \otimes \mathbf{Z}_{tr}(Y))$. The main result of this paper is Corollary 4.9 which asserts that for any Y the obvious map $\mathbf{Z}_{tr}(Y) \to F_Y$ defines a quasi-isomorphism of singular simplicial complexes

$$C_*(\mathbf{Z}_{tr}(Y)) \to C_*(F_Y)$$

as complexes of presheaves i.e. for any X the map of complexes of abelain groups

$$C_*(\mathbf{Z}_{tr}(Y))(X) \to C_*(F_Y)(X)$$

is a quasi-isomorphism. We then deduce from this result the "Cancellation Theorem" for triangulated motives which asserts that if k is a perfect field then for any K, L in $DM_-^{eff}(k)$ the map

$$Hom(K, K') \to Hom(K(1), K'(1))$$

is bijective.

This result was previously known in two particular situations. For varieties over a field k with resolution of singularities it was proved in [4]. For K' being the motivic complex $\mathbf{Z}(n)[m]$ and any field k it was proved in [5]. Both proofs are very long.

The main part of our argument does not use the assumption that we work with smooth schemes over a field and we give it for separated schemes of finite type over a noetherian base. To be able to do it we define in the first section the category of finite correspondences for separated schemes of finite type over a base. The definition is a straightforward generalization of the definition for schemes over a field based on the constructions of [2] and can be skipped. In the second section we define intersection of relative cycles with Cartier divisors and prove the properties of this construction which we need. In the third we prove our main theorem 4.6 and deduce from it the cancellation theorem over perfect fileds 4.10.

In this paper we say "a relative cycle" instead of "an equidimensional relative cycle". All schemes are separated. The letter S is typically reserved for the base scheme which is assumed to be noetherian. All the standard schemes $\mathbf{P}^1$, $\mathbf{A}^1$ etc. are over S. When no confusion is possible we write XY instead of $X \times_S Y$.

I would like to thank Pierre Deligne who explained to me how to compute the length function.

2 Finite correspondences

For a scheme X of finite type over a noetherian scheme S we denote by $c(X/S)$ the group of finite relative cycles on X over S. In [2] this group was denoted by $c_{equi}(X/S, 0)$. If S is regular or if S is normal and the characteristic of X is zero, $c(X/S)$ is the free abelian group generated by closed irreducible subsets of X which are finite over S and surjective over a connected component of S. For the general definition see [2, after Lemma 3.3.9]. A morphism $f : S' \to S$ defines the pull-back homomorphism $c(X/S) \to c(XS'/S')$ which we denote by $cycl(f)$.

For two schemes X, Y of finite type over S we define the group $c(X, Y)$ of *finite correspondences* from X to Y as $c(XY/X)$.

Let us recall the following construction from [2, §3.7]. Let $X' \to X \to S$ be morphisms of finite type, $\mathcal{W}$ a relative cycle on X' over X and $\mathcal{Z}$ a relative cycle on X over S. Then one defines a cycle $Cor(\mathcal{W}, \mathcal{Z})$ on X' as follows. Let Z_i be the components of the support of $\mathcal{Z}$ present with multiplicites n_i and $e_i : Z_i \to X$ the corresponding closed embeddings. Let $e'_i : Z_i \times_X X' \to X'$ denote the projections. We set

$$Cor(\mathcal{W}, \mathcal{Z}) = \sum_i n_i (e'_i)_* cycl(e_i)(\mathcal{W})$$

where $(e'_i)_*$ is the (proper) push-forward on cycles.
Let X, Y be schemes of finite type over S and

$$f \in c(X, Y) = c(XY/X)$$

$$g \in c(Y, Z) = c(YZ/Y)$$

finite correspondences. Let

$$p_X : XY \to Y$$

$$p_Y : XYZ \to XZ$$

be the projections. We define the composition $g \circ f$ by the formula:

$$g \circ f = (p_Y)_* Cor(cycl(p_X)(g), f) \tag{2.1}$$

This operation is linear in both arguments and thus defines a homomorphism of abelian groups

$$c(X, Y) \otimes c(Y, Z) \to c(X, Z)$$

The lemma below follows immediately from the definition of $Cor(-, -)$ and the fact that the (proper) push-forward commutes with the $cycl(-)$ homomorphisms ([2, Prop. 3.6.2]).

Lemma 2.1 *Let $Y \to X \to S$ be a sequence of morphisms of finite type, $p : Y \to Y'$ a morphism over X, $\mathcal{Y} \in Cycl(Y/X, r) \otimes \mathbf{Q}$ and $\mathcal{X} \in Cycl(X/S, s) \otimes \mathbf{Q}$. Assume that p is proper on the support of $\mathcal{Y}$. Then*

$$p_* Cor(\mathcal{Y}, \mathcal{X}) = Cor(p_*(\mathcal{Y}, \mathcal{X})).$$

Lemma 2.2 *For any $f \in c(X, Y)$, $g \in c(Y, Z)$, $h \in c(Z, T)$ one has*

$$(h \circ g) \circ f = h \circ (g \circ f).$$

Proof: Consider the following diagram

$$\begin{tikzcd}
XT & XYT \arrow[l, "4"'] \arrow[r] & YT & & \\
XZT \arrow[u, "7"] \arrow[d] & XYZT \arrow[l, "9"'] \arrow[r] \arrow[u, "8"] \arrow[d] & YZT \arrow[r] \arrow[u, "2"] \arrow[d] & ZT \arrow[r] \arrow[d] & T \\
XZ & XYZ \arrow[l, "5"'] \arrow[r, "9"] \arrow[d] & YZ \arrow[r, "1"] \arrow[d] & Z & \\
 & XY \arrow[r, "3"] \arrow[d] & Y & & \\
 & X & & &
\end{tikzcd}$$

where the morphisms are the obvious projections. Note that all the squares are cartesian. We will also use the projection $6 : XZ \to Z$.
We have $f \in c(XY/X)$, $g \in c(YZ/Y)$ and $h \in c(ZT/Z)$. The compositions are given by:

$$g \circ f = 5_* Cor(cycl(3)(g), f)$$

$$h \circ g = 2_* Cor(cycl(1)(h), g)$$

$$(h \circ g) \circ f = 4_* Cor(cycl(3)(h \circ g), f) = 4_* Cor(cycl(3)(2_* Cor(cycl(1)(h), g)), f)$$

$$h \circ (g \circ f) = 7_* Cor(cycl(6)(h), g \circ f) = 7_* Cor(cycl(6)(h), 5_* Cor(cycl(3)(g), f))$$

We have:

$$\begin{aligned}
&4_* Cor(cycl(3)(2_* Cor(cycl(1)(h), g)), f) = \\
&= 4_* Cor(8_* cycl(3) Cor(cycl(1)(h), g), f) = \\
&= 4_* 8_* Cor(cycl(3) Cor(cycl(1)(h), g), f) = \\
&= 4_* 8_* Cor(Cor(cycl(1 \circ 9)(h), cycl(3)(g)), f)
\end{aligned}$$

where the first equality holds by [2, Prop. 3.6.2], the second by Lemma 2.1 and the third by [2, Th. 3.7.3]. We also have:

$$\begin{aligned}
&7_* Cor(cycl(6)(h), 5_* Cor(cycl(3)(g), f)) = \\
&= 7_* 9_* Cor(cycl(6 \circ 5)(h), Cor(cycl(3)(g), f))
\end{aligned}$$

by [2, Lemma 3.7.1]. We conclude that $(h \circ g) \circ f = h \circ (g \circ f)$ by [2, Prop. 3.7.7].

We denote by $Cor(S)$ the category of finite correspondences whose objects are schemes of finite type over S, morphisms are finite correspondences and the composition of morphisms is defined by (2.1).

For a morphism of schemes $f : X \to Y$ let Γ_f be its graph considered as an element of $c(XY/X)$. One verifies easily that $\Gamma_{gf} = \Gamma_g \circ \Gamma_f$ and we get a functor $Sch/S \to Cor(S)$. Below we use the same symbol for a morphism of schemes and its graph considered as a finite correspondence.
The external product of cycles defines pairings

$$c(X,Y) \otimes c(X',Y') \to c(XX',YY')$$

and one verifies easily using the results of [2] that this pairing extends to a tensor structure on $Cor(S)$ with $X \otimes Y := XY$.

3 Intersecting relative cycles with divisors

Let X be a noetherian scheme and D a Cartier divisor on X i.e. a global section of the sheaf $\mathcal{M}^*/\mathcal{O}^*$. One defines the cycle $cycl(D)$ associated with D as follows. Let U_i be an open covering of X such that D_{U_i} is of the form $f_{i,+}/f_{i,-} \in \mathcal{M}^*(U_i)$. Then $cycl(D)$ is determined by the property that

$$cycl(D)_{|U_i} = cycl(f_{i,+}^{-1}(0)) - cycl(f_{i,-}^{-1}(0))$$

where on the right hand side one considers the cycles associated with closed subschemes ([2,]). One defines the support of D as the closed subset $supp(D) := supp(cycl(D))$.
We say that a cycle $\mathcal{Z} = \sum n_i z_i$ on X intersects D properly if the points z_i do not belong to $supp(D)$. Let Z_i be the closure of z_i considered as a reduced closed subscheme and $e_i : Z_i \to X$ the closed embedding. If $\mathcal{Z}$ and D intersect properly we define their intersection $(\mathcal{Z}, D)$ as the cycle

$$(\mathcal{Z}, D) := \sum n_i (e_i)_*(cycl(e_i^*(D)))$$

If $p : X \to S$ is a morphism of finite type and $\mathcal{Z}$ is a relative cycle of relative dimension d over S, we say that D intersects $\mathcal{Z}$ properly relative to p (or properly over S) if the dimension of fibers of $supp(D) \cap supp(\mathcal{Z})$ over S is $\leq d-1$. This clearly implies that $\mathcal{Z}$ intersects D properly and $(\mathcal{Z}, D)$ is defined.

Proposition 3.1 *Let $p : X \to S$ be a morphism of finite type, $\mathcal{Z}$ a relative equidimensional cycle of relative dimension d on X over S and D a Cartier divisor on X which intersects $\mathcal{Z}$ properly over S. Then:*

1. *$(\mathcal{Z}, D)$ is a relative cycle of relative dimension $d-1$ over S,*
2. *let $f : S' \to S$ be a morphism, $X' = (X \times_S S')_{red}$ and let $q_{red} : X' \to X$ be the restriction of the projection to X'. If $q_{red}^*(D)$ is well defined then*

$$f^*(cycl(Z), D) = (f^*(cycl(Z)), q_{red}^*(D)). \tag{3.1}$$

where f^* refers to the pull-back of relative cycles as defined in [2].

Proof: Let $\mathcal{Z} = \sum_i n_i z_i$ where z_i are points on the generic fibers of p and $n_i \neq 0$. As usually we denote by $[z_i]$ the reduced closed subschemes with generic points z_i.
Since our problem is local in the Zariski topology on X and additive in D we may assume that $D = D(f)$ where $f \in \mathcal{O}(X)$ is a function on X which is not zero divisor. The condition that D intersects $\mathcal{Z}$ properly over S is equivalent to the condition that for each i and each point y of S the restriction of f to $([z_i] \times_S Spec(k_y))_{red}$ is not a zero divisor. Localizing around $[z_i]$ we may assume that the restriction of f to $(X \times_S Spec(k_y))_{red}$ is not a zero divisor for any y. Under these assumptions $q^*_{red}(D)$ is well defined for any $f : S' \to S$. The proposition follows now from Lemma 3.2.

Lemma 3.2 *Let Z be an integral scheme, S a reduced scheme, $p : Z \to S$ an equidimensional morphism and $Spec(k) \stackrel{s_0}{\to} Spec(R) \stackrel{s_1}{\to} S$ a fat point over a point $s : Spec(k) \to S$ of S (see [2, p.23]). Let $Z_s = Z \times_S Spec(k)$ and let $q : Z_s \to Z$ be the projection. Let $f \in \mathcal{O}(Z)$ be a function such that the image of f in $\mathcal{O}(Z_s)_{red}$ is not a zero divisor. Then*

$$(s_0, s_1)^*(D(f)) = ((s_0, s_1)^*(\eta), f \circ q_{red}) \tag{3.2}$$

where η is the generic point of Z considered as a cycle on Z and $q_{red} : Z_{s,red} \to Z$ is the restriction of q to the maximal reduced subscheme of Z_s.

Proof: Observe first the cycles on both sides of (3.2) are supported in points of codimension 1 of Z_s. Let z be such a point. We want to show that the multiplicities of the left and right hand sides of (3.2) in z coincide.
To compute $(s_0, s_1)^*(\eta)$ one considers the surjection $\psi : \mathcal{O}_{Z_R} \to H$ such that $ker(\psi)$ is supported in the closed fiber of $Z_R \to Spec(R)$ and H is flat over R. Let p_j be the minimal prime ideals of $\mathcal{O}_{Z_s}$ and $A_i = \mathcal{O}_{Z_s}/p_i$. Then by definition (see [2, Lemma 3.1.2]),

$$(s_0, s_1)^*(\eta) = \sum_j length_{A_j}(q_0^*(H) \otimes A_j) p_j$$

Therefore, for a point z of codimension 1 on Z_s we have

$$mlt_z(((s_0, s_1)^*(\eta), f \circ q_{red})) =$$

$$= \sum_j length_{A_j}(q_0^*(H) \otimes A_j) length_{\mathcal{O}_{Z_s,z}}((A_j/f_j) \otimes \mathcal{O}_{Z_s,z})$$

where f_j is the restriction of $f \circ q_{red}$ to $[p_j]$.
Let $F = \mathcal{O}_Z/f\mathcal{O}_Z$. We have $D(f) = \sum_i length_{\mathcal{O}_{Z,y_i}}(F \otimes \mathcal{O}_{Z,y_i}) y_i$ where y_i are the generic points of the scheme $Y = f^{-1}(0)$. Let $F_i = F \otimes \mathcal{O}_{[y_i]}$. By definition, we have

$$(s_0, s_1)^*(D(f)) = \sum_i length_{\mathcal{O}_{Z,y_i}}(F \otimes \mathcal{O}_{Z,y_i})\, Cycl(q_0^*(G_i)).$$

where G_i is a quotient of $q_1^*(F_i)$ which is flat over R and such that the kernel of the projection $\phi_i : q_1^*(F_i) \to G_i$ is supported in the closed fiber of $Z_R \to Spec(R)$. Our conditions imply that this cycle is supported in points of codimension 1 of Z_s and for such a point z the multiplicity of $(s_0, s_1)^*(D(f))$ in z equals

$$mlt_z((s_0,s_1)^*(D(f))) =$$
$$\sum_i length_{\mathcal{O}_{Z,y_i}}(F \otimes \mathcal{O}_{Z,y_i})\, length_{\mathcal{O}_{Z_s,z}}(q_0^*(G_i) \otimes \mathcal{O}_{Z_s,z}) \tag{3.3}$$

Let $K_0^\vee(Z_s)$ be the Grothendieck group of the bounded derived category of complexes of coherent sheaves Z_s whose cohomology are supported in codimension ≥ 1. Then the formula

$$l_{Z_s,z}(M) = length_{\mathcal{O}_{Z_s,z}}(M \otimes \mathcal{O}_{Z_s,z})$$

defines an additive functional on this group and we need to show that

$$l_{Z_s,z}(\sum_i length_{\mathcal{O}_{Z,y_i}}(F \otimes \mathcal{O}_{Z,y_i})\, q_0^*(G_i)) =$$
$$= l_{Z_s,z}(\sum_j length_{A_j}(q_0^*(H) \otimes A_j) A_j/f_j)$$

Let f_s be the image of f in $\mathcal{O}_{Z_s}$ and let $K_s = cone(\mathcal{O}_{Z_s} \xrightarrow{\cdot f_s} \mathcal{O}_{Z_s})$. Since f_j are not zero divisors, we have $A_j/f_j = A_j \otimes_{\mathbf{L}} K$ and the additivity of length implies that $l_{Z_s,z}(M \otimes_{\mathbf{L}} K_s)$ is zero on any M which is supported in codimension ≥ 1. Since this condition holds for the difference $q_0^*(H) - (\sum_j length_{A_j}(q_0^*(H) \otimes A_j) A_j)$ we conclude that

$$l_{Z_s,z}(\sum_j length_{A_j}(q_0^*(H) \otimes A_j) A_j/f_j) = l_{Z_s,z}(q_0^*(H) \otimes_{\mathbf{L}} K_s) =$$
$$= l_{Z_s,z}(\mathbf{L}q_0^*(cone(H \xrightarrow{f} H)) = l_{Z_s,z}(cone(q_0^*(H) \xrightarrow{f} q_0^*(H))) \tag{3.4}$$

Let u be a generator of the maximal ideal of R. Then $ker(\phi_i)$ and $ker(\psi)$ are just the u-torsion elements in $q_1^*(F_i)$ and $\mathcal{O}_{Z_R}$ respectively. In particular, G_i are H-modules i.e. $G_i = G_i \otimes H$. Therefore, both (3.3) and (3.4) are zero if z does not belong to $W_s = Spec(q_0^*(H)) \subset Z_s$ and for $z \in W_s$ we have

$$mlt_z((s_0,s_1)^*(D(f))) = l_{W_s,z}(\sum_i length_{\mathcal{O}_{Z,y_i}}(F \otimes \mathcal{O}_{Z,y_i})\, \mathbf{L}q_W^*(h^*(G_i)))$$

and

$$mlt_z(((s_0,s_1)^*(\eta), f \circ q_{red})) = l_{W_s,z}(\mathbf{L}q_W^*(cone(H \xrightarrow{f} H)))$$

where $q_W : W_s \to Spec(H)$ and $h : Spec(H) \to Spec(Z_R)$ are the obvious morphisms. We claim that the difference

$$M = cone(H \xrightarrow{f} H) - (\sum_i length_{\mathcal{O}_{Z,y_i}}(F \otimes \mathcal{O}_{Z,y_i})\, h^*(G_i))$$

as an element of K_0 of H-modules is supported in points of $Spec(H)$ of codimension at ≥ 2 and therefore

$$l_{W_s,z}(\mathbf{L}q_W^*(M)) = 0$$

by Lemma 3.4. Indeed, both sides are zero in the generic points of the generic and of the closed fiber. The restriction of f to the generic fiber Z_K of Z_R is not a zero divisor since the map $q_K : Z_K \to Z$ is flat (because S is reduced) and since Z is integral f is not a zero divisor in $\mathcal{O}_Z$. Therefore, the generic fiber of $cone(H \stackrel{f}{\to} H)$ coincides with $q_K^*(F)$ which, as an element of K_0, coincides with $\sum_i length_{\mathcal{O}_{Z,y_i}}(F \otimes \mathcal{O}_{Z,y_i})\, q_K^*(F_i)$ up to codimension ≥ 2.

Lemma 3.3 *Let $p : W \to Spec(R)$ be a flat morphism such that R is a discrete valuation ring, let $s : Spec(k) \to Spec(R)$ be a morphism whose image is the closed point of $Spec(R)$, $W_s = W \times_{Spec(R)} Spec(k)$ and let $q_W : W_s \to W$ be the projection. Let further M be a coherent sheaf on W supported in the closed fiber of p. Then*

$$\mathbf{L}q_W^*(M) \cong q_W^*(M) \oplus q_W^*(M)[1]$$

Proof: Let $s = is'$ be the factorization of s where $i : Spec(R/m) \to Spec(R)$ is the closed embedding and $s' : Spec(k) \to Spec(R/m)$ a flat morphism and let $q_W = q_i'q'$ be the corresponding factorization of q_W. Then it is sufficient to show that $\mathbf{L}q_i^*(M) \cong q_i^*(M) \oplus q_i^*(M)[1]$. Since $(q_i)_*$ is an exact full embedding it is further sufficient to show that $(q_i)_*\mathbf{L}q_i^*(M) \cong (q_i)_*q_i^*(M) \oplus (q_i)_*q_i^*(M)[1]$. The functor $(q_i)_*q_i^*$ is isomorphic to the functor $(-)\otimes B$ where $B = \mathcal{O}_W/p^*(m)$. Therefore, $(q_i)_*\mathbf{L}q_i^*$ is isomorphic to the functor $(-)\otimes_{\mathbf{F}} B$. Since R is a discrete valuation ring m is a principal ideal. Let u be a generator of m. Since p is flat the image of u in $\mathcal{O}_W$ is not a zero divisor. Therefore

$$(-) \otimes_{\mathbf{L}} B = cone((-) \stackrel{u}{\to} (-))$$

If M is supported in the closed fiber of p then $M \otimes B = M$ and the multiplication by u on M equals zero.

Lemma 3.4 *Under the assumptions of Lemma 3.3 let M be a coherent sheaf on W supported in codimension ≥ 2 and let w be a point of codimension 1 on W_s. Then*

$$length_{\mathcal{O}_{W_s,w}}(\mathbf{L}q_W^*(M) \otimes \mathcal{O}_{W_s,w}) = 0 \tag{3.5}$$

Proof: It is sufficient to show that (3.5) holds for $M = \mathcal{O}_W/p$ where p is a prime ideal of codimension ≥ 2. There are two types of prime ideals satisfying this condition - the ideals lying over the generic point and the ideals lying over the closed point. If p lies over the generic point and has codimension ≥ 2 then the closed fiber of the corresponding closed subscheme has codimension at least 2 and $\mathbf{L}q_W^*(M) \otimes \mathcal{O}_{W_s,w} = 0$ since w is of codimension 1.

If p lies in the closed fiber an has codimension ≥ 1 there then $q_W^*(M)$ has finite length in w and (3.5) follows by additivity of length from Lemma 3.3.

Corollary 3.5 *Let $X' \xrightarrow{f} X \to S$ be morphisms of finite type, $\mathcal{Z}$ a relative cycle on X over S and $\mathcal{W}$ a relative cycle on X' over X of dimension* 0. *Let further D be a Cartier divisor on X' which intersects $\mathcal{W}$ properly over X. Then D intersects $Cor(\mathcal{W}, \mathcal{Z})$ properly over S and one has:*

$$(Cor(\mathcal{W}, \mathcal{Z}), D) = Cor((\mathcal{W}, D), \mathcal{Z}) \tag{3.6}$$

Proof: It is a straightforward corollary of the definition of $Cor(-,-)$ and (3.1).

Lemma 3.6 *Let $f : X' \to X$ be a morphism of schemes of finite type over S, $\mathcal{Z}$ a relative cycle on X' such that f is proper on $supp(\mathcal{Z})$ and D a Cartier divisor on X. Assume that $f^*(D)$ is defined and $\mathcal{Z}$ intersects $f^*(D)$ properly over S. Then $f_*(\mathcal{Z})$ intersects D properly over S and one has:*

$$f_*(\mathcal{Z}, f^*(D)) = (f_*(\mathcal{Z}), D) \tag{3.7}$$

Proof: Let d be the relative dimension of $\mathcal{Z}$ over S. To see that $f_*(\mathcal{Z})$ intersects D properly over S we need to check that the dimension of the fibers of $supp(D) \cap supp(f_*(\mathcal{Z}))$ over S is $\leq d-1$. This follows from our assumption and the inclusion

$$supp(D) \cap supp(f_*(\mathcal{Z})) \subset supp(D) \cap f(supp(\mathcal{Z})) =$$

$$= f(f^{-1}(supp(D)) \cap supp(\mathcal{Z})) = f(supp(f^*(D)) \cap supp(\mathcal{Z}))$$

To verify (3.7) it is sufficient to consider the situation locally around the generic points of $f(supp(f^*(D)) \cap supp(\mathcal{Z}))$. Therefore we may assume that $D = D(g)$ is the divisor of a regular function g and $\mathcal{Z} = z$ is just one point with the closure Z. Replacing X' by Z and X by $f(Z)$ we may assume that X, X' are integral, f is surjective and X is local of dimension 1. Let $A = \mathcal{O}(X)$, $B = \mathcal{O}(X')$. Consider the function $l_g : M \mapsto l_A(M \otimes^L A/g)$ on $K_0(A - mod)$. This function vanishes on modules with the support in the closed point which implies that

$$l_g(B) = deg(f) l_g(A) = deg(f) l_A(A/g)$$

On the other hand $l_g(A) = l_A(B/(f^*(g)))$. Let x'_i be the closed points of X', k'_i their residue fields and k the residue field of the closed point of X. Let further M_i be the part of $B/(f^*(g))$ supported in x'_i. One can easily see that $l_A(B/(f^*(g))) = \sum_i [k'_i : k] l_B(M_i)$. Combining our equalities we get:

$$deg(f) l_A(A/g) = \sum_i [k'_i : k] l_B(M_i) \tag{3.8}$$

which is equivalent to (3.7).

4 Cancellation theorem

Consider a finite correspondence

$$\mathcal{Z} \in c(\mathbf{G}_m X, \mathbf{G}_m Y) = c(\mathbf{G}_m X \mathbf{G}_m Y / \mathbf{G}_m X).$$

Let f_1, f_2 be the projections to the first and the second copy of $\mathbf{G}_m$ respectively and let g_n denote the rational function $(f_1^{n+1} - 1)/(f_1^{n+1} - f_2)$ on $\mathbf{G}_m X \mathbf{G}_m Y$.

Lemma 4.1 *For any $\mathcal{Z}$ there exists N such that for all $n \geq N$ the divisor of g_n intersects $\mathcal{Z}$ properly over X and the cycle $(\mathcal{Z}, D(g_n))$ is finite over X.*

Proof: Let $\bar{f}_1 \times \bar{q} : \bar{C} \to \mathbf{P}^1 X$ be a finite morphism which extends the projection $supp(\mathcal{Z}) \to \mathbf{G}_m X$. Let N be an integer such that the rational function $\bar{f}_1^N / f_2$ is regular in a neighborhood of $\bar{f}_1^{-1}(0)$ and the rational function $f_2/\bar{f}_1^N$ is regular in a neighborhood of $\bar{f}_1^{-1}(\infty)$. Then for any $n \geq N$ one has:

1. the restriction of $g_n f_2$ to $supp(\mathcal{Z})$ is regular on a neighborhood of $\bar{f}_1^{-1}(0)$ and equals 1 on $\bar{f}_1^{-1}(0)$
2. the restriction of g_n to $supp(\mathcal{Z})$ is regular a neighborhood of $\bar{f}_1^{-1}(\infty)$ and equals 1 on $\bar{f}_1^{-1}(\infty)$

Conditions (1),(2) imply that the divisor of g_n intersects $\mathcal{Z}$ properly over X and that the relative cycle $(\mathcal{Z}, D(g_n))$ is finite over X.

If $(\mathcal{Z}, D(g_n))$ is defined as a finite relative cycle we let $\rho_n(\mathcal{Z}) \in c(X, Y)$ denote the projection of $(\mathcal{Z}, D(g_n))$ to XY.

Remark 4.2 Note that we can define a finite correspondence $\rho_g(\mathcal{Z}) : X \to Y$ for any function g satisfying the conditions (1),(2) in the same way as we defined $\rho_n = \rho_{g_n}$. In particular, if n and m are large enough then the function $tg_n + (1-t)g_m$ defines a finite correspondence $h = h_{n,m} : X\mathbf{A}^1 \to Y$ such that $h_{|X \times \{0\}} = \rho_m(\mathcal{Z})$ and $h_{|X \times \{1\}} = \rho_n(\mathcal{Z})$, i.e. we get a canonical $\mathbf{A}^1$-homotopy from $\rho_m(\mathcal{Z})$ to $\rho_n(\mathcal{Z})$.

Lemma 4.3 *(i) For a finite correspondence $\mathcal{W} : X \to Y$ and any $n \geq 1$ one has $\rho_n(Id_{\mathbf{G}_m} \otimes \mathcal{W}) = \mathcal{W}$*

(ii) Let e_X be the composition $\mathbf{G}_m X \xrightarrow{pr} X \xrightarrow{\{1\} \times Id} \mathbf{G}_m X$. Then $\rho_n(e_X) = 0$ for any $n \geq 0$.

Proof: The cycle on $\mathbf{G}_m X \mathbf{G}_m Y$ over $\mathbf{G}_m X$ which represents $Id_{\mathbf{G}_m} \otimes \mathcal{W}$ is $\Delta_*(\mathbf{G}_m \times \mathcal{W})$ where Δ is the diagonal embedding $\mathbf{G}_m XY \to \mathbf{G}_m X \mathbf{G}_m Y$. The cycle $(\Delta_*(\mathbf{G}_m \times \mathcal{W}), g_n)$ is $\Delta_*(D \otimes \mathcal{W})$ where D is the divisor of the function $(t^{n+1} - 1)/(t^{n+1} - t)$ on $\mathbf{G}_m$. The push-forward of $\Delta_*(D \otimes \mathcal{W})$ to XY is the cycle $deg(D)\mathcal{W}$. Since $deg(D) = 1$ we get the first statement of the lemma.
The cycle $\mathcal{Z}$ on $\mathbf{G}_m X \mathbf{G}_m X$ representing e_X is the image of the embedding $\mathbf{G}_m X \to \mathbf{G}_m X \mathbf{G}_m X$ which is diagonal on X and of the form $t \mapsto (t, 1)$ on $\mathbf{G}_m$. This shows that the restriction of g_n to $supp(\mathcal{Z})$ equals 1 and $(\mathcal{Z}, D(g_n)) = 0$.

Lemma 4.4 *Let* $\mathcal{Z} : \mathbf{G}_m X \to \mathbf{G}_m Y$ *be a finite correspondence such that* $\rho_n(\mathcal{Z})$ *is defined. Then for any finite correspondence* $\mathcal{W} : X' \to X$, $\rho_n(\mathcal{Z} \circ (Id_{\mathbf{G}_m} \otimes \mathcal{W}))$ *is defined and one has*

$$\rho_n(\mathcal{Z} \circ (Id_{\mathbf{G}_m} \otimes \mathcal{W})) = \rho_n(\mathcal{Z}) \circ \mathcal{W} \tag{4.1}$$

Proof: Let us show that (4.1) holds. In the process it will become clear that the left hand side is defined. We can write $\rho_n(\mathcal{Z}) \circ \mathcal{W}$ as the composition

$$X' \xrightarrow{\mathcal{W}} X \xrightarrow{(\mathcal{Z}, D(g_n))} \mathbf{G}_m \mathbf{G}_m Y \xrightarrow{pr} Y$$

and $\rho_n(\mathcal{Z} \circ (Id_{\mathbf{G}_m} \otimes \mathcal{W}))$ as the composition

$$X' \xrightarrow{\mathcal{Y}} \mathbf{G}_m \mathbf{G}_m Y \xrightarrow{pr} Y$$

where $\mathcal{Y} = (\mathcal{Z} \circ (Id_{\mathbf{G}_m} \otimes \mathcal{W}), D(g_n))$. Consider the diagram

$$\begin{tikzcd}
\mathbf{G}_m X' \mathbf{G}_m Y & \mathbf{G}_m X' X \mathbf{G}_m Y \arrow[l, "p_1"'] \arrow[r] \arrow[d] & \mathbf{G}_m X \mathbf{G}_m Y \arrow[d] \\
& X' X \arrow[r, "p_2"] \arrow[d] & X \\
& X' &
\end{tikzcd}$$

where the arrows are the obvious projections. If we consider $\mathcal{Z}$ as a cycle of dimension 1 over X then the cycle $\mathcal{Z} \circ (Id_{\mathbf{G}_m} \otimes \mathcal{W})$, considered as a cycle over X', is $(p_1)_* Cor(cycl(p_2)(\mathcal{Z}), \mathcal{W})$ and we have

$$((p_1)_* Cor(cycl(p_2)(\mathcal{Z}), \mathcal{W}), D(g_n)) =$$

$$= (p_1)_*(Cor(cycl(p_2)(\mathcal{Z}), \mathcal{W}), D(g_n)) = (p_1)_* Cor((cycl(p_2)(\mathcal{Z}), D(g_n)), \mathcal{W}) =$$
$$= (p_1)_* Cor(cycl(p_2)(\mathcal{Z}, D(g_n)), \mathcal{W})$$

where the first equality holds by (3.7), the second by (3.6) and the third by (3.1).

The last expression represents the composition $\mathcal{W} \circ (\mathcal{Z}, D(g_n))$ and we conclude that

$$\rho_n(\mathcal{Z}) \circ \mathcal{W} = \rho_n(\mathcal{Z} \circ (Id_{\mathbf{G}_m} \otimes \mathcal{W}))$$

Lemma 4.5 *Let* $\mathcal{Z} : \mathbf{G}_m X \to \mathbf{G}_m Y$ *be a finite correspondence such that* $\rho_n(\mathcal{Z})$ *is defined. Then for any morphism of schemes* $f : X' \to Y'$, $\rho_n(\mathcal{Z} \otimes f)$ *is defined and one has*

$$\rho_n(\mathcal{Z} \otimes f) = \rho_n(\mathcal{Z}) \otimes f \tag{4.2}$$

Proof: Consider the diagram

$$\begin{array}{ccccc} \mathbf{G}_m XX'\mathbf{G}_m YY' & \xleftarrow{p_1} & \mathbf{G}_m XX'\mathbf{G}_m Y & \longrightarrow & \mathbf{G}_m X\mathbf{G}_m Y \\ & & \downarrow & & \downarrow \\ & & XX' & \xrightarrow{p_2} & X \end{array}$$

where p_1 is defined by the embedding $X' \xrightarrow{f\times Id} X'Y'$ and the rest of the morphisms are the obvious projections. Consider $\mathcal{Z}$ as a cycle over X. Then $\rho_n(\mathcal{Z}\otimes f)$ is given by the composition

$$\mathbf{G}_m XX' \xrightarrow{\mathcal{Y}_1} \mathbf{G}_m\mathbf{G}_m Y \xrightarrow{pr} YY'$$

where $\mathcal{Y}_1 = ((p_1)_* cycl(p_2)(\mathcal{Z}), g_n)$ and $\rho_n(\mathcal{Z})\otimes f$ by the composition

$$\mathbf{G}_m XX' \xrightarrow{\mathcal{Y}_2} \mathbf{G}_m\mathbf{G}_m Y \xrightarrow{pr} YY'$$

where $\mathcal{Y}_2 = (p_1)_*(cycl(p_2)((\mathcal{Z}, g_n)))$. The equality $\mathcal{Y}_1 = \mathcal{Y}_2$ follows from (3.7) and (3.1).

For our next result we need to use presheaves with transfers. A presheaf with transfers on Sch/S is an additive contravariant functor from $Cor(S)$ to the category of abelian groups. For X in Sch/S we let $\mathbf{Z}_{tr}(X)$ denote the functor represented by X on $Cor(S)$. One defines tensor product of presheaves with transfers in the usual way such that $\mathbf{Z}_{tr}(X)\otimes\mathbf{Z}_{tr}(Y) = \mathbf{Z}_{tr}(X\times Y)$. To simplify notations we will write X instead of $\mathbf{Z}_{tr}(X)$ and identify morphisms $\mathbf{Z}_{tr}(X)\to\mathbf{Z}_{tr}(Y)$ with finite correspondences $X\to Y$. Note in particular that $\mathbf{G}_m$ denotes the presheaf with transfers $\mathbf{Z}_{tr}(\mathbf{G}_m)$ not the presheaf with transfers represented by $\mathbf{G}_m$ as a scheme. To preserve compatibility with the notation XY for the product of X and Y we write FG for the tensor product of presheaves with transfers F and G.
Let S^1_t denote the presheaf with transfers $ker(\mathbf{G}_m\to S)$. We consider it as a direct summand of $\mathbf{G}_m$ with respect to the projection $Id-e$ where e is defined by the composition $\mathbf{G}_m\to S\xrightarrow{1}\mathbf{G}_m$. In the following theorem we let $f\cong g$ denote that the morphisms f and g are $\mathbf{A}^1$-homotopic.

Theorem 4.6 *Let F be a presheaf with transfers such that there is an epimorphism $X\to F$ for a scheme X. Let $\phi: S^1_t\otimes F\to S^1_t Y$ be a morphism. Then there exists a unique up to an $\mathbf{A}^1$-homotopy morphism $\rho(\phi): F\to Y$ such that $Id_{S^1_t}\otimes\rho(\phi)\cong\phi$.*

Proof: Let us fix an epimorphism $p: X\to F$. Then the morphism ϕ defines a finite correspondence $\mathcal{Z}:\mathbf{G}_m X\to\mathbf{G}_m Y$ and for n sufficiently large we may consider $\rho_n(\mathcal{Z}): X\to Y$. Lemma 4.4 implies immediately that $\rho_n(\mathcal{Z})$ vanishes on $ker(p)$ and therefore it defines a morphism $\rho_n(\phi): F\to X$.

Consider a morphism ϕ of the form $Id_{S^1_t} \otimes \psi$. Then $\mathcal{Z}$ is of the form $(Id_{\mathbf{G}_m} - e) \otimes \mathcal{W}$ where $\mathcal{W} : X \to Y$ corresponds to ψ. By Lemma 4.3 we have $\rho_n(\mathcal{Z}) = \mathcal{W}$ and therefore $\rho_n(Id_{S^1_t} \otimes \psi) = \psi$ for any $n \geq 1$. If ρ, ρ' are two morphims such that $Id_{S^1_t} \otimes \rho \cong \phi$ and $Id_{S^1_t} \otimes \rho' \cong \phi$ then for a sufficiently large n we have

$$\rho = \rho_n(Id_{S^1_t} \otimes \rho) \cong \rho_n(Id_{S^1_t} \otimes \rho') = \rho'$$

This implies the uniqueness part of the theorem.
To prove the existence let us show that for a sufficiently large n one has $Id_{S^1_t} \otimes \rho_n(\phi) \cong \phi$. Let $\widetilde{\phi}$ be the morphism $\mathbf{G}_m F \to \mathbf{G}_m Y$ defined by ϕ and let

$$\widetilde{\phi}^* : F\mathbf{G}_m \to Y\mathbf{G}_m$$

be the morphism obtained from $\widetilde{\phi}$ by the obvious permutation.

Lemma 4.7 *The morphisms $\widetilde{\phi} \otimes (Id_{\mathbf{G}_m} - e)$ and $(Id_{\mathbf{G}_m} - e) \otimes \widetilde{\phi^*}$ are $\mathbf{A}^1$-homotopic.*

Proof: One can easily see that these two morphisms are obtained from the morphisms

$$\phi \otimes Id_{S^1_t}, Id_{S^1_t} \otimes \phi^* : S^1_t F S^1_t \to S^1_t Y S^1_t$$

by using the standard direct sum decomposition. One can see further that $\phi \otimes Id_{S^1_t} = \sigma_Y(Id_{S^1_t} \otimes \phi^*)\sigma_F$ where σ_F and σ_Y are the permutations of the two copies of S^1_t in $S^1_t F S^1_t$ and $S^1_t Y S^1_t$ respectively. Lemma 4.8 below implies now that $\phi \otimes Id_{S^1_t} \cong Id_{S^1_t} \otimes \phi^*$.

Lemma 4.8 *The permutation on $S^1_t S^1_t$ is $\mathbf{A}^1$-homotopic to $\{-1\}Id \otimes Id$ where $\{-1\} : S^1_t \to S^1_t$ is defined by the morphism $\mathbf{G}_m \xrightarrow{x \mapsto x^{-1}} \mathbf{G}_m$.*

Proof: The same arguments as the ones used in [1, p.142] show that for any scheme X and any pair of invertible functions f, g on X the morphism $X \xrightarrow{f \otimes g} S^1_t S^1_t$ is $\mathbf{A}^1$-homotopic to the morphism $g \otimes f^{-1}$. This implies immediately that the permutation on $S^1_t S^1_t$ is $\mathbf{A}^1$-homotopic to the morphism $Id \otimes (\{-1\}Id)$ where $\{-1\}Id : S^1_t \to S^1_t$ is the morphism defined by the map $\mathbf{G}_m \xrightarrow{x \mapsto x^{-1}} \mathbf{G}_m$.

For a sufficiently large n we have

$$\rho_n(\phi \otimes (Id_{\mathbf{G}_m} - e)) = \rho_n(\phi) \otimes (Id_{\mathbf{G}_m} - e)$$

by Lemma 4.5. On the other hand

$$\rho_n((Id_{\mathbf{G}_m} - e) \otimes \phi^*) = \phi^*$$

by Lemma 4.3. By Lemma 4.7 we conclude that

$$\phi^* \cong \rho_n(\phi) \otimes (Id_{\mathbf{G}_m} - e)$$

which is equivalent to $Id_{S^1_t} \otimes \rho_n(\phi) \cong \phi$. Theorem 4.6 is proved.

Corollary 4.9 *Denote by F_Y the presheaf*

$$X \mapsto Hom(S^1_t X, S^1_t Y)$$

and consider the obvious map $Y \to F_Y$. Then for any X the corresponding map of complexes of abelian groups

$$C_*(Y)(X) \to C_*(F_Y)(X)$$

is a quasi-isomorphism

Proof: Let $\Delta^n \cong \mathbf{A}^n$ be the standard algebraic simplex and $\partial\Delta^n$ the sub-presheaf in Δ^n which is the union of the images of the face maps $\Delta^{n-1} \to \Delta^n$. Then the n-th homology group of the complex $C_*(F)(X)$ for any F is the group of homotopy classes of maps from $X \otimes (\Delta^n/\partial\Delta^n)$ to F. Our result now follows directly from 4.6.

Corollary 4.10 *Let k be a perfect field. Then for any K, L in $DM^{eff}_-(k)$ the map $Hom(K, L) \to Hom(K(1), L(1))$ is a bijection.*

Proof: Since DM^{eff}_- is generated by objects of the form X it is enough to check that for smooth schemes X, Y over k and $n \in \mathbf{Z}$ one has

$$Hom(S^1_t X, S^1_t Y[n]) = Hom(X, Y[n])$$

By Corollary 4.9 we know that the map

$$Y \to F_Y = \underline{Hom}(S^1_t, S^1_t Y)$$

is an isomorphism in DM. Let us show now that for any sheaf with transfers F and any X one has

$$Hom_{DM}(S^1_t X, F[n]) = Hom_{DM}(X, \underline{Hom}(S^1_t, F)[n]) \tag{4.3}$$

The left hand side of (4.3) is the hypercohomology group $\mathbf{H}^n(\mathbf{G}_m X, C_*(F))$ modulo the subgroup $\mathbf{H}^n(X, C_*(F))$. The right hand side is the hypercohomology group $\mathbf{H}^n(X, C_*\underline{Hom}(\mathbf{G}_m, F))$ modulo similar subgroup. Let $p : \mathbf{G}_m X \to X$ be the projection. It is easy to see that (4.3) asserts that $\mathbf{R}p_*(C_*(F)) \cong C_*(p_*(F))$. There is a spectral sequence which converges to the cohomology sheaves of $\mathbf{R}p_*(C_*(F))$ and starts with the higher direct images $R^i p_*(\underline{H}^j(C_*(F)))$. We need to verify that $R^i p_*(\underline{H}^j(C_*(F))) = 0$ for $i > 0$ and that $p_*(\underline{H}^j(C_*(F))) = \underline{H}^j(C_*(p_*(F)))$. Both statements follow from [3, Prop. 4.34, p.124] and the comparison of Zariski and Nisnevich cohomology for homotopy invariant presheaves with transfers.

References

[1] Andrei Suslin and Vladimir Voevodsky. Bloch-Kato conjecture and motivic cohomology with finite coefficients. In *The arithmetic and geometry of algebraic cycles (Banff, AB, 1998)*, volume 548 of *NATO Sci. Ser. C Math. Phys. Sci.*, pages 117–189. Kluwer Acad. Publ., Dordrecht, 2000.

[2] Andrei Suslin and Vladimir Voevodsky. Relative cycles and Chow sheaves. In *Cycles, transfers, and motivic homology theories*, volume 143 of *Ann. of Math. Stud.*, pages 10–86. Princeton Univ. Press, Princeton, NJ, 2000.

[3] Vladimir Voevodsky. Cohomological theory of presheaves with transfers. In *Cycles, transfers, and motivic homology theories*, volume 143 of *Ann. of Math. Stud.*, pages 87–137. Princeton Univ. Press, Princeton, NJ, 2000.

[4] Vladimir Voevodsky. Triangulated categories of motives over a field. In *Cycles, transfers, and motivic homology theories*, volume 143 of *Ann. of Math. Stud.*, pages 188–238. Princeton Univ. Press, Princeton, NJ, 2000.

[5] Vladimir Voevodsky. Motivic cohomology groups are isomorphic to higher Chow groups in any characteristic. *Int. Math. Res. Not.*, (7):351–355, 2002.

Vladimir Voevodsky
School of Mathematics
Institute for Advanced Study
Princeton NJ 08540
USA
vladimir@math.ias.edu

Bounds for the Dimensions of p-Adic Multiple L-Value Spaces

Dedicated to Professor Andrei Suslin

on the occasion of his 60th birthday

Go Yamashita[1]

Received: January 13, 2009
Revised: October 12, 2009

Abstract. First, we will define p-adic multiple L-values (p-adic MLV's), which are generalizations of Furusho's p-adic multiple zeta values (p-adic MZV's) in Section 2.

Next, we prove bounds for the dimensions of p-adic MLV-spaces in Section 3, assuming results in Section 4, and make a conjecture about a special element in the motivic Galois group of the category of mixed Tate motives, which is a p-adic analogue of Grothendieck's conjecture about a special element in the motivic Galois group. The bounds come from the rank of K-groups of ring of S-integers of cyclotomic fields, and these are p-adic analogues of Goncharov-Terasoma's bounds for the dimensions of (complex) MZV-spaces and Deligne-Goncharov's bounds for the dimensions of (complex) MLV-spaces. In the case of p-adic MLV-spaces, the gap between the dimensions and the bounds is related to spaces of modular forms similarly as the complex case.

In Section 4, we define the crystalline realization of mixed Tate motives and show a comparison isomorphism, by using p-adic Hodge theory.

2010 Mathematics Subject Classification: Primary 11R42; Secondary 11G55, 14F42, 14F30.
Keywords and Phrases: p-adic multiple zeta values, mixed Tate motives, algebraic K-theory, p-adic Hodge theory.

[1]Partially supported by JSPS Research Fellowships for Young Scientists. The author also thanks the financial support by the 21st Century COE Program in Kyoto University "Formation of an international center of excellence in the frontiers of mathematics and fostering of researchers in future generations", and EPSRC grant EP/E049109/1.
Supported by JSPS Research Fellowships for Young Scientists.

Contents

1 Introduction.

For the multiple zeta values (MZV's)

$$\zeta(k_1,\dots,k_d) := \sum_{n_1<\cdots<n_d} \frac{1}{n_1^{k_1}\cdots n_d^{k_d}} \left(= \lim_{\mathbb{C}\ni z\to 1} \mathrm{Li}_{k_1,\dots,k_d}(z)\right)$$

$(k_1,\dots,k_{d-1}\geq 1,\ k_d\geq 2)$, Zagier conjectures the dimension of the space of MZV's

$$Z_w := \langle \zeta(k_1,\dots,k_d) \mid d\geq 1, k_1+\cdots+k_d=w, k_1,\dots,k_{d-1}\geq 1, k_d\geq 2\rangle_{\mathbb{Q}} \subset \mathbb{R},$$

and $Z_0 := \mathbb{Q}$ (Here, $\langle\cdots\rangle_{\mathbb{Q}}$ means the $\mathbb{Q}$-vector space spanned by $\cdots$) as follows.

Conjecture 1 (Zagier) Let $D_{n+3} = D_{n+1} + D_n$, $D_0 = 1$, $D_1 = 0$, $D_2 = 1$ (that is, the generating function $\sum_{n=0}^{\infty} D_n t^n$ is $\dfrac{1}{1-t^2-t^3}$). Then, for $w\geq 0$ we have

$$\dim_{\mathbb{Q}} Z_w = D_w.$$

Terasoma, Goncharov, and Deligne-Goncharov proved the upper bound:

Theorem 1.1 *(Terasoma [T], Goncharov [G1], Deligne-Goncharov [DG]) For $w\geq 0$, we have*

$$\dim_{\mathbb{Q}} Z_w \leq D_w.$$

Deligne-Goncharov also proved an upper bound for dimensions of multiple L-value (MLV) spaces. ([DG])
On the other hand, Furusho defined p-adic MZV's [Fu1] by using Coleman's iterated integral theory:

$$\zeta_p(k_1,\dots,k_d) := \lim_{\mathbb{C}_p\ni z\to 1}{}' \mathrm{Li}^a_{k_1,\dots,k_d}(z).$$

where Li^a is the p-adic multiple polylogarithm defined by Coleman's iterated integral, and a is a branching parameter (For the notations $\lim'$, see [Fu1, Notation 2.12]). For $k_d \geq 2$, RHS converges, and the limit value is independent of a and lands in $\mathbb{Q}_p$ ([Fu1, Theorem 2.13, 2.18, 2.25]). Put

$$Z^p_w := \langle \zeta_p(k_1,\dots,k_d) \mid d\geq 1, k_1+\cdots+k_d = w, k_1,\dots,k_{d-1}\geq 1, k_d\geq 2\rangle_{\mathbb{Q}} \subset \mathbb{Q}_p,$$

and $Z^p_0 := \mathbb{Q}$. Note that for $k_d = 1$, p-adic MZV's may converge, however, these are $\mathbb{Q}$-linear combinations of p-adic MZV's corresponding to the same weight indices with $k_d \geq 2$ (See, [Fu1, Theorem 2.22]). The following conjecture is proposed.

Conjecture 2 (Furusho-Y.) Let $d_{n+3} = d_{n+1} + d_n$, $d_0 = 1$, $d_1 = 0$, $d_2 = 0$ (that is, the generating function $\sum_{n=0}^{\infty} d_n t^n$ is $\dfrac{1-t^2}{1-t^2-t^3}$). Then, for $w \geq 0$ we have

$$\dim_{\mathbb{Q}} Z^p_w = d_w.$$

From the fact $\zeta_p(2) = 0$ and the motivic point of views (see, Remark 3.7, p-adic analogue of Grothendieck's conjecture about an element of a motivic Galois group (Conjecture 4), and Proposition 3.12), it seems natural to conjecture as above.

Remark 1.2 The conjecture implies that $\dim_{\mathbb{Q}} Z^p_w$ is independent of p. On the other hand, $\zeta_p(2k+1) \neq 0$ is equivalent to the higher Leopoldt conjecture in the Iwasawa theory. For a regular prime p, or a prime p satisfying $(p-1) \mid 2k$, we have $\zeta_p(2k+1) \neq 0$. However, it is not known if $\zeta_p(2k+1)$ is zero or not in general. Thus, it is non-trivial that $\dim_{\mathbb{Q}} Z^p_w$ is independent of p (See also [Fu1, Example 2.19 (b)]). It seems that the above conjecture contains the "Leopoldt conjecture for higher depth".

For Conjecture 2, we will prove the following result.

Theorem 1.3 *For $w \geq 0$, we have*

$$\dim_{\mathbb{Q}} Z^p_w \leq d_w.$$

We can also define p-adic multiple L-values for N-th roots of unity $\zeta_1,\dots,\zeta_d$ and $k_1,\dots,k_d \geq 1$, $(k_d,\zeta_d) \neq (1,1)$ and a prime ideal $\mathfrak{p} \nmid N$ above p in the cyclotomoic field $\mathbb{Q}(\mu_N)$,

$$L_{\mathfrak{p}}(k_1,\dots,k_d;\zeta_1,\dots,\zeta_d) \in \mathbb{Q}(\mu_N)_{\mathfrak{p}},$$

by Coleman's iterated integral as Furusho did for MZV's (See, Section 2.1). Here, $\mathbb{Q}(\mu_N)_{\mathfrak{p}}$ is the completion of $\mathbb{Q}(\mu_N)$ at the finite place $\mathfrak{p}$. Put

$$\begin{aligned}Z^{\mathfrak{p}}_w[N] :=&\langle L_{\mathfrak{p}}(k_1,\ldots,k_d;\zeta_1,\ldots,\zeta_d) \mid d \geq 1, k_1+\cdots+k_d = w, k_1,\ldots,k_d \geq 1,\\ &\zeta_1^N = \cdots = \zeta_d^N = 1, (k_d,\zeta_d) \neq (1,1)\rangle_{\mathbb{Q}} \subset \mathbb{Q}(\mu_N)_{\mathfrak{p}},\end{aligned}$$

$$\text{and } Z^{\mathfrak{p}}_0[N] := \mathbb{Q}.$$

This $Z^p_w[1]$ is equal to the above Z^p_w. We will also prove bounds for the dimensions of p-adic MLV's.

Theorem 1.4 *For $w \geq 0$, we have*

$$\dim_{\mathbb{Q}} Z^{\mathfrak{p}}_w[N] \leq d[N]_w.$$

Here, $d[N]_w$ is defined as follows:

1. *For $N = 1$, $d[1]_{n+3} = d[1]_{n+1} + d[1]_n$ $(n \geq 0)$, $d[1]_0 = 1$, $d[1]_1 = 0$, $d[1]_2 = 0$, that is, the generating function is $\dfrac{1-t^2}{1-t^2-t^3}$ (This $d[1]_n$ is equal to the above d_n).*
2. *For $N = 2$, $d[2]_{n+2} = d[2]_{n+1} + d[2]_n$ $(n \geq 1)$, $d[2]_0 = 1$, $d[2]_1 = 1$, $d[2]_2 = 1$, that is, the generating function is $\dfrac{1-t^2}{1-t-t^2}$.*
3. *For $N \geq 3$, $d[N]_{n+2} = \left(\frac{\varphi(N)}{2}+\nu\right) d[N]_{n+1} - (\nu-1)d[N]_n$ $(n \geq 0)$, $d[N]_0 = 1$, $d[N]_1 = \frac{\varphi(N)}{2} + \nu - 1$, that is, the generating function is $\dfrac{1-t}{1-\left(\frac{\varphi(N)}{2}+\nu\right)t+(\nu-1)t^2}$. Here, $\varphi(N) := \#(\mathbb{Z}/N\mathbb{Z})^\times$, and ν is the number of prime divisors of N.*

Remark 1.5 It is not known that $\dim_{\mathbb{Q}} Z^{\mathfrak{p}}_w[N]$ is independent of p.

Remark 1.6 In the proof of the above bounds, we use some kinds of (pro-)varieties, which are related to the algebraic K-theory. For $N > 4$, the above bounds are not best possible in general, because in the proof, we use smaller varieties in general than varieties, which give the above bounds. The gap of dimensions is related to the space of cusp forms of weight 2 on $X_1(N)$ if N is a prime. See also [DG, 5.27][G2].

In the proof of the above theorem, we use a special element in motivic Galois group of the category of mixed Tate motives like in the complex case ([DG]). We also propose a p-adic analogue of Grothendieck's conjecture on this special element (see Section 3 for the details):

Conjecture 3 (= Conjecture 4 in Section 3, p-adic analogue of Grothendieck's conjecture) The element $\varphi_{\mathfrak{p}} \in U_\omega(\mathbb{Q}(\mu_N)_{\mathfrak{p}})$ is $\mathbb{Q}$-Zariski dense. That means that if a subvariety X of U_ω over $\mathbb{Q}$ satisfies $\varphi_{\mathfrak{p}} \in X(\mathbb{Q}(\mu_N)_{\mathfrak{p}})$, then $X = U_\omega$.

Finally, we will give the plan of this paper. First, we define the p-adic MLV's, twisted p-adic multiple polylogarithms (twisted p-adic MPL's), and p-adic Drinfel'd associator for twisted p-adic MPL's in Section 2. Next, assuming results of Section 4, we will show bounds for dimensions of p-adic MLV-spaces in the sense of Deligne [D1][DG], by using the motivic fundamental groupoid constructed in [DG] in Section 3.2. Lastly, we show bounds for dimensions of Furusho's p-adic MLV-spaces, by comparing the two p-adic MLV-spaces in the Tannakian interpretation in Section 3.3. In Section 4, we construct the crystalline realization of mixed Tate motives, and prove a comparison isomorphism, by using p-adic Hodge theory. In the end of this article, we propose some questions.

We fix conventions. We use the notation $\gamma'\gamma$ for a composition of paths, which means that γ followed by γ'. Similarly, we use the notation $g'g$ for a product of elements in a motivic Galois group, which means that the action of g followed by the one of g'.

Acknowledgement. In the proof of the main theorem, a crucial ingredient is the algebraic K-theory, the area of mathematics to which Professor Andrei A. Suslin greatly contributed. It is great pleasure for the author to dedicate this paper to Professor Andrei A. Suslin.

He sincerely thanks to Hidekazu Furusho for introducing to the author the theory of multiple zeta values and the theory of Grothendieck-Teichmüller group, and for helpful discussions. He also expresses his gratitude to Professor Pierre Deligne for helpful discussions for the crystalline realization. The last chapter of this paper is written during the author's staying at IHES from January/2006 to July/2006. He also thanks to the hospitality of IHES. Finally, he thanks the referee for kind comments.

2 p-adic Multiple L-values.

In this section, we define twisted p-adic multiple polylogarithms (twisted p-adic MPL), p-adic multiple L-values (p-adic MLV), p-adic KZ-equation for twisted p-adic MPL, and p-adic Drinfel'd associator for twisted p-adic MPL, similarly as Furusho's definitions in [Fu1]. We discuss the fundamental properties of them.

Fix a prime ideal $\mathfrak{p}$ in $\mathbb{Q}(\mu_N)$, and an embedding $\iota_{\mathfrak{p}} : \mathbb{Q}(\mu_N) \hookrightarrow \mathbb{C}_p$. Put $S := \{0, \infty\} \cup \mu_N$, $\mathbb{U}_N := \mathbb{P}^1_{\mathbb{Q}(\mu_N)} \setminus S$, and $\overline{\mathbb{U}_N} := \mathbb{U}_N \otimes_{\mathbb{Q}(\mu_N)} \mathbb{C}_p$ (The variety $\mathbb{U}_N$ is defined over $\mathbb{Q}$, however, we use $\mathbb{U}_N$ over $\mathbb{Q}(\mu_N)$ for the purpose of bounding dimensions in the next section).

2.1 The Twisted p-adic Multiple Polylogarithm.

We use the same notations as in [Fu1]: the tube $]x[\subset \mathbb{P}^1_{\mathbb{C}_p}$ of $x \in (\mathbb{U}_N)_{\mathbb{F}_p}(\overline{\mathbb{F}_p})$, the algebra $A(U)$ of rigid analytic functions on U, and the algebra A^a_{Col} of Coleman functions on $\overline{\mathbb{U}_N}$ with a branching parameter a.

Definition 2.1 *For $\mathfrak{p} \nmid N$, $k_1, \dots, k_d \geq 1$, and $\zeta_1 \dots, \zeta_d \in \mu_N$, we define the (one variable) twisted p-adic multiple polylogarithm (twisted p-adic MPL) $\mathrm{Li}^a_{(k_1,\dots,k_d;\zeta_1,\dots,\zeta_d)}(z) \in A^a_{\mathrm{Col}}$ attached to $a \in \mathbb{C}_p$ by the following integrals inductively:*

$$\mathrm{Li}^a_{(1;\zeta_1)}(z) := -\log^a(\iota_{\mathfrak{p}}(\zeta_1) - z) := \int_0^z \frac{dt}{\iota_{\mathfrak{p}}(\zeta_1) - t},$$

$$\mathrm{Li}^a_{(k_1,\dots,k_d;\zeta_1,\dots,\zeta_d)}(z) := \begin{cases} \displaystyle\int_0^z \frac{1}{t}\mathrm{Li}^a_{(k_1,\dots,(k_d-1);\zeta_1,\dots,\zeta_d)}(t)dt & k_d \neq 1, \\ \displaystyle\int_0^z \frac{1}{\iota_{\mathfrak{p}}(\zeta_d) - t}\mathrm{Li}^a_{(k_1,\dots,k_{(d-1)};\zeta_1,\dots,\zeta_{d-1})}(t)dt & k_d = 1. \end{cases}$$

Here, $\log^a$ is the logarithm with a branching parameter a, which means $\log^a(p) = a$.

Remark 2.2 For $|z|_p < 1$, it is easy to see that

$$\mathrm{Li}^a_{(k_1,\dots,k_d;\zeta_1,\dots,\zeta_d)}(z) = \sum_{0<n_1<\cdots<n_d} \frac{\iota_{\mathfrak{p}}(\zeta_1^{-n_1}\zeta_2^{n_1-n_2}\cdots\zeta_d^{n_{d-1}-n_d})z^{n_d}}{n_1^{k_1}\cdots n_d^{k_d}}.$$

Inductively, we can easily verify that $\mathrm{Li}^a_{(k_1,\dots,k_d;\zeta_1,\dots,\zeta_d)}(z)|_{]0[} \in A(]0[)$, $\mathrm{Li}^a_{(k_1,\dots,k_d;\zeta_1,\dots,\zeta_d)}(z)|_{]\infty[} \in A(]\infty[)[\log^a t^{-1}]$, and $\mathrm{Li}^a_{(k_1,\dots,k_d;\zeta_1,\dots,\zeta_d)}(z)|_{]\iota_{\mathfrak{p}}(\zeta)[} \in A(]\iota_{\mathfrak{p}}(\zeta)[)[\log^a(z - \iota_{\mathfrak{p}}(\zeta))]$ for $\zeta \in \mu_N$.

Proposition 2.3 *Fix $k_1, \dots, k_d \geq 1$, and N-th roots of unity $\zeta_1, \dots, \zeta_d \in \mu_N$. Then the convergence of $\lim_{\mathbb{C}_p \ni z \to 1}{}' \mathrm{Li}^a_{(k_1,\dots,k_d;\zeta_1,\dots,\zeta_d)}(z)$ is independent of branches $a \in \mathbb{C}_p$. Moreover, if it converges in $\mathbb{C}_p$, the limit value is independent of branches $a \in \mathbb{C}_p$ and lands in $\mathbb{Q}(\mu_N)_{\mathfrak{p}}$ (For the notation $\lim'$, see [Fu1, Notation 2.12]).*

Proof The same as [Fu1, Theorem 2.13, Theorem 2.25].

Definition 2.4 *When the limit $\lim'_{\mathbb{C}_p \ni z \to 1} \mathrm{Li}^a_{(k_1,\dots,k_d;\zeta_1,\dots,\zeta_d)}(z)$ converges, we define the corresponding p-adic multiple L-value to be its limit value:*

$$L_{\mathfrak{p}}(k_1, \dots, k_d; \zeta_1, \dots, \zeta_d) := \lim_{\mathbb{C}_p \ni z \to 1}{}' \mathrm{Li}^a_{(k_1,\dots,k_d;\zeta_1,\dots,\zeta_d)}(z)$$

For example, $L_{\mathfrak{p}}(1;\zeta) = -\log^a(\iota_{\mathfrak{p}}(\zeta) - 1)$ $(1 \neq \zeta \in \mu_N)$ is independent of a, since $\log^a(z)$ does not depend on a for $|z| = 1$. (Recall that we assume $\mathfrak{p} \nmid N$.)

2.2 The p-adic Drinfel'd Associator for Twisted p-adic Multiple Polylogarithms.

Let $A^{\wedge}_{\mathbb{C}_p} := \mathbb{C}_p\langle\langle A, B_\zeta \mid \zeta \in \mu_N\rangle\rangle$ be the non-commutative formal power series ring with $\mathbb{C}_p$ coefficients generated by variables A and B_ζ for $\zeta \in \mu_N$. For a word W consisting of A and $\{B_\zeta\}_{\zeta\in\mu_N}$, we call the sum of all exponents of A and $\{B_\zeta\}_{\zeta\in\mu_N}$ the weight of W, and the sum of all exponents of $\{B_\zeta\}_{\zeta\in\mu_N}$ the depth of W.

Definition 2.5 *Fix a prime ideal $\mathfrak{p}$ above p in $\mathbb{Q}(\mu_N)$ and an embedding $\iota_\mathfrak{p} : \mathbb{Q}(\mu_N) \hookrightarrow \mathbb{C}_p$. The p-adic Knizhnik-Zamolodchikov equation (p-adic KZ-equation) is the differential equation*

$$\frac{dG}{dz}(z) = \left(\frac{A}{z} + \sum_{\zeta\in\mu_N} \frac{B_\zeta}{z - \iota_\mathfrak{p}(\zeta)}\right) G(z),$$

where $G(z)$ is an analytic function in variable $z \in \overline{\mathbb{U}_N}$ with values in $A^{\wedge}_{\mathbb{C}_p}$. Here, $G = \sum_W G_W(z)W$ is 'analytic' means each of whose coefficient $G_W(z)$ is locally p-adically analytic.

Proposition 2.6 *Fix $a \in \mathbb{C}_p$. Then, there exist unique solutions $G_0^a(z), G_1^a(z) \in A^a_{\mathrm{Col}}\widehat{\otimes}A^{\wedge}_{\mathbb{C}_p}$, which are locally analytic on $\mathbb{P}^1(\mathbb{C}_p) \setminus S$ and satisfy $G_0^a(z) \approx z^A$ $(z \to 0)$, and $G_1^a(z) \approx (1-z)^{B_1}$ $(z \to 1)$.*

Here, the notations u^A means $\sum_{n=0}^\infty \frac{1}{n!}(A\log^a u)^n$. Note that it depends on a. For the notations $G_0^a(z) \approx z^A$ $(z \to 0)$, see [Fu1, Theorem 3.4].

Remark 2.7 We do not have the symmetry $z \mapsto 1-z$ on $\overline{\mathbb{U}_N}$. Thus, we do not have a simple relation between $G_0^a(z)$ and $G_1^a(z)$ as in [Fu1, Proposition 3.8]. On the other hand, we have the symmetry $z \mapsto z^{-1}$ on $\overline{\mathbb{U}_N}$. Thus, we have a unique locally analytic solution $G_\infty^a(z)$ with $G_\infty^a(z) \approx (z^{-1})^{-A-\sum_{\zeta\in\mu_N} B_\zeta}$ $(z \to \infty)$, and have a relation

$$G_\infty^a(A, \{B_\zeta\}_{\zeta\in\mu_N})(z) = G_0^a(-A - \sum_{\zeta\in\mu_N} B_\zeta, \{B_{\zeta^{-1}}\}_{\zeta\in\mu_N})(z^{-1}).$$

However, when we define a Drinfel'd associator by using G_0^a and G_∞^a similarly as below (Definition 2.8), there appears

$$\lim_{\mathbb{C}_p\ni z\to\infty} {}'\mathrm{Li}^a_{(k_1,\dots,k_d;\zeta_1,\dots,\zeta_d)}(z)$$

in the coefficient of that Drinfel'd associator. What we want is $\lim_{\mathbb{C}_p\ni z\to 1}{}'$. Thus, we use the boundary condition at $z = 1$.

Proof The uniqueness is easy. In [Fu1], he cites Drinfel'd's paper [Dr] for the existence of a solution of the KZ-equation. Here, we give an alternative proof of the existence without using the quasi-triangular quasi-Hopf algebra theory and the quasi-tensor category theory. In fact, we put $G_0^a(z)$ to be $\sum_W (-1)^{\mathrm{depth}(W)}\mathrm{Li}_W^a(z)W$. Here, for a word W, we define $\mathrm{Li}_W^a(z)$ inductively as following: $\mathrm{Li}_{A^n}^a(z) := \frac{1}{n!}(\log^a z)^n$, $\mathrm{Li}_{AW}^a(z) := \int_0^z \frac{1}{t}\mathrm{Li}_W^a(t)dt$, for $W \neq A^n$ $(n \geq 0)$, $\mathrm{Li}_{B_\zeta W}^a(z) := \int_0^z \frac{1}{\iota_{\mathfrak{p}}(\zeta)-t}\mathrm{Li}_W^a(t)dt$, for $\zeta \in \mu_N$. It is easy to verify that $\sum_W (-1)^{\mathrm{depth}(W)}\mathrm{Li}_W^a(z)W$ satisfies the p-adic KZ-equation. As for the boundary condition $G_0^a(z) \approx z^A$ $(z \to 0)$, it is easy to show that

$$\sum_{W:W\neq W'A,W'\neq\emptyset} (-1)^{\mathrm{depth}(W)}\mathrm{Li}_W^a(z)W$$

satisfies the above boundary condition.
Thus, it remains to show that $\mathrm{Li}_{W'A^n}^a(z) \to 0$ $(z \to 0)$ for $n > 0$, $W' \neq \emptyset$. For $\mathrm{Li}_{B_\zeta A^n}^a$,

$$\mathrm{Li}_{B_\zeta A^n}^a(z) = \int_0^z \frac{1}{\iota_{\mathfrak{p}}(\zeta)-t}\mathrm{Li}_{A^n}^a(t)dt = \frac{1}{n!}\int_0^z \zeta^{-1}\sum_{k=0}^{\infty}(\zeta^{-1}t)^k(\log^a t)^n dt,$$

in $|z| < 1$. Since $\int_0^z t^k \log^a t dt = \frac{z^{k+1}}{k+1}\log^a z - \frac{z^{k+1}}{(k+1)^2}$, we have $\int_0^z t^k \log^a t dt \to 0$ $(z \to 0)$. Inductively, we have $\int_0^z t^k(\log^a t)^n dt \to 0$ $(z \to 0)$. Thus, we showed $\mathrm{Li}_{B_\zeta A^n}^a(z) \to 0$ $(z \to 0)$. For general $\mathrm{Li}_{W'A}^a(z)$'s, we can inductively show $\mathrm{Li}_{W'A}^a(z) \to 0$ $(z \to 0)$ by using the following fact for $f(z) = \mathrm{Li}_{**}^a(z)$: For a locally analytic function $f(z)$ satisfying $f(0) = 0$, we have $\int_0^z \frac{1}{t}f(t)dt \to 0$ $(z \to 0)$, $\int_0^z \frac{1}{\iota_{\mathfrak{p}}(\zeta)-t}f(t)dt \to 0$ $(z \to 0)$.
As for $G_1^a(z)$, the same argument works, by replacing $\mathrm{Li}_{A^n}^a(z) := \frac{1}{n!}(\log^a z)^n$ by $\mathrm{Li}_{B_1^n}^a(z) := \frac{1}{n!}(\log^a(1-z))^n$, and $\int_0^z$ by $\int_1^z$.

Definition 2.8 *We define the p-adic Drinfel'd associator for twisted p-adic multiple polylogarithms to be* $\Phi_{\mathrm{KZ}}^{\mathfrak{p}}(A, \{B_\zeta\}_{\zeta\in\mu_N}) := G_1^a(z)^{-1}G_0^a(z)$. *It is in* $A_{\mathbb{C}_p}^{\wedge} = \mathbb{C}_p\langle\langle A, \{B_\zeta\}_{\zeta\in\mu_N}\rangle\rangle$, *and independent of a by the same argument in [Fu1, Remark 3.9, Theorem 3.10].*

By the same arguments as in [Fu1], we can show the following propositions.

Proposition 2.9 $\lim_{\mathbb{C}_p\ni z\to 1}{}'\mathrm{Li}_{(k_1,\ldots,k_d;\zeta_1,\ldots,\zeta_d)}^a(z)$ *converges when* $(k_d, \zeta_d) \neq (1, 1)$.

Proof See, [Fu1, Theorem 2.18] for the case where $N = 1$.

For W in $A \cdot A_{\mathbb{C}_p}^{\wedge} \cdot B_\zeta$ or $B_{\zeta'} \cdot A_{\mathbb{C}_p}^{\wedge} \cdot B_\zeta$ $(\zeta' \neq 1)$, we define $L_{\mathfrak{p}}(W)$ to be $\lim_{\mathbb{C}_p\ni z\to 1}{}'\mathrm{Li}_W^a(z)$.

Proposition 2.10 *(Explicit Formulae) The coefficient $I_{\mathfrak{p}}(W)$ of W in the p-adic Drinfel'd associator for twisted p-adic MPL's is the following: When W is written as $B_1^r V A^s$ for $(r,s \geq 0)$, V is in $A \cdot A^{\wedge}_{\mathbb{C}_p} \cdot B_\zeta$ or $B_{\zeta'} \cdot A^{\wedge}_{\mathbb{C}_p} \cdot B_\zeta$ $(\zeta' \neq 1)$,*

$$I_{\mathfrak{p}}(W) = (-1)^{\mathrm{depth}(W)}(-1)^{a+b} \sum_{0 \leq a \leq r, 0 \leq b \leq s} L_{\mathfrak{p}}(f(B_1^a \circ B_1^{r-a} V A^{s-b} \circ A^b)).$$

In particular, when W is in $A \cdot A^{\wedge}_{\mathbb{C}_p} \cdot B_\zeta$ or $B_{\zeta'} \cdot A^{\wedge}_{\mathbb{C}_p} \cdot B_\zeta$ $(\zeta' \neq 1)$, $I_{\mathfrak{p}}(W) = (-1)^{\mathrm{depth}(W)} L_{\mathfrak{p}}(W)$. Here, $f : A^{\wedge}_{\mathbb{C}_p} \to A^{\wedge}_{\mathbb{C}_p}$ is the composition of $A^{\wedge}_{\mathbb{C}_p} \twoheadrightarrow A^{\wedge}_{\mathbb{C}_p}/(B_1 \cdot A^{\wedge}_{\mathbb{C}_p} + A^{\wedge}_{\mathbb{C}_p} \cdot A)$, $A^{\wedge}_{\mathbb{C}_p}/(B_1 \cdot A^{\wedge}_{\mathbb{C}_p} + A^{\wedge}_{\mathbb{C}_p} \cdot A) \xrightarrow{\sim} \mathbb{C}_p \cdot 1 + A \cdot A^{\wedge}_{\mathbb{C}_p} \cdot B_1$, and $\mathbb{C}_p \cdot 1 + A \cdot A^{\wedge}_{\mathbb{C}_p} \cdot B_1 \hookrightarrow A^{\wedge}_{\mathbb{C}_p}$.

For the definition of the shuffle product $\circ$, see [Fu0, Definition 3.2.2].

Proof See, [Fu1, Theorem 3.28] for the case where $N = 1$. Note we use $G_i^a(A - \alpha, B_1 - \beta, \{B_\zeta\}_{\zeta \in \mu_N, \zeta \neq 1})(z) = z^{-\alpha}(1-z)^{-\beta} G_i^a(A, \{B_\zeta\}_{\zeta \in \mu_N})(z)$ for $i = 0, 1$.

Proposition 2.11 *Suppose $\lim_{\mathbb{C}_p \in z \to 1} {}'\mathrm{Li}^a_{(k_1,\dots,k_{d-1},1;\zeta_1,\dots,\zeta_{d-1},1)}(z)$ converges. Then, the limit value is a p-adic regularized MLV, that is, $L_{\mathfrak{p}}(k_1,\dots,k_{d-1},1;\zeta_1,\dots,\zeta_{d-1},1) = (-1)^{\mathrm{depth}(W)} I_{\mathfrak{p}}(W)$. In particular, $L_{\mathfrak{p}}(k_1,\dots,k_{d-1},1;\zeta_1,\dots,\zeta_{d-1},1)$ can be written as a $\mathbb{Q}$-linear combination of p-adic MLV's corresponding to the same weight indices with $(k_d,\zeta_d) \neq (1,1)$.*

Proof See, [Fu1, Theorem 2.22] for the case where $N = 1$.

Definition 2.12 *We define the p-adic multiple L-value space of weight w $Z^{\mathfrak{p}}_w[N]$ to be the finite dimensional $\mathbb{Q}$-linear subspace of $\mathbb{Q}(\mu_N)_{\mathfrak{p}}$ generated by the all p-adic MLV's of indices of weight w, $\zeta_1^N = \cdots = \zeta_d^N = 1$. Put $Z^{\mathfrak{p}}_0[N] := \mathbb{Q}$. We define $Z^{\mathfrak{p}}_\bullet[N]$ to be the formal direct sum of $Z^{\mathfrak{p}}_w[N]$ for $w \geq 0$.*

Remark 2.13 By Proposition 2.11, we see that

$$\begin{aligned} Z^{\mathfrak{p}}_w[N] := & \langle L_{\mathfrak{p}}(k_1,\dots,k_d;\zeta_1,\dots,\zeta_d) \mid d \geq 1, k_1 + \cdots + k_d = w, k_1,\dots,k_d \geq 1, \\ & \zeta_1^N = \cdots = \zeta_d^N = 1, (k_d,\zeta_d) \neq (1,1) \rangle_{\mathbb{Q}} \\ & = \langle I_{\mathfrak{p}}(W) \mid \text{the weight of } W \text{ is } w \rangle_{\mathbb{Q}} \subset \mathbb{Q}(\mu_N)_{\mathfrak{p}}. \end{aligned}$$

Proposition 2.14 *We have $\Delta(\Phi^{\mathfrak{p}}_{\mathrm{KZ}}) = \Phi^{\mathfrak{p}}_{\mathrm{KZ}} \widehat{\otimes} \Phi^{\mathfrak{p}}_{\mathrm{KZ}}$. In particular, the graded $\mathbb{Q}$-vector space $Z^{\mathfrak{p}}_\bullet[N]$ has a $\mathbb{Q}$-algebra structure, that is, $Z^{\mathfrak{p}}_a[N] \cdot Z^{\mathfrak{p}}_b[N] \subset Z^{\mathfrak{p}}_{a+b}[N]$ for $a, b \geq 0$.*

Proof See, [Fu1, Proposition 3.39, Theorem 2.28] for the case where $N = 1$.

Proposition 2.15 *(Shuffle Product Formulae) For* $W, W' \in (A \cdot A^{\wedge}_{\mathbb{C}_p} \cdot B_\zeta) \cup \cup_{\zeta' \neq 1} (B_{\zeta'} \cdot A^{\wedge}_{\mathbb{C}_p} \cdot B_\zeta)$*, we have*

$$L_{\mathfrak{p}}(W \circ W') = L_{\mathfrak{p}}(W) L_{\mathfrak{p}}(W').$$

Proof This follows from Proposition 2.10 and Proposition 2.14. See, [Fu1, Corollary 3.42] for the case where $N = 1$.

3 Bounds for Dimensions of p-adic Multiple L-value spaces.

In this section, we show Theorem 1.4, by the method of Deligne-Goncharov [DG], assuming results of Section 4. First, we recall some facts about the motivic fundamental groupoids in [DG]. Next, we show that bounds for dimensions of p-adic MLV-spaces in the sense of Deligne [D1][DG]. Lastly, we show that p-adic MLV-spaces in the previous section is equal to p-adic MLV-spaces in the sense of Deligne by the Tannakian interpretations.

3.1 The Motivic Fundamental Groupoids of $\mathbb{U}_N$.

Deligne-Goncharov constructed the category $\mathrm{MT}(\mathbb{Z}[\mu_N, \{\frac{1}{1-\zeta_w}\}_{w|N}])$ of mixed Tate motives over $\mathbb{Z}[\mu_N, \{\frac{1}{1-\zeta_w}\}_{w|N}]$, the fundamental $\mathrm{MT}(\mathbb{Z}[\mu_N, \{\frac{1}{1-\zeta_w}\}_{w|N}])$-group $\pi_1^{\mathcal{M}}(\mathbb{U}_N, x)$ and the fundamental $\mathrm{MT}(\mathbb{Z}[\mu_N, \{\frac{1}{1-\zeta_w}\}_{w|N}])$-groupoid $P^{\mathcal{M}}_{y,x}$ for $\mathbb{U}_N$ not only for rational base points x, y, but also for tangential base points x, y [DG, Theorem 4.4, Proposition 5.11]. Here, $w \mid N$ runs through primes w dividing N, and ζ_w is a w-th root of unity (Since $\mathbb{U}_N$ is defined over $\mathbb{Q}$, $\pi_1^{\mathcal{M}}(\mathbb{U}_N, x)$, $P^{\mathcal{M}}_{y,x}$ are also $\mathrm{MAT}(\mathbb{Q}(\mu_N)/\mathbb{Q})$-schemes. However, we do not use this fact. Here, $\mathrm{MAT}(\mathbb{Q}(\mu_N)/\mathbb{Q})$ is the category of mixed Artin-Tate motives for $\mathbb{Q}(\mu_N)/\mathbb{Q})$. For $\mathcal{T}$-schemes, $\mathcal{T}$-group schemes, and $\mathcal{T}$-groupoids for a Tannakian category $\mathcal{T}$, see [D1, §5, §6], [D2, 7.8], and [DG, 2.6].
First, we recall some facts about them. Let

$$G := \pi_1(\mathrm{MT}(\mathbb{Z}[\mu_N, \{\frac{1}{1-\zeta_w}\}_{w|N}])) \in \text{pro-}\mathrm{MT}(\mathbb{Z}[\mu_N, \{\frac{1}{1-\zeta_w}\}_{w|N}])$$

be the fundamental $\mathrm{MT}(\mathbb{Z}[\mu_N, \{\frac{1}{1-\zeta_w}\}_{w|N}])$-group [D1, §6][D2, Definition 8.13]. Then, by its action on $\mathbb{Q}(1)$, we have a surjection $G \twoheadrightarrow \mathbb{G}_m$ (Here, we regard $\mathbb{G}_m$ as an $\mathrm{MT}(\mathbb{Z}[\mu_N, \{\frac{1}{1-\zeta_w}\}_{w|N}])$-group). The kernel U of the map $G \to \mathbb{G}_m$ is a pro-unipotent group. Then, we have an isomorphism [DG, 2.8.2]:

$$\begin{aligned}\mathrm{Lie}(U^{\mathrm{ab}}) \cong \prod_n \mathrm{Ext}^1_{\mathrm{MT}(\mathbb{Z}[\mu_N, \{\frac{1}{1-\zeta_w}\}_{w|N}])}(\mathbb{Q}(0), \mathbb{Q}(n))^\vee \otimes \mathbb{Q}(n) \\ \in \text{pro-}\mathrm{MT}(\mathbb{Z}[\mu_N, \{\frac{1}{1-\zeta_w}\}_{w|N}]).\end{aligned}$$

The extension group is related to the algebraic K-theory [DG, 2.1.3]:

$$\mathrm{Ext}^1_{\mathrm{MT}(\mathbb{Z}[\mu_N,\{\frac{1}{1-\zeta_w}\}_{w|N}])}(\mathbb{Q}(0),\mathbb{Q}(n)) = \begin{cases} 0 & n\le 0, \\ \mathbb{Z}[\mu_N,\{\frac{1}{1-\zeta_w}\}_{w|N}]^\times \otimes_{\mathbb{Z}} \mathbb{Q} & n=1, \\ K_{2n-1}(\mathbb{Q}(\mu_N))\otimes_{\mathbb{Z}}\mathbb{Q} & n\ge 2. \end{cases}$$

Let ω be the canonical fiber functor $\omega : \mathrm{MT}(\mathbb{Z}[\mu_N,\{\frac{1}{1-\zeta_w}\}_{w|N}]) \to \mathrm{Vect}_{\mathbb{Q}}$, which sends a motive M to $\oplus_n \mathrm{Hom}(\mathbb{Q}(n), \mathrm{Gr}^W_{-2n}(M))$. Here, $W_m(M)$ is the weight filtration of M. Let $G_\omega := \omega(G) = \underline{\mathrm{Aut}}^{\otimes}(\mathrm{MT}(\mathbb{Z}[\mu_N,\{\frac{1}{1-\zeta_w}\}_{w|N}]),\omega)$ be the motivic Galois gruop of $\mathrm{MT}(\mathbb{Z}[\mu_N,\{\frac{1}{1-\zeta_w}\}_{w|N}])$ with respect to the canonical fiber functor ω (For the de Rham realization M_{dR} of a motive $M\in \mathrm{MT}(\mathbb{Q}(\mu_N))$, we have $M_{\mathrm{dR}} = \omega(M)\otimes_{\mathbb{Q}}\mathbb{Q}(\mu_N)$ [DG, Proposition 2.10]). Then, the ω-realization of the exact sequence $0\to U\to G\to \mathbb{G}_m\to 0$ is split by the action of $\mathbb{G}_m$, which gives the grading by weights,

$$G_\omega = \mathbb{G}_m \ltimes U_\omega.$$

Here, $U_\omega := \omega(U)$. Let τ denote the splitting $\mathbb{G}_m \to G_\omega$. The pro-unipotent group U_ω is equipped with the grading $\{(U_\omega)_n\}_n$. Put $(\mathrm{Lie}U_\omega)^{\mathrm{gr}} := \oplus_n (\mathrm{Lie}U_\omega)_n$. Then, $(\mathrm{Lie}U_\omega)^{\mathrm{gr}}$ is a free Lie algebra, since we have $\mathrm{Ext}^2_{\mathrm{MT}(\mathbb{Z}[\mu_N,\{\frac{1}{1-\zeta_w}\}_{w|N}])}(\mathbb{Q}(0),\mathbb{Q}(n)) = K_{2n-2}(\mathbb{Q}(\mu_N))\otimes_{\mathbb{Z}}\mathbb{Q} = 0$ [DG, Proposition 2.3]. Thus, the generating function of the universal envelopping algebra of $(\mathrm{Lie}U_\omega)^{\mathrm{gr}}$ is $\sum_{n=0}^{\infty} f(t)^n$, where

$$f(t) = \begin{cases} t^3+t^5+t^7+\cdots = \frac{t^3}{1-t^2} & N=1, \\ t+t^3+t^5+\cdots = \frac{t}{1-t^2} & N=2, \\ \left(\frac{\varphi(N)}{2}+\nu-1\right)t + \frac{\varphi(N)}{2}t^2 + \frac{\varphi(N)}{2}t^3 + \cdots = \frac{\varphi(N)}{2}\frac{t}{1-t} + (\nu-1)t & N\ge 3. \end{cases}$$

Therefore, we have

$$\sum_{n=0}^{\infty} f(t)^n = \frac{1}{1-f(t)} = \begin{cases} \dfrac{1-t^2}{1-t^2-t^3} & N=1, \\ \dfrac{1-t^2}{1-t-t^2} & N=2, \\ \dfrac{1-t}{1-\left(\frac{\varphi(N)}{2}+\nu\right)t + (\nu-1)t^2} & N\ge 3. \end{cases}$$

That is the generating function of $d[N]_n$'s in Section 1.

Let $P^{\mathcal{M}}_{y,x}$ be the fundamental $\mathrm{MT}(\mathbb{Z}[\mu_N,\{\frac{1}{1-\zeta_w}\}_{w|N}])$-groupoid of $\mathbb{U}_N$ at (tangential) base points x and y. We consider only tangential base points λ_x at $x\in S := \{0,\infty\}\cup\mu_N$ with tangent vectors λ in roots of unity under the identification the tangent space at x with $\mathbb{G}_a$. Then, $P^{\mathcal{M}}_{\lambda'_y,\lambda_x}$ depends only on x and

y, by the triviality of a Kummer $\mathbb{Q}(1)$-torsor [DG, 5.4]. Let $P^{\mathcal{M}}_{y,x}$ denote $P^{\mathcal{M}}_{\lambda'_y,\lambda_x}$. We have the following structures of the system of $\mathrm{MT}(\mathbb{Z}[\mu_N,\{\frac{1}{1-\zeta_w}\}_{w|N}])$-schemes $\{P^{\mathcal{M}}_{y,x}\}_{x,y\in S}$ [DG, 5.5, 5.7]:
[The system of groupoids in the level of motives]

$(1)^{\mathcal{M}}$ The Tate object $\mathbb{Q}(1)$,

$(2)^{\mathcal{M}}$ For $x,y\in S$, the fundamental $\mathrm{MT}(\mathbb{Z}[\mu_N,\{\frac{1}{1-\zeta_w}\}_{w|N}])$-groupoid $P^{\mathcal{M}}_{y,x}$,

$(3)^{\mathcal{M}}$ The composition of paths,

$(4)^{\mathcal{M}}$ For $x\in S$, a morphism of $\mathrm{MT}(\mathbb{Z}[\mu_N,\{\frac{1}{1-\zeta_w}\}_{w|N}])$-group scheme (the local monodromy around x):

$$\mathbb{Q}(1)\to P^{\mathcal{M}}_{x,x},$$

$(5)^{\mathcal{M}}$ An equivariance under the dihedral group $\mathbb{Z}/2\mathbb{Z}\ltimes\mu_N$.

By applying a fiber functor F to the category of K-vector spaces, where K is a field of characteristic 0, we get the following structure [DG, 5.8]:
[The system of groupoids under the fiber functor F]

$(1)^F$ A vector space $K(1)$ of dimension 1,

$(2)^F$ For $x,y\in S$, a scheme $P^F_{y,x}$ over K,

$(3)^F$ a system of morphisms of schemes $P^F_{z,y}\times P^F_{y,x}\to P^F_{z,x}$ making $P^F_{y,x}$'s a groupoid. The group schemes $P^F_{x,x}$ are pro-unipotent,

$(4)^F$ For $x\in S$, a morphism

$$(\text{additive group } K(1))\to P^F_{x,x}.$$

That is equivalent to giving $K(1)\to \mathrm{Lie}P^F_{x,x}$,

$(5)^F$ An $\mathbb{Z}/2\mathbb{Z}\ltimes\mu_N$-equivariance.

In particular, we take the canonical fiber functor ω as F, and we consider the following weakened structure (forgetting the conditions at infinity) [DG, 5.8]. Note that in the realization ω, the weight filtrations split and give the grading, and that all $\pi_1^{\omega}(\mathbb{U}_N,x)$-groupoids are trivial since $H^1(\mathbb{U}_N,\mathcal{O}_{\mathbb{U}_N})=0$. Let $\mathcal{L}$ be the Lie algebra freely generated by symbols A, and $\{B_\zeta\}_{\zeta\in\mu_N}$. Let Π be the pro-unipotent group

$$\Pi:=\varprojlim_n \exp(\mathcal{L}/\text{degree }\geq n).$$

Then, we have the following structure [DG, 5.8]:

[The (weakened) system of groupoids under the canonical fiber functor ω]

$(1)^\omega$ The vector space $\mathbb{Q}$,

$(2)^\omega$ A copy $\Pi_{0,0}$ of Π, and the trivial $\Pi_{0,0}$-torsor $\Pi_{1,0}$. The twist of $\Pi_{0,0}$ by this torsor is a new copy of Π, denoted by $\Pi_{1,1}$,

$(3)^\omega$ The group law of Π,

$(4)^\omega$ The morphism

$$\mathbb{Q} \to \mathcal{L}^\wedge : 1 \mapsto A, \ \mathbb{Q} \to \mathcal{L}^\wedge : 1 \mapsto B_1.$$

for $x = 0, 1$ respectively. Here, $\mathcal{L}^\wedge := \varprojlim_n \mathcal{L}/(\text{degree } \geq n)$,

$(5)^\omega$ The action μ_N on $\Pi_{0,0}$, which induces on the Lie algebra $B_\zeta \mapsto B_{\sigma\zeta}$.

Let H_ω be the group scheme of automorphisms of $\mathbb{Q}$ and Π preserving the above structure $(1)^\omega$-$(5)^\omega$. The action of H_ω on the one dimensional vector space $(1)^\omega$ gives a morphism $H_\omega \twoheadrightarrow \mathbb{G}_m$. Let V_ω be the kernel. The grading gives a splitting,

$$H_\omega = \mathbb{G}_m \ltimes V_\omega.$$

Also let τ denote the splitting $\mathbb{G}_m \to V_\omega$. The action G_ω on the above structure factors through H_ω, which sends U_ω to V_ω.

$$\begin{tikzcd} 1 \arrow[r] & U_\omega \arrow[r] \arrow[d] & G_\omega \arrow[r] \arrow[d] & \mathbb{G}_m \arrow[r] \arrow[d, "="] & 1 \\ 1 \arrow[r] & V_\omega \arrow[r] & H_\omega \arrow[r] & \mathbb{G}_m \arrow[r] & 1. \end{tikzcd}$$

Let ι denote both of $G_\omega \to H_\omega$, and $U_\omega \to V_\omega$. The above diagram comes from $\mathrm{MT}(\mathbb{Z}[\mu_N, \{\frac{1}{1-\zeta_w}\}_{w|N}])$-schemes (splitting does not come from $\mathrm{MT}(\mathbb{Z}[\mu_N, \{\frac{1}{1-\zeta_w}\}_{w|N}])$-schemes), however we do not use this fact (see, [DG, 5.12.1]). For the details of affine $\mathcal{T}$-schemes, where $\mathcal{T}$ is a Tannakian category, see [D1, §5, §6], [D2, 7.8], and [DG, 2.6].
By the Proposition 5.9 in [DG], the map

$$\eta : V_\omega \to \Pi_{1,0} \ (v \mapsto v(\gamma_{\mathrm{dR}}))$$

is bijective. Here, γ_{dR} is the neutral element of $\Pi_{1,0}$, that is, γ_{dR} is the canonical path from 0 to 1 in the realization of ω.

3.2 The p-adic MLV-space in the Sense of Deligne.

We will discuss the crystalline realization of mixed Tate motives, and now *we assume the results of Section* 4 (See, Remark 4.8). We use the word "crystalline", not "rigid" for the purpose of fixing terminologies.
In [D1], Deligne has found the p-adic zeta values (i.e., the p-adic MZV's of depth 1), and the p-adic differential equation of p-adic polylogarithms in the

study of crystalline aspects of the fundamental group of $\mathbb{U}_N$ modulo depth ≥ 2 [D1, 19.6]. Deligne-Goncharov proposed that the coefficients of the image of

$$\varphi_{\mathfrak{p}} := F_{\mathfrak{p}}^{-1}\tau(q)^{-1} \in U_\omega(\mathbb{Q}(\mu_N)_{\mathfrak{p}})$$

by the map

$$\eta \cdot \iota : U_\omega(\mathbb{Q}(\mu_N)_{\mathfrak{p}}) \to V_\omega(\mathbb{Q}(\mu_N)_{\mathfrak{p}}) \xrightarrow{\sim} \Pi(\mathbb{Q}(\mu_N)_{\mathfrak{p}}) \subset \mathbb{Q}(\mu_N)_{\mathfrak{p}}\langle\langle A, \{B_\zeta\}_{\zeta\in\mu_N}\rangle\rangle$$

"seem" to be p-adic analogies of MZV's [DG, 5.28]. Here, τ is the splitting $\mathbb{G}_m \to G_\omega$, $F_{\mathfrak{p}}$ is the Frobenius endomorphism at $\mathfrak{p}$, q is the cardinality of the residue field at $\mathfrak{p}$, and $\Pi(\mathbb{Q}(\mu_N)_{\mathfrak{p}})$ is the $\mathbb{Q}(\mu_N)_{\mathfrak{p}}$-valued points of Π in the previous subsection. Note that we have the Frobenius endomorphism on $M_\omega \otimes \mathbb{Q}(\mu_N)_{\mathfrak{p}} \cong M_{\mathrm{crys}}$ for $M \in \mathrm{MT}(\mathbb{Z}[\mu_N, \{\frac{1}{1-\zeta_w}\}_{w|N}])$ by Remark 4.8. Here, M_{crys} is the crystalline realization of M.

Definition 3.1 *We define the p-adic multiple L-values in the sense of Deligne of weight w to be the coefficients $I_{\mathfrak{p}}^{\mathrm{D}}(W)$ of words W of weight w in $\eta\iota(\varphi_{\mathfrak{p}}) \in \Pi(\mathbb{Q}(\mu_N)_{\mathfrak{p}}) \subset \mathbb{Q}(\mu_N)_{\mathfrak{p}}\langle\langle A, \{B_\zeta\}_{\zeta\in\mu_N}\rangle\rangle$. We define the p-adic L-value spaces in the sense of Deligne of weight w $Z_w^{\mathfrak{p},\mathrm{D}}[N]$ to be the finite dimensional $\mathbb{Q}$-linear subspace of $\mathbb{Q}(\mu_N)_{\mathfrak{p}}$ generated by all p-adic MLV's in the sense of Deligne of indices of weight w. By the definition, we have $Z_0^{\mathfrak{p},\mathrm{D}}[N] = \mathbb{Q}$. We define $Z_\bullet^{\mathfrak{p},\mathrm{D}}[N]$ to be the formal direct sum of $Z_w^{\mathfrak{p},\mathrm{D}}[N]$ for $w \geq 0$.*

On the othe hand, we call p-adic MLV's defined in Section 2.1 p-adic MLV's in the sense of Furusho.

Remark 3.2 If we calculate the action of Frobenius $F_{\mathfrak{p}}^{-1}$ on $(P_{1,0})_\omega$, we get the following KZ-like p-adic differential equation by the same arguments as in [D1, 19.6]:

$$\begin{aligned} dG(t) = \\ &- qG(t)\left(\frac{dt}{t}A + \sum_{\zeta\in\mu_N}\frac{dt}{t-\iota_{\mathfrak{p}}(\zeta)}\zeta(\Phi_D^{\mathfrak{p}})^{-1}B_\zeta\zeta(\Phi_D^{\mathfrak{p}})\right) \\ &+ \left(\frac{d(t^q)}{t^q}A + \sum_{\zeta\in\mu_N}\frac{d(t^q)}{t^q-\iota_{\mathfrak{p}}(\zeta)}B_\zeta\right)G(t). \end{aligned}$$

Here, $\zeta(\Phi_D^{\mathfrak{p}})$ means the action of ζ on $\Phi_D^{\mathfrak{p}}$ determined by $\zeta(A) = A$ and $\zeta(B_{\zeta'}) = B_{\zeta\zeta'}$. Here, $\Phi_D^{\mathfrak{p}}$ is the Deligne associator (See, the subsection of Tannakian interpretions, and Proposition 3.10).

The coefficient of a word W in the solution of the above p-adic differential equation is $q^{w(W)}I_{\mathfrak{p}}^{\mathrm{D}}(W)$ in the limit $t \to 1$, that is, p-adic MLV's in the sense of Deligne (multiplied by $q^{w(W)}$). (More precisely, we have to consider the effect $(1-t)^{-B_1}$ of the tangential base point in taking the limit). The first term in RHS is multiplied by G from the left, and the second term in RHS is multiplied by G from the right. Thus, the inductive procedure of determining coefficients is more complicated.

In [D1, 19.6], Deligne calculated the Frobenius action on $\pi_1^{\omega}(\mathbb{U}_N, 1_0) = (P_{1,0})_{\omega}$ modulo depth ≥ 2, however, we get the above p-adic differential equation by the same arguments. Here we give a sketch. We use some notations in [D1]. The above equation arises from the horizontality of Frobenius ([D1, 19.6.2]):

$$F_{\mathfrak{p}}^{-1}(e^{-1}\nabla e) = G^{-1}\nabla G.$$

Here, e is the identity element. The above $F_{\mathfrak{p}}^{-1}$ and G are F_* and v in [D1] respectively. On the LHS, we have [D1, 12.5, 12.12, 12.15]

$$e^{-1}\nabla e = -\alpha = -\left(\frac{dt}{t}A + \sum_{\zeta\in\mu_N}\frac{dt}{t-\iota_{\mathfrak{p}}(\zeta)}B_{\zeta}\right).$$

Here, α is the Maurer-Cartan form ([D1, 12.5.5]). On the RHS, since the connection is the one of $\tilde{F}^*(P_{1,0})_{\omega}$, we have $\nabla e = -\tilde{F}^*\alpha$, where $\tilde{F}^*$ means the Frobenius lift $t \mapsto t^q$. Combining these and $\nabla G = dG + (\nabla e)G$, we get

$$-qG\left(\frac{dt}{t}A + \sum_{\zeta\in\mu_N}\frac{dt}{t-\iota_{\mathfrak{p}}(\zeta)}F_{\mathfrak{p}}^{-1}(B_{\zeta})\right) =$$
$$dG - \left(\frac{d(t^q)}{t^q}A + \sum_{\zeta\in\mu_N}\frac{d(t^q)}{t^q-\iota_{\mathfrak{p}}(\zeta)}B_{\zeta}\right)G.$$

This gives the equation (For $F_{\mathfrak{p}}^{-1}(B_{\zeta})$, see the proof of Proposition 3.10).

Example 1 From the p-adic differential equation in the above Remark 3.2, the coefficient of $A^{k-1}B$ in $\eta\iota(F_p^{-1}\tau(p)^{-1})$ in the case where $N=1$ is the limit value at $z=1$ of the p-adic analytic continuation of the following analytic function on $|z|_p < 1$ [D1, 19.6]:

$$\sum_{p\nmid n}\frac{z^n}{n^k}.$$

That limit value is $(1-p^{-k})\zeta_p(k)$. From the condition $p \nmid n$ in the summation, we lose the Euler factor at p for p-adic MZV's of depth 1 in the sense of Deligne.

Proposition 3.3 *For $a, b \geq 0$, we have*

$$Z_a^{\mathfrak{p},\mathrm{D}}[N] \cdot Z_b^{\mathfrak{p},\mathrm{D}}[N] \subset Z_{a+b}^{\mathfrak{p},\mathrm{D}}[N].$$

Proof The group $\Pi(\mathbb{Q}(\mu_N)_{\mathfrak{p}})$ is the subgroup of group-like elements in $\mathbb{Q}(\mu_N)_{\mathfrak{p}}\langle\langle A, \{B_\zeta\}_{\zeta\in\mu_N}\rangle\rangle$, and $\eta\iota(\varphi_{\mathfrak{p}})$ is an element of $\Pi(\mathbb{Q}(\mu_N)_{\mathfrak{p}})$ by the definition. Thus, we have $\Delta(\eta\iota(\varphi_{\mathfrak{p}})) = \eta\iota(\varphi_{\mathfrak{p}})\widehat{\otimes}\eta\iota(\varphi_{\mathfrak{p}})$. This implies the proposition.

Proposition 3.4 *For $w \geq 0$, we have*

$$\dim_{\mathbb{Q}} Z_w^{\mathfrak{p},\mathrm{D}}[N] \leq d[N]_w.$$

Proof Let $U_\omega = \mathrm{Spec}R$ and $\eta\iota(U_\omega) = \mathrm{Spec}S$. The algebras $R = \prod_n R^n$ and $S = \prod_n S^n$ are graded algebras over $\mathbb{Q}$. Here, the grading of R and S come from the grading of U_ω. Then, $\eta\iota(\varphi_{\mathfrak{p}}) \in \eta\iota(U_\omega)(\mathbb{Q}(\mu_N)_{\mathfrak{p}})$ gives a homomorphism $\psi_{\mathfrak{p}} : S \to \mathbb{Q}(\mu_N)_{\mathfrak{p}}$. The coefficients of $\eta\iota(\varphi_{\mathfrak{p}})$ of weight w are contained in $\psi_{\mathfrak{p}}(S^w)$. Thus, we have $Z_w^{\mathfrak{p},\mathrm{D}}[N] \subset \psi_{\mathfrak{p}}(S^w)$. By the surjection $\iota : U_\omega \twoheadrightarrow \iota(U_\omega)(\subset V_\omega \overset{\eta}{\cong} \Pi)$, the dimension of S^w is at most the one of the w-th graded part of the universal envelopping algebra of $(\mathrm{Lie}U_\omega)^{\mathrm{gr}}$. That dimension is $d[N]_w$. We are done.

Remark 3.5 As remarked in [DG, 5.27], $\iota : \mathrm{Lie}U_\omega \to \mathrm{Lie}V_\omega$ is not injective for $N > 4$ in general. Thus, the above bounds are not best possible for $N > 4$ in general. The kernel is related to the space of cusp forms of weight 2 on $X_1(N)$ if N is a prime. See also [G2].

Remark 3.6 In the complex case [DG], $\mathrm{dch}(\sigma)$ is in $(P_{1,0})_\omega \otimes \mathbb{C} = \Pi(\mathbb{C}) \overset{\sim}{\leftarrow} V_\omega(\mathbb{C})$. (Here, $\mathrm{dch}(\sigma)$ is the "droit chemin" from 0 to 1 in the Betti realization with respect to $\sigma : \mathbb{Q}(\mu_N) \hookrightarrow \mathbb{C}$.) Thus, Deligne-Goncharov relate $\mathrm{dch}(\sigma)$ to the motivic Galois group U_ω for the purpose of bounds for the dimensions in [DG, Proposition 5.18, 5.19, 5.20, 5.21, 5.22]. (The point is that V_ω is too big, and U_ω is small enough.) However, in the p-adic situation, $\varphi_{\mathfrak{p}}$ is contained a priori in a small enough variety, i.e., we have $\varphi_{\mathfrak{p}} \in U_\omega(\mathbb{Q}(\mu_N)_{\mathfrak{p}})$ by the definition. Thus, the bounds from K-theory of p-adic MLV's in the sense of Deligne are almost trivial.

We give remarks on $\zeta_p(2)$.

Remark 3.7 By Proposition 3.4 and Example 1, we have $\zeta_p(2) = 0$, since $\dim_{\mathbb{Q}} Z_2^{p,\mathrm{D}}[1] = 0$. It is another proof of that well-known fact. To bound dimensions, Deligne-Goncharov used $\iota(U_\omega) \times \mathbb{A}^1$ in the complex case [DG, 5.20, 5.21, 5.22, 5.23, 5.24, 5.25]. This affine line corresponds to "π^2", and we need this affine line simply because π^2 is not in $\mathbb{Q}$. In the p-adic case, we do not need such an affine line, simply because the image of F_p^{-1} in $(\mathbb{G}_m)_\omega$ (i.e., p) is in $\mathbb{Q}$. This gives a motivic interpretation of $\zeta_p(2) = 0$.

Remark 3.8 It is well-known that $\zeta_p(2m) = 0$. However, it is non-trivial because we do not know how to show directly

$$\text{“}\sum_{\mathbb{C}_p \ni z \to 1} \frac{z^n}{n^{2m}} = 0\text{”}$$

(We add a double quotation in the above, since we have to take p-adic analytic continuation). The well-known proof of $\zeta_p(2m) = 0$ is following (also see, [Fu1, Example 2.19(a)]): By the Coleman's comparison [C], we have $\lim_{\mathbb{C}_p \ni z \to 1} \mathrm{Li}_k^a(z) = (1 - p^{-k})^{-1} L_p(k, \omega^{1-k})$ for $k \geq 2$. Here, L_p is the p-adic L-function of Kubota-Leopoldt, ω is the Teichmüller character. This is the values of the p-adic L-function at *positive* integers. On the other hand, the p-adic L-function interpolates the values of usual L-functions at *negative* integers, thus, $L_p(z, \omega^{1-k})$ is constantly zero for even k. Therefore, we have $\zeta_p(2m) = 0$. That proof is indirect.

Furusho informed to the author that 2-, and 3-cycle relations induce $\zeta_p(2m) = 0$ similarly as in [D1, §18] (In the notations in [D1, §18], we can take $\gamma =$(the unique Frobenius invariant path from 0 to 1) (see, the next subsection,) and $x = 0$). These relations come from the geometry of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. Thus, it seems that it comes from "the same origin" that '$\zeta_p(2) = 0$ from cycle relations' and '$\zeta_p(2) = 0$ from the bounds by K-theory'. Furusho also comments that we may translate '$\zeta_p(2m) = 0$ from cycle relations' into '$\zeta_p(2m) = 0$ from p-adic differential equation', i.e., we may show that $\zeta_p(2k) = 0$ directly from the p-adic analytic function $\sum_{n \geq 1} \frac{z^n}{n^{2m}}$.

3.3 The Tannakian Interpretations of Two p-adic MLV's.

Besser proved that there exists a unique Frobenius invariant path in the fundamental groupoids of certain p-adic analytic spaces [B, Corollary 3.2]. Furthermore, Besser showed the existence of Frobenius invariant path on p-adic analytic spaces is equivalent to the Coleman's integral theory [B, §5].

Let γ_{crys} be the unique Frobenius invariant path in $(P_{1,0})_{\mathrm{crys}}$. To a differential form ω, the path γ_{crys} associates the Colman integration $\int_0^1 \omega$. Let $\gamma_{\mathrm{dR}} \in (P_{1,0})_\omega$ be the canonical path from 0 to 1 under the realization ω. Furusho proved the path $\alpha_F := \gamma_{\mathrm{dR}}^{-1} \gamma_{\mathrm{crys}} \in \pi_1^{\mathrm{crys}}(\mathbb{U}_N, 1_0)$ is equal to the p-adic Drinfel'd associator Φ_{KZ}^p for p-adic MZV's, that is, for $N = 1$ in [Fu2]. By the same argument, we can verify that $\alpha_F = \Phi_{\mathrm{KZ}}^{\mathfrak{p}}$ for p-adic MLV's. Briefly, we review the argument. For details, see [Fu2] (See also [Ki, Proposition 4]). The coefficient of a word $A^{k_d - 1} B_{\zeta_d} \cdots A^{k_1 - 1} B_{\zeta_1}$ in $\alpha_F = \gamma_{\mathrm{dR}}^{-1} \gamma_{\mathrm{crys}} \in \pi_1^{\mathrm{crys}}(\mathbb{U}_N, 1_0) \subset \mathbb{Q}(\mu_N)_{\mathfrak{p}} \langle\langle A, \{B_\zeta\}_{\zeta \in \mu_N} \rangle\rangle$ for $(k_d, \zeta_d) \neq (1, 1)$ is an iterated integral

$$\int_0^1 \frac{dt}{t} \cdots \int_0^t \frac{dt}{t} \int_0^t \frac{dt}{t - \iota_{\mathfrak{p}}(\zeta_d)} \int_0^t \frac{dt}{t} \cdots \int_0^t \frac{dt}{t} \int_0^t \frac{dt}{t - \iota_{\mathfrak{p}}(\zeta_1)}$$

by the characterization of γ_{crys} with respect to Coleman's integration theory (Here, the succesive numbers of dt/t are $k_d - 1, k_{d-1} - 1, \cdots, k_2 - 1$ and $k_1 - 1$). For words beginning from A or ending B_1, the coefficients are regularized p-adic MLV's, because the coefficients in α_F are the one in $\lim'_{\mathbb{C}_p \ni z \to 1} (1 - z)^{-B_1} G_0(z)$ by using the tangential base point. Thus, α_F is the p-adic Drinfel'd associator

$\Phi_{\mathrm{KZ}}^{\mathfrak{p}}$ for twisted p-adic MPL's in Section 2.2:

$$\alpha_F := \gamma_{\mathrm{dR}}^{-1}\gamma_{\mathrm{crys}} = \Phi_{\mathrm{KZ}}^{\mathfrak{p}} = \sum_W I_{\mathfrak{p}}(W)W.$$

On the other hand, $\eta\iota(\varphi_{\mathfrak{p}}) \in \Pi_{0,0}(\mathbb{Q}(\mu_N)_{\mathfrak{p}}) = \pi_1^{\mathrm{crys}}(\mathbb{U}_N, 1_0)$ is $\gamma_{\mathrm{dR}}^{-1}\varphi_{\mathfrak{p}}(\gamma_{\mathrm{dR}})$ by the definition (Recall that $V_\omega \overset{\eta}{\cong} \Pi_{1,0}$ and $\Pi_{0,0} \cong \Pi_{1,0} : 1 \mapsto \gamma_{\mathrm{dR}}$). Briefly, p-adic MLV's in the sense of Furusho come from $\alpha_F = \gamma_{\mathrm{dR}}^{-1}\gamma_{\mathrm{crys}}$, and p-adic MLV's in the sense of Deligne come from $\alpha_D := \gamma_{\mathrm{dR}}^{-1}\varphi_{\mathfrak{p}}(\gamma_{\mathrm{dR}})$. That is the Tannakian interpretations of p-adic MLV's. In [Fu2], he calls $\Phi_D^{\mathfrak{p}} := \gamma_{\mathrm{dR}}^{-1}F_{\mathfrak{p}}^{-1}(\gamma_{\mathrm{dR}})$ the Deligne associator.

Remark 3.9 In both of complex and p-adic cases, the iterated integrals appear in the theory of MZV's. However, the iterated integrals come from different origins in the complex case and the p-adic case.
In the complex case, the iterated integrals appear in the comparison map between the Betti fundamental group $\pi_1^{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C}$ tensored by $\mathbb{C}$ of $\mathbb{P}^1 \setminus \{0,1,\infty\}$ and the de Rham fundamental group $\pi_1^{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{C}$ tensored by $\mathbb{C}$ of $\mathbb{P}^1 \setminus \{0,1,\infty\}$. The difference between the $\mathbb{Q}$-structure π_1^{B} and the $\mathbb{Q}$-structure π_1^{dR} under the comparison $\pi_1^{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C} \cong \pi_1^{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{C}$ is expressed by iterated integrals.
In the p-adic case, iterated integrals do not appear in the comparison map between the de Rham fundamental group $\pi_1^{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{Q}_p$ tensored by $\mathbb{Q}_p$ and the crystalline fundamental group π_1^{crys}. Furthermore, there is no $\mathbb{Q}$-structure on π_1^{crys}. For p-adic MZV's in the sense of Deligne, iterated integrals appear in the difference between the $\mathbb{Q}$-structure π_1^{dR} and the $\mathbb{Q}$-structure $F_{\mathfrak{p}}^{-1}(\pi_1^{\mathrm{dR}})$ in $P_{1,0}^{\mathrm{crys}}$ under the comparison $P_{1,0}^{\mathrm{crys}} \cong P_{1,0}^{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{Q}_p = \pi_1^{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{Q}_p$. For p-adic MZV's in the sense of Furusho, they appear in the difference between $\mathbb{Q}$-structure π_1^{dR} and the $\mathbb{Q}$-structure $\alpha\pi_1^{\mathrm{dR}}$ in π_1^{crys} under the comparison $\pi_1^{\mathrm{crys}} \cong \pi_1^{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{Q}_p$. Here, α is a unique element in π_1^{crys} such that $\gamma_{\mathrm{dR}} \cdot \alpha \in P_{1,0}^{\mathrm{crys}}$ is invariant under the Frobenius (Thus, α is equal to α_F).
From this, it seems difficult to find a "motivic Drinfel'd associator", which is an origin of both complex and p-adic MZV's, and a motivic element, which is an origin of linear relations of both complex and p-adic MZV's. Note also that roughly speaking, the complex Drinfel'd associator is the differenc between Betti and de Rham realizations ([DG, 5.19]), and the p-adic Drinfel'd associator is the Frobenius element at p.

Example 2 1. (Kummer torsor) Let $K(x)_\omega$ be the fundamental groupoid from 1 to x on $\mathbb{G}_m$ with respect to the realization ω. Deligne calculated in [D1, 2.10] the action of F_p^{-1} on $K(x)_\omega \subset K(x)_{\mathrm{crys}}$:

$$F_p^{-1}(\gamma_{\mathrm{dR}}) = \gamma_{\mathrm{dR}} + \log^a x^{1-p}.$$

Here, γ_{dR} is the canonical de Rham path from 1 to x, and $+$ means the right action of $\pi_1^{\mathrm{crys}}(\mathbb{G}_m, 1) = \mathbb{Q}(1)_{\mathrm{crys}} = \mathbb{Q}_p(1)$ on $K(x)_{\mathrm{crys}}$. From this, we have

$$F_p^{-1}(\gamma_{\mathrm{dR}} + \log^a x) = \gamma_{\mathrm{dR}} + \log^a x^{1-p} + p\log^a x = \gamma_{\mathrm{dR}} + \log^a x.$$

Thus, $\gamma_{\mathrm{dR}} + \log^a x$ is Frobenius invariant, that is, the unique crystalline path γ_{crys} from 1 to x.

2. (Polylogarithm torsor) Let $P_{1,k}(\zeta)_\omega$ be the k-th polylogarithm torsor with respect to the realization ω for $\zeta \in \mu_N$ (see, [D1, Definition 16.18]). The polylogarithm torsors are not fundamental groupoids, but quotients of fundamental groupoids. However, we use the terminology "$\mathbb{Z}(k)$-torsor of $\mathbb{Z}(k)$-paths from 0 to ζ" in [D1, 13.15]. Here, we consider as $\mathbb{Q}(k)_\omega$-torsor not as $\mathbb{Z}(k)_\omega$-torsor, and we do not multiply $\frac{1}{(k-1)!}$ on the integral structure unlike as [D1]. Deligne calculated in [D1, 19.6, 19.7] the action of F_p^{-1} on $P_{1,k}(\zeta)_\omega \subset P_{1,k}(\zeta)_{\mathrm{crys}}$:

$$F_p^{-1}(\gamma_{\mathrm{dR}}) = \gamma_{\mathrm{dR}} + p^k(1-p^{-k})N^{k-1}\mathrm{Li}_k^a(\zeta)$$

(That is, $F_p^{-1}\tau(p)^{-1}(\gamma_{\mathrm{dR}}) = \gamma_{\mathrm{dR}} + (1-p^{-k})N^{k-1}\mathrm{Li}_k^a(\zeta)$). Here, $+$ means the right action of $\mathbb{Q}(k)_{\mathrm{crys}} = \mathbb{Q}_p(k)$ on $P_{1,k}(\zeta)_{\mathrm{crys}}$. From this, we have

$$\begin{aligned} F_p^{-1}(\gamma_{\mathrm{dR}} - N^{k-1}\mathrm{Li}_k^a(\zeta)) &= \gamma_{\mathrm{dR}} + p^k(1-p^{-k})N^{k-1}\mathrm{Li}_k^a(\zeta) - p^kN^{k-1}\mathrm{Li}_k^a(\zeta) \\ &= \gamma_{\mathrm{dR}} - N^{k-1}\mathrm{Li}_k^a(\zeta). \end{aligned}$$

Thus, $\gamma_{\mathrm{dR}} - N^{k-1}\mathrm{Li}_k^a(\zeta)$ is Frobenius invariant, that is, the unique crystalline path γ_{crys} from 0 to ζ.

3. In the case where $N = 1$, the coefficient of $A^{k-1}B$ in $\Phi_{\mathrm{KZ}}^{\mathfrak{p}}$ is $-\zeta_p(k)$ and the one of $A^{k-1}B$ in $\eta\iota(F_p^{-1}\tau(p)^{-1})$ is $(1-p^{-k})\zeta_p(k)$, from the above example.

4. (Furusho) The coefficient of $A^{b-1}BA^{a-1}B$ in $F_p^{-1}\tau(p)^{-1}$ in the case where $N = 1$ is

$$\begin{aligned} &\left(\frac{1}{p^{a+b}} - 1\right)\zeta_p(a,b) - \left(\frac{1}{p^a} - 1\right)\zeta_p(a)\zeta_p(b) \\ &+ \sum_{r=0}^{a-1}(-1)^r\left(\frac{1}{p^{b+r}} - 1\right)\binom{b-1+r}{b-1}\zeta_p(a-r)\zeta_p(b+r) \\ &+ (-1)^{a+1}\sum_{s=0}^{b-1}\left(\frac{1}{p^{a+s}} - 1\right)\binom{a-1+s}{a-1}\zeta_p(a+s)\zeta_p(b-s), \end{aligned}$$

for $b > 1$.

The following proposition combined with Proposition 3.4 gives a proof of Theorem 1.4. The author learned the following proposition from Furusho's caluculation Example 2(4).

Proposition 3.10 *For $w \geq 0$, we have*

$$Z_w^{\mathfrak{p}}[N] = Z_w^{\mathfrak{p},\mathrm{D}}[N].$$

Proof The effect of $\tau(q)$ is the multiplication by q^w on p-adic MLV's of weight w in the sense of Deligne. Thus, $Z_w^{\mathfrak{p},\mathrm{D}}[N]$ is not changed when we replace $F_{\mathfrak{p}}^{-1} \in G_\omega(\mathbb{Q}(\mu_N)_{\mathfrak{p}})$ by $\varphi_{\mathfrak{p}} = F_{\mathfrak{p}}^{-1}\tau(q)^{-1} \in G_\omega(\mathbb{Q}(\mu_N)_{\mathfrak{p}})$ in $\alpha_D = \gamma_{\mathrm{dR}}^{-1}\varphi_{\mathfrak{p}}(\gamma_{\mathrm{dR}})$. Let $J_{\mathfrak{p}}^{\mathrm{D}}(W)$ be the coefficient of a word W in $\Phi_D^{\mathfrak{p}} := \gamma_{\mathrm{dR}}^{-1}F_{\mathfrak{p}}^{-1}(\gamma_{\mathrm{dR}})$. We have

$$Z_w^{\mathfrak{p},\mathrm{D}}[N] = \langle J_{\mathfrak{p}}^{\mathrm{D}}(W) \mid \text{the weight of } W \text{ is } w\rangle_{\mathbb{Q}} \subset \mathbb{Q}(\mu_N)_{\mathfrak{p}}$$

(We recall that the coefficient of a word W in α_F is $I_{\mathfrak{p}}(W)$). We have

$$\begin{aligned}\alpha_F &= \gamma_{\mathrm{dR}}^{-1}\gamma_{\mathrm{crys}} = \gamma_{\mathrm{dR}}^{-1}F_{\mathfrak{p}}^{-1}(\gamma_{\mathrm{dR}}) \cdot (F_{\mathfrak{p}}^{-1}(\gamma_{\mathrm{dR}}))^{-1}F_{\mathfrak{p}}^{-1}(\gamma_{\mathrm{crys}}) = \Phi_D^{\mathfrak{p}}F_{\mathfrak{p}}^{-1}(\alpha_F)\\ &= \left(\sum_W J_{\mathfrak{p}}^{\mathrm{D}}(W)W\right)\left(\sum_W I_{\mathfrak{p}}(W)F_{\mathfrak{p}}^{-1}(W)\right)\end{aligned}$$

(By a theorem of Besser [B, Theorem 3.1], we see that α_F and α_D determine each other from the above formula).
We compute the action $F_{\mathfrak{p}}^{-1}$ on a word W. Let $\gamma_{\mathrm{dR},\zeta}$ be the canonical path from 0 to ζ under the realization ω, that is, $\gamma_{\mathrm{dR},1} = \gamma_{\mathrm{dR}}$, $\gamma_{\mathrm{dR},\zeta} = \zeta(\gamma_{\mathrm{dR},1})$. Here, $\zeta(\gamma_{\mathrm{dR},1})$ is the action of $\zeta \in \mu_N$ on Π. Then, $B_\zeta = (\gamma_{\mathrm{dR},\zeta})^{-1}A \cdot \gamma_{\mathrm{dR},\zeta}$ ([DG, (5.11.3)]). Thus, we have $F_{\mathfrak{p}}^{-1}(A) = qA$ and

$$\begin{aligned}F_{\mathfrak{p}}^{-1}(B_\zeta) &= (F_{\mathfrak{p}}^{-1}(\gamma_{\mathrm{dR},\zeta}))^{-1}qAF_{\mathfrak{p}}^{-1}(\gamma_{\mathrm{dR},\zeta}) = q\zeta(\Phi_D^{\mathfrak{p}})^{-1}B_\zeta\zeta(\Phi_D^{\mathfrak{p}})\\ &= q\left(\sum_W J_{\mathfrak{p}}^{\mathrm{D}}(\zeta^{-1}(W))W\right)^{-1} B_\zeta \left(\sum_W J_{\mathfrak{p}}^{\mathrm{D}}(\zeta^{-1}(W))W\right).\end{aligned}$$

Here, the action of $\zeta \in \mu_N$ on words is given by $\zeta(A) = A$, and $\zeta(B_{\zeta'}) = B_{\zeta\zeta'}$. From the above formula about α_F, we have

$$\begin{aligned}\alpha_F &= \Phi_D^{\mathfrak{p}}F_{\mathfrak{p}}^{-1}(\alpha_F) = \left(\sum_W J_{\mathfrak{p}}^{\mathrm{D}}(W)W\right)\left(\sum_W I_{\mathfrak{p}}(W)F_{\mathfrak{p}}^{-1}(W)\right)\\ &= \left(\sum_W J_{\mathfrak{p}}^{\mathrm{D}}(W)W\right)\Bigg[\sum_{W=A^{k_d}B_{\zeta_d}\cdots A^{k_1}B_{\zeta_1}A^{k_0}} q^{k_0+\cdots+k_d+d}I_{\mathfrak{p}}(W)A^{k_d}\\ &\quad \cdot\left(\sum_W J_{\mathfrak{p}}^{\mathrm{D}}(\zeta_d^{-1}(W))W\right)^{-1} B_{\zeta_d}\left(\sum_W J_{\mathfrak{p}}^{\mathrm{D}}(\zeta_d^{-1}(W))W\right)\cdots\\ &\quad \cdot\left(\sum_W J_{\mathfrak{p}}^{\mathrm{D}}(\zeta_1^{-1}(W))W\right)^{-1} B_{\zeta_1}\left(\sum_W J_{\mathfrak{p}}^{\mathrm{D}}(\zeta_1^{-1}(W))W\right)A^{k_0}\Bigg],\end{aligned}$$

There, by using Proposition 2.14 and Proposition 3.3, for a word W of weight w we have

$$(1-q^w)I_{\mathfrak{p}}(W) - J_{\mathfrak{p}}^{\mathrm{D}}(W) \in \sum_{w=w'+w'':w'<w,w''<w} Z_{w'}^{\mathfrak{p}} \cdot Z_{w''}^{\mathfrak{p},\mathrm{D}}.$$

By induction, we have $Z_w^{\mathfrak{p}} = Z_w^{\mathfrak{p},\mathrm{D}}$.

Finally, we remark on some conjectures. The following conjecture is a p-adic analogue of Grothendieck's conjecture [DG, 5.20], which says that $a_\sigma \in G_\omega(\mathbb{C})$ is $\mathbb{Q}$-Zariski dense (weakly, $a_\sigma^0 := a_\sigma \tau(2\pi\sqrt{-1})^{-1} \in U_\omega(\mathbb{C})$ is $\mathbb{Q}$-Zariski dense). Here, a_σ is the "difference" between the Betti realization with respect to σ and the de Rham realization (For elements a_σ and a_σ^0, see [DG, Proposition 2.12] and [D1, 8.10 Proposition]).

Conjecture 4 The element $\varphi_{\mathfrak{p}} \in U_\omega(\mathbb{Q}(\mu_N)_{\mathfrak{p}})$ is $\mathbb{Q}$-Zariski dense. That means that if a subvariety X of U_ω over $\mathbb{Q}$ satisfies $\varphi_{\mathfrak{p}} \in X(\mathbb{Q}(\mu_N)_{\mathfrak{p}})$, then $X = U_\omega$.

Remark 3.11 We have the Chebotarev density theorem for usual Galois groups. So, the author expects that there may be "Chebotarev density like" theorem for the Frobenius element in the motivic Galois group varying the prime number p. It will be interesting to study for this "Chebotarev density like" theorem varying p, adèle valued points of the motivic Galois group, and possible relations among "Chebotarev density like" theorem varying p, Grothendieck's conjecture about the motivic element, and the above p-adic analogue of Grothendieck's conjecture about the Frobenius element.

The following conjecture in the case $N = 1$ (i.e. p-adic MZV's) is proposed by Furusho (non published).

Conjecture 5 All linear relations among p-adic MLV's are linear combinations of linear relations among p-adic MLV's with same weights.

The following proposition is obvious (cf. [DG, 5.27]).

Proposition 3.12 *We consider the following statements:*

1. *The inequality in Theorem 1.4 is an equality (For $N = 1$, this is Conjecture* 2*).*
2. *The map $\iota : U_\omega \to V_\omega$ is injective.*
3. *Conjecture* 4*.*
4. *Conjecture* 5*.*

Then, (1) *is equivalent to the combination of* (2) *and* (3)*, and implies* (4)*.*

Remark 3.13 The statement (2) is true for $N = 2, 3, 4$. For $N > 4$, the statement (2) is false in general. The kernel is related to the space of cusp forms of weight 2 on $X_1(N)$ if N is a prime. See, [DG, 5.27][G2].

4 Crystalline Realization of Mixed Tate Motives.

In this section, we consider the construction of the crystalline realization of mixed Tate motives, and Berthelot-Ogus isomorphism for the de Rham and crystalline realizations of mixed Tate motives.

4.1 Crystalline Realization.

Let k be a number field, v be a finite place of k, and G_k be the absolute Galois group of k. First, we define the crystalline inertia group at v. Let p be a prime divided by v. Let $\underline{\mathrm{Rep}}_{\mathbb{Q}_p}(G_k)$, and $\underline{\mathrm{Rep}}^{\mathrm{crys},v}_{\mathbb{Q}_p}(G_k)$ be the category of finite dimensional representations of G_k over $\mathbb{Q}_p$, and the subcategory of crystaline representations of G_k at v.

Definition 4.1 *(crystalline inertia group) The inclusion* $\underline{\mathrm{Rep}}^{\mathrm{crys},v}_{\mathbb{Q}_p}(G_k) \hookrightarrow \underline{\mathrm{Rep}}_{\mathbb{Q}_p}(G_k)$ *induces the map of Tannaka dual groups with respect to the forgetful fiber functor. We define a crystalline inertia group* $I_v^{\mathrm{crys}}(\subset G_{k,p} := \underline{\mathrm{Aut}}^{\otimes}(\underline{\mathrm{Rep}}_{\mathbb{Q}_p}(G_k)))$ *at v to be its kernel.*

Here, $G_{k,p}$ is the (algebraic group over $\mathbb{Q}_p$)-closure of G_k. The group I_v^{crys} is a pro-algebraic group over $\mathbb{Q}_p$. Note that by the definition, the action of G_k on M_p is crystalline at v if and only if the action of I_v^{crys} on M_p is trivial.
We recall Bloch-Kato's group H^1_f. Let $O_{(v)}$ be the localization at v of the ring of integers of k, and k_v be the completion of k with respect to v. For a finite dimensional representation V of G_{k_v} over $\mathbb{Q}_\ell$, they defined [BK, §3]

$$H^1_f(k_v, V) := \begin{cases} \ker(H^1(k_v, V) \to H^1(k_v^{\mathrm{ur}}, V)) & v \nmid \ell, \\ \ker(H^1(k_v, V) \to H^1(k_v, B_{\mathrm{crys}} \otimes V)) & v \mid \ell. \end{cases}$$

Here, k_v^{ur} is the maximal unramified extension of k_v, and B_{crys} is the Fontaine's p-adic period ring (See, [Fo1]). For a prime ℓ not divided by v, $\mathrm{Hom}_{\mathrm{Gal}(\overline{k_v}/k_v^{\mathrm{ur}})}(\mathbb{Q}_\ell(m), \mathbb{Q}_\ell(m+n))$ is trivial for $n \geq 2$. Thus, we have

$$H^1_f(k_v, \mathbb{Q}_\ell(n)) = \begin{cases} O_{(v)}^\times \otimes \mathbb{Q}_\ell & n = 1, \\ H^1(k_v, \mathbb{Q}_\ell(n)) & n \geq 2. \end{cases}$$

In the crystalline case, we have from the calculations

$$H^1_f(k_v, \mathbb{Q}_p(n)) = \begin{cases} O_{(v)}^\times \otimes \mathbb{Q}_p & n = 1, \\ H^1(k_v, \mathbb{Q}_p(n)) & n \geq 2, \end{cases} \tag{4.1}$$

(See, [BK, Example 3.9]) monodromy informaions of I_v^{crys} on mixed Tate motives. We recall that the fact $H^1_f(k_v, \mathbb{Q}_p(n)) = H^1(k_v, \mathbb{Q}_p(n))$ for $n \geq 2$, $v \mid p$ follows from

$$\begin{aligned}
&\dim_{\mathbb{Q}_p} H^1_f(k_v, \mathbb{Q}_p(n)) \\
&= \dim_{\mathbb{Q}_p} D_{\mathrm{dR}}(\mathbb{Q}_p(n))/\mathrm{Fil}^0 D_{\mathrm{dR}}(\mathbb{Q}_p(n)) + \dim_{\mathbb{Q}_p} H^0(k_v, \mathbb{Q}_p(n)) \\
&= [k_v : \mathbb{Q}_p] + 0 = -\chi(\mathbb{Q}_p(n)) = \dim_{\mathbb{Q}_p} H^1(k_v, \mathbb{Q}_p(n))
\end{aligned}$$

(See, [BK, Corollary 3.8.4, Example 3.9]). Here, D_{dR} is the Fontaine's functor ([Fo2]), and $\chi(V)$ is the Euler characteristic of V for a Galois representation V. Thus, it holds without assuming that k_v is unramified over $\mathbb{Q}_p$. Let $H^1_f(k, V)$ be the inverse image of $H^1_f(k_v, V)$ via the restriction map $H^1(k, V) \to H^1(k_v, V)$.

Theorem 4.2 *(cf. [DG, Proposition 1.8]) Let k be a number field, and v be a finite place of k. Take a mixed Tate motive M in $\mathrm{MT}(k)$. Then, the following statements are equivalent.*

1. *The motive M is unramified at v, that is, $M \in \mathrm{MT}(O_{(v)})$.*
2. *For a prime ℓ not divided by v, the ℓ-adic realization M_ℓ of M is an unramified representation at v.*
3. *For all prime ℓ not divided by v, the ℓ-adic realization M_ℓ of M is an unramified representation at v.*
4. *For the prime p divided by v, the p-adic realization M_p of M is a crystalline representation at v.*

Proof The equivalence of (1), (2), and (3) is proved in [DG, Proposition 1.8]. We show that (1) is equivalent to (4). The proof is a crystalline analogue of [DG, Proposition 1.8]. The Kummer torsor $K(a)$ for $a \in k^\times \otimes \mathbb{Q}$ is crystalline at v, if and only if $a \in O_{(v)}^\times \otimes \mathbb{Q}$ (See, the isomorphism (4.1) $H^1_f(k_v, \mathbb{Q}_p(1)) \cong O_{(v)}^\times \otimes \mathbb{Q}_p$). Since Kummer torsors generate $\mathrm{Ext}^1_{\mathrm{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(1))$, it suffices to show that the following statement: For a mixed Tate motive $M \in \mathrm{MT}(k)$, the action of I_v^{crys} on M_p is trivial if the action of I_v^{crys} on $W_{-2n}M_p/W_{-2(n+2)}M_p$ is trivial for each $n \in \mathbb{Z}$. Assume that the action of I_v^{crys} on $W_{-2n}M_p/W_{-2(n+2)}M_p$ is trivial for each $n \in \mathbb{Z}$. We show that the action of I_v^{crys} on $W_{-2n}M_p/W_{-2(n+r)}M_p$ is trivial by the induction on r. For $r = 2$, it is the hypothesis. For $r > 2$, the induction hypothesis assure that the action of I_v^{crys} is trivial on $W_{-2n}/W_{-2(n+r-1)}$ and $W_{-2(n+1)}/W_{-2(n+r)}$. Thus, the action of $\sigma \in I_v^{\mathrm{crys}}$ is of the form $1 + \nu(\sigma)$, where $\nu(\sigma)$ is the composite:

$$W_{-2n}/W_{-2(n+r)} \twoheadrightarrow \mathrm{Gr}^W_{-2n} \xrightarrow{\mu(\sigma)} \mathrm{Gr}^W_{-2(n+r-1)} \hookrightarrow W_{-2n}/W_{-2(n+r)}.$$

We have $\mu(\sigma_1\sigma_2) = \mu(\sigma_1) + \mu(\sigma_2)$. This μ is compatible with the action of $G_{k,p}$. It suffices to show that the map $\mu(\sigma) : \mathrm{Gr}^W_{-2n} \to \mathrm{Gr}^W_{-2(n+r-1)}$ is trivial.

This follows from

$$
\begin{aligned}
&\mathrm{Hom}_{G_{k,p}}(I_v^{\mathrm{crys}}, \mathrm{Hom}(\mathbb{Q}_p(n), \mathbb{Q}_p(n+r-1)))\\
&\cong \mathrm{Ext}^1_{\underline{\mathrm{Rep}}_{\mathbb{Q}_p}(I_v^{\mathrm{crys}})}(\mathbb{Q}_p(n), \mathbb{Q}_p(n+r-1))^{G_{k,p}/I_v^{\mathrm{crys}}}\\
&\cong \mathrm{Ext}^1_{\underline{\mathrm{Rep}}_{\mathbb{Q}_p}(G_{k,p})}(\mathbb{Q}_p(n), \mathbb{Q}_p(n+r-1))/\mathrm{Ext}^1_{\underline{\mathrm{Rep}}_{\mathbb{Q}_p}(G_{k,p}/I_v^{\mathrm{crys}})}\\
&\cong \mathrm{Ext}^1_{\underline{\mathrm{Rep}}_{\mathbb{Q}_p}(G_k)}(\mathbb{Q}_p(n), \mathbb{Q}_p(n+r-1))/\mathrm{Ext}^1_{\underline{\mathrm{Rep}}^{\mathrm{crys},v}_{\mathbb{Q}_p}(G_k)}\\
&\cong H^1(k, \mathbb{Q}_p(r-1))/H^1_f(k, \mathbb{Q}_p(r-1)) = 0,
\end{aligned}
$$

where we abbreviate $\mathrm{Ext}^1_{\underline{\mathrm{Rep}}_{\mathbb{Q}_p}(G_{k,p}/I_v^{\mathrm{crys}})}(\mathbb{Q}_p(n), \mathbb{Q}_p(n+r-1))$ and $\mathrm{Ext}^1_{\underline{\mathrm{Rep}}^{\mathrm{crys},v}_{\mathbb{Q}_p}(G_k)}(\mathbb{Q}_p(n), \mathbb{Q}_p(n+r-1))$ as $\mathrm{Ext}^1_{\underline{\mathrm{Rep}}_{\mathbb{Q}_p}(G_{k,p}/I_v^{\mathrm{crys}})}$ and $\mathrm{Ext}^1_{\underline{\mathrm{Rep}}^{\mathrm{crys},v}_{\mathbb{Q}_p}(G_k)}$ respectively by a typesetting reason. The second isomorphism follows from the fact that $\mathrm{Ext}^2_{\underline{\mathrm{Rep}}^{\mathrm{crys},v}_{\mathbb{Q}_p}(G_k)} = 0$, and the action of I_v^{crys} on $\mathbb{Q}_p(r-1)$ is trivial, and the last equality follows from the isomorphism (4.1). (We have $\mathrm{Ext}^2_{\underline{\mathrm{Rep}}^{\mathrm{crys},v}_{\mathbb{Q}_p}(G_k)} = 0$ from the elemental theory of the category of filtered φ-modules. In fact, $R\mathrm{Hom}$ is calculated by a complex, which is concentrated only in degree 0 and 1.)

Remark 4.3 If we have a full sub-Tannakian category $\mathrm{MT}(O_{(v)})^{\mathrm{good}}$ of $\mathrm{MT}(k)$ satisfying

$$
\mathrm{Ext}^1_{\mathrm{MT}(O_{(v)})^{\mathrm{good}}}(\mathbb{Q}(0), \mathbb{Q}(1)) \cong \begin{cases} O^\times_{(v)} \otimes \mathbb{Q}, & n = 1,\\ \mathrm{Ext}^1_{\mathrm{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(n)), & n \geq 2,\end{cases}
$$

and

$$
\mathrm{Ext}^2_{\mathrm{MT}(O_{(v)})^{\mathrm{good}}}(\mathbb{Q}(0), \mathbb{Q}(n)) = 0 \text{ for any } n,
$$

then by introducing the "motivic inertia group" at v

$$
I_v^{\mathcal{M}} := \ker\{\underline{\mathrm{Aut}}^{\otimes}(\omega_{\mathrm{MT}(k)}) \to \underline{\mathrm{Aut}}^{\otimes}(\omega_{\mathrm{MT}(O_{(v)})^{\mathrm{good}}})\},
$$

we can prove the similar result for $\mathrm{MT}(O_{(v)})^{\mathrm{good}}$, that is, M is in $\mathrm{MT}(O_{(v)})$ if and only if M is in $\mathrm{MT}(O_{(v)})^{\mathrm{good}}$ by the "motivic analogue" of the above proof.
In a naive way, we cannot define "$M \otimes_{O_{(v)}} k(v)$" the reduction at v of an object M in $\mathrm{MT}(O_{(v)})$, since $\mathrm{MT}(O_{(v)})$ is not defined by a "geometrical way". So, the author hopes that this remark will be useful to construct "the reduction at v" of object in $\mathrm{MT}(O_{(v)})$. If we "geometrically" construct a full sub-Tannakian category $\mathrm{MT}(O_{(v)})^{\mathrm{good}}$ of $\mathrm{MT}(k)$ satisfying the above conditions, then we can get a good definition of "the reduction at v". Here, the word "geometrically" means that returning the definition of Voevodsky's category $DM(k)$. See also the proof of Theorem 4.6.

Definition 4.4 *For a mixed Tate motive $M \in \mathrm{MT}(O_{(v)})$ unramified at v, we define the crystalline realization $M_{\mathrm{crys},v}$ to be $D_{\mathrm{crys}}(M_p)$. Here D_{crys} is the Fontaine's functor $(B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} -)^{G_{k_v}}$, and M_p is the p-adic realization of M.*

Note that M_p is a crystalline representaion of G_{k_v} by Theorem 4.2, so we have $\dim_{k_{0,v}} M_{\mathrm{crys},v} = \dim_{\mathbb{Q}_p} M_p = \dim_{\mathbb{Q}} M_\omega$. Here, $k_{0,v}$ is the fraction field of the ring of Witt vectors with coefficients in the residue field $k(v)$ of $O_{(v)}$. Note also that the pair $(M_{\mathrm{crys},v}, M_{crys,v} \otimes_{k_{0,v}} k_v)$ gives an admissible filtered φ-module in the sense of Fontaine ([Fo1], [Fo2]). The crystalline realization is functorial, and defines a fiber functor $\mathrm{MT}(O_{(v)}) \to \mathrm{Vect}_{k_{0,v}}$, which factors through the category of admissible filtered φ-modules $\mathrm{MF}^{\mathrm{ad}}_{k_{0,v}}(\varphi)$.

Remark 4.5 By using the fact that $H^1_{\mathrm{st}}(k_v, \mathbb{Q}_p(1)) = H^1(k_v, \mathbb{Q}_p(1))$ and introducing "semistable inertia group" at v, we can show that M_p is a semistable representation of G_{k_v} for any mixed Tate motive M in $\mathrm{MT}(k)$, similarly as the proof of Theorem 4.2. Thus, we can define the crystalline realization (or semistable realization) $M_{\mathrm{crys},v}$ to be $D_{\mathrm{st}}(M_p) = (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} M_p)^{G_{k_v}}$ for all $M \in \mathrm{MT}(k)$, and get a functor $\mathrm{MT}(k) \to \mathrm{MF}^{\mathrm{ad}}_{k_{0,v}}(\varphi, N)$ to the category of admissible filtered (φ, N)-modules.

4.2 Comparison Isomorphism.

In this subsection, we prove a "Berthelot-Ogus like" comparison isomorphism between the crystalline realization and the de Rham realization. We defined the crystalline realization by using Fontaine's functor, so we need another "geometrical" construction of the crystalline realization to compare it with the de Rham realization (it is not obvious that the other construction is functorial).
For preparing the following theorem, we briefly recall that Voevodsky's category $\mathrm{DM}(k)$ (see, [V]), Levine's category $\mathrm{MT}(k)$ (see, [L]), and Deligne-Goncharov's category $\mathrm{MT}(O_{(v)})$ (see, [DG]). Let k be a field. First, let $\mathrm{SmCor}(k)$ be the additive category whose objects are smooth separated scheme over k, and morphisms $\mathrm{Hom}(X, Y)$ are free abelian group generated by reduced irreducible closed subschemes Z of $X \times Y$, which are finite over X and dominate a connected component of X. Then, Voevodsky's tensor triangulated category $\mathrm{DM}(k)$ is constructed from the category of bounded complexes $K^b(\mathrm{SmCor}(k))$ of $\mathrm{SmCor}(k)$ by localizing the thick subcategory generated by $[X \times \mathbb{A}^1] \to [X]$ (homotopy invariance), and $[U \cap V] \to [U] \oplus [V] \to [X]$ for $X = U \cup V$ (Mayer-Vietoris), adding images of direct factors of idempotents, and inverting formally $\mathbb{Z}(1)$.
Let k be a number field. Then, the vanishing conjecture of Beilinson-Soulé holds for k. From the vanishing conjecture of Beilinson-Soulé, Levine constructed the Tannakian category of mixed Tate motives $\mathrm{MT}(k)$ from $\mathrm{DMT}(k)$ by taking a heart with respect to a t-structure. Here, $\mathrm{DMT}(k)$ is the sub-tensor triangulated category of $\mathrm{DM}(k)_{\mathbb{Q}}$ generated by $\mathbb{Q}(n)$'s.

For a finite place v of k, let $O_{(v)}$ denote the localization of k at v. Deligne-Goncharov defined the full subcategory $\mathrm{MT}(O_{(v)})$ of mixed Tate motives unramified at v in $\mathrm{MT}(k)$, whose objects are mixed Tate motives M in $\mathrm{MT}(k)$ such that for each subquotient E of M, which is an extension of $\mathbb{Q}(n)$ by $\mathbb{Q}(n+1)$, the extension class of E in

$$\mathrm{Ext}^1_{\mathrm{MT}(k)}(\mathbb{Q}(n), \mathbb{Q}(n+1)) \xleftarrow{\cong} \mathrm{Ext}^1_{\mathrm{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(1)) \cong k^\times \otimes \mathbb{Q}$$

is in $O^\times_{(v)} \otimes \mathbb{Q} (\subset k^\times \otimes \mathbb{Q})$. The following theorem is the comparison isomorphism between crystalline realization and de Rham realization. However, we defined the crystalline realization by using p-adic étale realization. So, the content of the following theorem is the comparison isomorphism between p-adic étale realization and the pair of crystalline and de Rham realizations.

Theorem 4.6 *(Berthelot-Ogus isomorphism) For any mixed Tate motive M in $\mathrm{MT}(O_{(v)})$, we have a canonical isomorphism*

$$k_v \otimes_{k_{0,v}} M_{\mathrm{crys},v} \cong k_v \otimes_k M_{\mathrm{dR}}.$$

Remark 4.7 (Hyodo-Kato isomorphism) After choosing a uniformizer π of k_v, we can prove a canonical isomorphism

$$k_v \otimes_{k_{0,v}} M_{\mathrm{crys},v} \cong k_v \otimes_k M_{\mathrm{dR}}$$

for any mixed Tate motive M in $\mathrm{MT}(k)$ by the same way (cf. Remark 4.5).

Remark 4.8 From the functorial isomorphism $M_{\mathrm{crys},v} \otimes_{k_{0,v}} k_v \cong M_{\mathrm{dR}} \otimes_k k_v$, we have $G_\omega \otimes_{\mathbb{Q}} k_v \cong G_{\mathrm{crys}} \otimes_{k_{0,v}} k_v$. Here, $G := \pi_1(\mathrm{MT}(O_{(v)})) \in \text{pro-}\mathrm{MT}(O_{(v)})$ is the fundamental $\mathrm{MT}(O_{(v)})$-group (See, [D1, §6][D2, Definition 8.13]). Thus, we can consider the Frobenius element $F_{\mathfrak{p}}^{-1} \in G_\omega(k_v)$ if $k_{0,v} = k_v$ (For example, in the case where k is $\mathbb{Q}(\mu_N)$ and v is a prime ideal not dividing (N)).

Proof First, we observe the following thing. Let X and Y be smooth schemes over k, and Γ be an integral closed subschemes of $X \times Y$, which is finite surjective over a component of X. Then, by using de Jong's alterations, there exists a finite extension k' of k, a prime ideal w over v, semistable pairs (cf. [dJ]) $(\mathcal{X}, \mathcal{D})$ and $(\mathcal{Y}, \mathcal{E})$ over $O_{(w)}$, such that $f_X : (\mathcal{X} \setminus \mathcal{D})_{k'} \to X$ and $f_Y : (\mathcal{Y} \setminus \mathcal{E})_{k'} \to Y$ are generically étale alterations of X, and Y, respectively. Put $[\widetilde{\Gamma}'_{k'}] := (f_X \times f_Y)^![\Gamma \otimes_k k']$. Here, $(f_X \times f_Y)^! : CH^*(\Gamma \otimes_k k') \to CH^*(\Gamma \otimes_k k' \times_{(X \times_k Y) \otimes_k k'} ((\mathcal{X} \setminus \mathcal{D})_{k'} \times (\mathcal{Y} \setminus \mathcal{E})_{k'}))$ is the Fulton-MacPherson's refined Gysin map. Let $\widetilde{\Gamma}_{k'}$ denote the closure of $\widetilde{\Gamma}'_{k'}$ in $\mathcal{X}_{k'} \times \mathcal{Y}_{k'}$, Then, we have $\widetilde{\Gamma}_{k'} \cap (\mathcal{X}_{k'} \times \mathcal{E}_{k'}) \subset \widetilde{\Gamma}_{k'} \cap (\mathcal{D}_{k'} \times \mathcal{Y}_{k'})$. After choosing a uniformizer $\pi' \in k'_w$, we have the comparison isomorphisms $B_{\mathrm{st}} \otimes_{k'_{0,w}} H^m_{\text{log-crys}}((\mathcal{X} \setminus \mathcal{D})_{k'(w)}) \cong B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}((\mathcal{X} \setminus \mathcal{D})_{\overline{k}})$, and $B_{\mathrm{st}} \otimes_{k'_{0,w}} H^m_{\text{log-crys}}((\mathcal{Y} \setminus \mathcal{E})_{k'(w)}) \cong B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}((\mathcal{Y} \setminus \mathcal{E})_{\overline{k}})$

proved in [Y]. By $\widetilde{\Gamma}_{k'} \cap (\mathcal{X}_{k'} \times \mathcal{E}_{k'}) \subset \widetilde{\Gamma}_{k'} \cap (\mathcal{D}_{k'} \times \mathcal{Y}_{k'})$, we can define the cycle classes (cf. [Y])

$$\mathrm{cl}(\widetilde{\Gamma}_{\overline{k}}) \in H^{2\dim Y}_{\text{ét}}(\mathcal{X}_{\overline{k}} \times \mathcal{Y}_{\overline{k}}, (\mathcal{X}_{\overline{k}} \times \mathcal{E}_{\overline{k}})_!, (\mathcal{D}_{\overline{k}} \times \mathcal{Y}_{\overline{k}})_*),$$

and

$$\mathrm{cl}(\widetilde{\Gamma}_{k'}) \in H^{2\dim Y}_{\mathrm{dR}}(\mathcal{X}_{k'} \times \mathcal{Y}_{k'}, (\mathcal{X}_{k'} \times \mathcal{E}_{k'})_!, (\mathcal{D}_{k'} \times \mathcal{Y}_{k'})_*).$$

Then, by using these cycle classes, we get a commutative diagram ([Y])

$$\begin{tikzcd}
k'_w \otimes_{k'_{0,w}} D_{\mathrm{st},k'_w}(H^m_{\text{ét}}((\mathcal{Y} \setminus \mathcal{E})_{\overline{k}})) \arrow[r, "\cong"] \arrow[d, "{[\widetilde{\Gamma}_{\overline{k}}]^*}"] & k'_w \otimes_{k'} H^m_{\mathrm{dR}}((\mathcal{Y} \setminus \mathcal{E})_{k'}) \arrow[d, "{[\widetilde{\Gamma}_{k'}]^*}"] \\
k'_w \otimes_{k'_{0,w}} D_{\mathrm{st},k'_w}(H^m_{\text{ét}}((\mathcal{X} \setminus \mathcal{D})_{\overline{k}})) \arrow[r, "\cong"] & k'_w \otimes_{k'} H^m_{\mathrm{dR}}((\mathcal{X} \setminus \mathcal{D})_{k'}),
\end{tikzcd}$$

where we used Hyodo-Kato isomorphism [Y].
Let $[\Xi_X] \in CH(X_{k'} \times_{(X_{k'} \times X_{k'})} (\mathcal{X} \times \mathcal{X}))$ be $(f_X \times f_X)^!([\Delta_{X_{k'}}])$, where f_X is the morphism $\mathcal{X}_{k'} \to X_{k'}$, $(f \times f)^!$ means Fulton-MacPherson's refined Gysin homomorphism, and $\Delta_{X_{k'}}$ is the diagonal class of $X_{k'}$. We define $[\Xi_Y]$ by the same way, then by using these cycle classes and the compatibility of the comparison isomorphism with cycle classes, we get commutative diagrams

$$\begin{tikzcd}
D_{\mathrm{st},k'_w}(H^m_{\text{ét}}((\mathcal{X} \setminus \mathcal{D})_{\overline{k}}))_{k'_w} \arrow[r, two heads, "f_{X*}"] \arrow[d, "\cong"] & D_{\mathrm{st},k'_w}(H^m_{\text{ét}}(X_{\overline{k}}))_{k'_w} \arrow[r, hook, "f_X^*"] & D_{\mathrm{st},k'_w}(H^m_{\text{ét}}((\mathcal{X} \setminus \mathcal{D})_{\overline{k}}))_{k'_w} \arrow[d, "\cong"] \\
H^m_{\mathrm{dR}}((\mathcal{X} \setminus \mathcal{D})_{k'})_{k'_w} \arrow[r, two heads, "f_{X*}"] & H^m_{\mathrm{dR}}(X_{k'})_{k'_w} \arrow[r, hook, "f_X^*"] & H^m_{\mathrm{dR}}((\mathcal{X} \setminus \mathcal{D})_{k'})_{k'_w},
\end{tikzcd}$$

$$\begin{tikzcd}
D_{\mathrm{st},k'_w}(H^m_{\text{ét}}((\mathcal{Y} \setminus \mathcal{E})_{\overline{k}}))_{k'_w} \arrow[r, two heads, "f_{Y*}"] \arrow[d, "\cong"] & D_{\mathrm{st},k'_w}(H^m_{\text{ét}}(Y_{\overline{k}}))_{k'_w} \arrow[r, hook, "f_Y^*"] & D_{\mathrm{st},k'_w}(H^m_{\text{ét}}((\mathcal{Y} \setminus \mathcal{E})_{\overline{k}}))_{k'_w} \arrow[d, "\cong"] \\
H^m_{\mathrm{dR}}((\mathcal{Y} \setminus \mathcal{E})_{k'})_{k'_w} \arrow[r, two heads, "f_{Y*}"] & H^m_{\mathrm{dR}}(Y_{k'})_{k'_w} \arrow[r, hook, "f_Y^*"] & H^m_{\mathrm{dR}}((\mathcal{Y} \setminus \mathcal{E})_{k'})_{k'_w}k,
\end{tikzcd}$$

where we abbreviate $k'_w \otimes_{k'_{0,w}} D_{\mathrm{st},k'_w}(-)$ and $k'_w \otimes_{k'} H^m_{\mathrm{dR}}(-)$ as $D_{\mathrm{st},k'_w}(-)_{k'_w}$ and $H^m_{\mathrm{dR}}(-)_{k'_w}$ respectively by a typesetting reason. So, we get isomorphisms

$$k'_w \otimes_{k'_{0,w}} D_{\mathrm{st},k'_w}(H^m_{\text{ét}}(X_{\overline{k}})) \cong k'_w \otimes_{k'} H^m_{\mathrm{dR}}(X_{k'}),$$

and

$$k'_w \otimes_{k'_{0,w}} D_{\mathrm{st},k'_w}(H^m_{\text{ét}}(Y_{\overline{k}})) \cong k'_w \otimes_{k'} H^m_{\mathrm{dR}}(Y_{k'}).$$

By using the following commutative diagrams

$$\begin{tikzcd}
H^m_{\text{ét}}((\mathcal{Y} \setminus \mathcal{E})_{\overline{k}}) \arrow[r, two heads, "f_{Y*}"] \arrow[d, "{[\widetilde{\Gamma}_{\overline{k}}]^*}"] & H^m_{\text{ét}}(Y_{\overline{k}}) \arrow[r, hook, "f_Y^*"] & H^m_{\text{ét}}((\mathcal{Y} \setminus \mathcal{E})_{\overline{k}}) \arrow[d, "{[\widetilde{\Gamma}_{\overline{k}}]^*}"] \\
H^m_{\text{ét}}((\mathcal{X} \setminus \mathcal{D})_{\overline{k}}) \arrow[r, two heads, "f_{X*}"] & H^m_{\text{ét}}(X_{\overline{k}}) \arrow[r, hook, "f_X^*"] & H^m_{\text{ét}}((\mathcal{X} \setminus \mathcal{D})_{\overline{k}}),
\end{tikzcd}$$

and

$$\begin{array}{ccccc} H^m_{\mathrm{dR}}((\mathcal{Y}\setminus\mathcal{E})_{k'}) & \xrightarrow{f_{Y*}} & H^m_{\mathrm{dR}}(Y_{k'}) & \xrightarrow{f^*_Y} & H^m_{\mathrm{dR}}((\mathcal{Y}\setminus\mathcal{E})_{k'}) \\ \downarrow{\scriptstyle [\widetilde{\Gamma}_{k'}]^*} & & & & \downarrow{\scriptstyle [\widetilde{\Gamma}_{k'}]^*} \\ H^m_{\mathrm{dR}}((\mathcal{X}\setminus\mathcal{D})_{k'}) & \xrightarrow{f_{X*}} & H^m_{\mathrm{dR}}(X_{k'}) & \xrightarrow{f^*_X} & H^m_{\mathrm{dR}}((\mathcal{X}\setminus\mathcal{D})_{k'}), \end{array}$$

we finally get a commutative diagram

$$\begin{array}{ccc} k'_w \otimes_{k'_{0,w}} D_{\mathrm{st},k'_w}(H^m_{\text{ét}}(Y_{\overline{k}})) & \xrightarrow{\cong} & k'_w \otimes_{k'} H^m_{\mathrm{dR}}(Y_{k'}) \\ {\scriptstyle \text{restriction of } [\widetilde{\Gamma}_{\overline{k}}]^*}\downarrow & & \downarrow{\scriptstyle \text{restriction of } [\widetilde{\Gamma}_{k'}]^*} \\ k'_w \otimes_{k'_{0,w}} D_{\mathrm{st},k'_w}(H^m_{\text{ét}}(X_{\overline{k}})) & \xrightarrow{\cong} & k'_w \otimes_{k'} H^m_{\mathrm{dR}}(X_{k'}). \end{array}$$

Now, take a triple $(X^\bullet, f, n)$ for the given motive M in $\mathrm{MT}(O_{(v)})$, such that $f(X^\bullet)(n)$ represents M, where $X^\bullet \in K^b(\mathrm{SmCor}(k))$, $n \in \mathbb{Z}$, and f is an idempotent in $K^b(\mathrm{SmCor}(k))$. We will proceed the above construction successively for the complex $X^\bullet$ in $\mathrm{SmCor}(k)$, by replacing the finite extension k' one by one (Here, $X^\bullet$ is bounded. So, we can start from the first non-empty place and make the above construction and the above commutative diagram. Next, we make the above construction and commutative diagram in the next place after a finite base extension. We replace the first place by the finite base extension...). By using $((\mathcal{X}^\bullet, \mathcal{D}^\bullet), \{\Gamma^\bullet_{j_\bullet}\}_{j_\bullet})$, we can define sequences $((C^\bullet)^\bullet_{\text{ét}}, d^\bullet_{\text{ét}})$, and $((C^\bullet)^\bullet_{\mathrm{dR}}, d^\bullet_{\mathrm{dR}})$ of cohomological complexes, where $(C^\bullet)^i_{\text{ét}}$ and $(C^\bullet)^i_{\mathrm{dR}}$ calculate the étale cohomology and de Rham cohomology of $\mathcal{X}^i_{\overline{k}}$ and $\mathcal{X}^i_k$ respectively, and $d^i_{\text{ét}}$ and d^i_{dR} are defined by $\{\Gamma^i_{\overline{k},j_\bullet}\}_{j_\bullet}$, and $\{\Gamma^i_{k,j_\bullet}\}_{j_\bullet}$ respectively. Note that we do not define the crystalline version $((C^\bullet)^i_{\mathrm{crys}}, d^\bullet_{\mathrm{crys}})$. Even if we define it by taking integral models of $\Gamma^\bullet_j$'s, we do not have $d^{i+1}_{\mathrm{crys}} \circ d^i_{\mathrm{crys}} = 0$ for the sequence of complexes $(C^\bullet)^\bullet_{\mathrm{crys}}$ in general, because of the lack of the uniqueness of the extensions $\Gamma^\bullet_j$'s (cf. [DG, Lemma 1.5.1]). So, we cannot define a crystalline realization by using $(C^\bullet)^\bullet_{\mathrm{crys}}$ at least in the present situation (Note that we do not need to get $d^\bullet_{\mathrm{crys}}$ by integral models of $\Gamma^\bullet_j$'s in this proof). On the other hand, we have $d^{i+1}_{\text{ét}/\mathrm{dR}} \circ d^i_{\text{ét}/\mathrm{dR}} = 0$ for the sequence of complexes $(C^\bullet)^\bullet_{\text{ét}/\mathrm{dR}}$, because they live on the generic fiber (cf. [DG, Lemma 1.5.1]) and we have the uniqueness (Note that the above construction and the above commutative diagram work after replacing H^m by $R\Gamma$, because we do not use integral models of $\Gamma^\bullet_j$'s, but only use the generic fiber of them. See [DG, 1.5] for the de Rham part $(C^\bullet)^\bullet_{\mathrm{dR}}$). Therefore, we get an isomorphism $k'_w \otimes_{k'_{0,w}} M_{\mathrm{crys},w} \cong k'_w \otimes_{k'_{0,w}} D_{\mathrm{st},k'_w}(M_p) \cong k'_w \otimes_{k'} M_{\mathrm{dR},k'} \cong k'_w \otimes_k M_{\mathrm{dR}}$.
Now, we use the condition that M is in $\mathrm{MT}(O_{(v)})$. The p-adic realization M_p is crystalline at v by Thoerem 4.2. So, we have $M_{\mathrm{crys},w} \cong k'_{0,w} \otimes_{k_{0,v}} M_{\mathrm{crys},v}$. Therefore, we have an isomorphism $k'_w \otimes_{k_{0,v}} M_{\mathrm{crys},v} \cong k'_w \otimes_k M_{\mathrm{dR}}$. In general, for any element $\tau \in \mathrm{Gal}(k'_w/k_v)$, we have an isomorphism $k'^{;\tau}_w \otimes_{k_{0,v}} M_{\mathrm{crys},v} \cong k'^{;\tau}_w \otimes_k M_{\mathrm{dR}}$ by using the triple $\{(\mathcal{X}^{\bullet,\tau}, \mathcal{D}^{\bullet,\tau}), f^\tau, n\}$. Thus, we have

an isomorphism $k_v \otimes_{k_{0,v}} M_{\mathrm{crys},v} \cong k_v \otimes_k M_{\mathrm{dR}}$ by the descent. Since M_p is crystalline at v, this isomorphism does not depend on the choice of π', and we can show that this isomorphism does not depend on the choice of good reduction models and this isomorphism is functorial by using the standard product argument.

4.3 Some Remarks and Questions.

The crystalline realization to the category of φ-modules (*not* to the category of admissible filtered φ-modules) is split, because we have

$$\mathrm{Ext}^1_{\mathrm{MT}(O_{(v)})}(\mathbb{Q}(0), \mathbb{Q}(n)) = 0$$

for $n \leq 0$ and $\mathrm{Ext}^1_{\mathrm{Mod}_{k_{0,v}}(\varphi)}(k_{0,v}(0), k_{0,v}(n)) = 0$ for $n > 0$.
So, we can expect that the crystalline realization $\mathrm{MT}(O_{(v)}) \to \mathrm{Vect}_{k_{0,v}}$ factors through $\mathrm{MT}(k(v))$. Note that the weight filtration of mixed Tate motives over a finite field is split by Quillen's calculations of K-groups of finite fields ([Q]). Thus, they are sums of $\mathbb{Q}(n)$'s.
The weight filtration is motivic, and both of the de Rham realization and the crystalline realization are split. However, the splittings do not coincide, that is, the splitting of the crystalline realization does not coincide to the splitting of the de Rham realization via the Berthelot-Ogus isomorphism of Theorem 4.6. The iterated integrals and p-adic MLV's appear in the difference of these splittings. See also Remark 3.9.

Remark 4.9 We have $\mathrm{Ext}^1_{\mathrm{Mod}_{k_{0,v}}(\varphi)}(k_{0,v}(0), k_{0,v}(0)) \cong \mathbb{Q}_p \neq 0$, and this gap corresponds to the "near critical strip case" of Beilinson's conjecture and Bloch-Kato's Tamagawa number conjecture, that is, we need not only regulator maps, but also Chow groups to formulate these conjectures near the critical strip case (that is, the case where the weight of motive is 0 or -2). In this case, this corresponds to the "dual" of the fact that the image of the Dirichlet regulator is not a lattice of $\mathbb{R}^{r_1+r_2}$, but a lattice of a hyperplane of $\mathbb{R}^{r_1+r_2}$. The author does not know a direct proof of the fact that the non-trivial extension in $\mathrm{Ext}^1_{\mathrm{Mod}_{k_{0,v}}(\varphi)}(k_{0,v}(0), k_{0,v}(0)) = \mathbb{Q}_p$ does not occur in the crystalline realization.

Example 3 (Kummer torsor) Let K be a finite extension of $\mathbb{Q}_p$, K_0 be the fraction field of the ring of Witt vectors with coefficient in the residue field of K. Let $z \in 1 + \pi O_K$. Let

$$0 \to \mathbb{Q}_p(1) \to V(z)_p \to \mathbb{Q}_p(0) \to 0$$

be the extension of p-adic realization corresponding to z. Fix e_0 a generator of $\mathbb{Q}_p(1)$ corresponding $\{\zeta_n\}_n$, and e_1 the generator of $\mathbb{Q}_p(0)$ corresponding 1.

Then, the action of Galois group is the following:

$$\begin{cases} ge_0 = \chi(g)e_0, \\ ge_1 = e_1 + \psi_z(g)e_0. \end{cases}$$

Here, χ is the p-adic cyclotomic character, and ψ_z is characterized by $g(z^{1/p^n}) = \zeta_n^{\psi_z(g)} z^{1/p^n}$.
Then, $V(z)_{\text{crys}} \cong (B_{\text{crys}} \otimes_{\mathbb{Q}_p} V(z)_p)^{G_K}$ has the following basis:

$$\begin{cases} t^{-1} \otimes e_0 =: x_0, \\ e_1 - t^{-1} \log[\underline{z}] \otimes e_0 =: x_1. \end{cases}$$

Here, $t := \log[\underline{\zeta}], \log[\underline{z}] \in B_{\text{crys}}$. Thus, the Frobenius action is the following:

$$\begin{cases} \phi(x_0) = \frac{1}{p} x_0, \\ \phi(x_1) = x_1. \end{cases}$$

The filtration after $K \otimes_{K_0}$ is the following:

$$\begin{cases} \text{Fil}^{-1} V(z)_{\text{dR}} = V(z)_{\text{dR}} = \langle x_0, x_1 \rangle_K, \\ \text{Fil}^0 V(z)_{\text{dR}} = \langle x_1 + (\log z) x_0 \rangle_K, \\ \text{Fil}^1 V(z)_{\text{dR}} = 0 \end{cases}$$

(In B_{dR}, we have $t^{-1} \log \frac{z}{[\underline{z}]} \in \text{Fil}^0 B_{\text{dR}}$). Thus, we have splittings:

$$V(z)_{\text{crys}} = \langle x_0 \rangle_{K_0} \oplus \langle x_1 \rangle_{K_0} = K_0(1) \oplus K_0(0),$$

$$V(z)_{\text{dR}} = \langle x_0 \rangle_K \oplus \langle x_1 + (\log z) x_0 \rangle_K = K(1) \oplus K(0).$$

These splittings do not coincide in general.
We will recover the calculation $\phi^{-1}(0) = \log z^{1-p}$ in [D1, 2.9, 2.10]. In this case, we assume $K = K_0$. By the above calculation, the Kummer torsor $K(z)_{\text{dR}}$ is

$$K(z)_{\text{dR}} = -(x_1 + (\log z) x_0) + K x_0$$

(For the purpose of making satisfy $\nabla(u) = du - \frac{dz}{z}$ in [D1, 2.10], we use the above sign convention). Then, we have

$$\begin{aligned} \phi^{-1}(0) \leftrightarrow \phi^{-1}(-(x_1 + (\log z) x_0) + 0) &= -(x_1 + p(\log z) x_0) \\ &= -(x_1 + (\log z) x_0) + (1-p)(\log z) x_0 \\ &= -(x_1 + (\log z) x_0) + (\log z^{1-p}) x_0 \\ &\leftrightarrow \log z^{1-p}. \end{aligned}$$

This coincides the calculation in [D1, 2.10]. Here, $\leftrightarrow$ is the identification via $K(z)_{\text{dR}} = -(x_1 + (\log z) x_0) + K x_0 \cong K$.

Next, we define polylogarithm extensions. In the following, we consider the case where k is a cyclotimic field $\mathbb{Q}(\mu_N)$ for $N \geq 1$. For $\zeta \in \mu_N$, let $U_\zeta \in$ pro-MT$(\mathbb{Q}(\mu_N))$ be the kernel of $\pi_1^{\mathcal{M}}(\mathbb{P}^1 \setminus \{0, 1\infty\}, \zeta) \to \pi_1^{\mathcal{M}}(\mathbb{G}_m, \zeta)$. We define $\mathcal{L}og_\zeta$ to be the abelianization of U_ζ Tate-twisted by (-1). We define $\mathcal{P}ol_\zeta$ with Tate twist (1) to be the push-out in the following diagram (see also, [D1, §16]):

$$\begin{tikzcd}
1 \arrow[r] & U_\zeta \arrow[r] \arrow[d] & \pi_1^{\mathcal{M}}(\mathbb{P}^1 \setminus \{0,1,\infty\}, \zeta) \arrow[r] \arrow[d] & \pi_1^{\mathcal{M}}(\mathbb{G}_m, \zeta) \arrow[r] \arrow[d, "="] & 1 \\
0 \arrow[r] & \mathcal{L}og_\zeta(1) \arrow[r] & \mathcal{P}ol_\zeta(1) \arrow[r] & \mathbb{Q}(1) \arrow[r] & 0.
\end{tikzcd} \tag{4.2}$$

For $n \geq 1$, we also define $\mathcal{P}ol_{n,\zeta}$ to be the push-out under $\mathcal{L}og_\zeta = \Pi_{n\geq 0}\mathbb{Q}(n) \to \mathbb{Q}(n)$ (see also, [D1, §16]):

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{L}og_\zeta \arrow[r] \arrow[d] & \mathcal{P}ol_\zeta \arrow[r] \arrow[d] & \mathbb{Q}(0) \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \mathbb{Q}(n) \arrow[r] & \mathcal{P}ol_{n,\zeta} \arrow[r] & \mathbb{Q}(0) \arrow[r] & 0.
\end{tikzcd}$$

The extension class $[\mathcal{P}ol_{n,\zeta}]$ lives in $\mathrm{Ext}^1_{\mathrm{MT}(\mathbb{Q}(\mu_N))}(\mathbb{Q}(0), \mathbb{Q}(n)) \cong K_{2n-1}(\mathbb{Q}(\mu_N))_{\mathbb{Q}}$. Let μ_N^0 be the group of primitive N-th roots of unity. Recall that Huber-Wildeshaus constructed motivic polylogarithm *classes* $\mathrm{pol}_\zeta \in \prod_{n\geq 2} K_{2n-1}(\mathbb{Q}(\mu_N))_{\mathbb{Q}}$ (not *extensions* of motives) in [HW].

Proposition 4.10 *Let n be an integer greater than or equal to 2, and ζ be an N-th root of unity. Then, the n-th component of Huber-Wildeshaus' motivic polylogarithm class* pol_ζ *(see, [HW, Definition 9.4]) is equal to* $(-1)^{n-1}\frac{n!}{N^{n-1}}[\mathcal{P}ol_{n,\zeta}]$ *under the identification*

$$K_{2n-1}(\mathbb{Q}(\mu_N))_{\mathbb{Q}} \cong \mathrm{Ext}^1_{\mathrm{MT}(\mathbb{Q}(\mu_N))}(\mathbb{Q}(0), \mathbb{Q}(n)).$$

In particular, the extension classes $\{[\mathcal{P}ol_{n,\zeta}]\}_{\zeta\in\mu_N^0}$ *generate* $K_{2n-1}(\mathbb{Q}(\mu_N))_{\mathbb{Q}}$.

Proof It is sufficient to show the equality after taking the Hodge realization. This follows from [D1, §3, §16, §19] and [HW, Theorem 9.5, Corolary 9.6]. Note that we consider as $\mathbb{Q}(n)_\omega$-torsor not as $\mathbb{Z}(n)_\omega$-torsor, and we do not multiply $\frac{1}{(n-1)!}$ on the integral structure unlike as [D1] (See also Example (2, 2)).

Fix a place $v \nmid N$ of $\mathbb{Q}(\mu_N)$. Put $K := \mathbb{Q}(\mu_N)_v$. Let p be the prime devied by v. Note that K is unramified over $\mathbb{Q}_p$. Let σ denote the Frobenius endomorphism on K. For a mixed Tate motive $[0 \to \mathbb{Q}(n) \to M \to \mathbb{Q}(0) \to 0] \in \mathrm{Ext}^1_{\mathrm{MT}(O_{(v)})}(\mathbb{Q}(0), \mathbb{Q}(n))$, the pair $M_{\mathrm{syn}} := (M_{\mathrm{crys},v}, M_{\mathrm{dR}} \otimes_{\mathbb{Q}(\mu_N)} K)$ defines a extension of filtered φ-modules:

$$0 \to K(n) \to M_{\mathrm{syn}} \to K(0) \to 0.$$

Here, $K(i)$ is the Tate object in the category of filtered φ-modules over K. Thus, we have a map

$$\begin{aligned} r_n : K_{2n-1}(O_{(v)})_{\mathbb{Q}} &\cong \mathrm{Ext}^1_{\mathrm{MT}(O_{(v)})}(\mathbb{Q}(0), \mathbb{Q}(n)) \\ &\to \mathrm{Ext}^1_{\mathrm{MF}^f_K}(K(0), K(n)) \cong H^1_{\mathrm{syn}}(K, K(n)). \end{aligned}$$

See, [Ba] for the last isomorphism. We call r_n the n-th *syntomic regulator map*. Recall that H^1_{syn} is a finite dimensional $\mathbb{Q}_p$-vector space, not a K-vector space. We fix an isomorphism $H^1_{\mathrm{syn}}(K, K(n)) \cong K$ as $\mathbb{Q}_p$-vector spaces for $n \geq 1$ as follows.

$$\begin{aligned} &H^1_{\mathrm{syn}}(K, K(n)) \\ &\cong \mathrm{coker}(K(n)_{\mathrm{crys}} \overset{a\mapsto(\bar{a},(1-\varphi)(a))}{\longrightarrow} (K(n)_{\mathrm{dR}}/\mathrm{Fil}^0 K(n)_{\mathrm{dR}}) \oplus K(n)_{\mathrm{crys}}) \\ &\cong \mathrm{coker}(K \overset{a\mapsto(a,(1-p^{-n}\sigma)(a))}{\longrightarrow} K \oplus K) \\ &\overset{[(a,b)]\mapsto b-(1-p^{-n}\sigma)(a)}{\cong} K. \end{aligned}$$

In general, note that for a filtered φ-module D and for

$$[(x,y)] \in \mathrm{coker}(D \overset{a\mapsto(\bar{a},(1-\varphi_D)(a))}{\longrightarrow} (D/\mathrm{Fil}^0 D) \oplus D) \cong \mathrm{Ext}^1_{\mathrm{MF}^f_K}(K(0), D),$$

the corresponding extension E of $K(0)$ by D is the following: $E = D \oplus Ke_0$

$$\begin{cases} \mathrm{Fil}^i E = \mathrm{Fil}^i D + \langle x + e_0 \rangle_K & \text{for } i \leq 0, \\ \mathrm{Fil}^i E = \mathrm{Fil}^i D & \text{for } i > 0, \end{cases}$$

$$\begin{cases} \varphi_E(a) = \varphi_D(a) & \text{for } a \in D, \\ \varphi_E(e_0) = e_0 + y. \end{cases}$$

PROPOSITION 4.11 *The syntomic regulator map*

$$r_1 : K_1(O_{(v)})_{\mathbb{Q}} \cong O^{\times}_{(v)} \otimes \mathbb{Q} \to H^1_{\mathrm{syn}}(K, K(1)) \cong K$$

is given by $z \mapsto -(1-\frac{1}{p})\log z$*. For* $n \geq 2$*, the syntomic regulator map*

$$r_n : K_{2n-1}(\mathbb{Q}(\mu_N))_{\mathbb{Q}} \to H^1_{\mathrm{syn}}(K, K(n)) \cong K$$

sends $[\mathcal{P}ol_{n,\zeta}]$ *to* $-N^{n-1}(1-\frac{1}{p^n})\mathrm{Li}^a_n(\zeta)$.

Note that Coleman's p-adic polylogarithm $(1-\frac{1}{p^n})\mathrm{Li}^a_n(\zeta)$ is often written by $\ell^{(p)}_n(\zeta)$, and does not depend on the chice of a.

Remark 4.12 In the above proposition, we used the homomorphism induced by crystalline realizations and the isomorphism between K-theory and $\mathrm{Ext}^1_{\mathrm{MT}}$ as a regulator. For a purely K-theoretic definition of a regulator and its calculation, see [BdJ].

Remark 4.13 If we use an identification

$$\mathrm{coker}(K \overset{a\mapsto(a,(1-p^{-n}\sigma)(a))}{\longrightarrow} K\oplus K) \overset{[(a,b)]\mapsto a-(1-p^{-n}\sigma)^{-1}(b)}{\cong} K$$

(note that $1-p^{-n}\sigma$ is a bijection on K for $n\geq 1$), then the above formula changes as the following: the map

$$r_1 : K_1(O_{(v)})_{\mathbb{Q}} \cong O^{\times}_{(v)}\otimes\mathbb{Q} \to H^1_{\mathrm{syn}}(K,K(1))\cong K$$

is given by $z\mapsto \log z$. For $n\geq 2$, the map

$$r_n : K_{2n-1}(\mathbb{Q}(\mu_N))_{\mathbb{Q}} \to H^1_{\mathrm{syn}}(K,K(n))\cong K$$

sends $[\mathcal{P}ol_{n,\zeta}]$ to $N^{n-1}\mathrm{Li}^a_n(\zeta)$.

Proof The first assertion follows from Example (3). The second assertion follows from the following structure of $(\mathcal{P}ol_{n,\zeta})_{\mathrm{syn}} = ((\mathcal{P}ol_{n,\zeta})_{\mathrm{crys}}, (\mathcal{P}ol_{n,\zeta})_{\mathrm{dR}})$: $(\mathcal{P}ol_{n,\zeta})_{\mathrm{crys}} = \langle x_0,x_1\rangle_K$

$$\begin{cases}\varphi(x_0)=\frac{1}{p^n}x_0,\\ \varphi(x_1)=x_1-N^{n-1}(1-p^{-n})\mathrm{Li}^a_n(\zeta),\end{cases}$$

$$\begin{cases}\mathrm{Fil}^{-n}(\mathcal{P}ol_{n,\zeta})_{\mathrm{dR}}=\langle x_0,x_1\rangle_K,\\ \mathrm{Fil}^{i}(\mathcal{P}ol_{n,\zeta})_{\mathrm{dR}}=\langle x_1\rangle_K \text{ for } -n<i\leq 0,\\ \mathrm{Fil}^{1}(\mathcal{P}ol_{n,\zeta})_{\mathrm{dR}}=0.\end{cases}$$

This structure follows from Example (2).

Remark 4.14 We have an isomorphism

$$B_{\mathrm{crys}}\otimes_{\mathbb{Q}_p}(P^{\mathcal{M}}_{y,x})_p \cong B_{\mathrm{crys}}\otimes_{K_0}(P^{\mathcal{M}}_{y,x})_{\mathrm{crys}}.$$

Here, $P^{\mathcal{M}}_{y,x}$ is a fundamental groupoid of $\mathbb{P}^1\setminus\{0,\infty\}\cup\mu_N$. This induces an isomorphism

$$B_{\mathrm{crys}}\otimes_{\mathbb{Q}_p}(\mathcal{P}ol_\zeta)_p \cong B_{\mathrm{crys}}\otimes_{K_0}(\mathcal{P}ol_\zeta)_{\mathrm{crys}}.$$

Thus, we have the following commutative diagram for $n\geq 2$:

$$\begin{tikzcd} K_{2n-1}(\mathbb{Q}(\mu_N))_{\mathbb{Q}} \arrow[r] \arrow[dr] & H^1(K,\mathbb{Q}_p(n)) \arrow[d, "\cong"] \\ & H^1_{\mathrm{syn}}(K,K(n)) \end{tikzcd} \qquad \begin{tikzcd} {[\mathcal{P}ol_{n,\zeta}]} \arrow[r, mapsto] \arrow[dr, mapsto] & {[(\mathcal{P}ol_{n,\zeta})_p]} \arrow[d, mapsto] \\ & {[(\mathcal{P}ol_{n,\zeta})_{\mathrm{syn}}].} \end{tikzcd}$$

Here, K denotes $\mathbb{Q}_p(\mu_N)$, ζ is in μ_N, and p does not divide N. The horizontal map sends the extension class $[\mathcal{P}ol_{n,\zeta}]$ to the one $[(\mathcal{P}ol_{n,\zeta})_p]$, and the oblique map sends the extension class $[\mathcal{P}ol_{n,\zeta}]$ to the one $[(\mathcal{P}ol_{n,\zeta})_{\mathrm{syn}}]$.
The fact that $[(\mathcal{P}ol_{n,\zeta})_p]$ is sent to $[(\mathcal{P}ol_{n,\zeta})_{\mathrm{syn}}]$ was first shown by T. Tsuji. Unfortunately, no preprint is available yet. His method is totally different. He does not use motivic theory or motivic π_1. He used the classical characterization (or the definition) of p-adic and syntomic polylogarithm sheaves as a specified extension (via residue isomorphisms etc.) of the constant sheaf by $\mathcal{L}og$, and checked the characterization coincides via the p-adic Hodge comparison isomorphism.

Finally, we'd like to propose some very vague questions. If we take the Hodge (resp. ℓ-adic) realization of the lower line of (4.2), and specialize it to the roots of unity, then we get the special values of polylogarithms (resp. the Soulé elements). This fact is important of Bloch-Kato's Tamagawa number conjecture ([BK]) for Tata motives. Furthermore, the Soulé elements form an Euler system, which has a power to show a half of Iwasawa main conjecture. The Soulé elements are sent to the Kubota-Leopoldt's p-adic L-function via Bloch-Kato's dual exponentioal map.

Question 1 Can we "suitably lift" this theory to the upper line of (4.2)?

More concretely:

Question 2 This will give a theory between non-commutative extensions of cyclotomic fields and multiple zeta values?

(It seems that Massey products play some roles instead of Ext^1.)

Question 3 This is related with Ihara's higher cyclotomic fields, Anderson-Ihara's higher circular units ([AI]), and Ozaki's non-commutative Iwasawa theory?

(Ozaki considered the maximal pro-p extensions unramified outside p of the cyclotmic fields, and its graded quotients of the lower central series, and he showed that Iwasawa class number formura for each graded quotient.)
Wojtkoviak studied ([W]) ℓ-adic iterated integrals, which specialize to the Soulé elements at the roots of unity in the case where the depth is one.

Question 4 What are the properties and axioms of "iterated integrals of Euler system"?

There are many difficulties to establish the above theory. It seems that the origin of the difficulties is that there are no good analytic properties for the zeta function in the higher depth cases. The above things are questions above "non-commutative Iwasawa theory in the mixed Tate sense".

Next, we propose some very vague questions about "non-commutative class field theory in the mixed Tate sense". We have the universal mixed Tate representation

$$\mathrm{Gal}(\overline{k}/k) \longrightarrow G_{\mathbb{A}_f}(\mathbb{A}_f)$$

for any number field k and ring of S-integers O_S, where $G_{\mathbb{A}_f}$ is the motivic Galois group of $\mathrm{MT}(O_S)$ with respect to the finite adele realization.

Question 5 Can we relate this with an automorphic representation of $G_{\mathbb{A}}$?

(We also note that $X^*(U_\omega) \cong X^*(U_\omega^{\mathrm{ab}}) \cong X^*(\mathrm{Lie}U_\omega^{\mathrm{ab}}) \cong \oplus_{n\geq 1}\mathrm{Ext}^1_{\mathrm{MT}(O_S)}(\mathbb{Q}(0),\mathbb{Q}(n)) \cong K_{2n-1}(O_S)_{\mathbb{Q}}$.) It seems that the concept of the automorphic representation is not good for unipotent groups. So, the author thinks that it will not be successful to consider automorphic representations. He also thinks that this corresponds that we cannot consider the L-factors and the functional equations in the higher depth cases. We modify the above question as follows (it becomes more vague):

Question 6 Can we find some "automorphy" in the lattice $G_\omega(\mathbb{Q}) \hookrightarrow G_\omega(\mathbb{A})$?

Manin studied the iterated integrals of modular forms ([M]). However, the analytic properties of them (e.g. "automorphy in the higher depth cases") are not clarified.

Question 7 Are there some kinds of relations among $a \in G_\omega(\mathbb{C})$, $F_{\mathfrak{p}}^{-1} \in G_\omega(k_{\mathfrak{p},0})$, and $\mathrm{Frob}_{\mathfrak{p}} \in G_{\mathbb{A}_f^p}(\mathbb{A}_f^p)$ for $\mathfrak{p} \notin S$?

The lower bounds of (p-adic) multiple zeta value spaces are (p-adic) transcendental number theoritic problem. The author thinks that we cannot show the lower bounds by using only algebraic arithmetic geometry, and that we need (p-adic) transcendental number theory (or ergodic theory) to show them. However, we might be able to attack the following weaker statement by using only algebraic arithmetic geometry.

Question 8 By finding some kinds of "automorphy" in the case where $k=\mathbb{Q}$, and $O_S=\mathbb{Z}$, can we show that the lower bounds of the dimensions of the p-adic multiple zeta value spaces for $p \leq \infty$ except p_0 are equivalent to the lower bounds of the dimensions of the p-adic multiple zeta value spaces for all $p \leq \infty$?

Take a 2-step unipotent quotient of U_ω. Then, we can consider the adelic theta theory for this group.

Question 9 Can we describe explicitly the theta theory for (p-adic) multiple L-values?

Question 10 By studying this, can we formulate a conjecture about the precise dimensions of (p-adic) multiple L-value spaces?

References

[AI] Anderson, G., Ihara, Y. *Pro-l branched coverings of P^1 and higher circular l-units.* Ann. of Math. (2) 128 (1988), no. 2, 271–293.

[B] Besser, A. *Coleman integration using the Tannakian formalism.* Math. Ann. 322 (2002), no. 1, 19–48.

[Ba] Bannai, K. *Syntomic cohomology as a p-adic absolute Hodge cohomology.* Math. Zeit. 242/3 (2002) 443–480.

[BdJ] Besser, A.; de Jeu, R. *The syntomic regulator for the K-theory of fields.* Ann. Sci. École Norm. Sup. (4) 36 (2003), no. 6, 867–924 (2004)

[BK] Bloch, S.; Kato, K. *L-functions and Tamagawa numbers of motives.* The Grothendieck Festschrift, Vol. I, 333–400.

[C] Coleman, R. F. *Dilogarithms, regulators and p-adic L-functions.* Invent. Math. 69 (1982), no. 2, 171–208.

[D1] Deligne, P. *Le groupe fondamental de la droite projective moins trois points.* Galois groups over $\mathbb{Q}$ (Berkeley, CA, 1987), 79–297, Math. Sci. Res. Inst. Publ., 16,

[D2] Deligne, P. *Catégories tannakiennes.* The Grothendieck Festschrift, Vol. II, Progr. Math., 87(1990) Birkhäuser Boston, Boston, MA, 111–195.

[DG] Deligne, P., Goncharov, B. *Groupes fondamentaux motiviques Tate mixte.* Ann. Sci. École Norm. Sup. (4) 38 (2005), no. 1, 1–56.

[dJ] de Jong, A. J. *Smoothness, semistability and alterations.* Inst. Hautes Etudes Sci. Publ. Math. No. 83 (1996), 51–93.

[Dr] Drinfel'd, V. G. *On quasitriangular quasi-Hopf algebras and a group closely connected woth* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Leningrad Math. J. 2(1991), no. 4, 829–860.

[Fa] Faltings, G. *F-isocrystals on open varieties: results and conjectures.* The Grothendieck Festschrift, Vol. II, Birkhäuser Boston, Boston, MA, (1990), 219–248.

[Fo1] Fontaine, J. -M. *Le corps des périodes p-adiques.* Periodes p-adiques (Bures-sur-Yvette, 1988). Astérisque 223 (1994), 59–111.

[Fo2] Fontaine, J.-M. *Repreésentations p-adiques semi-stables.* Periodes p-adiques (Bures-sur-Yvette, 1988). Asterisque No. 223 (1994), 113–184

[Fu0] Furusho, H. *The multiple zeta values and Grothendieck-Teichuüller groups.* to appear in AMS contemporary Math.

[Fu1] Furusho, H. *p-adic multiple zeta values. I. p-adic multiple polylogarithms and the p-dic KZ equation.* Invent. Math. 155(2004) , no. 2, 253–286.

[Fu2] Furusho, H. *p-adic multiple zeta values. II. Tannakian interpretation.* to appear in Amer. J. Math.

[G1] Goncharov, B. *Multiple ζ-Values, Galois Groupes, and Geometry of Modular Varieties.* preprint AG/0005069

[G2] Goncharov, B. *The dihedral Lie algebras and Galois symmetries of $\pi_1(\mathbb{P}^1 - \{0, \infty\} \cup \mu_N)$.* Duke. Math. J. 110, (2001), 397–487.

[HK] Hyodo, O.; Kato, K. *Semi-stable reduction and crystalline cohomology with logarithmic poles.* Periodes p-adiques (Bures-sur-Yvette, 1988). Astérisque 223, (1994), 221–268.

[HW] Huber, A.; Wildeshaus, J. *classical motivic polylogarithm according to Beilinson and Deligne.* Doc. Math. J. 3, (1998), 27–133.

[Ka] Kato, K. *Logarithmic structures of Fontaine-Illusie.* Algebraic analysis,geometry,and number theory. Johns Hopkins University Press, Baltimore (1989) , 191–224

[Ki] Kim, M. *The motivic fundamental group of* $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ *and the theorem of Siegel.* Invent. Math. 161 (2005), no. 3, 629–656.

[L] Levine, M. *Tate motives and the vanishing conjectures for algebraic K-theory.* Algebraic K-theory and algebraic topology (Lake Louis, 1991), NATO Adv. Sci. Inst. Ser. C Math. Phys. 407, Kluwer (1993) 167–188.

[M] Manin, Y. I. *Iterated integrals of modular forms and noncommutative modular symbols.* Algebraic geometry and number theory, 565–597, Progr. Math., 253, Birkhauser Boston, Boston, MA, 2006.

[Q] Quillen, D. *On the cohomology and K-theory of the general linear groups over a finite field.* Ann. Math. 96 No. 3 (1972) 552–586

[T] Terasoma, T. *Mixed Tate motives and multiple zeta values.* Invent. Math. 149(2002) 339–369.

[V] Voevodsky, V. *Triangulated categories of motives over a field.* Cycles, trasfers and motivic homology theories. Ann. of Math. Studies 143, Princeton University Press (2000) 188-238.

[W] Wojtkowiak, Z. *On l-adic iterated integrals. III. Galois actions on fundamental groups.* Nagoya Math. J. 178 (2005), 1–36.

[Y] Yamashita, G. *p-adic étale cohomology and crystalline cohomology for open varieties with semistable reduction.* submitted.

Go Yamashita
School of Mathematical Sciences
University of Nottingham
University Park
Nottingham
NG7 2RD, United Kingdom
Go.Yamashita@nottingham.ac.uk