A p-adic Regulator Map and Finiteness Results for Arithmetic Schemes

Dedicated to Andrei Suslin on his 60th birthday

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Abstract. A main theme of the paper is a conjecture of Bloch-Kato on the image of p-adic regulator maps for a proper smooth variety X over an algebraic number field k. The conjecture for a regulator map of particular degree and weight is related to finiteness of two arithmetic objects: One is the p-primary torsion part of the Chow group in codimension 2 of X. Another is an unramified cohomology group of X. As an application, for a regular model $\mathscr{X}$ of X over the integer ring of k, we prove an injectivity result on the torsion cycle class map of codimension 2 with values in a new p-adic cohomology of $\mathscr{X}$ introduced by the second author, which is a candidate of the conjectural étale motivic cohomology with finite coefficients of Beilinson-Lichtenbaum.

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1 Introduction

Let k be an algebraic number field and let G_k be the absolute Galois group $\mathrm{Gal}(\overline{k}/k)$, where $\overline{k}$ denotes a fixed algebraic closure of k. Let X be a projective smooth variety over k and put $\overline{X} := X \otimes_k \overline{k}$. Fix a prime p and integers $r, m \geq 1$. A main theme

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of this paper is a conjecture of Bloch and Kato concerning the image of the p-adic regulator map

$$\mathrm{reg}^{r,m} \ : \ \mathrm{CH}^r(X,m)\otimes\mathbb{Q}_p \longrightarrow H^1_{\mathrm{cont}}(k, H^{2r-m-1}_{\text{ét}}(\overline{X},\mathbb{Q}_p(r)))$$

from Bloch's higher Chow group to continuous Galois cohomology of G_k ([BK2] Conjecture 5.3). See §3 below for the definition of this map in the case $(r,m)=(2,1)$. This conjecture affirms that its image agrees with the subspace

$$H^1_g(k, H^{2r-m-1}_{\text{ét}}(\overline{X},\mathbb{Q}_p(r))) \subset H^1_{\mathrm{cont}}(k, H^2_{\text{ét}}(\overline{X},\mathbb{Q}_p(2)))$$

defined in loc. cit. (see §2.1 below), and plays a crucial role in the so-called Tamagawa number conjecture on special values of L-functions attached to X. In terms of Galois representations, the conjecture means that a 1-extension of continuous p-adic representations of G_k

$$0 \longrightarrow H^{2r-m-1}_{\text{ét}}(\overline{X},\mathbb{Q}_p(r)) \longrightarrow E \longrightarrow \mathbb{Q}_p \longrightarrow 0$$

arises from a 1-extension of motives over k

$$0 \longrightarrow h^{2r-m-1}(X)(r) \longrightarrow M \longrightarrow h(\mathrm{Spec}(k)) \longrightarrow 0,$$

if and only if E is a de Rham representation of G_k. There has been only very few known results on the conjecture. In this paper we consider the following condition, which is the Bloch-Kato conjecture in the special case $(r,m)=(2,1)$:

> **H1**: *The image of the regulator map*
>
> $$\mathrm{reg} := \mathrm{reg}^{2,1} : \mathrm{CH}^2(X,1)\otimes\mathbb{Q}_p \longrightarrow H^1_{\mathrm{cont}}(k, H^2_{\text{ét}}(\overline{X},\mathbb{Q}_p(2))).$$
>
> *agrees with* $H^1_g(k, H^2_{\text{ét}}(\overline{X},\mathbb{Q}_p(2)))$.

We also consider a variant:

> **H1**$*$: *The image of the regulator map with* $\mathbb{Q}_p/\mathbb{Z}_p$*-coefficients*
>
> $$\mathrm{reg}_{\mathbb{Q}_p/\mathbb{Z}_p} : \mathrm{CH}^2(X,1)\otimes\mathbb{Q}_p/\mathbb{Z}_p \longrightarrow H^1_{\mathrm{Gal}}(k, H^2_{\text{ét}}(\overline{X},\mathbb{Q}_p/\mathbb{Z}_p(2)))$$
>
> *agrees with* $H^1_g(k, H^2_{\text{ét}}(\overline{X},\mathbb{Q}_p/\mathbb{Z}_p(2)))_{\mathrm{Div}}$ (*see* §2.1 *for* $H^1_g(k,-)$). *Here for an abelian group* M, M_{Div} *denotes its maximal divisible subgroup.*

We will show that **H1** always implies **H1**$*$, which is not straight-forward. On the other hand the converse holds as well under some assumptions. See Remark 3.2.5 below for details.

Fact 1.1 *The condition* **H1** *holds in the following cases*:

(1) $H^2(X, \mathscr{O}_X) = 0$ ([CTR1], [CTR2], [Sal]).

(2) *X is the self-product of an elliptic curve over $k = \mathbb{Q}$ with square-free conductor and without complex multiplication, and $p \geq 5$* ([Md], [Fl], [LS], [La1]).

(3) *X is the elliptic modular surface of level 4 over $k = \mathbb{Q}$ and $p \geq 5$* ([La2]).

(4) *X is a Fermat quartic surface over $k = \mathbb{Q}$ or $\mathbb{Q}(\sqrt{-1})$ and $p \geq 5$* ([O]).

A main result of this paper relates the condition $\mathbf{H1}*$ to finiteness of two arithmetic objects. One is the p-primary torsion part of the Chow group $\mathrm{CH}^2(X)$ of algebraic cycles of codimension two on X modulo rational equivalence. Another is an unramified cohomology of X, which we are going to introduce in what follows.

Let $\mathfrak{o}_k$ be the integer ring of k, and put $S := \mathrm{Spec}(\mathfrak{o}_k)$. We assume the following:

ASSUMPTION 1.2 *There exists a regular scheme $\mathscr{X}$ which is proper and flat over S and whose generic fiber is X. Moreover, $\mathscr{X}$ has good or semistable reduction at each closed point of S of characteristic p.*

Let $K = k(X)$ be the function field of X. For an integer $q \geq 0$, let $\mathscr{X}^q$ be the set of all points $x \in \mathscr{X}$ of codimension q. Fix an integer $n \geq 0$. Roughly speaking, the unramified cohomology group $H^{n+1}_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(n))$ is defined as the subgroup of $H^{n+1}_{\text{ét}}(\mathrm{Spec}(K), \mathbb{Q}_p/\mathbb{Z}_p(n))$ consisting of those elements that are "unramified" along all $y \in \mathscr{X}^1$. For a precise definition, we need the p-adic étale Tate twist $\mathfrak{T}_r(n) = \mathfrak{T}_r(n)_{\mathscr{X}}$ introduced in [SH]. This object is defined in $D^b(\mathscr{X}_{\text{ét}}, \mathbb{Z}/p^r)$, the derived category of bounded complexes of étale sheaves of $\mathbb{Z}/p^r$-modules on $\mathscr{X}$, and expected to coincide with $\mathit{\Gamma}(2)^{\mathscr{X}}_{\text{ét}} \otimes^{\mathbb{L}} \mathbb{Z}/p^r$. Here $\mathit{\Gamma}(2)^{\mathscr{X}}_{\text{ét}}$ denotes the conjectural étale motivic complex of Beilinson-Lichtenbaum [Be], [Li1]. We note that the restriction of $\mathfrak{T}_r(n)$ to $\mathscr{X}[p^{-1}] := \mathscr{X} \otimes_{\mathbb{Z}} \mathbb{Z}[p^{-1}]$ is isomorphic to $\mu_{p^r}^{\otimes n}$, where μ_{p^r} denotes the étale sheaf of p^r-th roots of unity.Then $H^{n+1}_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(n))$ is defined as the kernel of the boundary map of étale cohomology groups

$$H^{n+1}_{\text{ét}}(\mathrm{Spec}(K), \mathbb{Q}_p/\mathbb{Z}_p(n)) \longrightarrow \bigoplus_{x \in \mathscr{X}^1} H^{n+2}_x(\mathrm{Spec}(\mathscr{O}_{\mathscr{X},x}), \mathfrak{T}_\infty(n)),$$

where $\mathfrak{T}_\infty(n)$ denotes $\varinjlim_{r \geq 1} \mathfrak{T}_r(n)$. There are natural isomorphisms

$$H^1_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(0)) \simeq H^1_{\text{ét}}(\mathscr{X}, \mathbb{Q}_p/\mathbb{Z}_p) \quad \text{and} \quad H^2_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(1)) \simeq \mathrm{Br}(\mathscr{X})_{p\text{-tors}},$$

where $\mathrm{Br}(\mathscr{X})$ denotes the Grothendieck-Brauer group $H^2_{\text{ét}}(\mathscr{X}, \mathbb{G}_{\mathrm{m}})$, and for an abelian group M, $M_{p\text{-tors}}$ denotes its p-primary torsion part. An intriguing question is as to whether the group $H^{n+1}_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(n))$ is finite, which is related to several significant theorems and conjectures in arithmetic geometry (see Remark 4.3.1 below). In this paper we are concerned with the case $n = 2$. A crucial role will be played by the following subgroup of $H^3_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$:

$$\begin{aligned} &H^3_{\mathrm{ur}}(K, X; \mathbb{Q}_p/\mathbb{Z}_p(2)) \\ &:= \mathrm{Im}\Big(H^3_{\text{ét}}(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \to H^3_{\text{ét}}(\mathrm{Spec}(K), \mathbb{Q}_p/\mathbb{Z}_p(2))\Big) \cap H^3_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(2)). \end{aligned}$$

It will turn out that $\mathrm{CH}^2(X)_{p\text{-tors}}$ and $H^3_{\mathrm{ur}}(K,X;\mathbb{Q}_p/\mathbb{Z}_p(2))$ are cofinitely generated over $\mathbb{Z}_p$ if $\mathrm{Coker}(\mathrm{reg}_{\mathbb{Q}_p/\mathbb{Z}_p})_{\mathrm{Div}}$ is cofinitely generated over $\mathbb{Z}_p$ (cf. Proposition 3.3.2, Lemma 5.2.3). Our main finiteness result is the following:

Theorem 1.3 *Let $\mathscr{X}$ be as in Assumption* 1.2*, and assume $p \geq 5$. Then*:

(1) $\mathbf{H1}{*}$ *implies that* $\mathrm{CH}^2(X)_{p\text{-tors}}$ *and* $H^3_{\mathrm{ur}}(K,X;\mathbb{Q}_p/\mathbb{Z}_p(2))$ *are finite.*

(2) *Assume that the reduced part of every closed fiber of $\mathscr{X}/S$ has simple normal crossings on $\mathscr{X}$, and that the Tate conjecture holds in codimension* 1 *for the irreducible components of those fibers* (*see the beginning of* §7 *for the precise contents of the last assumption*)*. Then the finiteness of* $\mathrm{CH}^2(X)_{p\text{-tors}}$ *and* $H^3_{\mathrm{ur}}(K,X;\mathbb{Q}_p/\mathbb{Z}_p(2))$ *implies* $\mathbf{H1}{*}$.

We do not need Assumption 1.2 to deduce the finiteness of $\mathrm{CH}^2(X)_{p\text{-tors}}$ from $\mathbf{H1}{*}$, by the alteration theorem of de Jong [dJ] (see also Remark 3.1.2 (3) below). However, we need a regular proper model $\mathscr{X}$ as above crucially in our computations on $H^3_{\mathrm{ur}}(K,X;\mathbb{Q}_p/\mathbb{Z}_p(2))$. The assertion (2) is a converse of (1) under the assumption of the Tate conjecture. We obtain the following result from Theorem 1.3 (1) (see also the proof of Theorem 1.6 in §5.1 below):

Corollary 1.4 $H^3_{\mathrm{ur}}(K,X;\mathbb{Q}_p/\mathbb{Z}_p(2))$ *is finite in the four cases in Fact* 1.1 (*under the assumption* 1.2).

We will also prove variants of Theorem 1.3 over local integer rings (see Theorems 3.1.1, 5.1.1 and 7.1.1 below). As for the finiteness of $H^3_{\mathrm{ur}}(K,\mathbb{Q}_p/\mathbb{Z}_p(2))$ over local integer rings, Spiess proved that $H^3_{\mathrm{ur}}(K,\mathbb{Q}_p/\mathbb{Z}_p(2)) = 0$, assuming that $\mathfrak{o}_k$ is an ℓ-adic local integer ring with $\ell \neq p$ and that either $H^2(X,\mathscr{O}_X)=0$ or $\mathscr{X}$ is a product of two smooth elliptic curves over S ([Spi] §4). In [SSa], the authors extended his vanishing result to a more general situation that $\mathfrak{o}_k$ is ℓ-adic local with $\ell \neq p$ and that $\mathscr{X}$ has generalized semistable reduction. Finally we have to remark that there exists a smooth projective surface X with $p_g(X) \neq 0$ over a local field k for which the condition $\mathbf{H1}{*}$ does not hold and such that $\mathrm{CH}^2(X)_{\mathrm{tors}}$ is infinite [AS].

We next explain an application of the above finiteness result to a cycle class map of arithmetic schemes. Let us recall the following fact due to Colliot-Thélène, Sansuc, Soulé and Gros:

Fact 1.5 ([CTSS], [Gr]) *Let X be a proper smooth variety over a finite field of characteristic $\ell > 0$. Let p be a prime number, which may be the same as ℓ. Then the cycle class map restricted to the p-primary torsion part*

$$\mathrm{CH}^2(X)_{p\text{-tors}} \longrightarrow H^4_{\text{ét}}(X,\mathbb{Z}/p^r(2))$$

is injective for a sufficiently large $r > 0$. Here $\mathbb{Z}/p^r(2)$ denotes $\mu_{p^r}^{\otimes 2}$ if $\ell \neq p$. Otherwise $\mathbb{Z}/p^r(2)$ denotes $W_r\Omega^2_{X,\log}[-2]$ with $W_r\Omega^2_{X,\log}$ the étale subsheaf of the logarithmic part of the Hodge-Witt sheaf $W_r\Omega^2_X$ ([Bl1], [Il]).

In this paper, we study an arithmetic variant of this fact. We expect that a similar result holds for proper regular arithmetic schemes, i.e., regular schemes which are proper flat of finite type over the integer ring of a number field or a local field. To be more precise, let k, $\mathfrak{o}_k$ and X be as before and let $\mathscr{X}$ be as in Assumption 1.2. The p-adic étale Tate twist $\mathfrak{T}_r(2) = \mathfrak{T}_r(2)_{\mathscr{X}}$ mentioned before replaces $\mathbb{Z}/p^r(2)$ in Fact 1.5, and there is a cycle class map

$$\varrho_r^2 : \mathrm{CH}^2(\mathscr{X})/p^r \longrightarrow H^4_{\text{ét}}(\mathscr{X}, \mathfrak{T}_r(2)).$$

We are concerned with the induced map

$$\varrho^2_{p\text{-tors},r} : \mathrm{CH}^2(\mathscr{X})_{p\text{-tors}} \longrightarrow H^4_{\text{ét}}(\mathscr{X}, \mathfrak{T}_r(2)).$$

It is shown in [SH] that the group on the right hand side is finite. So the injectivity of this map is closely related with the finiteness of $\mathrm{CH}^2(\mathscr{X})_{p\text{-tors}}$. The second main result of this paper concerns the injectivity of this map:

THEOREM 1.6 (§5) *Assume that $H^2(X, \mathscr{O}_X) = 0$. Then $\mathrm{CH}^2(\mathscr{X})_{p\text{-tors}}$ is finite and $\varrho^2_{p\text{-tors},r}$ is injective for a sufficiently large $r > 0$.*

The finiteness of $\mathrm{CH}^2(\mathscr{X})_{p\text{-tors}}$ in this theorem is originally due to Salberger [Sal], Colliot-Thélène and Raskind [CTR1], [CTR2]. Note that there exists a projective smooth surface V over a number field with $H^2(V, \mathscr{O}_V) = 0$ for which the map

$$\mathrm{CH}^2(V)_{p\text{-tors}} \longrightarrow H^4_{\text{ét}}(V, \mu_{p^r}^{\otimes 2})$$

is not injective for some bad prime p and any $r \geq 1$ [Su] (cf. [PS]). Our result suggests that we are able to recover the injectivity of torsion cycle class maps by considering a proper regular model of V over the ring of integers in k. The fundamental ideas of Theorem 1.6 are the following. A crucial point of the proof of Fact 1.5 in [CTSS] and [Gr] is Deligne's proof of the Weil conjecture [De2]. In the arithmetic situation, the role of the Weil conjecture is replaced by the condition $\mathbf{H1}$, which implies the finiteness of $\mathrm{CH}^2(X)_{p\text{-tors}}$ and $H^3_{\mathrm{ur}}(K, X; \mathbb{Q}_p/\mathbb{Z}_p(2))$ by Theorem 1.3 (1). The injectivity result in Theorem 1.6 is derived from the finiteness of those objects.

This paper is organized as follows. In §2, we will review some fundamental facts on Galois cohomology groups and Selmer groups which will be used frequently in this paper. In §3, we will prove the finiteness of $\mathrm{CH}^2(X)_{p\text{-tors}}$ in Theorem 1.3 (1). In §4, we will review p-adic étale Tate twists briefly and then provide some fundamental lemmas on cycle class maps and unramified cohomology groups. In §5, we will first reduce Theorem 1.6 to Theorem 1.3 (1), and then reduce the finiteness of $H^3_{\mathrm{ur}}(K, X; \mathbb{Q}_p/\mathbb{Z}_p(2))$ in Theorem 1.3 (1) to Key Lemma 5.4.1. In §6, we will prove that key lemma, which will complete the proof of Theorem 1.3 (1). §7 will be devoted to the proof of Theorem 1.3 (2). In the appendix A, we will include an observation that the finiteness of $H^3_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$ is deduced from the Beilinson–Lichtenbaum conjectures on motivic complexes.

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NOTATION

1.6. For an abelian group M and a positive integer n, ${}_nM$ and M/n denote the kernel and the cokernel of the map $M \xrightarrow{\times n} M$, respectively. See §2.3 below for other notation for abelian groups. For a field k, $\overline{k}$ denotes a fixed separable closure, and G_k denotes the absolute Galois group $\mathrm{Gal}(\overline{k}/k)$. For a discrete G_k-module M, $H^*(k, M)$ denote the Galois cohomology groups $H^*_{\mathrm{Gal}}(G_k, M)$, which are the same as the étale cohomology groups of $\mathrm{Spec}(k)$ with coefficients in the étale sheaf associated with M.

1.7. Unless indicated otherwise, all cohomology groups of schemes are taken over the étale topology. For a scheme X, an étale sheaf $\mathscr{F}$ on X (or more generally an object in the derived category of sheaves on $X_{\mathrm{\acute{e}t}}$) and a point $x \in X$, we often write $H^*_x(X, \mathscr{F})$ for $H^*_x(\mathrm{Spec}(\mathscr{O}_{X,x}), \mathscr{F})$. For a pure-dimensional scheme X and a non-negative integer q, let X^q be the set of all points on X of codimension q. For a point $x \in X$, let $\kappa(x)$ be its residue field. For an integer $n \geq 0$ and a noetherian excellent scheme X, $\mathrm{CH}_n(X)$ denotes the Chow group of algebraic cycles on X of dimension n modulo rational equivalence. If X is pure-dimensional and regular, we will often write $\mathrm{CH}^{\dim(X)-n}(X)$ for this group. For an integral scheme X of finite type over $\mathrm{Spec}(\mathbb{Q})$, $\mathrm{Spec}(\mathbb{Z})$ or $\mathrm{Spec}(\mathbb{Z}_\ell)$, we define $\mathrm{CH}^2(X, 1)$ as the cohomology group, at the middle, of the Gersten complex of Milnor K-groups

$$K_2^M(L) \longrightarrow \bigoplus_{y \in X^1} \kappa(y)^\times \longrightarrow \bigoplus_{x \in X^2} \mathbb{Z},$$

where L denotes the function field of X. As is well-known, this group coincides with a higher Chow group ([Bl3], [Le2]) by localization sequences of higher Chow groups ([Bl4], [Le1]) and the Nesterenko-Suslin theorem [NS] (cf. [To]).

1.8. In §§4–7, we will work under the following setting. Let k be an algebraic number field or its completion at a finite place. Let $\mathfrak{o}_k$ be the integer ring of k and put $S := \mathrm{Spec}(\mathfrak{o}_k)$. Let p be a prime number, and let $\mathscr{X}$ be a regular scheme which is proper flat of finite type over S and satisfies the following condition:

ASSUMPTION 1.8.1 *If p is not invertible in $\mathfrak{o}_k$, then $\mathscr{X}$ has good or semistable reduction at each closed point of S of characteristic p.*

This condition is the same as Assumption 1.2 when k is a number field.

1.9. Let k be an algebraic number field, and let $\mathscr{X} \to S = \mathrm{Spec}(\mathfrak{o}_k)$ be as in 1.8. In this situation, we will often use the following notation. For a closed point $v \in S$, let $\mathfrak{o}_v$ (resp. k_v) be the completion of $\mathfrak{o}_k$ (resp. k) at v, and let $\mathbb{F}_v$ be the residue field of k_v. We put

$$\mathscr{X}_v := \mathscr{X} \otimes_{\mathfrak{o}_k} \mathfrak{o}_v, \qquad X_v := \mathscr{X} \otimes_{\mathfrak{o}_k} k_v, \qquad Y_v := \mathscr{X} \otimes_{\mathfrak{o}_k} \mathbb{F}_v$$

and write $j_v : X_v \hookrightarrow \mathscr{X}_v$ (resp. $i_v : Y_v \hookrightarrow \mathscr{X}_v$) for the natural open (resp. closed) immersion. We put $\overline{Y_v} := Y_v \times_{\mathbb{F}_v} \overline{\mathbb{F}_v}$, and write Σ for the set of all closed point on S of characteristic p.

1.10. Let k be an ℓ-adic local field with ℓ a prime number, and let $\mathscr{X} \to S = \mathrm{Spec}(\mathfrak{o}_k)$ be as in 1.8. In this situation, we will often use the following notation. Let $\mathbb{F}$ be the residue field of k and put

$$X := \mathscr{X} \otimes_{\mathfrak{o}_k} k, \qquad Y := \mathscr{X} \otimes_{\mathfrak{o}_k} \mathbb{F}.$$

We write $j : X \hookrightarrow \mathscr{X}$ (resp. $i : Y \hookrightarrow \mathscr{X}$) for the natural open (resp. closed) immersion. Let k^{ur} be the maximal unramified extension of k, and let $\mathfrak{o}^{\mathrm{ur}}$ be its integer ring. We put

$$\mathscr{X}^{\mathrm{ur}} := \mathscr{X} \otimes_{\mathfrak{o}_k} \mathfrak{o}^{\mathrm{ur}}, \qquad X^{\mathrm{ur}} := \mathscr{X} \otimes_{\mathfrak{o}_k} k^{\mathrm{ur}}, \qquad \overline{Y} := Y \times_{\mathbb{F}} \overline{\mathbb{F}}.$$

2 Preliminaries on Galois Cohomology

In this section, we provide some preliminary lemmas which will be frequently used in this paper. Let k be an algebraic number field (global field) or its completion at a finite place (local field). Let $\mathfrak{o}_k$ be the integer ring of k, and put $S := \mathrm{Spec}(\mathfrak{o}_k)$. Let p be a prime number. If k is global, we often write Σ for the set of the closed points on S of characteristic p.

2.1 Selmer Group

Let X be a proper smooth variety over $\mathrm{Spec}(k)$, and put $\overline{X} := X \otimes_k \overline{k}$. If k is global, we fix a non-empty open subset $U_0 \subset S \setminus \Sigma$ for which there exists a proper smooth morphism $\mathscr{X}_{U_0} \to U_0$ with $\mathscr{X}_{U_0} \times_{U_0} k \simeq X$. For $v \in S^1$, let k_v and $\mathbb{F}_v$ be as in the notation 1.9. In this section we are concerned with G_k-modules

$$V := H^i(\overline{X}, \mathbb{Q}_p(n)) \quad \text{and} \quad A := H^i(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(n)).$$

For $M = V$ or A and a non-empty open subset $U \subset U_0$, let $H^*(U, M)$ denote the étale cohomology groups with coefficients in the smooth sheaf on $U_{\text{ét}}$ associated to M.

Definition 2.1.1 (1) *Assume that k is local. Let $H^1_f(k, V)$ and $H^1_g(k, V)$ be as defined in* [BK2] (3.7). *For $* \in \{f, g\}$, we define*

$$H^1_*(k, A) := \mathrm{Im}\big(H^1_*(k, V) \longrightarrow H^1(k, A)\big).$$

(2) *Assume that k is global. For $M \in \{V, A\}$ and a non-empty open subset $U \subset S$, we define the subgroup $H^1_{f,U}(k, M) \subset H^1_{\mathrm{cont}}(k, M)$ as the kernel of the natural map*

$$H^1_{\mathrm{cont}}(k, M) \longrightarrow \prod_{v \in U^1} \frac{H^1_{\mathrm{cont}}(k_v, M)}{H^1_f(k_v, M)} \times \prod_{v \in S \setminus U} \frac{H^1_{\mathrm{cont}}(k_v, M)}{H^1_g(k_v, M)}.$$

If $U \subset U_0$, we have

$$H^1_{f,U}(k, M) = \mathrm{Ker}\Big(H^1(U, M) \longrightarrow \prod\nolimits_{v \in S \setminus U} H^1_{\mathrm{cont}}(k_v, M)/H^1_g(k_v, M)\Big).$$

We define the group $H^1_g(k, M)$ and $H^1_{\mathrm{ind}}(k, M)$ as

$$H^1_g(k, M) := \varinjlim_{U \subset U_0} H^1_{f,U}(k, M), \qquad H^1_{\mathrm{ind}}(k, M) := \varinjlim_{U \subset U_0} H^1(U, M),$$

where U runs through all non-empty open subsets of U_0. These groups are independent of the choice of U_0 and $\mathscr{X}_{U_0}$ (cf. [EGA4] 8.8.2.5).

(3) *If k is local, we define $H^1_{\mathrm{ind}}(k, M)$ to be $H^1_{\mathrm{cont}}(k, M)$ for $M \in \{V, A\}$.*

Note that $H^1_{\mathrm{ind}}(k, A) = H^1(k, A)$.

2.2 p-adic Point of Motives

We provide a key lemma from p-adic Hodge theory which play crucial roles in this paper (see Corollary 2.2.3 below). Assume that k is a p-adic local field, and that there exists a regular scheme $\mathscr{X}$ which is proper flat of finite type over $S = \mathrm{Spec}(\mathfrak{o}_k)$ with $\mathscr{X} \otimes_{\mathfrak{o}_k} k \simeq X$ and which has semistable reduction. Let i and n be non-negative integers. Put

$$V^i := H^{i+1}(\overline{X}, \mathbb{Q}_p), \qquad V^i(n) := V^i \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(n),$$

and

$$H^{i+1}(\mathscr{X}, \tau_{\leq n} Rj_*\mathbb{Q}_p(n)) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_{r \geq 1} H^{i+1}(\mathscr{X}, \tau_{\leq n} Rj_*\mu_{p^r}^{\otimes n}),$$

where j denotes the natural open immersion $X \hookrightarrow \mathscr{X}$. There is a natural pull-back map

$$\alpha : H^{i+1}(\mathscr{X}, \tau_{\leq n} Rj_*\mathbb{Q}_p(n)) \longrightarrow H^{i+1}(X, \mathbb{Q}_p(n)).$$

Let $H^{i+1}(\mathscr{X}, \tau_{\leq r} Rj_*\mathbb{Q}_p(n))^0$ be the kernel of the composite map

$$\alpha' : H^{i+1}(\mathscr{X}, \tau_{\leq n} Rj_*\mathbb{Q}_p(n)) \xrightarrow{\alpha} H^{i+1}(X, \mathbb{Q}_p(n)) \longrightarrow \big(V^{i+1}(n)\big)^{G_k}.$$

For this group, there is a composite map

$$\overline{\alpha} : H^{i+1}(\mathscr{X}, \tau_{\leq n} Rj_*\mathbb{Q}_p(n))^0 \longrightarrow F^1H^{i+1}(X, \mathbb{Q}_p(n)) \longrightarrow H^1_{\mathrm{cont}}(k, V^i(n)).$$

Here the first arrow is induced by α, the second is an edge homomorphism in a Hochschild-Serre spectral sequence

$$E_2^{u,v} := H^u_{\mathrm{cont}}(k, V^v(n)) \Longrightarrow H^{u+v}_{\mathrm{cont}}(X, \mathbb{Q}_p(n))(\simeq H^{u+v}(X, \mathbb{Q}_p(n))),$$

and $F^\bullet$ denotes the filtration on $H^{i+1}(X, \mathbb{Q}_p(n))$ resulting from this spectral sequence. To provide with Corollary 2.2.3 below concerning the image of $\overline{\alpha}$, we need some strong results in p-adic Hodge theory. We first recall the following comparison theorem of log syntomic complexes and p-adic vanishing cycles due to Tsuji, which extends a comparison result of Kurihara [Ku] to semistable families. Let Y be the closed fiber of $\mathscr{X} \to S$ and let $\iota : Y \hookrightarrow \mathscr{X}$ be the natural closed immersion.

Theorem 2.2.1 ([Ts2] Theorem 5.1) *For integers n, r with $0 \le n \le p-2$ and $r \ge 1$, there is a canonical isomorphism*

$$\eta : s_r^{\log}(n) \xrightarrow{\ \sim\ } \iota_*\iota^*(\tau_{\le n} Rj_*\mu_{p^r}^{\otimes n}) \quad \text{in } D^b(\mathscr{X}_{\text{ét}}, \mathbb{Z}/p^r),$$

where $s_r^{\log}(n) = s_r^{\log}(n)_{\mathscr{X}}$ is the log syntomic complex defined by Kato [Ka2].

Put

$$H^*(\mathscr{X}, s^{\log}_{\mathbb{Q}_p}(n)) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_{r \ge 1} H^*(\mathscr{X}, s_r^{\log}(n)),$$

and define $H^{i+1}(\mathscr{X}, s^{\log}_{\mathbb{Q}_p}(n))^0$ as the kernel of the composite map

$$H^{i+1}(\mathscr{X}, s^{\log}_{\mathbb{Q}_p}(n)) \ \underset{\eta}{\xrightarrow{\ \sim\ }} H^{i+1}(\mathscr{X}, \tau_{\le n} Rj_*\mathbb{Q}_p(n)) \xrightarrow{\alpha'} \left(V^{i+1}(n)\right)^{G_k},$$

where we have used the properness of $\mathscr{X}$ over S. There is an induced map

$$\overline{\eta} : H^{i+1}(\mathscr{X}, s^{\log}_{\mathbb{Q}_p}(n))^0 \ \underset{\eta}{\xrightarrow{\ \sim\ }} H^{i+1}(\mathscr{X}, \tau_{\le n} Rj_*\mathbb{Q}_p(n))^0 \xrightarrow{\overline{\alpha}} H^1_{\mathrm{cont}}(k, V^i(n)).$$

Concerning this map, we have the following fact due to Langer and Nekovář:

Theorem 2.2.2 ([La3], [Ne2] Theorem 3.1) $\mathrm{Im}(\overline{\eta})$ *agrees with* $H^1_g(k, V^i(n))$.

As an immediate consequence of these facts, we obtain

Corollary 2.2.3 *Assume that $p \ge n+2$. Then* $\mathrm{Im}(\overline{\alpha}) = H^1_g(k, V^i(n))$.

Remark 2.2.4 (1) *Theorem* 2.2.2 *is an extension of the p-adic point conjecture raised by Schneider in the good reduction case* [Sch]. *This conjecture was proved by Langer-Saito* [LS] *in a special case and by Nekovář* [Ne1] *in the general case.*

(2) *Theorem* 2.2.2 *holds unconditionally on p, if we define $H^{i+1}(\mathscr{X}, s^{\log}_{\mathbb{Q}_p}(n))$ using Tsuji's version of log syntomic complexes $\mathscr{S}_r^{\sim}(n)$ $(r \ge 1)$ in* [Ts1] §2.

2.3 Elementary Facts on $\mathbb{Z}_p$-Modules

For an abelian group M, let M_{Div} be its maximal divisible subgroup. For a torsion abelian group M, let $\mathrm{Cotor}(M)$ be the cotorsion part M/M_{Div}.

Definition 2.3.1 *Let M be a $\mathbb{Z}_p$-module.*

(1) *We say that M is cofinitely generated over $\mathbb{Z}_p$ (or simply, cofinitely generated), if its Pontryagin dual $\mathrm{Hom}_{\mathbb{Z}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p)$ is a finitely generated $\mathbb{Z}_p$-module.*

(2) *We say that M is cofinitely generated up to a finite-exponent group, if M_{Div} is cofinitely generated and $\mathrm{Cotor}(M)$ has a finite exponent.*

(3) *We say that M is divisible up to a finite-exponent group, if $\mathrm{Cotor}(M)$ has a finite exponent.*

Lemma 2.3.2 *Let $0 \to L \to M \to N \to 0$ be a short exact sequence of $\mathbb{Z}_p$-modules.*

(1) *Assume that L, M and N are cofinitely generated. Then there is a positive integer r_0 such that for any $r \geq r_0$ we have an exact sequence of finite abelian p-groups*

$$0 \to {}_{p^r}L \to {}_{p^r}M \to {}_{p^r}N \to \mathrm{Cotor}(L) \to \mathrm{Cotor}(M) \to \mathrm{Cotor}(N) \to 0.$$

Consequently, taking the projective limit of this exact sequence with respect to $r \geq r_0$ there is an exact sequence of finitely generated $\mathbb{Z}_p$-modules

$$0 \to T_p(L) \to T_p(M) \to T_p(N) \to \mathrm{Cotor}(L) \to \mathrm{Cotor}(M) \to \mathrm{Cotor}(N) \to 0,$$

where for an abelian group A, $T_p(A)$ denotes its p-adic Tate module.

(2) *Assume that L is cofinitely generated up to a finite-exponent group. Assume further that M is divisible, and that N is cofinitely generated and divisible. Then L and M are cofinitely generated.*

(3) *Assume that L is divisible up to a finite-exponent group. Then for a divisible subgroup $D \subset N$ and its inverse image $D' \subset M$, the induced map $(D')_{\mathrm{Div}} \to D$ is surjective. In particular, the natural map $M_{\mathrm{Div}} \to N_{\mathrm{Div}}$ is surjective.*

(4) *If $L_{\mathrm{Div}} = N_{\mathrm{Div}} = 0$, then we have $M_{\mathrm{Div}} = 0$.*

Proof. (1) There is a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] \arrow[d, "\times p^r"'] & M \arrow[r] \arrow[d, "\times p^r"'] & N \arrow[r] \arrow[d, "\times p^r"'] & 0 \\ 0 \arrow[r] & L \arrow[r] & M \arrow[r] & N \arrow[r] & 0. \end{tikzcd}$$

One obtains the assertion by applying the snake lemma to this diagram, noting $\mathrm{Cotor}(A) \simeq A/p^r$ for a cofinitely generated $\mathbb{Z}_p$-module A and a sufficiently large $r \geq 1$.

(2) Our task is to show that $\mathrm{Cotor}(L)$ is finite. By a similar argument as for (1), there is an exact sequence for a sufficiently large $r \geq 1$

$$0 \longrightarrow {}_{p^r}L \longrightarrow {}_{p^r}M \longrightarrow {}_{p^r}N \longrightarrow \mathrm{Cotor}(L) \longrightarrow 0,$$

where we have used the assumptions on L and M. Hence the finiteness of $\mathrm{Cotor}(L)$ follows from the assumption that N is cofinitely generated.

(3) We have only to show the case $D = N_{\mathrm{Div}}$. For a $\mathbb{Z}_p$-module A, we have

$$A_{\mathrm{Div}} = \mathrm{Im}\left(\mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p, A) \to A\right)$$

by [J1] Lemma (4.3.a). Since $\mathrm{Ext}^1_{\mathbb{Z}_p}(\mathbb{Q}_p, L) = 0$ by the assumption on L, the following natural map is surjective:

$$\mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p, M) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p, N).$$

By these facts, the natural map $M_{\mathrm{Div}} \to N_{\mathrm{Div}}$ is surjective.

(4) For a $\mathbb{Z}_p$-module A, we have

$$A_{\mathrm{Div}} = 0 \Longleftrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p, A) = 0$$

by [J1] Remark (4.7). The assertion follows from this fact and the exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p, L) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p, M) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p, N).$$

This completes the proof of the lemma. □

2.4 DIVISIBLE PART OF $H^1(k, A)$

Let the notation be as in §2.1. We prove here the following general lemma, which will be used frequently in §§3–7:

LEMMA 2.4.1 *Under the notation in Definition* 2.1.1 *we have*

$$\mathrm{Im}\big(H^1_{\mathrm{ind}}(k, V) \to H^1(k, A)\big) = H^1(k, A)_{\mathrm{Div}},$$
$$\mathrm{Im}\big(H^1_g(k, V) \to H^1(k, A)\big) = H^1_g(k, A)_{\mathrm{Div}}.$$

Proof. The assertion is clear if k is local. Assume that k is global. Without loss of generality we may assume that A is divisible. We prove only the second equality and omit the first one (see Remark 2.4.9 (2) below). Let $U_0 \subset S$ be as in §2.1. We have

$$\mathrm{Im}\big(H^1_{f,U}(k, V) \to H^1(U, A)\big) = H^1_{f,U}(k, A)_{\mathrm{Div}} \tag{2.4.2}$$

for non-empty open $U \subset U_0$. This follows from a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & H^1_{f,U}(k,V) \arrow[r] \arrow[d] & H^1(U,V) \arrow[r] \arrow[d, "\alpha"'] & \prod_{v\in S\setminus U} H^1_{\mathrm{cont}}(k_v,V)/H^1_g(k_v,V) \arrow[d, "\beta"'] \\
0 \arrow[r] & H^1_{f,U}(k,A) \arrow[r] & H^1(U,A) \arrow[r] & \prod_{v\in S\setminus U} H^1(k_v,A)/H^1_g(k_v,A)
\end{tikzcd}$$

and the facts that $\mathrm{Coker}(\alpha)$ is finite and that $\mathrm{Ker}(\beta)$ is finitely generated over $\mathbb{Z}_p$. By (2.4.2), the second equality of the lemma is reduced to the following assertion:

$$\varinjlim_{U\subset U_0} \big(H^1_{f,U}(k,A)_{\mathrm{Div}}\big) = \left(\varinjlim_{U\subset U_0} H^1_{f,U}(k,A))\right)_{\mathrm{Div}}. \tag{2.4.3}$$

To show this equality, we will prove the following sublemma:

SUBLEMMA 2.4.4 *For an open subset $U \subset U_0$, put*

$$C_U := \mathrm{Coker}\big(H^1_{f,U_0}(k,A) \to H^1_{f,U}(k,A)\big).$$

Then there exists a non-empty open subset $U_1 \subset U_0$ such that the quotient C_U/C_{U_1} is divisible for any open subset $U \subset U_1$.

We first finish our proof of (2.4.3) admitting this sublemma. Let $U_1 \subset U_0$ be a non-empty open subset as in Sublemma 2.4.4. Noting that $H^1_{f,U}(k,A)$ is cofinitely generated, there is an exact sequence of finite groups

$$\mathrm{Cotor}\big(H^1_{f,U_1}(k,A)\big) \longrightarrow \mathrm{Cotor}\big(H^1_{f,U}(k,A)\big) \longrightarrow \mathrm{Cotor}(C_U/C_{U_1}) \longrightarrow 0$$

for open $U \subset U_1$ by Lemma 2.3.2 (1). By this exact sequence and Sublemma 2.4.4, the natural map $\mathrm{Cotor}(H^1_{f,U_1}(k,A)) \to \mathrm{Cotor}(H^1_{f,U}(k,A))$ is surjective for any open $U \subset U_1$, which implies that the inductive limit

$$\varinjlim_{U\subset U_0} \mathrm{Cotor}(H^1_{f,U}(k,A))$$

is a finite group. The equality (2.4.3) follows easily from this.

Proof of Sublemma 2.4.4. We need the following general fact:

SUBLEMMA 2.4.5 *Let $\underline{N} = \{N_\lambda\}_{\lambda\in\Lambda}$ be an inductive system of cofinitely generated $\mathbb{Z}_p$-modules indexed by a filtered set Λ such that* $\mathrm{Coker}(N_\lambda \to N_{\lambda'})$ *is divisible for any two $\lambda, \lambda' \in \Lambda$ with $\lambda' \geq \lambda$. Let L be a cofinitely generated $\mathbb{Z}_p$-module and $\{f_\lambda : N_\lambda \to L\}_{\lambda\in\Lambda}$ be $\mathbb{Z}_p$-homomorphisms compatible with the transition maps of $\underline{N}$. Then there exists $\lambda_0 \in \Lambda$ such that* $\mathrm{Coker}\big(\mathrm{Ker}(f_{\lambda_0}) \to \mathrm{Ker}(f_\lambda)\big)$ *is divisible for any $\lambda \geq \lambda_0$.*

Proof of Sublemma 2.4.5. Let $f_\infty : N_\infty \to L$ be the limit of f_λ. The assumption on $\underline{N}$ implies that for any two $\lambda, \lambda' \in \Lambda$ with $\lambda' \geq \lambda$, the quotient $\mathrm{Im}(f_{\lambda'})/\mathrm{Im}(f_\lambda)$ is divisible, so that

$$\mathrm{Cotor}(\mathrm{Im}(f_\lambda)) \to \mathrm{Cotor}(\mathrm{Im}(f_{\lambda'})) \text{ is surjective.} \tag{2.4.6}$$

By the equality $\mathrm{Im}(f_\infty) = \varinjlim_{\lambda \in \Lambda} \mathrm{Im}(f_\lambda)$, there is a short exact sequence

$$0 \longrightarrow \varinjlim_{\lambda \in \Lambda} \big(\mathrm{Im}(f_\lambda)_{\mathrm{Div}}\big) \longrightarrow \mathrm{Im}(f_\infty) \longrightarrow \varinjlim_{\lambda \in \Lambda} \mathrm{Cotor}(\mathrm{Im}(f_\lambda)) \longrightarrow 0,$$

and the last term is finite by the fact (2.4.6) and the assumption that L is cofinitely generated. Hence we get

$$\varinjlim_{\lambda \in \Lambda} \big(\mathrm{Im}(f_\lambda)_{\mathrm{Div}}\big) = \mathrm{Im}(f_\infty)_{\mathrm{Div}}.$$

Since $\mathrm{Im}(f_\infty)_{\mathrm{Div}}$ has finite corank, there exists an element $\lambda_0 \in \Lambda$ such that $\mathrm{Im}(f_\lambda)_{\mathrm{Div}} = \mathrm{Im}(f_\infty)_{\mathrm{Div}}$ for any $\lambda \geq \lambda_0$. This fact and (2.4.6) imply the equality

$$\mathrm{Im}(f_\lambda) = \mathrm{Im}(f_{\lambda_0}) \quad \text{for any } \lambda \geq \lambda_0. \tag{2.4.7}$$

Now let $\lambda \in \Lambda$ satisfy $\lambda \geq \lambda_0$. Applying the snake lemma to the commutative diagram

$$\begin{tikzcd}
 & N_{\lambda_0} \arrow[r] \arrow[d, "f_{\lambda_0}"'] & N_\lambda \arrow[r] \arrow[d, "f_\lambda"'] & N_\lambda/N_{\lambda_0} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L \arrow[r, equal] & L \arrow[r] & 0, &
\end{tikzcd}$$

we get an exact sequence

$$\mathrm{Ker}(f_{\lambda_0}) \longrightarrow \mathrm{Ker}(f_{\lambda_0}) \longrightarrow N_\lambda/N_{\lambda_0} \xrightarrow{0} \mathrm{Coker}(f_{\lambda_0}) \xrightarrow{\sim} \mathrm{Coker}(f_\lambda),$$

which proves Sublemma 2.4.5, beucase N_λ/N_{λ_0} is divisible by assumption. $\square$

We now turn to the proof of Sublemma 2.4.4. For non-empty open $U \subset U_0$, there is a commutative diagram with exact rows

$$\begin{tikzcd}
 & H^1(U_0, A) \arrow[r] \arrow[d, "r_{U_0}"'] & H^1(U, A) \arrow[r] \arrow[d, "r_U"'] & \bigoplus_{v \in U_0 \setminus U} A(-1)^{G_{\mathbb{F}_v}} \arrow[r, "\beta_U"] \arrow[d, "\alpha_U"'] & H^2(U_0, A) \\
0 \arrow[r] & \bigoplus_{v \in S \setminus U_0} H^1_{/g}(k_v, A) \arrow[r] & \bigoplus_{v \in S \setminus U} H^1_{/g}(k_v, A) \arrow[r] & \bigoplus_{v \in U_0 \setminus U} H^1_{/g}(k_v, A), &
\end{tikzcd}$$

where we put

$$H^1_{/g}(k_v, A) := H^1(k_v, A)/H^1_g(k_v, A)$$

for simplicity. The upper row is obtained from a localization exact sequence of étale cohomology and the isomorphism

$$H^2_v(U_0, A) \simeq H^1(k_v, A)/H^1(\mathbb{F}_v, A) \simeq A(-1)^{G_{\mathbb{F}_v}} \quad \text{for } v \in U_0 \setminus U,$$

where we have used the fact that the action of G_k on A is unramified at $v \in U_0$. The map α_U is obtained from the facts that $H^1_g(k_v, A) = H^1(k_v, A)_{\mathrm{Div}}$ if $v \notin \Sigma$ and that $H^1(\mathbb{F}_v, A)$ is divisible (recall that A is assumed to be divisible). It gives

$$\mathrm{Ker}(\alpha_U) = \bigoplus_{v \in U_0 \setminus U} \big(A(-1)^{G_{\mathbb{F}_v}}\big)_{\mathrm{Div}}. \tag{2.4.8}$$

Now let ϕ_U be the composite map

$$\phi_U : \mathrm{Ker}(\alpha_U) \hookrightarrow \bigoplus_{v \in U_0 \setminus U} A(-1)^{G_{\mathbb{F}_v}} \xrightarrow{\beta_U} H^2(U_0, A),$$

and let

$$\psi_U : \mathrm{Ker}(\phi_U) \longrightarrow \mathrm{Coker}(r_{U_0})$$

be the map induced by the above diagram. Note that

$$C_U \simeq \mathrm{Ker}(\psi_U), \text{ since } H^1_{f,U}(k, A) = \mathrm{Ker}(r_U).$$

By (2.4.8), the inductive system $\{\mathrm{Ker}(\alpha_U)\}_{U \subset U_0}$ and the maps $\{\phi_U\}_{U \subset U_0}$ satisfy the assumptions in Sublemma 2.4.5. Hence there exists a non-empty open subset $U' \subset U_0$ such that $\mathrm{Ker}(\phi_U)/\mathrm{Ker}(\phi_{U'})$ is divisible for any open $U \subset U'$. Then applying Sublemma 2.4.5 again to the inductive system $\{\mathrm{Ker}(\phi_U)\}_{U \subset U'}$ and the maps $\{\psi_U\}_{U \subset U'}$, we conclude that there exists a non-empty open subset $U_1 \subset U'$ such that the quotient

$$\mathrm{Ker}(\psi_U)/\mathrm{Ker}(\psi_{U_1}) = C_U/C_{U_1}$$

is divisible for any open subset $U \subset U_1$. This completes the proof of Sublemma 2.4.4 and Lemma 2.4.1. □

Remark 2.4.9 (1) *By the argument after Sublemma* 2.4.4, $\mathrm{Cotor}(H^1_g(k, A))$ *is finite if A is divisible.*

(2) *One obtains the first equality in Lemma* 2.4.1 *by replacing the local terms* $H^1_{/g}(k_v, A)$ *in the above diagram with* $\mathrm{Cotor}(H^1(k_v, A))$.

2.5 Cotorsion Part of $H^1(k, A)$

Assume that k is global, and let the notation be as in §2.1. We investigate here the boundary map

$$\delta_{U_0} : H^1(k, A) \longrightarrow \bigoplus_{v \in (U_0)^1} A(-1)^{G_{\mathbb{F}_v}}$$

arising from a localization exact sequence of étale cohomology and the purity for discrete valuation rings. Concerning this map, we prove the following standard lemma, which will be used in our proof of Theorem 1.3:

Lemma 2.5.1 (1) *The map*

$$\delta_{U_0,\mathrm{Div}} : H^1(k,A)_{\mathrm{Div}} \longrightarrow \bigoplus_{v\in(U_0)^1} \big(A(-1)^{G_{\mathbb{F}_v}}\big)_{\mathrm{Div}}$$

induced by δ_{U_0} has cofinitely generated cokernel.

(2) *The map*

$$\delta_{U_0,\mathrm{Cotor}} : \mathrm{Cotor}(H^1(k,A)) \longrightarrow \bigoplus_{v\in(U_0)^1} \mathrm{Cotor}\big(A(-1)^{G_{\mathbb{F}_v}}\big)$$

induced by δ_{U_0} has finite kernel and cofinitely generated cokernel.

We have nothing to say about the finiteness of the cokernel of these maps.

Proof. For a non-empty open $U \subset U_0$, there is a commutative diagram of cofinitely generated $\mathbb{Z}_p$-modules

$$\begin{tikzcd}
 & H^1(U,A)_{\mathrm{Div}} \arrow[r, "\gamma_U"] \arrow[d, hook] & \bigoplus_{v\in U_0\setminus U} \big((A(-1)^{G_{\mathbb{F}_v}}\big)_{\mathrm{Div}} \arrow[d, hook] & \\
H^1(U_0,A) \arrow[r] & H^1(U,A) \arrow[r, "\alpha_U"] & \bigoplus_{v\in U_0\setminus U} A(-1)^{G_{\mathbb{F}_v}} \arrow[r, "\beta_U"] & H^2(U_0,A),
\end{tikzcd}$$

where the lower row is obtained from a localization exact sequence of étale cohomology and the purity for discrete valuation rings, and γ_U is induced by α_U. Let

$$f_U : \mathrm{Cotor}(H^1(U,A)) \longrightarrow \bigoplus_{v\in U_0\setminus U} \mathrm{Cotor}\big(A(-1)^{G_{\mathbb{F}_v}}\big)$$

be the map induced by α_U. By a diagram chase, we obtain an exact sequence

$$\mathrm{Ker}(f_U) \longrightarrow \mathrm{Coker}(\gamma_U) \longrightarrow \mathrm{Coker}(\alpha_U) \longrightarrow \mathrm{Coker}(f_U) \longrightarrow 0.$$

Taking the inductive limit with respect to all non-empty open subsets $U \subset U_0$, we obtain an exact sequence

$$\mathrm{Ker}(\delta_{U_0,\mathrm{Cotor}}) \to \mathrm{Coker}(\delta_{U_0,\mathrm{Div}}) \to \varinjlim_{U\subset U_0} \mathrm{Coker}(\alpha_U) \to \mathrm{Coker}(\delta_{U_0,\mathrm{Cotor}}) \to 0,$$

where we have used Lemma 2.4.1 to obtain the equalities

$$\mathrm{Ker}(\delta_{U_0,\mathrm{Cotor}}) = \varinjlim_{U\subset U_0} \mathrm{Ker}(f_U) \quad \text{and} \quad \mathrm{Coker}(\delta_{U_0,\mathrm{Div}}) = \varinjlim_{U\subset U_0} \mathrm{Coker}(\gamma_U).$$

Since $\varinjlim_{U\subset U_0} \mathrm{Coker}(\alpha_U)$ is a subgroup of $H^2(U_0,A)$, it is cofinitely generated. Hence the assertions in Lemma 2.5.1 are reduced to showing that $\mathrm{Ker}(\delta_{U_0,\mathrm{Cotor}})$ is finite. We prove this finiteness assertion. The lower row of the above diagram yields exact sequences

$$\mathrm{Cotor}(H^1(U_0, A)) \longrightarrow \mathrm{Cotor}(H^1(U, A)) \longrightarrow \mathrm{Cotor}(\mathrm{Im}(\alpha_U)) \longrightarrow 0, \tag{2.5.2}$$

$$T_p(\mathrm{Im}(\beta_U)) \longrightarrow \mathrm{Cotor}(\mathrm{Im}(\alpha_U)) \longrightarrow \bigoplus_{v \in U_0 \setminus U} \mathrm{Cotor}\big(A(-1)^{G_{\mathbb{F}_v}}\big), \tag{2.5.3}$$

where the second exact sequence arises from the short exact sequence

$$0 \longrightarrow \mathrm{Im}(\alpha_U) \longrightarrow \bigoplus_{v \in U_0 \setminus U} A(-1)^{G_{\mathbb{F}_v}} \longrightarrow \mathrm{Im}(\beta_U) \longrightarrow 0$$

(cf. Lemma 2.3.2 (1)). Taking the inductive limit of (2.5.2) with respect to all non-empty open $U \subset U_0$, we obtain the finiteness of the kernel of the map

$$\mathrm{Cotor}(H^1(k, A)) \longrightarrow \varinjlim_{U \subset U_0} \mathrm{Cotor}(\mathrm{Im}(\alpha_U)).$$

Taking the inductive limit of (2.5.3) with respect to all non-empty open $U \subset U_0$, we see that the kernel of the map

$$\varinjlim_{U \subset U_0} \mathrm{Cotor}(\mathrm{Im}(\alpha_U)) \longrightarrow \bigoplus_{v \in (U_0)^1} \mathrm{Cotor}\big(A(-1)^{G_{\mathbb{F}_v}}\big),$$

is finite, because we have

$$\varinjlim_{U \subset U_0} T_p(\mathrm{Im}(\beta_U)) \subset T_p(H^2(U_0, A))$$

and the group on the right hand side is a finitely generated $\mathbb{Z}_p$-module. Thus $\mathrm{Ker}(\delta_{U_0,\mathrm{Cotor}})$ is finite and we obtain Lemma 2.5.1. $\square$

2.6 Local-Global Principle

Let the notation be as in §2.1. If k is local, then the Galois cohomological dimension $\mathrm{cd}(k)$ is 2 (cf. [Se] II.4.3). In the case that k is global, we have $\mathrm{cd}(k) = 2$ either if $p \geq 3$ or if k is totally imaginary. Otherwise, $H^q(k, A)$ is finite 2-torsion for $q \geq 3$ (cf. loc. cit. II.4.4 Proposition 13, II.6.3 Theorem B). As for the second Galois cohomology groups, the following local-global principle due to Jannsen [J2] plays a fundamental role in this paper (see also loc. cit. §7 Corollary 7):

Theorem 2.6.1 ([J2] §4 Theorem 4) *Assume that k is global and that $i \neq 2(n-1)$. Let P be the set of all places of k. Then the map*

$$H^2(k, H^i(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(n))) \longrightarrow \bigoplus_{v \in P} H^2(k_v, H^i(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(n)))$$

has finite kernel and cokernel, and the map

$$H^2(k, H^i(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(n))_{\mathrm{Div}}) \longrightarrow \bigoplus_{v \in P} H^2(k_v, H^i(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(n))_{\mathrm{Div}})$$

is bijective.

We apply these facts to the filtration $F^\bullet$ on $H^*(X, \mathbb{Q}_p/\mathbb{Z}_p(n))$ resulting from the Hochschild-Serre spectral sequence

$$E_2^{u,v} = H^u(k, H^v(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(n))) \Longrightarrow H^{u+v}(X, \mathbb{Q}_p/\mathbb{Z}_p(n)). \tag{2.6.2}$$

COROLLARY 2.6.3 *Assume that k is global and that $i \neq 2n$. Then*:

(1) *$F^2H^i(X, \mathbb{Q}_p/\mathbb{Z}_p(n))$ is cofinitely generated up to a finite-exponent group.*

(2) *For $v \in P$, put $X_v := X \otimes_k k_v$. Then the natural maps*

$$F^2H^i(X, \mathbb{Q}_p/\mathbb{Z}_p(n)) \longrightarrow \bigoplus_{v \in P} F^2H^i(X_v, \mathbb{Q}_p/\mathbb{Z}_p(n)),$$
$$F^2H^i(X, \mathbb{Q}_p/\mathbb{Z}_p(n))_{\mathrm{Div}} \longrightarrow \bigoplus_{v \in P} F^2H^i(X_v, \mathbb{Q}_p/\mathbb{Z}_p(n))_{\mathrm{Div}}$$

have finite kernel and cokernel (*and the second map is surjective*).

Proof. Let $\mathfrak{o}_k$ be the integer ring of k, and put $S := \mathrm{Spec}(\mathfrak{o}_k)$. Note that the set of all finite places of k agrees with S^1.

(1) The group $H^2(k_v, H^{i-2}(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(n))_{\mathrm{Div}})$ is divisible and cofinitely generated for any $v \in S^1$, and it is zero if $p \nmid v$ and X has good reduction at v, by the local Poitou-Tate duality [Se] II.5.2 Théorème 2 and Deligne's proof of the Weil conjecture [De2] (see [Sat2] Lemma 2.4 for details). The assertion follows from this fact and Theorem 2.6.1.

(2) We prove the assertion only for the first map. The assertion for the second map is similar and left to the reader. For simplicity, we assume that

($\sharp$) $p \geq 3$ or k is totally imaginary.

Otherwise one can check the assertion by repeating the same arguments as below in the category of abelian groups modulo finite abelian groups. By ($\sharp$), we have $\mathrm{cd}_p(k) = 2$ and there is a commutative diagram

$$\begin{tikzcd}
H^2(k, H^{i-2}(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(n))) \arrow[r] \arrow[d] & \bigoplus_{v \in S^1} H^2(k_v, H^{i-2}(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(n))) \arrow[d] \\
F^2H^i(X, \mathbb{Q}_p/\mathbb{Z}_p(n)) \arrow[r] & \bigoplus_{v \in S^1} F^2H^i(X_v, \mathbb{Q}_p/\mathbb{Z}_p(n)),
\end{tikzcd}$$

where the vertical arrows are edge homomorphisms of Hochschild-Serre spectral sequences and these arrows are surjective. Since

$$H^2(k_v, H^{i-2}(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(n))) = 0 \quad \text{for archimedean places } v$$

by ($\sharp$), the top horizontal arrow has finite kernel and cokernel by Theorem 2.6.1. Hence it is enough to show that the right vertical arrow has finite kernel. For any $v \in S^1$, the

v-component of this map has finite kernel by Deligne's criterion [De1] (see also [Sat2] Remark 1.2). If v is prime to p and X has good reduction at v, then the v-component is injective. Indeed, there is an exact sequence resulting from a Hochschild-Serre spectral sequence and the fact that $\mathrm{cd}(k_v) = 2$:

$$\begin{aligned} H^{i-1}(X_v, \mathbb{Q}_p/\mathbb{Z}_p(n)) &\xrightarrow{d} H^{i-1}(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(n)))^{G_{k_v}} \\ &\to H^2(k_v, H^{i-2}(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(n))) \to F^2H^i(X_v, \mathbb{Q}_p/\mathbb{Z}_p(n)). \end{aligned}$$

The edge homomorphism d is surjective by the commutative square

$$\begin{tikzcd} H^{i-1}(Y_v, \mathbb{Q}_p/\mathbb{Z}_p(n)) \arrow[r, two heads] \arrow[d] & H^{i-1}(\overline{Y_v}, \mathbb{Q}_p/\mathbb{Z}_p(n)))^{G_{\mathbb{F}_v}} \arrow[d, "\wr"] \\ H^{i-1}(X_v, \mathbb{Q}_p/\mathbb{Z}_p(n)) \arrow[r, "d"] & H^{i-1}(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(n)))^{G_{k_v}}. \end{tikzcd}$$

Here Y_v denotes the reduction of X at v and $\overline{Y_v}$ denotes $Y_v \otimes_{\mathbb{F}_v} \overline{\mathbb{F}_v}$. The left (resp. right) vertical arrow arises from the proper base-change theorem (resp. proper smooth base-change theorem), and the top horizontal arrow is surjective by the fact that $\mathrm{cd}(\mathbb{F}_v) = 1$. Thus we obtain the assertion. $\square$

3 Finiteness of Torsion in a Chow Group

Let k, S, p and Σ be as in the beginning of §2, and let X be a proper smooth geometrically integral variety over $\mathrm{Spec}(k)$. We introduce the following technical condition:

H0: *The group $H^3_{\text{ét}}(\overline{X}, \mathbb{Q}_p(2))^{G_k}$ is trivial.*

If k is global, **H0** always holds by Deligne's proof of the Weil conjecture [De2]. When k is local, **H0** holds if $\dim(X) = 2$ or if X has good reduction (cf. [CTR2] §6); it is in general a consequence of the monodromy-weight conjecture.

3.1 Finiteness of $\mathrm{CH}^2(X)_{p\text{-tors}}$

The purpose of this section is to show the following result, which is a generalization of a result of Langer [La4] Proposition 3 and implies the finiteness assertion on $\mathrm{CH}^2(X)_{p\text{-tors}}$ in Theorem 1.3 (1):

Theorem 3.1.1 *Assume* **H0**, **H1∗** *and either $p \geq 5$ or the equality*

$$H^1_g(k, H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)))_{\mathrm{Div}} = H^1(k, H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)))_{\mathrm{Div}}. \qquad (*_g)$$

Then $\mathrm{CH}^2(X)_{p\text{-tors}}$ is finite.

REMARK 3.1.2 (1) $(*_g)$ *holds if* $H^2(X, \mathscr{O}_X) = 0$ *or if* k *is* ℓ*-adic local with* $\ell \neq p$.

(2) *Crucial facts to this theorem are Lemmas* 3.2.2, 3.3.5 *and* 3.5.2 *below. The short exact sequence in Lemma* 3.2.2 *is an important consequence of the Merkur'ev-Suslin theorem* [MS].

(3) *In Theorem* 3.1.1*, we do not need to assume that* X *has good or semistable reduction at any prime of* k *dividing* p (*cf.* 1.8.1)*, because we do not need this assumption in Lemma* 3.5.2 *by the alteration theorem of de Jong* [dJ].

3.2 REGULATOR MAP

We recall here the definition of the regulator maps

$$\mathrm{reg}_\Lambda : \mathrm{CH}^2(X,1) \otimes \Lambda \longrightarrow H^1_{\mathrm{ind}}(k, H^2(\overline{X}, \Lambda(2))) \tag{3.2.1}$$

with $\Lambda = \mathbb{Q}_p$ or $\mathbb{Q}_p/\mathbb{Z}_p$, assuming **H0**. The general framework on étale Chern class maps and regulator maps is due to Soulé [So1], [So2]. We include here a more elementary construction of reg_Λ, which will be useful in this paper. Let $K := k(X)$ be the function field of X. Take an open subset $U_0 \subset S \setminus \Sigma = S[p^{-1}]$ and a smooth proper scheme $\mathscr{X}_{U_0}$ over U_0 satisfying $\mathscr{X}_{U_0} \times_{U_0} \mathrm{Spec}(k) \simeq X$. For an open subset $U \subset U_0$, put $\mathscr{X}_U := \mathscr{X}_{U_0} \times_{U_0} U$ and define

$$N^1H^3(\mathscr{X}_U, \mu_{p^r}^{\otimes 2}) := \mathrm{Ker}\big(H^3(\mathscr{X}_U, \mu_{p^r}^{\otimes 2}) \to H^3(K, \mu_{p^r}^{\otimes 2})\big).$$

LEMMA 3.2.2 *For an open subset* $U \subset U_0$*, there is an exact sequence*

$$0 \longrightarrow \mathrm{CH}^2(\mathscr{X}_U, 1)/p^r \longrightarrow N^1H^3(\mathscr{X}_U, \mu_{p^r}^{\otimes 2}) \longrightarrow {}_{p^r}\mathrm{CH}^2(\mathscr{X}_U) \longrightarrow 0$$

See §1.7 *for the definition of* $\mathrm{CH}^2(\mathscr{X}_U, 1)$.

Proof. The following argument is due to Bloch [Bl], Lecture 5. We recall it for the convenience of the reader. There is a localization spectral sequence

$$E_1^{u,v} = \bigoplus_{x \in (\mathscr{X}_U)^u} H_x^{u+v}(\mathscr{X}_U, \mu_{p^r}^{\otimes 2}) \Longrightarrow H^{u+v}(\mathscr{X}_U, \mu_{p^r}^{\otimes 2}). \tag{3.2.3}$$

By the relative smooth purity, there is an isomorphism

$$E_1^{u,v} \simeq \bigoplus_{x \in (\mathscr{X}_U)^u} H^{v-u}(x, \mu_{p^r}^{\otimes 2-u}), \tag{3.2.4}$$

which implies that $N^1H^3(\mathscr{X}_U, \mu_{p^r}^{\otimes 2})$ is isomorphic to the cohomology of the Bloch-Ogus complex

$$H^2(K, \mu_{p^r}^{\otimes 2}) \longrightarrow \bigoplus_{y \in (\mathscr{X}_U)^1} H^1(y, \mu_{p^r}) \longrightarrow \bigoplus_{x \in (\mathscr{X}_U)^2} \mathbb{Z}/p^r.$$

By Hilbert's theorem 90 and the Merkur'ev-Suslin theorem [MS], this complex is isomorphic to the Gersten complex

$$K_2^M(K)/p^r \longrightarrow \bigoplus_{y\in(\mathscr{X}_U)^1} \kappa(y)^\times/p^r \longrightarrow \bigoplus_{x\in(\mathscr{X}_U)^2} \mathbb{Z}/p^r.$$

On the other hand, there is an exact sequence obtained by a diagram chase

$$0 \longrightarrow \mathrm{CH}^2(\mathscr{X}_U,1)\otimes\mathbb{Z}/p^r \longrightarrow \mathrm{CH}^2(\mathscr{X}_U,1;\mathbb{Z}/p^r) \longrightarrow {}_{p^r}\mathrm{CH}^2(\mathscr{X}_U) \longrightarrow 0.$$

Here $\mathrm{CH}^2(\mathscr{X}_U,1;\mathbb{Z}/p^r)$ denotes the cohomology of the above Gersten complex and it is isomorphic to $N^1H^3(\mathscr{X}_U,\mu_{p^r}^{\otimes 2})$. Thus we obtain the lemma. □

Put

$$M^q := H^q(\overline{X},\Lambda(2)) \quad \text{with} \quad \Lambda\in\{\mathbb{Q}_p,\mathbb{Q}_p/\mathbb{Z}_p\}.$$

For an open subset $U\subset U_0$ let $H^*(U,M^q)$ be the étale cohomology with coefficients in the smooth sheaf associated with M^q. There is a Leray spectral sequence

$$E_2^{u,v}=H^u(U,M^v)\Longrightarrow H^{u+v}(\mathscr{X}_U,\Lambda(2)).$$

By Lemma 3.2.2, there is a natural map

$$\mathrm{CH}^2(\mathscr{X}_U,1)\otimes\Lambda \longrightarrow H^3(\mathscr{X}_U,\Lambda(2)).$$

Noting that $E_2^{0,3}$ is zero or finite by **H0**, we define the map

$$\mathrm{reg}_{\mathscr{X}_U,\Lambda} : \mathrm{CH}^2(\mathscr{X}_U,1)\otimes\Lambda \longrightarrow H^1(U,M^2)$$

as the composite of the above map with an edge homomorphism of the Leray spectral sequence. Finally we define reg_Λ in (3.2.1) by passing to the limit over all non-empty open $U\subset U_0$. Our construction of reg_Λ does not depend on the choice of U_0 or $\mathscr{X}_{U_0}$.

Remark 3.2.5 *By Lemma* 2.4.1*,* **H1** *always implies* **H1∗***. If k is local,* **H1∗** *conversely implies* **H1***. If k is global, one can check that* **H1∗** *implies* **H1***, assuming that the group* $\mathrm{Ker}(\mathrm{CH}^2(\mathscr{X}_{U_0})\to\mathrm{CH}^2(X))$ *is finitely generated up to torsion and that the Tate conjecture for divisors holds for almost all closed fibers of* $\mathscr{X}_{U_0}/U_0$.

3.3 Proof of Theorem 3.1.1

We start the proof of Theorem 3.1.1, which will be completed in §3.5 below. By Lemma 3.2.2, there is an exact sequence

$$0 \longrightarrow \mathrm{CH}^2(X,1)\otimes\mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\phi} N^1H^3(X,\mathbb{Q}_p/\mathbb{Z}_p(2)) \longrightarrow \mathrm{CH}^2(X)_{p\text{-tors}} \longrightarrow 0, \tag{3.3.1}$$

where we put

$$N^1H^3(X,\mathbb{Q}_p/\mathbb{Z}_p(2)) := \mathrm{Ker}(H^3(X,\mathbb{Q}_p/\mathbb{Z}_p(2))\to H^3(K,\mathbb{Q}_p/\mathbb{Z}_p(2))).$$

In view of (3.3.1), Theorem 3.1.1 is reduced to the following two propositions:

PROPOSITION 3.3.2 (1) *If k is local, then* $\mathrm{CH}^2(X)_{p\text{-tors}}$ *is cofinitely generated over* $\mathbb{Z}_p$.

(2) *Assume that k is global, and that* $\mathrm{Coker}\big(\mathrm{reg}_{\mathbb{Q}_p/\mathbb{Z}_p}\big)_{\mathrm{Div}}$ *is cofinitely generated over* $\mathbb{Z}_p$. *Then* $\mathrm{CH}^2(X)_{p\text{-tors}}$ *is cofinitely generated over* $\mathbb{Z}_p$.

PROPOSITION 3.3.3 *Assume* **H0**, **H1**$*$ *and either* $p \geq 5$ *or* $(*_g)$. *Then we have*

$$\mathrm{Im}(\phi) = N^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))_{\mathrm{Div}}.$$

We will prove Proposition 3.3.2 in §3.4 below, and Proposition 3.3.3 in §3.5 below.

REMARK 3.3.4 (1) *If k is local, then* $H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))$ *is cofinitely generated. Hence Proposition* 3.3.2 (1) *immediately follows from the exact sequence* (3.3.1).

(2) *When k is global, then* $H^1(k, A)_{\mathrm{Div}}/H^1_g(k, A)_{\mathrm{Div}}$ *with* $A := H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))$ *is cofinitely generated by Lemma* 2.4.1. *Hence* **H1**$*$ *implies the second assumption of Proposition* 3.3.2 (2).

Let $F^\bullet$ be the filtration on $H^*(X, \mathbb{Q}_p/\mathbb{Z}_p(2))$ resulting from the Hochschild-Serre spectral sequence (2.6.2). The following fact due to Salberger will play key roles in our proof of the above two propositions:

LEMMA 3.3.5 ([Sal] Main Lemma 3.9) *The following group has a finite exponent*:

$$N^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \cap F^2H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)).$$

3.4 PROOF OF PROPOSITION 3.3.2

For (1), see Remark 3.3.4 (1). We assume that k is global, and prove (2). Put

$$H^3 := H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \quad \text{and} \quad \Gamma := \phi(\mathrm{CH}^2(X,1) \otimes \mathbb{Q}_p/\mathbb{Z}_p) \subset H^3$$

(cf. (3.3.1)). Let $F^\bullet$ be the filtration on H^3 resulting from the spectral sequence (2.6.2), and put $N^1H^3 := N^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))$. We have $\Gamma \subset (F^1H^3)_{\mathrm{Div}} = (H^3)_{\mathrm{Div}}$ by **H0**, and there is a filtration on H^3

$$0 \subset \Gamma + (F^2H^3)_{\mathrm{Div}} \subset (F^1H^3)_{\mathrm{Div}} \subset H^3.$$

By (3.3.1), the inclusion $N^1H^3 \subset H^3$ induces an inclusion $\mathrm{CH}^2(X)_{p\text{-tors}} \subset H^3/\Gamma$. We show that the image of this inclusion is cofinitely generated, using the above filtration on H^3. It suffices to show the following lemma:

LEMMA 3.4.1 (1) *The kernel of* $\mathrm{CH}^2(X)_{p\text{-tors}} \to H^3/(\Gamma + (F^2H^3)_{\mathrm{Div}})$ *is finite.*

(2) *The image of* $\mathrm{CH}^2(X)_{p\text{-tors}} \to H^3/(F^1H^3)_{\mathrm{Div}}$ *is finite.*

(3) *The second assumption of Proposition* 3.3.2 (2) *implies that the group*

$$M := (F^1H^3)_{\mathrm{Div}}/(\varGamma + (F^2H^3)_{\mathrm{Div}})$$

is cofinitely generated.

Proof. (1) There is an exact sequence

$$0 \longrightarrow \frac{N^1H^3 \cap (F^2H^3)_{\mathrm{Div}}}{\varGamma \cap (F^2H^3)_{\mathrm{Div}}} \longrightarrow \mathrm{CH}^2(X)_{p\text{-tors}} \longrightarrow \frac{H^3}{\varGamma + (F^2H^3)_{\mathrm{Div}}}.$$

Hence (1) follows from Lemma 3.3.5 and Corollary 2.6.3 (1).

(2) Let U_0 and $\mathscr{X}_{U_0} \to U_0$ be as in §3.2. For non-empty open $U \subset U_0$, there is a commutative diagram up to a sign

$$\begin{tikzcd} N^1H^3(\mathscr{X}_U, \mathbb{Q}_p/\mathbb{Z}_p(2)) \arrow[r] \arrow[d, hook] & \mathrm{CH}^2(\mathscr{X}_U) \otimes \mathbb{Z}_p \arrow[d, "\varrho"] \\ H^3(\mathscr{X}_U, \mathbb{Q}_p/\mathbb{Z}_p(2)) \arrow[r] & H^4(\mathscr{X}_U, \mathbb{Z}_p(2)) \end{tikzcd}$$

by the same argument as for [CTSS], §1, Proposition 1. Here the top arrow is the composite of $N^1H^3(\mathscr{X}_U, \mathbb{Q}_p/\mathbb{Z}_p(2)) \to \mathrm{CH}^2(\mathscr{X}_U)_{p\text{-tors}}$ (cf. Lemma 3.2.2) with the natural inclusion. The bottom arrow is a Bockstein map and the right vertical arrow is the cycle class map of $\mathscr{X}_U$. Taking the inductive limit with respect to all non-empty $U \subset U_0$, we see that the left square of the following diagram commutes (up to a sign):

$$\begin{tikzcd} N^1H^3 \arrow[r] \arrow[d, hook] & \mathrm{CH}^2(X) \otimes \mathbb{Z}_p \arrow[d, "\varrho_{\mathrm{ind}}"] \arrow[dr, "\varrho_{\mathrm{cont}}"] & \\ H^3 \arrow[r] & H^4_{\mathrm{ind}}(X, \mathbb{Z}_p(2)) \arrow[r] & H^4_{\mathrm{cont}}(X, \mathbb{Z}_p(2)), \end{tikzcd}$$

where $H^*_{\mathrm{cont}}(X, \mathbb{Z}_p(2))$ denotes the continuous étale cohomology [J1] and the bottom right arrow is by definition the inductive limit, with respect to $U \subset U_0$, of the natural restriction map

$$H^4(\mathscr{X}_U, \mathbb{Z}_p(2)) = H^4_{\mathrm{cont}}(\mathscr{X}_U, \mathbb{Z}_p(2)) \longrightarrow H^4_{\mathrm{cont}}(X, \mathbb{Z}_p(2)).$$

The right triangle of the diagram commutes by the definition of cycle classes in loc. cit. Theorem (3.23). This diagram and the exact sequence (3.3.1) yield a commutative diagram (up to a sign)

$$\begin{tikzcd} \mathrm{CH}^2(X)_{p\text{-tors}} \arrow[r] \arrow[d] & \mathrm{CH}^2(X) \otimes \mathbb{Z}_p \arrow[d, "\varrho_{\mathrm{cont}}"] \\ H^3/(F^1H^3)_{\mathrm{Div}} \arrow[r, hook] & H^4_{\mathrm{cont}}(X, \mathbb{Z}_p(2)), \end{tikzcd}$$

where the bottom arrow is injective by **H0** and loc. cit. Theorem (5.14). Now the assertion follows from that fact that $\mathrm{Im}(\varrho_{\mathrm{cont}})$ is finitely generated over $\mathbb{Z}_p$ ([Sa] Theorem (4-4)).

(3) Put

$$N := (F^1H^3)_{\mathrm{Div}}/\{\varGamma + (F^2H^3 \cap (F^1H^3)_{\mathrm{Div}})\} = \mathrm{Coker}\big(\mathrm{reg}_{\mathbb{Q}_p/\mathbb{Z}_p}\big)_{\mathrm{Div}},$$

which is cofinitely generated by assumption and fits into an exact sequence

$$(F^2H^3 \cap (F^1H^3)_{\mathrm{Div}})/(F^2H^3)_{\mathrm{Div}} \longrightarrow M \longrightarrow N \longrightarrow 0.$$

The first group in this sequence has a finite exponent by Corollary 2.6.3 (1), N is divisible and cofinitely generated, and M is divisible. Hence M is cofinitely generated by Lemma 2.3.2 (2). This completes the proof of Lemma 3.4.1 and Proposition 3.3.2. $\square$

3.5 Proof of Proposition 3.3.3

We put

$$NF^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) := N^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \cap F^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)).$$

Note that $N^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))_{\mathrm{Div}} = NF^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))_{\mathrm{Div}}$ by **H0**. There is an edge homomorphism of the spectral sequence (2.6.2)

$$\psi : F^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \longrightarrow H^1(k, H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))). \tag{3.5.1}$$

The composite of ϕ in (3.3.1) and ψ agrees with $\mathrm{reg}_{\mathbb{Q}_p/\mathbb{Z}_p}$. Hence by Lemma 3.3.5, the assertion of Proposition 3.3.3 is reduced to the following lemma, which generalizes [LS] Lemma (5.7) and extends [La1] Lemma (3.3):

Lemma 3.5.2 *Assume either $p \geq 5$ or $(*_g)$ (but we do not assume* **H1∗**). *Then we have*

$$\psi(NF^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))_{\mathrm{Div}}) \subset H^1_g(k, H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))).$$

We start the proof of this lemma. The assertion is obvious under the assumption $(*_g)$. Hence we are done if k is ℓ-adic local with $\ell \neq p$ (cf. Remark 3.1.2 (1)). It remains to deal with the following two cases:

(1) k is p-adic local with $p \geq 5$.

(2) k is global and $p \geq 5$.

Put $A := H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))$ for simplicity. We first reduce the case (2) to the case (1). Suppose that k is global. Then there is a commutative diagram

$$\begin{CD}
NF^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))_{\mathrm{Div}} @>>> H^1(k, A) \\
@VVV @VVV \\
\prod_{v \in S^1} NF^1H^3(X_v, \mathbb{Q}_p/\mathbb{Z}_p(2))_{\mathrm{Div}} @>>> \prod_{v \in S^1} H^1(k_v, A),
\end{CD}$$

where the vertical arrows are natural restriction maps. By this diagram and the definition of $H^1_g(k,A)$, the case (2) is reduced to the case (1).

We prove the case (1). We first reduce the problem to the case where X has semistable reduction. By the alteration theorem of de Jong [dJ], there exists a proper flat generically finite morphism $X' \to X$ such that X' is projective smooth over k and has a proper flat regular model over the integral closure $\mathfrak{o}'$ of $\mathfrak{o}_k$ in $\Gamma(X', \mathscr{O}_{X'})$ with semistable reduction. Put

$$L := \mathrm{Frac}(\mathfrak{o}') \quad \text{and} \quad A' := H^2(X' \otimes_L \overline{k}, \mathbb{Q}_p/\mathbb{Z}_p(2)).$$

Then there is a commutative diagram whose vertical arrows are natural restriction maps

$$\begin{CD} NF^1H^3(X,\mathbb{Q}_p/\mathbb{Z}_p(2))_{\mathrm{Div}} @>>> H^1(k,A) @>>> H^1(k,A)/H^1_g(k,A) \\ @VVV @VVV @VVV \\ NF^1H^3(X',\mathbb{Q}_p/\mathbb{Z}_p(2))_{\mathrm{Div}} @>>> H^1(L,A') @>>> H^1(L,A')/H^1_g(L,A'). \end{CD}$$

Our task is to show that the composite of the upper row is zero. Because X' and X are proper smooth varieties over k, the restriction map $r : A \to A'$ has a quasi-section $s : A' \to A$ with $s \circ r = d \cdot \mathrm{id}_A$, where d denotes the extension degree of the function field of $X' \otimes_L \overline{k}$ over that of $\overline{X}$. Hence by the functoriality of $H^1_g(k,A)$ in A, the right vertical arrow in the above diagram has finite kernel, and the problem is reduced to showing that the composite of the lower row is zero. Thus we are reduced to the case that X has a proper flat regular model $\mathscr{X}$ over $S = \mathrm{Spec}(\mathfrak{o}_k)$ with semistable reduction. We prove this case in what follows.

Let $j : X \hookrightarrow \mathscr{X}$ be the natural open immersion. There is a natural injective map

$$\alpha_r : H^3(\mathscr{X}, \tau_{\leq 2}Rj_*\mu_{p^r}^{\otimes 2}) \hookrightarrow H^3(X, \mu_{p^r}^{\otimes 2})$$

induced by the natural morphism $\tau_{\leq 2}Rj_*\mu_{p^r}^{\otimes 2} \to Rj_*\mu_{p^r}^{\otimes 2}$. By Corollary 2.2.3, it suffices to show the following two lemmas (see also Remark 3.5.6 below):

Lemma 3.5.3 *$N^1H^3(X, \mu_{p^r}^{\otimes 2}) \subset \mathrm{Im}(\alpha_r)$ for any $r \geq 1$.*

Lemma 3.5.4 *Put*

$$H^3(\mathscr{X}, \tau_{\leq 2}Rj_*\mathbb{Q}_p/\mathbb{Z}_p(2)) := \varinjlim_{r\geq 1} H^3(\mathscr{X}, \tau_{\leq 2}Rj_*\mu_{p^r}^{\otimes 2}),$$

and define $H^3(\mathscr{X}, \tau_{\leq 2}Rj_*\mathbb{Q}_p/\mathbb{Z}_p(2))^0$ as the kernel of the natural map

$$H^3(\mathscr{X}, \tau_{\leq 2}Rj_*\mathbb{Q}_p/\mathbb{Z}_p(2)) \to H^3(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)).$$

Then the canonical map

$$H^3(\mathscr{X}, \tau_{\leq 2}Rj_*\mathbb{Q}_p(2))^0 \longrightarrow H^3(\mathscr{X}, \tau_{\leq 2}Rj_*\mathbb{Q}_p/\mathbb{Z}_p(2))^0$$

has finite cokernel, where $H^3(\mathscr{X}, \tau_{\leq 2}Rj_*\mathbb{Q}_p(2))^0$ is as we defined in §2.2.

To prove Lemma 3.5.3, we need the following fact due to Hagihara, whose latter vanishing will be used later in §6:

LEMMA 3.5.5 ([SH] A.2.4, A.2.6) *Let n, r and c be integers with $n \geq 0$ and $r, c \geq 1$. Then for any $q \leq n + c$ and any closed subscheme $Z \subset Y$ with* $\operatorname{codim}_{\mathscr{X}}(Z) \geq c$, *we have*

$$H^q_Z(\mathscr{X}, \tau_{\leq n} Rj_*\mu_{p^r}^{\otimes n}) = 0 = H^{q+1}_Z(\mathscr{X}, \tau_{\geq n+1} Rj_*\mu_{p^r}^{\otimes n}).$$

Proof of Lemma 3.5.3. We compute the local-global spectral sequence

$$E_1^{u,v} = \bigoplus_{x \in \mathscr{X}^a} H_x^{u+v}(\mathscr{X}, \tau_{\leq 2} Rj_*\mu_{p^r}^{\otimes 2}) \Longrightarrow H^{u+v}(\mathscr{X}, \tau_{\leq 2} Rj_*\mu_{p^r}^{\otimes 2}).$$

By the first part of Lemma 3.5.5 and the smooth purity for points on X, we have

$$E_1^{u,v} = \begin{cases} H^v(K, \mu_{p^r}^{\otimes 2}) & \text{(if } u = 0) \\ \bigoplus_{x \in X^u} H^{v-u}(x, \mu_{p^r}^{\otimes 2-u}) & \text{(if } v \leq 2). \end{cases}$$

Repeating the same computation as in the proof of Lemma 3.2.2, we obtain

$$N^1H^3(X, \mu_{p^r}^{\otimes 2}) \simeq E_2^{1,2} = E_\infty^{1,2} \hookrightarrow H^3(\mathscr{X}, \tau_{\leq 2} Rj_*\mu_{p^r}^{\otimes 2}),$$

which implies Lemma 3.5.3. □

REMARK 3.5.6 *Lemma* 3.5.3 *extends a result of Langer-Saito* ([LS] *Lemma* (5.4)) *to regular semistable families and removes the assumption in* [La1] *Lemma* (3.1) *concerning Gersten's conjecture for algebraic K-groups. Therefore the same assumption in loc. cit. Theorem* A *has been removed as well.*

Proof of Lemma 3.5.4. By the Bloch-Kato-Hyodo theorem on the structure of p-adic vanishing cycles ([BK1], [Hy]), there is a distinguished triangle of the following form in $D^b(\mathscr{X}_{\text{ét}})$ (cf. [SH], (4.3.3)):

$$\tau_{\leq 2} Rj_*\mu_{p^r}^{\otimes 2} \longrightarrow \tau_{\leq 2} Rj_*\mu_{p^{r+s}}^{\otimes 2} \longrightarrow \tau_{\leq 2} Rj_*\mu_{p^s}^{\otimes 2} \longrightarrow (\tau_{\leq 2} Rj_*\mu_{p^r}^{\otimes 2})[1]$$

Taking étale cohomology groups, we obtain a long exact sequence

$$\begin{aligned} \cdots \to H^q(\mathscr{X}, \tau_{\leq 2} Rj_*\mu_{p^r}^{\otimes 2}) \to H^q(\mathscr{X}, \tau_{\leq 2} Rj_*\mu_{p^{r+s}}^{\otimes 2}) \to H^q(\mathscr{X}, \tau_{\leq 2} Rj_*\mu_{p^s}^{\otimes 2}) \\ \to H^{q+1}(\mathscr{X}, \tau_{\leq 2} Rj_*\mu_{p^r}^{\otimes 2}) \to \cdots . \end{aligned} \tag{3.5.7}$$

We claim that $H^q(\mathscr{X}, \tau_{\leq 2} Rj_*\mu_{p^r}^{\otimes 2})$ is finite for any q and r. Indeed, the claim is reduced to the case $r = 1$ by the exactness of (3.5.7) and this case follows from the Bloch-Kato-Hyodo theorem mentioned above and the properness of $\mathscr{X}$ over S. Hence taking the projective limit of (3.5.7) with respect to r and then taking the inductive limit with respect to s we obtain a long exact sequence

$$\begin{aligned} \cdots \to H^q(\mathscr{X}, \tau_{\leq 2} Rj_*\mathbb{Z}_p(2)) \to H^q(\mathscr{X}, \tau_{\leq 2} Rj_*\mathbb{Q}_p(2)) \to H^q(\mathscr{X}, \tau_{\leq 2} Rj_*\mathbb{Q}_p/\mathbb{Z}_p(2)) \\ \to H^{q+1}(\mathscr{X}, \tau_{\leq 2} Rj_*\mathbb{Z}_p(2)) \to \cdots , \end{aligned}$$

where $H^q(\mathscr{X}, \tau_{\leq 2} Rj_*\mathbb{Z}_p(2))$ is finitely generated over $\mathbb{Z}_p$ for any q. The assertion in the lemma easily follows from this exact sequence and a similar long exact sequence of étale cohomology groups of $\overline{X}$. The details are straight-forward and left to the reader. □

This completes the proof of Lemma 3.5.2, Proposition 3.3.3 and Theorem 3.1.1.

4 Cycle Class Map and Unramified Cohomology

Let $k, S, p, \mathscr{X}$ and K be as in the notation 1.8. In particular, we always assume that $\mathscr{X}$ satisfies 1.8.1. In this section we give a brief review of p-adic étale Tate twists and provide some preliminary results on cycle class maps. The main result of this section is Corollary 4.4.3 below.

4.1 p-adic Étale Tate Twist

Let n and r be positive integers. We recall here the fundamental properties $(\mathbf{S1})$–$(\mathbf{S7})$ listed below of the object $\mathfrak{T}_r(n) = \mathfrak{T}_r(n)_{\mathscr{X}} \in D^b(\mathscr{X}_{\text{ét}}, \mathbb{Z}/p^r)$ introduced by the second author [SH]. The properties $(\mathbf{S1})$, $(\mathbf{S2})$, $(\mathbf{S3})$ and $(\mathbf{S4})$ characterizes $\mathfrak{T}_r(n)$ uniquely up to a unique isomorphism in $D^b(\mathscr{X}_{\text{ét}}, \mathbb{Z}/p^r)$.

$(\mathbf{S1})$ *There is an isomorphism* $t : \mathfrak{T}_r(n)|_V \simeq \mu_{p^r}^{\otimes n}$ *on* $V := \mathscr{X}[p^{-1}]$.

$(\mathbf{S2})$ $\mathfrak{T}_r(n)$ *is concentrated in* $[0, n]$.

$(\mathbf{S3})$ *Let* $Z \subset \mathscr{X}$ *be a locally closed regular subscheme of pure codimension* c *with* $\mathrm{ch}(Z) = p$. *Let* $i : Z \to \mathscr{X}$ *be the natural immersion. Then there is a canonical Gysin isomorphism*

$$\mathrm{Gys}_i^n : W_r\Omega_{Z,\log}^{n-c}[-n-c] \xrightarrow{\sim} \tau_{\leq n+c} Ri^!\mathfrak{T}_r(n) \quad \text{in } D^b(Z_{\text{ét}}, \mathbb{Z}/p^r),$$

where $W_r\Omega_{Z,\log}^q$ *denotes the étale subsheaf of the logarithmic part of the Hodge-Witt sheaf* $W_r\Omega_Z^q$ ([Bl1], [Il]).

$(\mathbf{S4})$ *For* $x \in \mathscr{X}$ *and* $q \in \mathbb{Z}_{\geq 0}$, *we define* $\mathbb{Z}/p^r(q) \in D^b(x_{\text{ét}}, \mathbb{Z}/p^r)$ *as*

$$\mathbb{Z}/p^r(q) := \begin{cases} \mu_{p^r}^{\otimes q} & (\textit{if } \mathrm{ch}(x) \neq p) \\ W_r\Omega_{x,\log}^q[-q] & (\textit{if } \mathrm{ch}(x) = p). \end{cases}$$

Then for $y, x \in \mathscr{X}$ *with* $c := \mathrm{codim}(x) = \mathrm{codim}(y) + 1$, *there is a commutative diagram*

$$\begin{CD}
H^{n-c+1}(y, \mathbb{Z}/p^r(n-c+1)) @>{-\partial^{\text{val}}}>> H^{n-c}(x, \mathbb{Z}/p^r(n-c)) \\
@V{\mathrm{Gys}_{i_y}^n}VV @VV{\mathrm{Gys}_{i_x}^n}V \\
H_y^{n+c-1}(\mathscr{X}, \mathfrak{T}_r(n)) @>{\delta^{\text{loc}}}>> H_x^{n+c}(\mathscr{X}, \mathfrak{T}_r(n)).
\end{CD}$$

Here for $z \in \mathscr{X}$, $\mathrm{Gys}^n_{i_z}$ is induced by the Gysin map in ($\mathbf{S3}$) (*resp. the absolute purity* [RZ], [Th], [FG]) *if* $\mathrm{ch}(z) = p$ (*resp.* $\mathrm{ch}(z) \neq p$). *The arrow δ^{loc} denotes the boundary map of a localization exact sequence and ∂^{val} denotes the boundary map of Galois cohomology groups due to Kato* [KCT] §1.

($\mathbf{S5}$) *Let Y be the union of the fibers of $\mathscr{X}/S$ of characteristic p. We define the étale sheaf $\nu^{n-1}_{Y,r}$ on Y as*

$$\nu^{n-1}_{Y,r} := \mathrm{Ker}\Big(\partial^{\mathrm{val}} : \bigoplus\nolimits_{y \in Y^0} i_{y*} W_r\Omega^{n-1}_{y,\log} \longrightarrow \bigoplus\nolimits_{x \in Y^1} i_{x*} W_r\Omega^{n-2}_{x,\log}\Big),$$

where for $y \in Y$, i_y denotes the canonical map $y \hookrightarrow Y$. Let i and j be as follows:

$$V = \mathscr{X}[p^{-1}] \xhookrightarrow{\ j\ } \mathscr{X} \xhookleftarrow{\ i\ } Y.$$

Then there is a distinguished triangle in $D^b(\mathscr{X}_{\text{ét}}, \mathbb{Z}/p^r)$

$$i_*\nu^{n-1}_{Y,r}[-n-1] \xrightarrow{\ g\ } \mathfrak{T}_r(n) \xrightarrow{\ t'\ } \tau_{\leq n} Rj_*\mu_{p^r}^{\otimes n} \xrightarrow{\ \sigma\ } i_*\nu^{n-1}_{Y,r}[-n],$$

where t' is induced by the isomorphism t in ($\mathbf{S1}$) *and the acyclicity property* ($\mathbf{S2}$). *The arrow g arises from the Gysin morphisms in* ($\mathbf{S3}$), *σ is induced by the boundary maps of Galois cohomology groups* (*cf.* ($\mathbf{S4}$)).

($\mathbf{S6}$) *There is a canonical distinguished triangle of the following form in $D^b(\mathscr{X}_{\text{ét}})$*:

$$\mathfrak{T}_{r+s}(n) \longrightarrow \mathfrak{T}_s(n) \xrightarrow{\ \delta_{s,r}\ } \mathfrak{T}_r(n)[1] \xrightarrow{\ \underline{p}^s\ } \mathfrak{T}_{r+s}(n)[1].$$

($\mathbf{S7}$) *$H^i(\mathscr{X}, \mathfrak{T}_r(n))$ is finite for any r and i (by the properness of $\mathscr{X}$).*

When k is p-adic local with $p \geq n+2$ and $\mathscr{X}$ is smooth over S, then $i^*\mathfrak{T}_r(n)$ is isomorphic to the syntomic complex $\mathscr{S}_r(n)$ of Kato [Ka1], which is the derived image of a syntomic sheaf of Fontaine-Messing [FM]. This fact follows from a result of Kurihara [Ku] and ($\mathbf{S5}$). Therefore our object $\mathfrak{T}_r(n)$ extends the syntomic complexes to the global situation. Note also that $i^*\mathfrak{T}_r(n)$ is not the log syntomic complex $s^{\log}_r(n)$ unless $n > \dim(\mathscr{X})$, because the latter object is isomorphic to $\tau_{\leq n} i^*Rj_*\mu_{p^r}^{\otimes n}$ by Theorem 2.2.1.

REMARK 4.1.1 *The above properties of $\mathfrak{T}_r(n)$ deeply rely on the computation on the étale sheaf of p-adic vanishing cycles due to Bloch-Kato* [BK1] *and Hyodo* [Hy].

LEMMA 4.1.2 *Put*

$$H^q(\mathscr{X}, \mathfrak{T}_{\mathbb{Z}_p}(n)) := \varprojlim_{r \geq 1} H^q(\mathscr{X}, \mathfrak{T}_r(n)), \quad H^q(\mathscr{X}, \mathfrak{T}_\infty(n)) := \varinjlim_{r \geq 1} H^q(\mathscr{X}, \mathfrak{T}_r(n))$$

and $H^q(\mathscr{X}, \mathfrak{T}_{\mathbb{Q}_p}(n)) := H^q(\mathscr{X}, \mathfrak{T}_{\mathbb{Z}_p}(n)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. *Then there is a long exact sequence of* $\mathbb{Z}_p$*-modules*

$$\cdots \longrightarrow H^q(\mathscr{X}, \mathfrak{T}_{\mathbb{Z}_p}(n)) \longrightarrow H^q(\mathscr{X}, \mathfrak{T}_{\mathbb{Q}_p}(n)) \longrightarrow H^q(\mathscr{X}, \mathfrak{T}_{\infty}(n)) \\ \longrightarrow H^{q+1}(\mathscr{X}, \mathfrak{T}_{\mathbb{Z}_p}(n)) \longrightarrow \cdots,$$

where $H^q(\mathscr{X}, \mathfrak{T}_{\mathbb{Z}_p}(n))$ *is finitely generated over* $\mathbb{Z}_p$, $H^q(\mathscr{X}, \mathfrak{T}_{\infty}(n))$ *is cofinitely generated over* $\mathbb{Z}_p$, *and* $H^q(\mathscr{X}, \mathfrak{T}_{\mathbb{Q}_p}(n))$ *is finite-dimensional over* $\mathbb{Q}_p$.

Proof. The assertions immediately follow from $(\mathbf{S6})$ and $(\mathbf{S7})$. The details are straight-forward and left to the reader. □

4.2 Cycle Class Map

Let us review the definition of the cycle map

$$\varrho_r^n : \mathrm{CH}^n(\mathscr{X})/p^r \longrightarrow H^{2n}(\mathscr{X}, \mathfrak{T}_r(n)).$$

Consider the local-global spectral sequence

$$E_1^{u,v} = \bigoplus_{x \in \mathscr{X}^u} H_x^{u+v}(\mathscr{X}, \mathfrak{T}_r(n)) \Longrightarrow H^{u+v}(\mathscr{X}, \mathfrak{T}_r(n)). \tag{4.2.1}$$

By $(\mathbf{S3})$ and the absolute cohomological purity [FG] (cf. [RZ], [Th]), we have

$$E_1^{u,v} \simeq \bigoplus_{x \in \mathscr{X}^u} H^{v-u}(x, \mathbb{Z}/p^r(n-u)) \qquad \text{for } v \le n. \tag{4.2.2}$$

This implies that there is an edge homomorphism $E_2^{n,n} \to H^{2n}(\mathscr{X}, \mathfrak{T}_r(n))$ with

$$E_2^{n,n} \simeq \mathrm{Coker}\Big(\partial^{\mathrm{val}} : \bigoplus\nolimits_{y \in \mathscr{X}^{n-1}} H^1(y, \mathbb{Z}/p^r(1)) \longrightarrow \bigoplus\nolimits_{x \in \mathscr{X}^n} H^0(x, \mathbb{Z}/p^r)\Big) \\ = \mathrm{CH}^n(\mathscr{X})/p^r,$$

where ∂^{val} is as in $(\mathbf{S4})$. We define ϱ_r^n as the composite map

$$\varrho_r^n : \mathrm{CH}^n(\mathscr{X})/p^r \simeq E_2^{n,n} \longrightarrow H^{2n}(\mathscr{X}, \mathfrak{T}_r(n)).$$

In what follows, we restrict our attention to the case $n = 2$.

Lemma 4.2.3 *Let* $Z \subset \mathscr{X}$ *be a closed subscheme of pure codimension* 1*, and let* K *be the function field of* $\mathscr{X}$. *Put*

$$N^1H^i(\mathscr{X}, \mathfrak{T}_r(2)) := \mathrm{Ker}\big(H^i(\mathscr{X}, \mathfrak{T}_r(2)) \to H^i(K, \mu_{p^r}^{\otimes 2})\big), \\ N^2H_Z^i(\mathscr{X}, \mathfrak{T}_r(2)) := \mathrm{Ker}\Big(H_Z^i(\mathscr{X}, \mathfrak{T}_r(2)) \to \bigoplus\nolimits_{z \in Z^0} H_z^i(\mathscr{X}, \mathfrak{T}_r(2))\Big).$$

(1) *$N^1H^3(\mathscr{X}, \mathfrak{T}_r(2))$ is isomorphic to the cohomology of the Gersten complex modulo p^r*

$$K_2^M(K)/p^r \longrightarrow \bigoplus_{y\in\mathscr{X}^1} \kappa(y)^\times/p^r \longrightarrow \bigoplus_{x\in\mathscr{X}^2} \mathbb{Z}/p^r,$$

and there is an exact sequence

$$0 \longrightarrow \mathrm{CH}^2(\mathscr{X}, 1)/p^r \longrightarrow N^1H^3(\mathscr{X}, \mathfrak{T}_r(2)) \longrightarrow {}_{p^r}\mathrm{CH}^2(\mathscr{X}) \longrightarrow 0.$$

See §1.7 *for the definition of* $\mathrm{CH}^2(\mathscr{X}, 1)$.

(2) *There are isomorphisms*

$$\begin{aligned} H^3_Z(\mathscr{X}, \mathfrak{T}_r(2)) &\simeq \mathrm{Ker}\Big(\partial^{\mathrm{val}} : \textstyle\bigoplus_{z\in Z^0} \kappa(z)^\times/p^r \to \bigoplus_{x\in Z^1} \mathbb{Z}/p^r\Big),\\ N^2H^4_Z(\mathscr{X}, \mathfrak{T}_r(2)) &\simeq \mathrm{Coker}\Big(\partial^{\mathrm{val}} : \textstyle\bigoplus_{z\in Z^0} \kappa(y)^\times/p^r \to \bigoplus_{x\in Z^1} \mathbb{Z}/p^r\Big)\\ &= \mathrm{CH}_{d-2}(Z)/p^r, \end{aligned}$$

where d denotes the Krull dimension of $\mathscr{X}$.

Proof. (1) follows from a similar argument as for the proof of Lemma 3.2.2, using the spectral sequence

$$E_1^{u,v} = \bigoplus_{x\in\mathscr{X}^u} H^{u+v}_x(\mathscr{X}, \mathfrak{T}_r(2)) \Longrightarrow H^{u+v}(\mathscr{X}, \mathfrak{T}_r(2)) \quad \text{((4.2.1) with } n = 2) \tag{4.2.4}$$

and the purity isomorphism

$$E_1^{u,v} \simeq \bigoplus_{x\in\mathscr{X}^u} H^{v-u}(x, \mathbb{Z}/p^r(2-u)) \qquad \text{for } v \le 2 \qquad \text{((4.2.1) with } n = 2). \tag{4.2.5}$$

More precisely, since $E_1^{u,v} = 0$ for (u, v) with $u > v$ and $v \le 2$, we have $N^1H^3(\mathscr{X}, \mathfrak{T}_r(2)) \simeq E_2^{1,2}$, which is isomorphic to the cohomology of the Gersten complex in the assertion by Hilbert's theorem 90, the Merkur'ev-Suslin theorem [MS] and $(\mathbf{S4})$. One can prove (2) in the same way as for (1), using the spectral sequence

$$E_1^{u,v} = \bigoplus_{x\in\mathscr{X}^u\cap Z} H^{u+v}_x(\mathscr{X}, \mathfrak{T}_r(2)) \Longrightarrow H^{u+v}_Z(\mathscr{X}, \mathfrak{T}_r(2))$$

and the purity isomorphism

$$E_1^{u,v} \simeq \bigoplus_{x\in Z^{u-1}} H^{v-u}(x, \mathbb{Z}/p^r(2-u)) \qquad \text{for } v \le 2$$

instead of (4.2.4) and (4.2.5). The details are straight-forward and left to the reader. □

Corollary 4.2.6 *${}_{p^r}\mathrm{CH}^2(\mathscr{X})$ is finite for any $r \geq 1$, and $\mathrm{CH}^2(\mathscr{X})_{p\text{-tors}}$ is cofinitely generated.*

Proof. The finiteness of ${}_{p^r}\mathrm{CH}^2(\mathscr{X})$ follows from the exact sequence in Lemma 4.2.3 (1) and (**S7**) in §4.1. The second assertion follows from Lemma 4.1.2 and the facts that $\mathrm{CH}^2(\mathscr{X})_{p\text{-tors}}$ is a subquotient of $H^3(\mathscr{X}, \mathfrak{T}_\infty(2))$. □

4.3 Unramified Cohomology

Let K be the function field of $\mathscr{X}$. We define the unramified cohomology groups $H^{n+1}_{\mathrm{ur}}(K, \mathbb{Z}/p^r(n))$ and $H^{n+1}_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(n))$ as follows:

$$H^{n+1}_{\mathrm{ur}}(K, \mathbb{Z}/p^r(n)) := \mathrm{Ker}\left(H^{n+1}(K, \mu_{p^r}^{\otimes n}) \to \bigoplus\nolimits_{y\in\mathscr{X}^1} H^{n+2}_y(\mathscr{X}, \mathfrak{T}_r(n))\right),$$
$$H^{n+1}_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(n)) := \varinjlim_{r\geq 1} H^{n+1}_{\mathrm{ur}}(K, \mathbb{Z}/p^r(n)).$$

We mention some remarks on these groups:

Remark 4.3.1 (1) *For $n = 0$, we have*

$$H^1_{\mathrm{ur}}(K, \mathbb{Z}/p^r(0)) = H^1(\mathscr{X}, \mathbb{Z}/p^r) \quad \textit{and} \quad H^1_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(0)) = H^1(\mathscr{X}, \mathbb{Q}_p/\mathbb{Z}_p).$$

If k is global, then $H^1_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(0))$ is finite by a theorem of Katz-Lang [KL].

(2) *For $n = 1$, we have*

$$H^2_{\mathrm{ur}}(K, \mathbb{Z}/p^r(1)) = {}_{p^r}\mathrm{Br}(\mathscr{X}) \quad \textit{and} \quad H^2_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(1)) = \mathrm{Br}(\mathscr{X})_{p\text{-tors}}.$$

If k is global, the finiteness of $H^2_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(1))$ is equivalent to the finiteness of the Tate-Shafarevich group of the Picard variety of X (*cf.* [G] III, [Ta1]).

(3) *For $n = d := \dim(\mathscr{X})$, $H^{d+1}_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(d))$ agrees with a group considered by Kato* [KCT]*, who conjectures that*

$$H^{d+1}_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(d)) = 0 \quad \textit{if } p \neq 2 \textit{ or } k \textit{ has no embedding into } \mathbb{R}.$$

His conjecture is a generalization, to higher-dimensional proper arithmetic schemes, of the corresponding classical fact on the Brauer groups of local and global integer rings. The $d = 2$ case is proved in [KCT] *and the $d = 3$ case is proved in* [JS].

We restrict our atttention to the case $n = 2$ in what follows. The following standard proposition relates $H^3_{\mathrm{ur}}(K, \mathbb{Z}/p^r(2))$ with the cycle class map ϱ^2_r, which will be useful later.

PROPOSITION 4.3.2 *For a positive integer r, there is an exact sequence*

$$\begin{aligned} 0 \longrightarrow N^1H^3(\mathscr{X}, \mathfrak{T}_r(2)) &\longrightarrow H^3(\mathscr{X}, \mathfrak{T}_r(2)) \longrightarrow H^3_{\text{ur}}(K, \mathbb{Z}/p^r(2)) \\ \longrightarrow \quad \mathrm{CH}^2(\mathscr{X})/p^r &\xrightarrow{\varrho_r^2} H^4(\mathscr{X}, \mathfrak{T}_r(2)) \,. \end{aligned}$$

Consequently, taking the inductive limit on $r \geq 1$, we get an exact sequence

$$\begin{aligned} 0 \longrightarrow N^1H^3(\mathscr{X}, \mathfrak{T}_\infty(2)) &\longrightarrow H^3(\mathscr{X}, \mathfrak{T}_\infty(2)) \longrightarrow H^3_{\text{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(2)) \\ \longrightarrow \mathrm{CH}^2(\mathscr{X}) \otimes \mathbb{Q}_p/\mathbb{Z}_p &\xrightarrow{\varrho^2_{\mathbb{Q}_p/\mathbb{Z}_p}} H^4(\mathscr{X}, \mathfrak{T}_\infty(2)) \,. \end{aligned} \tag{4.3.3}$$

Proof. Consider the spectral sequence (4.2.4). Since $E_1^{u,v} = 0$ for (u, v) with $u > v$ and $v \leq 2$ by (4.2.5), there is an exact sequnece

$$0 \to E_2^{1,2} \to H^3(\mathscr{X}, \mathfrak{T}_r(2)) \to E_2^{0,3} \to E_2^{2,2} \to H^4(\mathscr{X}, \mathfrak{T}_r(2)).$$

One obtains the assertion by rewriting these E_2-terms by similar arguments as for the proof of Lemma 4.2.3 (1). $\square$

REMARK 4.3.4 *Because the groups $H^*(\mathscr{X}, \mathfrak{T}_\infty(2))$ are cofinitely generated by Lemma* 4.1.2*, the sequence* (4.3.3) *implies that $H^3_{\text{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$ is cofinitely generated if and only if $\mathrm{CH}^2(\mathscr{X}) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ is cofinitely generated.*

We next prove that $H^3_{\text{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$ is related with the torsion part of the cokernel of a cycle class map, assuming its finiteness. This result will not be used in the rest of this paper, but shows an arithmetic meaning of $H^3_{\text{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$. See also Appendix B below for a zeta value formula for threefolds over finite fields using unramified cohomology.

PROPOSITION 4.3.5 *Assume that $H^3_{\text{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$ is finite. Then the order of*

$$\mathrm{Coker}\big(\varrho^2_{\mathbb{Z}_p} : \mathrm{CH}^2(\mathscr{X}) \otimes \mathbb{Z}_p \longrightarrow H^4(\mathscr{X}, \mathfrak{T}_{\mathbb{Z}_p}(2))\big)_{p\text{-tors}}$$

agrees with that of $H^3_{\text{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$.

Proof. Note that $H^4(\mathscr{X}, \mathfrak{T}_{\mathbb{Z}_p}(2))$ is finitely generated over $\mathbb{Z}_p$ by Lemma 4.1.2, so that $\mathrm{Coker}(\varrho^2_{\mathbb{Z}_p})_{p\text{-tors}}$ is finite. Consider the following commutative diagram with exact rows (cf. Lemma 4.1.2):

$$\begin{CD} 0 @>>> \mathrm{CH}^2(\mathscr{X})_{p\text{-tors}} @>>> \mathrm{CH}^2(\mathscr{X}) \otimes \mathbb{Z}_p @>{b}>> \mathrm{CH}^2(\mathscr{X}) \otimes \mathbb{Q}_p @>>> \mathrm{CH}^2(\mathscr{X}) \otimes \mathbb{Q}_p/\mathbb{Z}_p @>>> 0 \\ @. @V{a}VV @V{\varrho^2_{\mathbb{Z}_p}}VV @V{\varrho^2_{\mathbb{Q}_p}}VV @V{\varrho^2_{\mathbb{Q}_p/\mathbb{Z}_p}}VV \\ 0 @>>> \mathrm{Cotor}(H^3(\mathscr{X}, \mathfrak{T}_\infty(2))) @>>> H^4(\mathscr{X}, \mathfrak{T}_{\mathbb{Z}_p}(2)) @>{c}>> H^4(\mathscr{X}, \mathfrak{T}_{\mathbb{Q}_p}(2)) @>>> H^4(\mathscr{X}, \mathfrak{T}_\infty(2)), \end{CD}$$

where the arrow a denotes the map obtained from the short exact sequence in Lemma 4.2.3 (1) and the arrows b and c are natural maps. See Lemma 4.4.2 below for the commutativity of the left square. By the finiteness of $H^3_{\text{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$, we see that

$$\mathrm{Coker}(a) \simeq \mathrm{gr}^0_N H^3(\mathscr{X}, \mathfrak{T}_\infty(2)) := H^3(\mathscr{X}, \mathfrak{T}_\infty(2))/N^1H^3(\mathscr{X}, \mathfrak{T}_\infty(2))$$

(cf. Lemma 4.2.3 (1)) and that the natural map $\mathrm{Ker}(\varrho^2_{\mathbb{Q}_p}) \to \mathrm{Ker}(\varrho^2_{\mathbb{Q}_p/\mathbb{Z}_p})$ is zero (cf. (4.3.3)). The latter conclusion further implies that the kernel of the induced map $\mathrm{Im}(b) \to \mathrm{Im}(c)$ is divisible. Noting these facts, we obtain a short exact sequence

$$0 \longrightarrow \mathrm{gr}^0_N H^3(\mathscr{X}, \mathfrak{T}_\infty(2)) \longrightarrow \mathrm{Coker}(\varrho^2_{\mathbb{Z}_p})_{p\text{-tors}} \longrightarrow \mathrm{Ker}(\varrho^2_{\mathbb{Q}_p/\mathbb{Z}_p}) \longrightarrow 0$$

by a diagram chase on the above diagram. Comparing this sequence with (4.3.3), we obtain the assertion. □

4.4 Torsion Cycle Class Map of Codimension Two

We define $H^3_{\mathrm{ur}}(K, X; \mathbb{Q}_p/\mathbb{Z}_p(2))$ as the following subgroup of $H^3_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$:

$$\mathrm{Im}\big(H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \to H^3(K, \mathbb{Q}_p/\mathbb{Z}_p(2))\big) \cap H^3_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(2)).$$

In this subsection we relate the finiteness of this group with the injectivity of torsion cycle class maps of codimension two (see Corollary 4.4.3 below), which will be used in the proof of Theorem 1.6. We start with the following proposition.

Proposition 4.4.1 *Assume that the quotient*

$$\mathrm{gr}^0_N H^3(\mathscr{X}, \mathfrak{T}_\infty(2)) := H^3(\mathscr{X}, \mathfrak{T}_\infty(2))/N^1H^3(\mathscr{X}, \mathfrak{T}_\infty(2))$$

is finite. Then there exists a positive integer r_0 such that the kernel of the map

$$\varrho^2_{p\text{-tors},r} : \mathrm{CH}^2(\mathscr{X})_{p\text{-tors}} \longrightarrow H^4(\mathscr{X}, \mathfrak{T}_r(2))$$

agrees with $(\mathrm{CH}^2(\mathscr{X})_{p\text{-tors}})_{\mathrm{Div}}$ for any $r \geq r_0$.

We need the following lemma to prove this proposition (cf. [CTSS] Proposition 1):

Lemma 4.4.2 *For integers $r, s > 0$, there is a commutative diagram up to a sign*

$$\begin{array}{ccc} N^1H^3(\mathscr{X}, \mathfrak{T}_s(2)) & \xrightarrow{\alpha_s} & {}_{p^s}\mathrm{CH}^2(\mathscr{X}) \\ \big\downarrow & & \big\downarrow{\scriptstyle \varrho^2_{s,r}} \\ H^3(\mathscr{X}, \mathfrak{T}_s(2)) & \xrightarrow{\delta_{s,r}} & H^4(\mathscr{X}, \mathfrak{T}_r(2)), \end{array}$$

where α_s denotes the boundary map in the short exact sequence of Lemma 4.2.3 (1) *and $\varrho^2_{s,r}$ denotes the cycle class map ϱ^2_r restricted to ${}_{p^s}\mathrm{CH}^2(\mathscr{X})$. The arrow $\delta_{s,r}$ is the connecting morphism of the distinguished triangle in* (**S6**):

$$\mathfrak{T}_{r+s}(2) \longrightarrow \mathfrak{T}_s(2) \xrightarrow{\delta_{s,r}} \mathfrak{T}_r(2)[1] \xrightarrow{p^s} \mathfrak{T}_{r+s}(2)[1].$$

Proof of Lemma 4.4.2. Note that $N^1H^3(\mathscr{X}, \mathfrak{T}_s(2))$ is generated by the image of $H^3_Z(\mathscr{X}, \mathfrak{T}_s(2))$ for closed subsets $Z \subset \mathscr{X}$ of pure codimension 1. We fix such a

Z, and endow it with the reduced subscheme structure. There is a diagram which commutes obviously

$$\begin{tikzcd} H^3_Z(\mathscr{X}, \mathfrak{T}_s(2)) \arrow[r, "\delta_{s,r}"] \arrow[d, "\text{canonical}"'] & H^4_Z(\mathscr{X}, \mathfrak{T}_r(2)) \arrow[d, "\text{canonical}"] \\ H^3(\mathscr{X}, \mathfrak{T}_s(2)) \arrow[r, "\delta_{s,r}"] & H^4(\mathscr{X}, \mathfrak{T}_r(2)). \end{tikzcd}$$

We show that the image of the upper $\delta_{s,r}$ lies in the subgroup

$$\begin{aligned} N^2H^4_Z(\mathscr{X}, \mathfrak{T}_r(2)) &= \mathrm{Ker}\Big(H^4_Z(\mathscr{X}, \mathfrak{T}_r(2)) \to \bigoplus\nolimits_{z \in Z^0} H^4_z(\mathscr{X}, \mathfrak{T}_r(2))\Big) \\ &\simeq \mathrm{Coker}\Big(\partial^{\mathrm{val}} : \bigoplus\nolimits_{z \in Z^0} \kappa(z)^\times/p^r \to \bigoplus\nolimits_{x \in Z^1} \mathbb{Z}/p^r\Big) \\ &= \mathrm{CH}_{d-2}(Z)/p^r \qquad\qquad (d := \dim(\mathscr{X})) \end{aligned}$$

(cf. Lemma 4.2.3). Indeed, there is a commutative diagram with exact bottom row

$$\begin{tikzcd} & H^3_Z(\mathscr{X}, \mathfrak{T}_s(2)) \arrow[r, "\delta_{s,r}"] \arrow[d] & H^4_Z(\mathscr{X}, \mathfrak{T}_r(2)) \arrow[d] \\ \displaystyle\bigoplus_{z \in Z^0} H^3_z(\mathscr{X}, \mathfrak{T}_{r+s}(2)) \arrow[r, two heads] & \displaystyle\bigoplus_{z \in Z^0} H^3_z(\mathscr{X}, \mathfrak{T}_s(2)) \arrow[r, "\delta_{s,r}"] & \displaystyle\bigoplus_{z \in Z^0} H^4_z(\mathscr{X}, \mathfrak{T}_r(2)), \end{tikzcd}$$

whose bottom left arrow is surjective by the purity in ($\mathbf{S3}$) and Hilbert's theorem 90:

$$H^3_z(\mathscr{X}, \mathfrak{T}_t(2)) \simeq H^1(z, \mathbb{Z}/p^t(1)) \simeq \kappa(z)^\times/p^t \quad \text{ for } \quad t = r+s,\ s.$$

Hence the lower $\delta_{s,r}$ is the zero map and the image of the upper $\delta_{s,r}$ is contained in $N^2H^4_Z(\mathscr{X}, \mathfrak{T}_r(2))$. Now the composite map

$$H^3_Z(\mathscr{X}, \mathfrak{T}_s(2)) \xrightarrow{\delta_{s,r}} N^2H^4_Z(\mathscr{X}, \mathfrak{T}_r(2)) \simeq \mathrm{CH}_{d-2}(Z)/p^r \longrightarrow \mathrm{CH}^2(\mathscr{X})/p^r$$

agrees, up to a sign, with the composite map

$$H^3_Z(\mathscr{X}, \mathfrak{T}_s(2)) \to N^1H^3(\mathscr{X}, \mathfrak{T}_s(2)) \xrightarrow{\alpha_s} {}_{p^s}\mathrm{CH}^2(\mathscr{X}) \hookrightarrow \mathrm{CH}^2(\mathscr{X}) \twoheadrightarrow \mathrm{CH}^2(\mathscr{X})/p^r$$

by ($\mathbf{S4}$) and computations on boudary maps (see [CTSS] Proof of Proposition 1, Step 6). We obtain the commutativity in question from these facts, because the cycle class map ϱ^2_r for cycles on Z is given by the composite map

$$\mathrm{CH}_{d-2}(Z)/p^r \simeq N^2H^4_Z(\mathscr{X}, \mathfrak{T}_r(2)) \hookrightarrow H^4_Z(\mathscr{X}, \mathfrak{T}_r(2)) \to H^4(\mathscr{X}, \mathfrak{T}_r(2))$$

by definition. □

Proof of Proposition 4.4.1. The following argument is essentially the same as the proof of [CTSS] Corollaire 3. Taking the inductive limit of the diagram in Lemma

4.4.2 with respect to $s \geq 1$, we obtain a diagram whose square commutes up to a sign and whose bottom row is exact

$$\begin{tikzcd}
 & N^1H^3(\mathscr{X}, \mathfrak{T}_\infty(2)) \arrow[r, "\alpha_\infty"] \arrow[d, hook] & \mathrm{CH}^2(\mathscr{X})_{p\text{-tors}} \arrow[d, "\varrho^2_{p\text{-tors},r}"] \\
H^3(\mathscr{X}, \mathfrak{T}_\infty(2)) \arrow[r, "\times p^r"] & H^3(\mathscr{X}, \mathfrak{T}_\infty(2)) \arrow[r, "\delta_{\infty,r}"] & H^4(\mathscr{X}, \mathfrak{T}_r(2)).
\end{tikzcd}$$

Since Ker(α_∞) is divisible by Lemma 4.2.3 (1), this diagram induces the following commutative diagram up to a sign:

$$\begin{tikzcd}
 & \mathrm{Cotor}(N^1H^3(\mathscr{X}, \mathfrak{T}_\infty(2))) \arrow[r, "\overline{\alpha_\infty}", "\sim"'] \arrow[d, hook] & \mathrm{Cotor}(\mathrm{CH}^2(\mathscr{X})_{p\text{-tors}}) \arrow[d, "\overline{\varrho^2_{p\text{-tors},r}}"] \\
\mathrm{Cotor}(H^3(\mathscr{X}, \mathfrak{T}_\infty(2))) \arrow[r, "\times p^r"] & \mathrm{Cotor}(H^3(\mathscr{X}, \mathfrak{T}_\infty(2))) \arrow[r, "\overline{\delta_{\infty,r}}"] & H^4(\mathscr{X}, \mathfrak{T}_r(2)),
\end{tikzcd}$$

where the bottom row remains exact, and the injectivity of the central vertical arrow follows from the finiteness of $\mathrm{gr}^0_N H^3(\mathscr{X}, \mathfrak{T}_\infty(2))$. Because $\mathrm{Cotor}(H^3(\mathscr{X}, \mathfrak{T}_\infty(2)))$ is finite by $(\mathbf{S7})$ and Lemma 4.1.2, the map $\overline{\delta_{\infty,r}}$ is injective for any r for which p^r annihilates $\mathrm{Cotor}(H^3(\mathscr{X}, \mathfrak{T}_\infty(2)))$. Thus we obtain Proposition 4.4.1. □

Corollary 4.4.3 *If $H^3_{\mathrm{ur}}(K, X; \mathbb{Q}_p/\mathbb{Z}_p(2))$ is finite, then there is a positive integer r_0 such that* $\mathrm{Ker}(\varrho^2_{p\text{-tors},r}) = (\mathrm{CH}^2(\mathscr{X})_{p\text{-tors}})_{\mathrm{Div}}$ *for any $r \geq r_0$.*

Proof. Since $\mathrm{gr}^0_N H^3(\mathscr{X}, \mathfrak{T}_\infty(2))$ is a subgroup of $H^3_{\mathrm{ur}}(K, X; \mathbb{Q}_p/\mathbb{Z}_p(2))$ (cf. (4.3.3)), the assumption implies that $\mathrm{gr}^0_N H^3(\mathscr{X}, \mathfrak{T}_\infty(2))$ is finite. Hence the assertion follows from Proposition 4.4.1. □

Remark 4.4.4 *If k is ℓ-adic local with $\ell \neq p$, then we have $\mathfrak{T}_\infty(2) = \mathbb{Q}_p/\mathbb{Z}_p(2)$ by definition and*

$$H^3(\mathscr{X}, \mathfrak{T}_\infty(2)) = H^3(\mathscr{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)) \simeq H^3(Y, \mathbb{Q}_p/\mathbb{Z}_p(2))$$

by the proper base-change theorem, where Y denotes the closed fiber of $\mathscr{X}/S$. The last group is finite by Deligne's proof of the Weil conjecture [De2]. *Hence $\varrho^2_{p\text{-tors},r}$ for $\mathscr{X}$ is injective for a sufficiently large $r \geq 1$ by Proposition* 4.4.1. *On the other hand, if k is global or p-adic local, then $H^3(\mathscr{X}, \mathfrak{T}_\infty(2))$ is not in general finite. Therefore we need to consider the finiteness of the group $H^3_{\mathrm{ur}}(K, X; \mathbb{Q}_p/\mathbb{Z}_p(2))$ to investigate the injectivity of $\varrho^2_{p\text{-tors},r}$.*

5 Finiteness of an Unramified Cohomology Group

Let $k, S, p, \mathscr{X}$ and K be as in the notation 1.8. We always assume 1.8.1 throughout this and the next section.

5.1 FINITENESS OF $H^3_{\mathrm{ur}}(K, X; \mathbb{Q}_p/\mathbb{Z}_p(2))$

In this and the next section, we prove the following result, which implies the finiteness of $H^3_{\mathrm{ur}}(K, X; \mathbb{Q}_p/\mathbb{Z}_p(2))$ in Theorem 1.3 (1). See the beginning of §3 for **H0**.

THEOREM 5.1.1 *Assume* **H0**, **H1**$*$ *and either* $p \geq 5$ *or the equality*

$$H^1_g(k, H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)))_{\mathrm{Div}} = H^1(k, H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)))_{\mathrm{Div}}. \qquad (*_g)$$

Then $H^3_{\mathrm{ur}}(K, X; \mathbb{Q}_p/\mathbb{Z}_p(2))$ *is finite.*

In this section we reduce Theorem 5.1.1 to Key Lemma 5.4.1 stated in §5.4 below. We will prove the key lemma in §6. We first prove Theorem 1.6 admitting Theorem 5.1.1.

Proof of Theorem 1.6. The assumption $H^2(X, \mathscr{O}_X) = 0$ implies **H1**$*$ and $(*_g)$ (cf. Fact 1.1, Remark 3.2.5, Remark 3.1.2 (1)). Hence $H^3_{\mathrm{ur}}(K, X; \mathbb{Q}_p/\mathbb{Z}_p(2))$ is finite by Theorem 5.1.1. By Corollary 4.4.3, there is a positive integer r_0 such that

$$\mathrm{Ker}(\varrho^2_{p\text{-tors},r}) = (\mathrm{CH}^2(\mathscr{X})_{p\text{-tors}})_{\mathrm{Div}} \quad \text{for any } r \geq r_0.$$

Thus it remains to check that $\mathrm{CH}^2(\mathscr{X})_{p\text{-tors}}$ is finite, which follows from the finiteness of $\mathrm{CH}^2(X)_{p\text{-tors}}$ (cf. Theorem 3.1.1) and [CTR2] Lemma 3.3. This completes the proof. □

5.2 PROOF OF THEOREM 5.1.1, STEP 1

We reduce Theorem 5.1.1 to Proposition 5.2.2 below. Let $N^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))$ (resp. $\mathrm{gr}^0_N H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))$) be the kernel (resp. the image) of the natural map

$$H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \longrightarrow H^3(K, \mathbb{Q}_p/\mathbb{Z}_p(2)).$$

In view of Lemma 4.2.3, there is a commutative diagram with exact rows

$$\begin{array}{ccccc} N^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) & \hookrightarrow & H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) & \to & \mathrm{gr}^0_N H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \\ {\scriptstyle \delta_1}\downarrow & & {\scriptstyle \delta_2}\downarrow & & {\scriptstyle \overline{\mathfrak{d}}}\downarrow \\ \bigoplus_{v \in S^1} N^2H^4_{Y_v}(\mathscr{X}, \mathfrak{T}_\infty(2)) & \hookrightarrow & \bigoplus_{v \in S^1} H^4_{Y_v}(\mathscr{X}, \mathfrak{T}_\infty(2)) & \to & \bigoplus_{v \in S^1} \bigoplus_{y \in Y_v^0} H^4_y(\mathscr{X}, \mathfrak{T}_\infty(2)), \end{array} \qquad (5.2.1)$$

where the arrows δ_2 and $\overline{\mathfrak{d}}$ arise from boundary maps of localization exact sequences and δ_1 is induced by the right square. Note that we have

$$\mathrm{Ker}(\overline{\mathfrak{d}}) = H^3_{\mathrm{ur}}(K, X; \mathbb{Q}_p/\mathbb{Z}_p(2)).$$

PROPOSITION 5.2.2 *Assume* **H0**, **H1**$*$ *and either* $p \geq 5$ *or* $(*_g)$*. Then we have*

$$\mathrm{Ker}(\overline{\mathfrak{d}})_{\mathrm{Div}} = 0.$$

The proof of this proposition will be started in §5.3 below and finished in the next section. We first finish the proof of Theorem 5.1.1, admitting Proposition 5.2.2. It suffices to show the following lemma (see also Remark 3.3.4 (2)):

Lemma 5.2.3 (1) *If k is local, then* $\mathrm{Ker}(\overline{\eth})$ *is cofinitely generated over* $\mathbb{Z}_p$.

(2) *Assume that k is global, and that* $\mathrm{Coker}\big(\mathrm{reg}_{\mathbb{Q}_p/\mathbb{Z}_p}\big)_{\mathrm{Div}}$ *is cofinitely generated over $\mathbb{Z}_p$, where* $\mathrm{reg}_{\mathbb{Q}_p/\mathbb{Z}_p}$ *denotes the regulator map* (3.2.1). *Then* $\mathrm{Ker}(\overline{\eth})$ *is cofinitely generated over* $\mathbb{Z}_p$.

Proof. (1) is obvious, because $H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))$ is cofinitely generated. We prove (2). We use the notation fixed in 1.9. By Lemma 4.1.2, $H^3(\mathscr{X}, \mathfrak{T}_\infty(2))$ is cofinitely generated. Hence it suffices to show $\mathrm{Coker}(\delta_1)$ is cofinitely generated, where δ_1 is as in (5.2.1). There is a commutative diagram

$$\begin{CD}
\mathrm{CH}^2(X,1)\otimes\mathbb{Q}_p/\mathbb{Z}_p @>>> N^1H^3(X,\mathbb{Q}_p/\mathbb{Z}_p(2))\\
@V{\partial}VV @VV{\delta_1}V\\
\bigoplus_{v\in S^1}\mathrm{CH}_{d-2}(Y_v)\otimes\mathbb{Q}_p/\mathbb{Z}_p @>{\sim}>> \bigoplus_{v\in S^1} N^2H^4_{Y_v}(\mathscr{X},\mathfrak{T}_\infty(2)),
\end{CD}$$

where the bottom isomorphism follows from Lemma 4.2.3 (2) and ∂ is the boundary map of the localization sequence of higher Chow groups. See (3.3.1) for the top arrow. Since $N^2H^4_{Y_v}(\mathscr{X}, \mathfrak{T}_\infty(2))$ is cofinitely generated for any $v \in S^1$, it suffices to show that for a sufficiently small non-empty open subset $U \subset S$, the cokernel of the boundary map

$$\partial_U : \mathrm{CH}^2(X,1)\otimes\mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \bigoplus_{v\in(U)^1}\mathrm{CH}_{d-2}(Y_v)\otimes\mathbb{Q}_p/\mathbb{Z}_p$$

is cofinitely generated. Note that $\mathrm{CH}_{d-2}(Y_v) = \mathrm{CH}^1(Y_v)$ if Y_v is smooth. Now let U be a non-empty open subset of $S \setminus \Sigma$ for which $\mathscr{X} \times_S U \to U$ is smooth. Put $A := H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))$, viewed as a smooth sheaf on $U_{\text{ét}}$. There is a commutative diagram up to a sign

$$\begin{CD}
\mathrm{CH}^2(X,1)\otimes\mathbb{Q}_p/\mathbb{Z}_p @>{\mathrm{reg}_{\mathbb{Q}_p/\mathbb{Z}_p}}>> H^1(k,A)\\
@V{\partial_U}VV @VV{\delta_U}V\\
\bigoplus_{v\in U^1}\mathrm{CH}^1(Y_v)\otimes\mathbb{Q}_p/\mathbb{Z}_p @>{\tau_U}>> \bigoplus_{v\in U^1} A(-1)^{G_{\mathbb{F}_v}}.
\end{CD}$$

See §2.5 for δ_U. The bottom arrow τ_U is defined as the composite map

$$\mathrm{CH}^1(Y_v)\otimes\mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow H^2(Y_v,\mathbb{Q}_p/\mathbb{Z}_p(1)) \xrightarrow{\epsilon} H^2(\overline{Y_v},\mathbb{Q}_p/\mathbb{Z}_p(1))^{G_{\mathbb{F}_v}} = A(-1)^{G_{\mathbb{F}_v}},$$

where the first injective map is the cycle class map for divisors on Y_v. Note that $\mathrm{Coker}(\partial_U)$ is divisible and that $\mathrm{Ker}(\tau_U)$ has a finite exponent by the isomorphisms

$$\mathrm{Ker}(\epsilon) \simeq H^1(\mathbb{F}_v, H^1(\overline{Y_v}, \mathbb{Q}_p/\mathbb{Z}_p(1))) \simeq H^1(\mathbb{F}_v, \mathrm{Cotor}(H^1(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(1))))$$

for $v \in U^1$, where the first isomorphism follows from the Hochschild-Serre spectral sequence for Y_v, and the second follows from Deligne's proof of the Weil conjecture [De2] and the proper smooth base-change theorem $H^1(\overline{Y_v}, \mathbb{Q}_p/\mathbb{Z}_p(1)) \simeq H^1(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(1))$. Hence to prove that $\mathrm{Coker}(\partial_U)$ is cofinitely generated, it suffices to show that the map

$$\partial' := \tau_U \circ \partial_U : \mathrm{CH}^2(X,1) \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \bigoplus_{v \in U^1} \big(A(-1)^{G_{\mathbb{F}_v}}\big)_{\mathsf{Div}}$$

has cofinitely generated cokernel (cf. Lemma 2.3.2 (2)). Finally $\mathrm{Coker}(\mathrm{reg}_{\mathbb{Q}_p/\mathbb{Z}_p})_{\mathsf{Div}}$ is cofinitely generated by assumption, which implies that ∂' has cofinitely generated cokernel by Lemma 2.5.1 (1). Thus we obtain Lemma 5.2.3. □

5.3 Proof of Theorem 5.1.1, Step 2

We construct a key commutative diagram (5.3.3) below and prove Lemma 5.3.5, which play key roles in our proof of Proposition 5.2.2. We need some preliminaries. We suppose that k is global until the end of Lemma 5.3.1. Let $\Sigma \subset S$ be the set of the closed points on S of characteristic p. For non-empty open $U \subset S$, put

$$\mathscr{X}_U := \mathscr{X} \times_S U \quad \text{and} \quad \mathscr{X}_U[p^{-1}] := \mathscr{X}_U \times_S (S \setminus \Sigma).$$

Let $j_U : \mathscr{X}_U[p^{-1}] \to \mathscr{X}_U$ be the natural open immersion. There is a natural injective map

$$\alpha_{U,r} : H^3(\mathscr{X}_U, \tau_{\leq 2} Rj_{U*}\mu_{p^r}^{\otimes 2}) \hookrightarrow H^3(\mathscr{X}_U[p^{-1}], \mu_{p^r}^{\otimes 2})$$

induced by the canonical morphism $\tau_{\leq 2} Rj_{U*}\mu_{p^r}^{\otimes 2} \to Rj_{U*}\mu_{p^r}^{\otimes 2}$.

LEMMA 5.3.1 *We have* $N^1H^3(\mathscr{X}_U[p^{-1}], \mu_{p^r}^{\otimes 2}) \subset \mathrm{Im}(\alpha_{U,r})$.

Proof. We compute the local-global spectral sequence

$$E_1^{u,v} = \bigoplus_{x \in (\mathscr{X}_U)^u} H_x^{u+v}(\mathscr{X}_U, \tau_{\leq 2} Rj_{U*}\mu_{p^r}^{\otimes 2}) \Longrightarrow H^{u+v}(\mathscr{X}_U, \tau_{\leq 2} Rj_{U*}\mu_{p^r}^{\otimes 2}).$$

By the absolute cohomological purity [FG] and Lemma 3.5.5 (1), we have

$$E_1^{u,v} \simeq \begin{cases} H^v(K, \mu_{p^r}^{\otimes 2}) & (\text{if } u = 0) \\ \bigoplus_{x \in (\mathscr{X}_U[p^{-1}])^u} H^{v-u}(x, \mu_{p^r}^{\otimes 2-u}) & (\text{if } v \leq 2). \end{cases}$$

Repeating the same computation as in the proof of Lemma 3.2.2, we obtain

$$N^1H^3(\mathscr{X}_U[p^{-1}], \mu_{p^r}^{\otimes 2}) \simeq E_2^{1,2} = E_\infty^{1,2} \hookrightarrow H^3(\mathscr{X}_U, \tau_{\leq 2} Rj_{U*}\mu_{p^r}^{\otimes 2}),$$

which completes the proof of Lemma 5.3.1. □

Now we suppose that k is either local or global, and put

$$\mathscr{W} := \begin{cases} H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) & \text{(if } k \text{ is } \ell\text{-adic local with } \ell \neq p) \\ H^3(\mathscr{X}, \tau_{\leq 2} Rj_* \mathbb{Q}_p/\mathbb{Z}_p(2)) & \text{(if } k \text{ is } p\text{-adic local)} \\ \varinjlim_{\Sigma \subset U \subset S} H^3(\mathscr{X}_U, \tau_{\leq 2} Rj_{U*} \mathbb{Q}_p/\mathbb{Z}_p(2)) & \text{(if } k \text{ is global),} \end{cases} \tag{5.3.2}$$

where j in the second case denotes the natural open immersion $X \hookrightarrow \mathscr{X}$, and the limit in the last case is taken over all non-empty open subsets $U \subset S$ which contain Σ. By Lemma 3.5.3 and Lemma 5.3.1, there are inclusions

$$N^1 H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \subset \mathscr{W} \subset H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))$$

and a commutative diagram

$$\begin{tikzcd} NF^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))_{\mathrm{Div}} \arrow[r, hook] \arrow[dr, "\nu"'] & (\mathscr{W}^0)_{\mathrm{Div}} \arrow[d, "\omega"] \\ & H^1(k, H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))). \end{tikzcd} \tag{5.3.3}$$

Here $NF^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))$ is as we defined in §3.5, and we put

$$\mathscr{W}^0 := \mathrm{Ker}\big(\mathscr{W} \longrightarrow H^3(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))\big). \tag{5.3.4}$$

The arrows ω and ν are induced by the edge homomorphism (3.5.1). We show here the following lemma, which extends Lemma 3.5.2 under Assumption 1.8.1:

Lemma 5.3.5 *Assume either $p \geq 5$ or $(*_g)$. Then we have*

$$\mathrm{Im}(\omega) \subset H^1_g(k, H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))).$$

Remark 5.3.6 *We will prove the equality* $\mathrm{Im}(\omega) = H^1_g(k, H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)))_{\mathrm{Div}}$ *under the same assumptions, later in Lemma* 7.2.2.

The following corollary of Lemma 5.3.5 will be used later in §5.4:

Corollary 5.3.7 *Assume* **H0**, **H1**$*$ *and either $p \geq 5$ or $(*_g)$. Then we have*

$$\mathrm{Im}(\nu) = \mathrm{Im}(\omega) = H^1_g(k, H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)))_{\mathrm{Div}}.$$

Proof of Lemma 5.3.5. The assertion under the second condition is rather obvious. In particular, we are done if k is ℓ-adic local with $\ell \neq p$ (cf. Remark 3.1.2 (1)). If k is p-adic local with $p \geq 5$, the assertion follows from Corollary 2.2.3 and Lemma 3.5.4. Before proving the global case, we show the following sublemma:

SUBLEMMA 5.3.8 *Let k be an ℓ-adic local field with $\ell \neq p$. Let $\mathscr{X}$ be a proper smooth scheme over $S := \mathrm{Spec}(\mathfrak{o}_k)$. Put $A := H^i(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(n))$ and*

$$H^{i+1}_{\mathrm{ur}}(X, \mathbb{Q}_p/\mathbb{Z}_p(n)) := \mathrm{Im}\big(H^{i+1}(\mathscr{X}, \mathbb{Q}_p/\mathbb{Z}_p(n)) \to H^{i+1}(X, \mathbb{Q}_p/\mathbb{Z}_p(n))\big).$$

Then we have

$$H^1_f(k, A) \subset \mathrm{Im}\big(F^1H^{i+1}(X, \mathbb{Q}_p/\mathbb{Z}_p(n)) \cap H^{i+1}_{\mathrm{ur}}(X, \mathbb{Q}_p/\mathbb{Z}_p(n)) \to H^1(k, A)\big)$$

and the quotient is annihilated by $\#(A/A_{\mathrm{Div}})$, where $F^\bullet$ denotes the filtration induced by the Hochschild-Serre spectral sequence (2.6.2)*.*

Proof. Put $\Lambda := \mathbb{Q}_p/\mathbb{Z}_p$, and let $\mathbb{F}$ be the residue field of k. By the proper smooth base-change theorem, G_k acts on A through the quotient $G_{\mathbb{F}}$. It suffices to show the following two claims:

(i) *We have*

$$\mathrm{Im}\big(F^1H^{i+1}(X, \Lambda(n)) \cap H^{i+1}_{\mathrm{ur}}(X, \Lambda(n)) \to H^1(k, A)\big) = H^1(\mathbb{F}, A),$$

where $H^1(\mathbb{F}, A)$ is regarded as a subgroup of $H^1(k, A)$ by an inflation map.

(ii) *We have*

$$H^1_f(k, A) \subset H^1(\mathbb{F}, A)$$

and the quotient is annihilated by $\#(A/A_{\mathrm{Div}})$.

We show these claims. Let Y be the closed fiber of $\mathscr{X}/S$, and consider a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & H^1(\mathbb{F}, H^i(\overline{Y}, \Lambda(n))) \arrow[r] \arrow[d, hook, "\sigma_1"'] & H^{i+1}(Y, \Lambda(n)) \arrow[r] \arrow[d, "\sigma_2"'] & H^{i+1}(\overline{Y}, \Lambda(n)))^{G_{\mathbb{F}}} \arrow[d, "\sigma_3", "\wr"'] \\
0 \arrow[r] & H^1(k, A) \arrow[r] & H^{i+1}(X, \Lambda(n))/F^2 \arrow[r] & H^{i+1}(\overline{X}, \Lambda(n)))^{G_k},
\end{tikzcd}$$

where the exactness of the upper (resp. lower) row follows from the fact that $\mathrm{cd}(G_{\mathbb{F}}) = 1$ (resp. $\mathrm{cd}(G_k) = 2$). The arrows σ_1 and σ_3 are induced by the isomorphism

$$H^*(\overline{Y}, \Lambda(n)) \simeq H^*(\overline{X}, \Lambda(n)) \quad \text{(proper smooth base-change theorem).}$$

The arrow σ_2 is induced by

$$\sigma_2' : H^{i+1}(Y, \Lambda(n)) \simeq H^{i+1}(\mathscr{X}, \Lambda(n)) \longrightarrow H^{i+1}(X, \Lambda(n)).$$

Since $\mathrm{Im}(\sigma_2') = H^{i+1}_{\mathrm{ur}}(X, \Lambda(n))$ by definition, the claim (i) follows from the above diagram. The second assertion immediately follows from the fact that $H^1_f(k, A) = \mathrm{Im}(H^1(\mathbb{F}, A)_{\mathrm{Div}} \to H^1(k, A))$. This completes the proof of Sublemma 5.3.8. $\square$

We prove Lemma 5.3.5 in the case that k is global with $p \geq 5$. Let $\mathscr{W}$ and $\mathscr{W}^0$ be as in (5.3.2) and (5.3.4), respectively, and put

$$A := H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)).$$

Note that $(\mathscr{W}^0)_{\mathrm{Div}} = \mathscr{W}_{\mathrm{Div}}$ by **H0**. By a similar argument as for Lemma 2.4.1, we have

$$\mathscr{W}_{\mathrm{Div}} = \varinjlim_{\Sigma \subset U \subset S} H^3(\mathscr{X}_U, \tau_{\leq 2} Rj_{U*}\mathbb{Q}_p/\mathbb{Z}_p(2))_{\mathrm{Div}}.$$

Here the limit is taken over all non-empty open subsets $U \subset S$ which contain Σ, and j_U denotes the natural open immersion $\mathscr{X}_U[p^{-1}] \hookrightarrow \mathscr{X}_U$. By this equality and the definition of $H^1_g(k, A)$ (cf. Definition 2.1.1), it suffices to show the following sublemma:

Sublemma 5.3.9 *Let U be an open subset of S containing Σ, and fix an open subset U' of $U \setminus \Sigma$ for which $\mathscr{X}_{U'} \to U'$ is smooth (and proper). Put $\mathscr{W}_U := H^3(\mathscr{X}_U, \tau_{\leq 2} Rj_{U*}\mathbb{Q}_p/\mathbb{Z}_p(2))$. Then for any $x \in (\mathscr{W}_U)_{\mathrm{Div}}$, its diagonal image*

$$\overline{x} = (\overline{x}_v)_{v \in S^1} \in \prod_{v \in (U')^1} \frac{H^1(k_v, A)}{H^1_f(k_v, A)} \times \prod_{v \in S \setminus U'} \frac{H^1(k_v, A)}{H^1_g(k_v, A)}$$

is zero.

Proof. Since $(\mathscr{W}_U)_{\mathrm{Div}}$ is divisible, it suffices to show that $\overline{x}$ is killed by a positive integer independent of x. By Lemma 5.3.8, $\overline{x}_v$ with $v \in (U')^1$ is killed by $\#(A/A_{\mathrm{Div}})$. Next we compute $\overline{x}_v$ with $v \in \Sigma$. Let $\mathscr{X}_v$ and $j_v : X_v \hookrightarrow \mathscr{X}_v$ be as in 1.9, and put

$$\mathscr{W}_v := H^3(\mathscr{X}_v, \tau_{\leq 2} Rj_{v*}\mathbb{Q}_p/\mathbb{Z}_p(2)).$$

By **H0** over k, we have

$$\mathrm{Im}\big((\mathscr{W}_U)_{\mathrm{Div}} \to \mathscr{W}_v\big) \subset \mathrm{Ker}\big(\mathscr{W}_v \to H^3(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))\big)_{\mathrm{Div}}.$$

Hence Corollary 2.2.3 and Lemma 3.5.4 imply that $\overline{x}_v = 0$ for $v \in \Sigma$. Finally, because the product of the other components

$$\prod_{v \in S \setminus (U' \cup \Sigma)} \frac{H^1(k_v, A)}{H^1_g(k_v, A)}$$

is a finite group, we see that all local components of $\overline{x}$ are annihilated by a positive integer independent of x. Thus we obtain the sublemma and Lemma 5.3.5. $\square$

5.4 Proof of Theorem 5.1.1, Step 3

We reduce Proposition 5.2.2 to Key Lemma 5.4.1 below. We replace the conditions in Proposition 5.2.2 with another condition

> **N1**: *We have* $\mathrm{Im}(\omega) = \mathrm{Im}(\nu)$ *in* (5.3.3)*, and* $\mathrm{Coker}\big(\mathrm{reg}_{\mathbb{Q}_p/\mathbb{Z}_p}\big)_{\mathrm{Div}}$ *is cofinitely generated over* $\mathbb{Z}_p$*. Here* $\mathrm{reg}_{\mathbb{Q}_p/\mathbb{Z}_p}$ *denotes the regulator map* (3.2.1).

Indeed, assuming $\mathbf{H0}$, $\mathbf{H1}*$ and either $p \geq 5$ or $(*_g)$, we see that $\mathbf{N1}$ holds by Corollary 5.3.7 and the fact that the quotient $H(k, A)_{\mathrm{Div}}/H^1_g(k, A)_{\mathrm{Div}}$, with $A = H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))$, is cofinitely generated over $\mathbb{Z}_p$ for (cf. Lemma 2.4.1). Thus Proposition 5.2.2 is reduced to the following:

KEY LEMMA 5.4.1 *Assume* $\mathbf{H0}$ *and* $\mathbf{N1}$*. Then we have* $\mathrm{Ker}(\overline{\mathfrak{d}})_{\mathrm{Div}} = 0$.

This lemma will be proved in the next section.

6 PROOF OF THE KEY LEMMA

The notation remains as in the previous section. We always assume 1.8.1 throughout this section. The aim of this section is to prove Key Lemma 5.4.1.

6.1 PROOF OF KEY LEMMA 5.4.1

Let

$$\mathfrak{d} : H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \longrightarrow \bigoplus_{v \in S^1} \bigoplus_{y \in (Y_v)^0} H^4_y(\mathscr{X}, \mathfrak{T}_\infty(2))$$

be the map induced by $\overline{\mathfrak{d}}$ in (5.2.1). Put

$$\Theta := H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \,/\big(N^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))_{\mathrm{Div}}\big)$$

and let $\widetilde{\Theta} \subset \Theta$ be the image of $\mathrm{Ker}(\mathfrak{d})$. Note that we have

$$\widetilde{\Theta} = \mathrm{Ker}\Big(\Theta \to \mathrm{gr}^0_N H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \xrightarrow{\overline{\mathfrak{d}}} \bigoplus\nolimits_{v \in S^1} \bigoplus\nolimits_{y \in (Y_v)^0} H^4_y(\mathscr{X}, \mathfrak{T}_\infty(2))\Big)$$

and a short exact sequence

$$0 \longrightarrow \mathrm{Cotor}(N^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))) \longrightarrow \widetilde{\Theta} \longrightarrow \mathrm{Ker}(\overline{\mathfrak{d}}) \longrightarrow 0.$$

If k is global, the assumption of Proposition 3.3.2 (2) is satisfied by the condition $\mathbf{N1}$. Hence $\mathrm{Cotor}(N^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)))$ is finite in both cases k is local and global (cf. Proposition 3.3.2, (3.3.1)). By the above short exact sequence and Lemma 2.3.2 (3), our task is to show

$$\widetilde{\Theta}_{\mathrm{Div}} = 0, \quad \text{assuming } \ \mathbf{H0} \ \text{ and } \ \mathbf{N1}.$$

Let $F^\bullet$ be the filtration on $H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))$ resulting from the Hochschild-Serre spectral sequence (2.6.2). We define the filtration $F^\bullet$ on Θ as that induced by $F^\bullet H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))$, and define the filtration $F^\bullet\widetilde{\Theta} \subset \widetilde{\Theta}$ as the pull-back of $F^\bullet\Theta$. Since $\mathbf{H0}$ implies the finiteness of $\mathrm{gr}^0_F\widetilde{\Theta}$, it suffices to show

$$(F^1\widetilde{\Theta})_{\mathrm{Div}} = 0, \quad \text{assuming } \ \mathbf{N1}. \tag{6.1.1}$$

The following lemma will play key roles:

Lemma 6.1.2 *Suppose that k is local. Then the following composite map has finite kernel*:

$$\mathfrak{d}_2 : H^2(k, H^1(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)) \longrightarrow H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \xrightarrow{\mathfrak{d}} \bigoplus_{y \in Y^0} H^4_y(\mathscr{X}, \mathfrak{T}_\infty(2)).$$

Here the first map is obtained by the Hochschild-Serre spectral sequence (2.6.2) *and the fact that* $\mathrm{cd}(k) = 2$ (*cf.* §2.6). *Consequently, the group* $F^2H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \cap \mathrm{Ker}(\mathfrak{d})$ *is finite.*

Admitting this lemma, we will prove (6.1.1) in §§6.2–6.3. We will prove Lemma 6.1.2 in §6.4.

6.2 Proof of (6.1.1) in the Local Case

We prove (6.1.1) assuming that k is local and that Lemma 6.1.2 holds. Let $\mathbb{F}$ be the residue field of k. By Lemma 6.1.2, $F^2\widetilde{\Theta}$ is finite. We prove that $\mathrm{Im}(F^1\widetilde{\Theta} \to \mathrm{gr}^1_F\Theta)$ is finite, which is exactly the finiteness of $\mathrm{gr}^1_F\widetilde{\Theta}$ and implies (6.1.1). Let $\mathscr{W}$ and $\mathscr{W}^0$ be as in (5.3.2) and (5.3.4), respectively. $\mathbf{N1}$ implies

$$\mathrm{gr}^1_F\Theta \simeq F^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))\big/\big((\mathscr{W}^0)_{\mathrm{Div}} + F^2H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))\big). \qquad (6.2.1)$$

If $p \neq \mathrm{ch}(\mathbb{F})$, then the group on the right hand side is clearly finite. If $p = \mathrm{ch}(\mathbb{F})$, then Lemma 6.2.2 below implies that the image of $F^1\widetilde{\Theta} \to \mathrm{gr}^1_F\Theta$ is a subquotient of $\mathrm{Cotor}(\mathscr{W}^0)$, which is finite by the proof of Lemma 3.5.4. Thus we are reduced to

Lemma 6.2.2 *If* $p = \mathrm{ch}(\mathbb{F})$, *then* $\mathrm{Ker}(\mathfrak{d}) \subset \mathscr{W}$.

We do not need to assume $\mathbf{H0}$ or $\mathbf{N1}$ in this lemma.

Proof. Let the notation be as in 1.10. Note that $\mathscr{W} = H^3(\mathscr{X}, \tau_{\leq 2}Rj_*\mathbb{Q}_p/\mathbb{Z}_p(2))$ by definition. There is a commutative diagram with distinguished rows in $D^b(\mathscr{X}, \mathbb{Z}/p^r)$

$$\begin{array}{ccccccc}
\mathfrak{T}_r(2) & \longrightarrow & Rj_*\mu_{p^r}^{\otimes 2} & \longrightarrow & Ri_*Ri^!\mathfrak{T}_r(2)[1] & \longrightarrow & \mathfrak{T}_r(2)[1] \\
\downarrow{\scriptstyle t} & & \| & & \downarrow{\scriptstyle Ri_*Ri^!(t)[1]} & & \downarrow{\scriptstyle t[1]} \\
\tau_{\leq 2}Rj_*\mu_{p^r}^{\otimes 2} & \longrightarrow & Rj_*\mu_{p^r}^{\otimes 2} & \longrightarrow & Ri_*Ri^!(\tau_{\leq 2}Rj_*\mu_{p^r}^{\otimes 2})[1] & \longrightarrow & (\tau_{\leq 2}Rj_*\mu_{p^r}^{\otimes 2})[1],
\end{array}$$

where t is as in ($\mathbf{S5}$) in §4.1. The central square of this diagram gives rise to the left square of the following commutative diagram (whose rows are not exact):

$$\begin{array}{ccccc}
H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) & \longrightarrow & H^4_Y(\mathscr{X}, \mathfrak{T}_\infty(2)) & \longrightarrow & \bigoplus_{y \in Y^0} H^4_y(\mathscr{X}, \mathfrak{T}_\infty(2)) \\
\| & & \downarrow & & \downarrow \\
H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) & \xrightarrow{\epsilon_1} & H^0(Y, i^*R^3j_*\mathbb{Q}_p/\mathbb{Z}_p(2)) & \xrightarrow{\epsilon_2} & \bigoplus_{y \in Y^0} H^0(y, i^*R^3j_*\mathbb{Q}_p/\mathbb{Z}_p(2)).
\end{array} \qquad (6.2.3)$$

Here the middle and the right vertical arrow is obtained from the composite morphism

$$Ri_*Ri^!\mathfrak{T}_r(2)[1] \xrightarrow{Ri_*Ri^!(t)[1]} Ri_*Ri^!\tau_{\leq 2}Rj_*\mu_{p^r}^{\otimes 2}[1] \xleftarrow{\ \sim\ } \tau_{\geq 3}Rj_*\mu_{p^r}^{\otimes 2},$$

and the right square commutes by the functoriality of restriction maps. The composite of the upper row is $\mathfrak{d}$. We have $\mathrm{Ker}(\epsilon_1) = \mathscr{W}$ obviously, and ϵ_2 is injective by the second assertion of Lemma 3.5.5. Hence we have $\mathrm{Ker}(\mathfrak{d}) \subset \mathrm{Ker}(\epsilon_2 \circ \epsilon_1) = \mathrm{Ker}(\epsilon_1) = \mathscr{W}$. □

6.3 PROOF OF (6.1.1) IN THE GLOBAL CASE

We prove (6.1.1) assuming that k is global and that Lemma 6.1.2 holds. Let $\mathscr{W}$ and $\mathscr{W}^0$ be as in (5.3.2) and (5.3.4), respectively. $\mathbf{N1}$ implies

$$\mathrm{gr}^1_F\Theta \simeq F^1H^3(X,\mathbb{Q}_p/\mathbb{Z}_p(2))\big/\big((\mathscr{W}^0)_{\mathrm{Div}} + F^2H^3(X,\mathbb{Q}_p/\mathbb{Z}_p(2))\big). \qquad (6.3.1)$$

We first prove the following lemma, where we do not assume $\mathbf{H0}$ or $\mathbf{N1}$:

LEMMA 6.3.2 $\mathrm{Ker}(\mathfrak{d}) \subset \mathscr{W}$.

Proof. We use the notation in 1.9. By the same argument as for the proof of Lemma 6.2.2, we obtain a commutative diagram analogous to (6.2.3)

$$\begin{array}{ccccc}
H^3(X,\mathbb{Q}_p/\mathbb{Z}_p(2)) & \longrightarrow & \bigoplus_{v\in S^1} H^4_{Y_v}(\mathscr{X},\mathfrak{T}_\infty(2)) & \longrightarrow & \bigoplus_{v\in S^1}\bigoplus_{y\in(Y_v)^0} H^4_y(\mathscr{X},\mathfrak{T}_\infty(2)) \\
\| & & \downarrow & & \downarrow \\
H^3(X,\mathbb{Q}_p/\mathbb{Z}_p(2)) & \xrightarrow{\epsilon_1} & \bigoplus_{v\in\Sigma} H^0(Y_v, i_v^*R^3j_{v*}\mathbb{Q}_p/\mathbb{Z}_p(2)) & \xrightarrow{\epsilon_2} & \bigoplus_{v\in\Sigma}\bigoplus_{y\in(Y_v)^0} H^0(y, i_v^*R^3j_{v*}\mathbb{Q}_p/\mathbb{Z}_p(2))
\end{array}$$

The composite of the upper row is $\mathfrak{d}$. The assertion follows from the facts that $\mathrm{Ker}(\epsilon_1) = \mathscr{W}$ and that ϵ_2 is injective (cf. Lemma 3.5.5). □

We prove (6.1.1). By Lemma 2.3.2 (4), it suffcies to show that

$$(F^2\widetilde{\Theta})_{\mathrm{Div}} = 0 = (\mathrm{gr}^1_F\widetilde{\Theta})_{\mathrm{Div}}.$$

Since $F^2H^3(X,\mathbb{Q}_p/\mathbb{Z}_p(2))\cap\mathrm{Ker}(\mathfrak{d})$ has a finite exponent (Corollary 2.6.3 (2), Lemma 6.1.2), we have $(F^2\widetilde{\Theta})_{\mathrm{Div}} = 0$. We show $(\mathrm{gr}^1_F\widetilde{\Theta})_{\mathrm{Div}} = 0$. By (6.3.1) and Lemma 6.3.2, we have

$$\mathrm{gr}^1_F\widetilde{\Theta} \subset \varXi := \mathscr{W}^0/((\mathscr{W}^0)_{\mathrm{Div}} + Z) \quad \text{with } Z := \mathscr{W}^0 \cap F^2H^3(X,\mathbb{Q}_p/\mathbb{Z}_p(2)).$$

By Corollary 2.6.3 (1), $\mathrm{Cotor}(Z)$ has a finite exponent, which implies

$$(\mathrm{gr}^1_F\widetilde{\Theta})_{\mathrm{Div}} \subset \varXi_{\mathrm{Div}} = \mathrm{Cotor}(\mathscr{W}^0)_{\mathrm{Div}} = 0$$

(cf. Lemma 2.3.2 (3)). Thus we obtain (6.1.1).

6.4 PROOF OF LEMMA 6.1.2

The case that k is p-adic local follows from [Sat1] Theorem 3.1, Lemma 3.2 (1) (cf. [Ts3]). More precisely, $\mathscr{X}$ is assumed in [Sat1] §3 to have strict semistable reduction, but one can remove the strictness assumption easily. The details are left to the reader.

We prove Lemma 6.1.2 assuming that k is ℓ-adic local with $\ell \neq p$. Note that in this case $\mathscr{X}/S$ may not have semistable reduction. If $\mathscr{X}/S$ has strict semistable reduction, then the assertion is proved in [Sat1] Theorem 2.1. We prove the general case. Put

$$\Lambda := \mathbb{Q}_p/\mathbb{Z}_p$$

for simplicity. By the alteration theorem of de Jong [dJ], we take a proper generically finite morphism $f : \mathscr{X}' \to X$ such that $\mathscr{X}'$ has strict semistable reduction over the normalization $S' = \mathrm{Spec}(\mathfrak{o}_{k'})$ of S in $\mathscr{X}'$. sNote that $\mathfrak{d}_2$ is the composite of a composite map

$$\mathfrak{d}_3 : H^2(k, H^1(\overline{X}, \Lambda(2)) \longrightarrow H^3(X, \Lambda(2)) \xrightarrow{\delta^{\mathrm{loc}}} H^4_Y(\mathscr{X}, \Lambda(2))$$

with a pull-back map

$$H^4_Y(\mathscr{X}, \Lambda(2)) \longrightarrow \bigoplus_{y \in Y^0} H^4_y(\mathscr{X}, \Lambda(2)). \tag{6.4.1}$$

Here the arrow δ^{loc} is the boundary map of a localization exact sequence. There is a commutative diagram

$$\begin{tikzcd} H^2(k, H^1(\overline{X}, \Lambda(2))) \arrow[r, "\mathfrak{d}_3"] \arrow[d, "f^*"'] & H^4_Y(\mathscr{X}, \Lambda(2)) \arrow[d, "f^*"] \\ H^2(k', H^1(\overline{X}', \Lambda(2))) \arrow[r, "\mathfrak{d}_3'"] & H^4_{Y'}(\mathscr{X}', \Lambda(2)), \end{tikzcd}$$

where $\overline{X}' := \mathscr{X}' \otimes_{\mathfrak{o}_{k'}} \overline{k}$ and Y' denotes the closed fiber of $\mathscr{X}'/S'$. We have already shown that $\mathrm{Ker}(\mathfrak{d}_3')$ is finite, and a standard norm argument shows that the left vertical arrow has finite kernel. Thus $\mathrm{Ker}(\mathfrak{d}_3)$ is finite as well. It remains to show

LEMMA 6.4.2 $\mathrm{Im}(\mathfrak{d}_3) \cap N^2H^4_Y(\mathscr{X}, \Lambda(2))$ *is finite, where* $N^2H^4_Y(\mathscr{X}, \Lambda(2))$ *denotes the kernel of the map* (6.4.1)*.*

Proof. First we note that

$$\mathrm{Im}(\mathfrak{d}_3) \subset \mathrm{Im}\big(H^1(\mathbb{F}, H^3_{\overline{Y}}(\mathscr{X}^{\mathrm{ur}}, \Lambda(2))) \to H^4_Y(\mathscr{X}, \Lambda(2))\big).$$

Indeed, this follows from the fact that $\mathfrak{d}_3$ factors as follows:

$$\begin{aligned} &H^2(k, H^1(\overline{X}, \Lambda(2))) \simeq H^1(\mathbb{F}, H^1(k^{\mathrm{ur}}, H^1(\overline{X}, \Lambda(2)))) \\ &\longrightarrow H^1(\mathbb{F}, H^2(X^{\mathrm{ur}}, \Lambda(2))) \longrightarrow H^1(\mathbb{F}, H^3_{\overline{Y}}(\mathscr{X}^{\mathrm{ur}}, \Lambda(2))) \longrightarrow H^4_Y(\mathscr{X}, \Lambda(2)). \end{aligned}$$

Hence it suffices to show the finiteness of the kernel of the composite map

$$\upsilon : H^1(\mathbb{F}, H^3_{\overline{Y}}(\mathscr{X}^{\mathrm{ur}}, \Lambda(2))) \longrightarrow H^4_Y(\mathscr{X}, \Lambda(2)) \longrightarrow \bigoplus_{y \in Y^0} H^4_y(\mathscr{X}, \Lambda(2)).$$

There is a commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & \mathrm{CH}_{d-2}(Y) \otimes \Lambda \arrow[r, "\iota"] \arrow[d, hook] & \mathrm{CH}_{d-2}(\overline{Y}) \otimes \Lambda \arrow[d, hook] \\
H^1(\mathbb{F}, H^3_{\overline{Y}}(\mathscr{X}^{\mathrm{ur}}, \Lambda(2))) \arrow[r, hook] \arrow[dr, "\upsilon"'] & H^4_Y(\mathscr{X}, \Lambda(2)) \arrow[r] \arrow[d] & H^4_{\overline{Y}}(\mathscr{X}^{\mathrm{ur}}, \Lambda(2)) \arrow[d] \\
 & \displaystyle\bigoplus_{y \in Y^0} H^4_y(\mathscr{X}, \Lambda(2)) \arrow[r] & \displaystyle\bigoplus_{\eta \in (\overline{Y})^0} H^4_\eta(\mathscr{X}^{\mathrm{ur}}, \Lambda(2)),
\end{tikzcd}$$

whose columns arise from the isomorphisms

$$N^2 H^4_Y(\mathscr{X}, \Lambda(2)) \simeq \mathrm{CH}_{d-2}(Y) \otimes \Lambda, \quad N^2 H^4_{\overline{Y}}(\mathscr{X}^{\mathrm{ur}}, \Lambda(2)) \simeq \mathrm{CH}_{d-2}(\overline{Y}) \otimes \Lambda$$

with $d := \dim(\mathscr{X})$ (see Lemma 4.2.3 (2), noting that $\mathfrak{T}_\infty(2) = \Lambda(2)$ in this situation). The middle exact row arises from the Hochschild-Serre spectral sequence for the covering $\mathscr{X}^{\mathrm{ur}} \to \mathscr{X}$. A diagram chase shows that $\mathrm{Ker}(\upsilon) \simeq \mathrm{Ker}(\iota)$, and we are reduced to showing the finiteness of $\mathrm{Ker}(\iota)$. Because the natural restriction map

$$\mathrm{CH}_{d-2}(Y)/\mathrm{CH}_{d-2}(Y)_{\mathrm{tors}} \to \mathrm{CH}_{d-2}(\overline{Y})/\mathrm{CH}_{d-2}(\overline{Y})_{\mathrm{tors}}$$

is injective by the standard norm argument, the finiteness of $\mathrm{Ker}(\iota)$ follows from the following general lemma:

LEMMA 6.4.3 *Let e be a positive integer and let Z be a scheme which is separated of finite type over $F := \overline{\mathbb{F}}$ with $\dim(Z) \le e$. Then the group $\mathrm{CH}_{e-1}(Z)/\mathrm{CH}_{e-1}(Z)_{\mathrm{tors}}$ is a finitely generated abelian group.*

Proof of Lemma 6.4.3. Obviously we may suppose that Z is reduced. We first reduced the problem to the case where Z is proper. Take a dense open immersion $Z \hookrightarrow Z'$ with $\overline{Z}'$ is proper. Writing Z'' for $Z' \setminus Z$, there is an exact sequence

$$\mathrm{CH}_{e-1}(Z'') \longrightarrow \mathrm{CH}_{e-1}(Z') \longrightarrow \mathrm{CH}_{e-1}(Z) \longrightarrow 0,$$

where $\mathrm{CH}_{e-1}(Z'')$ is finitely generated free abelian group because $\dim(Z'') \le e-1$. Let $f : \widetilde{Z} \to Z$ be the normalization. Since f is birational and finite, one easily sees that the cokernel of $f_* : \mathrm{CH}_{e-1}(\widetilde{Z}) \to \mathrm{CH}_{e-1}(Z)$ is finite. Thus we may assume Z is a proper normal variety of dimension e over F. Since F is algebraically closed, Z has an F-rational point. Now the theory of Picard functor (cf. [Mu] §5) implies the functorial isomorphisms $\mathrm{CH}_{e-1}(Z) \simeq \mathrm{Pic}_{Z/F}(F)$, where $\mathrm{Pic}_{Z/F}$ denotes the Picard functor for Z/F. This functor is representable by a group scheme and fits into the exact sequence of group schemes

$$0 \longrightarrow \mathrm{Pic}^{\tau}_{Z/F} \longrightarrow \mathrm{Pic}_{Z/F} \longrightarrow \mathrm{NS}_{Z/F} \longrightarrow 0,$$

where $\mathrm{Pic}^{\tau}_{Z/F}$ is quasi-projective over F and the reduce part of $\mathrm{NS}_{Z/F}$ is associated with a finitely generated abelian group. Since F is the algebraic closure of a finite field, the group $\mathrm{Pic}^{\tau}_{Z/F}(F)$ is torsion. Lemma 6.4.3 follows immediately from these facts. □

This completes the proof of Lemma 6.4.2, Lemma 6.1.2 and the key lemma 5.4.1. □

7 Cokernel of the Regulator Map

Let $k, S, p, \mathscr{X}$ and K be as in the notation 1.8. We always assume 1.8.1 throughout this section. For a proper smooth geometrically integral variety Z over a finite field $\mathbb{F}$. we say that *the Tate conjecture holds in codimension* 1 *for* Z, if the étale cycle class map

$$\mathrm{CH}^1(Z) \otimes \mathbb{Q}_\ell \longrightarrow H^2(Z \otimes_{\mathbb{F}} \overline{\mathbb{F}}, \mathbb{Q}_\ell(1))^{G_{\mathbb{F}}}$$

is surjective for a prime number $\ell \neq \mathrm{ch}(\mathbb{F})$ ([Ta1], [Ta2]). By [Mi1] Theorems 4.1 and 6.1, this condition is independent of $\ell \neq \mathrm{ch}(\mathbb{F})$ and equivalent to that the Grothendieck-Brauer group $\mathrm{Br}(Z) = H^2(Z, \mathbb{G}_{\mathrm{m}})$ is finite.

7.1 Statement of the Result

Let $\mathscr{C}$ be the category of $\mathbb{Z}_p$-modules modulo the Serre subcategory consisting of p-primary torsion abelian groups of finite-exponent. In this section, we prove the following result, which implies Theorem 1.3 (2) (see the beginning of §3 for **H0**):

Theorem 7.1.1 *Assume* **H0** *and either* $p \geq 5$ *or the equality*

$$H^1_g(k, H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)))_{\mathrm{Div}} = H^1(k, H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)))_{\mathrm{Div}}. \qquad (*_g)$$

Assume further the following conditions:

- **T**: *The reduced part of every closed fiber of* $\mathscr{X}/S$ *has simple normal crossings on* $\mathscr{X}$*, and the Tate conjecture holds in codimension* 1 *for the irreducible components of those fibers.*
- **F**: $\mathrm{Cotor}(\mathrm{CH}^2(X)_{p\text{-tors}})$ *has a finite exponent.*

Then there exists a short exact sequence in $\mathscr{C}$

$$0 \longrightarrow \mathrm{CH}^2(X)_{p\text{-tors}} \longrightarrow \frac{H^1_g(k, H^2_{\text{ét}}(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)))}{\mathrm{Im}(\mathrm{reg}_{\mathbb{Q}_p/\mathbb{Z}_p})} \longrightarrow H^3_{\mathrm{ur}}(K, X; \mathbb{Q}_p/\mathbb{Z}_p(2)).$$

Moreover the image of the last map contains $H^3_{\mathrm{ur}}(K, X; \mathbb{Q}_p/\mathbb{Z}_p(2))_{\mathrm{Div}}$.

This result is a generalization of Theorems 3.1.1 and 5.1.1 under the assumption **T**. The formulation of Theorem 7.1.1 in this final version was much inspired by discussions with Masanori Asakura.

7.2 Proof of Theorem 7.1.1

Let

$$\mathfrak{d} : H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \longrightarrow \bigoplus_{v \in S} \bigoplus_{y \in (Y_v)^0} H^4_y(\mathscr{X}, \mathfrak{T}_\infty(2)).$$

be the map induced by $\overline{\mathfrak{d}}$ in (5.2.1). Let $\mathscr{W}$ be as in (5.3.2) and let $\mathscr{W}^0$ be as in (5.3.4). We need the following two lemmas:

Lemma 7.2.1 *Assume that* $\mathbf{T}$ *holds. Then we have*

$$\mathscr{W}_{\mathrm{Div}} \subset \mathrm{Ker}(\mathfrak{d}) + F^2H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)).$$

Lemma 7.2.2 *Assume either* $p \geq 5$ *or* $(*_g)$. *Then we have*

$$\mathrm{Im}(\omega) = H^1_g(k, H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)))_{\mathrm{Div}},$$

where ω *is as in* (5.3.3)*.*

Lemma 7.2.1 will be proved in §§7.4–7.5 below, and Lemma 7.2.2 will be proved in §7.3 below. We prove Theorem 7.1.1 in this subsection, admitting these lemmas. Let

$$H^3 := H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)), \quad N^1H^3 \subset H^3 \quad \text{and} \quad F^iH^3 \subset H^3$$

be as in §3.4, and put

$$NF^1H^3 := N^1H^3 \cap F^1H^3 \quad \text{and} \quad A := H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)).$$

We see that

$$\mathrm{Cotor}(N^1H^3) \text{ has a finite exponent} \tag{7.2.3}$$

by the exact sequence (3.3.1) and the assumption $\mathbf{F}$, and moreover that

$$\mathrm{Cotor}(NF^1H^3) \text{ has a finite exponent} \tag{7.2.4}$$

by $\mathbf{H0}$.

We work in $\mathscr{C}$ for a while to simplify the arguments. By $\mathbf{H0}$, (3.3.1) and Lemma 3.3.5, there is a short exact sequence in $\mathscr{C}$

$$0 \longrightarrow \mathrm{CH}^2(X)_{p\text{-tors}} \longrightarrow H^1(k, A)/\mathrm{Im}(\mathrm{reg}_{\mathbb{Q}_p/\mathbb{Z}_p}) \longrightarrow H^1(k, A)/NF \longrightarrow 0,$$

where NF denotes the image of NF^1H^3 in $H^1(k, A)$. By the assumption '$p \geq 5$ or $(*_g)$' and Lemma 3.5.2, the composite map $(NF^1H^3)_{\mathrm{Div}} \to F^1H^3 \to H^1(k, A)$ factors through $H^1_g(k, A)$. Therefore we obtain a short exact sequence in $\mathscr{C}$

$$0 \longrightarrow \mathrm{CH}^2(X)_{p\text{-tors}} \longrightarrow H^1_g(k, A)/\mathrm{Im}(\mathrm{reg}_{\mathbb{Q}_p/\mathbb{Z}_p}) \longrightarrow H^1_g(k, A)/NF_{\mathrm{Div}} \longrightarrow 0,$$

whose exactness follows from (7.2.4) and the fact that the natural map $(NF^1H^3)_{\mathrm{Div}} \to NF_{\mathrm{Div}}$ is surjective (cf. Corollary 2.6.3 (1) and Lemma 2.3.2 (3)). It remains to construct an injective map

$$\alpha : H^1_g(k, A)/NF_{\mathrm{Div}} \hookrightarrow H^3_{\mathrm{ur}}(K, X; \mathbb{Q}_p/\mathbb{Z}_p(2)) \quad \text{in } \mathscr{C}$$

whose image contains $H^3_{\mathrm{ur}}(K,X;\mathbb{Q}_p/\mathbb{Z}_p(2))_{\mathrm{Div}}$.

We work in the usual category $\mathbb{Z}_p$-modules in what follows. There are $\mathbb{Z}_p$-homomorphisms

$$\begin{tikzcd} H^3_{\mathrm{ur}}(K,X;\mathbb{Q}_p/\mathbb{Z}_p(2)) \arrow[r, hook, "\iota"] & H^3/N^1H^3 \arrow[d, "\pi"', "\text{(finite-exponent kernel)}"] \\ & H^3/(N^1H^3+F^2H^3), \end{tikzcd} \tag{7.2.5}$$

where ι is induced by the definition of $H^3_{\mathrm{ur}}(K,X;\mathbb{Q}_p/\mathbb{Z}_p(2))$ and $\mathrm{Ker}(\pi)$ has a finite exponent by Lemma 3.3.5 (i.e., π is an isomorphism in $\mathscr{C}$). On the other hand, there is a diagram of $\mathbb{Z}_p$-submodules of H^3

$$\begin{tikzcd} (N^1H^3)_{\mathrm{Div}} \arrow[r, hook] \arrow[d, hook] & N^1H^3\cap\mathscr{W}_{\mathrm{Div}} \arrow[r, hook] \arrow[dl, hook] & \mathscr{W}_{\mathrm{Div}} \arrow[r, hook, "7.2.1"] \arrow[d, hook] & \mathrm{Ker}(\mathfrak{d})+F^2H^3 \\ N^1H^3 \arrow[r, hook] & \mathrm{Ker}(\mathfrak{d}) \arrow[r, hook, "6.2.2 \text{ and } 6.3.2"] & \mathscr{W}, & \end{tikzcd}$$

where the inclusion $\mathrm{Ker}(\mathfrak{d})\subset\mathscr{W}$ is obvious when k is ℓ-adic local with $\ell\neq p$. Since $N^1H^3\cap\mathscr{W}_{\mathrm{Div}}$ is divisible up to a finite-exponent group by (7.2.3), we have

$$\mathscr{W}_{\mathrm{Div}}/(N^1H^3\cap\mathscr{W}_{\mathrm{Div}})\simeq(\mathscr{W}/N^1H^3)_{\mathrm{Div}} \tag{7.2.6}$$

by Lemma 2.3.2 (3). Now for a subgroup $M\subset H^3$, let $\overline{M}$ be its image into $H^3/(N^1H^3+F^2H^3)$. Then we have $\overline{\mathscr{W}_{\mathrm{Div}}}\subset\overline{\mathrm{Ker}(\mathfrak{d})}\subset\overline{\mathscr{W}}$ by the above diagram, and moreover

$$\overline{\mathscr{W}_{\mathrm{Div}}}=\left(\overline{\mathrm{Ker}(\mathfrak{d})}\right)_{\mathrm{Div}}=(\overline{\mathscr{W}})_{\mathrm{Div}} \tag{7.2.7}$$

by (7.2.6) and the fact that $\mathrm{Ker}(\pi)$ has a finite exponent (cf. Lemma 2.3.2 (3)). Noting that $\mathrm{Coker}(H^1_g(k,A))$ has a finite exponent (Remark 2.4.9 (1)), we define the desired map α in $\mathscr{C}$ as that induced by the following diagram:

$$\begin{tikzcd} H^1_g(k,A)/NF_{\mathrm{Div}} \arrow[rr, dotted, "\alpha"] & & H^3_{\mathrm{ur}}(K,X;\mathbb{Q}_p/\mathbb{Z}_p(2)) \arrow[d, "\pi\circ\iota"] \\ H^1_g(k,A)_{\mathrm{Div}}/NF_{\mathrm{Div}} \arrow[u, hook] \arrow[r, two heads, "\tau"] & (\overline{\mathscr{W}})_{\mathrm{Div}} \arrow[r, hook] & \overline{\mathrm{Ker}(\mathfrak{d})}, \end{tikzcd}$$

where the right vertical map is surjective by the definition of $H^3_{\mathrm{ur}}(K,X;\mathbb{Q}_p/\mathbb{Z}_p(2))$ and has finite-exponent kernel by the diagram (7.2.5). The arrow τ is obtained from Lemma 7.2.2, which is surjective by **H0** and has finite-exponent kernel by (7.2.4). By the last fact, α is injective in $\mathscr{C}$. Finally $\mathrm{Im}(\alpha)$ contains $H^3_{\mathrm{ur}}(K,X;\mathbb{Q}_p/\mathbb{Z}_p(2))_{\mathrm{Div}}$ by the surjectivity of τ and (7.2.7). Thus we obtain Theorem 7.1.1 assuming Lemmas 7.2.1 and 7.2.2.

7.3 PROOF OF LEMMA 7.2.2

If k is local, then the assertion follows from Corollary 2.2.3 and Lemma 3.5.4. We show the inclusion $\mathrm{Im}(\omega) \supset H^1_g(k, H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)))_{\mathrm{Div}}$, assuming that k is global (the inclusion in the other direction has been proved in Lemma 5.3.5). Put $A := H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))$. By Lemma 2.3.2 (3), it is enough to show the following:

(i) *The image of the composite map*

$$\mathscr{W}^0 \hookrightarrow F^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \xrightarrow{\psi} H^1(k, A)$$

contains $H^1_g(k, A)_{\mathrm{Div}}$*, where the arrow* ψ *is as in* (3.5.1)*.*

(ii) *The kernel of this composite map is cofinitely generated up to a finite-exponent group.*

(ii) follows from Corollary 2.6.3 (1). We prove (i) in what follows. We use the notation fixed in 1.9. Let $U \subset S$ be a non-empty open subset which contains Σ and for which $\mathscr{X}_U \to U$ is smooth outside of Σ. Let $j_U : \mathscr{X}_U[p^{-1}] \to \mathscr{X}_U$ be the natural open immersion. Put $U' := U \setminus \Sigma$ and $\Lambda := \mathbb{Q}_p/\mathbb{Z}_p$. For $v \in \Sigma$, put

$$M_v := F^1H^3(X_v, \Lambda(2))/(H^3(\mathscr{X}_v, \tau_{\leq 2}R(j_v)_*\Lambda(2))^0)_{\mathrm{Div}},$$

where the superscript 0 means the subgroup of elements which vanishes in

$$H^3(\overline{X}, \Lambda(2)) \simeq H^3(X_v \otimes_{k_v} \overline{k_v}, \Lambda(2)).$$

We construct a commutative diagram with exact rows

$$\begin{CD}
0 @>>> \mathrm{Ker}(r_U) @>>> F^1H^3(\mathscr{X}_U[p^{-1}], \Lambda(2)) @>{r_U}>> \bigoplus_{v\in\Sigma} M_v \\
@. @V{c_U}VV @V{\psi_U}VV @V{b_\Sigma}VV \\
0 @>>> \mathrm{Ker}(a_U) @>>> H^1(U, A) @>{a_U}>> \bigoplus_{v\in\Sigma} H^1_{/g}(k_v, A),
\end{CD}$$

where F^1 on $H^3(\mathscr{X}_U[p^{-1}], \Lambda(2))$ means the filtration resulting from the Hochschild-Serre spectral sequence (3.2.3) for $\mathscr{X}_U[p^{-1}]$, and ψ_U is an edge homomorphism of that spectral sequence. The arrows r_U and a_U are natural pull-back maps, and we put

$$H^1_{/g}(k_v, A) := H^1(k_v, A)/H^1_g(k_v, A).$$

The existence of b_Σ follows from the local case of Lemma 7.2.2, and c_U denotes the map induced by the right square. Note that $\mathrm{Ker}(a_U)$ contains $H^1_{f,U}(k, A)$. Now let

$$c : \mathscr{W}^\dagger := \varinjlim_{\Sigma\subset U\subset S} \mathrm{Ker}(r_U) \longrightarrow \varinjlim_{\Sigma\subset U\subset S} \mathrm{Ker}(a_U)$$

be the inductive limit of c_U, where U runs through all non-empty open subsets of S which contains Σ and for which $\mathscr{X}_U \to U$ is smooth outside of Σ. Because the group on the right hand side contains $H^1_g(k, A)$, it remains to show that

(iii) Coker(c) *has a finite exponent.*

(iv) $\mathscr{W}^\dagger$ *is contained in* $\mathscr{W}^0$.

(iv) is rather straight-forward and left to the reader. We prove (iii). For $U \subset S$ as above, applying the snake lemma to the above diagram, we see that the kernel of the natural map

$$\mathrm{Coker}(c_U) \longrightarrow \mathrm{Coker}(\psi_U)$$

is a subquotient of Ker(b_Σ). By the local case of Lemma 7.2.2, we have

$$\mathrm{Ker}(b_\Sigma) \simeq \bigoplus_{v\in\Sigma} \mathrm{Im}(F^2H^3(X_v, \Lambda(2)) \to M_v)$$

and the group on the right hand side is finite by Lemma 7.5.1 below. On the other hand Coker(ψ_U) is zero if $p \geq 3$, and killed by 2 if $p = 2$. Hence passing to the limit, we see that Coker(c) has a finite exponent. This completes the proof of Lemma 7.2.2.

7.4 Proof of Lemma 7.2.1, Step 1

We start the proof of Lemma 7.2.1. Our task is to show the inclusion

$$\mathfrak{d}(\mathscr{W}_{\mathrm{Div}}) \subset \mathfrak{d}(F^2H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))). \tag{7.4.1}$$

If k is global, then the assertion is reduced to the local case, because the natural map

$$F^2H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \longrightarrow \bigoplus_{v\in S^1} F^2H^3(X_v, \mathbb{Q}_p/\mathbb{Z}_p(2))$$

has finite cokernel by Corollary 2.6.3 (2).

Assume now that k is local. In this subsection, we treat the case that k is ℓ-adic local with $\ell \neq p$. We use the notation fixed in 1.10. Recall that Y has simple normal crossings on $\mathscr{X}$ by the assumption $\mathbf{T}$. Note that $\mathfrak{d}$ factors as

$$H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \longrightarrow H^4_Y(\mathscr{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)) \xrightarrow{\ \iota\ } \bigoplus_{y\in Y^0} H^4_y(\mathscr{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)),$$

and that Im$(\mathfrak{d}) \subset$ Im(ι). There is a short exact sequence

$$\begin{aligned} 0 \to H^1(\mathbb{F}, H^3_{\overline{Y}}(\mathscr{X}^{\mathrm{ur}}, \mathbb{Q}_p/\mathbb{Z}_p(2))) &\to H^4_Y(\mathscr{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)) \\ &\to H^4_{\overline{Y}}(\mathscr{X}^{\mathrm{ur}}, \mathbb{Q}_p/\mathbb{Z}_p(2))^{G_{\mathbb{F}}} \to 0 \end{aligned}$$

arising from a Hochschild-Serre spectral sequence. We have Ker$(\iota) \simeq \mathrm{CH}_{d-2}(Y) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ with $d := \dim(\mathscr{X})$ by Lemma 4.2.3 (2). Hence to show the inclusion (7.4.1), it suffices to prove

PROPOSITION 7.4.2 (1) *Assume that* $\mathbf{T}$ *holds. Then the composite map*

$$\mathrm{CH}_{d-2}(Y) \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow H^4_Y(\mathscr{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)) \longrightarrow H^4_{\overline{Y}}(\mathscr{X}^{\mathrm{ur}}, \mathbb{Q}_p/\mathbb{Z}_p(2))^{G_{\mathbb{F}}} \tag{7.4.3}$$

is an isomorphism up to finite groups. Consequently, we have

$$\mathrm{Im}(\iota) \simeq H^1(\mathbb{F}, H^3_{\overline{Y}}(\mathscr{X}^{\mathrm{ur}}, \mathbb{Q}_p/\mathbb{Z}_p(2)))$$

up to finite groups.

(2) *The image of the composite map*

$$H^2(k, H^1(X, \mathbb{Q}_p/\mathbb{Z}_p(2))) \longrightarrow H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \longrightarrow H^4_Y(\mathscr{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))$$

contains $H^1(\mathbb{F}, H^3_{\overline{Y}}(\mathscr{X}^{\mathrm{ur}}, \mathbb{Q}_p/\mathbb{Z}_p(2)))_{\mathrm{Div}}$.

We first show the following lemma:

LEMMA 7.4.4 (1) *Consider the Mayer-Vietoris spectral sequence obtained from the absolute purity* (*cf.* [RZ], [Th], [FG])

$$E_1^{u,v} = H^{2u+v-2}(\overline{Y}^{(-u+1)}, \mathbb{Q}_p/\mathbb{Z}_p(u+1)) \Longrightarrow H^{u+v}_{\overline{Y}}(\mathscr{X}^{\mathrm{ur}}, \mathbb{Q}_p/\mathbb{Z}_p(2)), \tag{7.4.5}$$

where $\overline{Y}^{(q)}$ *denotes the disjoint union of* q*-fold intersections of distinct irreducible components of the reduced part of* $\overline{Y}$. *Then there are isomorphisms up to finite groups*

$$H^1(\mathbb{F}, H^3_{\overline{Y}}(\mathscr{X}^{\mathrm{ur}}, \mathbb{Q}_p/\mathbb{Z}_p(2))) \simeq H^1(\mathbb{F}, E_2^{-1,4}),$$
$$H^4_{\overline{Y}}(\mathscr{X}^{\mathrm{ur}}, \mathbb{Q}_p/\mathbb{Z}_p(2))^{G_{\mathbb{F}}} \simeq (E_2^{0,4})^{G_{\mathbb{F}}}.$$

(2) *As a* $G_{\mathbb{F}}$*-module,* $H^0(k^{\mathrm{ur}}, H^2(\overline{X}, \mathbb{Q}_p))$ *has weight* ≤ 2.

Proof of Lemma 7.4.4. (1) Since $E_1^{u,v} = 0$ for any (u, v) with $u > 0$ or $2u + v < 2$, there is a short exact sequence

$$0 \longrightarrow E_2^{0,3} \longrightarrow H^3_{\overline{Y}}(\mathscr{X}^{\mathrm{ur}}, \mathbb{Q}_p/\mathbb{Z}_p(2))) \longrightarrow E_2^{-1,4} \longrightarrow 0, \tag{7.4.6}$$

and the edge homomorphism

$$E_2^{0,4} \hookrightarrow H^4_{\overline{Y}}(\mathscr{X}^{\mathrm{ur}}, \mathbb{Q}_p/\mathbb{Z}_p(2))), \tag{7.4.7}$$

where we have $E_2^{-1,4} = \mathrm{Ker}(d_1^{-1,4})$ and $E_2^{0,4} = \mathrm{Coker}(d_1^{-1,4})$ and $d_1^{-1,4}$ is the Gysin map $H^0(\overline{Y}^{(2)}, \mathbb{Q}_p/\mathbb{Z}_p) \to H^2(\overline{Y}^{(1)}, \mathbb{Q}_p/\mathbb{Z}_p(1)))$. Note that $E_1^{u,v}$ is pure of weight $v - 4$ by Deligne's proof of the Weil conjecture [De2], so that $H^i(\mathbb{F}, E_\infty^{u,v})$ $(i = 0, 1)$ is finite unless $v = 4$. The assertions immediately follow from these facts.

(2) By the alteration theorem of de Jong [dJ], we may assume that $\mathscr{X}$ is projective and has semistable reduction over S. If X is a surface, then the assertion is proved in [RZ]. Otherwise, take a closed immersion $\mathscr{X} \hookrightarrow \mathbb{P}^N_S =: \mathbb{P}$. By [JS] Proposition 4.3 (b), there exists a hyperplane $H \subset \mathbb{P}$ which is flat over S and for which $\mathscr{Z} := \mathscr{X} \times_{\mathbb{P}} H$ is regular with semistable reduction over S. The restriction map

$H^2(\overline{X}, \mathbb{Q}_p) \to H^2(\overline{Z}, \mathbb{Q}_p)$ ($\overline{Z} := \mathscr{X} \times_{\mathfrak{o}_k} \overline{k}$) is injective by the weak and hard Lefschetz theorems. Hence the claim is reduced to the case of surfaces. This completes the proof of the lemma. $\square$

Proof of Proposition 7.4.2. (1) Note that the composite map (7.4.3) in question has finite kernel by Lemma 6.4.3 and the arguments in the proof of Lemma 6.4.2. We prove that (7.4.3) has finite cokernel, assuming $\mathbf{T}$. By the Kummer theory, there is a short exact sequence

$$0 \longrightarrow \mathrm{Pic}(\overline{Y}^{(1)}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow H^2(\overline{Y}^{(1)}, \mathbb{Q}_p/\mathbb{Z}_p(1)) \longrightarrow \mathrm{Br}(\overline{Y}^{(1)})_{p\text{-tors}} \longrightarrow 0 \tag{7.4.8}$$

and the differential map $d_1^{-1,4}$ of the spectral sequence (7.4.5) factors through the Gysin map

$$H^0(\overline{Y}^{(2)}, \mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow \mathrm{Pic}(\overline{Y}^{(1)}) \otimes \mathbb{Q}_p/\mathbb{Z}_p,$$

whose cokernel is $\mathrm{CH}_{d-2}(\overline{Y}) \otimes \mathbb{Q}_p/\mathbb{Z}_p$. Hence in view of the computations in the proof of Lemma 7.4.4 (1), the Gysin map

$$\mathrm{CH}_{d-2}(\overline{Y}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to H^4_{\overline{Y}}(\mathscr{X}^{\mathrm{ur}}, \mathbb{Q}_p/\mathbb{Z}_p(2))$$

(cf. Lemma 4.2.3 (2)) factors through the map (7.4.7) and we obtain a commutative diagram

$$\begin{array}{ccc}
\mathrm{CH}_{d-2}(Y) \otimes \mathbb{Q}_p/\mathbb{Z}_p & \xrightarrow{(7.4.3)} & H^4_{\overline{Y}}(\mathscr{X}^{\mathrm{ur}}, \mathbb{Q}_p/\mathbb{Z}_p(2))^{G_{\mathbb{F}}} \\
\big\downarrow & & \big\uparrow{\scriptstyle (7.4.7)} \\
(\mathrm{CH}_{d-2}(\overline{Y}) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^{G_{\mathbb{F}}} & \longrightarrow & (E_2^{0,4})^{G_{\mathbb{F}}},
\end{array}$$

where the left vertical arrow has finite cokernel (and kernel) by Lemma 6.4.3 and a standard norm argument. The right vertical arrow has finite cokernel (and is injective) by Lemma 7.4.4 (1). Thus it suffices to show that the bottom horizontal arrow has finite cokernel. By the exact sequence (7.4.8), there is a short exact sequence

$$0 \longrightarrow \mathrm{CH}_{d-2}(\overline{Y}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow E_2^{0,4} \longrightarrow \mathrm{Br}(\overline{Y}^{(1)})_{p\text{-tors}} \longrightarrow 0.$$

Our task is to show that $\big(\mathrm{Br}(\overline{Y}^{(1)})_{p\text{-tors}}\big)^{G_{\mathbb{F}}}$ is finite, which follows from the assumption $\mathbf{T}$ and the finiteness of the kernel of the natural map

$$H^1(G_{\mathbb{F}}, \mathrm{Pic}(\overline{Y}^{(1)}) \otimes \mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow H^1(G_{\mathbb{F}}, H^2(\overline{Y}^{(1)}, \mathbb{Q}_p/\mathbb{Z}_p(1)))$$

(cf. Lemma 7.6.2 in §7.6 below). Thus we obtain the assertion.

(2) Since $\mathrm{cd}(k^{\mathrm{ur}}) = 1$, there is a short exact sequence

$$0 \to H^1(k^{\mathrm{ur}}, H^1(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))) \to H^2(X^{\mathrm{ur}}, \mathbb{Q}_p/\mathbb{Z}_p(2)) \to H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))^{G_{k^{\mathrm{ur}}}} \to 0$$

arising from a Hochschild-Serre spectral sequence. By Lemma 7.4.4 the last group has weight ≤ -2, and we have isomorphisms up to finite groups

$$\begin{aligned} H^2(k, H^1(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))) &\simeq H^1(\mathbb{F}, H^1(k^{\mathrm{ur}}, H^1(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)))) \\ &\simeq H^1(\mathbb{F}, H^2(X^{\mathrm{ur}}, \mathbb{Q}_p/\mathbb{Z}_p(2))). \end{aligned} \tag{7.4.9}$$

Now we plug the short exact sequence (7.4.6) into the localization exact sequence

$$H^2(X^{\mathrm{ur}}, \mathbb{Q}_p/\mathbb{Z}_p(2)) \longrightarrow H^3_{\overline{Y}}(\mathscr{X}^{\mathrm{ur}}, \mathbb{Q}_p/\mathbb{Z}_p(2))) \xrightarrow{\alpha} H^3(\mathscr{X}^{\mathrm{ur}}, \mathbb{Q}_p/\mathbb{Z}_p(2)).$$

Note that $H^3(\mathscr{X}^{\mathrm{ur}}, \mathbb{Q}_p/\mathbb{Z}_p(2)) \simeq H^3(\overline{Y}, \mathbb{Q}_p/\mathbb{Z}_p(2))$, so that it has weight ≤ -1 (cf. [De2]). Let $E_2^{u,v}$ be as in (7.4.5). Since $E_2^{-1,4}$ is pure of weight 0, the induced map

$$E_2^{-1,4} \longrightarrow H^3(\mathscr{X}^{\mathrm{ur}}, \mathbb{Q}_p/\mathbb{Z}_p(2)))/\alpha(E_2^{0,3})$$

has finite image. Hence the composite map

$$H^2(X^{\mathrm{ur}}, \mathbb{Q}_p/\mathbb{Z}_p(2)) \longrightarrow H^3_{\overline{Y}}(\mathscr{X}^{\mathrm{ur}}, \mathbb{Q}_p/\mathbb{Z}_p(2))) \longrightarrow E_2^{-1,4}$$

has finite cokernel, and the following map has finite cokernel as well:

$$H^1(\mathbb{F}, H^2(X^{\mathrm{ur}}, \mathbb{Q}_p/\mathbb{Z}_p(2))) \longrightarrow H^1(\mathbb{F}, E_2^{-1,4}).$$

Now Proposition 7.4.2 (2) follows from this fact together with (7.4.9) and the first isomorphism in Lemma 7.4.4 (1). □

REMARK 7.4.10 *Let J be the set of the irreducible components of $Y^{(2)}$ and put*

$$\varDelta := \mathrm{Ker}\big(g' : \mathbb{Z}^J \to \mathrm{NS}(Y^{(1)})\big) \quad \text{with} \quad \mathrm{NS}(Y^{(1)}) := \bigoplus_{y \in Y^0} \mathrm{NS}(Y_y),$$

where for $y \in Y^0$, Y_y denotes the closure $\overline{\{y\}} \subset Y$ and $\mathrm{NS}(Y_y)$ denotes its Néron-Severi group. The arrow g' arises from the Gysin map $\mathbb{Z}^J \to \mathrm{Pic}(Y^{(1)})$. One can easily show, assuming $\mathbf{T}$ and using Lemma 7.6.2 in §7.6 below, that the corank of $H^1(\mathbb{F}, E_2^{-1,4})$ over $\mathbb{Z}_p$ is equal to the rank of $\varDelta$ over $\mathbb{Z}$. Hence Proposition 7.4.2 (2) implies the inequality

$$\dim_{\mathbb{Q}_p}(H^2(k, H^1(\overline{X}, \mathbb{Q}_p(2))) \geq \dim_{\mathbb{Q}}(\varDelta \otimes \mathbb{Q}), \tag{7.4.11}$$

which will be used in the next subsection.

7.5 PROOF OF LEMMA 7.2.1, STEP 2

We prove Lemma 7.2.1, assuming that k is p-adic local (see 1.10 for notation). We first show the following lemma:

LEMMA 7.5.1 *We have*

$$F^2H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \subset H^3(\mathscr{X}, \tau_{\leq 2}Rj_*\mathbb{Q}_p/\mathbb{Z}_p(2)) \; (= \mathscr{W}).$$

Proof of Lemma 7.5.1. There is a distinguished triangle in $D^b(\mathscr{X}_{\text{ét}}, \mathbb{Z}/p^r)$ by §4.1 (**S5**)

$$i_*\nu^1_{Y,r}[-3] \xrightarrow{\ g\ } \mathfrak{T}_r(2) \xrightarrow{\ t'\ } \tau_{\leq 2}Rj_*\mu_{p^r}^{\otimes 2} \xrightarrow{\ \sigma\ } i_*\nu^1_{Y,r}[-2]. \qquad (7.5.2)$$

Applying $Ri^!$ to this triangle, we obtain a distinguished triangle in $D^b(Y_{\text{ét}}, \mathbb{Z}/p^r)$

$$\nu^1_{Y,r}[-3] \xrightarrow{\mathrm{Gys}^2_i} Ri^!\mathfrak{T}_r(2) \xrightarrow{Ri^!(t)} i^*(\tau_{\geq 3}Rj_*\mu_{p^r}^{\otimes 2})[-1] \longrightarrow \nu^1_{Y,r}[-2], \qquad (7.5.3)$$

where $\mathrm{Gys}^2_i := Ri^!(g)$ and we have used the natural isomorphism

$$i^*(\tau_{\geq 3}Rj_*\mu_{p^r}^{\otimes 2})[-1] \simeq Ri^!(\tau_{\leq 2}Rj_*\mu_{p^r}^{\otimes 2}).$$

Now let us recall the commutative diagram (6.2.3):

$$\begin{array}{ccccc}
H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) & \longrightarrow & H^4_Y(\mathscr{X}, \mathfrak{T}_\infty(2)) & \longrightarrow & \bigoplus_{y\in Y^0} H^4_y(\mathscr{X}, \mathfrak{T}_\infty(2)) \\
\| & & \downarrow & & \downarrow{\scriptstyle\lambda} \\
H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) & \xrightarrow{\epsilon_1} & H^0(Y, i^*R^3j_*\mathbb{Q}_p/\mathbb{Z}_p(2)) & \xrightarrow{\epsilon_2} & \bigoplus_{y\in Y^0} H^0(y, i^*R^3j_*\mathbb{Q}_p/\mathbb{Z}_p(2)),
\end{array}$$

where the middle and the right vertical arrows are induced by $Ri^!(t)$ in (7.5.3). By the proof of Lemma 6.2.2, we have $H^3(\mathscr{X}, \tau_{\leq 2}Rj_*\mathbb{Q}_p/\mathbb{Z}_p(2)) = \mathrm{Ker}(\epsilon_2\epsilon_1)$. Hence it suffices to show the image of the composite map

$$\mathfrak{d}_2 : H^2(k, H^1(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)) \longrightarrow H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \xrightarrow{\ \mathfrak{d}\ } \bigoplus_{y\in Y^0} H^4_y(\mathscr{X}, \mathfrak{T}_\infty(2))$$

is contained in $\mathrm{Ker}(\lambda)$. By the distinguished triangle (7.5.3), $\mathrm{Ker}(\lambda)$ agrees with the image of the Gysin map

$$\mathrm{Gys} := \bigoplus_{y\in Y^0} \mathrm{Gys}^2_{i_y} : \bigoplus_{y\in Y^0} H^1(y, W_\infty\Omega^1_{y,\log}) \hookrightarrow \bigoplus_{y\in Y^0} H^4_y(\mathscr{X}, \mathfrak{T}_\infty(2)).$$

On the other hand, $\mathfrak{d}_2$ factors into the following maps up to a sign, by the commutativity of the central square in [SH] (4.4.2):

$$\begin{aligned}
&H^2(k, H^1(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))) \simeq H^1(\mathbb{F}, H^1(k^{\mathrm{ur}}, H^1(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)))) \\
&\longrightarrow H^1(\mathbb{F}, H^2(X^{\mathrm{ur}}, \mathbb{Q}_p/\mathbb{Z}_p(2))) \xrightarrow{\ \sigma\ } H^1(\mathbb{F}, H^0(\overline{Y}, \nu^1_{\overline{Y},\infty})) \\
&\longrightarrow H^1\Big(\mathbb{F}, \textstyle\bigoplus_{\eta\in\overline{Y}^0} H^0(\eta, W_\infty\Omega^1_{\eta,\log})\Big) \longrightarrow \textstyle\bigoplus_{y\in Y^0} H^1(y, W_\infty\Omega^1_{y,\log}) \\
&\xrightarrow{\mathrm{Gys}} \textstyle\bigoplus_{y\in Y^0} H^4_y(\mathscr{X}, \mathfrak{T}_\infty(2)),
\end{aligned}$$

where $\nu^1_{\overline{Y},\infty} := \varinjlim_r \nu^1_{\overline{Y},r}$. Thus we obtain the assertion. □

We start the proof of Lemma 7.2.1, i.e., the inclusion (7.4.1), assuming that k is p-adic local. The triangle (7.5.2) gives rise to the upper exact row of the following commutative diagram with exact rows (cf. [SH] (4.4.2)):

$$\begin{tikzcd} H^3(\mathscr{X}, \mathfrak{T}_r(2)) \arrow[r] \arrow[d, equal] & H^3(\mathscr{X}, \tau_{\leq 2} Rj_* \mu_{p^r}^{\otimes 2}) \arrow[r, "-\sigma"] \arrow[d] & H^1(Y, \nu_{Y,r}^1) \arrow[d, "\mathrm{Gys}_i^2"] \\ H^3(\mathscr{X}, \mathfrak{T}_r(2)) \arrow[r] & H^3(X, \mu_{p^r}^{\otimes 2}) \arrow[r] & H^4_Y(\mathscr{X}, \mathfrak{T}_r(2)), \end{tikzcd}$$

where σ (resp. Gys_i^2) is as in (7.5.2) (resp. (7.5.3)). Hence the map $\mathfrak{d}$ restricted to $\mathscr{W} = H^3(\mathscr{X}, \tau_{\leq 2} Rj_* \mathbb{Q}_p/\mathbb{Z}_p(2))$ factors as

$$\mathscr{W} \longrightarrow H^1(Y, \nu_{Y,\infty}^1) \longrightarrow H^4_Y(\mathscr{X}, \mathfrak{T}_\infty(2)) \longrightarrow \bigoplus_{y \in Y^0} H^4_y(\mathscr{X}, \mathfrak{T}_\infty(2)),$$

where $\nu_{Y,\infty}^1 := \varinjlim_r \nu_{Y,r}^1$. By Lemmas 7.5.1 and 6.1.2, it suffices to show that the corank of

$$\mathrm{Im}\Big(H^1(Y, \nu_{Y,\infty}^1) \to \bigoplus\nolimits_{y \in Y^0} H^4_y(\mathscr{X}, \mathfrak{T}_\infty(2))\Big) \tag{7.5.4}$$

is not greater than $\dim_{\mathbb{Q}_p} H^2(k, H^1(\overline{X}, \mathbb{Q}_p(2)))$. We pursue an analogy to the case $p \neq \mathrm{ch}(\mathbb{F})$ by replacing $H^4_Y(\mathscr{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))$ with $H^1(Y, \nu_{Y,\infty}^1)$. There is an exact sequence

$$0 \longrightarrow H^1(\mathbb{F}, H^0(\overline{Y}, \nu_{\overline{Y},\infty}^1)) \longrightarrow H^1(Y, \nu_{Y,\infty}^1) \longrightarrow H^1(\overline{Y}, \nu_{\overline{Y},\infty}^1)^{G_{\mathbb{F}}} \longrightarrow 0$$

arising from a Hochschild-Serre spectral sequence. By [Sat3] Corollary 2.2.7, there is a Mayer-Vietoris spectral sequence

$$E_1^{a,b} = H^{a+b}(\overline{Y}^{(1-a)}, W_\infty \Omega^{1+a}_{\overline{Y}^{(1-a)},\log}) \Longrightarrow H^{a+b}(\overline{Y}, \nu_{\overline{Y},\infty}^1).$$

Note that $E_1^{a,b}$ is of weight $b-1$ so that $H^i(\mathbb{F}, E_1^{a,b})$ is finite unless $b = 1$. Thus we obtain isomorphisms up to finite groups

$$H^1(\mathbb{F}, H^0(\overline{Y}, \nu_{\overline{Y},\infty}^1)) \simeq H^1(\mathbb{F}, E_2^{-1,1}), \tag{7.5.5}$$

$$H^1(\overline{Y}, \nu_{\overline{Y},\infty}^1)^{G_{\mathbb{F}}} \simeq (E_2^{0,1})^{G_{\mathbb{F}}} \tag{7.5.6}$$

with $E_2^{-1,1} = \mathrm{Ker}(d_1^{-1,1})$ and $E_2^{0,1} = \mathrm{Coker}(d_1^{-1,1})$, where $d_1^{-1,1}$ is the Gysin map

$$H^0(\overline{Y}^{(2)}, \mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow H^1(\overline{Y}^{(1)}, W_\infty \Omega^1_{\overline{Y}^{(1)},\log}).$$

There is an exact sequence of $G_{\mathbb{F}}$-modules (cf. (7.6.1) below)

$$0 \longrightarrow \mathrm{Pic}(\overline{Y}^{(1)}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow H^1(\overline{Y}^{(1)}, W_\infty \Omega^1_{\overline{Y}^{(1)},\log}) \longrightarrow \mathrm{Br}(\overline{Y}^{(1)})_{p\text{-tors}} \longrightarrow 0.$$

Hence we see that the group (7.5.4) coincides with the image of $H^1(\mathbb{F}, H^0(\overline{Y}, \nu^1_{Y,\infty}))$ up to finite groups by the condition $\mathbf{T}$, the same computation as for Proposition 7.4.2 (1) and [CTSS] p. 782 Théorème 3. Now we are reduced to showing

$$\begin{aligned}\dim_{\mathbb{Q}_p} H^2(k, H^1(\overline{X}, \mathbb{Q}_p(2))) &\geq \mathrm{corank}\big(H^1(\mathbb{F}, H^0(\overline{Y}, \nu^1_{\overline{Y},\infty}))\big)\\ &= \mathrm{corank}\big(H^1(\mathbb{F}, E_2^{-1,1})\big),\end{aligned}$$

where the last equality follows from (7.5.5). As is seen in Remark 7.4.10, the right hand side is equal to $\dim_{\mathbb{Q}}(\varDelta \otimes \mathbb{Q})$ under the condition $\mathbf{T}$. On the other hand, by [J2] Corollary 7, the left hand side does not change when one replaces p with another prime p'. Thus the desired inequality follows from (7.4.11). This completes the proof of Lemma 7.2.1 and Theorem 7.1.1. $\square$

7.6 Appendix to Section 7

Let Z be a proper smooth variety over a finite field $\mathbb{F}$. For a positive integer m, we define the object $\mathbb{Z}/m\mathbb{Z}(1) \in D^b(Z_{\text{ét}}, \mathbb{Z}/m\mathbb{Z})$ as

$$\mathbb{Z}/m\mathbb{Z}(1) := \mu_{m'} \oplus (W_r\Omega^1_{Z,\log}[-1])$$

where we factorized m as $m' \cdot p^r$ with $(p, m') = 1$. There is a distinguished triangle of Kummer theory for $\mathbb{G}_{\mathrm{m}} := \mathbb{G}_{\mathrm{m},Z}$ in $D^b(Z_{\text{ét}})$

$$\mathbb{Z}/m\mathbb{Z}(1) \longrightarrow \mathbb{G}_{\mathrm{m}} \xrightarrow{\times m} \mathbb{G}_{\mathrm{m}} \longrightarrow \mathbb{Z}/m\mathbb{Z}(1)[1].$$

So there is a short exact sequence of $G_{\mathbb{F}}$-modules

$$0 \longrightarrow \mathrm{Pic}(\overline{Z})/m \longrightarrow H^2(\overline{Z}, \mathbb{Z}/m\mathbb{Z}(1)) \longrightarrow {}_m\mathrm{Br}(\overline{Z}) \longrightarrow 0,$$

where $\overline{Z} := Z \otimes_{\mathbb{F}} \overline{\mathbb{F}}$. Taking the inductive limit with respect to $m \geq 1$, we obtain a short exact sequence of $G_{\mathbb{F}}$-modules

$$0 \longrightarrow \mathrm{Pic}(\overline{Z}) \otimes \mathbb{Q}/\mathbb{Z} \xrightarrow{\alpha} H^2(\overline{Z}, \mathbb{Q}/\mathbb{Z}(1)) \longrightarrow \mathrm{Br}(\overline{Z}) \longrightarrow 0. \tag{7.6.1}$$

Concerning the arrow α, we prove the following lemma, which has been used in this section.

LEMMA 7.6.2 *The map $H^1(\mathbb{F}, \mathrm{Pic}(\overline{Z}) \otimes \mathbb{Q}/\mathbb{Z}) \to H^1(\mathbb{F}, H^2(\overline{Z}, \mathbb{Q}/\mathbb{Z}(1))$ induced by α has finite kernel.*

Proof. Note that $\mathrm{Pic}(\overline{Z}) \otimes \mathbb{Q}/\mathbb{Z} \simeq (\mathrm{NS}(\overline{Z})/\mathrm{NS}(\overline{Z})_{\mathrm{tors}}) \otimes \mathbb{Q}/\mathbb{Z}$. By a theorem of Matsusaka [Ma] Theorem 4, the group $\mathrm{Div}(\overline{Z})/\mathrm{Div}(\overline{Z})_{\mathrm{num}}$ is isomorphic to $\mathrm{NS}(\overline{Z})/\mathrm{NS}(\overline{Z})_{\mathrm{tors}}$, where $\mathrm{Div}(\overline{Z})$ denotes the group of Weil divisors on $\overline{Z}$, $\mathrm{Div}(Z)_{\mathrm{num}}$ denotes the subgroup of Weil divisors numerically equivalent to zero. By this fact and the fact that $\mathrm{NS}(\overline{Z})$ is finitely generated, there exists a finite family $\{C_i\}_{i\in I}$ of proper smooth curves over $\mathbb{F}$ which are finite over Z and for which the kernel of the natural map $\mathrm{NS}(\overline{Z}) \to \bigoplus_{i\in I} \mathrm{NS}(\overline{C}_i)$ with $\overline{C}_i := C_i \otimes_{\mathbb{F}} \overline{\mathbb{F}}$ is torsion. Now consider a commutative diagram

$$\begin{array}{ccc} H^1(\mathbb{F}, \mathrm{NS}(\overline{Z}) \otimes \mathbb{Q}/\mathbb{Z}) & \longrightarrow & H^1(\mathbb{F}, H^2(\overline{Z}, \mathbb{Q}/\mathbb{Z}(1))), \\ \downarrow & & \downarrow \\ \bigoplus_{i\in I} H^1(\mathbb{F}, \mathrm{NS}(\overline{C}_i) \otimes \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\sim} & \bigoplus_{i\in I} H^1(\mathbb{F}, H^2(\overline{C}_i, \mathbb{Q}/\mathbb{Z}(1))). \end{array}$$

By a standard norm argument, one can easily show that the left vertical map has finite kernel. The bottom horizontal arrow is bijective, because $\mathrm{Br}(\overline{C}_i) = 0$ for any $i \in I$ by Tsen's theorem (cf. [Se] II.3.3). Hence the top horizontal arrow has finite kernel and we obtain the lemma. $\square$

A Relation with Conjectures of Beilinson and Lichtenbaum

Let $k, p, S, \mathscr{X}$ and K be as in the notation 1.8. We always assume 1.8.1 in what follows. The Zariski site Z_{Zar} on a scheme Z always means $(\text{ét}/Z)_{\mathrm{Zar}}$, and $Z_{\text{ét}}$ means the usual small étale site. The main result of this appendix is Proposition A.1.3 below.

A.1 Motivic Complex and Conjectures

Let $\mathbb{Z}(2)_{\mathrm{Zar}} = \mathbb{Z}(2)^{\mathscr{X}}_{\mathrm{Zar}}$ be the motivic complex on $\mathscr{X}_{\mathrm{Zar}}$ defined by using Bloch's cycle complex, and let $\mathbb{Z}(2)_{\text{ét}}$ be its étale sheafification, which are, by works of Levine ([Le1], [Le2]), considered as strong candidates for motivic complexes of Beilinson-Lichtenbaum ([Be], [Li1]) in Zariski and étale topology, respectively (see also [Li2], [Li3]). We put

$$H^*_{\mathrm{Zar}}(\mathscr{X}, \mathbb{Z}(2)) := H^*_{\mathrm{Zar}}(\mathscr{X}, \mathbb{Z}(2)_{\mathrm{Zar}}), \qquad H^*_{\text{ét}}(\mathscr{X}, \mathbb{Z}(2)) := H^*_{\text{ét}}(\mathscr{X}, \mathbb{Z}(2)_{\text{ét}}).$$

In this appendix, we observe that the finiteness of $H^3_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$ is deduced from the following conjectures on motivic complexes:

Conjecture A.1.1 *Let $\epsilon : \mathscr{X}_{\text{ét}} \to \mathscr{X}_{\mathrm{Zar}}$ be the natural continuous map of sites. Then*:

(1) (*Beilinson-Lichtenbaum conjecture*). *We have*

$$\mathbb{Z}(2)_{\mathrm{Zar}} \xrightarrow{\sim} \tau_{\leq 2} R\epsilon_* \mathbb{Z}(2)_{\text{ét}} \quad \textit{in } D(\mathscr{X}_{\mathrm{Zar}}).$$

(2) (*Hilbert's theorem* 90). *We have $R^3\epsilon_*\mathbb{Z}(2)_{\text{ét}} = 0$.*

(3) (*Kummer theory on $\mathscr{X}[p^{-1}]_{\text{ét}}$*). *We have $(\mathbb{Z}(2)_{\text{ét}})|_{\mathscr{X}[p^{-1}]} \otimes^{\mathbb{L}} \mathbb{Z}/p^r \simeq \mu_{p^r}^{\otimes 2}$.*

This conjecture holds if $\mathscr{X}$ is smooth over S by a result of Geisser [Ge1] Theorem 1.2 and the Merkur'ev-Suslin theorem [MS] (see also [GL2] Remark 5.9).

Conjecture A.1.2 *Let γ^2 be the canonical map*

$$\gamma^2 : \mathrm{CH}^2(\mathscr{X}) = H^4_{\mathrm{Zar}}(\mathscr{X}, \mathbb{Z}(2)) \longrightarrow H^4_{\text{ét}}(\mathscr{X}, \mathbb{Z}(2)).$$

Then the p-primary torsion part of $\mathrm{Coker}(\gamma^2)$ *is finite.*

This conjecture is based on Lichtenbaum's conjecture [Li1] that $H^4_{\text{ét}}(\mathscr{X}, \mathbb{Z}(2))$ is a finitely generated abelian group (by the properness of $\mathscr{X}/S$). The aim of this appendix is to prove the following:

Proposition A.1.3 *If Conjectures* A.1.1 *and* A.1.2 *hold, then* $H^3_{\mathrm{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$ *is finite.*

This proposition is reduced to the following lemma:

Lemma A.1.4 (1) *If Conjecture* A.1.1 *holds, then for $r \geq 1$ there is an exact sequence*

$$0 \longrightarrow \mathrm{Coker}(\alpha_r) \longrightarrow H^3_{\mathrm{ur}}(K, \mathbb{Z}/p^r(2)) \longrightarrow \mathrm{Ker}(\varrho^2_r) \longrightarrow 0,$$

where α_r denotes the map

$$\alpha_r : {}_{p^r}\mathrm{CH}^2(\mathscr{X}) \longrightarrow {}_{p^r}H^4_{\text{ét}}(\mathscr{X}, \mathbb{Z}(2))$$

induced by γ^2, and ϱ^2_r denotes the cycle class map

$$\varrho^2_r : \mathrm{CH}^2(\mathscr{X})/p^r \longrightarrow H^4_{\text{ét}}(\mathscr{X}, \mathfrak{T}_r(2)).$$

(2) *If Conjectures* A.1.1 *and* A.1.2 *hold, then* $\mathrm{Coker}(\alpha_{\mathbb{Q}_p/\mathbb{Z}_p})$ *and* $\mathrm{Ker}(\varrho^2_{\mathbb{Q}_p/\mathbb{Z}_p})$ *are finite, where $\alpha_{\mathbb{Q}_p/\mathbb{Z}_p} := \varinjlim_{r\geq 1} \alpha_r$ and $\varrho^2_{\mathbb{Q}_p/\mathbb{Z}_p} := \varinjlim_{r \geq 1} \varrho^2_r$.*

To prove this lemma, we need the following sublemma, which is a variant of Geisser's arguments in [Ge1] §6 and provides a Kummer theory on the whole $\mathscr{X}$ extending Conjecture A.1.1 (3):

Sublemma A.1.5 *Put $\mathbb{Z}/p^r(2)_{\text{ét}} := \mathbb{Z}(2)_{\text{ét}} \otimes^{\mathbb{L}} \mathbb{Z}/p^r$. If Conjecture* A.1.1 *holds, then there is a unique isomorphism*

$$\mathbb{Z}/p^r(2)_{\text{ét}} \xrightarrow{\sim} \mathfrak{T}_r(2) \quad \text{in } D(\mathscr{X}_{\text{ét}}, \mathbb{Z}/p^r)$$

that extends the isomorphism in Conjecture A.1.1 (3).

We prove Sublemma A.1.5 in §A.2 below and Lemma A.1.4 in §A.3 below.

A.2 Proof of Sublemma A.1.5

By Conjecture A.1.1 (3), we have only to consider the case where p is not invertible on S. Let us note that

$(*)\quad \mathbb{Z}/p^r(2)_{\text{ét}}$ is concentrated in degrees ≤ 2

by Conjecture A.1.1 (1) and (2). Let V, Y, i and j be as follows:

$$V := \mathscr{X}[p^{-1}] \xhookrightarrow{\ j\ } \mathscr{X} \xhookleftarrow{\ i\ } Y,$$

where Y denotes the union of the fibers of $\mathscr{X}/S$ of characteristic p. In étale topology, we define $Ri^!$ and Rj_* for unbounded complexes by the method of Spaltenstein [Spa]. We will prove

$$\tau_{\leq 3} Ri^! \mathbb{Z}/p^r(2)_{\text{ét}} \simeq \nu^1_{Y,r}[-3] \quad \text{in } D(Y_{\text{ét}}, \mathbb{Z}/p^r), \tag{A.2.1}$$

using $(*)$ (see (**S5**) in §4.1 for $\nu^1_{Y,r}$). We first prove Sublemma A.1.5 admitting this isomorphism. Since $(\mathbb{Z}(2)_{\text{ét}})|_V \otimes^{\mathbb{L}} \mathbb{Z}/p^r \simeq \mu_{p^r}^{\otimes 2}$ by Conjecture A.1.1 (3), we obtain a distinguished triangle from (A.2.1) and $(*)$

$$i_*\nu^1_{Y,r}[-3] \longrightarrow \mathbb{Z}/p^r(2)_{\text{ét}} \longrightarrow \tau_{\leq 2} Rj_*\mu_{p^r}^{\otimes 2} \longrightarrow i_*\nu^1_{Y,r}[-2].$$

Hence comparing this distinguished triangle with that of (**S5**) in §4.1, we obtain the desired isomorphism in the sublemma, whose uniqueness follows from [SH] Lemmas 1.1 and 1.2 (1).

In what follows, we prove (A.2.1). Put $\mathscr{K} := \mathbb{Z}(2)_{\text{Zar}} \otimes^{\mathbb{L}} \mathbb{Z}/p^r$ and $\mathscr{L} := \mathbb{Z}/p^r(2)_{\text{ét}}$ for simplicity. Let $\epsilon : \mathscr{X}_{\text{ét}} \to \mathscr{X}_{\text{Zar}}$ be as in Conjecture A.1.1. In Zariski topology, we define $Ri^!_{\text{Zar}}$ and $Rj_{\text{Zar}*}$ for unbounded complexes in the usual way by the finiteness of cohomological dimension. Because $\mathscr{L} = \epsilon^*\mathscr{K}$ is concentrated in degrees ≤ 2 by $(*)$, there is a commutative diagram with distinguished rows in $D^b(\mathscr{X}_{\text{ét}}, \mathbb{Z}/p^r)$

$$\begin{tikzcd}
\epsilon^*\mathscr{K} \arrow[r] \arrow[d, equal] & \tau_{\leq 2}\epsilon^* Rj_{\text{Zar}*}j^*_{\text{Zar}}\mathscr{K} \arrow[r] \arrow[d, "\alpha"'] & (\tau_{\leq 3}\epsilon^* i_{\text{Zar}*}Ri^!_{\text{Zar}}\mathscr{K})[1] \arrow[r] \arrow[d, "\beta"'] & \epsilon^*\mathscr{K}[1] \arrow[d, equal] \\
\mathscr{L} \arrow[r] & \tau_{\leq 2} Rj_{\text{ét}*}j^*_{\text{ét}}\mathscr{L} \arrow[r] & (\tau_{\leq 3} i_{\text{ét}*}Ri^!_{\text{ét}}\mathscr{L})[1] \arrow[r] & \mathscr{L}[1],
\end{tikzcd}$$

where the upper (resp. lower) row is obtained from the localization triangle in the Zariski (resp. étale) topology and the arrows α and β are canonical base-change morphisms. Since α is an isomorphism ([MS], [SV], [GL2]), β is an isomorphism as well. Hence (A.2.1) is reduced to showing

$$\tau_{\leq 3} Ri^!_{\text{Zar}}\mathscr{K} \simeq \epsilon_{Y*}\nu^1_{Y,r}[-3] \quad \text{in} \in D(Y_{\text{Zar}}, \mathbb{Z}/p^r), \tag{A.2.2}$$

where $\epsilon_Y : Y_{\text{ét}} \to Y_{\text{Zar}}$ denotes the natural continuous map of sites and we have used the base-change isomorphism $\epsilon^* i_{\text{Zar}*} = i_{\text{ét}*}\epsilon^*_Y$ ([Ge1] Proposition 2.2 (a)). Finally we show (A.2.2). Consider the local-global spectral sequence in the Zariski topology

$$E_1^{u,v} = \bigoplus_{x \in \mathscr{X}^u \cap Y} R^{u+v} i_{x*}(Ri^!_x Ri^!_{\text{Zar}}\mathscr{K}) \Longrightarrow R^{u+v} i^!_{\text{Zar}}\mathscr{K},$$

where for $x \in Y$, i_x denotes the natural map $x \to Y$. We have

$$E_1^{u,v} \simeq \begin{cases} \bigoplus_{x \in \mathscr{X}^u \cap Y} i_{x*}\epsilon_{x*} W_r\Omega^{2-u}_{x,\log} & \text{(if } v = 2\text{)} \\ 0 & \text{(otherwise)} \end{cases}$$

by the localization sequence of higher Chow groups [Le1] and results of Geisser-Levine ([GL1] Proposition 3.1, Theorem 7.1), where for $x \in Y$, ϵ_x denotes the natural continuous map $x_{\text{ét}} \to x_{\text{Zar}}$ of sites. By this description of E_1-terms and the compatibility of boundary maps ([GL2] Lemma 3.2, see also [Sz] Appendix), we obtain (A.2.2). This completes the proof of Sublemma A.1.5.

A.3 Proof of Lemma A.1.4

(1) By Sublemma A.1.5, there is an exact sequence

$$0 \longrightarrow H^3_{\text{ét}}(\mathscr{X}, \mathbb{Z}(2))/p^r \longrightarrow H^3_{\text{ét}}(\mathscr{X}, \mathfrak{T}_r(2)) \longrightarrow {}_{p^r}H^4_{\text{ét}}(\mathscr{X}, \mathbb{Z}(2)) \longrightarrow 0.$$

By Conjecture A.1.1 (1) and (2), we have

$$H^3_{\text{ét}}(\mathscr{X}, \mathbb{Z}(2)) \simeq H^3_{\text{Zar}}(\mathscr{X}, \mathbb{Z}(2)) \simeq \text{CH}^2(\mathscr{X}, 1).$$

Thus we get an exact sequence

$$0 \longrightarrow \text{CH}^2(\mathscr{X}, 1)/p^r \longrightarrow H^3_{\text{ét}}(\mathscr{X}, \mathfrak{T}_r(2)) \longrightarrow {}_{p^r}H^4_{\text{ét}}(\mathscr{X}, \mathbb{Z}(2)) \longrightarrow 0.$$

On the other hand, there is an exact sequence

$$0 \to N^1H^3_{\text{ét}}(\mathscr{X}, \mathfrak{T}_r(2)) \to H^3_{\text{ét}}(\mathscr{X}, \mathfrak{T}_r(2)) \to H^3_{\text{ur}}(K, \mathbb{Z}/p^r(2)) \to \text{Ker}(\varrho^2_r) \to 0$$

by Proposition 4.3.2 (see Lemma 4.2.3 for N^1). In view of these facts and the short exact sequence in Lemma 4.2.3 (1), we get the desired exact sequence.

(2) By Conjecture A.1.1 (1) and (2), the map γ^2 in Conjecture A.1.2 is injective. Hence we get an exact sequence

$$0 \longrightarrow \text{Coker}(\alpha_r) \longrightarrow {}_{p^r}\text{Coker}(\gamma^2) \longrightarrow \text{CH}^2(\mathscr{X})/p^r \xrightarrow{\gamma^2/p^r} H^4_{\text{ét}}(\mathscr{X}, \mathbb{Z}(2))/p^r.$$

Noting that the composite of γ^2/p^r and the injective map

$$H^4_{\text{ét}}(\mathscr{X}, \mathbb{Z}(2))/p^r \hookrightarrow H^4_{\text{ét}}(\mathscr{X}, \mathfrak{T}_r(2))$$

obtained from Sublemma A.1.5 coincides with ϱ^2_r, we get a short exact sequence

$$0 \longrightarrow \text{Coker}(\alpha_r) \longrightarrow {}_{p^r}\text{Coker}(\gamma^2) \longrightarrow \text{Ker}(\varrho^2_r) \longrightarrow 0,$$

which implies the finiteness of $\text{Coker}(\alpha_{\mathbb{Q}_p/\mathbb{Z}_p})$ and $\text{Ker}(\varrho^2_{\mathbb{Q}_p/\mathbb{Z}_p})$ under Conjecture A.1.2. This completes the proof of Lemma A.1.4 and Proposition A.1.3. □

B Zeta Value of Threefolds over Finite Fields

In this appendix B, all cohomology groups of schemes are taken over the étale topology. Let X be a projective smooth geometrically integral threefold over a finite field $\mathbb{F}_q$, and let K be the function field of X (the case of fourfolds is treated in a recent paper of Kohmoto [Ko]). We define the unramified cohomology $H^{n+1}_{\mathrm{ur}}(K, \mathbb{Q}/\mathbb{Z}(n))$ in the same way as in 1.8. We show that the groups

$$H^2_{\mathrm{ur}}(K, \mathbb{Q}/\mathbb{Z}(1)) = \mathrm{Br}(X) \quad \text{and} \quad H^3_{\mathrm{ur}}(K, \mathbb{Q}/\mathbb{Z}(2))$$

are related with the value of the Hasse-Weil zeta function $\zeta(X, s)$ at $s = 2$:

$$\zeta^*(X, 2) := \lim_{s\to 2} \zeta(X, s)(1 - q^{2-s})^{-\varrho_2}, \qquad \text{where } \varrho_2 := \mathrm{ord}_{s=2}\, \zeta(X, s).$$

Let

$$\theta \;:\; \mathrm{CH}^2(X) \longrightarrow \mathrm{Hom}(\mathrm{CH}^1(X), \mathbb{Z})$$

be the map induced by the intersection pairing and the degree map

$$\mathrm{CH}^2(X) \times \mathrm{CH}^1(X) \longrightarrow \mathrm{CH}^3(X) = \mathrm{CH}_0(X) \xrightarrow{\deg} \mathbb{Z}.$$

The map θ has finite cokernel by a theorem of Matsusaka [Ma] Theorem 4. We define

$$\mathscr{R} := |\mathrm{Coker}(\theta)|.$$

We prove the following formula (compare with the formula in [Ge2]):

Theorem B.1 *Assume that* $\mathrm{Br}(X)$ *and* $H^3_{\mathrm{ur}}(K, \mathbb{Q}/\mathbb{Z}(2))$ *are finite. Then* $\zeta^*(X, 2)$ *equals the following rational number up to a sign*:

$$q^{\chi(X,\mathscr{O}_X,2)} \cdot \frac{|H^3_{\mathrm{ur}}(K, \mathbb{Q}/\mathbb{Z}(2))|}{|\mathrm{Br}(X)| \cdot \mathscr{R}} \cdot \prod_{i=0}^{3} |\mathrm{CH}^2(X, i)_{\mathrm{tors}}|^{(-1)^i} \cdot \prod_{i=0}^{1} |\mathrm{CH}^1(X, i)_{\mathrm{tors}}|^{(-1)^i},$$

where $\mathrm{CH}^2(X, i)$ *and* $\mathrm{CH}^1(X, i)$ *denote Bloch's higher Chow groups* [Bl3] *and* $\chi(X, \mathscr{O}_X, 2)$ *denotes the following integer*:

$$\chi(X, \mathscr{O}_X, 2) := \sum_{i,j} (-1)^{i+j}\,(2 - i)\, \dim_{\mathbb{F}_q} H^j(X, \Omega^i_X) \quad (0 \le i \le 2,\; 0 \le j \le 3).$$

This theorem follows from a theorem of Milne ([Mi2] Theorem 0.1) and Proposition B.2 below. For integers $i, n \ge 0$, we define

$$H^i(X, \widehat{\mathbb{Z}}(n)) := \prod_{\text{all } \ell} H^i(X, \mathbb{Z}_\ell(n)),$$

where ℓ runs through all prime numbers, and $H^i(X, \mathbb{Z}_p(n))$ ($p := \mathrm{ch}(\mathbb{F}_q)$) is defined as

$$H^i(X, \mathbb{Z}_p(n)) := \varprojlim_{r\ge 1} H^{i-n}(X, W_r\Omega^n_{X,\log}).$$

Proposition B.2 (1) *We have*

$$H^i(X,\widehat{\mathbb{Z}}(2)) \simeq \begin{cases} \mathrm{CH}^2(X,4-i)_{\mathrm{tors}} & (i=0,1,2,3) \\ (\mathrm{CH}^1(X,i-6)_{\mathrm{tors}})^* & (i=6,7), \end{cases} \tag{B.3}$$

where for an abelian group M, we put

$$M^* := \mathrm{Hom}(M,\mathbb{Q}/\mathbb{Z}).$$

Furthermore, $\mathrm{CH}^1(X,j)_{\mathrm{tors}}$ *and* $\mathrm{CH}^2(X,j)_{\mathrm{tors}}$ *are finite for any $j \geq 0$, and we have*

$$\mathrm{CH}^1(X,j)_{\mathrm{tors}} = 0 \ \textit{for} \ j \geq 2 \quad \textit{and} \quad \mathrm{CH}^2(X,j)_{\mathrm{tors}} = 0 \ \textit{for} \ j \geq 4.$$

(2) *Assume that* $\mathrm{Br}(X)$ *is finite. Then we have*

$$H^5(X,\widehat{\mathbb{Z}}(2))_{\mathrm{tors}} \simeq \mathrm{Br}(X)^*,$$

and the cycle class map

$$\mathrm{CH}^2(X)\otimes\mathbb{Z}_\ell \longrightarrow H^4(X,\mathbb{Z}_\ell(2))$$

has finite cokernel for any prime number ℓ.

(3) *Assume that* $\mathrm{Br}(X)$ *and* $H^3_{\mathrm{ur}}(K,\mathbb{Q}/\mathbb{Z}(2))$ *are finite. Then the following map given by the cup product with the canonical element* $1 \in \widehat{\mathbb{Z}} \simeq H^1(\mathbb{F}_q,\widehat{\mathbb{Z}})$ *has finite kernel and cokernel*:

$$\epsilon^4 : H^4(X,\widehat{\mathbb{Z}}(2)) \longrightarrow H^5(X,\widehat{\mathbb{Z}}(2)),$$

and we have the following equality of rational numbers:

$$\frac{|\mathrm{Ker}(\epsilon^4)|}{|\mathrm{Coker}(\epsilon^4)|} = \frac{|H^3_{\mathrm{ur}}(K,\mathbb{Q}/\mathbb{Z}(2))|\cdot|\mathrm{CH}^2(X)_{\mathrm{tors}}|}{|\mathrm{Br}(X)|\cdot\mathscr{R}}.$$

Proof of Proposition B.2. (1) By standard arguments on limits, there is a long exact sequence

$$\begin{aligned}\cdots \longrightarrow H^i(X,\widehat{\mathbb{Z}}(2)) \longrightarrow H^i(X,\widehat{\mathbb{Z}}(2))\otimes_{\mathbb{Z}}\mathbb{Q} \longrightarrow H^i(X,\mathbb{Q}/\mathbb{Z}(2)) \\ \longrightarrow H^{i+1}(X,\widehat{\mathbb{Z}}(2)) \longrightarrow \cdots .\end{aligned}$$

By [CTSS] p. 780 Théorème 2, p. 782 Théorème 3, we see that

$$H^i(X,\widehat{\mathbb{Z}}(2)) \text{ and } H^i(X,\mathbb{Q}/\mathbb{Z}(2)) \text{ are finite for } i \neq 4,5.$$

Hence we have

$$H^i(X,\widehat{\mathbb{Z}}(2)) \simeq H^{i-1}(X,\mathbb{Q}/\mathbb{Z}(2)) \quad \text{for } \ i \neq 4,5,6.$$

On the other hand, there is an exact sequence

$$0 \longrightarrow \mathrm{CH}^2(X,5-i)\otimes\mathbb{Q}/\mathbb{Z} \longrightarrow H^{i-1}(X,\mathbb{Q}/\mathbb{Z}(2)) \longrightarrow \mathrm{CH}^2(X,4-i)_{\mathrm{tors}} \longrightarrow 0$$

for $i \leq 3$ ([MS], [SV], [GL1], [GL2]), where $\mathrm{CH}^2(X,5-i)\otimes\mathbb{Q}/\mathbb{Z}$ must be zero because it is divisible and finite. Thus we get the isomorphism (B.3) for $i \leq 3$, the finiteness of $\mathrm{CH}^2(X,j)_{\mathrm{tors}}$ for $j \geq 1$ and the vanishing of $\mathrm{CH}^2(X,j)_{\mathrm{tors}}$ for $j \geq 4$. The finiteness of $\mathrm{CH}^2(X,0)_{\mathrm{tors}} = \mathrm{CH}^2(X)_{\mathrm{tors}}$ (cf. [CTSS] p. 780 Théorème 1) follows from the exact sequence

$$0 \longrightarrow \mathrm{CH}^2(X,1)\otimes\mathbb{Q}/\mathbb{Z} \longrightarrow N^1H^3(X,\mathbb{Q}/\mathbb{Z}(2)) \longrightarrow \mathrm{CH}^2(X)_{\mathrm{tors}} \longrightarrow 0$$

(cf. Lemma 3.2.2), where we put

$$N^1H^3(X,\mathbb{Q}/\mathbb{Z}(2)) := \mathrm{Ker}(H^3(X,\mathbb{Q}/\mathbb{Z}(2)) \to H^3(K,\mathbb{Q}/\mathbb{Z}(2))).$$

As for the case $i = 6,7$ of (B.3), we have

$$H^i(X,\widehat{\mathbb{Z}}(2))^* \simeq H^{7-i}(X,\mathbb{Q}/\mathbb{Z}(1))$$

by a theorem of Milne [Mi2] Theorem 1.14 (a). It remains to show

$$\mathrm{CH}^1(X,j)_{\mathrm{tors}} \simeq H^{1-j}(X,\mathbb{Q}/\mathbb{Z}(1)) \quad \text{ for } \ j \geq 0,$$

which can be checked by similar arguments as before.

(2) We have $H^5(X,\widehat{\mathbb{Z}}(2))^* \simeq H^2(X,\mathbb{Q}/\mathbb{Z}(1))$ and an exact sequence

$$0 \longrightarrow \mathrm{CH}^1(X)\otimes\mathbb{Q}/\mathbb{Z} \longrightarrow H^2(X,\mathbb{Q}/\mathbb{Z}(1)) \longrightarrow \mathrm{Br}(X) \longrightarrow 0. \tag{B.4}$$

Hence we have $(H^5(X,\widehat{\mathbb{Z}}(2))_{\mathrm{tors}})^* \simeq \mathrm{Br}(X)$, assuming $\mathrm{Br}(X)$ is finite. To show the second assertion for $\ell \neq \mathrm{ch}(\mathbb{F}_q)$, it is enough to show that the cycle class map

$$\mathrm{CH}^2(X)\otimes\mathbb{Q}_\ell \longrightarrow H^4(\overline{X},\mathbb{Q}_\ell(2))^{\Gamma}$$

is surjective, where $\Gamma := \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$. The assumption on $\mathrm{Br}(X)$ implies the bijectivity of the cycle class map

$$\mathrm{CH}^1(X)\otimes\mathbb{Q}_\ell \xrightarrow{\ \sim\ } H^2(\overline{X},\mathbb{Q}_\ell(1))^{\Gamma}$$

by [Ta2] Proposition (4.3) (see also [Mi1] Theorem 4.1), and the assertion follow from [Ta2] Proposition (5.1). As for the case $\ell = \mathrm{ch}(\mathbb{F}_q)$, one can easily pursue an analogy using crystalline cohomology, whose details are left to the reader.

(3) The finiteness assumption on $\mathrm{Br}(X)$ implies the condition $\boldsymbol{SS}(X,1,\ell)$ in [Mi2] for all prime numbers ℓ by loc. cit. Proposition 0.3. Hence $\boldsymbol{SS}(X,2,\ell)$ holds by the Poincaré duality, and ϵ^4 has finite kernel and cokernel by loc. cit. Theorem 0.1.

To show the equality assertion, we put

$$\widehat{\mathrm{CH}}^2(X) := \varprojlim_{n\geq 1} \mathrm{CH}^2(X)/n,$$

and consider the following commutative square (cf. [Mi3] Lemma 5.4):

$$\begin{tikzcd}
\widehat{\mathrm{CH}}^2(X) \arrow[r, "\Theta"] \arrow[d, "\alpha"'] & \mathrm{Hom}(\mathrm{CH}^1(X), \widehat{\mathbb{Z}}) \\
H^4(X, \widehat{\mathbb{Z}}(2)) \arrow[r, "\epsilon^4"] & H^5(X, \widehat{\mathbb{Z}}(2)), \arrow[u, "\beta"']
\end{tikzcd}$$

where the top arrow Θ denotes the map induced by θ. The arrow α denotes the cycle class map of codimension 2, and β denotes the Pontryagin dual of the cycle class map with $\mathbb{Q}/\mathbb{Z}$-coefficients in (B.4). The arrow α is injective (cf. (4.3.3)) and we have

$$|\mathrm{Coker}(\alpha)| = |H^3_{\mathrm{ur}}(K, \mathbb{Q}/\mathbb{Z}(2))|$$

by the finiteness assumption on $H^3_{\mathrm{ur}}(K, \mathbb{Q}/\mathbb{Z}(2))$ and (2) (cf. Proposition 4.3.5). The arrow β is surjective and we have

$$\mathrm{Ker}(\beta) = H^5(X, \widehat{\mathbb{Z}}(2))_{\mathrm{tors}} \overset{(2)}{\simeq} \mathrm{Br}(X)^*,$$

by Milne's lemma ([Mi3] Lemma 5.3) and the isomorphism $\mathrm{CH}^1(X) \otimes \widehat{\mathbb{Z}} \simeq H^2(X, \widehat{\mathbb{Z}}(1))$ (cf. [Ta2] Proposition (4.3)), where we have used again the finiteness assumption on $\mathrm{Br}(X)$. Therefore in view of the finiteness of $\mathrm{Ker}(\epsilon^4)$, the map Θ has finite kernel and we obtain

$$\mathrm{Ker}(\Theta) = \widehat{\mathrm{CH}}^2(X)_{\mathrm{tors}} = \mathrm{CH}^2(X)_{\mathrm{tors}},$$

where we have used the finiteness of $\mathrm{CH}^2(X)_{\mathrm{tors}}$ in (1). Finally the assertion follows from the following equality concerning the above diagram:

$$\frac{|\mathrm{Ker}(\Theta)|}{|\mathrm{Coker}(\Theta)|} = \frac{|\mathrm{Ker}(\alpha)|}{|\mathrm{Coker}(\alpha)|} \cdot \frac{|\mathrm{Ker}(\epsilon^4)|}{|\mathrm{Coker}(\epsilon^4)|} \cdot \frac{|\mathrm{Ker}(\beta)|}{|\mathrm{Coker}(\beta)|}$$

This completes the proof of Proposition B.2 and Theorem B.1. □

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