On Equivariant Dedekind Zeta-Functions at $s = 1$

Dedicated to Professor Andrei Suslin

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Abstract. We study a refinement of an explicit conjecture of Tate concerning the values at $s = 1$ of Artin L-functions. We reinterpret this refinement in terms of Tamagawa number conjectures and then use this connection to obtain some important (and unconditional) evidence for our conjecture.

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1. Introduction and statement of main results

This article studies a refinement of a conjecture of Tate concerning the values at $s = 1$ of Artin L-functions. We recall that Tate's conjecture was originally formulated in [26, Chap. I, Conj. 8.2] as an analogue of (Tate's reformulation of) the main conjecture of Stark on the leading terms at $s = 0$ of Artin L-functions and that the precise form of the 'regulators' and 'periods' that Tate introduced in this context were natural generalisations of earlier constructions of Serre in [24].

The refinement of Tate's conjecture that we study here was formulated by the present authors in [5, Conj. 3.3] and predicts an explicit formula for the leading term at $s = 1$ of the zeta-function of a finite Galois extension of number fields L/K in terms of the Euler characteristic of a certain perfect complex of $\mathrm{Gal}(L/K)$-modules (see (3) for a statement of this formula). In comparison to Tate's conjecture, this refinement predicts not only that the quotient by Tate's regulator of the leading term at $s = 1$ of the Artin L-function of a complex character χ of $\mathrm{Gal}(L/K)$ is an algebraic number but also that as χ varies these algebraic numbers should be related by certain types of integral congruence relations. We further recall that [5, Conj. 3.3] is also known to imply the '$\Omega(1)$-Conjecture' that was formulated by Chinburg in [13].

In the sequel we write $\mathbb{Q}(1)_L$ for the motive $h^0(\operatorname{Spec} L)(1)$, considered as defined over K and endowed with the natural action of the group ring $\mathbb{Q}[\operatorname{Gal}(L/K)]$. We recall that the 'equivariant Tamagawa number conjecture' applies in particular to pairs of the form $(\mathbb{Q}(1)_L, \mathbb{Z}[\operatorname{Gal}(L/K)])$ and was formulated by Flach and the second named author in [9] as a natural refinement of the seminal 'Tamagawa number conjecture' of Bloch and Kato [3]. The main technical result of the present article is then the following

Theorem 1.1. *Let L be a finite complex Galois extension of $\mathbb{Q}$. If Leopoldt's Conjecture is valid for L, then* [5, Conj. 3.3] *is equivalent to the equivariant Tamagawa number conjecture of* [9, Conj. 4] *for the pair* $(\mathbb{Q}(1)_L, \mathbb{Z}[\operatorname{Gal}(L/\mathbb{Q})])$.

Corollary 1.2. *If Leopoldt's Conjecture is valid for every number field, then for every Galois extension of number fields L/K the conjecture* [5, Conj. 3.3] *is equivalent to the conjecture* [9, Conj. 4] *for the pair* $(\mathbb{Q}(1)_L, \mathbb{Z}[\operatorname{Gal}(L/K)])$.

These results connect the explicit leading term formula of [5, Conj. 3.3] to a range of interesting results and conjectures. For example, [9, Conj. 4(iv)] is known to be a consequence of the 'main conjecture of non-commutative Iwasawa theory' that is formulated by Fukaya and Kato in [18, Conj. 2.3.2] and also of the 'main conjecture of non-commutative Iwasawa theory for Tate motives' that is formulated by Venjakob and the second named author in [12, Conj. 7.1]. Corollary 1.2 therefore allows one to regard the study of the explicit conjecture [5, Conj. 3.3] as an attempt to provide supporting evidence for these more general conjectures. Indeed, when taken in conjunction with the philosophy described by Huber and Kings in [19, §3.3] and by Fukaya and Kato in [18, §2.3.5], Corollary 1.2 suggests that, despite its comparatively elementary nature, [5, Conj. 3.3] may well play a particularly important role in the context of the very general conjecture of Fukaya and Kato.

In addition to the above consequences, our proof of Theorem 1.1 also answers an explicit question posed by Flach and the second named author in [7] (see Remark 5.1) and combines with previous work to give new evidence in support of the conjectures made in [5] including the following unconditional results.

Corollary 1.3. *If L is abelian over $\mathbb{Q}$, and K is any subfield of L, then both* [5, Conj. 3.3] *and* [5, Conj. 4.1] *are valid for the extension L/K.*

Corollary 1.4. *There exists a natural infinite family of quaternion extensions $L/\mathbb{Q}$ with the property that, if K is any subfield of L, then both* [5, Conj. 3.3] *and* [5, Conj. 4.1] *are valid for the extension L/K.*

We recall (from [5, Prop. 4.4(i)]) that [5, Conj. 4.1] is a natural refinement of the 'main conjecture of Stark at $s = 0$'. For details of connections between [5, Conj. 3.3 and Conj. 4.1] and other interesting conjectures of Chinburg, of Gruenberg, Ritter and Weiss and of Solomon see [5, Prop. 3.6 and Prop. 4.4] and the recent thesis of Jones [20].

The main contents of this article is as follows. In §2 we recall the explicit statement of [5, Conj. 3.3] and in §3 we review (and clarify) certain constructions in étale cohomology that are made in [8]. In §4 we make a detailed analysis of the p-adic completion of the perfect complex that occurs in [5, Conj. 3.3]. In §5 we prove Theorem 1.1 and in §6 we use Theorem 1.1 to prove Corollaries 1.2, 1.3 and 1.4.

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2. The explicit leading term conjecture

In this section we quickly review [5, Conj. 3.3]. To do this it is necessary to summarise some background about K-theory and homological algebra.

2.1. K-theory. Let R be an integral domain of characteristic 0, E an extension of the field of fractions of R, and G a finite group. We denote the relative algebraic K-group associated to the ring homomorphism $R[G] \to E[G]$ by $K_0(R[G], E)$; a description of $K_0(R[G], E)$ in terms of generators and relations is given in [25, p. 215]. The group $K_0(R[G], E)$ is functorial in the pair (R, E) and also sits inside a long exact sequence of relative K-theory. In this paper we will use the homomorphisms $\partial^1_{R[G],E} : K_1(E[G]) \to K_0(R[G], E)$ and $\partial^0_{R[G],E} : K_0(R[G], E) \to K_0(R[G])$ from the latter sequence.

Let $\mathrm{Z}(E[G])^\times$ denote the multiplicative group of the centre of the finite dimensional semisimple E-algebra $E[G]$. The reduced norm induces a homomorphism $\mathrm{nr} : K_1(E[G]) \to \mathrm{Z}(E[G])^\times$ and we denote its image by $\mathrm{Z}(E[G])^{\times+}$. In this paper E will always be either $\mathbb{R}$ or $\mathbb{C}_p$ for some prime number p. In both cases the map nr is injective and hence we can use it to identify $K_1(E[G])$ and $\mathrm{Z}(E[G])^{\times+}$. In particular we will consider $\partial^1_{R[G],E}$ as a map $\mathrm{Z}(E[G])^{\times+} \to K_0(R[G], E)$. If $E = \mathbb{C}_p$ then $\mathrm{Z}(E[G])^{\times+} = \mathrm{Z}(E[G])^\times$.

For every prime p and embedding $j : \mathbb{R} \to \mathbb{C}_p$ there are induced homomorphisms $j_* : K_0(\mathbb{Z}[G], \mathbb{R}) \to K_0(\mathbb{Z}_p[G], \mathbb{C}_p)$ and $j_* : \mathrm{Z}(\mathbb{R}[G])^\times \to \mathrm{Z}(\mathbb{C}_p[G])^\times$. In [5, §2.1.2] it is shown that there exists a (unique) homomorphism $\hat{\partial}^1_G : \mathrm{Z}(\mathbb{R}[G])^\times \to K_0(\mathbb{Z}[G], \mathbb{R})$ which coincides with $\partial^1_{\mathbb{Z}[G],\mathbb{R}}$ on $\mathrm{Z}(\mathbb{R}[G])^{\times+}$ and is such that for every prime p and embedding $j : \mathbb{R} \to \mathbb{C}_p$ one has $j_* \circ \hat{\partial}^1_G = \partial^1_{\mathbb{Z}_p[G],\mathbb{C}_p} \circ j_* : \mathrm{Z}(\mathbb{R}[G])^\times \to K_0(\mathbb{Z}_p[G], \mathbb{C}_p)$.

2.2. Homological algebra. For our homological algebra constructions in this paper we use the same notations and sign conventions as in [5]. So in particular by a complex we mean a cochain complex of left R-modules for a ring R, we use the phrase 'distinguished triangle' in the sense specified in [5, §2.2.1] and by a perfect complex we mean a complex that in the derived category $\mathcal{D}(R)$ is isomorphic to a bounded complex of finitely generated projective left R-modules. The full triangulated subcategory of $\mathcal{D}(R)$ consisting of the perfect complexes will be denoted by $\mathcal{D}^{\mathrm{perf}}(R)$.

Now let R, E and G be as in §2.1. For any object C of $\mathcal{D}(R[G])$ we write $H^{\mathrm{ev}}(C)$ and $H^{\mathrm{od}}(C)$ for the direct sums $\oplus_{i\,\mathrm{even}} H^i(C)$ and $\oplus_{i\,\mathrm{odd}} H^i(C)$ where

i runs over all even and all odd integers respectively. A trivialisation t (over E) of a complex C in $\mathcal{D}^{\mathrm{perf}}(R[G])$ is an isomorphism of $E[G]$-modules of the form $t : H^{\mathrm{ev}}(C) \otimes_R E \xrightarrow{\cong} H^{\mathrm{od}}(C) \otimes_R E$. We write $\chi_{R[G],E}(C,t)$ for the Euler characteristic in $K_0(R[G],E)$ defined in [4, Definition 5.5]. To simplify notation in the sequel we write χ_G for $\chi_{\mathbb{Z}[G],\mathbb{R}}$.

We shall interpret certain complexes in the derived category in terms of Yoneda extension classes as in [8, p. 1353]. To be specific, for any complex E that is acyclic outside degrees 0 and $n \geq 1$ we associate the class in $\mathrm{Ext}_R^{n+1}(H^n(E), H^0(E))$ given by the truncated complex $E' := \tau^{\leq n}\tau^{\geq 0}E$ with the induced maps $H^0(E) \xrightarrow{\cong} H^0(E') \to (E')^0$ and $(E')^n \to H^n(E') \xrightarrow{\cong} H^n(E)$ considered as a Yoneda extension.

2.3. Notation for number fields. Let L be a number field. We write $\mathcal{O}_L$ for the ring of integers of L and $S(L)$ for the set of all places of L. For any place $w \in S(L)$ we denote the completion of L at w by L_w. For a non-archimedean place w we write $\mathcal{O}_w$ for the ring of integers of L_w, $\mathfrak{m}_w$ for the maximal ideal of $\mathcal{O}_w$ and $U_{L_w}^{(1)}$ for the group $1 + \mathfrak{m}_w$ of principal units in L_w.

If L is an extension of K and $v \in S(K)$ then $S_v(L)$ is the set of all places of L above v. We also use the notation $S_f(L)$ and $S_\infty(L)$ for the sets of all non-archimedean and archimedean places, $S_\mathbb{R}(L)$ for the set of real archimedean places and $S_\mathbb{C}(L)$ for the set of complex archimedean places.

From now on let L/K be a Galois extension of number fields with Galois group G. For w in $S(L)$ we let G_w denote the decomposition group of w. For any place v in $S(K)$ we set $L_v := \prod_{w \in S_v(L)} L_w$ and (if $v \in S_f(K)$) $\mathcal{O}_{L,v} := \prod_{w \in S_v(L)} \mathcal{O}_w$ and $\mathfrak{m}_{L,v} := \prod_{w \in S_v(L)} \mathfrak{m}_w$. Note that L_v, $\mathcal{O}_{L,v}$ and $\mathfrak{m}_{L,v}$ are G-modules in an obvious way.

Let S be a finite subset of $S(K)$. The G-stable set of places of L that lie above a place in S will also be denoted by S. This should not cause any confusion because places of K will be called v and places of L will be called w. For a finite subset S of $S(K)$ which contains all archimedean places we let $\mathcal{O}_{L,S}$ be the ring of S-integers in L. Note that $\mathcal{O}_{L,S}$ is a G-module and that if $S = S_\infty(K)$, then $\mathcal{O}_L = \mathcal{O}_{L,S}$.

2.4. The conjecture. Let L/K be a Galois extension of number fields with Galois group G. Let S be a finite subset of $S(K)$ which contains all archimedean places and all places which ramify in L/K and is such that $\mathrm{Pic}(\mathcal{O}_{L,S}) = 0$. In [5, Lemma 2.7(ii)] it is shown that the leading term $\zeta_{L/K,S}^*(1)$ at $s = 1$ of the S-truncated zeta-function of L/K belongs to $\mathrm{Z}(\mathbb{R}[G])^{\times +}$. In this subsection we recall the explicit conjectural description of $\hat{\partial}_G^1(\zeta_{L/K,S}^*(1))$ formulated in [5, Conj. 3.3].

For each $v \in S_\infty(K)$ we let $\exp : L_v \to L_v^\times$ denote the product of the (real or complex) exponential maps $L_w \to L_w^\times$ for $w \in S_v(L)$. If $v \in S_f(K)$, then for sufficiently large i the exponential map $\exp : \mathfrak{m}_{L,v}^i \to L_v^\times$ is the product of the p-adic exponential maps $\mathfrak{m}_w^i \to L_w^\times$ for $w \in S_v(L)$.

To state [5, Conj. 3.3] we need to choose certain lattices. For each $v \in S_f := S \cap S_f(K)$, with residue characteristic p, we choose a full projective $\mathbb{Z}_p[G]$-lattice $\mathcal{L}_v \subseteq \mathcal{O}_{L,v}$ which is contained in a sufficiently large power of $\mathfrak{m}_{L,v}$ to ensure that the exponential map is defined on $\mathcal{L}_v$. Let $\mathcal{L}$ be the full projective $\mathbb{Z}[G]$-sublattice of $\mathcal{O}_L$ which has p-adic completions

$$\mathcal{L} \otimes_{\mathbb{Z}} \mathbb{Z}_p = \Bigg(\prod_{v \in S_p(K) \setminus S} \mathcal{O}_{L,v} \Bigg) \times \Bigg(\prod_{v \in S_p(K) \cap S} \mathcal{L}_v \Bigg). \tag{1}$$

We set $L_S := \prod_{v \in S} L_v$ and $\mathcal{L}_S := \prod_{v \in S} \mathcal{L}_v$ (where $\mathcal{L}_v := L_v$ for each $v \in S_\infty(K)$) and we let $\exp_S$ denote the map $\mathcal{L}_S \to L_S^\times$ that is induced by the product of the respective exponential maps. We also write Δ_S for the natural diagonal embedding from $L^\times$ to $L_S^\times$.

Following the notation of [23, Chap. VIII] we write I_L for the group of idèles of L and regard $L^\times$ as embedded diagonally in I_L. The idèle class group is $C_L := I_L/L^\times$ and the S-idèle class group is $C_S(L) := I_L/(L^\times U_{L,S})$, where $U_{L,S} := \prod_{w \in S}\{1\} \times \prod_{w \notin S} \mathcal{O}_w^\times$. We remark that since $\mathrm{Pic}(\mathcal{O}_{L,S}) = 0$, the natural map $L_S^\times \to C_S(L)$ is surjective with kernel $\Delta_S(\mathcal{O}_{L,S}^\times)$. There is also a canonical invariant isomorphism $\mathrm{inv}_{L/K,S} : H^2(G, C_S(L)) \xrightarrow{\cong} \frac{1}{|G|}\mathbb{Z}/\mathbb{Z}$ and we write e_S^{glob} for the element of $\mathrm{Ext}^2_{\mathbb{Z}[G]}(\mathbb{Z}, C_S(L)) = H^2(G, C_S(L))$ that is sent by $\mathrm{inv}_{L/K,S}$ to $\frac{1}{|G|}$.

Let E_S be a complex in $\mathcal{D}(\mathbb{Z}[G])$ which corresponds (in the sense of the last paragraph of §2.2) to e_S^{glob}. Then by [5, Lemma 2.4] there is a unique morphism $\alpha_S : \mathcal{L}_S[0] \oplus \mathcal{L}[-1] \to E_S$ in $\mathcal{D}(\mathbb{Z}[G])$ for which $H^0(\alpha_S)$ is the composite $\mathcal{L}_S \xrightarrow{\exp_S} L_S^\times \to C_S(L)$ and $H^1(\alpha_S)$ is the restriction of the trace map $\mathrm{tr}_{L/\mathbb{Q}} : L \to \mathbb{Q}$ to $\mathcal{L}$. Let $E_S(\mathcal{L})$ be any complex which lies in a distinguished triangle in $\mathcal{D}(\mathbb{Z}[G])$ of the form

$$\mathcal{L}_S[0] \oplus \mathcal{L}[-1] \xrightarrow{\alpha_S} E_S \xrightarrow{\beta_S} E_S(\mathcal{L}) \xrightarrow{\gamma_S} . \tag{2}$$

To describe the cohomology of $E_S(\mathcal{L})$ we set $L_\infty := \prod_{w \in S_\infty(L)} L_w$ and write L_∞^0 for the kernel of the map $L_\infty \to \mathbb{R}$ defined by $(l_w)_{w \in S_\infty(L)} \mapsto \sum_{w \in S_\infty(L)} \mathrm{tr}_{L_w/\mathbb{R}}(l_w)$. We write $\exp_\infty$ for the product of the exponential maps $L_\infty \to L_\infty^\times$, Δ_∞ for the diagonal embedding $L^\times \to L_\infty^\times$ and $\log_\infty(\mathcal{O}_L^\times)$ for the full sublattice of L_∞^0 comprising elements x of L_∞ with $\exp_\infty(x) \in \Delta_\infty(\mathcal{O}_L^\times)$. In [5, Lemma 3.1] it is shown that $E_S(\mathcal{L})$ is a perfect complex of G-modules, that $E_S(\mathcal{L}) \otimes \mathbb{Q}$ is acyclic outside degrees -1 and 0, that $H^{-1}(\gamma_S)$ induces an identification of $H^{-1}(E_S(\mathcal{L}))$ with $\{x \in \mathcal{L}_S : \exp_S(x) \in \Delta_S(\mathcal{O}_L^\times)\}$ and that $H^0(\gamma_S)$ induces an identification of $H^0(E_S(\mathcal{L})) \otimes \mathbb{Q}$ with $\ker(\mathrm{tr}_{L/\mathbb{Q}})$. In addition, the projection $\mathcal{L}_S \to L_\infty$ induces an isomorphism of $\mathbb{Q}[G]$-modules from $\{x \in \mathcal{L}_S : \exp_S(x) \in \Delta_S(\mathcal{O}_L^\times)\} \otimes \mathbb{Q}$ to $\log_\infty(\mathcal{O}_L^\times) \otimes \mathbb{Q}$. With these identifications the isomorphism $\ker(\mathrm{tr}_{L/\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\cong} L_\infty^0 = \log_\infty(\mathcal{O}_L^\times) \otimes \mathbb{R}$ which is obtained by restricting the natural isomorphism $L \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\cong} L_\infty$ to $\ker(\mathrm{tr}_{L/\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{R}$ gives a trivialisation $\mu_L : H^0(E_S(\mathcal{L})) \otimes \mathbb{R} \xrightarrow{\cong} H^{-1}(E_S(\mathcal{L})) \otimes \mathbb{R}$ of $E_S(\mathcal{L})$. In [5, Conj.

3.3] it is conjectured that

$$\hat{\partial}^1_G(\zeta^*_{L/K,S}(1)) = -\chi_G(E_S(\mathcal{L}), \mu_L). \tag{3}$$

For a discussion of the basic properties of this conjecture see [5, §3]. In particular for a proof of the fact that this conjecture refines Tate's conjecture [26, Chap. I, Conj. 8.2] see [5, Prop. 3.6(i)].

3. Preliminaries concerning étale cohomology

To relate the conjectural equality (3) to [9, Conj. 4] we will use constructions in étale cohomology that are made in [8]. However, to do this certain aspects of the exposition in [8] require clarification and so in this section we review the relevant parts of these constructions.

We fix L/K and S as in §2.4 but for simplicity we also assume henceforth that S contains at least one non-archimedean place. For each $w \in S(L)$ we denote the algebraic closure of L in L_w by L^h_w. For $w \in S_f(L)$ we let $\mathcal{O}^h_w$ be the ring of integers in L^h_w; note that $\mathcal{O}^h_w$ is the henselization of (the localization of) $\mathcal{O}_L$ at w (compare [21, Chap. I, Exam. 4.10(a)]) and that L^h_w is the field of fractions of $\mathcal{O}^h_w$.

Similarly, for a place $v \in S(K)$ we define K^h_v as the algebraic closure of K in K_v. The inclusions $\mathcal{O}_{K,S} \subset K^h_v \subset K_v$ induce canonical maps $g^h_v : \operatorname{Spec} K^h_v \to \operatorname{Spec} \mathcal{O}_{K,S}$, $f_v : \operatorname{Spec} K_v \to \operatorname{Spec} K^h_v$ and $g_v = g^h_v \circ f_v : \operatorname{Spec} K_v \to \operatorname{Spec} \mathcal{O}_{K,S}$.

3.1. General conventions.

Let X be any scheme and $\mathcal{F}$ an étale sheaf on X, i.e. a sheaf on the étale site X_{et}. If Y is an étale X-scheme then we denote by $R\Gamma(Y, \mathcal{F})$ the complex in the derived category $\mathcal{D}(\mathbb{Z})$ which is obtained by applying the right derived functor of the section functor $\Gamma(Y, -)$ to the sheaf $\mathcal{F}$; thus $R\Gamma(Y, \mathcal{F})$ is defined up to canonical isomorphism in $\mathcal{D}(\mathbb{Z})$. If $Y = \operatorname{Spec} R$ for some commutative ring R, then we will write $R\Gamma(R, \mathcal{F})$ for $R\Gamma(\operatorname{Spec} R, \mathcal{F})$ and $H^i(R, \mathcal{F})$ for the cohomology groups $H^i(R\Gamma(R, \mathcal{F}))$.

Now let $v \in S(K)$, $w \in S_v(L)$ and let $\mathcal{F}$ be an étale sheaf on $\operatorname{Spec} K^h_v$. The G_w-action on $\operatorname{Spec} L^h_w$ induces a G_w-action on the sections $\Gamma(\operatorname{Spec} L^h_w, \mathcal{F})$ and hence the complex $R\Gamma(L^h_w, \mathcal{F})$ naturally lies in $\mathcal{D}(\mathbb{Z}[G_w])$. Similarly, if $\mathcal{F}$ is an étale sheaf on $\operatorname{Spec} \mathcal{O}_{K,S}$, then $R\Gamma(\mathcal{O}_{L,S}, \mathcal{F})$ belongs to $\mathcal{D}(\mathbb{Z}[G])$. Finally for $v \in S(K)$ and $\mathcal{F}$ an étale sheaf on $\operatorname{Spec} \mathcal{O}_{K,S}$ we can consider $\bigoplus_{w \in S_v(L)} R\Gamma(L^h_w, (g^h_v)^*\mathcal{F})$ as a complex in $\mathcal{D}(\mathbb{Z}[G])$. This is possible because there is a canonical isomorphism

$$\bigoplus_{w \in S_v(L)} R\Gamma(L^h_w, (g^h_v)^*\mathcal{F}) \cong R\Gamma\Big(\coprod_{w \in S_v(L)} \operatorname{Spec} L^h_w, (g^h_v)^*\mathcal{F}\Big),$$

and $\coprod_{w \in S_v(L)} \operatorname{Spec} L^h_w$ is a Galois covering of $\operatorname{Spec} K^h_v$ with group G. Of course the same is true with L^h_w and g^h_v replaced by L_w and g_v respectively.

3.2. Local cohomology. Let v be a place of K and $w \in S_v(L)$. Recall that $f_v : \operatorname{Spec} K_v \to \operatorname{Spec} K_v^h$ corresponds to the inclusion $K_v^h \to K_v$. For any étale sheaf $\mathcal{F}$ on $\operatorname{Spec} K_v^h$ the canonical map $R\Gamma(L_w^h, \mathcal{F}) \longrightarrow R\Gamma(L_w, f_v^*\mathcal{F})$ is an isomorphism in $\mathcal{D}(\mathbb{Z}[G_w])$. Indeed, if $\overline{L_w}$ is an algebraic closure of L_w and $\overline{L_w^h}$ is the algebraic closure of L_w^h in $\overline{L_w}$, then the restriction map gives an isomorphism $\operatorname{Gal}(\overline{L_w}/K_v) \xrightarrow{\cong} \operatorname{Gal}(\overline{L_w^h}/K_v^h)$. Thus, upon identifying étale cohomology and Galois cohomology the claimed isomorphism follows.
If $\mathcal{F} = \mathbb{G}_m$ on $(\operatorname{Spec} K_v^h)_{\mathrm{et}}$, then $f_v^*\mathbb{G}_m$ is not isomorphic to the sheaf $\mathbb{G}_m$ on $(\operatorname{Spec} K_v)_{\mathrm{et}}$. However the complexes $R\Gamma(L_w^h, \mathbb{G}_m) \cong R\Gamma(L_w, f_v^*\mathbb{G}_m)$ and $R\Gamma(L_w, \mathbb{G}_m)$ are related as follows.

Lemma 3.1. *There is a distinguished triangle in* $\mathcal{D}(\mathbb{Z}[G_w])$

$$R\Gamma(L_w^h, \mathbb{G}_m) \longrightarrow R\Gamma(L_w, \mathbb{G}_m) \longrightarrow (L_w^\times/(L_w^h)^\times)[0] \longrightarrow,$$

whose cohomology sequence in degree 0 *identifies with the canonical short exact sequence* $0 \longrightarrow (L_w^h)^\times \longrightarrow L_w^\times \longrightarrow L_w^\times/(L_w^h)^\times \longrightarrow 0$. *The* G_w*-module* $L_w^\times/(L_w^h)^\times$ *is uniquely divisible and hence cohomologically trivial.*

Proof. There is a canonical injection $f_v^*\mathbb{G}_m \to \mathbb{G}_m$ of sheaves on $(\operatorname{Spec} K_v)_{\mathrm{et}}$ such that the sequence

$$0 \longrightarrow f_v^*\mathbb{G}_m \longrightarrow \mathbb{G}_m \longrightarrow \mathbb{G}_m/f_v^*\mathbb{G}_m \longrightarrow 0$$

corresponds to the exact sequence $0 \to \overline{L_w^h}^\times \to \overline{L_w}^\times \to \overline{L_w}^\times/\overline{L_w^h}^\times \to 0$ of $\operatorname{Gal}(\overline{L_w}/K_v)$-modules. Now $\overline{L_w}^\times/\overline{L_w^h}^\times$ is uniquely divisible. Also, the isomorphism $\operatorname{Gal}(\overline{L_w}/K_v) \cong \operatorname{Gal}(\overline{L_w^h}/K_v^h)$ combines with Hilbert's Theorem 90 to imply $H^0(\operatorname{Gal}(\overline{L_w}/L_w), \overline{L_w}^\times/\overline{L_w^h}^\times) = L_w^\times/(L_w^h)^\times$ as G_w-modules. It follows that $L_w^\times/(L_w^h)^\times$ is uniquely divisible and hence cohomologically trivial (as a G_w-module). In addition, by applying $R\Gamma(L_w, -)$ to the displayed exact sequence we obtain the claimed distinguished triangle. □

Lemma 3.2. *There are canonical isomorphisms of* G_w*-modules*

$$H^i(L_w, \mathbb{G}_m) \cong \begin{cases} L_w^\times & \text{if } i = 0, \\ 0 & \text{if } i = 1, \\ \operatorname{Br}(L_w) & \text{if } i = 2. \end{cases}$$

If w is non-archimedean then $H^i(L_w, \mathbb{G}_m) = 0$ for $i \geq 3$ and the local invariant isomorphism gives a canonical identification $\operatorname{Br}(L_w) \cong \mathbb{Q}/\mathbb{Z}$. *With respect to this identification the class of* $R\Gamma(L_w, \mathbb{G}_m)$ *in* $\operatorname{Ext}^3_{\mathbb{Z}[G_w]}(\mathbb{Q}/\mathbb{Z}, L_w^\times) \cong H^2(G_w, L_w^\times)$ *is the local canonical class.*

Proof. This is [8, Prop. 3.5.(a)]. □

3.3. Cohomology with compact support. For any étale sheaf $\mathcal{F}$ on $\operatorname{Spec} \mathcal{O}_{K,S}$ we define the complex $R\Gamma_c(\mathcal{O}_{L,S}, \mathcal{F})$ in $\mathcal{D}(\mathbb{Z}[G])$ by

$$R\Gamma_c(\mathcal{O}_{L,S}, \mathcal{F}) := \operatorname{cone}\left(R\Gamma(\mathcal{O}_{L,S}, \mathcal{F}) \to \bigoplus_{w \in S} R\Gamma(L_w^h, (g_{v(w)}^h)^*\mathcal{F})\right)[-1], \tag{4}$$

where, for every $w \in S$, $v(w)$ denotes the place of K below w. Thus this complex lies in a distinguished triangle

$$R\Gamma_c(\mathcal{O}_{L,S}, \mathcal{F}) \longrightarrow R\Gamma(\mathcal{O}_{L,S}, \mathcal{F}) \longrightarrow \bigoplus_{w \in S} R\Gamma(L^h_w, (g^h_{v(w)})^*\mathcal{F}) \longrightarrow . \tag{5}$$

In [8, (3)] a complex $R\Gamma_c(\mathcal{O}_{L,S}, \mathcal{F})$ is defined just as in (4) but with L^h_w and $g^h_{v(w)}$ replaced by L_w and $g_{v(w)}$ respectively. However, the observation made at the beginning of §3.2 ensures that this definition coincides with that given above.

3.3.1. *The complex $R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{G}_m)$.* We define a G-module $C^h_S(L)$ in the same way as $C_S(L)$ is defined in §2.4 but with L_w replaced by L^h_w for each $w \in S(L)$ and $\mathcal{O}_w$ replaced by $\mathcal{O}^h_w$ for each $w \in S_f(L)$. Then, since we assume $\mathrm{Pic}(\mathcal{O}_{L,S}) = 0$, the natural map $\prod_{w \in S}(L^h_w)^\times \to C^h_S(L)$ is surjective with kernel $\mathcal{O}^\times_{L,S}$.

LEMMA 3.3. *There are canonical isomorphisms of G-modules*

$$H^i(R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{G}_m)) \cong \begin{cases} C^h_S(L) & \text{if } i = 1, \\ \mathbb{Q}/\mathbb{Z} & \text{if } i = 3, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We first note that there are canonical isomorphisms of G-modules

$$H^i(\mathcal{O}_{L,S}, \mathbb{G}_m) \cong \begin{cases} \mathcal{O}^\times_{L,S} & \text{if } i = 0, \\ 0 & \text{if } i = 1, \\ \ker\left(\mathrm{Br}(L) \to \bigoplus_{w \notin S} \mathrm{Br}(L_w)\right) & \text{if } i = 2, \\ \bigoplus_{w \in S_{\mathbb{R}}(L)} H^i(L_w, \mathbb{G}_m) & \text{if } i \geq 3, \end{cases}$$

(cf. [22, Chap. II, Prop. 2.1, Rem. 2.2] and recall that $\mathrm{Pic}(\mathcal{O}_{L,S}) = 0$ and $S_f \neq \emptyset$). Now, for every $w \in S$ one has $(g^h_{v(w)})^*\mathbb{G}_m = \mathbb{G}_m$ on $(\mathrm{Spec}\, K^h_{v(w)})_{\mathrm{et}}$ because $K^h_{v(w)}$ is an algebraic extension of K. The cohomology sequence of the distinguished triangle (5) with $\mathcal{F} = \mathbb{G}_m$ thus combines with Lemmas 3.1 and 3.2 and the above displayed isomorphisms to give exact sequences

$$0 \to H^0(R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{G}_m)) \to \mathcal{O}^\times_{L,S} \to \bigoplus_{w \in S}(L^h_w)^\times \\ \to H^1(R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{G}_m)) \to 0$$

and

$$0 \to H^2(R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{G}_m)) \to \ker\left(\mathrm{Br}(L) \to \bigoplus_{w \notin S} \mathrm{Br}(L_w)\right) \\ \to \bigoplus_{w \in S} \mathrm{Br}(L_w) \to H^3(R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{G}_m)) \to 0$$

and an equality $H^i(R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{G}_m)) = 0$ for each $i \geq 4$. All maps here are the canonical ones, thus for $i = 0$ and $i = 1$ the claimed description follows immediately and for $i = 2$ and $i = 3$ it follows by using the canonical exact sequence $0 \to \mathrm{Br}(L) \to \bigoplus_{w \in S(L)} \mathrm{Br}(L_w) \to \mathbb{Q}/\mathbb{Z} \to 0$. $\square$

3.3.2. *The complex* $\widehat{R\Gamma}_c(\mathcal{O}_{L,S}, \mathbb{G}_m)$. Recall that for every $w \in S$ there is a canonical map $g_{v(w)} : \operatorname{Spec} K_{v(w)} \to \operatorname{Spec} \mathcal{O}_{K,S}$ of schemes and an inclusion $g_{v(w)}^* \mathbb{G}_m \to \mathbb{G}_m$ of étale sheaves on $\operatorname{Spec} K_{v(w)}$. Thus we can consider the composite morphism

$$R\Gamma(\mathcal{O}_{L,S}, \mathbb{G}_m) \longrightarrow \bigoplus_{w \in S} R\Gamma(L_w, g_{v(w)}^* \mathbb{G}_m) \longrightarrow \bigoplus_{w \in S} R\Gamma(L_w, \mathbb{G}_m)$$

in $\mathcal{D}(\mathbb{Z}[G])$. We then define the complex $\widehat{R\Gamma}_c(\mathcal{O}_{L,S}, \mathbb{G}_m)$ by setting

$$\widehat{R\Gamma}_c(\mathcal{O}_{L,S}, \mathbb{G}_m) := \operatorname{cone}\left(R\Gamma(\mathcal{O}_{L,S}, \mathbb{G}_m) \longrightarrow \bigoplus_{w \in S} R\Gamma(L_w, \mathbb{G}_m) \right)[-1].$$

Lemma 3.4. *There are canonical isomorphisms of G-modules*

$$H^i(\widehat{R\Gamma}_c(\mathcal{O}_{L,S}, \mathbb{G}_m)) \cong \begin{cases} C_S(L) & \text{if } i = 1, \\ \mathbb{Q}/\mathbb{Z} & \text{if } i = 3, \\ 0 & \text{otherwise.} \end{cases}$$

The class of $\widehat{R\Gamma}_c(\mathcal{O}_{L,S}, \mathbb{G}_m)[1]$ in $\operatorname{Ext}^3_{\mathbb{Z}[G]}(\mathbb{Q}/\mathbb{Z}, C_S(L)) \cong H^2(G, C_S(L))$ is the global canonical class.

Proof. The computation of the cohomology is similar to the proof of Lemma 3.3, except that the role of (5) is now played by the distinguished triangle

$$\widehat{R\Gamma}_c(\mathcal{O}_{L,S}, \mathbb{G}_m) \longrightarrow R\Gamma(\mathcal{O}_{L,S}, \mathbb{G}_m) \longrightarrow \bigoplus_{w \in S} R\Gamma(L_w, \mathbb{G}_m) \longrightarrow \tag{6}$$

that is induced by the definition of $\widehat{R\Gamma}_c(\mathcal{O}_{L,S}, \mathbb{G}_m)$. In degree 1 we also use the fact that, since $\operatorname{Pic}(\mathcal{O}_{L,S}) = 0$, $C_S(L)$ is canonically isomorphic to the cokernel of the diagonal embedding $\mathcal{O}_{L,S}^\times \to \prod_{w \in S} L_w^\times$. For the extension class see [8, Prop. 3.5(b)] (but note that the result and proof in [8] apply to $\widehat{R\Gamma}_c(\mathcal{O}_{L,S}, \mathbb{G}_m)$ rather than to $R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{G}_m)$ as incorrectly stated in loc. cit.). □

Lemma 3.5. *There is a distinguished triangle in $\mathcal{D}(\mathbb{Z}[G])$*

$$R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{G}_m) \longrightarrow \widehat{R\Gamma}_c(\mathcal{O}_{L,S}, \mathbb{G}_m) \longrightarrow \bigoplus_{w \in S} (L_w^\times / (L_w^h)^\times)[-1] \longrightarrow$$

which on cohomology in degree 1 induces the canonical exact sequence

$$0 \to C_S^h(L) \to C_S(L) \to \prod_{w \in S} L_w^\times / (L_w^h)^\times \to 0$$

and on cohomology in degree 3 induces the identity map $\mathbb{Q}/\mathbb{Z} \xrightarrow{=} \mathbb{Q}/\mathbb{Z}$.

Proof. This follows upon combining the distinguished triangle in Lemma 3.1 for each $w \in S$ with the distinguished triangle (5) with $\mathcal{F} = \mathbb{G}_m$ and the distinguished triangle (6). □

4. Pro-p-completion

Let L/K be a Galois extension of number fields, $G = \mathrm{Gal}(L/K)$, and S a set of places of K as in §2.4. We will assume throughout this section that L is totally complex. We fix a prime number p and also assume henceforth that S contains all places of residue characteristic p. As in §2.4 we choose lattices $\mathcal{L}_v$ for $v \in S_f$ and define $\mathcal{L}$ by (1). We fix an algebraic closure $\overline{K}$ of K containing L and write K_S for the maximal extension of K inside $\overline{K}$ which is unramified outside S. For each natural number n we write μ_{p^n} for the group of p^n-th roots of unity in $\overline{K}$ and let $\mathbb{Z}_p(1)$ denote the continuous $\mathrm{Gal}(K_S/K)$-module $\varprojlim_n \mu_{p^n}$ where the limit is taken with respect to p-th power maps. In this section we relate $E_S(\mathcal{L}) \otimes \mathbb{Z}_p$ to the explicit complex $R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(1))$ that is defined in [9, p. 522]. For convenience we often abbreviate $R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(1))$ to $R\Gamma_c(\mathbb{Z}_p(1))$.

For any abelian group A and natural number m we write $A_{[m]}$ for the kernel of the endomorphism given by multiplication by m. For each natural number n we consider the $\mathbb{Z}/p^n[G]$-module $\prod_{w \in S_\infty(L)} (L_w^\times)_{[p^n]} \subset L_\infty^\times$. We then define a $\mathbb{Z}_p[G]$-module by setting $L(1)_p := \varprojlim_n \left(\prod_{w \in S_\infty(L)} (L_w^\times)_{[p^n]}\right)$ where the transition morphisms are the p-th power maps. We set $L_p := \prod_{w \in S_p(L)} L_w$ and note that $\mathcal{L}_p := \prod_{v \in S_p(K)} \mathcal{L}_v$ is a full projective $\mathbb{Z}_p[G]$-sublattice of L_p. We write λ_p for the natural localization map $\mathcal{O}_L^\times \otimes \mathbb{Z}_p \to \prod_{w \in S_p(L)} U_{L_w}^{(1)}$. Recall that Leopoldt's Conjecture for the field L and prime number p is the statement that λ_p is injective. With these notations we can now describe the cohomology of the complex $R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Lemma 4.1. *If λ_p is injective (as predicted by Leopoldt's Conjecture for the field L and prime p), then there are canonical isomorphisms*

$$H^i(R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(1))) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong \begin{cases} L(1)_p \otimes_{\mathbb{Z}_p} \mathbb{Q}_p & \text{if } i = 1, \\ \mathrm{cok}(\lambda_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p & \text{if } i = 2, \\ \mathbb{Q}_p & \text{if } i = 3, \\ 0 & \text{otherwise.} \end{cases}$$

Before proving Lemma 4.1 we first state the main result of this section and introduce some further notation.

Proposition 4.2. *There is a distinguished triangle in $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}_p[G])$ of the form*

$$\mathcal{L}_p[0] \oplus \mathcal{L}_p[-1] \longrightarrow R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(1))[2] \longrightarrow E_S(\mathcal{L}) \otimes \mathbb{Z}_p \longrightarrow . \tag{7}$$

Now assume that λ_p is injective (as predicted by Leopoldt's Conjecture for the field L and prime p). With respect to the isomorphisms in Lemma 4.1 and the description of the cohomology groups $H^i(E_S(\mathcal{L})) \otimes \mathbb{Q}$ given in §2.4, the image *under* $- \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ *of the cohomology sequence of (7) is equal to*

$$
\begin{aligned}
&0 \longrightarrow L(1)_p \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\theta_1} H^{-1}(E_S(\mathcal{L})) \otimes \mathbb{Q}_p \\
&\xrightarrow{\theta_2} L_p \xrightarrow{\exp_p} \mathrm{cok}(\lambda_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{0} H^0(E_S(\mathcal{L})) \otimes \mathbb{Q}_p \\
&\xrightarrow{\subset} L_p \xrightarrow{\mathrm{tr}_{L_p/\mathbb{Q}_p}} \mathbb{Q}_p \longrightarrow 0
\end{aligned} \tag{8}
$$

where θ_1 *sends an element* $(r_w \cdot \{\exp(2\pi\sqrt{-1}/p^n)\}_{n\geq 0})_{w\in S_\infty(L)}$ *of* $L(1)_p \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ *to the element* $(r_w \cdot 2\pi\sqrt{-1})_{w\in S_\infty(L)}$ *of* $\ker(\exp_\infty) \otimes \mathbb{Q}_p \subset H^{-1}(E_S(\mathcal{L})) \otimes \mathbb{Q}_p$ *and* θ_2 *is induced by the projection* $L_S \to L_p$.

In the proofs of Lemma 4.1 and Proposition 4.2 we will need the complex $R\Gamma_c(\mu_{p^n}) := R\Gamma_c(\mathcal{O}_{L,S}, \mu_{p^n})$ for each natural number n. This complex can be considered in two different ways. On the one hand, since μ_{p^n} is a continuous $\mathrm{Gal}(K_S/K)$-module, we can consider $R\Gamma_c(\mu_{p^n})$ as the concrete complex of $\mathbb{Z}/p^n[G]$-modules that is constructed using continuous cochains in [9, p. 522]. On the other hand, there is a natural étale sheaf μ_{p^n} on $\mathrm{Spec}\, \mathcal{O}_{K,S}$ and we can consider the cohomology with compact support as defined in §3.3. However this will not cause any confusion because it can be shown that these two possible definitions of $R\Gamma_c(\mu_{p^n})$ agree (up to canonical isomorphism), and whenever it is necessary to distinguish between these two constructions of $R\Gamma_c(\mu_{p^n})$ we will emphasize which one we are using.

Proof of Lemma 4.1. Recall that the complex $R\Gamma_c(\mathbb{Z}_p(1))$ defined in [9, p. 522] is equal to $\varprojlim_n R\Gamma_c(\mu_{p^n})$, where $R\Gamma_c(\mu_{p^n})$ denotes the complex constructed using continuous cochains and the transition morphisms are induced by the p-th power map $\mu_{p^{n+1}} \to \mu_{p^n}$. From the exact sequence $0 \to \mu_{p^n} \to \mathbb{G}_m \xrightarrow{p^n} \mathbb{G}_m \to 0$ of étale sheaves on $\mathrm{Spec}\, \mathcal{O}_{K,S}$ we obtain the distinguished triangle

$$
R\Gamma_c(\mu_{p^n}) \xrightarrow{\theta} R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{G}_m) \xrightarrow{p^n} R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{G}_m) \longrightarrow \tag{9}
$$

in $\mathcal{D}(\mathbb{Z}[G])$. To compute the modules $H^i(R\Gamma_c(\mathbb{Z}_p(1)))$ explicitly we combine the cohomology sequence of (9) with the identifications of Lemma 3.3 and then pass to the inverse limit over n. In particular, since each module $L_w^\times/(L_w^h)^\times$ is uniquely divisible (by Lemma 3.1), one obtains in this way canonical isomorphisms

$$
H^i(R\Gamma_c(\mathbb{Z}_p(1))) \cong \begin{cases} \varprojlim_n C_S(L)_{[p^n]} & \text{if } i = 1, \\ \varprojlim_n C_S(L)/p^n & \text{if } i = 2, \\ \mathbb{Z}_p & \text{if } i = 3, \\ 0 & \text{otherwise.} \end{cases} \tag{10}
$$

To describe this cohomology more explicitly we use the natural exact sequence of finite G-modules

$$(11)\quad 0 \to (\mathcal{O}_{L,S}^\times)_{[p^n]} \xrightarrow{\Delta_S} \prod_{w\in S} (L_w^\times)_{[p^n]} \to C_S(L)_{[p^n]} \\ \to \mathcal{O}_{L,S}^\times/p^n \xrightarrow{\Delta_S/p^n} \prod_{w\in S} L_w^\times/p^n \to C_S(L)/p^n \to 0.$$

For each place (resp. finite place) w of L we write $L_w^\times\hat{\otimes}\mathbb{Z}_p$ (resp. $\mathcal{O}_{L_w}^\times\hat{\otimes}\mathbb{Z}_p$) for the pro-$p$-completion of $L_w^\times$ (resp. $\mathcal{O}_{L_w}^\times$). Note that $\mathcal{O}_{L_w}^\times\hat{\otimes}\mathbb{Z}_p \cong U_{L_w}^{(1)}$ if $w \in S_p(L)$, and that $\mathcal{O}_{L_w}^\times\hat{\otimes}\mathbb{Z}_p$ is finite if $w \in S_f(L) \setminus S_p(L)$. Hence from the commutative diagram

$$\begin{tikzcd} \mathcal{O}_L^\times \otimes \mathbb{Z}_p \arrow[r] \arrow[d, "\subset"] & \prod_{w\in S_f} \mathcal{O}_{L_w}^\times\hat{\otimes}\mathbb{Z}_p \arrow[d, "\subset"] \\ \mathcal{O}_{L,S}^\times \otimes \mathbb{Z}_p \arrow[r, "\varprojlim_n \Delta_S/p^n"] & \prod_{w\in S} L_w^\times\hat{\otimes}\mathbb{Z}_p \end{tikzcd}$$

we can deduce that the map $\varprojlim_n \Delta_S/p^n$ is injective (since $\lambda_p : \mathcal{O}_L^\times \otimes \mathbb{Z}_p \to \prod_{w\in S_p(L)} U_{L_w}^{(1)}$ is injective by assumption), and that $\mathrm{cok}\big(\varprojlim_n \Delta_S/p^n\big)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p = \mathrm{cok}(\lambda_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Now the limit $\varprojlim_n(\mathcal{O}_{L,S}^\times)_{[p^n]}$ vanishes and one has $\varprojlim_n \prod_{w\in S}(L_w^\times)_{[p^n]} = \varprojlim_n \prod_{w\in S_\infty(L)}(L_w^\times)_{[p^n]} = L(1)_p$. By passing to the inverse limit over n the sequence (11) thus induces identifications $\varprojlim_n C_S(L)_{[p^n]} = L(1)_p$ and $\varprojlim_n C_S(L)/p^n = \mathrm{cok}\big(\varprojlim_n \Delta_S/p^n\big)$. The explicit description of $H^i(R\Gamma_c(\mathbb{Z}_p(1)))\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ given in Lemma 4.1 therefore follows from (10) and the identification $\mathrm{cok}\big(\varprojlim_n \Delta_S/p^n\big) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \mathrm{cok}(\lambda_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ described above. □

The proof of Proposition 4.2 will occupy the rest of this section. As the first step in this proof we introduce a useful auxiliary complex.

LEMMA 4.3. *There exists a complex Q in $\mathcal{D}(\mathbb{Z}[G])$ which corresponds (in the sense of the third paragraph of §2.2) to the extension class e_S^{glob} and also possesses all of the following properties.*

(i) *Q is a complex of $\mathbb{Z}$-torsion-free G-modules of the form $Q^{-1} \to Q^0 \to Q^1$ (where the first term is placed in degree -1).*

(ii) *The morphism α_S used in the distinguished triangle (2) is represented by a morphism of complexes of G-modules $\alpha : \mathcal{L}_S[0] \oplus \mathcal{L}[-1] \to Q$.*

(iii) *For each natural number n the complex Q/p^n consists of finite projective $\mathbb{Z}/p^n[G]$-modules.*

Proof. At the outset we fix a representative of e_S^{glob} of the form $A \xrightarrow{\delta} B$ as in [5, Rem. 3.2] with B a finitely generated projective $\mathbb{Z}[G]$-module. We write d^{-1} for the composite of $\exp_S : \mathcal{L}_S \to C_S(L)$ and the inclusion $C_S(L) \subset A$. Since $\mathrm{cok}(\exp_S)$ is finite we may choose a finitely generated free $\mathbb{Z}[G]$-module F and a homomorphism $\pi : F \to A$ such that the morphism $(d^{-1}, \pi) : \mathcal{L}_S \oplus F \to A$ is surjective. We take Q to be the complex $\ker((d^{-1}, \pi)) \xrightarrow{\subseteq} \mathcal{L}_S \oplus F \xrightarrow{\delta\circ(d^{-1},\pi)} B$ where the first term is placed in degree -1. Then (d^{-1}, π) restricts to give a surjection $\ker(\delta \circ (d^{-1}, \pi)) \to C_S(L)$ which induces an identification of $H^0(Q)$ with $C_S(L)$. Via this identification, the morphism from Q to $A \to B$ that is equal to (d^{-1}, π) in degree 0 and to the identity map in degree 1 induces the identity map on cohomology in each degree and so Q represents e_S^{glob}. Further, we obtain a morphism α as in claim (ii) by defining α^0 to be the inclusion $\mathcal{L}_S \subset \mathcal{L}_S \oplus F$ and α^1 to be any lift $\mathcal{L} \xrightarrow{\mathrm{tr}'} B$ of $\mathcal{L} \xrightarrow{\mathrm{tr}} \mathbb{Z}$ through the given surjection $B \to \mathbb{Z}$.

It is easy to see that $(\mathcal{L}_S \oplus F)/p^n$ and B/p^n are finite and projective as $\mathbb{Z}/p^n[G]$-modules. So to prove claim (iii) it remains to show that $\ker((d^{-1}, \pi))/p^n$ is a finite projective $\mathbb{Z}/p^n[G]$-module. The proof of [5, Lemma 3.1] shows that $\ker(\mathcal{L}_S \to C_S(L))$ is finitely generated, from which we can deduce that $\ker((d^{-1}, \pi))$ is finitely generated. Since furthermore $\ker((d^{-1}, \pi))$ is $\mathbb{Z}$-torsion-free, it follows that $\ker((d^{-1}, \pi))$ is in fact $\mathbb{Z}$-free. But the exact sequence $0 \to \ker((d^{-1}, \pi)) \to \mathcal{L}_S \oplus F \to A \to 0$ implies that the G-module $\ker((d^{-1}, \pi))$ is cohomologically trivial, and any cohomologically trivial $\mathbb{Z}$-free $\mathbb{Z}[G]$-module is a projective $\mathbb{Z}[G]$-module. From this it immediately follows that $\ker((d^{-1}, \pi))/p^n$ is finite and projective as $\mathbb{Z}/p^n[G]$-module, as required. $\square$

We now fix a complex Q as in Lemma 4.3, and set $Q_{\lim} := \varprojlim_n Q/p^n$ where the inverse limit is taken with respect to the natural transition morphisms. To compute the cohomology $H^i(Q_{\lim}) = \varprojlim_n H^i(Q/p^n)$ we use the short exact sequence $0 \to Q \xrightarrow{p^n} Q \to Q/p^n \to 0$ together with the identifications $H^0(Q) = C_S(L)$ and $H^1(Q) = \mathbb{Z}$ to compute the cohomology of Q/p^n and then pass to the inverse limit over n. We find that (similar to the proof of Lemma 4.1) $H^{-1}(Q_{\lim}) = \varprojlim_n C_S(L)_{[p^n]}$, $H^0(Q_{\lim}) = \varprojlim_n C_S(L)/p^n$, $H^1(Q_{\lim}) = \mathbb{Z}_p$, and $H^i(Q_{\lim}) = 0$ otherwise. Hence, if we assume that Leopoldt's Conjecture is valid for L at the prime p and use the same identifications as in the proof of Lemma 4.1, then we obtain isomorphisms

$$H^i(Q_{\lim}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong \begin{cases} L(1)_p \otimes_{\mathbb{Z}_p} \mathbb{Q}_p & \text{if } i = -1, \\ \mathrm{cok}(\lambda_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p & \text{if } i = 0, \\ \mathbb{Q}_p & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

Lemma 4.4. *There exists an isomorphism $Q_{\lim} \cong R\Gamma_c(\mathbb{Z}_p(1))[2]$ in $\mathcal{D}(\mathbb{Z}_p[G])$. Further, if Leopoldt's Conjecture is valid for L at the prime p and we use the isomorphisms in Lemma 4.1 and (12) to identify the cohomology groups of $R\Gamma_c(\mathbb{Z}_p(1))[2] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $Q_{\lim} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ respectively, then this isomorphism induces the identity map in each degree of cohomology after tensoring with $\mathbb{Q}_p$.*

Proof. Applying $R\Gamma_c$ to the short exact sequence $0 \to \mu_{p^n} \to \mathbb{G}_m \xrightarrow{p^n} \mathbb{G}_m \to 0$ and combining the resulting distinguished triangle with the triangle of Lemma 3.5 and the fact that each module $L_w^\times/(L_w^h)^\times$ is uniquely divisible (by Lemma 3.1) one obtains the following commutative diagram of distinguished triangles

$$
\begin{tikzcd}
R\Gamma_c(\mu_{p^n}) \arrow[r] \arrow[d, equal] & R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{G}_m) \arrow[r, "p^n"] \arrow[d] & R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{G}_m) \arrow[r] \arrow[d] & {} \\
R\Gamma_c(\mu_{p^n}) \arrow[r] & \widehat{R\Gamma}_c(\mathcal{O}_{L,S}, \mathbb{G}_m) \arrow[r, "p^n"] & \widehat{R\Gamma}_c(\mathcal{O}_{L,S}, \mathbb{G}_m) \arrow[r] & {}
\end{tikzcd}
\tag{13}
$$

Rotating the lower row of (13) (without changing the signs of the maps) gives the distinguished triangle

$$\widehat{R\Gamma}_c(\mathcal{O}_{L,S}, \mathbb{G}_m)[1] \xrightarrow{p^n} \widehat{R\Gamma}_c(\mathcal{O}_{L,S}, \mathbb{G}_m)[1] \xrightarrow{\varrho_n'} R\Gamma_c(\mu_{p^n})[2] \to .$$

It is not difficult to see that one obtains the same identifications for $H^i(R\Gamma_c(\mu_{p^n}))$ (and hence also for $H^i(R\Gamma_c(\mathbb{Z}_p(1))) = \varprojlim_n H^i(R\Gamma_c(\mu_{p^n})))$ if one computes the cohomology using this distinguished triangle instead of the first row of (13).

Let $\widehat{Q}$ denote the complex

$$Q^{-1} \to Q^0 \to Q^1 \to \mathbb{Q}$$

where Q^{-1} is placed in degree -1, the first two arrows are the differentials of Q and the third is the natural map $Q^1 \to H^1(Q) = \mathbb{Z} \subset \mathbb{Q}$. Associated to the natural short exact sequence $0 \to \widehat{Q} \xrightarrow{p^n} \widehat{Q} \to Q/p^n \to 0$ is a distinguished triangle

$$\widehat{Q} \xrightarrow{p^n} \widehat{Q} \xrightarrow{\varrho_n} Q/p^n \to .$$

It is easy to see that one obtains the same identifications for $H^i(Q/p^n)$ (and hence also for $H^i(Q_{\lim}) = \varprojlim_n H^i(Q/p^n))$ if one computes the cohomology using this distinguished triangle instead of the short exact sequence $0 \to Q \xrightarrow{p^n} Q \to Q/p^n \to 0$.

The second assertion of Lemma 3.4 combines with the fact that Q corresponds to e_S^{glob} to imply the existence of an isomorphism $\xi : \widehat{Q} \cong \widehat{R\Gamma}_c(\mathcal{O}_{L,S}, \mathbb{G}_m)[1]$ in $\mathcal{D}(\mathbb{Z}[G])$ which induces the identity map on each degree of cohomology.

We now consider the following diagram in $\mathcal{D}(\mathbb{Z}[G])$

$$
\begin{tikzcd}
\widehat{Q} \arrow[r, "p^n"] \arrow[d, "\xi"] & \widehat{Q} \arrow[r, "\varrho_n"] \arrow[d, "\xi"] & Q/p^n \arrow[r] & {} \\
\widehat{R\Gamma}_c(\mathcal{O}_{L,S}, \mathbb{G}_m)[1] \arrow[r, "p^n"] & \widehat{R\Gamma}_c(\mathcal{O}_{L,S}, \mathbb{G}_m)[1] \arrow[r, "\varrho_n'"] & R\Gamma_c(\mu_{p^n})[2] \arrow[r] & {}
\end{tikzcd}
\tag{14}
$$

Since the left hand square of (14) commutes there exists an isomorphism

$$\xi_n : Q/p^n \to R\Gamma_c(\mu_{p^n})[2]$$

in $\mathcal{D}(\mathbb{Z}[G])$ that makes the diagram into an isomorphism of distinguished triangles. In fact the isomorphisms ξ_n can be chosen to be compatible with the inverse systems over n, i.e. such that for every n the square

$$\begin{tikzcd} Q/p^n \arrow[r, "\xi_n"] \arrow[d] & R\Gamma_c(\mu_{p^n})[2] \arrow[d] \\ Q/p^{n-1} \arrow[r, "\xi_{n-1}"] & R\Gamma_c(\mu_{p^{n-1}})[2] \end{tikzcd}$$

commutes in $\mathcal{D}(\mathbb{Z}[G])$. This can be seen for example as follows: if we compute $\widehat{R\Gamma}_c(\mathcal{O}_{L,S}, \mathbb{G}_m)$ and $R\Gamma_c(\mu_{p^n})$ using the concrete realisation of all chain complexes given by the Godement resolution of the sheaves (as described, for example, in [21, Chap. III, Rem. 1.20(c)]), then we obtain a short exact sequence

$$0 \to R\Gamma_c(\mu_{p^n}) \to \widehat{R\Gamma}_c(\mathcal{O}_{L,S}, \mathbb{G}_m) \xrightarrow{p^n} \widehat{R\Gamma}_c(\mathcal{O}_{L,S}, \mathbb{G}_m) \to 0.$$

Then both the top and the bottom row of (14) are canonically isomorphic to the distinguished triangles coming from short exact sequences (i.e. the distinguished triangles which are constructed using mapping cones), and for such distinguished triangles the statement is easy to see.

To be able to pass to the inverse limit we must replace the maps ξ_n in $\mathcal{D}(\mathbb{Z}[G])$ by actual maps of complexes. Since both Q/p^n and $R\Gamma_c(\mu_{p^n})[2]$ are cohomologically bounded complexes of $\mathbb{Z}/p^n[G]$-modules, the natural restriction of scalars homomorphism

$$\text{(15)} \qquad \mathrm{Hom}_{\mathcal{D}(\mathbb{Z}_p[G])}(Q/p^n, R\Gamma_c(\mu_{p^n})[2]) \to \mathrm{Hom}_{\mathcal{D}(\mathbb{Z}[G])}(Q/p^n, R\Gamma_c(\mu_{p^n})[2])$$

is bijective (cf. [8, Lemma 17]). Thus for each n the map $\xi_n : Q/p^n \to R\Gamma_c(\mu_{p^n})[2]$ can be represented as $Q/p^n \xleftarrow{\sim} T_n \xrightarrow{\sim} R\Gamma_c(\mu_{p^n})[2]$ where T_n is a complex of $\mathbb{Z}_p[G]$-modules and $Q/p^n \xleftarrow{\sim} T_n$ and $T_n \xrightarrow{\sim} R\Gamma_c(\mu_{p^n})[2]$ are quasi-isomorphisms of complexes of $\mathbb{Z}_p[G]$-modules. By choosing a projective resolution we can assume that T_n is a bounded above complex of projective $\mathbb{Z}_p[G]$-modules. There exists a morphism $T_n \to T_{n-1}$ in $\mathcal{D}(\mathbb{Z}_p[G])$ such that the diagram

$$\begin{tikzcd} Q/p^n \arrow[d] & T_n \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, dashed] & R\Gamma_c(\mu_{p^n})[2] \arrow[d] \\ Q/p^{n-1} & T_{n-1} \arrow[l, "\sim"'] \arrow[r, "\sim"] & R\Gamma_c(\mu_{p^{n-1}})[2] \end{tikzcd}$$

commutes in $\mathcal{D}(\mathbb{Z}_p[G])$. Since T_n is a bounded above complex of projective $\mathbb{Z}_p[G]$-modules, the morphism $T_n \to T_{n-1}$ in $\mathcal{D}(\mathbb{Z}_p[G])$ can be realised by an actual map of complexes, and the above diagram will commute up to homotopy. The same argument as in [8, p. 1367] shows that after modifying the horizontal maps in this diagram by homotopies we can assume that the diagram is commutative. Finally, we can add suitable acyclic complexes to the T_n to guarantee that the maps $T_n \to T_{n-1}$ are surjective.

To summarise, we have constructed morphisms of inverse systems of complexes of $\mathbb{Z}_p[G]$-modules $(Q/p^n) \leftarrow (T_n) \rightarrow (R\Gamma_c(\mu_{p^n})[2])$ such that for each n the composite $Q/p^n \xleftarrow{\sim} T_n \xrightarrow{\sim} R\Gamma_c(\mu_{p^n})[2]$ considered as a map in $\mathcal{D}(\mathbb{Z}[G])$ is equal to ξ_n. Furthermore the transition maps in each inverse system are surjective. Passing to the inverse limit gives morphisms of complexes of $\mathbb{Z}_p[G]$-modules

$$Q_{\lim} = \varprojlim_n Q/p^n \longleftarrow \varprojlim_n T_n \longrightarrow \varprojlim_n R\Gamma_c(\mu_{p^n})[2] = R\Gamma_c(\mathbb{Z}_p(1))[2].$$

Now [8, Lemma 9] implies that these morphisms are quasi-isomorphisms and that the resulting map $Q_{\lim} \rightarrow R\Gamma_c(\mathbb{Z}_p(1))[2]$ in $\mathcal{D}(\mathbb{Z}_p[G])$ has the required properties. □

We now fix a morphism α as in Lemma 4.3(ii). Then, for each natural number n one has a commutative diagram of morphisms of complexes of G-modules

$$\text{(16)}\qquad \begin{tikzcd} \mathcal{L}_S[0] \oplus \mathcal{L}[-1] \arrow[r, "\alpha"] \arrow[d, hook, "p^n"] & Q \arrow[r, "\beta"] \arrow[d, hook, "p^n"] & \mathrm{cone}(\alpha) \arrow[r, "\gamma"] \arrow[d, hook, "p^n"] & {} \\ \mathcal{L}_S[0] \oplus \mathcal{L}[-1] \arrow[r, "\alpha"] \arrow[d, two heads] & Q \arrow[r, "\beta"] \arrow[d, two heads] & \mathrm{cone}(\alpha) \arrow[r, "\gamma"] \arrow[d, two heads] & {} \\ \mathcal{L}_S/p^n[0] \oplus \mathcal{L}/p^n[-1] \arrow[r, "\alpha/p^n"] & Q/p^n \arrow[r, "\beta/p^n"] & \mathrm{cone}(\alpha/p^n) \arrow[r, "\gamma/p^n"] & {} \end{tikzcd}$$

In this diagram the maps β and γ come from the definition of cone(α) and so the first (and second) row is an explicit representative of the triangle (2). Also, the columns are the short exact sequences which result from the fact that $\mathcal{L}_S$, $\mathcal{L}$ and all terms of Q (and hence also of cone(α)) are $\mathbb{Z}$-torsion-free. Now $\mathcal{L}_p$ is canonically isomorphic to both $\varprojlim_n \mathcal{L}_S/p^n$ and $\varprojlim_n \mathcal{L}/p^n$. Furthermore, as cone$(\alpha)$ is a perfect complex of $\mathbb{Z}$-torsion-free modules, there is a natural isomorphism $\mathrm{cone}(\alpha)\otimes\mathbb{Z}_p \cong \varprojlim_n \mathrm{cone}(\alpha)/p^n$ in $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}_p[G])$, and clearly $\varprojlim_n \mathrm{cone}(\alpha)/p^n \cong \varprojlim_n \mathrm{cone}(\alpha/p^n) \cong \mathrm{cone}(\varprojlim_n \alpha/p^n)$ (where in all cases the limits are taken with respect to the natural transition morphisms). Hence, upon passing to the inverse limit of the lower row of (16), we obtain a distinguished triangle in $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}_p[G])$ of the form

$$\text{(17)}\qquad \mathcal{L}_p[0] \oplus \mathcal{L}_p[-1] \xrightarrow{\varprojlim_n \alpha/p^n} Q_{\lim} \xrightarrow{\varprojlim_n \beta/p^n} \mathrm{cone}(\alpha)\otimes\mathbb{Z}_p \xrightarrow{\varprojlim_n \gamma/p^n} .$$

The distinguished triangle (17) together with the isomorphism $Q_{\lim} \cong R\Gamma_c(\mathbb{Z}_p(1))[2]$ from Lemma 4.4 show the existence of a triangle of the form (7).

It remains to show that if Leopoldt's Conjecture is valid for L at the prime p and we use the identifications of the cohomology of $Q_{\lim}$ given in (12), then after tensoring with $\mathbb{Q}_p$ the long exact sequence of cohomology of the triangle (17) is equal to (8). Now the identifications of the cohomology of the three terms in (17) come from the columns in (16). In particular we have natural isomorphisms $H^i(\mathcal{L}_p[0] \oplus \mathcal{L}_p[-1]) \cong \varprojlim_n H^i(\mathcal{L}_S[0] \oplus \mathcal{L}_S[-1])/p^n$ and $H^i(\mathrm{cone}(\alpha)\otimes\mathbb{Z}_p) \cong$

$\varprojlim_n H^i(\mathrm{cone}(\alpha))/p^n$ for all i, and $H^i(Q_{\lim}) \cong \varprojlim_n H^i(Q)/p^n$ for $i = 0$ and $i = 1$. Therefore by considering the cohomology sequences of the second and third rows in (16), we can easily deduce the explicit description of all maps in (8) except for the map $L(1)_p \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = H^{-1}(Q_{\lim}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to H^{-1}(E_S(\mathcal{L})) \otimes \mathbb{Q}_p = \log_\infty(\mathcal{O}_L^\times) \otimes \mathbb{Q}_p$.
To compute this map we consider the following diagram.

$$\begin{tikzcd}
 & & H^{-1}(Q/p^n) \arrow[r, "H^{-1}(\beta/p^n)"] \arrow[d, hook] & H^{-1}(\mathrm{cone}(\alpha)/p^n) \\
 & H^0(\mathcal{L}_S[0]\oplus\mathcal{L}[-1]) \arrow[r, "H^0(\alpha)"] \arrow[d, "p^n"] & H^0(Q) \arrow[d, "p^n"] & \\
H^{-1}(\mathrm{cone}(\alpha)) \arrow[r, hook, "H^{-1}(\gamma)"] \arrow[d] & H^0(\mathcal{L}_S[0]\oplus\mathcal{L}[-1]) \arrow[r, "H^0(\alpha)"] & H^0(Q) & \\
H^{-1}(\mathrm{cone}(\alpha)/p^n) & & &
\end{tikzcd}$$

By an easy computation with cochains one shows that if an element of $H^0(\mathcal{L}_S[0] \oplus \mathcal{L}[-1])$ lies in the kernel of $H^0(\mathcal{L}_S[0] \oplus \mathcal{L}[-1]) \xrightarrow{p^n \cdot H^0(\alpha)} H^0(Q)$, then its images under the two maps

$$H^0(\mathcal{L}_S[0] \oplus \mathcal{L}[-1]) \xrightarrow{H^0(\alpha)} H^0(Q) \leftarrow H^{-1}(Q/p^n) \xrightarrow{H^{-1}(\beta/p^n)} H^{-1}(\mathrm{cone}(\alpha)/p^n)$$

and

$$H^0(\mathcal{L}_S[0] \oplus \mathcal{L}[-1]) \xrightarrow{p^n} H^0(\mathcal{L}_S[0] \oplus \mathcal{L}[-1]) \xleftarrow{H^{-1}(\gamma)} H^{-1}(\mathrm{cone}(\alpha)) \\ \to H^{-1}(\mathrm{cone}(\alpha)/p^n)$$

coincide (note that the inverse arrows make sense in this context). By considering the elements $(r_w \cdot 2\pi\sqrt{-1}/p^n)_{w \in S_\infty} \in L_\infty \subseteq \mathcal{L}_S = H^0(\mathcal{L}_S[0] \oplus \mathcal{L}[-1])$ for $r_w \in \mathbb{Z}$ we see that the map $H^{-1}(Q/p^n) \to H^{-1}(\mathrm{cone}(\alpha)/p^n)$ sends the image of $(r_w \cdot \exp(2\pi\sqrt{-1}/p^n))_{w \in S_\infty(L)} \in (L_S^\times)_{[p^n]} \subset L_S^\times$ in $C_S(L)_{[p^n]} = H^{-1}(Q/p^n)$ to the image of the element $(r_w \cdot 2\pi\sqrt{-1})_{w \in S_\infty(L)} \in \ker(\exp_\infty) \subseteq H^{-1}(\mathrm{cone}(\alpha))$ in $H^{-1}(\mathrm{cone}(\alpha)/p^n)$. Passing to the inverse limit gives the desired description of θ_1. This completes the proof of Proposition 4.2.
q

5. The proof of Theorem 1.1

In this section we prove Theorem 1.1. Let L/K be a Galois extension of number fields with Galois group G. We define an element of $K_0(\mathbb{Z}[G], \mathbb{R})$ by setting

$$T\Omega(L/K, 1) := \hat{\partial}_G^1(\zeta_{L/K,S}^*(1)) + \chi_G(E_S(\mathcal{L}), \mu_L)$$

where the terms on the right hand side are as in §2.4. The element $T\Omega(L/K, 1)$ depends only upon L/K (see [5, Prop. 3.4]), and the conjectural equality (3) asserts that $T\Omega(L/K,1)$ vanishes. We also recall that [9, Conj. 4(iv)] for the pair $(\mathbb{Q}(1)_L,\mathbb{Z}[G])$ asserts the vanishing of an element $T\Omega(\mathbb{Q}(1)_L,\mathbb{Z}[G])$ of $K_0(\mathbb{Z}[G],\mathbb{R})$ that is defined (unconditionally) in [9, Conj. 4(iii)]. To prove Theorem 1.1 it is therefore enough to prove the following result.

Proposition 5.1. *Let L be a complex Galois extension of $\mathbb{Q}$ and $G = \mathrm{Gal}(L/\mathbb{Q})$. If Leopoldt's Conjecture is valid for L and all prime numbers p, then $T\Omega(L/\mathbb{Q},1) = T\Omega(\mathbb{Q}(1)_L,\mathbb{Z}[G])$.*

Remark 5.1. Recall that we write $\partial^0_{\mathbb{Z}[G],\mathbb{R}}$ for the natural homomorphism of K-groups $K_0(\mathbb{Z}[G],\mathbb{R}) \to K_0(\mathbb{Z}[G])$. The argument of [5, Prop. 3.6(ii)] combines with the equality of Proposition 5.1 to imply that if Leopoldt's Conjecture is valid, then $\partial^0_{\mathbb{Z}[G],\mathbb{R}}(T\Omega(\mathbb{Q}(1)_L,\mathbb{Z}[G]))$ is equal to the element $\Omega(L/K,1)$ of $K_0(\mathbb{Z}[G])$ defined by Chinburg in [13]. Proposition 5.1 therefore answers the question raised in [7, Question 1.54].

5.1. Preliminaries. From now on let $L/\mathbb{Q}$ be a complex Galois extension with Galois group G. For each p and each embedding $j:\mathbb{R}\to\mathbb{C}_p$ there is an induced homomorphism $j_*: K_0(\mathbb{Z}[G],\mathbb{R}) \to K_0(\mathbb{Z}_p[G],\mathbb{C}_p)$ and it is known that $\bigcap_{p,j}\ker(j_*) = \{0\}$ where p runs over all primes and j over all embeddings $\mathbb{R}\to\mathbb{C}_p$ (cf. [5, Lemma 2.1]). To prove Proposition 5.1 it is thus enough to prove that for all p and j one has

$$j_*(T\Omega(L/\mathbb{Q},1)) = j_*(T\Omega(\mathbb{Q}(1)_L,\mathbb{Z}[G])). \tag{18}$$

The proof of this equality will occupy the rest of this section.

We fix a prime p and in the sequel assume that Leopoldt's Conjecture is valid for L and p. We also fix an embedding $j:\mathbb{R}\to\mathbb{C}_p$ and often suppress it from our notation; so in particular in a tensor product of the form $-\otimes_{\mathbb{R}}\mathbb{C}_p$ we consider $\mathbb{C}_p$ as an $\mathbb{R}$-module via j. Just as in §4 we will always assume that S contains all places of residue characteristic p.

In the following we will need to use the language of virtual objects. To this end we consider the Picard categories $\mathcal{V}(\mathbb{Z}_p[G])$, $\mathcal{V}(\mathbb{C}_p[G])$ and $\mathcal{V}(\mathbb{Z}_p[G],\mathbb{C}_p[G])$ discussed in [4, §5]. We fix a unit object $\mathbf{1}_{\mathcal{V}(\mathbb{C}_p[G])}$ of $\mathcal{V}(\mathbb{C}_p[G])$ and for each object X of $\mathcal{V}(\mathbb{C}_p[G])$ we fix an inverse, i.e. an object X^{-1} of $\mathcal{V}(\mathbb{C}_p[G])$ together with an isomorphism $X\otimes X^{-1}\cong \mathbf{1}_{\mathcal{V}(\mathbb{C}_p[G])}$ in $\mathcal{V}(\mathbb{C}_p[G])$. We also write ι : $\pi_0\mathcal{V}(\mathbb{Z}_p[G],\mathbb{C}_p[G])\cong K_0(\mathbb{Z}_p[G],\mathbb{C}_p)$ for the group isomorphism described in [4, Lemma 5.1].

We need to slightly generalise the definition of a trivialised complex and its Euler characteristic. If P is a perfect complex of $\mathbb{Z}_p[G]$-modules and $\tau:[H^{\mathrm{ev}}(P\otimes_{\mathbb{Z}_p}\mathbb{C}_p)]\to[H^{\mathrm{od}}(P\otimes_{\mathbb{Z}_p}\mathbb{C}_p)]$ an isomorphism in $\mathcal{V}(\mathbb{C}_p[G])$, then we will sometimes call the pair (P,τ) a trivialised complex. Its Euler characteristic $\chi_{\mathbb{Z}_p[G],\mathbb{C}_p}(P,\tau)$ is defined as in [4, Definition 5.5] except that $[t]$ is replaced by τ. Clearly any trivialised complex (P,t) as in §2.2 gives rise to the trivialised complex $(P,[t])$ in the new sense, but in general not every trivialisation $\tau:[H^{\mathrm{ev}}(P\otimes_{\mathbb{Z}_p}\mathbb{C}_p)]\to[H^{\mathrm{od}}(P\otimes_{\mathbb{Z}_p}\mathbb{C}_p)]$ of P will be of the form $[t]$ for some isomorphism $t:H^{\mathrm{ev}}(P\otimes_{\mathbb{Z}_p}\mathbb{C}_p)\to H^{\mathrm{od}}(P\otimes_{\mathbb{Z}_p}\mathbb{C}_p)$.

5.2. The element $j_*(T\Omega(L/\mathbb{Q},1))$. We set $R\Gamma_c(\mathbb{Z}_p(1)) := R\Gamma_c(\mathcal{O}_{L,S},\mathbb{Z}_p(1))$ and also $H^i_c(\mathbb{C}_p(1)) := H^i(R\Gamma_c(\mathbb{Z}_p(1))\otimes_{\mathbb{Z}_p}\mathbb{C}_p)$. Furthermore we write $H^{\mathrm{ev}}_c(\mathbb{C}_p(1))$ and $H^{\mathrm{od}}_c(\mathbb{C}_p(1))$ for the direct sums $\oplus_{i \text{ even}}H^i_c(\mathbb{C}_p(1))$ and $\oplus_{i \text{ odd}}H^i_c(\mathbb{C}_p(1))$ respectively.
We start by defining an isomorphism

$$\psi : [H^{\mathrm{ev}}_c(\mathbb{C}_p(1))]\otimes[\mathrm{im}(\theta_2)\otimes_{\mathbb{Q}_p}\mathbb{C}_p] \xrightarrow{\cong} [H^{\mathrm{od}}_c(\mathbb{C}_p(1))]\otimes[\mathrm{im}(\theta_2)\otimes_{\mathbb{Q}_p}\mathbb{C}_p]$$

in $\mathcal{V}(\mathbb{C}_p[G])$ which is induced by the identifications from Lemma 4.1, the exact sequence (8) in Proposition 4.2, and μ_L. More precisely, we let ψ be the following composite map.

$$\begin{aligned}
&[H^2_c(\mathbb{C}_p(1))]\otimes[\mathrm{im}(\theta_2)\otimes_{\mathbb{Q}_p}\mathbb{C}_p]\\
&\qquad\xrightarrow{\alpha_1}[L_p\otimes_{\mathbb{Q}_p}\mathbb{C}_p]\\
&\qquad\xrightarrow{\alpha_2}[H^0(E_S(\mathcal{L}))\otimes\mathbb{C}_p]\otimes[\mathbb{C}_p]\\
&\qquad\xrightarrow{\alpha_3}[H^{-1}(E_S(\mathcal{L}))\otimes\mathbb{C}_p]\otimes[\mathbb{C}_p]\\
&\qquad\xrightarrow{\alpha_4}[L(1)_p\otimes_{\mathbb{Z}_p}\mathbb{C}_p]\otimes[\mathrm{im}(\theta_2)\otimes_{\mathbb{Q}_p}\mathbb{C}_p]\otimes[\mathbb{C}_p]\\
&\qquad\xrightarrow{\alpha_5}[H^1_c(\mathbb{C}_p(1))\oplus H^3_c(\mathbb{C}_p(1))]\otimes[\mathrm{im}(\theta_2)\otimes_{\mathbb{Q}_p}\mathbb{C}_p].
\end{aligned}$$

Here α_1 is induced by the isomorphism $H^2_c(\mathbb{C}_p(1))\cong\mathrm{cok}(\lambda_p)\otimes_{\mathbb{Z}_p}\mathbb{C}_p$ and the short exact sequence

$$\mathrm{im}(\theta_2)\otimes_{\mathbb{Q}_p}\mathbb{C}_p \overset{\subset}{\hookrightarrow} L_p\otimes_{\mathbb{Q}_p}\mathbb{C}_p \xrightarrow{\exp_p} \mathrm{cok}(\lambda_p)\otimes_{\mathbb{Z}_p}\mathbb{C}_p, \tag{19}$$

α_2 and α_4 are induced by the short exact sequences

$$H^0(E_S(\mathcal{L}))\otimes\mathbb{C}_p \hookrightarrow L_p\otimes_{\mathbb{Q}_p}\mathbb{C}_p \xrightarrow{\mathrm{tr}} \mathbb{C}_p \tag{20}$$

and

$$L(1)_p\otimes_{\mathbb{Z}_p}\mathbb{C}_p \xrightarrow{\theta_1\otimes_{\mathbb{Q}_p}\mathbb{C}_p} H^{-1}(E_S(\mathcal{L}))\otimes\mathbb{C}_p \xrightarrow{\theta_2\otimes_{\mathbb{Q}_p}\mathbb{C}_p} \mathrm{im}(\theta_2)\otimes_{\mathbb{Q}_p}\mathbb{C}_p \tag{21}$$

respectively, $\alpha_3 = [\mu_L\otimes_{\mathbb{R}}\mathbb{C}_p]\otimes\mathrm{id}$, and α_5 is induced by the isomorphisms $H^1_c(\mathbb{C}_p(1))\cong L(1)_p\otimes_{\mathbb{Z}_p}\mathbb{C}_p$ and $H^3_c(\mathbb{C}_p(1))\cong\mathbb{C}_p$.
Now by the properties of a Picard category there exists a unique isomorphism

$$\nu : [H^{\mathrm{ev}}_c(\mathbb{C}_p(1))] \xrightarrow{\cong} [H^{\mathrm{od}}_c(\mathbb{C}_p(1))]$$

in $\mathcal{V}(\mathbb{C}_p[G])$ such that $\psi = \nu\otimes\mathrm{id}$. We will consider this isomorphism as a trivialisation of the complex $R\Gamma_c(\mathbb{Z}_p(1))$.

Lemma 5.2. *In $K_0(\mathbb{Z}_p[G],\mathbb{C}_p)$ one has*

$$j_*(T\Omega(L/\mathbb{Q},1)) = \partial^1_{\mathbb{Z}_p[G],\mathbb{C}_p}\big(j_*(\zeta^*_{L/\mathbb{Q},S}(1))\big) + \chi_{\mathbb{Z}_p[G],\mathbb{C}_p}(R\Gamma_c(\mathbb{Z}_p(1)),\nu).$$

Proof. To simplify the notation we will abbreviate '$\chi_{\mathbb{Z}_p[G],\mathbb{C}_p}$' to '$\chi_p$'.
It is clear that $j_*(\hat\partial^1_G(\zeta^*_{L/\mathbb{Q},S}(1))) = \partial^1_{\mathbb{Z}_p[G],\mathbb{C}_p}(j_*(\zeta^*_{L/\mathbb{Q},S}(1)))$ (compare §2.1) and also $j_*(\chi_{\mathbb{Z}[G],\mathbb{R}}(E_S(\mathcal{L}),\mu_L)) = \chi_p(E_S(\mathcal{L})\otimes\mathbb{Z}_p,\mu_L\otimes_{\mathbb{R}}\mathbb{C}_p)$. Moreover it follows

from [4, Prop. 5.6.3] that $\chi_p(R\Gamma_c(\mathbb{Z}_p(1)),\nu) = \chi_p(R\Gamma_c(\mathbb{Z}_p(1))[2],\nu)$. It is thus enough to prove that in $K_0(\mathbb{Z}_p[G],\mathbb{C}_p)$ one has

$$\chi_p(E_S(\mathcal{L})\otimes\mathbb{Z}_p,\mu_L\otimes_{\mathbb{R}}\mathbb{C}_p) = \chi_p(R\Gamma_c(\mathbb{Z}_p(1))[2],\nu). \tag{22}$$

To do this we will apply the additivity criterion of [4, Theorem 5.7] to the exact triangle (7) in Proposition 4.2. On the complex $\mathcal{L}_p[0]\oplus\mathcal{L}_p[-1]$ we consider the trivialisation given by the identity map $\mathrm{id}:\mathcal{L}_p\otimes_{\mathbb{Z}_p}\mathbb{C}_p\to\mathcal{L}_p\otimes_{\mathbb{Z}_p}\mathbb{C}_p$, on $R\Gamma_c(\mathbb{Z}_p(1))[2]$ we consider the trivialisation ν, and on $E_S(\mathcal{L})\otimes\mathbb{Z}_p$ we consider the trivialisation $\mu_L\otimes_{\mathbb{R}}\mathbb{C}_p$. Note that the additivity criterion in [4] is only stated for trivialisations as defined in §2.2, however it is easy to check that it remains valid for generalised trivialisations as defined in §5.1.
In our context, the map a in [4, Theorem 5.7] is the map $\mathcal{L}_p[0]\oplus\mathcal{L}_p[-1]\to R\Gamma_c(\mathbb{Z}_p(1))[2]$ in the distinguished triangle (7), and $\Sigma = \mathbb{C}_p[G]$. Therefore $\ker(H^{\mathrm{ev}}a_\Sigma) = \mathrm{im}(\theta_2)\otimes_{\mathbb{Q}_p}\mathbb{C}_p$ and $\ker(H^{\mathrm{od}}a_\Sigma) = L_p^0\otimes_{\mathbb{Q}_p}\mathbb{C}_p$ where $L_p^0 = \ker(\mathrm{tr}_{L_p/\mathbb{Q}_p}: L_p\to\mathbb{Q}_p)$. To apply the additivity criterion we must show that the following diagram commutes in $\mathcal{V}(\mathbb{C}_p[G])$.

$$\begin{tikzcd}
\begin{matrix}[\mathrm{cok}(\lambda_p)\otimes_{\mathbb{Z}_p}\mathbb{C}_p]\\ \otimes[\mathrm{im}(\theta_2)\otimes_{\mathbb{Q}_p}\mathbb{C}_p]\otimes[L_p^0\otimes_{\mathbb{Q}_p}\mathbb{C}_p]\end{matrix} \arrow[r, "s^{\mathrm{ev}}"] \arrow[d, "\nu\otimes\mathrm{id}\otimes[-\mathrm{id}]"] & {[L_p\otimes_{\mathbb{Q}_p}\mathbb{C}_p]\otimes[H^0(E_S(\mathcal{L}))\otimes\mathbb{C}_p]} \arrow[d, "\mathrm{id}\otimes[\mu_L\otimes_{\mathbb{R}}\mathbb{C}_p]"] \\
\begin{matrix}[L(1)_p\otimes_{\mathbb{Z}_p}\mathbb{C}_p\oplus\mathbb{C}_p]\\ \otimes[\mathrm{im}(\theta_2)\otimes_{\mathbb{Q}_p}\mathbb{C}_p]\otimes[L_p^0\otimes_{\mathbb{Q}_p}\mathbb{C}_p]\end{matrix} \arrow[r, "s^{\mathrm{od}}"] & {[L_p\otimes_{\mathbb{Q}_p}\mathbb{C}_p]\otimes[H^{-1}(E_S(\mathcal{L}))\otimes\mathbb{C}_p]}
\end{tikzcd}$$

Here the horizontal maps are induced by the even respectively odd part of the cohomology sequence (8) after tensoring with $\mathbb{C}_p$, i.e. the top horizontal map s^{ev} is induced by the short exact sequence (19) and the isomorphism

$$H^0(E_S(\mathcal{L}))\otimes\mathbb{C}_p\cong L_p^0\otimes_{\mathbb{Q}_p}\mathbb{C}_p, \tag{23}$$

and the bottom horizontal map s^{od} is induced by (21) and

$$L_p^0\otimes_{\mathbb{Q}_p}\mathbb{C}_p\xhookrightarrow{\subset} L_p\otimes_{\mathbb{Q}_p}\mathbb{C}_p\xrightarrow{\mathrm{tr}}\mathbb{C}_p. \tag{24}$$

To see the commutativity of the above diagram we will show that the automorphism

$$\kappa := (\mathrm{id}\otimes[\mu_L\otimes_{\mathbb{R}}\mathbb{C}_p])^{-1}\circ(s^{\mathrm{od}})\circ(\nu\otimes\mathrm{id}\otimes[-\mathrm{id}])\circ(s^{\mathrm{ev}})^{-1}$$

of $[L_p\otimes_{\mathbb{Q}_p}\mathbb{C}_p]\otimes[H^0(E_S(\mathcal{L}))\otimes\mathbb{C}_p]$ is the identity map. For this we use the isomorphism

$$[L_p\otimes_{\mathbb{Q}_p}\mathbb{C}_p]\otimes[H^0(E_S(\mathcal{L}))\otimes\mathbb{C}_p]\cong[L_p^0\otimes_{\mathbb{Q}_p}\mathbb{C}_p]\otimes[\mathbb{C}_p]\otimes[L_p^0\otimes_{\mathbb{Q}_p}\mathbb{C}_p]$$

which is induced by the short exact sequence (24) and the isomorphism (23). Using $\nu\otimes\mathrm{id}\otimes[-\mathrm{id}] = \psi\otimes[-\mathrm{id}]$ and the definition of ψ, it is easy to see that then κ becomes the automorphism of $[L_p^0\otimes_{\mathbb{Q}_p}\mathbb{C}_p]\otimes[\mathbb{C}_p]\otimes[L_p^0\otimes_{\mathbb{Q}_p}\mathbb{C}_p]$ which is given by (using the obvious abuse of notation)

$$a\otimes b\otimes c\mapsto[-\mathrm{id}](c)\otimes b\otimes a,$$

i.e. the morphism in $\mathcal{V}(\mathbb{C}_p[G])$ which swaps the two copies of $[L_p^0 \otimes_{\mathbb{Q}_p} \mathbb{C}_p]$ composed with the map $[-\mathrm{id}]$ on one of the two copies. It now follows from the general properties of a determinant functor (see e.g. [15, §4.9]), that this automorphism (and hence also κ) is the identity morphism as required.
The additivity criterion [4, Theorem 5.7] now implies that

$$\chi_p(R\Gamma_c(\mathbb{Z}_p(1))[2], \nu) = \chi_p(\mathcal{L}_p[0] \oplus \mathcal{L}_p[-1], \mathrm{id}) + \chi_p(E_S(\mathcal{L}) \otimes \mathbb{Z}_p, \mu_L \otimes_{\mathbb{R}} \mathbb{C}_p).$$

Since clearly $\chi_p(\mathcal{L}_p[0] \oplus \mathcal{L}_p[-1], \mathrm{id}) = 0$ this completes the proof of (22) and hence of Lemma 5.2. $\square$

5.3. The element $j_*(T\Omega(\mathbb{Q}(1)_L, \mathbb{Z}[G]))$. The motive $\mathbb{Q}(1)_L$ is pure of weight -2. The argument of [10, §2] therefore shows that

$$j_*(T\Omega(\mathbb{Q}(1)_L, \mathbb{Z}[G])) = \partial^1_{\mathbb{Z}_p[G],\mathbb{C}_p}\big(j_*(\zeta^*_{L/\mathbb{Q},S}(1))\big) + \iota\big(([R\Gamma_c(\mathbb{Z}_p(1))], \omega)\big) \tag{25}$$

with ω the composite morphism

$$[R\Gamma_c(\mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} \mathbb{C}_p] \xrightarrow{\tilde{\vartheta}_p \otimes_{\mathbb{Q}_p} \mathbb{C}_p} [\Xi(\mathbb{Q}(1)_L) \otimes_{\mathbb{Q}} \mathbb{C}_p] \xrightarrow{\vartheta_\infty \otimes_{\mathbb{R}} \mathbb{C}_p} \mathbf{1}_{\mathcal{V}(\mathbb{C}_p[G])}$$

where $\tilde{\vartheta}_p$ and ϑ_∞ are as defined in [10, p. 479, resp. p. 477]. Indeed, whilst the argument of [10, §2] is phrased solely in terms of abelian groups G it extends immediately to the general case upon replacing graded determinants by virtual objects and then (25) is the non-abelian generalisation of the equality [10, (11)]. Given the observations of [7, §1.1, §1.3] it is also a straightforward exercise to explicate the space $\Xi(\mathbb{Q}(1)_L)$ and the morphisms $\tilde{\vartheta}_p$ and ϑ_∞. To describe the result we introduce further notation. We write $\Sigma(L)$ for the set of all complex embeddings $L \to \mathbb{C}$ and consider $\bigoplus_{\Sigma(L)} \mathbb{C}$ as a $G \times \mathrm{Gal}(\mathbb{C}/\mathbb{R})$-module where G acts via L and $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts diagonally. We write H_B for the $G \times \mathrm{Gal}(\mathbb{C}/\mathbb{R})$-submodule $\bigoplus_{\Sigma(L)} 2\pi\sqrt{-1} \cdot \mathbb{Z}$ of $\bigoplus_{\Sigma(L)} \mathbb{C}$ and let H_B^+ and $\big(\bigoplus_{\Sigma(L)} \mathbb{C}\big)^+$ denote the G-submodules comprising elements invariant under the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. We also set $H^1_f := \mathrm{im}(\lambda_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Then ω is equal to the composite

$$\begin{aligned}
[R\Gamma_c(\mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} \mathbb{C}_p] &\cong [H^1_c(\mathbb{C}_p(1))]^{-1} \otimes [H^2_c(\mathbb{C}_p(1))] \otimes [H^3_c(\mathbb{C}_p(1))]^{-1} \\
&\cong [H^1_c(\mathbb{C}_p(1))]^{-1} \otimes \big([H^1_f \otimes_{\mathbb{Q}_p} \mathbb{C}_p] \otimes [H^2_c(\mathbb{C}_p(1))]\big) \\
&\qquad \otimes [H^1_f \otimes_{\mathbb{Q}_p} \mathbb{C}_p]^{-1} \otimes [H^3_c(\mathbb{C}_p(1))]^{-1} \\
&\cong \big([H_B^+ \otimes \mathbb{C}_p]^{-1} \otimes [L \otimes_{\mathbb{Q}} \mathbb{C}_p]\big) \\
&\qquad \otimes \big([\mathcal{O}_L^\times \otimes \mathbb{C}_p]^{-1} \otimes [\mathbb{C}_p]^{-1}\big) \\
&\cong \Big[\prod\nolimits_{S_\infty(L)} \mathbb{C}_p\Big] \otimes \Big[\prod\nolimits_{S_\infty(L)} \mathbb{C}_p\Big]^{-1} \\
&\cong \mathbf{1}_{\mathcal{V}(\mathbb{C}_p[G])}
\end{aligned} \tag{26}$$

where the maps are defined as follows. The first, second and fifth maps are clear. The third map is induced by the exact sequence

$$0 \to L(1)_p \otimes_{\mathbb{Z}_p} \mathbb{C}_p \xrightarrow{\cong} H^1_c(\mathbb{C}_p(1)) \xrightarrow{0} H^1_f \otimes_{\mathbb{Q}_p} \mathbb{C}_p \xrightarrow{\subset} \prod_{w \in S_p(L)} U^{(1)}_{L_w} \otimes_{\mathbb{Z}_p} \mathbb{C}_p$$
$$\xrightarrow{\pi} H^2_c(\mathbb{C}_p(1)) \to 0 \to 0 \to H^3_c(\mathbb{C}_p(1)) \xrightarrow{\cong} \mathbb{C}_p \to 0,$$

where π is induced by the identification $H^2_c(\mathbb{C}_p(1)) \cong \mathrm{cok}(\lambda_p) \otimes_{\mathbb{Z}_p} \mathbb{C}_p$ from Lemma 4.1 (this sequence is the cohomology sequence of the distinguished triangle of [10, (3)] with $M = \mathbb{Q}(1)_L$ and $A = \mathbb{Q}[G]$), together with the isomorphism $L(1)_p \cong H^+_B \otimes \mathbb{Z}_p$ that sends an element $\big(n_w \cdot \{\exp(2\pi\sqrt{-1}/p^n)\}_{n \geq 0}\big)_{w \in S_\infty(L)}$ in $L(1)_p$ to the element $\big(n_{w_\sigma} \cdot \hat{\sigma}(2\pi\sqrt{-1})\big)_{\sigma \in \Sigma(L)}$ in $H^+_B \otimes \mathbb{Z}_p$ (where w_σ denotes the place of L corresponding to σ, and $\hat{\sigma} : L_{w_\sigma} \to \mathbb{C}$ is the unique continuous extension of σ), the isomorphism

$$\prod_{w \in S_p(L)} U^{(1)}_{L_w} \otimes_{\mathbb{Z}_p} \mathbb{C}_p \cong \Big(\prod_{w \in S_p(L)} L_w\Big) \otimes_{\mathbb{Q}_p} \mathbb{C}_p = L \otimes_{\mathbb{Q}} \mathbb{C}_p \tag{27}$$

induced by the p-adic logarithm maps $U^{(1)}_{L_w} \to L_w$, and the isomorphism $\lambda_p \otimes_{\mathbb{Z}_p} \mathbb{C}_p : \mathcal{O}^\times_L \otimes \mathbb{C}_p \cong H^1_f \otimes_{\mathbb{Q}_p} \mathbb{C}_p$. The fourth map is induced by (the image under $- \otimes_{\mathbb{R}} \mathbb{C}_p$ of) the short exact sequence

$$\mathcal{O}^\times_L \otimes \mathbb{R} \xhookrightarrow{\mathrm{Reg}} \prod\nolimits_{S_\infty(L)} \mathbb{R} \twoheadrightarrow \mathbb{R} \tag{28}$$

where $\mathrm{Reg} : \mathcal{O}^\times_L \otimes \mathbb{R} \to \prod_{S_\infty(L)} \mathbb{R}$ denotes the usual regulator map $u \otimes r \mapsto r \cdot (2\log|\sigma_w(u)|)_{w \in S_\infty(L)}$ (here σ_w is a complex embedding of L corresponding to the place w), the natural isomorphism

$$\Big(\bigoplus\nolimits_{\Sigma(L)} \mathbb{C}\Big)^+ \cong L \otimes_{\mathbb{Q}} \mathbb{R} \tag{29}$$

and (the image under $- \otimes_{\mathbb{R}} \mathbb{C}_p$ of) the short exact sequence

$$H^+_B \otimes \mathbb{R} \xhookrightarrow{\subset} \Big(\bigoplus\nolimits_{\Sigma(L)} \mathbb{C}\Big)^+ \twoheadrightarrow \prod\nolimits_{S_\infty(L)} \mathbb{R} \tag{30}$$

in which the second arrow sends each element $(z_\sigma)_{\sigma \in \Sigma(L)}$ of $\big(\bigoplus_{\Sigma(L)} \mathbb{C}\big)^+$ to $(z_{\sigma_w} + z_{\overline{\sigma_w}})_{w \in S_\infty(L)}$ in $\prod_{S_\infty(L)} \mathbb{R}$ (where σ_w and $\overline{\sigma_w}$ denote the two complex embeddings of L corresponding to the place w).

5.4. Completion of the proof. Let

$$\psi' : [H^{\mathrm{ev}}_c(\mathbb{C}_p(1))] \otimes [H^1_f \otimes_{\mathbb{Q}_p} \mathbb{C}_p] \xrightarrow{\cong} [H^{\mathrm{od}}_c(\mathbb{C}_p(1))] \otimes [H^1_f \otimes_{\mathbb{Q}_p} \mathbb{C}_p]$$

denote the composite isomorphism

$$
\begin{aligned}
[H^2_c(\mathbb{C}_p(1))] \otimes [H^1_f \otimes_{\mathbb{Q}_p} \mathbb{C}_p] &\xrightarrow{\alpha'_1} [L \otimes_{\mathbb{Q}} \mathbb{C}_p] \\
&\xrightarrow{\alpha'_2} \Big[\Big(\bigoplus\nolimits_{\Sigma(L)} \mathbb{C}\Big)^+ \otimes_{\mathbb{R}} \mathbb{C}_p\Big] \\
&\xrightarrow{\alpha'_3} [H^+_B \otimes \mathbb{C}_p] \otimes \Big[\prod\nolimits_{S_\infty(L)} \mathbb{C}_p\Big] \\
&\xrightarrow{\alpha'_4} [L(1)_p \otimes_{\mathbb{Z}_p} \mathbb{C}_p] \otimes [\mathcal{O}_L^\times \otimes \mathbb{C}_p] \otimes [\mathbb{C}_p] \\
&\xrightarrow{\alpha'_5} [H^1_c(\mathbb{C}_p(1)) \oplus H^3_c(\mathbb{C}_p(1))] \otimes [H^1_f \otimes_{\mathbb{Q}_p} \mathbb{C}_p]
\end{aligned}
$$

where α'_1 is induced by the short exact sequence

$$H^1_f \otimes_{\mathbb{Q}_p} \mathbb{C}_p \hookrightarrow \prod\nolimits_{w \in S_p(L)} U^{(1)}_{L_w} \otimes_{\mathbb{Z}_p} \mathbb{C}_p \twoheadrightarrow H^2_c(\mathbb{C}_p(1))$$

and the isomorphism (27), the map α'_2 is induced by the isomorphism (29), the map α'_3 is induced by (the image under $-\otimes_{\mathbb{R}} \mathbb{C}_p$ of) the short exact sequence (30), α'_4 is induced by (the image under $-\otimes_{\mathbb{R}} \mathbb{C}_p$ of) the short exact sequence (28) and the isomorphism $H^+_B \otimes \mathbb{C}_p \cong L(1)_p \otimes_{\mathbb{Z}_p} \mathbb{C}_p$, and α'_5 is induced by the isomorphisms $H^1_c(\mathbb{C}_p(1)) \cong L(1)_p \otimes_{\mathbb{Z}_p} \mathbb{C}_p$, $H^3_c(\mathbb{C}_p(1)) \cong \mathbb{C}_p$ and $\mathcal{O}_L^\times \otimes \mathbb{C}_p \cong H^1_f \otimes_{\mathbb{Q}_p} \mathbb{C}_p$.

Let $\nu' : [H^{\mathrm{ev}}_c(\mathbb{C}_p(1))] \xrightarrow{\cong} [H^{\mathrm{od}}_c(\mathbb{C}_p(1))]$ be the unique isomorphism in $\mathcal{V}(\mathbb{C}_p[G])$ such that $\nu' \otimes \mathrm{id} = \psi'$. We recall that the Euler characteristic $\chi_{\mathbb{Z}_p[G],\mathbb{C}_p}(R\Gamma_c(\mathbb{Z}_p(1)), \nu')$ is defined to be $\iota\big(([R\Gamma_c(\mathbb{Z}_p(1))], \lambda)\big)$, where λ is the composite isomorphism

$$
\begin{aligned}
[R\Gamma_c(\mathbb{C}_p(1))] &\cong [H^{\mathrm{ev}}_c(\mathbb{C}_p(1))] \otimes [H^{\mathrm{od}}_c(\mathbb{C}_p(1))]^{-1} \\
&\xrightarrow{\nu' \otimes \mathrm{id}} [H^{\mathrm{od}}_c(\mathbb{C}_p(1))] \otimes [H^{\mathrm{od}}_c(\mathbb{C}_p(1))]^{-1} \cong \mathbf{1}_{\mathcal{V}(\mathbb{C}_p[G])}
\end{aligned}
$$

in $\mathcal{V}(\mathbb{C}_p[G])$ (compare [4, Definition 5.5]). Now by comparing ω and λ one can show that

$$\iota\big(([R\Gamma_c(\mathbb{Z}_p(1))], \omega)\big) = \chi_{\mathbb{Z}_p[G],\mathbb{C}_p}(R\Gamma_c(\mathbb{Z}_p(1)), \nu'). \tag{31}$$

The isomorphism (27) restricts to an isomorphism

$$\varphi : H^1_f \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong \mathrm{im}(\theta_2) \otimes_{\mathbb{Q}_p} \mathbb{C}_p$$

of $\mathbb{C}_p[G]$-modules and we will show below that the following diagram in $\mathcal{V}(\mathbb{C}_p[G])$ is commutative.

$$
\begin{array}{ccc}
[H^{\mathrm{ev}}_c(\mathbb{C}_p(1))] \otimes [H^1_f \otimes_{\mathbb{Q}_p} \mathbb{C}_p] & \xrightarrow{\mathrm{id} \otimes [\varphi]} & [H^{\mathrm{ev}}_c(\mathbb{C}_p(1))] \otimes [\mathrm{im}(\theta_2) \otimes_{\mathbb{Q}_p} \mathbb{C}_p] \\
\Big\downarrow{\scriptstyle \nu' \otimes \mathrm{id}} & & \Big\downarrow{\scriptstyle \nu \otimes \mathrm{id}} \\
[H^{\mathrm{od}}_c(\mathbb{C}_p(1))] \otimes [H^1_f \otimes_{\mathbb{Q}_p} \mathbb{C}_p] & \xrightarrow{\mathrm{id} \otimes [\varphi]} & [H^{\mathrm{od}}_c(\mathbb{C}_p(1))] \otimes [\mathrm{im}(\theta_2) \otimes_{\mathbb{Q}_p} \mathbb{C}_p]
\end{array}
$$

From this diagram it follows that $\nu = \nu'$. In view of Lemma 5.2 and equations (25) and (31) this implies the required equality (18) and hence Proposition 5.1. It now only remains to show that the above diagram in $\mathcal{V}(\mathbb{C}_p[G])$ is commutative. For this we consider the following diagram.

$$\begin{array}{ccc}
[H^2_c(\mathbb{C}_p(1))] \otimes [H^1_f \otimes_{\mathbb{Q}_p} \mathbb{C}_p] & \xrightarrow{\mathrm{id}\otimes[\varphi]} & [H^2_c(\mathbb{C}_p(1))] \otimes [\mathrm{im}(\theta_2) \otimes_{\mathbb{Q}_p} \mathbb{C}_p] \\
\big\downarrow{\scriptstyle \alpha'_1} & & \big\downarrow{\scriptstyle \alpha_1} \\
[L \otimes_{\mathbb{Q}} \mathbb{C}_p] & = & [L \otimes_{\mathbb{Q}} \mathbb{C}_p] \\
\big\downarrow{\scriptstyle \alpha'_3 \circ \alpha'_2} & & \big\downarrow{\scriptstyle \alpha_3 \circ \alpha_2} \\
[H^+_B \otimes \mathbb{C}_p] \otimes \left[\left(\prod_{S_\infty(L)} \mathbb{R}\right) \otimes_{\mathbb{R}} \mathbb{C}_p\right] & & [H^{-1}(E_S(\mathcal{L})) \otimes \mathbb{C}_p] \otimes [\mathbb{C}_p] \\
\big\downarrow{\scriptstyle \beta_1} & & \big\downarrow{\scriptstyle \alpha_4} \\
\begin{array}{c}[L(1)_p \otimes_{\mathbb{Z}_p} \mathbb{C}_p] \\ \otimes[\mathrm{im}(\theta_2) \otimes_{\mathbb{Q}_p} \mathbb{C}_p] \otimes [\mathbb{C}_p]\end{array} & = & \begin{array}{c}[L(1)_p \otimes_{\mathbb{Z}_p} \mathbb{C}_p] \\ \otimes[\mathrm{im}(\theta_2) \otimes_{\mathbb{Q}_p} \mathbb{C}_p] \otimes [\mathbb{C}_p]\end{array} \\
\big\downarrow{\scriptstyle \beta_2} & & \big\downarrow{\scriptstyle \alpha_5} \\
\begin{array}{c}[H^1_c(\mathbb{C}_p(1)) \oplus H^3_c(\mathbb{C}_p(1))] \\ \otimes[H^1_f \otimes_{\mathbb{Q}_p} \mathbb{C}_p]\end{array} & \xrightarrow{\mathrm{id}\otimes[\varphi]} & \begin{array}{c}[H^1_c(\mathbb{C}_p(1)) \oplus H^3_c(\mathbb{C}_p(1))] \\ \otimes[\mathrm{im}(\theta_2) \otimes_{\mathbb{Q}_p} \mathbb{C}_p]\end{array}
\end{array}$$

Here the maps α_i and α'_i are as above. The map β_1 is induced by the isomorphism $L(1)_p \otimes_{\mathbb{Z}_p} \mathbb{C}_p \cong H^+_B \otimes \mathbb{C}_p$ and the short exact sequence

$$\mathrm{im}(\theta_2) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \hookrightarrow \left(\prod_{S_\infty(L)} \mathbb{R}\right) \otimes_{\mathbb{R}} \mathbb{C}_p \twoheadrightarrow \mathbb{C}_p \tag{32}$$

which is obtained by applying $- \otimes_{\mathbb{R}} \mathbb{C}_p$ to the short exact sequence (28) and using the identification $\mathcal{O}_L^\times \otimes \mathbb{C}_p \cong \mathrm{im}(\theta_2) \otimes_{\mathbb{Q}_p} \mathbb{C}_p$, and the map β_2 is induced by the isomorphisms $H^1_c(\mathbb{C}_p(1)) \cong L(1)_p \otimes_{\mathbb{Z}_p} \mathbb{C}_p$, $H^3_c(\mathbb{C}_p(1)) \cong \mathbb{C}_p$ and φ.
By definition the composite of the right vertical maps is $\psi = \nu \otimes \mathrm{id}$. Furthermore it is not difficult to see that $\beta_2 \circ \beta_1 = \alpha'_5 \circ \alpha'_4$, hence the composite of the left vertical maps is $\psi' = \nu' \otimes \mathrm{id}$.
Clearly the bottom square is commutative. The isomorphism of short exact sequences

$$\begin{array}{ccccc}
H^1_f \otimes_{\mathbb{Q}_p} \mathbb{C}_p & \hookrightarrow & \prod_{w \in S_p(L)} U^{(1)}_{L_w} \otimes_{\mathbb{Z}_p} \mathbb{C}_p & \twoheadrightarrow & H^2_c(\mathbb{C}_p(1)) \\
\big\downarrow{\scriptstyle \varphi} & & \big\downarrow{\scriptstyle \cong} & & \big\downarrow{\scriptstyle \cong} \\
\mathrm{im}(\theta_2) \otimes_{\mathbb{Q}_p} \mathbb{C}_p & \hookrightarrow & L \otimes_{\mathbb{Q}} \mathbb{C}_p & \twoheadrightarrow & \mathrm{cok}(\lambda_p) \otimes_{\mathbb{Z}_p} \mathbb{C}_p
\end{array}$$

implies that the top square is commutative. The commutativity of the middle rectangle follows from the properties of a determinant functor applied to the following commutative diagram of short exact sequences.

$$\begin{tikzcd}
L(1)_p \otimes_{\mathbb{Z}_p} \mathbb{C}_p \arrow[r, hook, "\theta_1 \otimes_{\mathbb{Q}_p} \mathbb{C}_p"] \arrow[d, "\cong"] & H^{-1}(E_S(\mathcal{L})) \otimes \mathbb{C}_p \arrow[r, two heads, "\theta_2 \otimes_{\mathbb{Q}_p} \mathbb{C}_p"] \arrow[d, hook] & \mathrm{im}(\theta_2) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \arrow[d, hook] \\
H_B^+ \otimes \mathbb{C}_p \arrow[r, hook] & L \otimes_{\mathbb{Q}} \mathbb{C}_p \arrow[r, two heads] \arrow[d, "\mathrm{tr}"] & \left(\prod_{S_\infty(L)} \mathbb{R}\right) \otimes_{\mathbb{R}} \mathbb{C}_p \arrow[d] \\
 & \mathbb{C}_p \arrow[r, equal] & \mathbb{C}_p
\end{tikzcd}$$

Here the top horizontal and right vertical short exact sequences are (21) and (32) respectively. The middle horizontal short exact sequence comes from combining (30) with the isomorphism (29), and the middle vertical short exact sequence comes from combining (20) with the isomorphism $\mu_L \otimes_{\mathbb{R}} \mathbb{C}_p$. The commutativity of this diagram is easily checked.

6. The proofs of Corollaries 1.2, 1.3 and 1.4

In this section we use Theorem 1.1 to prove Corollaries 1.2, 1.3 and 1.4.

6.1. The proof of Corollary 1.2. Let F/E be a Galois extension of number fields and set $\Gamma := \mathrm{Gal}(F/E)$. Let L be a totally complex finite Galois extension of $\mathbb{Q}$ containing F and set $G := \mathrm{Gal}(L/\mathbb{Q})$. We write π for the natural composite homomorphism $K_0(\mathbb{Z}[G], \mathbb{R}) \to K_0(\mathbb{Z}[\mathrm{Gal}(L/E)], \mathbb{R}) \to K_0(\mathbb{Z}[\Gamma], \mathbb{R})$ where the first arrow is restriction and the second projection. Then it is known that $\pi(T\Omega(L/\mathbb{Q}, 1)) = T\Omega(F/E, 1)$ and $\pi(T\Omega(\mathbb{Q}(1)_L, \mathbb{Z}[G])) = T\Omega(\mathbb{Q}(1)_F, \mathbb{Z}[\Gamma])$ (see [5, Prop. 3.5] and [9, Prop. 4.1]). In particular, to prove that $T\Omega(F/E, 1) = T\Omega(\mathbb{Q}(1)_F, \mathbb{Z}[\Gamma])$ it is enough to prove that $T\Omega(L/\mathbb{Q}, 1) = T\Omega(\mathbb{Q}(1)_L, \mathbb{Z}[G])$. Given this observation, Corollary 1.2 is an immediate consequence of Theorem 1.1.

6.2. The proof of Corollary 1.3. By the functorial properties of the conjectures (see [5, Prop. 3.5 and Rem. 4.2]) it suffices to consider the case $K = \mathbb{Q}$ and L totally complex. Since L is abelian over $\mathbb{Q}$, Leopoldt's Conjecture is known to be valid for L and all primes p [6]. In addition, the validity of [9, Conj. 4(iv)] for the pair $(\mathbb{Q}(1)_L, \mathbb{Z}[\mathrm{Gal}(L/\mathbb{Q})])$ has been proved by Flach and the second named author in [11, Cor. 1.2]. (The proof of [11, Cor. 1.2] relies on certain 2-adic results of Flach in [16] and unfortunately the relevant results in [16] are now known to contain errors. However, in [17] Flach has recently provided the necessary corrections so that, in particular, the result of [11, Cor. 1.2] is valid as stated.) Given the validity of [9, Conj. 4(iv)] for $(\mathbb{Q}(1)_L, \mathbb{Z}[\mathrm{Gal}(L/\mathbb{Q})])$, the first assertion of Corollary 1.3 follows immediately from Theorem 1.1.

We now assume that [5, Conj. 3.3] is valid for $L/\mathbb{Q}$. Then [5, Theorem 5.2] implies that [5, Conj. 4.1] is valid for $L/\mathbb{Q}$ if and only if [5, Conj. 5.3] is valid for $L/\mathbb{Q}$. Also, in [5, Rem. 5.4] it is shown that [5, Conj. 5.3] is equivalent to

the earlier conjecture [2, Conj. 4.1]. To prove the second assertion of Corollary 1.3 we therefore need only note that [2, Conj. 4.1] is proved for abelian extensions $L/\mathbb{Q}$ of odd conductor in [2, Cor. 6.2] and for abelian extensions $L/\mathbb{Q}$ of arbitrary conductor in [11, Theorem 1.1] (see in particular the discussion at the end of [11, §3.1]).
This completes the proof of Corollary 1.3.

Remark 6.1. By using the main result of Bley in [1] one can prove an analogue of Corollary 1.3 for certain classes of abelian extensions of imaginary quadratic fields.

6.3. The proof of Corollary 1.4. Let p, q and r be distinct (odd) rational primes which satisfy $p \equiv r \equiv -q \equiv 3 \pmod 4$ and are such that the Legendre symbols $(\frac{p}{q})$ and $(\frac{r}{q})$ are both equal to -1. Then if ℓ is any odd prime such that $(\frac{\ell}{pr}) = -(\frac{\ell}{q}) = 1$ Chinburg has shown that there exists a unique totally complex field $L_{p,q,r,\ell}$ which contains $\mathbb{Q}(\sqrt{pr}, \sqrt{q})$, is Galois over $\mathbb{Q}$ with group isomorphic to the quaternion group of order 8 and is such that $L_{p,q,r,\ell}/\mathbb{Q}$ is ramified precisely at p, q, r, ℓ and infinity (cf. [14, Prop. 4.1.3]). We observe that the primes $p = 3, q = 5$ and $r = 7$ satisfy the congruence conditions described above and will now prove that the conjectures [5, Conj. 3.3] and [5, Conj. 4.1] are both valid for any extension of the form $L_{3,5,7,\ell}/K$. To do this we set $L_\ell := L_{3,5,7,\ell}$ and $G_\ell := \mathrm{Gal}(L_{3,5,7,\ell}/\mathbb{Q})$.
We note first that L_ℓ/K is tamely ramified and we recall that for any tamely ramified extension of number fields F/E the element $T\Omega^{\mathrm{loc}}(F/E, 1)$ that is defined in [5, §5.1.1] vanishes (by [5, Prop. 5.7(i)]) and hence that the conjectures [5, Conj. 3.3] and [5, Conj. 4.1] are equivalent for F/E (by [5, Theorem 5.2]). It therefore suffices for us to prove that [5, Conj. 3.3] is valid for all extensions L_ℓ/K. We recall that this is equivalent to asserting that the element $T\Omega(L_\ell/K, 1)$ of $K_0(\mathbb{Z}[\mathrm{Gal}(L_\ell/K)], \mathbb{R})$ that is defined in [5, §3.2] vanishes. Taking account of the functorial behaviour described in [5, Prop. 3.5(i)] it is therefore enough to prove that each element $T\Omega(L_\ell/\mathbb{Q}, 1)$ vanishes.
We claim next that $T\Omega(L_\ell/\mathbb{Q}, 1)$ belongs to the subgroup $K_0(\mathbb{Z}[G_\ell], \mathbb{Q})_{\mathrm{tor}}$ of $K_0(\mathbb{Z}[G_\ell], \mathbb{R})$. Indeed, since $T\Omega^{\mathrm{loc}}(L_\ell/\mathbb{Q}, 1)$ vanishes the equality of [5, Theorem 5.2] implies $T\Omega(L_\ell/\mathbb{Q}, 1) = \psi^*_{G_\ell}(T\Omega(L_\ell/\mathbb{Q}, 0))$ where $\psi^*_{G_\ell}$ is the involution of $K_0(\mathbb{Z}[G_\ell], \mathbb{R})$ defined in [5, §2.1.4] and $T\Omega(L_\ell/\mathbb{Q}, 0)$ the element of $K_0(\mathbb{Z}[G_\ell], \mathbb{R})$ defined in [5, §4]. Now $\psi^*_{G_\ell}$ preserves the subgroup $K_0(\mathbb{Z}[G_\ell], \mathbb{Q})_{\mathrm{tor}}$ and from [5, Prop. 4.4(ii)] one knows that $T\Omega(L_\ell/\mathbb{Q}, 0)$ belongs to $K_0(\mathbb{Z}[G_\ell], \mathbb{Q})_{\mathrm{tor}}$ if the 'strong Stark conjecture' of Chinburg is valid for $L_\ell/\mathbb{Q}$. It thus suffices to recall that, since every complex character of G_ℓ is rational valued, the strong Stark conjecture for $L_\ell/\mathbb{Q}$ has been proved by Tate in [26, Chap. II].
We write F_ℓ for the maximal abelian extension of $\mathbb{Q}$ in L_ℓ (and note that $F_\ell/\mathbb{Q}$ is biquadratic). Then, since the element $T\Omega(L_\ell/\mathbb{Q}, 1)$ belongs to $K_0(\mathbb{Z}[G_\ell], \mathbb{Q})_{\mathrm{tor}}$, the result of [10, Lemma 4] implies $T\Omega(L_\ell/\mathbb{Q}, 1)$ vanishes if it belongs to the kernels of both the natural projection homomorphism

$q : K_0(\mathbb{Z}[G_\ell], \mathbb{R}) \to K_0(\mathbb{Z}[\mathrm{Gal}(F_\ell/\mathbb{Q})], \mathbb{R})$ and the connecting homomorphism $\partial^0_{\mathbb{Z}[G_\ell],\mathbb{R}} : K_0(\mathbb{Z}[G_\ell], \mathbb{R}) \to K_0(\mathbb{Z}[G_\ell])$.

Now from [5, Prop. 3.6(ii)] one knows that $T\Omega(L_\ell/\mathbb{Q}, 1)$ belongs to $\ker(\partial^0_{\mathbb{Z}[G_\ell],\mathbb{R}})$ if Chinburg's 'Ω_1-Conjecture' [13, Question 3.2] is valid for $L_\ell/\mathbb{Q}$. In addition, the equality of [13, (3.2)] shows that the Ω_1-Conjecture is valid for $L_\ell/\mathbb{Q}$ if the 'Ω_3-Conjecture' [13, Conj. 3.1] and 'Ω_2-Conjecture' [13, Question 3.1] are both valid for $L_\ell/\mathbb{Q}$. But Chinburg proves the Ω_3-Conjecture for $L_\ell/\mathbb{Q}$ in [14] and, since $L_\ell/\mathbb{Q}$ is tamely ramified, the validity of the Ω_2-Conjecture for $L_\ell/\mathbb{Q}$ follows directly from [13, Theorems 3.2 and 3.3].

At this stage it suffices to prove that $T\Omega(L_\ell/\mathbb{Q}, 1)$ belongs to $\ker(q)$. But, by [5, Prop. 3.5(ii)], this is equivalent to asserting that [5, Conj. 3.3] is valid for the extension $F_\ell/\mathbb{Q}$ and since $F_\ell/\mathbb{Q}$ is abelian this follows from Corollary 1.3. This completes the proof of Corollary 1.4.

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