Power Reductivity over an Arbitrary Base

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Vincent Franjou[1] and Wilberd van der Kallen[2]

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Abstract. Our starting point is Mumford's conjecture, on representations of Chevalley groups over fields, as it is phrased in the preface of *Geometric Invariant Theory*. After extending the conjecture appropriately, we show that it holds over an arbitrary commutative base ring. We thus obtain the first fundamental theorem of invariant theory (often referred to as Hilbert's fourteenth problem) over an arbitrary Noetherian ring. We also prove results on the Grosshans graded deformation of an algebra in the same generality. We end with tentative finiteness results for rational cohomology over the integers.

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1 Introduction

The following statement may seem quite well known:

Theorem 1. *Let* $\mathbf{k}$ *be a Dedekind ring and let* G *be a Chevalley group scheme over* $\mathbf{k}$*. Let* A *be a finitely generated commutative* $\mathbf{k}$*-algebra on which* G *acts rationally through algebra automorphisms. The subring of invariants* A^G *is then a finitely generated* $\mathbf{k}$*-algebra.*

[1]LMJL - Laboratoire de Mathématiques Jean Leray, CNRS/Université de Nantes. The author acknowledges the hospitality and support of CRM Barcelona during the tuning of the paper.

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Indeed, R. Thomason proved [21, Theorem 3.8] the statement for any Noetherian Nagata ring $\mathbf{k}$. Thomason's paper deals with quite a different theme, that is the existence of equivariant resolutions by free modules. Thomason proves that equivariant sheaves can be resolved by equivariant vector bundles. He thus solves a conjecture of Seshadri [19, question 2 p.268]. The affirmative answer to Seshadri's question is explained to yield Theorem 1 in the same paper [19, Theorem 2 p.263]. The finesse only illustrates that the definition of geometric reductivity in [19] does not suit well an arbitrary base. Indeed, Seshadri does not follow the formulation in Mumford's book's introduction [GIT, Preface], and uses polynomials instead [19, Theorem 1 p.244]. This use of a dual in the formulation seems to be why one requires Thomason's result [21, Corollary 3.7]. One can rather go back to the original formulation in terms of symmetric powers as illustrated by the following:

Definition 2. Let $\mathbf{k}$ be a ring and let G be an algebraic group over $\mathbf{k}$. The group G is *power-reductive* over $\mathbf{k}$ if the following holds.

Property (Power reductivity). *Let L be a cyclic $\mathbf{k}$-module with trivial G-action. Let M be a rational G-module, and let φ be a G-module map from M onto L. Then there is a positive integer d such that the d-th symmetric power of φ induces a surjection:*

$$(S^d M)^G \to S^d L.$$

We show in Section 3 that power-reductivity holds for Chevalley group schemes G, without assumption on the commutative ring $\mathbf{k}$. Note that this version of reductivity is exactly what is needed in Nagata's treatment of finite generation of invariants. We thus obtain:

Theorem 3. *Let $\mathbf{k}$ be a Noetherian ring and let G be a Chevalley group scheme over $\mathbf{k}$. Let A be a finitely generated commutative $\mathbf{k}$-algebra on which G acts rationally through algebra automorphisms. The subring of invariants A^G is then a finitely generated $\mathbf{k}$-algebra.*

There is a long history of cohomological finite generation statements as well, where the algebra of invariants $A^G = \mathrm{H}^0(G, A)$ is replaced by the whole algebra $\mathrm{H}^*(G, A)$ of the derived functors of invariants. Over fields, Friedlander and Suslin's solution in the case of finite group schemes [8] lead to the conjecture in [13], now a theorem of Touzé [22]. In Section 5, we generalize to an arbitrary (Noetherian) base Grosshans' results on his filtration [9]. These are basic tools for obtaining finite generation statements on cohomology. In Section 6, we apply our results in an exploration of the case when the base ring is $\mathbb{Z}$. Section 4 presents results of use in Section 5 and Section 3. Our results support the hope that Touzé's theorem extends to an arbitrary base.

2 Power reductivity and Hilbert's 14th

2.1 Power surjectivity

To deal with the strong form of integrality we encounter, we find it convenient to make the following definition.

Definition 4. A morphism of $\mathbf{k}$-algebras: $\phi : S \to R$ is *power-surjective* if every element of R has a power in the image of ϕ. It is *universally power-surjective* if for every $\mathbf{k}$-algebra A, the morphism of $\mathbf{k}$-algebras $A \otimes \phi$ is power-surjective, that is: for every $\mathbf{k}$-algebra A, for every x in $A \otimes R$, there is a positive integer n so that x^n lies in the image of $A \otimes \phi$.

If $\mathbf{k}$ contains a field, one does not need arbitrary positive exponents n, but only powers of the characteristic exponent of $\mathbf{k}$ (compare [20, Lemma 2.1.4, Exercise 2.1.5] or Proposition 41 below). Thus if $\mathbf{k}$ is a field of characteristic zero, any universally power-surjective morphism of $\mathbf{k}$-algebras is surjective.

2.2 Consequences

We start by listing consequences of power reductivity, as defined in the introduction (Definition 2).

Convention 5. An algebraic group over our commutative ring $\mathbf{k}$ is always assumed to be a flat affine group scheme over $\mathbf{k}$. Flatness is essential, as we tacitly use throughout that the functor of taking invariants is left exact.

Proposition 6 (Lifting of invariants). *Let $\mathbf{k}$ be a ring and let G be a power-reductive algebraic group over $\mathbf{k}$. Let A be a finitely generated commutative $\mathbf{k}$-algebra on which G acts rationally through algebra automorphisms. If J is an invariant ideal in A, the map induced by reducing mod J:*

$$A^G \to (A/J)^G$$

is power-surjective.

For an example over $\mathbb{Z}$, see 2.3.2.

Remark 7. Let G be power reductive and let $\phi : A \to B$ be a power-surjective G-map of $\mathbf{k}$-algebras. One easily shows that $A^G \to B^G$ is power-surjective.

Theorem 8 (Hilbert's fourteenth problem). *Let $\mathbf{k}$ be a Noetherian ring and let G be an algebraic group over $\mathbf{k}$. Let A be a finitely generated commutative $\mathbf{k}$-algebra on which G acts rationally through algebra automorphisms. If G is power-reductive, then the subring of invariants A^G is a finitely generated $\mathbf{k}$-algebra.*

Proof. We apply [20, p. 23–26]. It shows that Theorem 8 relies entirely on the conclusion of Proposition 6, which is equivalent to the statement [20, Lemma 2.4.7 p. 23] that, for a surjective G-map $\phi : A \to B$ of $\mathbf{k}$-algebras, the induced map on invariants $A^G \to B^G$ is power-surjective. To prove that power reductivity implies this, consider an invariant b in B, take for L the cyclic module $\mathbf{k}.b$ and for M any submodule of A such that $\phi(M) = L$. We conclude with a commuting diagram:

$$\begin{tikzcd}
(S^dM)^G \arrow[r] \arrow[d, "S^d\phi"] & (S^dA)^G \arrow[r] \arrow[d] & A^G \arrow[d, "\phi^G"] \\
S^dL \arrow[r] & (S^dB)^G \arrow[r] & B^G.
\end{tikzcd}$$

□

Theorem 9 (Hilbert's fourteenth for modules). *Let* $\mathbf{k}$ *be a Noetherian ring and let* G *be a power-reductive algebraic group over* $\mathbf{k}$*. Let* A *be a finitely generated commutative* $\mathbf{k}$*-algebra on which* G *acts rationally, and let* M *be a Noetherian* A*-module, with an equivariant structure map* $A \otimes M \to M$*. If* G *is power-reductive, then the module of invariants* M^G *is Noetherian over* A^G*.*

Proof. As in [14, 2.2], consider either the symmetric algebra of M on A, or the 'semi-direct product ring' $A \ltimes M$ as in Proposition 57, whose underlying G-module is $A \oplus M$, with product given by $(a_1, m_1)(a_2, m_2) = (a_1a_2, a_1m_2 + a_2m_1)$. □

2.3 Examples

2.3.1

Let $\mathbf{k} = \mathbb{Z}$. Consider the group SL_2 acting on 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ by conjugation. Let L be the line of homotheties in $M := M_2(\mathbb{Z})$. Write $V^\#$ to indicate the dual module $\mathrm{Hom}_{\mathbb{Z}}(V, \mathbb{Z})$ of a $\mathbb{Z}$-module V. The restriction: $M^\# \to L^\#$ extends to

$$\mathbb{Z}[M] = \mathbb{Z}[a, b, c, d] \to \mathbb{Z}[\lambda] = \mathbb{Z}[L].$$

Taking SL_2-invariants:

$$\mathbb{Z}[a, b, c, d]^{\mathrm{SL}_2} = \mathbb{Z}[t, D] \to \mathbb{Z}[\lambda],$$

the trace $t = a + d$ is sent to 2λ, so λ does not lift to an invariant in $M^\#$. The determinant $D = ad - bc$ is sent to λ^2 however, illustrating power reductivity of SL_2.

2.3.2

Similarly, the adjoint action of SL_2 on $\mathfrak{sl}_2$ is such that $u(a) := \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ sends $X, H, Y \in \mathfrak{sl}_2$ respectively to $X + aH - a^2Y, H - 2aY, Y$. This action extends to the symmetric algebra $S^*(\mathfrak{sl}_2)$, which is a polynomial ring in variables X, H, Y. Take $\mathbf{k} = \mathbb{Z}$ again. The mod 2 invariant H does not lift to an integral invariant, but $H^2 + 4XY$ is an integral invariant, and it reduces mod 2 to the square H^2 in $\mathbb{F}_2[X, H, Y]$. This illustrates power reductivity with modules that are not flat, and the strong link between integral and modular invariants.

2.3.3

Consider the group U of 2×2 upper triangular matrices with diagonal 1: this is just an additive group. Let it act on M with basis $\{x, y\}$ by linear substitutions: $u(a)$ sends x, y respectively to $x, ax + y$. Sending x to 0 defines $M \to L$, and since $(S^*M)^U = \mathbf{k}[x]$, power reductivity fails.

2.4 Equivalence of power reductivity with property (Int)

Following [14], we say that a group G satisfies (Int) if $(A/J)^G$ is integral over the image of A^G for every A and J with G action. Note that if $(A/J)^G$ is a Noetherian A^G-module (compare Theorem 9), it must be integral over the image of A^G. As explained in [14, Theorem 2.8], when $\mathbf{k}$ is a field, the property (Int) is equivalent to geometric reductivity, which is equivalent to power-reductivity by [20, Lemma 2.4.7 p. 23]. In general, property (Int) is still equivalent to power-reductivity. But geometric reductivity in the sense of [19] looks too weak.

Proposition 10. *An algebraic group G has property (Int) if, and only if, it is power-reductive.*

Proof. By Proposition 6, power reductivity implies property (Int). We prove the converse. Let $\phi : M \to L$ be as in the formulation of power reductivity in Definition 2. Choose a generator b of L. Property (Int) gives a polynomial $t^n + a_1 t^{n-1} + \cdots + a_n$ with b as root, and with a_i in the image of $S^*(\varphi) : (S^*M)^G \to S^*L$. As b is homogeneous of degree one, we may assume $a_i \in S^i\varphi((S^iM)^G)$. Write a_i as $r_i b^i$ with $r_i \in \mathbf{k}$. Put $r = 1 + r_1 + \cdots r_n$. Then $rb^n = 0$, and $r^{(n-1)!}$ annihilates $b^{n!}$. Since $a_i^{n!/i}$ lies in the image of $S^{n!}\varphi : (S^{n!}M)^G \to S^{n!}L$, the cokernel of this map is annihilated by $r_i^{n!/i}$. Together $r^{(n-1)!}$ and the $r_i^{n!/i}$ generate the unit ideal. So the cokernel vanishes. $\square$

Example 11. Let G be a finite group, viewed as an algebraic group over $\mathbf{k}$. Then A is integral over A^G, because a is a root of $\prod_{g \in G}(x - g(a))$. (This goes back to Emmy Noether [18].) Property (Int) follows easily. Hence G is power reductive.

3 Mumford's conjecture over an arbitrary base

This section deals with the following generalization of the Mumford conjecture.

Theorem 12 (Mumford conjecture). *A Chevalley group scheme is power-reductive for every base.*

By a Chevalley group scheme over $\mathbb{Z}$, we mean a connected split reductive algebraic $\mathbb{Z}$-group $G_{\mathbb{Z}}$, and, by a Chevalley group scheme over a ring $\mathbf{k}$, we mean an algebraic $\mathbf{k}$-group $G = G_{\mathbf{k}}$ obtained by base change from such a $G_{\mathbb{Z}}$. We want to establish the following:

Property. *Let $\mathbf{k}$ be a commutative ring. Let L be a cyclic $\mathbf{k}$-module with trivial G-action. Let M be a rational G-module, and let φ be a G-module map from M onto L. Then there is a positive integer d such that the d-th symmetric power of φ induces a surjection:*

$$(S^d M)^G \to S^d L.$$

3.1 Reduction to local rings

We first reduce to the case of a local ring. For each positive integer d, consider the ideal in $\mathbf{k}$ formed by those scalars which are hit by an invariant in $(S^d M)^G$, and let:

$$\mathfrak{I}_d(\mathbf{k}) := \{x \in \mathbf{k} \mid \exists m \in \mathbb{N},\ x^m.S^d L \subset S^d\varphi((S^d M)^G)\}$$

be its radical. Note that these ideals form a monotone family: if d divides d', then $\mathfrak{I}_d$ is contained in $\mathfrak{I}_{d'}$. We want to show that $\mathfrak{I}_d(\mathbf{k})$ equals $\mathbf{k}$ for some d. To that purpose, it is enough to prove that for each maximal ideal $\mathfrak{M}$ in $\mathbf{k}$, the localized $\mathfrak{I}_d(\mathbf{k})_{(\mathfrak{M})}$ equals the local ring $\mathbf{k}_{(\mathfrak{M})}$ for some d. Notice that taking invariants commutes with localization. Indeed the whole Hochschild complex does and localization is exact. As a result, the localized $\mathfrak{I}_d(\mathbf{k})_{(\mathfrak{M})}$ is equal to the ideal $\mathfrak{I}_d(\mathbf{k}_{(\mathfrak{M})})$. This shows that it is enough to prove the property for a local ring $\mathbf{k}$.

For the rest of this proof, $\mathbf{k}$ denotes a local ring with residual characteristic p.

3.2 Reduction to cohomology

As explained in Section 3.5, we may assume that G is semisimple simply connected. Replacing M if necessary by a submodule that still maps onto L, we may assume that M is finitely generated.
We then reduce the desired property to cohomological algebra. To that effect, if X is a G-module, consider the evaluation map on the identity id_X: $\mathrm{Hom}_{\mathbf{k}}(X,X)^{\#} \to \mathbf{k}$ (we use $V^{\#}$ to indicate the dual module $\mathrm{Hom}_{\mathbf{k}}(V,\mathbf{k})$ of a module V). If X is $\mathbf{k}$-free of finite rank d, then $S^d(\mathrm{Hom}_{\mathbf{k}}(X,X)^{\#})$ contains the determinant. The determinant is G-invariant, and its evaluation at id_X is equal to 1. Let b a $\mathbf{k}$-generator of L and consider the composite:

$$\psi : \mathrm{Hom}_{\mathbf{k}}(X,X)^{\#} \to \mathbf{k} \to \mathbf{k}.b = L.$$

Its d-th power $S^d\psi$ sends the determinant to b^d. Suppose further that ψ lifts to M by a G-equivariant map. Then, choosing d to be the $\mathbf{k}$-rank of X, the d-th power of the resulting map $S^d(\mathrm{Hom}_{\mathbf{k}}(X,X)^\#) \to S^dM$ sends the determinant to a G-invariant in S^dM, which is sent to b^d through $S^d\varphi$. This would establish the property.

$$\begin{tikzcd} & \mathrm{Hom}_{\mathbf{k}}(X,X)^\# \arrow[dl, dashed] \arrow[d, "\psi"] \\ M \arrow[r, "\varphi"'] & L \end{tikzcd}$$

The existence of a lifting would follow from the vanishing of the extension group:

$$\mathrm{Ext}^1_G((\mathrm{Hom}_{\mathbf{k}}(X,X)^\#, \mathrm{Ker}\varphi),$$

or, better, from acyclicity, *i.e.* the vanishing of all positive degree Ext-groups. Inspired by the proof of the Mumford conjecture in [6, (3.6)], we choose X to be an adequate Steinberg module. To make this choice precise, we need notations, essentially borrowed from [6, 2].

3.3 Notations

We decide as in [11], and contrary to [12] and [6], that the roots of the standard Borel subgroup B are negative. The opposite Borel group B^+ of B will thus have positive roots. We also fix a Weyl group invariant inner product on the weight lattice $X(T)$. Thus we can speak of the length of a weight.
For a weight λ in the weight lattice, we denote by λ as well the corresponding one-dimensional rational B-module (or sometimes B^+-module), and by ∇_λ the costandard module (Schur module) $\mathrm{ind}_B^G\lambda$ induced from it. Dually, we denote by Δ_λ the standard module (Weyl module) of highest weight λ. So $\Delta_\lambda = \mathrm{ind}_{B^+}^G(-\lambda)^\#$. We shall use that, over $\mathbb{Z}$, these modules are $\mathbb{Z}$-free [11, II Ch. 8].
We let ρ be half the sum of the positive roots of G. It is also the sum of the fundamental weights. As G is simply connected, the fundamental weights are weights of B.
Let p be the characteristic of the residue field of the local ring $\mathbf{k}$. When p is positive, for each positive integer r, we let the weight σ_r be $(p^r-1)\rho$. When p is 0, we let σ_r be $r\rho$. Let St_r be the G-module $\nabla_{\sigma_r} = \mathrm{ind}_B^G\sigma_r$; it is a usual Steinberg module when $\mathbf{k}$ is a field of positive characteristic.

3.4

We shall use the following combinatorial lemma:

Lemma 13. *Let R be a positive real number. If r is a large enough integer, for all weights μ of length less than R, $\sigma_r+\mu$ is dominant.*

So, if r is a large enough integer to satisfy the condition in Lemma 13, for all G-modules M with weights that have length less than R, all the weights in $\sigma_r \otimes M$ are dominant. Note that in the preceding discussion, the G-module M is finitely generated. Thus the weights of M, and hence of $\mathrm{Ker}\varphi$, are bounded. Thus, Theorem 12 is implied by the following proposition.

Proposition 14. *Let R be a positive real number, and let r be as in Lemma 13 . For all local rings $\mathbf{k}$ with characteristic p residue field, for all G-module N with weights of length less than R, and for all positive integers n:*

$$\mathrm{Ext}^n_G((\mathrm{Hom}_{\mathbf{k}}(St_r, St_r)^{\#}, N) = 0 \ .$$

Proof. First, the result is true when $\mathbf{k}$ is a field. Indeed, we have chosen St_r to be a self-dual Steinberg module, so, for each positive integer n:

$$\mathrm{Ext}^n_G((\mathrm{Hom}_{\mathbf{k}}(St_r, St_r)^{\#}, N) = \mathrm{H}^n(G, St_r \otimes St_r \otimes N) = \mathrm{H}^n(B, St_r \otimes \sigma_r \otimes N).$$

Vanishing follows by [6, Corollary (3.3')] or the proof of [6, Corollary (3.7)].
Suppose now that N is defined over $\mathbb{Z}$ by a free $\mathbb{Z}$-module, in the following sense: $N = N_{\mathbb{Z}} \otimes_{\mathbb{Z}} V$ for a $\mathbb{Z}$-free $G_{\mathbb{Z}}$-module $N_{\mathbb{Z}}$ and a $\mathbf{k}$-module V with trivial G action. We then use the universal coefficient theorem [4, A.X.4.7] (see also [11, I.4.18]) to prove acyclicity in this case.
Specifically, let us note $Y_{\mathbb{Z}} := \mathrm{Hom}_{\mathbb{Z}}((St_r)_{\mathbb{Z}}, (St_r)_{\mathbb{Z}}) \otimes N_{\mathbb{Z}}$, so that, using base change (Proposition 16 for $\lambda = \sigma_r$):

$$\mathrm{Ext}^n_G((\mathrm{Hom}_{\mathbf{k}}(St_r, St_r)^{\#}, N) = \mathrm{H}^n(G, Y_{\mathbb{Z}} \otimes V).$$

This cohomology is computed [7, II.3] (see also [11, I.4.16]) by taking the homology of the Hochschild complex $C(G, Y_{\mathbb{Z}} \otimes V)$. This complex is isomorphic to the complex obtained by tensoring with V the integral Hochschild complex $C(G_{\mathbb{Z}}, Y_{\mathbb{Z}})$. Since the latter is a complex of torsion-free abelian groups, we deduce, by the universal coefficient theorem applied to tensoring with a characteristic p field k, and the vanishing for the case of such a field, that: $\mathrm{H}^n(G_{\mathbb{Z}}, Y_{\mathbb{Z}}) \otimes k = 0$, for all positive n. We apply this when k is the residue field of $\mathbb{Z}_{(p)}$; note that if $p = 0$ the residue field k is just $\mathbb{Q}$. Since the cohomology $\mathrm{H}^n(G_{\mathbb{Z}}, Y_{\mathbb{Z}})$ is finitely generated [11, B.6], the Nakayama lemma implies that: $\mathrm{H}^n(G_{\mathbb{Z}}, Y_{\mathbb{Z}}) \otimes \mathbb{Z}_{(p)} = 0$, for all positive n. And $\mathrm{H}^n(G_{\mathbb{Z}}, Y_{\mathbb{Z}}) \otimes \mathbb{Z}_{(p)} = \mathrm{H}^n(G_{\mathbb{Z}}, Y_{\mathbb{Z}} \otimes \mathbb{Z}_{(p)})$ because localization is exact. The complex $C(G_{\mathbb{Z}}, Y_{\mathbb{Z}} \otimes \mathbb{Z}_{(p)})$ is a complex of torsion-free $\mathbb{Z}_{(p)}$-modules, we thus can apply the universal coefficient theorem to tensoring with V. The vanishing of $\mathrm{H}^n(G, Y_{\mathbb{Z}} \otimes \mathbb{Z}_{(p)} \otimes V) = \mathrm{H}^n(G, Y_{\mathbb{Z}} \otimes V)$ follows.
For the general case, we proceed by descending induction on the highest weight of N. To perform the induction, we first choose a total order on weights of length less than R, that refines the usual dominance order of [11, II 1.5]. Initiate the induction with $N = 0$. For the induction step, consider the highest weight μ in N and let N_μ be its weight space. By the preceding case, we obtain vanishing for $\Delta_{\mu_{\mathbb{Z}}} \otimes_{\mathbb{Z}} N_\mu$. Now, by Proposition 21, $\Delta_{\mu_{\mathbb{Z}}} \otimes_{\mathbb{Z}} N_\mu$ maps to

N, and the kernel and the cokernel of this map have lower highest weight. By induction, they give vanishing cohomology. Thus $\mathrm{Hom}_{\mathbf{k}}(St_r, St_r) \otimes N$ is in an exact sequence where three out of four terms are acyclic, hence it is acyclic. □

This concludes the proof of Theorem 12.

3.5 Reduction to simply connected group schemes

Let $Z_{\mathbb{Z}}$ be the center of $G_{\mathbb{Z}}$ and let Z be the corresponding subgroup of G. It is a diagonalisable group scheme, so $M^Z \to L$ is also surjective. We may replace M with M^Z and G with G/Z, in view of the general formula $M^G = (M^Z)^{G/Z}$, see [11, I 6.8(3)]. So now G has become semisimple, but of adjoint type rather than simply connected type. So choose a simply connected Chevalley group scheme $\tilde{G}_{\mathbb{Z}}$ with center $\tilde{Z}_{\mathbb{Z}}$ so that $\tilde{G}_{\mathbb{Z}}/\tilde{Z}_{\mathbb{Z}} = G_{\mathbb{Z}}$. We may now replace G with $\tilde{G}$.

Remark 15. Other reductions are possible, to enlarge the supply of power reductive algebraic groups. For instance, if G has a normal subgroup N so that both N and G/N are power reductive, then so is G (for a proof, use Remark 7). And if $\mathbf{k} \to R$ is a faithfully flat extension so that G_R is power reductive, then G is already power reductive. So twisted forms are allowed, compare the discussion in [19, p. 239].

4 Generalities

This section collects known results over an arbitrary base, their proof, and correct proofs of known results over fields, for use in the other sections. The part up to subsection 4.3 is used, and referred to, in the previous section.

4.1 Notations

Throughout this paper, we let G be a semisimple Chevalley group scheme over the commutative ring $\mathbf{k}$. We keep the notations of Section 3.3. In particular, the standard parabolic B has negative roots. Its standard torus is T, its unipotent radical is U. The opposite Borel B^+ has positive roots and its unipotent radical is U^+. For a standard parabolic subgroup P its unipotent radical is $R_u(P)$. For a weight λ in $X(T)$, $\nabla_\lambda = \mathrm{ind}_B^G \lambda$ and $\Delta_\lambda = \mathrm{ind}_{B^+}^G(-\lambda)^\#$.

4.2

We first recall base change for costandard modules.

Proposition 16. *Let λ be a weight, and denote also by $\lambda = \lambda_{\mathbb{Z}} \otimes \mathbf{k}$ the B-module $\mathbf{k}$ with action by λ. For any ring $\mathbf{k}$, there is a natural isomorphism:*

$$\mathrm{ind}_{B_{\mathbb{Z}}}^{G_{\mathbb{Z}}} \lambda_{\mathbb{Z}} \otimes \mathbf{k} \cong \mathrm{ind}_B^G \lambda$$

In particular, $\mathrm{ind}_B^G \lambda$ is nonzero if and only if λ is dominant.

Proof. First consider the case when λ is not dominant. Then $\mathrm{ind}_B^G\lambda$ vanishes when $\mathbf{k}$ is a field [11, II.2.6], so both $\mathrm{ind}_{B_\mathbb{Z}}^{G_\mathbb{Z}}\lambda_\mathbb{Z}$ and the torsion in $R^1\mathrm{ind}_{B_\mathbb{Z}}^{G_\mathbb{Z}}\lambda_\mathbb{Z}$ must vanish. Then $\mathrm{ind}_B^G\lambda$ vanishes as well for a general $\mathbf{k}$ by the universal coefficient theorem.
In the case when λ is dominant, $R^1\mathrm{ind}_{B_\mathbb{Z}}^{G_\mathbb{Z}}\lambda_\mathbb{Z}$ vanishes by Kempf's theorem [11, II 8.8(2)]. Thus, by [11, I.4.18b)]: $\mathrm{ind}_{B_\mathbb{Z}}^{G_\mathbb{Z}}\lambda_\mathbb{Z}\otimes\mathbf{k}\cong\mathrm{ind}_B^G\lambda$. □

Proposition 17 (Tensor identity for weights). *Let λ be a weight, and denote again by λ the B-module $\mathbf{k}$ with action by λ. Let N be a G-module. There is a natural isomorphism:*

$$\mathrm{ind}_B^G(\lambda\otimes N)\cong(\mathrm{ind}_B^G\lambda)\otimes N.$$

Remark 18. The case when N is $\mathbf{k}$-flat is covered by [11, I.4.8]. We warn the reader against Proposition I.3.6 in the 1987 first edition of the book. Indeed, suppose we always had $\mathrm{ind}_B^G(M\otimes N)\cong(\mathrm{ind}_B^G M)\otimes N$. Take $\mathbf{k}=\mathbb{Z}$ and $N=\mathbb{Z}/p\mathbb{Z}$. The universal coefficient theorem would then imply that $R^1\mathrm{ind}_B^G M$ never has torsion. Thus $R^i\mathrm{ind}_B^G M$ would never have torsion for positive i. It would make [1, Cor. 2.7] contradict the Borel–Weil–Bott theorem.

Proof. Recall that for a B-module M one may define $\mathrm{ind}_B^G(M)$ as $(\mathbf{k}[G]\otimes M)^B$, where $\mathbf{k}[G]\otimes M$ is viewed as a $G\times B$-module with G acting by left translation on $\mathbf{k}[G]$, B acting by right translation on $\mathbf{k}[G]$, and B acting the given way on M. Let N_{triv} denote N with trivial B action. There is a B-module isomorphism $\psi:\mathbf{k}[G]\otimes\lambda\otimes N\to\mathbf{k}[G]\otimes\lambda\otimes N_{\mathrm{triv}}$, given in non-functorial notation by:

$$\psi(f\otimes 1\otimes n):x\mapsto f(x)\otimes 1\otimes xn.$$

So ψ is obtained by first applying the comultiplication $N\to\mathbf{k}[G]\otimes N$, then the multiplication $\mathbf{k}[G]\otimes\mathbf{k}[G]\to\mathbf{k}[G]$. It sends $(\mathbf{k}[G]\otimes\lambda\otimes N)^B$ to $(\mathbf{k}[G]\otimes\lambda\otimes N_{\mathrm{triv}})^B=(\mathbb{Z}[G_\mathbb{Z}]\otimes_\mathbb{Z}\lambda_\mathbb{Z}\otimes_\mathbb{Z}N_{\mathrm{triv}})^B$. Now recall from the proof of Proposition 16 that the torsion in $R^1\mathrm{ind}_{B_\mathbb{Z}}^{G_\mathbb{Z}}\lambda_\mathbb{Z}$ vanishes. By the universal coefficient theorem we get that $(\mathbb{Z}[G_\mathbb{Z}]\otimes_\mathbb{Z}\lambda_\mathbb{Z}\otimes_\mathbb{Z}N_{\mathrm{triv}})^B$ equals $(\mathbf{k}[G]\otimes\lambda)^B\otimes N_{\mathrm{triv}}$. To make these maps into G-module maps, one must use the given G-action on N as the action on N_{triv}. So B acts on N, but not N_{triv}, and for G it is the other way around. One sees that $(\mathbf{k}[G]\otimes\lambda)^B\otimes N_{\mathrm{triv}}$ is just $(\mathrm{ind}_B^G\lambda)\otimes N$. □

Proposition 19. *For a G-module M, there are only dominant weights in M^{U^+}.*

Proof. Let λ be a nondominant weight. Instead we show that $-\lambda$ is no weight of M^U, or that $\mathrm{Hom}_B(-\lambda,M)$ vanishes. By the tensor identity of Proposition 17: $\mathrm{Hom}_B(-\lambda,M)=\mathrm{Hom}_B(k,\lambda\otimes M)=\mathrm{Hom}_G(k,\mathrm{ind}_B^G(\lambda\otimes M))=\mathrm{Hom}_G(k,\mathrm{ind}_B^G\lambda\otimes M)$ which vanishes by Proposition 16. □

Proposition 20. *Let λ be a dominant weight. The restriction (or evaluation) map $\mathrm{ind}_B^G\lambda\to\lambda$ to the weight space of weight λ is a T-module isomorphism.*

Proof. Over fields of positive characteristic this is a result of Ramanathan [12, A.2.6]. It then follows over $\mathbb{Z}$ by the universal coefficient theorem applied to the complex $\mathrm{ind}_{B_{\mathbb{Z}}}^{G_{\mathbb{Z}}}\lambda_{\mathbb{Z}} \to \lambda_{\mathbb{Z}} \to 0$. For a general $\mathbf{k}$, apply proposition 16. □

Proposition 21 (Universal property of Weyl modules). *Let λ be a dominant weight. For any G-module M, there is a natural isomorphism*

$$\mathrm{Hom}_G(\Delta_\lambda, M) \cong \mathrm{Hom}_{B^+}(\lambda, M).$$

In particular, if M has highest weight λ, then there is a natural map from $\Delta_{\lambda_{\mathbb{Z}}} \otimes_{\mathbb{Z}} M_\lambda$ to M, its kernel has lower weights, and λ is not a weight of its cokernel.

Proof. By the tensor identity Proposition 17: $\mathrm{ind}_{B^+}^G(-\lambda \otimes M) \cong \mathrm{ind}_{B^+}^G(-\lambda) \otimes M$. Thus $\mathrm{Hom}_G(\Delta_\lambda, M) = \mathrm{Hom}_G(\mathbf{k}, \mathrm{ind}_{B^+}^G(-\lambda) \otimes M) = \mathrm{Hom}_{B^+}(\mathbf{k}, -\lambda \otimes M) = \mathrm{Hom}_{B^+}(\lambda, M)$. If M has highest weight λ, then $M_\lambda = \mathrm{Hom}_{B^+}(\lambda, M)$. Tracing the maps, the second part follows from Proposition 20. □

4.3 Notations

We now recall the notations from [13, §2.2]. Let the *Grosshans height function* $\mathrm{ht} : X(T) \to \mathbb{Z}$ be defined by:

$$\mathrm{ht}\,\gamma = \sum_{\alpha>0} \langle \gamma, \alpha^\vee \rangle.$$

For a G-module M, let $M_{\leq i}$ denote the largest G-submodule with weights λ that all satisfy: $\mathrm{ht}\,\lambda \leq i$. Similarly define $M_{<i} = M_{\leq i-1}$. For instance, $M_{\leq 0} = M^G$. We call the filtration

$$0 \subseteq M_{\leq 0} \subseteq M_{\leq 1} \cdots$$

the *Grosshans filtration*, and we call its associated graded the *Grosshans graded* $\mathrm{gr}\,M$ of M. We put: $\mathrm{hull}_\nabla(\mathrm{gr}\,M) = \mathrm{ind}_B^G M^{U^+}$.
Let A be a commutative $\mathbf{k}$-algebra on which G acts rationally through $\mathbf{k}$-algebra automorphisms. The Grosshans graded algebra $\mathrm{gr}\,A$ is given in degree i by:

$$\mathrm{gr}_i\,A = A_{\leq i}/A_{<i}.$$

4.4 Erratum

When $\mathbf{k}$ is a field, one knows that $\mathrm{gr}\,A$ embeds in a good filtration hull, which Grosshans calls R in [10], and which we call $\mathrm{hull}_\nabla(\mathrm{gr}\,A)$:

$$\mathrm{hull}_\nabla(\mathrm{gr}\,A) = \mathrm{ind}_B^G A^{U^+}.$$

When $\mathbf{k}$ is a field of positive characteristic p, it was shown by Mathieu [16, Key Lemma 3.4] that this inclusion is power-surjective: for every $b \in \mathrm{hull}_\nabla(\mathrm{gr}\,A)$, there is an r so that b^{p^r} lies in the subalgebra $\mathrm{gr}\,A$.

This result's exposition in [13, Lemma 2.3] relies on [12, Sublemma A.5.1]. Frank Grosshans has pointed out that the proof of this sublemma is not convincing beyond the reduction to the affine case. Later A. J. de Jong actually gave a counterexample to the reasoning. The result itself is correct and has been used by others. As power surjectivity is a main theme in this paper, we take the opportunity to give a corrected treatment. Mathieu's result will be generalized to an arbitrary base $\mathbf{k}$ in Section 5.

Proposition 22. *Let $\mathbf{k}$ be an algebraically closed field of characteristic $p > 0$. Let both A and B be commutative $\mathbf{k}$-algebras of finite type over $\mathbf{k}$, with B finite over A. Put $Y = \operatorname{Spec}(A)$, $X = \operatorname{Spec}(B)$. Assume $X \to Y$ gives a bijection between $\mathbf{k}$ valued points. Then for all $b \in B$ there is an m with $b^{p^m} \in A$.*

Proof. The result follows easily from [15, Lemma 13]. We shall argue instead by induction on the Krull dimension of A.
Say B as an A-module is generated by d elements $b_1, \ldots, b_d$. Let $\mathfrak{p}_1, \ldots \mathfrak{p}_s$ be the minimal prime ideals of A.
Suppose we can show that for every i, j we have $m_{i,j}$ so that $b_j^{p^{m_{i,j}}} \in A + \mathfrak{p}_i B$. Then for every i we have m_i so that $b^{p^{m_i}} \in A + \mathfrak{p}_i B$ for every $b \in B$. Then $b^{p^{m_1+\cdots m_s}} \in A + \mathfrak{p}_1 \cdots \mathfrak{p}_s B$ for every $b \in B$. As $\mathfrak{p}_1 \cdots \mathfrak{p}_s$ is nilpotent, one finds m with $b^{p^m} \in A$ for all $b \in B$. The upshot is that it suffices to prove the sublemma for the inclusion $A/\mathfrak{p}_i \subset B/\mathfrak{p}_i B$. [It is an inclusion because there is a prime ideal $\mathfrak{q}_i$ in B with $A \cap \mathfrak{q}_i = \mathfrak{p}_i$.] Therefore we further assume that A is a domain.
Let $\mathfrak{r}$ denote the nilradical of B. If we can show that for all $b \in B$ there is m with $b^{p^m} \in A + \mathfrak{r}$, then clearly we can also find a u with $b^{p^u} \in A$. So we may as well replace $A \subset B$ with $A \subset B/\mathfrak{r}$ and assume that B is reduced. But then at least one component of $\operatorname{Spec}(B)$ must map onto $\operatorname{Spec}(A)$, so bijectivity implies there is only one component. In other words, B is also a domain.
Choose t so that the field extension $\operatorname{Frac}(A) \subset \operatorname{Frac}(AB^{p^t})$ is separable. (So it is the separable closure of $\operatorname{Frac}(A)$ in $\operatorname{Frac}(B)$.) As $X \to \operatorname{Spec}(AB^{p^t})$ is also bijective, we have that $\operatorname{Spec}(AB^{p^t}) \to \operatorname{Spec}(A)$ is bijective. It clearly suffices to prove the proposition for $A \subset AB^{p^t}$. So we replace B with AB^{p^t} and further assume that $\operatorname{Frac}(B)$ is separable over $\operatorname{Frac}(A)$.
Now $X \to Y$ has a degree which is the degree of the separable field extension. There is a dense subset U of Y so that this degree is the number of elements in the inverse image of a point of U. [Take a primitive element of the field extension, localize to make its minimum polynomial monic over A, invert the discriminant.] Thus the degree must be one because of bijectivity. So we must now have that $\operatorname{Frac}(B) = \operatorname{Frac}(A)$.
Let $\mathfrak{c} = \{\ b \in B \mid bB \subset A\ \}$ be the conductor of $A \subset B$. We know it is nonzero. If it is the unit ideal then we are done. Suppose it is not. By induction applied to $A/\mathfrak{c} \subset B/\mathfrak{c}$ (we need the induction hypothesis for the original problem without any of the intermediate simplifications) we have that for each $b \in B$ there is an m so that $b^{p^m} \in A + \mathfrak{c} = A$. $\square$

4.5

This subsection prepares the ground for the proof of the theorems in Section 5. We start with the ring of invariants $\mathbf{k}[G/U]$ of the action of U by right translation on $\mathbf{k}[G]$.

Lemma 23. *The* $\mathbf{k}$*-algebra* $\mathbf{k}[G/U]$ *is finitely generated.*

Proof. We have:

$$k[G/U] = \bigoplus_{\lambda \in X(T)} k[G/U]_{-\lambda} = \bigoplus_{\lambda \in X(T)} (k[G] \otimes \lambda)^B = \bigoplus_{\lambda \in X(T)} \nabla_\lambda.$$

By Proposition 16, this equals the sum $\oplus_\lambda \nabla_\lambda$ over dominant weights λ only. When G is simply connected, every fundamental weight is a weight, and the monoid of dominant λ is finitely generated. In general, some multiple of a fundamental weight is in $X(T)$ and there are only finitely many dominant weights modulo these multiples. So the monoid is still finitely generated by Dickson's Lemma [5, Ch. 2 Thm. 7]. The maps $\nabla_\lambda \otimes \nabla_\mu \to \nabla_{\lambda+\mu}$ are surjective for dominant λ, μ, because this is so over $\mathbb{Z}$, by base change and surjectivity for fields [11, II, Proposition 14.20]. This implies the result. □

In the same manner one shows:

Lemma 24. *If the* $\mathbf{k}$*-algebra* A^U *is finitely generated, so is* $\operatorname{hull}_\nabla \operatorname{gr} A = \operatorname{ind}_B^G A^{U^+}$.

Proof. Use that A^{U^+} is isomorphic to A^U as $\mathbf{k}$-algebra. □

Lemma 25. *Suppose* $\mathbf{k}$ *is Noetherian. If the* $\mathbf{k}$*-algebra* A *with* G *action is finitely generated, then so is* A^U.

Proof. By the transfer principle [9, Ch. Two]:

$$A^U = \operatorname{Hom}_U(k, A) = \operatorname{Hom}_G(k, \operatorname{ind}_U^G A) = (A \otimes k[G/U])^G.$$

Now apply Lemma 23 and Theorem 3. □

Lemma 26. *If* M *is a* G*-module, there is a natural injective map*

$$\operatorname{gr} M \hookrightarrow \operatorname{hull}_\nabla(\operatorname{gr} M) = \operatorname{ind}_B^G M^{U^+}.$$

Proof. By Lemma 19, the weights of M^{U^+} are dominant. If one of them, say λ, has Grosshans height i, the universal property of Weyl modules (Proposition 21) shows that $(M^{U^+})_\lambda$ is contained in a G-submodule with weights that do not have a larger Grosshans height. So the weight space $(M^{U^+})_\lambda$ is contained in $M_{\leq i}$, but not $M_{<i}$. We conclude that the T-module $\oplus_i (\operatorname{gr}_i M)^{U^+}$ may be identified with the T-module M^{U^+}. It remains to embed $\operatorname{gr}_i M$ into $\operatorname{ind}_B^G (\operatorname{gr}_i M)^{U^+}$.

The T-module projection of $\mathrm{gr}_i M$ onto $(\mathrm{gr}_i M)^{U^+}$ may be viewed as a B-module map, and then, it induces a G-module map $\mathrm{gr}_i M \to \mathrm{ind}_B^G((\mathrm{gr}_i M)^{U^+})$, which restricts to an isomorphism on $(\mathrm{gr}_i M)^{U^+}$ by Proposition 20. So its kernel has weights with lower Grosshans height, and must therefore be zero. □

In the light of Lemma 26, one may write:

Definition 27. A G-module M has *good Grosshans filtration* if the embedding of $\mathrm{gr}\, M$ into $\mathrm{hull}_\nabla(\mathrm{gr}\, M)$ is an isomorphism.

It is worth recording the following characterization, just like in the case where $\mathbf{k}$ is a field.

Proposition 28 (Cohomological criterion). *For a G-module M, the following are equivalent.*

i. *M has good Grosshans filtration.*

ii. $\mathrm{H}^1(G, M \otimes \mathbf{k}[G/U])$ *vanishes.*

iii. $\mathrm{H}^n(G, M \otimes \mathbf{k}[G/U])$ *vanishes for all positive n.*

Proof. Let M have good Grosshans filtration. We have to show that $M \otimes \mathbf{k}[G/U]$ is acyclic. First, for each integer i, $\mathrm{gr}_i M \otimes \mathbf{k}[G/U]$ is a direct sum of modules of the form $\mathrm{ind}_B^G\lambda \otimes \mathrm{ind}_B^G\mu \otimes N$, where G acts trivially on N. Such modules are acyclic by [11, B.4] and the universal coefficient theorem. As each $\mathrm{gr}_i M \otimes \mathbf{k}[G/U]$ is acyclic, so is each $M_{\leq i} \otimes \mathbf{k}[G/U]$, and thus $M \otimes \mathbf{k}[G/U]$ is acyclic.
Conversely, suppose that M does not have good Grosshans filtration. Choose i so that $M_{<i}$ has good Grosshans filtration, but $M_{\leq i}$ does not. Choose λ so that $\mathrm{Hom}(\Delta_\lambda, \mathrm{hull}(\mathrm{gr}_i M)/\mathrm{gr}_i M)$ is nonzero. Note that λ has Grosshans height below i. As $\mathrm{Hom}(\Delta_\lambda, \mathrm{hull}(\mathrm{gr}_i M))$ vanishes, $\mathrm{Ext}^1_G(\Delta_\lambda, \mathrm{gr}_i M) = \mathrm{H}^1(G, \mathrm{gr}_i M \otimes \nabla_\lambda)$ does not. Since $M_{<i} \otimes \mathbf{k}[G/U] = \oplus_{\mu \text{ dominant}} M_{<i} \otimes \nabla_\mu$ is acyclic, $\mathrm{H}^1(G, M_{\leq i} \otimes \nabla_\lambda)$ is nonzero as well. Now use that $\mathrm{Hom}(\Delta_\lambda, M/M_{\leq i})$ vanishes, and conclude that $\mathrm{H}^1(G, M \otimes \mathbf{k}[G/U])$ does not vanish. □

5 Grosshans graded, Grosshans hull and powers

5.1

When G is a semisimple group over a field $\mathbf{k}$, Grosshans has introduced a filtration on G-modules. As recalled in Section 4.3, it is the filtration associated to the function defined on $X(T)$ by: $\mathrm{ht}\,\gamma = \sum_{\alpha>0}\langle\gamma, \alpha^\vee\rangle$. Grosshans has proved some interesting results about its associated graded, when the G-module is a $\mathbf{k}$-algebra A with rational G action. We now show how these results generalize to an arbitrary Noetherian base $\mathbf{k}$, and we draw some conclusions about $\mathrm{H}^*(G, A)$. All this suggests that the finite generation conjecture of [13] (see also [14]) deserves to be investigated in the following generality.

Problem. *Let* $\mathbf{k}$ *be a Noetherian ring and let* G *be a Chevalley group scheme over* $\mathbf{k}$*. Let* A *be a finitely generated commutative* $\mathbf{k}$*-algebra on which* G *acts rationally through algebra automorphisms. Is the cohomology ring* $\mathrm{H}^*(G, A)$ *a finitely generated* $\mathbf{k}$*-algebra?*

Let $\mathbf{k}$ be an arbitrary commutative ring.

Theorem 29 (Grosshans hull and powers). *The natural embedding of* $\operatorname{gr} A$ *in* $\operatorname{hull}_\nabla(\operatorname{gr} A)$ *is power-surjective.*

This will then be used to prove:

Theorem 30 (Grosshans hull and finite generation). *If the ring* $\mathbf{k}$ *is Noetherian, then the following are equivalent.*

i. The $\mathbf{k}$*-algebra* A *is finitely generated;*

ii. For every standard parabolic P*, the* $\mathbf{k}$*-algebra of invariants* $A^{R_u(P)}$ *is finitely generated;*

iii. The $\mathbf{k}$*-algebra* $\operatorname{gr} A$ *is finitely generated;*

iv. The $\mathbf{k}$*-algebra* $\operatorname{hull}_\nabla(\operatorname{gr} A)$ *is finitely generated.*

Remark 31. Consider a reductive Chevalley group scheme G. As the Grosshans height is formulated with the help of coroots $\alpha^\vee$, only the semisimple part of G is relevant for it. But of course everything is compatible with the action of the center of G also. We leave it to the reader to reformulate our results for reductive G. We return to the assumption that G is semisimple.

Theorem 32. *Let* A *be a finitely generated commutative* $\mathbf{k}$*-algebra. If* $\mathbf{k}$ *is Noetherian, there is a positive integer* n *so that:*

$$n \operatorname{hull}_\nabla(\operatorname{gr} A) \subseteq \operatorname{gr} A.$$

In particular $\mathrm{H}^i(G, \operatorname{gr} A)$ *is annihilated by* n *for positive* i*.*

This is stronger than the next result.

Theorem 33 (generic good Grosshans filtration). *Let* A *be a finitely generated commutative* $\mathbf{k}$*-algebra. If* $\mathbf{k}$ *is Noetherian, there is a positive integer* n *so that* $A[1/n]$ *has good Grosshans filtration. In particular* $\mathrm{H}^i(G, A) \otimes \mathbb{Z}[1/n] = \mathrm{H}^i(G, A[1/n])$ *vanishes for positive* i*.*

Remark 34. Of course $A[1/n]$ may vanish altogether, as we are allowed to take the characteristic for n, when that is positive.

Theorem 35. *Let* A *be a finitely generated commutative* $\mathbf{k}$*-algebra. If* $\mathbf{k}$ *is Noetherian, for each prime number* p*, the algebra map* $\operatorname{gr} A \to \operatorname{gr}(A/pA)$ *is power-surjective.*

5.2

We start with a crucial special case. Let $\mathbf{k} = \mathbb{Z}$. Let $\lambda \in X(T)$ be dominant. Let S' be the graded algebra with degree n part:

$$S'_n = \nabla_{n\lambda} = \Gamma(G/B, \mathcal{L}(n\lambda)).$$

Let us view Δ_λ as a submodule of ∇_λ with common λ weight space (the 'minimal admissible lattice' embedded in the 'maximal admissible lattice'). Let S be the graded subalgebra generated by Δ_λ in the graded algebra S'. If we wish to emphasize the dependence on λ, we write $S'(\lambda)$ for S', $S(\lambda)$ for S. Consider the map

$$G/B \to \mathbb{P}_{\mathbb{Z}}(\Gamma(G/B, \mathcal{L}(\lambda))^{\#})$$

given by the 'linear system' ∇_λ on G/B. The projective scheme $\mathrm{Proj}(S')$ corresponds with its image, which, by direct inspection, is isomorphic to G/P, where P is the stabilizer of the weight space with weight $-\lambda$ of $\nabla_\lambda^{\#}$. Indeed that weight space is the image of B/B, compare Proposition 20 and [11, II.8.5]. The inclusion $\phi : S \hookrightarrow S'$ induces a morphism from an open subset of $\mathrm{Proj}(S')$ to $\mathrm{Proj}(S)$. This open subset is called $G(\phi)$ in [EGA II, 2.8.1].

Lemma 36. *The morphism* $\mathrm{Proj}(S') \to \mathrm{Proj}(S)$ *is defined on all of* $G/P = \mathrm{Proj}(S')$.

Proof. As explained in [EGA II, 2.8.1], the domain $G(\phi)$ contains the principal open subset $D_+(s)$ of $\mathrm{Proj}(S')$ for any $s \in S_1$. Consider in particular a generator s of the λ weight space of ∇_λ. It is an element in S_1, and, by Lemma 20, it generates the free $\mathbf{k}$-module $\Gamma(P/P, \mathcal{L}(\lambda))$. Thus, the minimal Schubert variety P/P is contained in $D_+(s)$. We then conclude by homogeneity: s is also U^+ invariant, so in fact the big cell $\Omega = U^+P/P$ is contained in $D_+(s)$, and the domain $G(\phi)$ contains the big cell Ω. Then it also contains the Weyl group translates $w\Omega$, and thus it contains all of G/P. ☐

Lemma 37. *The graded algebra S' is integral over its subalgebra S.*

Proof. We also put a grading on the polynomial ring $S'[z]$, by assigning degree one to the variable z. One calls $\mathrm{Proj}(S'[z])$ the projective cone of $\mathrm{Proj}(S')$ [EGA II, 8.3]. By [EGA II, 8.5.4], we get from Lemma 36 that $\hat{\Phi} : \mathrm{Proj}(S'[z]) \to \mathrm{Proj}(S[z])$ is everywhere defined. Now by [EGA II, Th (5.5.3)], and its proof, the maps $\mathrm{Proj}(S'[z]) \to \mathrm{Spec}\,\mathbb{Z}$ and $\mathrm{Proj}(S[z]) \to \mathrm{Spec}\,\mathbb{Z}$ are proper and separated, so $\hat{\Phi}$ is proper by [EGA II, Cor (5.4.3)]. But now the principal open subset $D_+(z)$ associated to z in $\mathrm{Proj}(S'[z])$ is just $\mathrm{Spec}(S')$, and its inverse image is the principal open subset associated to z in $\mathrm{Proj}(S[z])$, which is $\mathrm{Spec}(S)$ (compare [EGA II, 8.5.5]). So $\mathrm{Spec}(S) \to \mathrm{Spec}(S')$ is proper, and S' is a finitely generated S-module by [EGA III, Prop (4.4.2)]. ☐

Lemma 38. *There is a positive integer t so that tS' is contained in S.*

Proof. Clearly $S' \otimes \mathbb{Q} = S \otimes \mathbb{Q}$, so the result follows from Lemma 37. ☐

Let p be a prime number. Recall from 4.4 the result of Mathieu [16, Key Lemma 3.4] that, for every element b of S'/pS', there is a positive r so that $b^{p^r} \in (S + pS')/(pS') \subseteq S'/pS'$.
By Lemma 38 and Proposition 41 below this implies

Lemma 39. *The inclusion $S \to S'$ is universally power-surjective.*

5.3

We briefly return to power surjectivity for a general commutative ring $\mathbf{k}$.

Definition 40. Let t be a positive integer and let $f : Q \to R$ a $\mathbf{k}$-algebra homomorphism. We say that f is t-power-surjective if for every $x \in R$ there is a power t^n with $x^{t^n} \in f(Q)$.

Proposition 41. *Let $f : Q \to R$ be a $\mathbf{k}$-algebra homomorphism and Y a variable.*

- *If $f \otimes \mathbf{k}[Y] : Q[Y] \to R[Y]$ is power-surjective, then $Q \to R/pR$ is p-power-surjective for every prime p;*
- *Assume t is a positive integer such that $tR \subset f(Q)$. If $Q \to R/pR$ is p-power-surjective for every prime p dividing t, then f is universally power-surjective.*

Proof. First suppose $f \otimes \mathbf{k}[Y] : Q[Y] \to R[Y]$ is power-surjective. Let $x \in R/pR$. We have to show that x^{p^n} lifts to Q for some n. As $R[Y] \to (R/pR)[Y]$ is surjective, the composite $Q[Y] \to (R/pR)[Y]$ is also power-surjective. Choose n prime to p and m so that $(x+Y)^{np^m}$ lifts to $Q[Y]$. Rewrite $(x+Y)^{np^m}$ as $(x^{p^m} + Y^{p^m})^n$ and note that the coefficient nx^{p^m} of $Y^{(n-1)p^m}$ must lift to Q. Now use that n is invertible in $\mathbf{k}/p\mathbf{k}$.
Next suppose $tR \subset f(Q)$ and $Q \to R/pR$ is p-power-surjective for every prime p dividing t. Let C be a $\mathbf{k}$-algebra. We have to show that $f \otimes C : Q \otimes C \to R \otimes C$ is power-surjective. Since $f \otimes C : Q \otimes C \to R \otimes C$ satisfies all the conditions that $f : Q \to R$ does, we may as well simplify notation and suppress C. For $x \in R$ we have to show that some power lifts to Q. By taking repeated powers we can get x in $f(Q) + pR$ for every prime p dividing t. So if $p_1, \dots, p_m$ are the primes dividing t, we can arrange that x lies in the intersection of the $f(Q) + p_i R$, which is $f(Q) + p_1 \cdots p_m R$. Now by taking repeated $p_1 \cdots p_m$-th powers, one pushes it into $f(Q) + (p_1 \cdots p_m)^n R$ for any positive n, eventually into $f(Q) + tR \subseteq f(Q)$. □

5.4

We come back to the $\mathbf{k}$-algebra A, and consider the inclusion $\operatorname{gr} A \hookrightarrow \operatorname{hull}_\nabla(\operatorname{gr} A)$, as in Theorem 29.

Notations 42. Let λ be a dominant weight and let $b \in A^{U^+}$ be a weight vector of weight λ. Then we define $\psi_b : S'(\lambda) \otimes \mathbf{k} \to \mathrm{hull}_\nabla(\mathrm{gr}\, A)$ as the algebra map induced by the B-algebra map $S'(\lambda) \otimes \mathbf{k} \to A^{U^+}$ which sends the generator (choose one) of the λ weight space of ∇_λ to b.

Lemma 43. *For each c in the image of ψ_b, there is a positive integer s so that $c^s \in \mathrm{gr}\, A$.*

Proof. The composite of $S \otimes \mathbf{k} \to S' \otimes \mathbf{k}$ with ψ_b factors through $\mathrm{gr}\, A$, so this follows from Lemma 39. □

Proof of Theorem 29. For every $b \in \mathrm{hull}_\nabla(\mathrm{gr}\, A)$, there are $b_1, \ldots, b_s$ of respective weights $\lambda_1, \ldots, \lambda_s$ so that b lies in the image of $\psi_{b_1} \otimes \cdots \otimes \psi_{b_s}$. As $\bigotimes_{i=1}^s S(\lambda_i) \to \bigotimes_{i=1}^s S'(\lambda_i)$ is universally power-surjective by lemma 39, lemma 43 easily extends to tensor products. □

Lemma 44. *Suppose* $\mathbf{k}$ *is Noetherian. If* $\mathrm{hull}_\nabla(\mathrm{gr}\, A)$ *is a finitely generated* $\mathbf{k}$*-algebra, so is* $\mathrm{gr}\, A$*.*

Proof. Indeed, $\mathrm{hull}_\nabla(\mathrm{gr}\, A)$ is integral over $\mathrm{gr}\, A$ by Theorem 29. Then it is integral over a finitely generated subalgebra of $\mathrm{gr}\, A$, and it is a Noetherian module over that subalgebra. □

Lemma 45. *If* $\mathrm{gr}\, A$ *is finitely generated as a* $\mathbf{k}$*-algebra, then so is A.*

Proof. Say $j_1, \ldots, j_n$ are nonnegative integers and $a_i \in A_{\leq j_i}$ are such that the classes $a_i + A_{<j_i} \in \mathrm{gr}_{j_i} A$ generate $\mathrm{gr}\, A$. Then the a_i generate A. □

Lemma 46. *Suppose* $\mathbf{k}$ *is Noetherian. If A^U is a finitely generated* $\mathbf{k}$*-algebra, so is A.*

Proof. Combine Lemmas 24, 44, 45. □

Lemma 47. *Let P be a standard parabolic subgroup. Suppose* $\mathbf{k}$ *is Noetherian. Then A is a finitely generated* $\mathbf{k}$*-algebra if and only if $A^{R_u(P)}$ is one.*

Proof. Let V be the intersection of U with the semisimple part of the standard Levi subgroup of P. Then $U = VR_u(P)$ and $A^U = (A^{R_u(P)})^V$. Suppose that A is a finitely generated $\mathbf{k}$-algebra. Then $A^U = (A^{R_u(P)})^V$ is one also by Lemma 25, and so is $A^{R_u(P)}$ by Lemma 46 (applied with a different group and a different algebra).
Conversely, if $A^{R_u(P)}$ is a finitely generated $\mathbf{k}$-algebra, Lemma 25 (with that same group and algebra) implies that $A^U = (A^{R_u(P)})^V$ is finitely generated, and thus A is as well, by Lemma 46. □

Proof of Theorem 30. Combine Lemmas 47, 25, 24, 44, 45. □

Proof of Theorem 32. Let $\mathbf{k}$ be Noetherian and let A be a finitely generated $\mathbf{k}$-algebra. By Theorem 30, the $\mathbf{k}$-algebra $\mathrm{hull}_\nabla(\mathrm{gr}\, A)$ is finitely generated. So we may choose $b_1,\dots,b_s$, so that $\psi_{b_1} \otimes \cdots \otimes \psi_{b_s}$ has image $\mathrm{hull}_\nabla(\mathrm{gr}\, A)$. By extending Lemma 38 to tensor products, we can argue as in the proof of Lemma 43 and Theorem 29, and see that there is a positive integer n so that $n\,\mathrm{hull}_\nabla(\mathrm{gr}\, A) \subseteq \mathrm{gr}\, A$. Now, $\mathrm{hull}_\nabla(\mathrm{gr}\, A) \otimes \mathbf{k}[G/U]$ is acyclic by Proposition 28, and thus its summand $\mathrm{hull}_\nabla(\mathrm{gr}\, A)$ is acyclic as well. It follows that $\mathrm{H}^i(G, \mathrm{gr}\, A)$ is a quotient of $\mathrm{H}^{i-1}(G, \mathrm{hull}_\nabla(\mathrm{gr}\, A)/\,\mathrm{gr}\, A)$, which is annihilated by n. ☐

Proof of Theorem 33. Take n as in Theorem 32, and use that localization is exact. ☐

Proof of Theorem 35. It suffices to show that the composite:

$$\mathrm{gr}\, A \to \mathrm{gr}(A/pA) \to \mathrm{hull}_\nabla(\mathrm{gr}(A/pA))$$

is power-surjective. It coincides with the composite

$$\mathrm{gr}\, A \to \mathrm{hull}_\nabla(\mathrm{gr}(A)) \to \mathrm{hull}_\nabla(\mathrm{gr}(A/pA)).$$

Now $A^{U^+} \to (A/pA)^{U^+}$ is p-power-surjective by a combination of Theorem 12, Proposition 6, Proposition 41 and the transfer principle [9, Ch. Two] as used in 25. After inducing up, $\mathrm{hull}_\nabla(\mathrm{gr}(A)) \to \mathrm{hull}_\nabla(\mathrm{gr}(A/pA))$ is still p-power-surjective, indeed the same p-power is sufficient. And $\mathrm{gr}\, A \to \mathrm{hull}_\nabla(\mathrm{gr}(A))$ is power-surjective by Theorem 29. ☐

6 Finiteness properties of cohomology algebras

In this section we study finiteness properties of $\mathrm{H}^*(G, A)$, primarily when the base ring $\mathbf{k}$ is $\mathbb{Z}$. We shall always assume that the commutative algebra A is finitely generated over the ring $\mathbf{k}$, with rational action of a Chevalley group scheme G. Further, M will be a noetherian A-module with compatible G-action. Torsion will refer to torsion as an abelian group, not as an A-module. We say that V has bounded torsion if there is a positive integer that annihilates the torsion subgroup V_{tors}.

Lemma 48. *A noetherian module over a graded commutative ring has bounded torsion.* ☐

Recall that we call a homomorphism $f : R \to S$ of graded commutative algebras *noetherian* if f makes S into a noetherian left R-module. Recall that CFG refers to cohomological finite generation. The main result of this section is the following.

Theorem 49 (Provisional CFG). *Suppose* $\mathbf{k} = \mathbb{Z}$.

- *Every* $\mathrm{H}^m(G, M)$ *is a noetherian* A^G*-module.*

- *If* $\mathrm{H}^*(G,A)$ *is a finitely generated algebra, then* $\mathrm{H}^*(G,M)$ *is a noetherian* $\mathrm{H}^*(G,A)$*-module.*
- $\mathrm{H}^*(G,\operatorname{gr} A)$ *is a finitely generated algebra.*
- *If* $\mathrm{H}^*(G,A)$ *has bounded torsion, then the reduction* $\mathrm{H}^{\mathrm{even}}(G,A) \to \mathrm{H}^{\mathrm{even}}(G,A/pA)$ *is power-surjective for every prime number* p.
- *If* $\mathrm{H}^{\mathrm{even}}(G,A) \to \mathrm{H}^{\mathrm{even}}(G,A/pA)$ *is noetherian for every prime number* p*, then* $\mathrm{H}^*(G,A)$ *is a finitely generated algebra.*

Remark 50. Note that the first statement would fail badly, by [11, I 4.12], if one replaced G with the additive group scheme $\mathbb{G}_a$. This may explain why our proof is far from elementary.

We hope to show in the future that $\mathrm{H}^{\mathrm{even}}(G,A) \to \mathrm{H}^{\mathrm{even}}(G,A/pA)$ is noetherian for every prime number p. The theorem suggests to ask:

Problem. *Let* $\mathbf{k}$ *be a Noetherian ring and let* G *be a Chevalley group scheme over* $\mathbf{k}$*. Let* A*,* Q *be finitely generated commutative* $\mathbf{k}$*-algebras on which* G *acts rationally through algebra automorphisms. Let* $f : A \to Q$ *be a power-surjective equivariant homomorphism. Is* $\mathrm{H}^*(G,A) \to \mathrm{H}^*(G,Q)$ *power-surjective?*

We will need the recent theorem of Touzé [22, Thm 1.1], see also [22, Thm 1.5],

Theorem 51 (CFG over a field). *If* $\mathbf{k}$ *is a field, then* $\mathrm{H}^*(G,A)$ *is a finitely generated* $\mathbf{k}$*-algebra and* $\mathrm{H}^*(G,M)$ *is a noetherian* $\mathrm{H}^*(G,A)$*-module.*

Remark 52. If $\mathbf{k}$ is a commutative ring and V is a $G_{\mathbf{k}}$-module, then the comultiplication $V \to V \otimes_{\mathbf{k}} \mathbf{k}[G]$ gives rise to a comultiplication $V \to V \otimes_{\mathbb{Z}} \mathbb{Z}[G]$ through the identification $V \otimes_{\mathbf{k}} \mathbf{k}[G] = V \otimes_{\mathbb{Z}} \mathbb{Z}[G]$. So one may view V as a $G_{\mathbb{Z}}$-module. Further $\mathrm{H}^*(G_{\mathbf{k}},V)$ is the same as $\mathrm{H}^*(G_{\mathbb{Z}},V)$, because the Hochschild complexes are the same. So if $\mathbf{k}$ is finitely generated over a field F, then the conclusions of the (CFG) theorem still hold, because $\mathrm{H}^*(G,A) = \mathrm{H}^*(G_F,A)$. We leave it to the reader to try a limit argument to deal with the case where $\mathbf{k}$ is essentially of finite type over a field.

First let the ring $\mathbf{k}$ be noetherian. We are going to imitate arguments of Benson–Habegger [3]. We thank Dave Benson for the reference.

Lemma 53. *Let* $m > 1$*,* $n > 1$*. The reduction* $\mathrm{H}^{\mathrm{even}}(G,A/mnA) \to \mathrm{H}^{\mathrm{even}}(G,A/nA)$ *is power-surjective.*

Proof. We may assume m is prime. By the Chinese Remainder Theorem we may then also assume that n is a power of that same prime. (If n is prime to m the Lemma is clear.) Let $x \in \mathrm{H}^{\mathrm{even}}(G,A/nA)$. We show that some power x^{m^r} of x lifts. Arguing as in the proof of Proposition 41 we may assume x is homogeneous. Let I be the kernel of $A/mnA \to A/nA$. Note that m annihilates I, hence also $\mathrm{H}^*(G,I)$. Further I is an A/nA-module and the connecting homomorphism $\partial : \mathrm{H}^i(G,A/nA) \to \mathrm{H}^{i+1}(G,I)$ satisfies the Leibniz rule. So $\partial(x^m) = mx^{m-1}\partial(x) = 0$ and x^m lifts. ☐

Proposition 54. *If* $\mathrm{H}^*(G,A)$ *has bounded torsion, then* $\mathrm{H}^{\mathrm{even}}(G,A) \to \mathrm{H}^{\mathrm{even}}(G,A/pA)$ *is power-surjective for every prime number* p.

Proof. Assume $\mathrm{H}^*(G,A)$ has bounded torsion. Write $\mathrm{H}^{\mathrm{pos}}$ for $\bigoplus_{i>0} \mathrm{H}^i$. Let p be a prime number. Choose a positive multiple n of p so that $n\mathrm{H}^{\mathrm{pos}}(G,A) = 0$ and $nA_{\mathrm{tors}} = 0$. We have an exact sequence

$$\cdots \to \mathrm{H}^i(G,A_{\mathrm{tors}}) \to \mathrm{H}^i(G,A) \to \mathrm{H}^i(G,A/A_{\mathrm{tors}}) \to \cdots .$$

Multiplication by n^2 is zero on $\mathrm{H}^{\mathrm{pos}}(G,A/A_{\mathrm{tors}})$, so $\mathrm{H}^{\mathrm{pos}}(G,A/A_{\mathrm{tors}}) \hookrightarrow \mathrm{H}^{\mathrm{pos}}(G,A/n^2A + A_{\mathrm{tors}})$. We have exact sequences

$$0 \to A/A_{\mathrm{tors}} \xrightarrow{\times n^2} A \to A/n^2A \to 0$$

and

$$0 \to A/n^2A + A_{\mathrm{tors}} \xrightarrow{\times n^2} A/n^4A \to A/n^2A \to 0.$$

Consider the diagram

$$\begin{tikzcd}
\mathrm{H}^{2i}(G,A) \arrow[r] \arrow[d] & \mathrm{H}^{2i}(G,A/n^2A) \arrow[r, "\partial_1"] \arrow[d, equal] & \mathrm{H}^{2i+1}(G,A/A_{\mathrm{tors}}) \arrow[d, hook] \\
\mathrm{H}^{2i}(G,A/n^4A) \arrow[r] & \mathrm{H}^{2i}(G,A/n^2A) \arrow[r, "\partial_2"] & \mathrm{H}^{2i+1}(G,A/n^2A{+}A_{\mathrm{tors}})
\end{tikzcd}$$

If $x \in \mathrm{H}^{2j}(G,A/n^2A)$, put $i = jn^2$. The image $n^2x^{n^2-1}\partial_2(x)$ in $\mathrm{H}^{2i+1}(G,A/n^2A + A_{\mathrm{tors}})$ of x^{n^2} vanishes, hence $\partial_1(x^{n^2})$ vanishes in $\mathrm{H}^{2i+1}(G,A/A_{\mathrm{tors}})$, and x^{n^2} lifts to $\mathrm{H}^{2i}(G,A)$. As $\mathrm{H}^{\mathrm{even}}(G,A/n^2A) \to \mathrm{H}^{\mathrm{even}}(G,A/pA)$ is power-surjective by Lemma 53, we conclude that for every homogeneous $y \in \mathrm{H}^{\mathrm{even}}(G,A/pA)$ some power lifts all the way to $\mathrm{H}^{\mathrm{even}}(G,A)$. We want to show more, namely that $\mathrm{H}^{\mathrm{even}}(G,A) \to \mathrm{H}^{\mathrm{even}}(G,A/pA)$ is universally power-surjective. By Proposition 41 we need to show that the power of y may be taken of the form y^{p^r}. Localize with respect to the multiplicative system $S = (1 + p\mathbb{Z})$ in $\mathbb{Z}$. The p-primary torsion is not affected and all the other torsion disappears, so n may be taken a power of p. The proofs then produce that some y^{p^r} lifts to $\mathrm{H}^{\mathrm{even}}(G,S^{-1}A)$. Now just remove the denominator, which acts trivially on y. □

We now restrict to the case $\mathbf{k} = \mathbb{Z}$. (More generally, one could take for $\mathbf{k}$ a noetherian ring so that for every prime number p the ring $\mathbf{k}/p\mathbf{k}$ is essentially of finite type over a field.)

Proposition 55. *Suppose* $\mathbf{k} = \mathbb{Z}$. *If* $\mathrm{H}^*(G,A)$ *has bounded torsion, then* $\mathrm{H}^*(G,A)$ *is a finitely generated algebra.*

Proof. By Theorem 33 we may choose a prime number p and concentrate on the p-primary part. Say by tensoring $\mathbb{Z}$ and A with $\mathbb{Z}_{(p)}$. So now $\mathrm{H}^{\mathrm{pos}}(G,A)$

is p-primary torsion and A is a $\mathbb{Z}_{(p)}$-algebra. We know that $\mathrm{H}^{\mathrm{even}}(G, A) \to \mathrm{H}^{\mathrm{even}}(G, A/pA)$ is power-surjective. By power surjectivity and the (CFG) Theorem 51, we choose an A^G-subalgebra C of $\mathrm{H}^*(G, A)$, generated by finitely many homogeneous elements, so that $C \to \mathrm{H}^*(G, A/pA)$ is noetherian. Again by the (CFG) Theorem 51 it follows that $\mathrm{H}^*(G, A/pA) \to \mathrm{H}^*(G, A/pA + A_{\mathrm{tors}})$ is noetherian, so that $C \to \mathrm{H}^*(G, A/pA + A_{\mathrm{tors}})$ is also noetherian.
Let N be the image of $\mathrm{H}^{\mathrm{pos}}(G, A/A_{\mathrm{tors}})$ in $\mathrm{H}^{\mathrm{pos}}(G, A/pA + A_{\mathrm{tors}}))$. As a C-module, it is isomorphic to $\mathrm{H}^{\mathrm{pos}}(G, A/A_{\mathrm{tors}})/p\mathrm{H}^{\mathrm{pos}}(G, A/A_{\mathrm{tors}})$. Choose homogeneous $v_i \in \mathrm{H}^{\mathrm{pos}}(G, A/A_{\mathrm{tors}})$ so that their images generate N. Say V is the C-span of the v_i. We have $\mathrm{H}^{\mathrm{pos}}(G, A/A_{\mathrm{tors}}) + V \subseteq p\mathrm{H}^{\mathrm{pos}}(G, A/A_{\mathrm{tors}}) + V$. Iterating this we get $\mathrm{H}^{\mathrm{pos}}(G, A/A_{\mathrm{tors}}) + V \subseteq p^r\mathrm{H}^{\mathrm{pos}}(G, A/A_{\mathrm{tors}}) + V$ for any $r > 0$. But $\mathrm{H}^*(G, A)$ and $\mathrm{H}^*(G, A_{\mathrm{tors}})$ have bounded torsion, so $\mathrm{H}^*(G, A/A_{\mathrm{tors}})$ also has bounded torsion. It follows that $\mathrm{H}^{\mathrm{pos}}(G, A/A_{\mathrm{tors}}) = V$. We conclude that $\mathrm{H}^{\mathrm{pos}}(G, A/A_{\mathrm{tors}})$ is a noetherian C-module.
Now let us look at $\mathrm{H}^{\mathrm{pos}}(G, A_{\mathrm{tors}})$. Filter $A_{\mathrm{tors}} \supseteq pA_{\mathrm{tors}} \supseteq p^2A_{\mathrm{tors}} \supseteq \cdots \supseteq 0$. By the (CFG) theorem $\mathrm{H}^{\mathrm{pos}}(G, p^iA_{\mathrm{tors}}/p^{i+1}A_{\mathrm{tors}})$ is a noetherian $\mathrm{H}^*(G, A/pA)$-module, hence a noetherian C-module. So $\mathrm{H}^{\mathrm{pos}}(G, A_{\mathrm{tors}})$ is also a noetherian C-module and thus $\mathrm{H}^*(G, A)$ is one. It follows that $\mathrm{H}^*(G, A)$ is a finitely generated A^G-algebra. And A^G itself is finitely generated by Theorem 3. □

Proposition 56. *Let* $\mathbf{k} = \mathbb{Z}$. *Then* $\mathrm{H}^*(G, \mathrm{gr}\, A)$ *is a finitely generated algebra.*

Proof. By Theorem 32 the algebra $\mathrm{H}^*(G, \mathrm{gr}\, A)$ has bounded torsion, so Proposition 55 applies. □

Proposition 57. *Let* $\mathbf{k} = \mathbb{Z}$. *Then* $\mathrm{H}^m(G, M)$ *is a noetherian* A^G*-module.*

Proof. Form the 'semi-direct product ring' $A \ltimes M$ whose underlying G-module is $A \oplus M$, with product given by $(a_1, m_1)(a_2, m_2) = (a_1a_2, a_1m_2 + a_2m_1)$. It suffices to show that $\mathrm{H}^m(G, A \ltimes M)$ is a noetherian $\mathrm{H}^0(G, A \ltimes M)$-module. In other words, we may forget M and just ask if $\mathrm{H}^m(G, A)$ is a noetherian A^G-module. Now $\mathrm{H}^*(G, \mathrm{gr}\, A)$ is a finitely generated algebra and $\mathrm{H}^0(G, \mathrm{gr}\, A) = \mathrm{gr}_0\, A$, so in the spectral sequence

$$E(A) : E_1^{ij} = \mathrm{H}^{i+j}(G, \mathrm{gr}_{-i}\, A) \Rightarrow \mathrm{H}^{i+j}(G, A)$$

the $\bigoplus_{i+j=t} E_1^{ij}$ are noetherian A^G-modules for each t. So for fixed t there are only finitely many nonzero $E_1^{i,t-i}$ and the result follows. □

Proposition 58. *Let* $\mathbf{k} = \mathbb{Z}$. *If* $\mathrm{H}^{\mathrm{even}}(G, A) \to \mathrm{H}^{\mathrm{even}}(G, A/pA)$ *is noetherian for every prime number* p*, then* $\mathrm{H}^*(G, A)$ *is a finitely generated algebra.*

Proof. We argue as in the proof of Proposition 55. We may no longer know that $\mathrm{H}^*(G, A)$ has bounded torsion, but for every $m > 0$ we know that $\mathrm{H}^m(G, A/A_{\mathrm{tors}})$ is a noetherian A^G-module, hence has bounded torsion. Instead of $\mathrm{H}^{\mathrm{pos}}(G, A/A_{\mathrm{tors}}) + V \subseteq p\mathrm{H}^{\mathrm{pos}}(G, A/A_{\mathrm{tors}}) + V$, we use $\mathrm{H}^m(G, A/A_{\mathrm{tors}}) + V \subseteq p\mathrm{H}^m(G, A/A_{\mathrm{tors}}) + V$. We find that $\mathrm{H}^m(G, A/A_{\mathrm{tors}}) \subseteq V$ for all $m > 0$ and thus $\mathrm{H}^{\mathrm{pos}}(G, A/A_{\mathrm{tors}}) = V$ again. Finish as before. □

Corollary 59. *Let* $\mathbf{k} = \mathbb{Z}$. *If* $\mathrm{H}^{\mathrm{even}}(G, A) \to \mathrm{H}^{\mathrm{even}}(G, A/pA)$ *is power-surjective for every prime number* p*, then* $\mathrm{H}^*(G, A)$ *is a finitely generated algebra.*

Proposition 60. *Let* $\mathbf{k} = \mathbb{Z}$. *If* $\mathrm{H}^*(G, A)$ *is a finitely generated algebra, then* $\mathrm{H}^*(G, M)$ *is a noetherian* $\mathrm{H}^*(G, A)$*-module.*

Proof. Let $\mathrm{H}^*(G, A)$ be a finitely generated algebra. So it has bounded torsion and $\mathrm{H}^{\mathrm{even}}(G, A) \to \mathrm{H}^{\mathrm{even}}(G, A/pA)$ is power-surjective for every prime number p. We argue again as in the proof of Proposition 55.
By Theorem 33, applied to $A \ltimes M$, we may choose a prime number p and concentrate on the p-primary part, so $\mathrm{H}^{\mathrm{pos}}(G, M)$ is p-primary torsion and A is a $\mathbb{Z}_{(p)}$-algebra. Write $C = \mathrm{H}^*(G, A)$. By power surjectivity and the (CFG) Theorem 51, $C \to \mathrm{H}^*(G, A/pA)$ is noetherian. Again by the (CFG) Theorem 51 it follows that $\mathrm{H}^*(G, M/pM + M_{\mathrm{tors}})$ is a noetherian $\mathrm{H}^*(G, A/pA)$-module, hence a noetherian C-module.
Let N be the image of $\mathrm{H}^{\mathrm{pos}}(G, M/M_{\mathrm{tors}})$ in $\mathrm{H}^{\mathrm{pos}}(G, M/pM + M_{\mathrm{tors}}))$. As a C-module, it is isomorphic to $\mathrm{H}^{\mathrm{pos}}(G, M/M_{\mathrm{tors}})/p\mathrm{H}^{\mathrm{pos}}(G, M/M_{\mathrm{tors}})$. Choose homogeneous $v_i \in \mathrm{H}^{\mathrm{pos}}(G, M/M_{\mathrm{tors}})$ so that their images generate N. Say V is the C-span of the v_i. We have $\mathrm{H}^m(G, M/M_{\mathrm{tors}}) + V \subseteq p\mathrm{H}^m(G, M/M_{\mathrm{tors}}) + V$ for $m > 0$. Iterating this we get $\mathrm{H}^m(G, M/M_{\mathrm{tors}}) + V \subseteq p^r\mathrm{H}^m(G, M/M_{\mathrm{tors}}) + V$ for any $r > 0$, $m > 0$. But $\mathrm{H}^m(G, M/M_{\mathrm{tors}})$ is a noetherian A^G-module, hence has bounded torsion. It follows that $\mathrm{H}^m(G, M/M_{\mathrm{tors}}) \subseteq V$ for all $m > 0$, and $\mathrm{H}^{\mathrm{pos}}(G, M/M_{\mathrm{tors}}) = V$. So $\mathrm{H}^{\mathrm{pos}}(G, M/M_{\mathrm{tors}})$ is a noetherian C-module.
Now let us look at $\mathrm{H}^{\mathrm{pos}}(G, M_{\mathrm{tors}})$. Filter $M_{\mathrm{tors}} \supseteq pM_{\mathrm{tors}} \supseteq p^2M_{\mathrm{tors}} \supseteq \cdots \supseteq 0$. By the (CFG) theorem $\mathrm{H}^{\mathrm{pos}}(G, p^iM_{\mathrm{tors}}/p^{i+1}M_{\mathrm{tors}})$ is a noetherian $\mathrm{H}^*(G, A/pA)$-module, hence a noetherian C-module. So $\mathrm{H}^{\mathrm{pos}}(G, M_{\mathrm{tors}})$ is also a noetherian C-module and thus $\mathrm{H}^*(G, M)$ is one. $\square$

Theorem 49 has been proven.

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Vincent Franjou
Mathématiques
UFR Sciences & Techniques
BP 92208
F-44322 Nantes Cedex 3
Vincent.Lastname@univ-nantes.fr

Wilberd van der Kallen
Mathematisch Instituut
Universiteit Utrecht
P.O. Box 80.010
NL-3508 TA Utrecht
Initial.vanderKallen@uu.nl