Motivically Functorial Coniveau Spectral Sequences; Direct Summands of Cohomology of Function Fields

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Abstract. The goal of this paper is to prove that coniveau spectral sequences are motivically functorial for all cohomology theories that could be factorized through motives. To this end the motif of a smooth variety over a countable field k is decomposed (in the sense of Postnikov towers) into twisted (co)motives of its points; this is generalized to arbitrary Voevodsky's motives. In order to study the functoriality of this construction, we use the formalism of weight structures (introduced in the previous paper). We also develop this formalism (for general triangulated categories) further, and relate it with a new notion of a *nice duality* (pairing) of (two distinct) triangulated categories; this piece of homological algebra could be interesting for itself.

We construct a certain *Gersten* weight structure for a triangulated category of *comotives* that contains DM_{gm}^{eff} as well as (co)motives of function fields over k. It turns out that the corresponding *weight spectral sequences* generalize the classical coniveau ones (to cohomology of arbitrary motives). When a cohomological functor is represented by a $Y \in \operatorname{Obj} DM_{-}^{eff}$, the corresponding coniveau spectral sequences can be expressed in terms of the (homotopy) t-truncations of Y; this extends to motives the seminal coniveau spectral sequence computations of Bloch and Ogus.

We also obtain that the comotif of a smooth connected semi-local scheme is a direct summand of the comotif of its generic point; comotives of function fields contain twisted comotives of their residue fields (for all geometric valuations). Hence similar results hold for any cohomology of (semi-local) schemes mentioned.

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Introduction

Let k be our perfect base field.

We recall two very important statements concerning coniveau spectral sequences. The first one is the calculation of E_2 of the coniveau spectral sequence for cohomological theories that satisfy certain conditions; see [5] and [8]. It was proved by Voevodsky that these conditions are fulfilled by any theory H represented by a motivic complex C (i.e. an object of DM_-^{eff}; see [25]); then the E_2-terms of the spectral sequence could be calculated in terms of the (homotopy t-structure) cohomology of C. This result implies the second one: H-cohomology of a smooth connected semi-local scheme (in the sense of §4.4 of [26]) injects into the cohomology of its generic point; the latter statement was extended to all (smooth connected) primitive schemes by M. Walker.

The main goal of the present paper is to construct (motivically) functorial coniveau spectral sequences converging to cohomology of arbitrary motives; there should exist a description of these spectral sequences (starting from E_2) that is similar to the description for the case of cohomology of smooth varieties (mentioned above).

A related objective is to clarify the nature of the injectivity result mentioned; it turned our that (in the case of a countable k) the cohomology of a smooth connected semi-local (more generally, primitive) scheme is actually a direct summand of the cohomology of its generic point. Moreover, the (twisted) cohomology of a residue field of a function field K/k (for any geometric valuation of K) is a direct summand of the cohomology of K. We actually prove more in §4.3.

Our main homological algebra tool is the theory of *weight structures* (in triangulated categories; we usually denote a weight structure by w) introduced in the previous paper [6]. In this article we develop it further; this part of the paper could be interesting also to readers not acquainted with motives (and could be read independently from the rest of the paper). In particular, we study *nice dualities* (certain pairings) of (two distinct) triangulated categories; it seems that this subject was not previously considered in the literature at all. This allows us to generalize the concept of *adjacent* weight and t-structures (t) in a triangulated category (developed in §4.4 of [6]): we introduce the notion of *orthogonal structures* in (two possibly distinct) triangulated categories. If Φ is a nice duality of triangulated $\underline{C}, \underline{D}$, $X \in Obj\underline{C}$, $Y \in Obj\underline{D}$, t is orthogonal to w, then the spectral sequence S converging to $\Phi(X,Y)$ that comes from the t-truncations of Y is naturally isomorphic (starting from E_2) to the *weight spectral sequence* T for the functor $\Phi(-,Y)$. T comes from *weight truncations* of X (note that those generalize stupid truncations for complexes). Our approach yields an abstract alternative to the method of comparing similar spectral sequences using filtered complexes (developed by Deligne and Paranjape, and used in [22], [11], and [6]). Note also that we relate t-truncations in $\underline{D}$ with *virtual t-truncations* of cohomological functors on $\underline{C}$. Virtual t-truncations for cohomological functors are defined for any $(\underline{C}, w)$ (we do not need any triangulated 'categories of functors' or t-structures for them here); this notion was introduced in §2.5 of [6] and is studied further in the current paper.
Now, we explain why we really need a certain new category of *comotives* (containing Voevodsky's DM^{eff}_{gm}), and so the theory of adjacent structures (i.e. orthogonal structures in the case $\underline{C} = \underline{D}$, $\Phi = \underline{C}(-,-)$) is not sufficient for our purposes. It was already proved in [6] that weight structures provide a powerful tool for constructing spectral sequences; they also relate the cohomology of objects of triangulated categories with t-structures adjacent to them. Unfortunately, a weight structure corresponding to coniveau spectral sequences cannot exist on $DM^{eff}_{-} \supset DM^{eff}_{gm}$ since these categories do not contain (any) motives for function fields over k (as well as motives of other schemes not of finite type over k; still cf. Remark 4.5.4(5)). Yet these motives should generate the *heart* of this weight structure (since the objects of this heart should corepresent covariant exact functors from the category of homotopy invariant sheaves with transfers to Ab).
So, we need a category that would contain certain homotopy limits of objects of DM^{eff}_{gm}. We succeed in constructing a triangulated category $\mathfrak{D}$ (of *comotives*) that allows us to reach the objectives listed. Unfortunately, in order to control morphisms between homotopy limits mentioned we have to assume k to be countable. In this case there exists a large enough triangulated category $\mathfrak{D}_s$ ($DM^{eff}_{gm} \subset \mathfrak{D}_s \subset \mathfrak{D}$) endowed with a certain *Gersten weight structure* w; its heart is 'generated' by comotives of function fields. w is (left) orthogonal to the homotopy t-structure on DM^{eff}_{-} and (so) is closely connected with coniveau spectral sequences and Gersten resolutions for sheaves. Note still: we need k to be countable only in order to construct the Gersten weight structure. So

those readers who would just want to have a category that contains reasonable homotopy limits of geometric motives (including comotives of function fields and of smooth semi-local schemes), and consider cohomology theories for this category, may freely ignore this restriction. Moreover, for an arbitrary k one can still pass to a countable homotopy limit in the Gysin distinguished triangle (as in Proposition 3.6.1). Yet for an uncountable k countable homotopy limits don't seem to be interesting; in particular, they definitely do not allow to construct a Gersten weight structure (in this case).
So, we consider a certain triangulated category $\mathfrak{D} \supset DM_{gm}^{eff}$ that (roughly!) 'consists of' (covariant) homological functors $DM_{gm}^{eff} \to Ab$. In particular, objects of $\mathfrak{D}$ define covariant functors $SmVar \to Ab$ (whereas another 'big' motivic category DM_{-}^{eff} defined by Voevodsky is constructed from certain sheaves i.e. contravariant functors $SmVar \to Ab$; this is also true for all motivic homotopy categories of Voevodsky and Morel). Besides, DM_{gm}^{eff} yields a family of (weak) cocompact cogenerators for $\mathfrak{D}$. This is why we call objects of $\mathfrak{D}$ comotives. Yet note that the embedding $DM_{gm}^{eff} \to \mathfrak{D}$ is covariant (actually, we invert the arrows in the corresponding 'category of functors' in order to make the Yoneda embedding functor covariant), as well as the functor that sends a smooth scheme U (not necessarily of finite type over k) to its comotif (which coincides with its motif if U is a smooth variety).
We also recall the Chow weight structure w'_{Chow} introduced in [6]; the corresponding *Chow-weight* spectral sequences are isomorphic to the classical (i.e. Deligne's) weight spectral sequences when the latter are defined. w'_{Chow} could be naturally extended to a weight structure w_{Chow} for $\mathfrak{D}$. We always have a natural comparison morphism from the Chow-weight spectral sequence for (H, X) to the corresponding coniveau one; it is an isomorphism for any birational cohomology theory. We consider the category of birational comotives $\mathfrak{D}_{bir}$ i.e. the localization of $\mathfrak{D}$ by $\mathfrak{D}(1)$ (that contains the category of birational geometric motives introduced in [15]; though some of the results of this unpublished preprint are erroneous, this makes no difference for the current paper). It turns our that w and w_{Chow} induce the same weight structure w'_{bir} on $\mathfrak{D}_{bir}$. Conversely, starting from w'_{bir} one can 'glue' (from *slices*) the weight structures induced by w and w_{Chow} on $\mathfrak{D}/\mathfrak{D}(n)$ for all $n > 0$. Moreover, these structures belong to an interesting family of weight structures indexed by a single integral parameter! It could be interesting to consider other members of this family. We relate briefly these observations with those of A. Beilinson (in [3] he proposed a 'geometric' characterization of the conjectural motivic t-structure).
Now we describe the connection of our results with related results of F. Deglise (see [9], [10], and [11]; note that the two latter papers are not published at the moment yet). He considers a certain category of pro-motives whose objects are naive inverse limits of objects of DM_{gm}^{eff} (this category is not triangulated, though it is *pro-triangulated* in a certain sense). This approach allows to obtain (in a universal way) classical coniveau spectral sequences for cohomology of motives of smooth varieties; Deglise also proves their relation with the homotopy t-truncations for cohomology represented by an object of DM_{-}^{eff}. Yet for

cohomology theories not coming from motivic complexes, this method does not seem to extend to (spectral sequences for cohomology of) arbitrary motives; motivic functoriality is not obvious also. Moreover, Deglise didn't prove that the pro-motif of a (smooth connected) semi-local scheme is a direct summand of the pro-motif of its generic point (though this is true, at least in the case of a countable k). We will tell much more about our strategy and on the relation of our results with those of Deglise in §1.5 below. Note also that our methods are much more convenient for studying functoriality (of coniveau spectral sequences) than the methods applied by M. Rost in the related context of cycle modules (see [24] and §4 of [10]).

The author would like to indicate the interdependencies of the parts of this text (in order to simplify reading for those who are not interested in all of it). Those readers who are not (very much) interested in (coniveau) spectral sequences, may avoid most of section 2 and read only §§2.1 –2.2 (Remark 2.2.2 could also be ignored). Moreover, in order to prove our direct summands results (i.e. Theorem 4.2.1, Corollary 4.2.2, and Proposition 4.3.1) one needs only a small portion of the theory of weight structures; so a reader very reluctant to study this theory may try to derive them from the results of §3 'by hand' without reading §2 at all. Still, for motivic functoriality of coniveau spectral sequences and filtrations (see Proposition 4.4.1 and Remark 4.4.2) one needs more of weight structures. On the other hand, those readers who are more interested in the (general) theory of triangulated categories may restrict their attention to §§1.1– 1.2 and §2; yet note that the rest of the paper describes in detail an important (and quite non-trivial) example of a weight structure which is orthogonal to a t-structure with respect to a nice duality (of triangulated categories). Moreover, much of section §4 could also be extended to a general setting of a triangulated category satisfying properties similar to those listed in Proposition 3.1.1; yet the author chose not to do this in order to make the paper somewhat less abstract.

Now we list the contents of the paper. More details could be found at the beginnings of sections.

We start §1 with the recollection of t-structures, idempotent completions, and Postnikov towers for triangulated categories. We describe a method for extending cohomological functors from a full triangulated subcategory to the whole $\underline{C}$ (after H. Krause). Next we recall some results and definitions for Voevodsky's motives (this includes certain properties of Tate twists for motives and cohomological functors). Lastly, we define pro-motives (following Deglise) and compare them with our triangulated category $\mathfrak{D}$ of comotives. This allows to explain our strategy step by step.

§2 is dedicated to weight structures. First we remind the basics of this theory (developed in §[6]). Next we recall that a cohomological functor H from an (arbitrary triangulated category) $\underline{C}$ endowed with a weight structure w could be 'truncated' as if it belonged to some triangulated category of functors (from $\underline{C}$) that is endowed with a t-structure; we call the corresponding pieces of H its *virtual t-truncations*. We recall the notion of a weight spectral sequence (introduces in ibid.). We prove that the derived exact couple for a weight spectral sequence could be described in terms of virtual t-truncations. Next we introduce the definition a *(nice) duality* $\Phi : \underline{C}^{op} \times \underline{D} \to \underline{A}$ (here $\underline{D}$ is triangulated, $\underline{A}$ is abelian), and of *orthogonal* weight and t-structures (with respect to Φ). If w is orthogonal to t, then the virtual t-truncations (corresponding to w) of functors of the type $\Phi(-,Y)$, $Y \in Obj\underline{D}$, are exactly the functors 'represented via Φ' by the actual t-truncations of Y (corresponding to t). Hence if w and t are orthogonal with respect to a nice duality, the weight spectral sequence converging to $\Phi(X,Y)$ (for $X \in Obj\underline{C}$, $Y \in Obj\underline{D}$) is naturally isomorphic (starting from E_2) to the one coming from t-truncations of Y. We also mention some alternatives and predecessors of our results. Lastly we compare weight decompositions, virtual t-truncations, and weight spectral sequences corresponding to distinct weight structures (in possibly distinct triangulated categories).

In §3 we describe the main properties of $\mathfrak{D} \supset DM^{eff}_{gm}$. The exact choice of $\mathfrak{D}$ is not important for most of this paper; so we just list the main properties of $\mathfrak{D}$ (and its certain *enhancement* $\mathfrak{D}'$) in §3.1. We construct $\mathfrak{D}$ using the formalism of differential graded modules in §5 later. Next we define comotives for (certain) schemes and ind-schemes of infinite type over k (we call them pro-schemes). We recall the notion of a primitive scheme. All (smooth) semi-local pro-schemes are primitive; primitive schemes have all nice 'motivic' properties of semi-local pro-schemes. We prove that there are no $\mathfrak{D}$-morphisms of positive degrees between comotives of primitive schemes (and also between certain Tate twists of those). In §3.6 we prove that the Gysin distinguished triangle for motives of smooth varieties (in DM^{eff}_{gm}) could be naturally extended to comotives of pro-schemes. This allows to construct certain Postnikov towers for comotives of pro-schemes; these towers are closely related with classical coniveau spectral sequences for cohomology.

§4 is central in this paper. We introduce a certain *Gersten weight structure* for a certain triangulated category $\mathfrak{D}_s$ ($DM^{eff}_{gm} \subset \mathfrak{D}_s \subset \mathfrak{D}$). We prove that Postnikov towers constructed in §3.6 are actually *weight Postnikov towers* with respect to w. We deduce our (interesting) results on direct summands of comotives of function fields. We translate these results to cohomology in the obvious way.

Next we prove that weight spectral sequences for the cohomology of X (corresponding to the Gersten weight structure) are naturally isomorphic (starting from E_2) to the classical coniveau spectral sequences if X is the motif of a smooth variety; so we call these spectral sequence coniveau ones in the general case also. We also prove that the Gersten weight structure w (on $\mathfrak{D}_s$) is orthogonal to the homotopy t-structure t on DM^{eff}_{-} (with respect to a certain Φ). It follows that for an arbitrary $X \in ObjDM^s$, for a cohomology theory represented by $Y \in ObjDM^{eff}_{-}$ (any choice of) the coniveau spectral sequence that converges to $\Phi(X,Y)$ could be described in terms of the t-truncations of Y (starting from E_2).

We also define coniveau spectral sequences for cohomology of motives over uncountable base fields as the limits of the corresponding coniveau spectral

sequences over countable perfect subfields of definition. This definition is compatible with the classical one; so we establish motivic functoriality of coniveau spectral sequences in this case also.
We also prove that the *Chow weight structure* for DM_{gm}^{eff} (introduced in §6 of [6]) could be extended to a weight structure w_{Chow} on $\mathfrak{D}$. The corresponding *Chow-weight* spectral sequences are isomorphic to the classical (i.e. Deligne's) ones when the latter are defined (this was proved in [6] and [7]). We compare coniveau spectral sequences with Chow-weight ones: we always have a comparison morphism; it is an isomorphism for a *birational* cohomology theory. We consider the category of birational comotives $\mathfrak{D}_{bir}$ i.e. the localization of $\mathfrak{D}$ by $\mathfrak{D}(1)$. w and w_{Chow} induce the same weight structure w'_{bir} on $\mathfrak{D}_{bir}$; one almost can glue w and w_{Chow} from copies of w'_{bir} (one may say that these weight structures could almost be glued from the same slices with distinct shifts).
§5 is dedicated to the construction of $\mathfrak{D}$ and the proof of its properties. We apply the formalism of differential graded categories, modules over them, and of the corresponding derived categories. A reader not interested in these details may skip (most of) this section. In fact, the author is not sure that there exists only one $\mathfrak{D}$ suitable for our purposes; yet the choice of $\mathfrak{D}$ does not affect cohomology of (comotives of) pro-schemes and of Voevodsky's motives.
We also explain how the differential graded modules formalism can be used to define base change (extension and restriction of scalars) for comotives. This allows to extend our results on direct summands of comotives (and cohomology) of function fields to pro-schemes obtained from them via base change. We also define tensoring of comotives by motives (in particular, this yields Tate twist for $\mathfrak{D}$), as well as a certain cointernal Hom (i.e. the corresponding left adjoint functor).
§6 is dedicated to properties of comotives that are not (directly) related with the main results of the paper; we also make several comments. We recall the definition of the additive category $\mathfrak{D}^{gen}$ of generic motives (studied in [9]). We prove that the exact conservative *weight complex* functor corresponding to w (that exists by the general theory of weight structures) could be modified to an exact conservative $WC : \mathfrak{D}_s \to K^b(\mathfrak{D}^{gen})$. Next we prove that a cofunctor $\underline{Hw} \to Ab$ is representable by a homotopy invariant sheaf with transfers whenever is converts all products into direct sums.
We also note that our theory could be easily extended to (co)motives with coefficients in an arbitrary ring. Next we note (after B. Kahn) that reasonable motives of pro-schemes with compact support do exist in DM_{-}^{eff}; this observation could be used for the construction of an alternative model for $\mathfrak{D}$. Lastly we describe which parts of our argument do not work (and which do work) in the case of an uncountable k.
A caution: the notion of a weight structure is quite a general formalism for triangulated categories. In particular, one triangulated category can support several distinct weight structures (note that there is a similar situation with t-structures). In fact, we construct an example for such a situation in this paper (certainly, much simpler examples exist): we define the Gersten weight

structure w for $\mathfrak{D}_s$ and a Chow weight structure w_{Chow} for $\mathfrak{D}$. Moreover, we show in §4.9 that these weight structures are compatible with certain weight structures defined on the localizations $\mathfrak{D}/\mathfrak{D}(n)$ (for all $n > 0$). These two series of weight structures are definitely distinct: note that w yields coniveau spectral sequences, whereas w_{Chow} yields Chow-weight spectral sequences, that generalize Deligne's weight spectral sequences for étale and mixed Hodge cohomology (see [6] and [7]). Also, the weight complex functor constructed in [7] and [6] is quite distinct from the one considered in §6.1 below (even the targets of the functors mentioned are completely distinct).

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Notation. For a category C, $A, B \in ObjC$, we denote by $C(A, B)$ the set of A-morphisms from A into B.

For categories C, D we write $C \subset D$ if C is a full subcategory of D.

For additive C, D we denote by $\mathrm{AddFun}(C, D)$ the category of additive functors from C to D (we will ignore set-theoretic difficulties here since they do not affect our arguments seriously).

Ab is the category of abelian groups. For an additive B we will denote by B^* the category $\mathrm{AddFun}(B, Ab)$ and by B_* the category $\mathrm{AddFun}(B^{op}, Ab)$. Note that both of these are abelian. Besides, Yoneda's lemma gives full embeddings of B into B_* and of B^{op} into B^* (these send $X \in ObjB$ to $X_* = B(-, X)$ and to $X^* = B(X, -)$, respectively).

For a category C, $X, Y \in ObjC$, we say that X is a *retract* of Y if id_X could be factorized through Y. Note that when C is triangulated or abelian then X is a retract of Y if and only if X is its direct summand. For any $D \subset C$ the subcategory D is called *Karoubi-closed* in C if it contains all retracts of its objects in C. We will call the smallest Karoubi-closed subcategory of C containing D the *Karoubization* of D in C; sometimes we will use the same term for the class of objects of the Karoubization of a full subcategory of C (corresponding to some subclass of $ObjC$).

For a category C we denote by C^{op} its opposite category.

For an additive $\underline{C}$ an object $X \in Obj\underline{C}$ is called cocompact if $\underline{C}(\prod_{i\in I} Y_i, X) = \bigoplus_{i\in I} \underline{C}(Y_i, X)$ for any set I and any $Y_i \in Obj\underline{C}$ such that the product exists (here we don't need to demand all products to exist, though they actually will exist below).

For $X, Y \in Obj\underline{C}$ we will write $X \perp Y$ if $\underline{C}(X, Y) = \{0\}$. For $D, E \subset Obj\underline{C}$ we will write $D \perp E$ if $X \perp Y$ for all $X \in D$, $Y \in E$. For $D \subset \underline{C}$ we will denote by $D^{\perp}$ the class

$$\{Y \in Obj\underline{C} : \ X \perp Y \ \forall X \in D\}.$$

Sometimes we will denote by $D^{\perp}$ the corresponding full subcategory of $\underline{C}$. Dually, ${}^{\perp}D$ is the class $\{Y \in Obj\underline{C} : \ Y \perp X \ \forall X \in D\}$. This convention is opposite to the one of §9.1 of [21].

In this paper all complexes will be cohomological i.e. the degree of all differentials is $+1$; respectively, we will use cohomological notation for their terms.

For an additive category B we denote by $C(B)$ the category of (unbounded) complexes over it. $K(B)$ will denote the homotopy category of complexes. If B is also abelian, we will denote by $D(B)$ the derived category of B. We will also need certain bounded analogues of these categories (i.e. $C^b(B)$, $K^b(B)$, $D^-(B)$).

$\underline{C}$ and $\underline{D}$ will usually denote some triangulated categories. We will use the term 'exact functor' for a functor of triangulated categories (i.e. for a for a functor that preserves the structures of triangulated categories).

$\underline{A}$ will usually denote some abelian category. We will call a covariant additive functor $\underline{C} \to \underline{A}$ for an abelian $\underline{A}$ *homological* if it converts distinguished triangles into long exact sequences; homological functors $\underline{C}^{op} \to \underline{A}$ will be called *cohomological* when considered as contravariant functors $\underline{C} \to \underline{A}$.

$H : \underline{C}^{op} \to \underline{A}$ will always be additive; it will usually be cohomological.

For $f \in \underline{C}(X,Y)$, $X,Y \in Obj\underline{C}$, we will call the third vertex of (any) distinguished triangle $X \xrightarrow{f} Y \to Z$ a cone of f. Note that different choices of cones are connected by non-unique isomorphisms, cf. IV.1.7 of [13]. Besides, in $C(B)$ we have canonical cones of morphisms (see section §III.3 of ibid.).

We will often specify a distinguished triangle by two of its morphisms.

When dealing with triangulated categories we (mostly) use conventions and auxiliary statements of [13]. For a set of objects $C_i \in Obj\underline{C}$, $i \in I$, we will denote by $\langle C_i \rangle$ the smallest strictly full triangulated subcategory containing all C_i; for $D \subset \underline{C}$ we will write $\langle D \rangle$ instead of $\langle Obj D \rangle$.

We will say that C_i generate $\underline{C}$ if $\underline{C}$ equals $\langle C_i \rangle$. We will say that C_i *weakly cogenerate* $\underline{C}$ if for $X \in Obj\underline{C}$ we have $\underline{C}(X, C_i[j]) = \{0\}\ \forall i \in I,\ j \in \mathbb{Z} \implies X = 0$ (i.e. if ${}^{\perp}\{C_i[j]\}$ contains only zero objects).

We will call a partially ordered set L a (filtered) *projective system* if for any $x, y \in L$ there exists some maximum i.e. a $z \in L$ such that $z \geq x$ and $z \geq y$. By abuse of notation, we will identify L with the following category D: $Obj D = L$; $D(l', l)$ is empty whenever $l' < l$, and consists of a single morphism otherwise; the composition of morphisms is the only one possible. If L is a projective system, C is some category, $X : L \to C$ is a covariant functor, we will denote $X(l)$ for $l \in L$ by X_l. We will write $Y = \varprojlim_{l \in L} X_l$ for the limit of this functor; we will call it the inverse limit of X_l. We will denote the colimit of a contravariant functor $Y : L \to C$ by $\varinjlim_{l \in L} Y_l$ and call it the direct limit. Besides, we will sometimes call the categorical image of L with respect to such an Y an *inductive system*.

Below I, L will often be projective systems; we will usually require I to be countable.

A subsystem L' of L is a partially ordered subset in which maximums exist (we will also consider the corresponding full subcategory of L). We will call L' unbounded in L if for any $l \in L$ there exists an $l' \in L'$ such that $l' \geq l$.

k will be our perfect base field. Below we will usually demand k to be countable. Note: this yields that for any variety the set of its closed (or open) subschemes is countable.

We also list central definitions and main notation of this paper.

First we list the main (general) homological algebra definitions. t-structures, t-truncations, and Postnikov towers in triangulated categories are defined in §1.1; weight structures, weight decompositions, weight truncations, weight Postnikov towers, and weight complexes are considered in §2.1; virtual t-truncations and nice exact complexes of functors are defined in §2.3; weight spectral sequences are studied in §2.4; (nice) dualities and orthogonal weight and t-structures are defined in Definition 2.5.1; right and left weight-exact functors are defined in Definition 2.7.1.

Now we list notation (and some definitions) for motives. $DM^{eff}_{gm} \subset DM^{eff}_{-}$, HI and the homotopy t-structure for DM^{eff}_{gm} are defined in §1.3; Tate twists are considered in §1.4; $\mathfrak{D}^{naive}$ is defined in §1.5; comotives ($\mathfrak{D}$ and $\mathfrak{D}'$) are defined in §3.1; in §3.2 we discuss pro-schemes and their comotives; in §3.3 we recall the definition of a primitive scheme; in §4.1 we define the Gersten weight structure w on a certain triangulated $\mathfrak{D}_s$; we consider w_{Chow} in §4.7; $\mathfrak{D}_{bir}$ and w'_{bir} are defined in §4.9; several differential graded constructions (including extension and restriction of scalars for comotives) are considered in §5; we define $\mathfrak{D}^{gen}$ and $WC : \mathfrak{D}_s \to K^b(\mathfrak{D}^{gen})$ in §6.1.

1 Some preliminaries on triangulated categories and motives

§1.1 we recall the notion of a t-structure (and introduce some notation for it), recall the notion of an idempotent completion of an additive category; we also recall that any small abelian category could be faithfully embedded into Ab (a well-known result by Mitchell).

In §1.2 we describe (following H. Krause) a natural method for extending cohomological functors from a full triangulated $\underline{C}' \subset \underline{C}$ to $\underline{C}$.

In §1.3 we recall some definitions and results of Voevodsky.

In §1.4 we recall the notion of a Tate twist; we study the properties of Tate twists for motives and homotopy invariant sheaves.

In §1.5 we define pro-motives (following [9] and [10]). These are not necessary for our main result; yet they allow to explain our methods step by step. We also describe in detail the relation of our constructions and results with those of Deglise.

1.1 t-structures, Postnikov towers, idempotent completions, and an embedding theorem of Mitchell

To fix the notation we recall the definition of a t-structure.

Definition 1.1.1. A pair of subclasses $\underline{C}^{t\ge 0}, \underline{C}^{t\le 0} \subset Obj\underline{C}$ for a triangulated category $\underline{C}$ will be said to define a t-structure t if $(\underline{C}^{t\ge 0}, \underline{C}^{t\le 0})$ satisfy the following conditions:

(i) $\underline{C}^{t\geq 0}, \underline{C}^{t\leq 0}$ are strict i.e. contain all objects of $\underline{C}$ isomorphic to their elements.
(ii) $\underline{C}^{t\geq 0} \subset \underline{C}^{t\geq 0}[1]$, $\underline{C}^{t\leq 0}[1] \subset \underline{C}^{t\leq 0}$.
(iii) **Orthogonality**. $\underline{C}^{t\leq 0}[1] \perp \underline{C}^{t\geq 0}$.
(iv) t-**decomposition**. For any $X \in Obj\underline{C}$ there exists a distinguished triangle

$$A \to X \to B[-1] \to A[1] \tag{1}$$

such that $A \in \underline{C}^{t\leq 0}, B \in \underline{C}^{t\geq 0}$.

We will need some more notation for t-structures.

Definition 1.1.2. 1. A category $\underline{Ht}$ whose objects are $\underline{C}^{t=0} = \underline{C}^{t\geq 0} \cap \underline{C}^{t\leq 0}$, $\underline{Ht}(X,Y) = \underline{C}(X,Y)$ for $X, Y \in \underline{C}^{t=0}$, will be called the *heart* of t. Recall (cf. Theorem 1.3.6 of [2]) that $\underline{Ht}$ is abelian (short exact sequences in $\underline{Ht}$ come from distinguished triangles in $\underline{C}$).
2. $\underline{C}^{t\geq l}$ (resp. $\underline{C}^{t\leq l}$) will denote $\underline{C}^{t\geq 0}[-l]$ (resp. $\underline{C}^{t\leq 0}[-l]$).

Remark 1.1.3. 1. The axiomatics of t-structures is self-dual: if $\underline{D} = \underline{C}^{op}$ (so $Obj\underline{C} = Obj\underline{D}$) then one can define the (opposite) weight structure t' on $\underline{D}$ by taking $\underline{D}^{t'\leq 0} = \underline{C}^{t\geq 0}$ and $\underline{D}^{t'\geq 0} = \underline{C}^{t\leq 0}$; see part (iii) of Examples 1.3.2 in [2].
2. Recall (cf. Lemma IV.4.5 in [13]) that (1) defines additive functors $\underline{C} \to \underline{C}^{t\leq 0} : X \to A$ and $C \to \underline{C}^{t\geq 0} : X \to B$. We will denote A, B by $X^{t\leq 0}$ and $X^{t\geq 1}$, respectively.
3. (1) will be called the *t-decomposition* of X. If $X = Y[i]$ for some $Y \in Obj\underline{C}$, $i \in \mathbb{Z}$, then we will denote A by $Y^{t\leq i}$ (it belongs to $\underline{C}^{t\leq 0}$) and B by $Y^{t\geq i+1}$ (it belongs to $\underline{C}^{t\geq 0}$), respectively. Sometimes we will denote $Y^{t^{\leq}i}[-i]$ by $t_{\leq i}Y$; $t_{\geq i+1}Y = Y^{t^{\geq}i+1}[-i-1]$. Objects of the type $Y^{t^{\leq}i}[j]$ and $Y^{t^{\geq}i}[j]$ (for $i, j \in \mathbb{Z}$) will be called *t-truncations of* Y.
4. We denote by $X^{t=i}$ the i-th cohomology of X with respect to t i.e. $(Y^{t\leq i})^{t\geq 0}$ (cf. part 10 of §IV.4 of [13]).
5. The following statements are obvious (and well-known): $\underline{C}^{t\leq 0} = {}^{\perp}\underline{C}^{t\geq 1}$; $\underline{C}^{t\geq 0} = \underline{C}^{t\leq -1\perp}$.

Now we recall the notion of idempotent completion.

Definition 1.1.4. An additive category B is said to be *idempotent complete* if for any $X \in ObjB$ and any idempotent $p \in B(X,X)$ there exists a decomposition $X = Y \bigoplus Z$ such that $p = i \circ j$, where i is the inclusion $Y \to Y \bigoplus Z$, j is the projection $Y \bigoplus Z \to Y$.

Recall that any additive B can be canonically idempotent completed. Its idempotent completion is (by definition) the category B' whose objects are (X,p) for $X \in ObjB$ and $p \in B(X,X) :\ p^2 = p$; we define

$$A'((X,p),(X',p')) = \{f \in B(X,X') :\ p'f = fp = f\}.$$

It can be easily checked that this category is additive and idempotent complete, and for any idempotent complete $C \supset B$ we have a natural full embedding $B' \to C$.
The main result of [1] (Theorem 1.5) states that an idempotent completion of a triangulated category $\underline{C}$ has a natural triangulation (with distinguished triangles being all retracts of distinguished triangles of $\underline{C}$).
Below we will need the notion of a Postnikov tower in a triangulated category several times (cf. §IV2 of [13])).

Definition 1.1.5. Let $\underline{C}$ be a triangulated category.
1. Let $l \le m \in \mathbb{Z}$.
We will call a bounded Postnikov tower for $X \in Obj\underline{C}$ the following data: a sequence of $\underline{C}$-morphisms $(0 =)Y_l \to Y_{l+1} \to \cdots \to Y_m = X$ along with distinguished triangles
$$Y_i \to Y_{i+1} \to X_i \qquad (2)$$
for some $X_i \in Obj\underline{C}$; here $l \le i < m$.
2. An unbounded Postnikov tower for X is a collection of Y_i for $i \in \mathbb{Z}$ that is equipped (for all $i \in \mathbb{Z}$) with: connecting arrows $Y_i \to Y_{i+1}$ (for $i \in \mathbb{Z}$), morphisms $Y_i \to X$ such that all the corresponding triangles commute, and distinguished triangles (2).
In both cases, we will denote $X_{-p}[p]$ by X^p; we will call X^p the *factors* of out Postnikov tower.

Remark 1.1.6. 1. Composing (and shifting) arrows from triangles (2) for two subsequent i one can construct a complex whose terms are X^p (it is easily seen that this is a complex indeed, cf. Proposition 2.2.2 of [6]). This observation will be important for us below when we will consider certain weight complex functors.
2. Certainly, a bounded Postnikov tower could be easily completed to an unbounded one. For example, one could take $Y_i = 0$ for $i < l$, $Y_i = X$ for $i > m$; then $X^i = 0$ if $i < l$ or $i \ge m$.
Lastly, we recall the following (well-known) result.

Proposition 1.1.7. *For any small abelian category $\underline{A}$ there exists an exact faithful functor $\underline{A} \to Ab$.*

Proof. By the Freyd-Mitchell's embedding theorem, any small $\underline{A}$ could be fully faithfully embedded into $R-\mathrm{mod}$ for some (associative unital) ring R. It remains to apply the forgetful functor $R-\mathrm{mod} \to Ab$. □

Remark 1.1.8. 1. We will need this statement below in order to assume that objects of $\underline{A}$ 'have elements'; this will considerably simplify diagram chase. Note that we can assume the existence of elements for a not necessarily small $\underline{A}$ in the case when a reasoning deals only with a finite number of objects of $\underline{A}$ at a time.
2. In the proof it suffices to have a faithful embedding $\underline{A} \to R-\mathrm{mod}$; this weaker assertion was also proved by Mitchell.

1.2 Extending cohomological functors from a triangulated subcategory

We describe a method for extending cohomological functors from a full triangulated $\underline{C}' \subset \underline{C}$ to $\underline{C}$ (after H. Krause). Note that below we will apply some of the results of [17] in the dual form. The construction requires $\underline{C}'$ to be skeletally small i.e. there should exist a (proper) subset $D \subset Obj\underline{C}'$ such that any object of $\underline{C}'$ is isomorphic to some element of D. For simplicity, we will sometimes (when writing sums over $Obj\underline{C}'$) assume that $Obj\underline{C}'$ is a set itself. Since the distinction between small and skeletally small categories will not affect our arguments and results, we will ignore it in the rest of the paper.
If $\underline{A}$ is an abelian category, then $\mathrm{AddFun}(\underline{C}'^{op}, \underline{A})$ is abelian also; complexes in it are exact whenever they are exact componentwisely.
Suppose that $\underline{A}$ satisfies AB5 i.e. it is closed with respect to all small coproducts, and filtered direct limits of exact sequences in $\underline{A}$ are exact.
Let $H' \in \mathrm{AddFun}(\underline{C}'^{op}, \underline{A})$ be an additive functor (it will usually be cohomological).

Proposition 1.2.1. *I Let $\underline{A}, H'$ be fixed.*
1. There exists an extension of H' to an additive functor $H : \underline{C} \to \underline{A}$. It is cohomological whenever H is. The correspondence $H' \to H$ defines an additive functor $\mathrm{AddFun}(\underline{C}'^{op}, \underline{A}) \to \mathrm{AddFun}(\underline{C}^{op}, \underline{A})$.
2. Moreover, suppose that in $\underline{C}$ we have a projective system X_l, $l \in L$, equipped with a compatible system of morphisms $X \to X_l$, such that the latter system for any $Y \in Obj\underline{C}'$ induces an isomorphism $\underline{C}(X, Y) \cong \varinjlim \underline{C}(X_l, Y)$. Then we have $H(X) \cong \varinjlim H(X_l)$.
II Let $X \in Obj\underline{C}$ be fixed.
1. One can choose a family of $X_l \in Obj\underline{C}$ and $f_l \in \underline{C}(X, X_l)$ such that (f_l) induce a surjection $\oplus H'(X_l) \to H(X)$ for any $H', \underline{A}$, and H as in assertion I1.
2. Let $F' \xrightarrow{f'} G' \xrightarrow{g'} H'$ be a (three-term) complex in $\mathrm{AddFun}(\underline{C}'^{op}, \underline{A})$ *that is exact in the middle; suppose that H' is cohomological. Then the complex $F \xrightarrow{f} G \xrightarrow{g} H$ (here F, G, H, f, g are the corresponding extensions) is exact in the middle also.*

Proof. I1. Following §1.2 of [17] (and dualizing it), we consider the abelian category $C = \underline{C}'^* = \mathrm{AddFun}(\underline{C}', Ab)$ (this is Mod $\underline{C}'^{op}$ in the notation of Krause). The definition easily implies that direct limits in C are exactly componentwise direct limits of functors. We have the Yoneda's functor $i' : \underline{C}^{op} \to C$ that sends $X \in Obj\underline{C}$ to the functor $X^* = (Y \mapsto \underline{C}(X, Y),\ Y \in Obj\underline{C}')$; it is obviously cohomological. We denote by i the restriction of i' to $\underline{C}'$ (i is opposite to a full embedding).
By Lemma 2.2 of [17] (applied to the category $\underline{C}'^{op}$) we obtain that there exists an exact functor $G : C \to \underline{A}$ that preserves all small coproducts and satisfies $G \circ i = H'$. It is constructed in the following way: if for $X \in Obj\underline{C}$ we have an

exact sequence (in C)

$$\oplus_{j\in J}\, X_j^* \to \oplus_{l\in L} X_l^* \to X^* \to 0 \tag{3}$$

for $X_j, X_l \in C'$, then we set

$$G(X) = \operatorname{Coker} \oplus_{j\in J} H'(X_j) \to \oplus_{l\in L} H'(X_l). \tag{4}$$

We define $H = G \circ i'$; it was proved in loc.cit. that we obtain a well-defined functor this way. As was also proved in loc.cit., the correspondence $H' \mapsto H$ yields a functor; H is cohomological if H' is.
2. The proof of loc.cit. shows (and mentions) that G respects (small) filtered inverse limits. Now note that our assertions imply: $X^* = \varinjlim X_l^*$ in C.
II 1. This is immediate from (4).
2. Note that the assertion is obviously valid if $X \in Obj\underline{C}'$. We reduce the general statement to this case.
Applying Yoneda's lemma to (3) is we obtain (canonically) some morphisms $f_l : X \to X_l$ for all $l \in L$ and $g_{lj} : X_l \to X_j$ for all $l \in L$, $j \in J$, such that: for any $l \in L$ almost all g_{lj} are 0; for any $j \in J$ almost all g_{lj} is 0; for any $j \in J$ we have $\sum_{l\in L} g_{lj} \circ f_l = 0$.
Now, by Proposition 1.1.7, we may assume that $\underline{A} = Ab$ (see Remark 1.1.8). We should check: if for $a \in G(X)$ we have $g_*(a) = 0$, then $a = f_*(b)$ for some $b \in F(X)$.
Using additivity of $\underline{C}'$ and $\underline{C}$, we can gather finite sets of X_l and X_j into single objects. Hence we can assume that $a = G(f_{l_0})(c)$ for some $c \in G(X_l)$ ($= G'(X_l)$), $l_0 \in L$ and that $g_*(c) \in H(g_{l_0 j_0})(H(X_{j_0}))$ for some $j_0 \in J$, whereas $g_{l_0 j_0} \circ f_{l_0} = 0$. We complete $X_{l_0} \to X_{j_0}$ to a distinguished triangle $Y \stackrel{\alpha}{\to} X_{l_0} \stackrel{g_{l_0 j_0}}{\to} X_{j_0}$; we can assume that $B \in Obj\underline{C}'$. We obtain that f_{l_0} could be presented as $\alpha \circ \beta$ for some $\beta \in \underline{C}(X, Y)$. Since H' is cohomological, we obtain that $H(\alpha)(g_*(c)) = 0$. Since $Y \in Obj\underline{C}$, the complex $F(Y) \to G(Y) \to H(Y)$ is exact in the middle; hence $G(\alpha)(c) = f_*(d)$ for some $d \in F(Y)$. Then we can take $b = F(\beta)(d)$.

□

1.3 Some definitions of Voevodsky: reminder

We use much notation from [25]. We recall (some of) it here for the convenience of the reader, and introduce some notation of our own.
$Var \supset SmVar \supset SmPrVar$ will denote the class of all varieties over k, resp. of smooth varieties, resp. of smooth projective varieties.
We recall that for categories of geometric origin (in particular, for $SmCor$ defined below) the addition of objects is defined via the disjoint union of varieties operation.
We define the category $SmCor$ of smooth correspondences. $ObjSmCor = SmVar$, $SmCor(X, Y) = \bigoplus_U \mathbb{Z}$ for all integral closed $U \subset X \times Y$ that are finite over X and dominant over a connected component of X; the composition

of correspondences is defined in the usual way via intersections (yet, we do not need to consider correspondences up to an equivalence relation).

We will write $\dots \to X^{i-1} \to X^i \to X^{i+1} \to \dots$, for $X^l \in SmVar$, for the corresponding complex over $SmCor$.

$PreShv(SmCor)$ will denote the (abelian) category of additive cofunctors $SmCor \to Ab$; its objects are usually called *presheaves with transfers*.

$Shv(SmCor) = Shv(SmCor)_{Nis} \subset PreShv(SmCor)$ is the abelian category of additive cofunctors $SmCor \to Ab$ that are sheaves in the Nisnevich topology (when restricted to the category of smooth varieties); these sheaves are usually called *sheaves with transfers*.

$D^-(Shv(SmCor))$ will be the bounded above derived category of $Shv(SmCor)$.

For $Y \in SmVar$ (more generally, for $Y \in Var$, see §4.1 of [25]) we consider $L(Y) = SmCor(-,Y) \in Shv(SmCor)$. For a bounded complex $X = (X^i)$ (as above) we will denote by $L(X)$ the complex $\dots \to L(X^{i-1}) \to L(X^i) \to L(X^{i+1}) \to \dots \in C^b(Shv(SmCor))$.

$S \in Shv(SmCor)$ is called homotopy invariant if for any $X \in SmVar$ the projection $\mathbb{A}^1 \times X \to X$ gives an isomorphism $S(X) \to S(\mathbb{A}^1 \times X)$. We will denote the category of homotopy invariant sheaves (with transfers) by HI; it is an exact abelian subcategory of $SmCor$ by Proposition 3.1.13 of [25].

$DM_-^{eff} \subset D^-(Shv(SmCor))$ is the full subcategory of complexes whose cohomology sheaves are homotopy invariant; it is triangulated by loc.cit. We will need the *homotopy* t-structure on DM_-^{eff}: it is the restriction of the canonical t-structure on $D^-(Shv(SmCor))$ to DM_-^{eff}. Below (when dealing with DM_-^{eff}) we will denote it by just by t. We have $\underline{Ht} = HI$.

We recall the following results of [25].

Proposition 1.3.1. *1. There exists an exact functor* $RC : D^-(Shv(SmCor)) \to DM_-^{eff}$ *right adjoint to the embedding* $DM_-^{eff} \to D^-(Shv(SmCor))$.

2. $DM_-^{eff}(M_{gm}(Y)[-i], F) = \mathbb{H}^i(F)(Y)$ *(the i-th Nisnevich hypercohomology of F computed in Y) for any $Y \in SmVar$.*

3. Denote $RC \circ L$ by M_{gm}. Then the corresponding functor $K^b(SmCor) \to DM_-^{eff}$ could be described as a certain localization of $K^b(SmCor)$.

Proof. See §3 of [25]. □

Remark 1.3.2. 1. In [25] (Definition 2.1.1) the triangulated category DM_{gm}^{eff} (of *effective geometric motives*) was defined as the idempotent completion of a certain localization of $K^b(SmCor)$. This definition is compatible with a *differential graded enhancement* for DM_{gm}^{eff}; cf. §5.3 below. Yet in Theorem 3.2.6 of [25] it was shown that DM_{gm}^{eff} is isomorphic to the idempotent completion of (the categorical image) $M_{gm}(C^b(SmCor))$; this description of DM_{gm}^{eff} will be sufficient for us till §5.

2. In fact, RC could be described in terms of so-called Suslin complexes (see loc.cit.). We will not need this below. Instead, we will just note that RC sends $D^-(Shv(SmCor))^{t\le 0}$ to $DM_-^{eff\,t\le 0}$.

1.4 Some properties of Tate twists

Tate twisting in $DM_-^{eff} \supset DM_{gm}^{eff}$ is given by tensoring by the object $\mathbb{Z}(1)$ (it is often denoted just by $-(1)$). Tate twist has several descriptions and nice properties. We will only need a few of them; our main source is §3.2 of [25]; a more detailed exposition could be found in [20] (see §§8–9).

In order to calculate the tensor product of $X, Y \in ObjDM_-^{eff}$ one should take any preimages X', Y' of X, Y in $ObjD^-(Shv(SmCor))$ with respect to RC (for example, one could take $X' = X$, $Y' = Y$); next one should resolve X, Y by direct sums of $L(Z_i)$ for $Z_i \in SmVar$; lastly one should tensor these resolutions using the identity $L(Z)\otimes L(T) = L(Z\times T)$ for $Z, T \in SmVar$, and apply RC to the result. This tensor product is compatible with the natural tensor product for $K^b(SmCor)$.

We note that any object $D^-(Shv(SmCor))^{t\le 0}$ has a resolution concentrated in negative degrees (the canonical resolution of the beginning of §3.2 of [25]). It follows that $DM_-^{eff\,t\le 0} \otimes DM_-^{eff\,t\le 0} \subset DM_-^{eff\,t\le 0}$ (see Remark 1.3.2(2); in fact, there is an equality since $\mathbb{Z} \in ObjHI$).

Next, we denote $\mathbb{A}^1 \setminus \{0\}$ by G_m. The morphisms $pt \to G_m \to pt$ (the point is mapped to 1 in G_m) induce a splitting $M_{gm}(G_m) = \mathbb{Z} \oplus \mathbb{Z}(1)[1]$ for a certain (*Tate*) motif $\mathbb{Z}(1)$; see Definition 3.1 of [20]. For $X \in ObjDM_-^{eff}$ we denote $X \otimes \mathbb{Z}(1)$ by $X(1)$.

One could also present $\mathbb{Z}(1)$ as $\mathrm{Cone}(pt \to G_m)[-1]$; hence the Tate twist functor $X \mapsto X(1)$ is compatible with the functor $-\otimes(\mathrm{Cone}(pt \to G_m)[-1])$ on $C^b(SmCor)$ via M_{gm}. We also obtain that $DM_-^{eff\,t\le 0}(1) \subset DM_-^{eff\,t\le 1}$.

Now we define certain twists for functors.

Definition 1.4.1. For an $G \in \mathrm{AddFun}(DM_{gm}^{eff}, Ab)$, $n \ge 0$, we define $G_{-n}(X) = G(X(n)[n])$.

Note that this definition is compatible with those of §3.1 of [26]. Indeed, for $X \in SmVar$ we have $G_{-1}(M_{gm}(X)) = G(M_{gm}(X \times G_m))/G(M_{gm}(X)) = \mathrm{Ker}(G(M_{gm}(X \times G_m)) \to G(M_{gm}(X)))$ (with respect to natural morphisms $X \times pt \to X \times G_m \to X \times pt$); G_{-n} for larger n could be defined by iterating $-_{-1}$.

Below we will extend this definition to (co)motives of pro-schemes.

For $F \in ObjDM_-^{eff}$ we will denote by F_* the functor $X \mapsto DM_-^{eff}(X, F) : DM_{gm}^{eff} \to Ab$.

Proposition 1.4.2. *Let $X \in SmVar$, $n \ge 0$, $i \in \mathbb{Z}$.*

1. For any $F \in ObjDM_-^{eff}$ we have: $F_{*-n}(M_{gm}(X)[-i])$ is a retract of $\mathbb{H}^i(F)(X \times G_m^{\times n})$ (which can be described explicitly).

2. There exists a t-exact functor $T_n : DM_-^{eff} \to DM_-^{eff}$ such that for any $F \in Obj DM_-^{eff}$ we have $F_{*-n} \cong (T_n(F))_*$.

Proof. 1. Proposition 1.3.1 along with our description of $\mathbb{Z}(1)$ yields the result.
2. For F represented by a complex of $F^i \in Obj Shv(SmCor)$ ($i \in \mathbb{Z}$) we define $T_n(F)$ as the complex of $T_n(F^i)$, where $T_n : PreShv(SmCor) \to PreShv(SmCor)$ is defined similarly to $-_{-n}$ in Definition 1.4.1. $T_n(F^i)$ are sheaves since $T_n(F_i)(X)$, $X \in SmVar$, is a functorial retract of $F_i(X \times G_m^n)$. In order to check that we actually obtain a well-defined a t-exact functor this way, it suffices to note that the restriction of T_n to $Shv(SmCor)$ is an exact functor by Proposition 3.4.3 of [9].
Now, it suffices to check that T_n defined satisfies the assertion for $n = 1$. In this case the statement follows easily from Proposition 4.34 of [26] (note that it is not important whether we consider Zariski or Nisnevich topology by Theorem 5.7 of ibid.).

□

1.5 Pro-motives vs. comotives; the description of our strategy

Below we will embed DM_{gm}^{eff} into a certain triangulated category $\mathfrak{D}$ of *comotives*. Its construction (and computations in it) is rather complicated; in fact, the author is not sure whether the main properties of $\mathfrak{D}$ (described below) specify it up to an isomorphism. So, before working with co-motives we will (following F. Deglise) describe a simpler category of *pro-motives*. The latter is not needed for our main results (so the reader may skip this subsection); yet the comparison of the categories mentioned would clarify the nature of our methods.
Following §3.1 of [9], we define the category $\mathfrak{D}^{naive}$ as the additive category of naive i.e. formal (filtered) pro-objects of DM_{gm}^{eff}. This means that for any $X : L \to DM_{gm}^{eff}$, $Y : J \to DM_{gm}^{eff}$ we define

$$\mathfrak{D}^{naive}(\varprojlim_{l \in L} X_l, \varprojlim_{j \in J} Y_j) = \varprojlim_{j \in J} (\varinjlim_{l \in L} DM_{gm}^{eff}(X_l, Y_j)). \qquad (5)$$

The main disadvantage of $\mathfrak{D}^{naive}$ is that it is not triangulated. Still, one has the obvious shift for it; following Deglise, one can define pro-distinguished triangles as (filtered) inverse limits of distinguished triangles in DM_{gm}^{eff}. This allows to construct a certain motivic coniveau exact couple for a motif of a smooth variety in §4.2 of [10] (see also §5.3 of [9]). This construction is parallel to the classical construction of coniveau spectral sequences (see §1 of [8]). One starts with certain 'geometric' Postnikov towers in DM_{gm}^{eff} (Deglise calls them *triangulated exact couples*). For $Z \in SmVar$ we consider filtrations $\varnothing = Z_{d+1} \subset Z_d \subset Z_{d-1} \subset \cdots \subset Z_0 = Z$; Z_i is everywhere of codimension $\geq i$ in Z for all i. Then we have a system of distinguished triangles relating $M_{gm}(Z \setminus Z_i)$ and $M_{gm}(Z \setminus Z_i \to Z \setminus Z_{i+1})$; this yields a Postnikov tower. Then one passes to the inverse limit of these towers in $\mathfrak{D}^{naive}$ (here the connecting morphisms are induced by the corresponding open embeddings). Lastly, the functorial form of the Gysin distinguished triangle for motives allows Deglise to identify $X_i = \varprojlim(M_{gm}(Z \setminus Z_i \to Z \setminus Z_{i+1}))$ with the product of shifted Tate twists of pro-motives of all points of Z of codimension i. Using the results of see §5.2 of [9] (the relation of pro-motives with cycle modules of M. Rost, see [24]) one can also compute the morphisms that connect X^i with X^{i+1}.

Next, for any cohomological $H : DM_{gm}^{eff} \to \underline{A}$, where $\underline{A}$ is an abelian category satisfying AB5, one can extend H to $\mathfrak{D}^{naive}$ via the corresponding direct limits. Applying H to the motivic coniveau exact couple one gets the classical coniveau spectral sequence (that converges to the H-cohomology of Z). This allows to extend the seminal results of §6 of [5] to a comprehensive description of the coniveau spectral sequence in the case when H is represented by $Y \in ObjDM_{-}^{eff}$ (in terms of the homotopy t-truncations of Y; see Theorem 6.4 of [11]).

Now suppose that one wants to apply a similar procedure for an arbitrary $X \in ObjDM_{gm}^{eff}$; say, $X = M_{gm}(Z^1 \xrightarrow{f} Z^2)$ for $Z^1, Z^2 \in SmVar$, $f \in SmCor(Z^1, Z^2)$. One would expect that the desired exact couple for X could be constructed from those for Z^j, $j = 1, 2$. This is indeed the case when f satisfies certain codimension restrictions; cf. §7.4 of [6]. Yet for a general f it seems to be quite difficult to relate the filtrations of distinct Z^j (by the corresponding Z_i^j). On the other hand, the formalism of weight structures and weight spectral sequences (developed in [6]) allows to 'glue' certain *weight* Postnikov towers for objects of a triangulated categories equipped with a weight structure; see Remark 4.1.2(3) below.

So, we construct a certain triangulated category $\mathfrak{D}$ that is somewhat similar to $\mathfrak{D}^{naive}$. Certainly, we want distinguished triangles in $\mathfrak{D}$ to be compatible with inverse limits that come from 'geometry'. A well-known recipe for this is: one should consider some category $\mathfrak{D}'$ where (certain) cones of morphisms are functorial and pass to (inverse) limits in $\mathfrak{D}'$; $\mathfrak{D}$ should be a localization of $\mathfrak{D}'$. In fact, $\mathfrak{D}'$ constructed in §5.3 below could be endowed with a certain (Quillen) model structure such that $\mathfrak{D}$ is its homotopy category. We will never use this fact below; yet we will sometimes call inverse limits coming from $\mathfrak{D}'$ homotopy limits (in $\mathfrak{D}$).

Now, in Proposition 4.3.1 below we will prove that cohomological functors $H : DM_{gm}^{eff} \to \underline{A}$ could be extended to $\mathfrak{D}$ in a way that is compatible with homotopy limits (those coming from $\mathfrak{D}'$). So one may say that objects of $\mathfrak{D}$ have the same cohomology as those of $\mathfrak{D}^{naive}$. On the other hand, we have to pay the price for $\mathfrak{D}$ being triangulated: (5) does not compute morphisms between homotopy limits in $\mathfrak{D}$. The 'difference' could be described in terms of certain higher projective limits (of the corresponding morphism groups in DM_{gm}^{eff}).

Unfortunately, the author does not know how to control the corresponding $\varprojlim^2$ (and higher ones) in the general case; this does not allow to construct a weight structure on a sufficiently large triangulated subcategory of $\mathfrak{D}$ if k

is uncountable (yet see §6.5, especially the last paragraph of it). In the case of a countable k only $\varprojlim^1$ is non-zero. In this case the morphisms between homotopy limits in $\mathfrak{D}$ are expressed by the formula (28) below. This allows to prove that there are no morphisms of positive degrees between certain Tate twists of comotives of function fields (over k). This immediately yields that one can construct a certain weight structure on the triangulated subcategory $\mathfrak{D}_s$ of $\mathfrak{D}$ generated by products of Tate twists of comotives of function fields (in fact, we also idempotent complete $\mathfrak{D}_s$). Now, in order to prove that $\mathfrak{D}_s$ contains DM_{gm}^{eff} it suffices to prove that the motif of any smooth variety X belongs to $\mathfrak{D}_s$. To this end it clearly suffices to decompose $M_{gm}(X)$ into a Postnikov tower whose factors are products of Tate twists of comotives of function fields. So, we lift the motivic coniveau exact couple (constructed in [10]) from $\mathfrak{D}^{naive}$ to $\mathfrak{D}$. Since cones in $\mathfrak{D}'$ are compatible with inverse limits, we can construct a tower whose terms are the homotopy limits of the corresponding terms of the geometric towers mentioned. In fact, this could be done for an uncountable k also; the difficulty is to identify the analogues of X_i in $\mathfrak{D}$. If k is countable, the homotopy limits corresponding to our tower are countable also. Hence (by an easy well-known result) the isomorphism classes of these homotopy limits could be computed in terms of the corresponding objects and morphisms in DM_{gm}^{eff}. This means: it suffices to compute X^i in $\mathfrak{D}^{naive}$ (as was done in [10]); this yields the result needed. Note that we cannot (completely) compute the $\mathfrak{D}$-morphisms $X^i \to X^{i+1}$; yet we know how they act on cohomology.

The most interesting application of the results described is the following one. We prove that there are no positive $\mathfrak{D}$-morphisms between (certain) Tate twists of comotives of smooth semi-local schemes (or *primitive schemes*, see below); this generalizes the corresponding result for function fields. It follows that these twists belong to the *heart* of the weight structure on $\mathfrak{D}_s$ mentioned. Therefore comotives of (connected) primitive schemes are retracts of comotives of their generic points. Hence the same is true for the cohomology of the comotives mentioned and also for the corresponding pro-motives. Also, the comotif of a function field contains as retracts twisted comotives of its residue fields (for all geometric valuations); this also implies the corresponding results for cohomology and pro-motives.

Remark 1.5.1. In fact, Deglise mostly considers pro-objects for Voevodsky's DM_{gm} and of DM_{-}^{eff}; yet the distinctions are not important since the full embeddings $DM_{gm}^{eff} \to DM_{gm}$ and $DM_{gm}^{eff} \to DM_{-}^{eff}$ obviously extend to full embedding of the corresponding categories of pro-objects. Still, the embeddings mentioned allow Deglise to extend several nice results for Voevodsky's motives to pro-motives.

2. One of the advantages of the results of Deglise is that he never requires k to be countable. Besides, our construction of weight Postnikov towers mentioned heavily relies on the functoriality of the Gysin distinguished triangle for motives (proved in [10]; see also Proposition 2.4.5 of [9]).

2 Weight structures: reminder, truncations, weight spectral sequences, and duality with t-structures

In §2.1 we recall basic definitions of the theory of weight structures (it was developed in [6]; the concept was also independently introduced in [23]). Note here that weight structures (usually denoted by w) are natural counterparts of t-structures. Weight structures yield weight truncations; those (vastly) generalize stupid truncations in $K(B)$: in particular, they are not canonical, yet any morphism of objects could be extended (non-canonically) to a morphism of their weight truncations. We recall several properties of weight structures in §2.2.

We recall *virtual t-truncations* for a (cohomological) functor $H : \underline{C} \to \underline{A}$ (for $\underline{C}$ endowed with a weight structure) in §2.3 (these truncations are defined in terms of weight truncations). Virtual t-truncations were introduced in §2.5 of [6]; they yield a way to present H (canonically) as an extension of a cohomological functor that is positive in a certain sense by a 'negative' one (as if H belonged to some triangulated category of functors $\underline{C} \to \underline{A}$ endowed with a t-structure). We study this notion further here, and prove that virtual t-truncations for a cohomological H could be characterized up to a unique isomorphism by their properties (see Theorem 2.3.1(III4)). In order to give some characterization also for the 'dimension shift' (connecting the positive and the negative virtual t-truncations of H), we introduce the notion of a *nice (strongly exact)* complex of functors. We prove that complexes of representable functors coming from distinguished triangles in $\underline{C}$ are nice, as well as those complexes that could be obtained from nice strongly exact complexes of functors $\underline{C}' \to \underline{A}$ for some small triangulated $\underline{C}' \subset \underline{C}$ (via the extension procedure given by Proposition 1.2.1).

In §2.4 we consider weight spectral sequences (introduced in §§2.3–2.4 of [6]). We prove that the derived exact couple for the weight spectral sequence $T(H)$ (for $H : \underline{C} \to \underline{A}$) could be naturally described in terms of virtual t-truncations of H. So, one can express $T(H)$ starting from E_2 (as well as the corresponding filtration of H^*) in these terms also. This is an important result, since the basic definition of $T(H)$ is given in terms of *weight Postnikov towers* for objects of $\underline{C}$, whereas the latter are not canonical. In particular, this result yields canonical functorial spectral sequences in classical situations (considered by Deligne; cf. Remark 2.4.3 of [6]; note that we do not need rational coefficients here).

In §2.5 we introduce the definition a *(nice) duality* $\Phi : \underline{C}^{op} \times \underline{D} \to \underline{A}$, and of (left) *orthogonal* weight and t-structures (with respect to Φ). The latter definition generalizes the notion of *adjacent* structures introduced in §4.4 of [6] (this is the case $\underline{C} = \underline{D}$, $\underline{A} = Ab$, $\Phi = \underline{C}(-,))$. If w is orthogonal to t then the virtual t-truncations (corresponding to w) of functors of the type $\Phi(-, Y)$, $Y \in Obj\underline{D}$, are exactly the functors 'represented via Φ' by the actual t-truncations of Y (corresponding to t). We also prove that (nice) dualities could be extended from $\underline{C}'$ to $\underline{C}$ (using Proposition 1.2.1). Note here that (to the knowledge of the author) this paper is the first one which considers 'pairings' of triangulated categories.

In §2.6 we prove: if w and t are orthogonal with respect to a nice duality, the weight spectral sequence converging to $\Phi(X,Y)$ (for $X \in Obj\underline{C}$, $Y \in Obj\underline{D}$) is naturally isomorphic (starting from E_2) to the one coming from t-truncations of Y. Moreover even when the duality is not nice, all E_r^{pq} for $r \geq 2$ and the filtrations corresponding to these spectral sequences are still canonically isomorphic. Here niceness of a duality (defined in §2.5) is a somewhat technical condition (defined in terms of nice complexes of functors). Niceness generalizes to pairings ($\underline{C} \times \underline{D} \to \underline{A}$) the axiom TR3 (of triangulated categories: any commutative square in $\underline{C}$ could be completed to a morphism of distinguished triangles; note that this axiom could be described in terms of the functor $\underline{C}(-,-) : \underline{C} \times \underline{C} \to Ab$). We also discuss some alternatives and predecessors of our methods and results.
In §2.7 we compare weight decompositions, virtual t-truncations, and weight spectral sequences corresponding to distinct weight structures (in possibly distinct triangulated categories, connected by an exact functor).

2.1 Weight structures: basic definitions

We recall the definition of a weight structure (see [6]; in [23] D. Pauksztello introduced weight structures independently and called them co-t-structures).

Definition 2.1.1 (Definition of a weight structure). A pair of subclasses $\underline{C}^{w\leq 0}, \underline{C}^{w\geq 0} \subset Obj\underline{C}$ for a triangulated category $\underline{C}$ will be said to define a weight structure w for $\underline{C}$ if they satisfy the following conditions:
(i) $\underline{C}^{w\geq 0}, \underline{C}^{w\leq 0}$ are additive and Karoubi-closed (i.e. contain all retracts of their objects that belong to $Obj\underline{C}$).
(ii) **"Semi-invariance" with respect to translations.**
$\underline{C}^{w\geq 0} \subset \underline{C}^{w\geq 0}[1]$; $\underline{C}^{w\leq 0}[1] \subset \underline{C}^{w\leq 0}$.
(iii) **Orthogonality.**
$\underline{C}^{w\geq 0} \perp \underline{C}^{w\leq 0}[1]$.
(iv) **Weight decomposition**.
For any $X \in Obj\underline{C}$ there exists a distinguished triangle

$$B[-1] \to X \to A \xrightarrow{f} B \tag{6}$$

such that $A \in \underline{C}^{w\leq 0}, B \in \underline{C}^{w\geq 0}$.

A simple example of a category with a weight structure is $K(B)$ for any additive B: positive objects are complexes that are homotopy equivalent to those concentrated in positive degrees; negative objects are complexes that are homotopy equivalent to those concentrated in negative degrees. Here one could also consider the subcategories of complexes that are bounded from above, below, or from both sides.
The triangle (6) will be called a *weight decomposition* of X. A weight decomposition is (almost) never unique; still we will sometimes denote any pair (A,B) as in (6) by $X^{w\leq 0}$ and $X^{w\geq 1}$. Besides, we will call objects of the type

$(X[i])^{w\le 0}[j]$ and $(X[i])^{w\ge 0}[j]$ (for $i,j\in\mathbb{Z}$) *weight truncations* of X. A shift of the distinguished triangle (6) by $[i]$ for any $i\in\mathbb{Z}$, $X\in Obj\underline{C}$ (as well as any its rotation) will sometimes be called a *shifted weight decomposition*.
In $K(B)$ (shifted) weight decompositions come from stupid truncations of complexes.
We will also need the following definitions and notation.

Definition 2.1.2. Let $X\in Obj\underline{C}$.

1. The category $\underline{Hw}\subset\underline{C}$ whose objects are $\underline{C}^{w=0}=\underline{C}^{w\ge 0}\cap\underline{C}^{w\le 0}$, $\underline{Hw}(Z,T)=\underline{C}(Z,T)$ for $Z,T\in\underline{C}^{w=0}$, will be called the *heart* of the weight structure w.
2. $\underline{C}^{w\ge l}$ (resp. $\underline{C}^{w\le l}$, resp. $\underline{C}^{w=l}$) will denote $\underline{C}^{w\ge 0}[-l]$ (resp. $\underline{C}^{w\le 0}[-l]$, resp. $\underline{C}^{w=0}[-l]$).
3. We denote $\underline{C}^{w\ge l}\cap\underline{C}^{w\le i}$ by $\underline{C}^{[l,i]}$.
4. $X^{w\le l}$ (resp. $X^{w\ge l}$) will denote $(X[l])^{w\le 0}$ (resp. $(X[l-1])^{w\ge 1}$).
5. $w_{\le i}X$ (resp. $w_{\ge i}X$) will denote $X^{w\le i}[-i]$ (resp. $X^{w\ge i}[-i]$).
6. w will be called *non-degenerate* if
$$\cap_l\underline{C}^{w\ge l}=\cap_l\underline{C}^{w\le l}=\{0\}.$$
7. We consider $\underline{C}^b=(\cup_{i\in\mathbb{Z}}\underline{C}^{w\le i})\cap(\cup_{i\in\mathbb{Z}}\underline{C}^{w\ge i})$ and call it the class of *bounded* objects of $\underline{C}$.

   For $X\in\underline{C}^b$ we will usually take $w_{\le i}X=0$ for i small enough, $w_{\ge i}X=0$ for i large enough.

   We will also denote by $\underline{C}^b$ the corresponding full subcategory of $\underline{C}$.
8. We will say that $(\underline{C},w)$ is bounded if $\underline{C}^b=\underline{C}$.
9. We will call a Postnikov tower for X (see Definition 1.1.5) a *weight Postnikov tower* if all Y_i are some choices for $w_{\ge 1-i}X$. In this case we will call the complex whose terms are X^p (see Remark 1.1.6) a *weight complex* for X.

   We will call a weight Postnikov tower for X *negative* if $X\in\underline{C}^{w\le 0}$ and we choose $w_{\ge j}X$ to be 0 for all $j>0$ here.
10. $D\subset Obj\underline{C}$ will be called extension-stable if for any distinguished triangle $A\to B\to C$ in $\underline{C}$ we have: $A,C\in D\implies B\in D$.

    We will also say that the corresponding full subcategory is extension-stable.
11. $D\subset Obj\underline{C}$ will be called *negative* if for any $i>0$ we have $D\perp D[i]$.

Remark 2.1.3. 1. One could also dualize our definition of a weight Postnikov tower i.e. build a tower from $w_{\le l}X$ instead of $w_{\ge l}X$. Our definition of a weight Postnikov tower is more convenient for our purposes since in §3.6 below we will consider $Y_i = j(Z_0 \setminus Z_i)$ instead of $= j(Z_0 \setminus Z_i \to Z_0)[-1]$. Yet this does not make much difference; see §1.5 of [6] and Theorem 2.2.1(12) below. In particular, our definition of the weight complex for X coincides with Definition 2.2.1 of ibid. Note also, that Definition 1.5.8 of ibid (of a weight Postnikov tower) contained both 'our' part of the data and the dual part.
2. Weight Postnikov towers for objects of $\underline{C}$ are far from being unique; their morphisms (provided by Theorem 2.2.1(15) below) are not unique also (cf. Remark 1.5.9 of [6]). Yet the corresponding weight spectral sequences for cohomology are unique and functorial starting from E_2; see Theorem 2.4.2 of ibid. and Theorem 2.4.2 below for more detail. In particular, all possible choices of a weight complex for X are homotopy equivalent (see Theorem 3.2.2(II) and Remark 3.1.7(3) in [6]).

2.2 Basic properties of weight structures

Now we list some basic properties of notions defined. In the theorem below we will assume that $\underline{C}$ is endowed with a fixed weight structure w everywhere except in assertions 18 – 20.

Theorem 2.2.1. *1. The axiomatics of weight structures is self-dual: if $\underline{D} = \underline{C}^{op}$ (so $Obj\underline{C} = Obj\underline{D}$) then one can define the (opposite) weight structure w' on $\underline{D}$ by taking $\underline{D}^{w'\le 0} = \underline{C}^{w\ge 0}$ and $\underline{D}^{w'\ge 0} = \underline{C}^{w\le 0}$.*

2. *We have*
$$\underline{C}^{w\le 0} = \underline{C}^{w\ge 1\perp} \tag{7}$$
and
$$\underline{C}^{w\ge 0} = {}^{\perp}\underline{C}^{w\le -1}. \tag{8}$$

3. *For any $i \in \mathbb{Z}$, $X \in Obj\underline{C}$ we have a distinguished triangle $w_{\ge i+1}X \to X \to w_{\le i}X$ (given by a shifted weight decomposition).*

4. *$\underline{C}^{w\le 0}$, $\underline{C}^{w\ge 0}$, and $\underline{C}^{w=0}$ are extension-stable.*

5. *All $\underline{C}^{w\le i}$ are closed with respect to arbitrary (small) direct products (those, which exist in $\underline{C}$); all $\underline{C}^{w\ge i}$ and $\underline{C}^{w=i}$ are additive.*

6. *For any weight decomposition of $X \in \underline{C}^{w\ge 0}$ (see (6)) we have $A \in \underline{C}^{w=0}$.*

7. *If $A \to B \to C \to A[1]$ is a distinguished triangle and $A, C \in \underline{C}^{w=0}$, then $B \cong A \oplus C$.*

8. *If we have a distinguished triangle $A \to B \to C$ for $B \in \underline{C}^{w=0}$, $C \in \underline{C}^{w\le -1}$ then $A \cong B \bigoplus C[-1]$.*

9. *If $X \in \underline{C}^{w=0}$, $X[-1] \to A \stackrel{f}{\to} B$ is a weight decomposition (of $X[-1]$), then $B \in \underline{C}^{w=0}$; $B \cong A \bigoplus X$.*

10. *Let $l \le m \in \mathbb{Z}$, $X, X' \in Obj\underline{C}$; let weight decompositions of $X[m]$ and $X'[l]$ be fixed. Then any morphism $g : X \to X'$ can be completed to a morphism of distinguished triangles*

$$\begin{tikzcd}
w_{\ge m+1}X \arrow[r] \arrow[d, "a"] & X \arrow[r, "c"] \arrow[d, "g"] & w_{\le m}X \arrow[d, "b"] \\
w_{\ge l+1}X' \arrow[r] & X' \arrow[r, "d"] & w_{\le l}X'
\end{tikzcd} \tag{9}$$

This completion is unique if $l < m$.

11. *Consider some completion of a commutative triangle $w_{\ge m+1}X \to w_{\ge l+1}X \to X$ (that is uniquely determined by the morphisms $w_{\ge m+1}X \to X$ and $w_{\ge l+1}X \to X$ coming from the corresponding shifted weight decompositions; see the previous assertion) to an octahedral diagram:*

$$\begin{tikzcd}
w_{\le l}X \arrow[dr, "{[1]}"] \arrow[dd, "{[1]}"'] & & X \arrow[ll] \\
& w_{\ge l+1}X \arrow[ur] \arrow[dl] & \\
w_{[l+1,m]}X \arrow[rr, "{[1]}"] & & w_{\ge m+1}X \arrow[ul] \arrow[uu]
\end{tikzcd}$$

$$\begin{tikzcd}
w_{\le l}X \arrow[dd, "{[1]}"'] & & X \arrow[ll] \arrow[dl] \\
& w_{\le m}X \arrow[ul] \arrow[dr, "{[1]}"] & \\
w_{[l+1,m]}X \arrow[ur] \arrow[rr, "{[1]}"] & & w_{\ge m+1}X \arrow[uu]
\end{tikzcd}$$

Then $w_{[l+1,m]}X \in \underline{C}^{[l+1,m]}$; all the distinguished triangles of this octahedron are shifted weight decompositions.

12. *For $X, X' \in Obj\underline{C}$, $l, l', m, m' \in \mathbb{Z}$, $l < m$, $l' < m'$, $l > l'$, $m > m'$, consider two octahedral diagrams: (11) and a similar one corresponding to the commutative triangle $w_{\ge m+1}X \to w_{\ge l+1}X \to X$ and $w_{\ge m'+1}X' \to w_{\ge l'+1}X \to X$ (i.e. we fix some choices of these diagrams). Then any $g \in \underline{C}(X, X')$ could be uniquely extended to a morphism of these diagrams. The corresponding morphism $h : w_{[l+1,m]}X \to w_{[l'+1,m']}X'$ is characterized uniquely by any of the following conditions:*

(i) there exists a $\underline{C}$-morphism i that makes the squares

$$\begin{tikzcd} w_{\geq l+1}X \arrow[r] \arrow[d, "i"] & X \arrow[d, "g"] \\ w_{\geq l'+1}X' \arrow[r] & X' \end{tikzcd} \tag{10}$$

and

$$\begin{tikzcd} w_{\geq l+1}X \arrow[r] \arrow[d, "i"] & w_{[l+1,m]}X \arrow[d, "h"] \\ w_{\geq l'+1}X' \arrow[r] & w_{[l'+1,m']}X' \end{tikzcd} \tag{11}$$

commutative.

(ii) there exists a $\underline{C}$-morphism j that makes the squares

$$\begin{tikzcd} X \arrow[r] \arrow[d, "g"] & w_{\leq m}X \arrow[d, "j"] \\ X' \arrow[r] & w_{\leq m'}X' \end{tikzcd} \tag{12}$$

and

$$\begin{tikzcd} w_{[l+1,m]}X \arrow[r] \arrow[d, "h"] & w_{\leq m}X \arrow[d, "j"] \\ w_{[l'+1,m']}X' \arrow[r] & w_{\leq m'}X' \end{tikzcd} \tag{13}$$

commutative.

13. *For any choice of $w_{\geq i}X$ there exists a weight Postnikov tower for X (see Definition 2.1.2(9)). For any weight Postnikov tower we have* $\operatorname{Cone}(Y_i \to X) \in \underline{C}^{w\leq -i}$*; $X^i \in \underline{C}^{w=0}$.*

14. *Conversely, any bounded Postnikov tower (for X) with $X^i \in \underline{C}^{w=0}$ is a weight Postnikov tower for it.*

15. *For $X, X' \in Obj\underline{C}$ and arbitrary weight Postnikov towers for them, any $g \in \underline{C}(X, X')$ can be extended to a morphism of Postnikov towers (i.e. there exist morphisms $Y_i \to Y'_i$, $X^i \to X'^i$, such that the corresponding squares commute).*

16. *For $X, X' \in \underline{C}^{w\leq 0}$, suppose that $f \in \underline{C}(X, X')$ can be extended to a morphism of (some of) their negative Postnikov towers that establishes an isomorphism $X^0 \to X'^0$. Suppose also that $X' \in \underline{C}^{w=0}$. Then f yields a projection of X onto X' (i.e. X' is a retract of X via f).*

17. *$\underline{C}^b$ is a Karoubi-closed triangulated subcategory of $\underline{C}$. w induces a non-degenerate weight structure for it, whose heart equals $\underline{Hw}$.*

18. *For a triangulated idempotent complete $\underline{C}$ let $D \subset Obj\underline{C}$ be negative. Then there exists a unique weight structure w on the Karoubization T of $\langle D\rangle$ in $\underline{C}$ such that $D \subset T^{w=0}$. Its heart is the Karoubization of the closure of D in $\underline{C}$ with respect to (finite) direct sums.*

19. *For the weight structure mentioned in the previous assertion, $T^{w\le 0}$ is the Karoubization of the smallest extension-stable subclass of $Obj\underline{C}$ containing $\cup_{i\ge 0}D[i]$; $T^{w\ge 0}$ is the Karoubization of the smallest extension-stable subclass of $Obj\underline{C}$ containing $\cup_{i\le 0}D[i]$.*

20. *For the weight structure mentioned in two previous assertions we also have*
$$T^{w\le 0} = (\cup_{i<0}D[i])^{\perp};\ T^{w\ge 0} = {}^{\perp}(\cup_{i>0}D[i]).$$

Proof. 1. Obvious; cf. Remark 1.1.3 of [6] (and Remark 1.1.2 of ibid. for more detail).

2. These are parts 1 and 2 of Proposition 1.3.3 of ibid.

3. Obvious (since $[i]$ is exact up to change of signs of morphisms); cf. Remark 1.2.2 of ibid.

4. This is part 3 of Proposition 1.3.3 of ibid.

5. Obvious from the definition and parts 4 of loc.cit.

6. This is part 6 of Proposition 1.3.3 of ibid.

7. This is part 7 of loc.cit.

8. It suffices to note that $\underline{C}(B, C) = 0$, hence the triangle splits.

9. This is part 8 of loc.cit.

10. This is Lemma 1.5.1 of ibid.

11. The only non-trivial statement here is that $w_{[l+1,m]}X \in \underline{C}^{[l+1,m]}$ (it easily implies: the left hand side of the lower cap in (11) also yields a shifted weight decomposition). (11) yields distinguished triangles: $T_1 = (w_{\ge l+1}X \to w_{[l+1,m]}X \to w_{\ge m+1}X[1])$ and $T_2 = (w_{\le l}X \to w_{[l+1,m]}X[1] \to w_{\le m}X[1])$. Hence assertion 4 yields the result.

12. By assertion 10, g extends uniquely to a morphism of the following distinguished triangles: from $T_3 = (w_{\ge m+1}X \to X \to w_{\le m}X)$ to $T_3' = (w_{\ge m'+1}X' \to X' \to w_{\le m'}X)$, and from $T_4 = (w_{\ge l+1}X \to X \to w_{\le l}X)$ to $T_4' = (w_{\ge l'+1}X' \to X' \to w_{\le l'}X)$; next we also obtain a unique morphism from T_1 (as defined in the proof of the previous assertion) to its analogue T_1'. Putting all of this together: we obtain unique morphisms of all of the vertices of our octahedra, which are compatible with all the edges of the octahedra expect (possibly) those that belong to T_2 (as

defined above). We also obtain that there exists unique i and h that complete (10) and (11) to commutative squares.

Now, the morphism $w_{\leq l}X \to w_{[l+1,m]}X$ could be decomposed into the composition of morphisms belonging to T_1 and T_3. Hence in order to verify that we have actually constructed a morphism of octahedral diagrams, it remains to verify the commutativity of the squares

$$\begin{CD} w_{\leq m}X @>>> w_{\leq l}X \\ @VV{g}V @VV{j}V \\ w_{\leq m'}X' @>>> w_{\leq l'}X' \end{CD} \tag{14}$$

and (13) i.e. we should check that the two possible compositions of arrows for each of the squares are equal. Now, assertion 10 implies: the compositions in question for (14) both equal the only morphism q that makes the square

$$\begin{CD} X @>>> w_{\leq m}X \\ @VV{g}V @VV{q}V \\ X' @>>> w_{\leq l'}X' \end{CD}$$

commutative. Similarly, the compositions for (13) both equal the only morphism r that makes the square

$$\begin{CD} w_{\geq l+1}X @>>> w_{[l+1,m]}X \\ @VVV @VV{r}V \\ X' @>>> w_{\leq m'}X' \end{CD}$$

commutative. Here we use the part of the octahedral axiom that says that the square

$$\begin{CD} w_{\geq l+1}X @>>> w_{[l+1,m]}X \\ @VVV @VVV \\ X @>>> w_{\leq m}X \end{CD}$$

is commutative (as well as the corresponding square for (X', l', m')).

Lastly, as we have already noted, the condition (i) characterizes h uniquely; for similar (actually, exactly dual) reasons the same is true for (ii). Since the morphism $w_{[l+1,m]}X \to w_{[l'+1,m']}X'$ coming from the morphism of the octahedra constructed satisfies both of these conditions, it is characterized by any of them uniquely.

13. Immediate from part 2 of (Proposition 1.5.6) of loc.cit (and also from assertion 11).

14. Immediate from Remark 1.5.9(2) of ibid.

15. Immediate from part 1 (of Remark 1.5.9) of loc.cit.

16. It suffices to prove that $\operatorname{Cone} f \in \underline{C}^{w\leq -1}$. Indeed, then the distinguished triangle $X \xrightarrow{f} X' \to \operatorname{Cone} f$ necessarily splits.

    We complete the commutative triangle $X^{w\leq -1} \to X'^{w\leq -1} \to X^0 (= X'^0)$ to an octahedral diagram. Then we obtain $\operatorname{Cone} f \cong \operatorname{Cone}(X^{w\leq -1} \to X'^{w\leq -1})[1]$; hence $\operatorname{Cone} f \in \underline{C}^{w\leq -1}$ indeed.

17. This is Proposition 1.3.6 of ibid.

18. By Theorem 4.3.2(II1) of ibid., there exists a unique weight structure on $\langle D \rangle$ such that $D \subset \langle D \rangle^{w=0}$. Next, Proposition 5.2.2 of ibid. yields that w can be extended to the whole T; along with part Theorem 4.3.2(II2) of loc.cit. it also allows to calculate $T^{w=0}$ in this case.

19. Immediate from Proposition 5.2.2 of ibid. and the description of $\langle H \rangle^{w\leq 0}$ and $\langle H \rangle^{w\geq 0}$ in the proof of Theorem 4.3.2(II1) of ibid.

20. If $X \in T^{w\leq 0}$ then the orthogonality condition for w immediately yields: $Y \perp X$ for any $Y \in \cup_{i<0} D[i]$.

    Conversely, suppose that for some $X \in ObjT$ we have $Y \perp X$ for all $Y \in \cup_{i<0} D[i]$. Then $Y \perp X$ also for all Y belonging to the smallest extension-stable subclass of $Obj\underline{C}$ containing $\cup_{i<0} D[i]$. Hence this is also true for all $Y \in T^{w\geq 1}$ (see the previous assertion). Hence (7) yields: $X \in T^{w\leq 0}$. We obtain the first part of the assertion.

    The second part of the assertion is dual to the first one (and easy from (8)).

□

Remark 2.2.2. 1. In the notation of assertion 10, for any a (resp. b) such that the left (resp. right) hand square in (9) commutes there exists some b (resp. some a) that makes (9) a morphism of distinguished triangles (this is just axiom TR3 of triangulated categories). Hence for $l < m$ the left (resp. right) hand side of (9) characterizes a (resp. b) uniquely.

2. Assertions 10 and 12 yield mighty tools for proving that a construction described in terms of weight decompositions is functorial (in a certain sense). In particular, the proofs of functoriality of weight filtration and virtual t-truncations for cohomology (we will consider these notions below) in [6] were based on assertion 10.

   Now we explain what kind of functoriality could be obtained using assertion loc.cit. Actually, such an argument was already used in the proof of assertion 12.

   In the notation of assertion 10 we will say that a and b are compatible with g (with respect to the corresponding weight decompositions). Now suppose that for some $X'' \in Obj\underline{C}$, some $n \leq l$, $g' \in \underline{C}(X', X'')$, and a distinguished triangle $w_{\geq n+1}X'' \to X' \to w_{\leq n}X'$ we have morphisms $a' : w_{\geq l+1}X' \to w_{\geq n+1}X''$ and $b' : w_{\leq l}X' \to w_{\leq n}X''$ compatible with g'. Then $a' \circ a$ and $b' \circ b$ are compatible with $g' \circ g$ (with respect to the corresponding weight decompositions)! Moreover, if $n < m$ then $(a' \circ a, b' \circ b)$ is exactly the (unique!) pair of morphisms compatible with $g' \circ g$.

3. In the notation of assertion 12 we will (also) say that $h : w_{[l+1,m]}X \to w_{[l'+1,m']}X'$ is compatible with g. Note that h is uniquely characterized by (i) (or (ii)) of loc.cit.; hence in order to characterize it uniquely it suffices to fix g and all the rows in (10) and (11) (or in (12) and (13)). Besides, we obtain that h is functorial in a certain sense (cf. the reasoning above).

4. Assertion 11 immediately implies: for any $l < m$ the class of all possible $w_{\leq l}X$ coincides with the class of possible $w_{\leq l}(w_{\leq m}X)$, whereas the class of possible $w_{\geq m}X$ coincides with those of $w_{\geq m}(w_{\geq l}X)$.

   Besides, assertion 11 also allows to construct weight Postnikov towers (cf. §1.5 of [6]). Hence $w_{[i,i]}X$ is just $X^i[-i]$ (for any $i \in \mathbb{Z},\ X \in Obj\underline{C}$), and a weight complex for any $w_{[l+1,m]}X$ can be assumed to be the corresponding stupid truncation of the weight complex of X.

5. Assertions 10 and 15 will be generalized in §2.7 below to the situation when there are two distinct weight structures; this will also clarify the proofs of these statements. Besides, note that our remarks on functoriality are also actual for this setting.

   Some of the proofs in §2.7 may also help to understand the concept of virtual t-truncations (that we will start to study just now) better.

2.3 Virtual t-truncations of (cohomological) functors

Till the end of the section $\underline{C}$ will be endowed with a fixed weight structure w; $H : \underline{C} \to \underline{A}$ ($\underline{A}$ is an abelian category) will be a contravariant (usually, cohomological) functor. We will not consider covariant (homological) functors here; yet certainly, dualization is absolutely no problem.
Now we recall the results of §2.5 of [6] and develop the theory further.

Theorem 2.3.1. *Let $H : \underline{C} \to \underline{A}$ be a contravariant functor, $k \in \mathbb{Z}$, $j > 0$.*
I The assignments $H_1 = H_1^{kj} : X \to \operatorname{Im}(H(w_{\leq k}X) \to H(w_{\leq k+j}X))$ and $H_2 = H_2^{kj} : X \to \operatorname{Im}(H(w_{\geq k}X) \to H(w_{\geq k+j}X))$ define contravariant functors $\underline{C} \to \underline{A}$ that do not depend (up to a canonical isomorphism) from the choice of weight decompositions. We have natural transformations $H_1 \to H \to H_2$.
II Let $k' \in \mathbb{Z}$, $j' > 0$. Then there exist the following natural isomorphisms.
1. $(H_1^{kj})_1^{k'j'} \cong H_1^{\min(k,k'),\max(k+j,k'+j')-\min(k,k')}$.
2. $(H_2^{kj})_2^{k'j'} \cong H_2^{\min(k,k'),\max(k+j,k'+j')-\min(k,k')}$.

3. $(H_1^{kj})_2^{k'j'} \cong (H_2^{k'j'})_1^{kj} \cong \operatorname{Im}(H(w_{[k,k']}X) \to H(w_{[k+j,k'+j']}X))$*. Here the last term is defined using the connection morphism* $w_{[k+j,k'+j']}X \to w_{[k,k']}X$ *that is compatible with* id_X *in the sense of Remark 2.2.2(3); the last isomorphism is functorial in the sense described in loc.cit.*

III Let H *be cohomological,* $j = 1$*; let* k *be fixed.*

1. H_l *(*$l = 1, 2$*) are also cohomological; the transformations* $H_1 \to H \to H_2$ *extend canonically to a long exact sequence of functors*

$$\cdots \to H_2 \circ [1] \to H_1 \to H \to H_2 \to H_1 \circ [-1] \to \dots \tag{15}$$

(i.e. the sequence is exact when applied to any $X \in Obj\underline{C}$*).*

2. $H_1 \cong H$ *whenever* H *vanishes on* $\underline{C}^{w \geq k+1}$*.*

3. $H \cong H_2$ *whenever* H *vanishes on* $\underline{C}^{w \leq k}$*.*

4. Let $H' \xrightarrow{f} H \xrightarrow{g} H''$ *be a (three-term) complex of functors exact in the middle such that:*

(i) H', H'' *are cohomological.*

(ii) for any $X \in Obj\underline{C}$ *we have* $\operatorname{Coker} g(X) \cong \operatorname{Ker} f(X[-1])$ *(we do not fix these isomorphisms).*

(iii) H' *vanishes on* $\underline{C}^{w \geq k+1}$*;* H'' *vanishes on* $\underline{C}^{w \leq k}$*.*

Then $H' \xrightarrow{f} H$ *is canonically isomorphic to* $H_1 \to H$*;* $H \xrightarrow{g} H''$ *is canonically isomorphic to* $H \to H_2$*.*

Proof. I This is Proposition 2.5.1(III1) of [6].

II Easily follows from Theorem 2.2.1, parts 11 and 12; see Remark 2.2.2.

III1. This is Proposition 2.5.1(III2) of [6].

2. If H vanishes on $\underline{C}^{w \geq k+1}$ then for any X we have $w_{\geq k+1}X = 0$; hence H_2 vanishes. Therefore in the long exact sequence $\cdots \to H_2(X[1]) \to H_1 \to H \to H_2(X) \to \dots$ given by assertion II1 we have $H_2(X[1]) \cong 0 \cong H_2(X)$; we obtain $H_1 \cong H$.

Conversely, suppose that $H_1 \cong H$. Let $X \in Obj\underline{C}^{w \geq k+1}$; we can assume that $w_{\leq k}X = 0$. Then we have $H(X) \cong H_1(X) = \operatorname{Im} H(w_{\leq k}X) \to H(w_{\leq k+1}X)) = 0$.

3. It suffices to apply assertion II1 to the dual functor $\underline{C}^{op} \to \underline{A}^{op}$; note that the axiomatics of abelian categories, triangulated categories, and weight structures are self-dual (see Remark 1.1.3(1) and Theorem 2.2.1(1)).

4. We should check that in the diagram

$$\begin{array}{ccc} H_1' & \xrightarrow{g} & H_1 \\ \downarrow{\scriptstyle h} & & \downarrow \\ H' & \longrightarrow & H \end{array}$$

g and h are isomorphisms. Then $g \circ h^{-1}$ will yield the first isomorphism desired, whereas dualization will yield the remaining half of the statement.

Now, assertion III2 yields that g in isomorphism.

Next, for an $X \in Obj\underline{C}$ we choose some weight decompositions for $X[k]$ and $X[k+1]$ and consider the diagram

$$\begin{CD}
H''((w_{\le k}X)[1]) @>>> H'(w_{\le k}X) @>{l}>> H(w_{\le k}X) @>>> H''(w_{\le k}X)\\
@. @VV{a}V @VV{b}V @.\\
H''((w_{\le k+1}X)[1]) @>>> H'(w_{\le k+1}X) @>{m}>> H(w_{\le k+1}X) @>>> H''(w_{\le k+1}X).
\end{CD}$$

By our assumptions, $H''((w_{\le k}X)[1]) \cong H''(w_{\le k}X) \cong H''((w_{\le k+1}X)[1]) \cong 0$; hence l is an isomorphism and m is a monomorphism. Hence the induced map $\operatorname{Im} a \to \operatorname{Im} b$ is an isomorphism; so h is an isomorphism (since its application to any $X \in Obj\underline{C}$ is an isomorphism).

$\square$

Definition 2.3.2. [virtual t-truncations of H]
Let $k, m \in \mathbb{Z}$. For a (co)homological H we will call H_l^{k1}, $l = 1, 2$, $k \in \mathbb{Z}$, *virtual t-truncations* of H. We will often denote them simply by H_l; in this case we will assume $k = 0$ unless k is specified explicitly.
We denote the following functors $\underline{C} \to \underline{A}$: H_1^{k1}, $H_2^{k-1,1}$, $(H_2^{m1})_1^{k1}$, and $X \mapsto (H_1^{01})_2^{-11}(X[k])$ by $\tau_{\le k}H$, $\tau_{\ge k}H$, $\tau_{[m+1,k]}H$, and $H^{\tau=k}$, respectively. Note that all of these functors are cohomological if H is.

Remark 2.3.3. 1. Note that H often lies in a certain triangulated 'category of functors' $\underline{D}$ (whose objects are certain cohomological functors $\underline{C} \to \underline{A}$). We will axiomatize this below by introducing the notion of a duality $\Phi : \underline{C}^{op} \times \underline{D} \to \underline{A}$: if Φ is a duality then for any $Y \in Obj\underline{D}$ we have a cohomological functor $\Phi(-, Y) : \underline{C} \to \underline{A}$. It is also often the case when the virtual t-truncations defined are compatible with actual t-truncations with respect to some t-structure t on $\underline{D}$ (see below). Still, it is very amusing that these t-truncated functors as well as their transformations corresponding to t-decompositions (see Definition 1.1.1) can be described without specifying any $\underline{D}$ and Φ!
2. Below we will need an explicit description of the connecting morphisms in (15). We give it here (following the proof of Proposition 2.5.1 of [6]).
The transformation $H_1 \to H$ (resp. $H \to H_2$) for any k, j can be calculated by applying H to any possible choice either of $X \to w_{\le k}X$ or of $X \to w_{\le k+j}X$ (resp. of $w_{\ge k}X \to X$ or of $w_{\ge k+j}X \to X$) that comes from any possible choice the corresponding weight decomposition. The transformation $H_2 \to H_1 \circ [-1]$ for $j = 1$ is given by applying H to any possible choice either of the morphism $w_{\le k+1}X \to w_{\ge k+2}X[1]$ or of the morphism $w_{\le k}X \to w_{\ge k+1}X[1]$ that comes from any possible choice of a weight decomposition of $X[k]$.
Here we use the following trivial observation: for $\underline{A}$-morphisms $X_1 \stackrel{f_1}{\to} Y_1$ and $X_2 \stackrel{f_2}{\to} Y_2$ any $g : X_1 \to X_2$ (resp. $h : Y_1 \to Y_2$) is compatible with at most one morphism $i : \operatorname{Im} f_1 \to \operatorname{Im} f_2$; if such an i exists, we will say that it is induced by g (resp. by h). Certainly, here f_1 could be equal to id_{X_1} or f_2 could be equal to id_{X_2}.

3. For any k, j, and any $\underline{C}$-morphism $g : X \to Y$ the morphism $H_1(X) \to H_1(Y)$ (resp. $H_2(X) \to H_2(Y)$) is induced by any choice of either of the morphism $w_{\le k}X \to w_{\le k}Y$ or of $w_{\le k+j}X \to w_{\le k+j}Y$ (resp. of the morphism $w_{\ge k}X \to w_{\ge k}Y$ or of $w_{\ge k+j}X \to w_{\ge k+j}Y$) that is compatible with g with respect to the corresponding weight decomposition (in the sense of Remark 2.2.2(2)); see the proof of Proposition 2.5.1 of [6].

We would like to extend assertion III4 of Theorem 2.3.1 to a statement on a (canonical) isomorphism of long exact sequences of functors. To this end we need the following definition.

Definition 2.3.4. 1. We will call a sequence of functors $C = \cdots \to H'' \circ [1] \stackrel{[1](h)}{\to} H' \stackrel{f}{\to} H \stackrel{g}{\to} H'' \stackrel{h}{\to} H' \circ [-1] \to \dots$ of contravariant functors $\underline{C} \to Ab$ a *strongly exact complex* if H', H, H'' are cohomological and $C(X)$ is a long exact sequence for any $X \in Obj\underline{C}$; here $[1](h)$ is the transformation induced by h.
2. We will also say that a strongly exact complex C is *nice* in H if the following condition is fulfilled:
For any distinguished triangle $T = A \stackrel{l}{\to} B \stackrel{m}{\to} C \stackrel{n}{\to} A[1]$ in $\underline{C}$ the natural morphism p:

$$\begin{gathered}\operatorname{Ker}((H'(A) \bigoplus H(B) \bigoplus H''(C)) \xrightarrow{\begin{pmatrix} f(A) & -H(l) & 0 \\ 0 & g(B) & -H''(m) \\ -H'([-1](n)) & 0 & h(C) \end{pmatrix}} \\ (H(A) \bigoplus H''(B) \bigoplus H'(C[-1]))) \stackrel{p}{\to} \operatorname{Ker}((H'(A) \bigoplus H(B)) \\ \xrightarrow{f(A) \oplus -H(l)} H(A)) \text{ is epimorphic.}\end{gathered} \tag{16}$$

Now we describe the connection of (16) with truncated realizations; our arguments will also somewhat clarify the meaning of this condition.

Theorem 2.3.5. *1. Let C be a strongly exact complex of functors that is nice in H; let $H' \stackrel{f}{\to} H \stackrel{g}{\to} H''$ (a 'piece' of C) satisfy the conditions of assertion III4 of Theorem 2.3.1. Then C is canonically isomorphic to (15).*
2. Let $X \to Y \to Z$ be a distinguished triangle in $\underline{C}$. Then $C = \cdots \to \underline{C}(-, X) \to \underline{C}(-, Y) \to \underline{C}(-, Z) \to \dots$ is a strongly exact complex of functors $\underline{C} \to Ab$; it is nice in $\underline{C}(-, Y)$.
3. Let there exist a (skeletally) small full triangulated $\underline{C'} \subset \underline{C}$ such that the restriction of a strongly exact complex C to $\underline{C'}$ is nice in H. For $D \in Obj\underline{C}$ we consider the projective system $L(D)$ whose elements are $(E, i) : E \in Obj\underline{C'}$, $i \in \underline{C}(D, E)$; we set $(E, i) \ge (E', i')$ if $(E, i) = (E' \bigoplus E'', i' \oplus i'')$ for some $(E'', i'') \in L(D)$.
Suppose that for any $D \in \underline{C}$ and for $G = H'$ and $G = H$ we have

$$\varinjlim_{L(D)} (\operatorname{Im} G(i) : G(E) \to G(D)) = G(D); \tag{17}$$

here we also assume that these limits exist. Then C is nice on $\underline{C}$ also.
4. Let $\underline{C}' \subset \underline{C}$ be a (skeletally) small triangulated subcategory, let $\underline{A}$ satisfy AB5. Let $C' = \dots \to H' \to H \to H'' \to \dots$ be a strongly exact complex of functors $\underline{C}' \to \underline{A}$. We extend all its terms from $\underline{C}'$ to $\underline{C}$ by the method of Proposition 1.2.1 and denote the complex obtained by C; we carry on the notation for the terms and arrows from C' to C. Then C is a strongly exact complex also (and its terms are cohomological functors).
It is nice in H whenever C' is.

Proof. 1. It suffices to check that the isomorphism provided by Theorem 2.3.1(III4) is compatible with the coboundaries if (16) is fulfilled. We can assume $\underline{A} = Ab$; see Remark 1.1.8. Then (16) transfers into: for any $(x, y): \ x \in H'(A),\ y \in H(B),\ f(A)(x) = H(l)(y)$ there exists a

$$z \in H''(C) \text{ such that } g(B)(y) = H''(z) \text{ and } H([-1](n))(x) = h(C)(z). \quad (18)$$

We should prove: if the images of $x \in H_2(X)$ and of $y \in H''(X)$ in $H_2''(X)$ coincide, $w \in H_1(X[-1])$ and $t = H(X)(y) \in H'(X[-1])$ are their coboundaries, then w and t come from some (single) $u \in H_1'(X[-1])$.
We lift x to some $x' \in H(w_{\geq k+1}X)$. Then (16) (if we substitute $w_{\geq k+1}$ for A and X for B in it) implies the existence of some $v \in H'((w_{\leq k}X)[-1])$ whose image in $H'(X[-1])$ (resp. in $H(w_{\leq k}X[-1])$) coincides with t (resp. with the coboundary of x'). Hence we can take u being the image of v (in $H_1'(X[-1])$).
2. Since the bi-functor $\underline{C}(-,-)$ is (co)homological with respect to both arguments, C is a strongly exact complex indeed. It remains to note: (16) in this case just means that any commutative square can be completed to a morphism of distinguished triangles; so it follows from the corresponding axiom (TR3) of triangulated categories.
3. First suppose that $\underline{A} = Ab$ (or any other abelian category equipped with an exact faithful functor $\underline{A} \to Ab$ that respects small direct limits; note that below we will only need $\underline{A} = Ab$). Then we should check (18).
Now note: it suffices to prove that there exist $A', B' \in Obj\underline{C}',\ l' \in \underline{C}(A', B')$, $\alpha \in \underline{C}(A, A'),\ \beta \in \underline{C}(B, B'),\ x' \in H'(A'),\ g' \in H(B')$ such that:

$$x = H'(\alpha)(x'),\ y = H(\beta)(y'),\ l' \circ \alpha = \beta \circ l,\ f(A')(x') = H(l')(y'). \quad (19)$$

Indeed, denote $C' = \operatorname{Cone}(l')$; denote by γ some element of $\underline{C}(C, C')$ that completes

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' \end{array}$$

to a morphism of triangles. Let $z' \in H''(C')$ be some element satisfying the obvious analogue of (18). Then $h = H''(\gamma)(h')$ is easily seen to satisfy (18).
Now we construct $A', B', \dots$ as desired. Note that in this case the assumption (17) is equivalent to: for any $t \in G(D)$ there exist $E \in Obj\underline{C}'$, $s \in G(D)$, and

$r \in \underline{C}(D, E)$, such that $t = G(r)(s)$ (since $\underline{C}'$ is additive). So, we can choose $A' \in Obj\underline{C}'$, $\alpha \in \underline{C}(A, A')$, $x' \in H'(A')$ such that $x = H'(\alpha)(x')$. We complete $q = \alpha \oplus l \in \underline{C}(A, A' \bigoplus B)$ to a distinguished triangle $A \to A' \bigoplus B \stackrel{p=p_1 \oplus p_2}{\to} D$. Since $H(q)((-H'(f(A')(x'), y)) = 0$, there exists an $s \in H(D)$ such that $H(p)(s) = (-H'(f(A')(x'), y)$ (recall that H is cohomological on $\underline{C}$). So, we have $H(p_2)(s) = y$, $-H(p_1)(s) = f(A')(X')$, $p_2 \circ l = -p_1 \circ \alpha$.
D fits for B' if it lies in $Obj\underline{C}'$. In the general case using (17) again, we choose $B' \in Obj\underline{C}'$, $\delta \in \underline{C}(D, B')$, $g' \in H(Y)$, such that $s = H(\delta)(g')$. Then it is easily seen that taking $l' = -\delta \circ p_1$, $\beta = \delta \circ p_2$, we complete the choice of a set of data satisfying (19).
This argument can be modified to work for a general $\underline{A}$. To this end we separate those parts of the reasoning where we used the fact that H is cohomological from those where we deal with limits; this allows us to 'work as if $\underline{A} = Ab$'.
We denote $\operatorname{Ker}(H'(A) \bigoplus H(B)) \to H(A))$ (with respect to the morphism in (16) by $S(A, B)$, and $\operatorname{Ker}(H'(A) \bigoplus H(B) \bigoplus H''(C)) \to H(A) \bigoplus H''(B) \bigoplus H'(C[-1])$ by $T(A, B, C)$.
Then we have a commutative diagram

$$\begin{array}{ccc}
\varinjlim(\operatorname{Im}(T(A', B', C') \to T(A, B, C))) & \xrightarrow{t'} & \varinjlim(\operatorname{Im}(S(A', B') \to S(A, B))) \\
\downarrow & & \downarrow{\scriptstyle i} \\
T(A, B, C) & \xrightarrow{t} & S(A, B)
\end{array}$$

here the first direct limit above is taken with respect to morphisms of triangles $(A \to B \to C) \to (A' \to B' \to C')$ for $A', B', C' \in Obj\underline{C}'$ (the ordering is similar to those of (17)); the second limit is taken similarly with respect to morphisms $(A \to B) \to (A' \to B')$ for $A', B' \in Obj\underline{C}'$. Since the restriction of C to $\underline{C}'$ is nice in H, for all A', B', C' the morphism $T(A', B', C') \to S(A', B')$ is epimorphic; hence t' is epimorphic. Therefore, it suffices to prove that i is epimorphic.
Now let us fix $A' = A_0$ and $\alpha = \alpha_0$. We use the notation introduced above; denote the preimage of $\operatorname{Im}(H'(\alpha) : H'(A') \to H'(A))$ with respect to the natural morphism $S(A, B) \to H'(A)$ by J. Then J equals $\operatorname{Im}(H'(A') \times H(D) \to S(A, B))$. Indeed, here we can apply Proposition 1.1.7 (see Remark 1.1.8) and then apply the reasoning 'with elements' used above.
In any $\underline{A}$ we obtain: since $\Phi(D, Y) = \varinjlim(\operatorname{Im}(\Phi(B', Y) \to \Phi(D, Y)))$, we obtain that $G = \varinjlim(\operatorname{Im}(S(A_0, B', X, Y) \to S(A, B, X, Y)))$. Here we use the following fact (valid in any abelian $\underline{A}$): if $J_i \subset J' \in Obj\underline{A}$, $\varinjlim J_i = J$ (for some projective system), $u : J' \to J$ is an $\underline{A}$-epimorphism, then $\varinjlim u(J_i) = J$.
Now, passing to the limit with respect to (A_0, α_0) (using (17)) finishes the proof.
4. C is a complex indeed since the extension procedure is functorial.
By Proposition 1.2.1(I1), all the terms of C are cohomological on $\underline{C}$. Also, part II2 of loc.cit. immediately implies that C is exact (i.e. $C(X)$ is exact for any $X \in Obj\underline{C}$). Hence C is a strongly exact complex.

Obviously, if C is nice in H then C' also is.
Conversely, let C' be nice in H. Then Proposition 1.2.1(II1) implies that H' and H satisfy (17) (for all D). Hence C is nice in H by assertion 3.

□

2.4 Weight spectral sequences and filtrations; relation with virtual t-truncations

Definition 2.4.1. For an arbitrary $(\underline{C}, w)$ let $H : \underline{C} \to \underline{A}$ be a cohomological functor ($\underline{A}$ is any abelian category).
We define $W^i(H) : \underline{C} \to \underline{A}$ as $X \to \operatorname{Im}(H(w_{\leq i}X) \to H(X))$.

By Proposition 2.1.2(2) of [6], $W^i(H)(X)$ does not depend on the the choice of the weight decomposition of $X[i]$; it also defines a (canonical) subfunctor of $H(X)$.
Now recall that Postnikov towers yield spectral sequences for cohomology. We will denote $H(X[-i])$ by $H^i(X)$ (for $X \in Obj\underline{C}$). We will also use the notation of Definition 2.3.2.

Theorem 2.4.2. *Let $k, m \in \mathbb{Z}$.*
I1. For any weight Postnikov tower for X (see Definition 2.1.2(9)) there exists a spectral sequence $T = T(H, X)$ with $E_1^{pq}(T) = H^q(X^{-p})$ such that the map $E_1^{pq} \to E_1^{p+1q}$ is induced by the morphism $X^{-p-1} \to X^{-p}$ (coming from the tower). We have $T(H, X) \implies H^{p+q}(X)$ for any $X \in \underline{C}^b$.
One can construct it using the following exact couple: $E_1^{pq} = H^q(X^{-p})$, $D_1^{pq} = H^q(X^{w\geq 1-p})$.
2. T is (covariantly) functorial in H; it is contravariantly $\underline{C}$-functorial in X starting from E_2.
3. Denote the step of filtration given by $(E_1^{l,m-l} : l \geq -k)$ on $H^m(X)$ by $F^{-k}H^m(X)$. Then $F^{-k}H^m(X) = (W^kH^m)(X)$.
II The derived exact couple for $T(H, X)$ can be naturally calculated in terms of virtual t-truncations of H in the following way: $E_2^{pq} \cong E_2'^{pq} = (H^q)^{\tau=-p}(X)$, $D_2^{pq} = D_2'^{pq} = (\tau_{\geq q}H)(X[1-p])$; the connecting morphisms of the couple $((E_2', D_2'))$ come from (15).
III1. $F^{-k}H^m(X) = \operatorname{Im}((\tau_{\leq k}H^m)(X) \to H^m(X))$ (with respect to the connecting morphism mentioned in Theorem 2.3.1(I)).
2. For any $r \geq 2$, $p, q \in \mathbb{Z}$ there exists a functorial isomorphism $E_r^{pq} \cong (F^p(\tau_{[-p+2-r,-p+r-2]}H)^q)^p / F^{p+1}(\tau_{[-p+2-r,-p+r-2]}H)^q)^p$.

Proof. I This is Theorem 2.4.2 of [6]; see also Remark 2.4.1 of ibid. for the discussion of exact couples.
In fact, assertion 1 follows easily from well known properties of Postnikov towers and of related spectral sequences.
II Since virtual t-truncations are functorial, the exact couple (D_2', E_2') is functorial also.

The definitions of the derived exact couple and of the virtual t-truncations imply immediately that D_2^{pq} and their connecting maps are exactly $D_2'^{pq}$ (and their connecting morphisms) specified in the assertion.

It remains to compare E_2 with E'_2, and also the connecting maps of exact couples starting and ending in E_2 with those for E'_2. It suffices to consider $p = q = 0$. Our strategy is the following one. First we construct an isomorphism $E_2^{00} \to E_2'^{00}$; our construction depends on some choices. Then we prove that the isomorphism constructed is actually natural (in particular, it does not depend on the choices made). Lastly we verify that the isomorphisms of the terms of the exact couples constructed is compatible with the connecting morphisms of these couples. Note that in this (last) part of the argument we can make those choices (of certain weight decompositions) that we like.

By the definition of the derived exact couple we have: E_2^{00} is the 0-th cohomology of the complex $(H(X^{-j}))$ (for any choice of the weight complex (X^i)). $E_2'^{00}$ is the image of $H(k)$ where $k \in \underline{C}(w_{[0,1]}X, w_{[-1,0]}X)$ is any morphism that is compatible with id_X with respect to the corresponding weight decompositions (see see Theorem 2.3.1(II3) and Remark 2.2.2(3)). So, we should compare a subfactor of $H(X^0)$ with a subobject of $H(w_{[0,1]}X)$.

Now suppose that we are given an octahedral diagram containing a commutative triangle $w_{[1,1]}X \to w_{[0,1]}X \to w_{[-1,1]}X$ (see Theorem 2.2.1(11)). We could obtain it as follows: fix some $w_{[-1,1]}X$; then choose certain $w_{[0,1]}X = w_{\geq 0}(w_{[-1,1]}X)$ and $w_{[1,1]}X = w_{\geq 1}(w_{[-1,1]}X)$ (see Remark 2.2.2(4)). For any possible completion of the commutative triangle $w_{[1,1]}X \to w_{[0,1]}X \to w_{[-1,1]}X$ to an octahedral diagram, the remaining vertices of the octahedron are certain $w_{[-1,0]}X$, $w_{[0,0]}X = X^0$, and $w_{[-1,-1]}X = X^{-1}[1]$ (by Theorem 2.2.1(11)). We obtain morphisms $w_{[0,1]}X \xrightarrow{i} X^0 \xrightarrow{j} w_{[-1,0]}X$ such that $k = j \circ i$. Moreover, $\operatorname{Im}(H(X^1) \to H(X^0)) = \operatorname{Ker} H(i)$. Hence $H(i)$ induces some monomorphism $\alpha : H(X^0)/\operatorname{Im}(H(X^1) \to H(X^0))$ to $H(w_{[0,1]}X)$. Besides, $\operatorname{Ker}(H(X^0) \to H(X^{-1})) = \operatorname{Im} H(j)$; therefore the restriction of α to $\alpha^{-1}(\operatorname{Im} H(k))$ yields an isomorphism $\beta : E_2^{00} \to E_2'^{00}$.

Now we verify that the isomorphism constructed is natural.

Note that it actually depends only on $w_{[0,1]}X \xrightarrow{i} X^0$ and $\operatorname{Im} H(k)$ (we used the remaining data only in order to verify that we actually obtain an isomorphism). So, suppose that we have $X' \in Obj\underline{C}$, $g \in \underline{C}(X, X')$, and some choice of $w_{\geq 0}X'$, $w_{\geq 1}X'$, and $w_{\geq 2}X'$. We have canonical connecting morphisms $w_{\geq 0}X' \to w_{\geq 1}X' \to w_{\geq 2}X'$ that are compatible with $id_{X'}$ with respect to the morphisms $w_{\geq l}X' \to X'$ ($l = 0, 1, 2$). Applying Theorem 2.2.1(11), we obtain a choice of $w_{[0,1]}X' \xrightarrow{i'} X'^0$. We also fix some choice of $H(k')$ (in order to do this we fix some choice of $w_{\leq -1}X$ and of $w_{[-1,0]}X$). Note that all of these choices are necessarily compatible with some choice of the isomorphism $\beta' : E_2^{00}(X') \to E_2'^{00}(X')$ constructed as above (see 2.2.2(2)).

Now we choose some morphisms $g_l : w_{\geq l}X \to w_{\geq l}X'$, for $-1 \leq l \leq 2$, compatible with g (see Remark 2.2.2(2)). These choices could be extended to some morphisms $a : w_{[0,1]}X \to w_{[0,1]}X'$ and $b : X^0 \to X'^0$ (by extending morphisms

of arrows to morphism of distinguished triangles).
Now we verify the commutativity of the diagram

$$\begin{tikzcd} w_{[0,1]}X \arrow[r, "i"] \arrow[d, "a"] & X^0 \arrow[d, "b"] \\ w_{[0,1]}X' \arrow[r, "i'"] & X'^0 \end{tikzcd}$$

It follows from Theorem 2.2.1(10) applied to the morphism $g_0 : w_{\geq 0}X \to w_{\geq 0}X'$, $l = 1$, $m = 2$ (since both $b \circ i$ and $i' \circ a$ are compatible with g_0). Moreover, Remark 2.2.2(3) yields that $H(a)$ sends $H(k)$ to $H(k')$. We obtain a commutative diagram

$$\begin{tikzcd} E_2^{00} \arrow[r, "\beta"] \arrow[d] & E_2'^{00} \arrow[d] \\ E_2^{00}(H, X') \arrow[r, "\beta'"] & E_2'^{00}(H, X') \end{tikzcd}$$

Since $E_2^{00}(H, -)$ and $E_2'^{00}(H, -)$ are $\underline{C}^{op}$-functorial (and the vertical arrows in the diagram are exactly those that yield this functoriality; see Remark 2.3.3(3)), we obtain the naturality in question.

Now it remains to prove that the isomorphisms of terms of exact couples constructed above is compatible with the (two remaining) connecting morphisms of these couples.

First consider the morphisms $E_2^{00} \to D_2^{10}$. Recall (by the definition of the derived exact couple) that it is induced by any morphism $w_{\geq 0}X \to X^0$ that extends to a weight decomposition of $w_{\geq 0}X$ (here we consider E_2^{00} as a subfactor of $H(X^0)$). On the other hand, the morphism $E_2'^{00} \to D_2'^{10} = \operatorname{Im}(H(w_{\geq -1}X) \to H(w_{\geq 0}X))$ is induced by any possible choice of a morphism $w_{\geq 0}X \to w_{[0,1]}X$ that yields a weight decomposition of $w_{\geq 0}X[1]$ (by Remark 2.3.3(2); see also Remark 2.2.2(3)). Hence it suffices to note that the triangle $w_{\geq 0}X \to w_{[0,1]}X \xrightarrow{i} X^0$ is necessarily commutative by Remark 2.2.2.

It remains consider the morphism $D_2^{1,-1} \to E_2^{00}$. It is induced by the morphism $X^0 \to w_{\geq 1}X$ (that yields a weight decomposition of $w_{\geq 0}X$). The morphism $D_2'^{1,-1} (= \operatorname{Im}(H(w_{\geq 1}X)[1]) \to H(w_{\geq 2}X)[1])) \to E_2'^{00}$ is induced by the morphism $w_{[0,1]}X \to w_{\geq 2}X[1]$. Hence it suffices to construct a commutative square

$$\begin{tikzcd} w_{[0,1]}X \arrow[r, "i"] \arrow[d] & X^0 \arrow[d] \\ w_{\geq 2}X[1] \arrow[r] & w_{\geq 1}X[1] \end{tikzcd}$$

By applying Theorem 2.2.1(11) to the commutative triangle $w_{\geq 2}X \to w_{\geq 1}X \to w_{\geq 0}X$ we obtain that there exists such a commutative square with a certain i_0

instead of i. Note that (by loc.cit.) i_0 yields a weight decomposition of $w_{[0,1]}X$. It suffices to verify that we may take i_0 for i i.e. that i_0 could be completed to an octahedral diagram one of whose faces yields some choice of the commutative triangle $w_{[1,1]}X \to w_{[0,1]}X \to w_{[-1,1]}X$. We take $w_{[1,1]}X = \operatorname{Cone} i_0[-1]$, choose some $w_{[-1,1]}X$ (coming from the same $w_{\leq 1}X$ as $w_{[0,1]}X$). By Remark 2.2.2(2) we obtain a unique commutative triangle $w_{[1,1]}X \to w_{[0,1]}X \to w_{[-1,1]}X$ that is compatible with $id_{w_{\leq 1}X}$ respect to the corresponding weight decompositions. It remains to apply Theorem 2.2.1(11).
III We can assume $k = m = 0$.
1. In the notation of Theorem 2.3.1 we consider the morphism of spectral sequences $M : T(H_1, X) \to T(H, X)$ (induced by $H_1 \to H$). Part II of loc.cit. implies: M is an isomorphism on E_2^{pq} for $p \geq -k$ and $E_2^{pq}(T(H_1, X)) = 0$ otherwise. The assertion follows immediately.
2. Similarly to the the previous reasoning, we have natural isomorphisms: $E_2^{pq}(T(\tau_{[2-r,r-2]}H, X) \cong E_2^{pq}(T(H, X))$ for $2-r \leq p \leq r-2$ and $= 0$ otherwise. It easily follows that $E_\infty^{pq}(T(\tau_{[2-r,r-2]}H, X) \cong E_r^{pq}(T(\tau_{[-p+2-r,-p+r-2]}H, X)$. The result follows immediately.

□

Remark 2.4.3. 1. The dual of assertion II is: if we consider the alternative exact couple for our weight spectral sequence (see Remark 2.1.3) then the derived exact couple can also be described in terms of virtual t-truncations (in a way that is dual in an appropriate sense to that of Theorem 2.4.2).
2. Possibly, at least a part of (assertion II of) the theorem could be proved by studying the functoriality of the derived exact couple (and applying Theorem 2.3.5(1)).

2.5 Dualities of triangulated categories; orthogonal weight and t-structures

Let $\underline{C}, \underline{D}$ be triangulated categories. We study certain pairings of triangulated categories $\underline{C}^{op} \times \underline{D} \to \underline{A}$. In the following definition we consider a general $\underline{A}$, yet below we will mainly need $\underline{A} = Ab$.

Definition 2.5.1. 1. We will call a (covariant) bi-functor $\Phi : \underline{C}^{op} \times \underline{D} \to \underline{A}$ a *duality* if it is bi-additive, homological with respect to both arguments; and is equipped with a (bi)natural transformation $\Phi(X, Y) \cong \Phi(X[1], Y[1])$.
2. We will say that Φ is *nice* if for any distinguished triangle $X \to Y \to Z$ the corresponding (strongly exact) complex of functors

$$\cdots \to \Phi(-, X) \to \Phi(-, Y) \to \Phi(-, Z) \xrightarrow{f} \Phi([-1](-), X) \to \dots \tag{20}$$

is nice in $\Phi(-, Y)$ (see Definition 2.3.4); here f is obtained from the natural morphism $\Phi(-, Z) \to \Phi(-, X[1])$ by applying the (bi)natural transformation mentioned above.

3. Suppose that $\underline{C}$ is endowed with a weight structure w, $\underline{D}$ is endowed with a t-structure t. Then we will say that w is (left) *orthogonal* to t with respect to Φ if the following *orthogonality condition* is fulfilled:

$$\Phi(X,Y)=0 \text{ if: } X\in \underline{C}^{w\le 0} \text{ and } Y\in \underline{D}^{t\ge 1}, \text{ or } X\in \underline{C}^{w\ge 0} \text{ and } Y\in \underline{D}^{t\le -1}. \tag{21}$$

4. If w is defined on $\underline{C}^{op}$, t is defined on $\underline{D}^{op}$, w is left orthogonal to t (with respect to some duality); then we will say that the corresponding opposite weight structure on $\underline{C}$ is *right orthogonal* to the opposite t-structure for $\underline{D}$.

Remark 2.5.2. 1. The axioms of Φ immediately imply that (20) is a strongly exact complex of functors indeed (whether Φ is nice or not).
2. Certainly, if Φ is nice then (20) is nice at any term (since we can 'rotate' distinguished triangles in $\underline{D}$).

First we prove a statement that will simplify checking the orthogonality of weight and t-structures.

Proposition 2.5.3. *Let $\Phi:\underline{C}^{op}\times\underline{D}\to\underline{A}$ be some duality; let $(\underline{C},w)$ be bounded. Then w is (left) orthogonal to t whenever there exists a $D\subset\underline{C}^{w=0}$ such that any object of $\underline{C}^{w=0}$ is a retract of a finite direct sum of elements of D and*

$$\Phi(X,Y)=0\ \forall\ X\in D,\ Y\in\underline{D}^{t\ge 1}\cup\underline{D}^{t\le -1}. \tag{22}$$

Proof. If w is is left orthogonal to t, then (22) for $D=\underline{C}^{w=0}$ follows immediately from the orthogonality condition.
Conversely, let D satisfy the assumptions of our assertion. Hence we have: $\Phi(X,Y)=0$ if $X\in D[i],\ i\ge 0, Y\in\underline{D}^{t\ge 1}$, or if $X\in D[i],\ i\le 0, Y\in\underline{D}^{t\le -1}$.
Now suppose that for some $E,F\subset Obj\underline{C}$ we have: any object of $\underline{C}^{w\le 0}$ is a retract of an object of E, any object of $\underline{C}^{w\ge 0}$ is a retract of an object of F. Then it obviously suffices to check that $\Phi(X,Y)=0$ if either $X\in E$ and $Y\in\underline{D}^{t\ge 1}$ or $X\in F$ and $Y\in\underline{D}^{t\le -1}$.
Now by Theorem 2.2.1(19), we can take E being the smallest extension-stable subcategory of $\underline{C}$ containing $D[i]$, $i\ge 0$; and F being the smallest extension-stable subcategory of $\underline{C}$ containing $D[i]$, $i\le 0$. To conclude the proof it remains to note that for a distinguished triangle $X\to Y\to Z$ in $\underline{C}$, $O\in Obj\underline{D}$ we have: $\Phi(X,O)=0=\Phi(Z,O)\implies\Phi(Y,O)=0$. $\square$

When (weight and t-) structures are orthogonal, virtual t-truncations of $\Phi(-,Y)$ are given by t-truncations in $\underline{D}$. We use the notation of Definition 2.3.2.

Proposition 2.5.4. *1. Let t be orthogonal to w with respect to Φ, $k\in\mathbb{Z}$. For $Y\in Obj\underline{D}$ denote the functor $\Phi(-,Y):\underline{C}\to\underline{A}$ by H. Then we have an isomorphism of complexes $(\tau_{\le k}H\to H\to\tau_{\ge k}H)\cong(\Phi(-,t_{\le k}Y)\to H\to\Phi(-,t_{\ge k+1}Y))$ (where the connecting maps of the second complex are induced by t-truncations); this isomorphism is natural in Y.*

2. Suppose also that Φ is nice. Then the (strongly exact) complex of functors that sends X to

$$\dots \to \Phi(X, t_{\leq k}Y) \to \Phi(X,Y) \to \Phi(X, t_{\geq k+1}Y) \to \Phi(X[-1], t_{\leq k}Y) \to \dots \tag{23}$$

(constructed as in the definition of a nice duality) is naturally isomorphic to (15).

Proof. 1. Since t and w orthogonal, $\Phi(-, t_{\leq k}Y)$ vanishes on $\underline{C}^{w\geq k+1}$, whereas $\Phi(-, t_{\geq k+1}Y)$ vanishes on $\underline{C}^{w\leq k}$. Moreover, (23) yields that $H' = \Phi(-, t_{\leq k}Y)$ and $H'' = \Phi(-, t_{\geq k+1}Y)$ also satisfy the condition (iii) of Theorem 2.3.1(III4). Hence the theorem yields the claim.
2. Immediate from the previous assertion and Theorem 2.3.5(1).

□

Remark 2.5.5. Note that we actually need quite a partial case of the 'niceness condition' for Φ in order to prove assertion 2. Hence here (and so, in all the applications below) we will not need the niceness condition in its full generality. Possibly, the corresponding partial case of the condition is weaker than the whole assertion; yet checking it does not seem to be much easier.
Also, it seems quite possible that for an arbitrary (not necessarily nice) duality there exists some isomorphism of (15) with (23) if we modify the boundary maps of the second complex. Yet there seems to be no way to choose such a modification canonically.

'Natural' dualities are nice; we will justify this thesis now.

Proposition 2.5.6. *1. If $\underline{A} = Ab$, $\underline{D} = \underline{C}$, then $\Phi : (X,Y) \mapsto \underline{C}(X,Y)$ is a nice duality.*
2. For some duality $\Phi : \underline{C}^{op} \times \underline{D} \to \underline{A}$ let there exist a (skeletally) small full triangulated $\underline{C}' \subset \underline{C}$ such that: the restriction of Φ to $\underline{C}'^{op} \times \underline{D}$ is a nice duality (of $\underline{C}'$ with $\underline{D}$); for any $X \in Obj\underline{D}$ the functor $G = \Phi(-, X)$, $\underline{C}^{op} \to \underline{A}$, satisfies (17). Then Φ is nice also.
3. For $\underline{D}$, $\underline{C}' \subset \underline{C}$ as above, $\underline{A}$ satisfying AB5, let $\Phi' : \underline{C}'^{op} \times \underline{D} \to \underline{A}$ be a duality. For any $Y \in Obj\underline{D}$ we extend the functor $\Phi'(-, Y)$ from $\underline{C}'$ to $\underline{C}$ by the method of Proposition 1.2.1; we denote the functor obtained by $\Phi(-, Y)$. Then the corresponding bi-functor Φ is a duality ($\underline{C}^{op} \times \underline{D} \to \underline{A}$). It is nice whenever Φ' is.

Proof. Immediate from parts 2–4 of Theorem 2.3.5.

□

Remark 2.5.7. 1. Proposition 2.5.6(1) yields an important family of nice dualities; this case was thoroughly studied in [6] (in sections 4 and 7). We will say that w is left (resp. right) *adjacent* to t if it is left (resp. right) orthogonal to it with respect to $\Phi(X,Y) = \underline{C}(X,Y)$. Note that for w left (resp. right) adjacent to t with respect to this definition we necessarily have $\underline{C}^{w\leq 0} = \underline{C}^{t\leq 0}$ (resp.

$\underline{C}^{w\geq 0}=\underline{C}^{t\geq 0}$) by Theorem 2.2.1(2) and Remark 1.1.3(2); so this definition is actually compatible with Definition 4.4.1 of [6].
One can generalize this family as in §8.3 of ibid.: for $\underline{A}=Ab$ and an exact $F:\underline{D}\to\underline{C}$ we define $\Phi(X,Y)=\underline{C}(X,F(Y))$. Certainly, one could also dualize this construction (in a certain sense) and consider $F:\underline{C}\to\underline{D}$ and $\Phi(X,Y)=\underline{C}(F(X),Y)$.
2. Another (general) family of dualities is mentioned in Remark 6.4.1(2) of ibid. All the families of dualities mentioned can be expanded using part 3 of the proposition.
3. It is also easy to construct a duality that is not nice. To this end one can start with $\underline{C}=\underline{D}$, $\Phi=\underline{C}(-,-)$ and then modify the choice of distinguished triangles in $\underline{D}$ (without changing the shift in $\underline{D}$, and changing nothing in $\underline{C}$) in a way that would not affect the properties of functors to be cohomological. The simplest way to do this is to proclaim a triangle $X\stackrel{f}{\to}Y\stackrel{g}{\to}Z\stackrel{h}{\to}X[1]$ to be distinguished in $\underline{D}$ if $X\stackrel{-f}{\to}Y\stackrel{-g}{\to}Z\stackrel{-h}{\to}X[1]$ is distinguished in $\underline{C}$. Certainly, such a modification is not very 'serious'; in particular, one can 'fix the problem' by multiplying the isomorphism $\Phi(X,Y)\cong\Phi(X[1],Y[1])$ by -1.
The author does not know whether any duality can be made nice by modifying the choice of the class of distinguished triangles (in $\underline{D}$), or by modifying the isomorphism mentioned. Note also that the question whether there exists a $\underline{D}$ for which such a modification can change the 'equivalence class' of triangulations is well-known to be open.

2.6 Comparison of weight spectral sequences with those coming from (orthogonal) t-truncations

Now we describe the relation of weight spectral sequences with orthogonal structures.

Theorem 2.6.1. *Let w for $\underline{C}$ and t for $\underline{D}$ be orthogonal with respect to a duality Φ; let $i,j\in\mathbb{Z}$, $X\in Obj\underline{C}$, $Y\in Obj\underline{D}$.*

1. *Consider the spectral sequence S coming from the following exact couple: $D_2^{pq}(S)=\Phi(X,Y^{t\geq q}[p-1])$, $E_2^{pq}(S)=\Phi(X,Y^{t=q}[p])$ (we start S from E_2). It naturally converges to $\Phi(X,Y[p+q])$ if $X\in\underline{C}^b$.*

2. *Let T be the weight spectral sequence given by Theorem 2.4.2 for the functor $H: Z\mapsto\Phi(Z,Y)$. Then for all $r\geq 2$ we have natural isomorphisms $E_r^{pq}(T(H,X))\cong E_r^{pq}(S)$. There is also an equality $F^{-k}H^m(X)=\operatorname{Im}(\Phi(X,t_{\leq k}Y[m])\to H^m(X))$ (here we use the notation of part I4 of loc.cit.) compatible with this isomorphism.*

3. *Suppose that Φ is also nice. Then the isomorphism mentioned in the previous assertion extends naturally to the isomorphism of of T with S (starting from E_2).*

4. *Let* $\dots \to X^{-j-1} \to X^{-j} \to X^{1-j} \to \dots$ *denote an arbitrary choice of the weight complex for* X. *Then we have a functorial isomorphism*

$$\Phi(X, Y^{t=i}[j]) \cong$$
$$(\operatorname{Ker}(\Phi(X^{-j}, Y[i]) \to \Phi(X^{-1-j}, Y[i]))/\operatorname{Im}(\Phi(X^{1-j}, Y[i]) \to \Phi(X^{-j}, Y[i])). \tag{24}$$

Proof. 1. The theory of t-structures easily yields: $Y^{t\geq q}$ and $Y^{t=q}$ can be functorially organized into a certain Postnikov tower for Y. Hence the usual results on spectral sequences coming from Postnikov towers (see §IV2, Exercise 2, of [13]) yield the assertion easily.

2. Immediate from Proposition 2.5.4(1) and Theorem 2.4.2(III). Note that the latter assertion does not use the 'dimension shift' in (15).

3. Proposition 2.5.4(2) and Theorem 2.4.2(II) imply: there is a natural isomorphism of the derived exact couple for T with the exact couple of S ('at level 2'). The result follows immediately.

4. This is just assertion 2 for E_2-terms.

□

Remark 2.6.2. 1. So, we justified parts 4 and 5 of Remark 4.4.3 of [6].

2. Note that the spectral sequence denoted by S in (Remark 4.4.3(4) and §6.4 of) ibid. started from E_1; so it differed from our S and T by a certain shift of indices.

3. So, we developed an 'abstract triangulated alternative' to the method of comparing similar spectral sequences that was developed by Deligne and Paranjape. The latter method used filtered complexes; it was applied in [22], [11], and in §6.4 of [6]. The disadvantage of this approach is that one needs extra information in order to construct the corresponding filtered complexes; this makes difficult to study the naturality of the isomorphism constructed. Moreover, in some cases the complexes required cannot exist at all; this is the case for the *spherical weight structure* and its adjacent Postnikov t-structure for $\underline{C} = \underline{D} = SH$ (the topological stable homotopy category; see §4.6 of [6]; yet in this case one can compare the corresponding spectral sequences using topology).

4. One could modify our reasoning to prove a version of the theorem that does not mention weight and t-structures. To this end instead of considering a weight Postnikov tower for X and the Postnikov tower coming from t-truncations of Y one should just compare spectral sequences coming from some Postnikov towers for X and Y in the case when these Postnikov towers satisfy those 'orthogonality' conditions (with respect to a (nice) duality Φ) that are implied by the orthogonality of structures

condition in our situation. Yet it seems difficult to obtain the naturality of the isomorphisms in Theorem 2.6.1(3) using this approach.

5. Even more generally, it suffices to have an inductive system of Postnikov towers in $\underline{D}$ and a projective system of Postnikov towers in $\underline{C}$ such that the orthogonality conditions required are satisfied in the (double) limit. Then the comparison statements for the double limits of the corresponding spectral sequences are valid also. A very partial (yet rather important) example of a reasoning of this sort is described in §7.4 of [6]. Besides, this approach could possibly yield the comparison result of §6 of [11] (even without assuming k to be countable as we do here).

6. A simple (yet important) case of (24) is: for any $i \in \mathbb{Z}$

$$X \in \underline{C}^{w=i} \implies \forall Y \in Obj\underline{D} \text{ we have } \Phi(X,Y) \cong \Phi(X, Y^{t=i}). \qquad (25)$$

2.7 'Change of weight structures'; comparing weight spectral sequences

Now we compare weight decompositions, virtual t-truncations, and weight spectral sequences corresponding to distinct weight structures. In order make our results more general (and to apply them below) we will assume that these structures are defined on distinct triangulated categories; yet the case when both are defined on $\underline{C}$ is also interesting.
So, till the end of the section we will assume: $\underline{C}, \underline{D}$ are triangulated categories endowed with weight structures w and v, respectively; $F : \underline{C} \to \underline{D}$ is an exact functor.

Definition 2.7.1. 1. We will say that F is *right weight-exact* if $F(\underline{C}^{w\ge 0}) \subset \underline{D}^{v\ge 0}$.
2. If F is fully faithful and right weight-exact, we will say that v *dominates* w.
3. We will say that F is *left weight-exact* if $F(\underline{C}^{w\le 0}) \subset \underline{D}^{v\le 0}$.
4. F will be called *weight-exact* if it is both right and left weight-exact.
We will say that w *induces* v (via F) if F is a weight-exact localization functor.

Proposition 2.7.2. *Let F be a right weight-exact functor; let $l \ge m \in \mathbb{Z}$, $X \in Obj\underline{D}$, $X' \in Obj\underline{C}$, $g \in \underline{D}(F(X'), X)$.*
1. Let weight decompositions of $X[m]$ with respect to v and $X'[l]$ with respect to w be fixed. Then g can be completed to a morphism of distinguished triangles

$$\begin{CD} F(w_{\ge l+1}X') @>>> F(X') @>>> F(w_{\le l}X') \\ @VaVV @VgVV @VbVV \\ v_{\ge m+1}X @>>> X @>>> v_{\le m}X \end{CD} \qquad (26)$$

This completion is unique if $l > m$.

2. For arbitrary weight Postnikov towers $Po_v(X)$ for X (with respect to v) and Po_wX' for X' (with respect to w), g can be extended to a morphism $F_*(Po_wX') \to Po_v(X)$.
3. Let $H : \underline{D} \to \underline{A}$ be any functor, $k \in \mathbb{Z}$, $j > 0$. Denote $H \circ F$ by G. Then (26) allows to extend $H(g)$ naturally to a diagram

$$\begin{array}{ccccc} H_1^v(X) & \longrightarrow & H(X) & \longrightarrow & H_2^v(X) \\ \downarrow & & \downarrow{\scriptstyle H(g)} & & \downarrow \\ G_1^w(X') & \longrightarrow & G(X') & \longrightarrow & G_2^w(X') \end{array}$$

here we add the weight structure chosen as an index to the notation of Theorem 2.3.1(I).

Proof. 1. Since F is right weight-exact, $\underline{D}(F(w_{\geq n+1}X'), v_{\leq m}X) = \{0\}$ for any $n \geq m$. Hence the composition morphism $F(w_{\geq l+1}X') \to v_{\leq m}X$ is zero; if $l > m$ then $\underline{D}(F(w_{\geq l+1}X'), (v_{\leq m}X)[-1]) = \{0\}$. The result follows easily; see Proposition 1.1.9 of [2].
2. Assertion 1 (in the case $l = m$) yields that there exists a system of morphisms $f_i \in \underline{D}(F(w_{\geq i}X'), v_{\geq i}X)$ compatible with g; we fix such a system. Applying the same assertion for any pair of $l, m : \ l > m$, we obtain that f_l is compatible with f_m (here we use arguments similar to those described in Remark 2.2.2). Finally, since any commutative square can be extended to a morphism of the corresponding distinguished triangles (an axiom of triangulated categories), we obtain that we can complete (uniquely up to a non-canonical isomorphism) the data chosen to a morphism of Postnikov towers (i.e. choose a compatible system of morphisms $F(X'^i) \to X^i$).
3. Easy from assertion 1; note that for any commutative square in $\underline{A}$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle h} & & \downarrow \\ Z & \xrightarrow{g} & T \end{array}$$

if we fix the rows then the morphism $g \circ h : X \to T$ completely determines the morphism $\operatorname{Im} f \to \operatorname{Im} g$ induced by h.

□

We easily obtain a comparison morphism of weight spectral sequences.

Proposition 2.7.3. *I Let F, X', G be as in the previous proposition; suppose also that H is cohomological. Set $X = F(X')$, $g = id_X$.*
1. There exists some comparison morphism of the corresponding weight spectral sequences $M : T_v(H, X) \to T_w(G, X')$. Moreover, this morphism is unique and additively functorial (in g) starting from E_2.
2. Let there exist $D \subset \underline{C}^{w=0}$ such that any $Y \in \underline{C}^{w=0}$ is a retract of some $Z \in D$, and that for any $Z \in D$ there exists a choice of $Z^{w\geq 1}$ such that

$E_2^{pq}T_v(H, F(Z^{w\geq 1})) = \{0\}$ for all $p, q \in \mathbb{Z}$. Then (any choice of) M yields an isomorphism of the spectral sequence functors starting from E_2.
3. Let $\underline{E}$ be a triangulated category endowed with a weight structure u, $F' : \underline{D} \to \underline{E}$ a right weight-exact functor; suppose that $H = E \circ F'$ for some cohomological functor $E : \underline{E} \to \underline{A}$. Then we have the following associativity property for comparison of weight spectral sequences: the composition of M with (any choice of) a comparison morphisms $M' : T_u(E, F'(X)) \to T_v(H, X)$ constructed as in assertion 1, starting from E_2 is canonically isomorphic to (any choice of a similarly constructed) comparison morphism $T_u(E, F'(X)) \to T_w(G, X')$.
II Let H, X', X, G be as above, but suppose that $F : \underline{C} \to \underline{D}$ is left weight-exact. Then a method dual to the one for assertion I1 yields a transformation $N : T_w(G, X') \to T_v(H, X)$; this construction satisfies the duals for all properties of M described in assertion I.

Proof. I 1. In order to construct some comparison morphism, it suffices to construct a morphism of the corresponding exact couples that is compatible with id_X. Hence it suffices to use Proposition 2.7.2(2) to obtain a morphism of the corresponding Postnikov towers, and then apply H to it.
Theorem 2.4.2(II) yields that weight spectral sequences could be described in terms of the corresponding virtual t-truncations. Hence Proposition 2.7.2(3) implies all the functoriality properties of M listed.
2. It suffices to prove that M is an isomorphism on $E_2^{**}T_w(G, Y)$ for all $Y \in Obj\underline{C}$.
Since $D \subset \underline{C}^{w\geq 0}$, this assertion is true for any $Y \in D$. Since $Z \mapsto E_2(T(G, Z))$ is a cohomological functor for any weight structure (see Theorem 2.4.2 and the remark at Definition 2.3.2), the assertion is also true for any $Y \in Obj\underline{C}^b$. To conclude it suffices to note that for any H, any $Y \in Obj\underline{C}$, any finite 'piece' of $E_2^{**}T_w(G, Y)$ coincides with the corresponding piece of $E_2^{**}T_w(G, w_{[i,j]}Y)$ (for any choice of $w_{[i,j]}Y$) if i is small enough and j is large enough, and this isomorphism is compatible with M.
3. We recall that comparison morphisms for weight spectral sequences were constructed using Proposition 2.7.2(1). It easily follows that $M' \circ M$ is one of the possible choices for a comparison morphism $T_u(E, F' \circ F(X)) \to T_w(G, X')$. It suffices to apply assertion I1 to conclude that this fixed choice of a comparison morphism coincides with any other possible choice starting from E_2.
II We obtain the assertion from assertion I immediately by dualization (see Theorem 2.2.1(1)); here one should consider the duals of $\underline{C}$, $\underline{D}$, and $\underline{A}$ (and also 'dualize' the connecting functors). $\square$

Remark 2.7.4. M is an isomorphism (starting from E_2) also if: there exists a localization of $\underline{D}$ such that H factorizes through it, v induces a weight structure v' on it, w induces a weight structure on the categorical image of $\underline{C}$ that coincides with the restriction of v' to it (since both weight spectral sequences are isomorphic to the spectral sequence corresponding to this new weight structure).

Yet this conditions are somewhat restrictive since weight structures do not 'descend' to localizations in general (since for an exact $F' : \underline{C} \to \underline{E}$ the classes $F'_*(\underline{C}^{w\geq 1})$ and $F'_*(\underline{C}^{w\leq 0})$ are not necessarily orthogonal in $\underline{E}$).

In order to simplify checking right and left weight-exactness of functors, we will need the following easy statement.

Lemma 2.7.5. *Let w be bounded.*

1. An exact $J : \underline{C} \to \underline{D}$ is a right weight-exact whenever there exists a $D \subset \underline{C}^{w=0}$ such that any $Y \in \underline{C}^{w=0}$ is a retract of some $X \in D$, and that for any $X \in D$ we have $J(Y) \in \underline{D}^{v\geq 0}$.

2. An exact $J : \underline{C} \to \underline{D}$ is a left weight-exact whenever there exists a $D \subset \underline{C}^{w=0}$ such that any $Y \in \underline{C}^{w=0}$ is a retract of some $X \in D$, and that for any $X \in D$ we have $J(Y) \in \underline{D}^{v\leq 0}$.

Proof. It suffices to prove assertion 1, since assertion 2 is exactly its dual.

If J is right weight-exact functor, then we can take $D = \underline{C}^{w=0}$

Now we prove the converse statement. Since $\underline{D}^{v\geq 0}$ is Karoubi-closed and extension-stable in $\underline{D}$, Theorem 2.2.1(19) yields that $J(\underline{C}^{w\geq 0})$ indeed belongs to $\underline{D}^{v\geq 0}$.

□

3 Categories of comotives (main properties)

We embed DM^{eff}_{gm} into a certain big triangulated motivic category $\mathfrak{D}$; we will call it objects *comotives*. We will need several properties of $\mathfrak{D}$; yet we will never use its description directly. For this reason in §3.1 we only list the main properties of $\mathfrak{D}$.

In §3.2 we associate certain comotives to (disjoint unions of) 'infinite intersections' of smooth varieties over k (we call those pro-schemes). We also introduce certain Tate twists for these comotives.

In §3.3 we recall the definition of a primitive scheme (note that in the case of a finite k we call a scheme primitive whenever it is smooth semi-local). The main motivic property of primitive schemes (proved by M. Walker) is: $F(S)$ injects into $F(S_0)$ if S is primitive connected, S_0 is its generic point, and F is a homotopy invariant presheaf with transfers.

In §3.4 we study the relation of (comotives of) primitive schemes with the homotopy t-structure for DM^{eff}_-.

In §3.5 we prove that there are no $\mathfrak{D}$-morphisms of positive degrees between comotives of primitive schemes (and also certain Tate twists of those); this is also true for products of comotives mentioned.

In §3.6 we prove that one can pass to countable homotopy limits in Gysin distinguished triangles; this yields Gysin distinguished triangles for comotives of pro-schemes. This allows to construct certain Postnikov towers for comotives of pro-schemes (and their Tate twists), whose factors are twisted products of comotives of function fields (over k). The construction of the tower is parallel to the classical construction of coniveau spectral sequences (see §1 of [8]).

3.1 Comotives: an 'axiomatic description'

We will define $\mathfrak{D}$ below as the derived category of differential graded functors $J \to B(Ab)$; here J yields a differential graded enhancement of DM_{gm}^{eff} (cf. [4], [19], or [7]), $B(Ab)$ is the differential graded category of complexes over Ab. We will also need some category $\mathfrak{D}'$ that projects to $\mathfrak{D}$ (a certain model of $\mathfrak{D}$). Derived categories of differential graded functors were studied in detail in [12] and [16]. We will define and study them in §5 below; now we will only list their properties that are needed for the proofs of main statements.
Below we will also need certain (filtered) inverse limits several times. $\mathfrak{D}$ is a triangulated category; so it is no wonder that there are no nice limits in it. So we consider a certain additive $\mathfrak{D}'$ endowed with an additive functor $p : \mathfrak{D}' \to \mathfrak{D}$. We will call (the images of) inverse limits from $\mathfrak{D}'$ homotopy limits in $\mathfrak{D}$.
The relation of $\mathfrak{D}'$ with $\mathfrak{D}$ is similar to the relation of $C(\underline{A})$ with $D(\underline{A})$. In particular, $\mathfrak{D}'$ is closed with respect to all (small filtered) inverse limits; we have functorial cones of morphisms in $\mathfrak{D}'$ that are compatible with inverse limits.
We will need some conventions and definitions introduced in Notation; in particular, I, L will be projective systems; I is countable.

Proposition 3.1.1. *1. There exists a triangulated category $\mathfrak{D} \supset DM_{gm}^{eff}$; all objects of DM_{gm}^{eff} are cocompact in $\mathfrak{D}$.*

2. *There exists an additive category $\mathfrak{D}'$ closed with respect to arbitrary (small filtered) inverse limits, and an additive functor $p : \mathfrak{D}' \to \mathfrak{D}$ that preserves (small) products. Moreover, p is surjective on objects.*

3. *$\mathfrak{D}'$ is endowed with a certain invertible shift functor* $[1]$ *that is compatible with the shift on $\mathfrak{D}$ and respects inverse limits.*

4. *There is a functorial cone of morphisms in $\mathfrak{D}'$ defined; it is compatible with* $[1]$ *and respects inverse limits.*

5. *Any triangle of the form $X \xrightarrow{f} Y \to \mathrm{Cone}(f) \to X[1]$ in $\mathfrak{D}'$ becomes distinguished in $\mathfrak{D}$.*

6. *The composition functor $M_{gm} : C^b(SmCor) \to DM_{gm}^{eff} \to \mathfrak{D}$ can be canonically factorized through an additive functor $j : C^b(SmCor) \to \mathfrak{D}'$. Shifts and cones of morphisms in $C^b(SmCor)$ are compatible with those in $\mathfrak{D}'$ via j.*

7. *For any $X \in M_{gm}(C^b(SmCor)) \subset Obj\mathfrak{D}$, any $Y : L \to \mathfrak{D}'$ we have $\mathfrak{D}(p(\varprojlim_{l\in L} Y_l), X) = \varinjlim_{l\in L} \mathfrak{D}(p(Y_l), X)$.*

8. *DM_{gm}^{eff} weakly cogenerates $\mathfrak{D}$ (i.e. we have ${}^{\perp}DM_{gm}^{eff} = \{0\}$, see Notation).*

9. *Let a sequence $i_n \in I$, $n > 0$, be increasing (i.e. $i_{n+1} > i_n$ for any $n > 0$) unbounded (see Notation). Then for all functors $X : I \to \mathfrak{D}'$, we have functorial distinguished triangles in $\mathfrak{D}$:*
$$p(\varprojlim_{i\in I} X_i) \to p(\prod X_{i_n}) \xrightarrow{e} p(\prod X_{i_n}); \tag{27}$$
e is the product of $id_{X_{i_n}} \oplus -\phi_n : X_{i_{n+1}} \to X_{i_n}$; here ϕ_n are the morphisms coming from I via X.

10. *There exists a differential graded enhancement for $\mathfrak{D}$; see §5.1 below.*

Remark 3.1.2. 1. Since below we will prove some statements for $\mathfrak{D}$ only using its 'axiomatics' (i.e. the properties listed in Proposition 3.1.1), these results would also be valid in any other category that fulfills these properties. This could be useful, since the author is not sure at all that all possible $\mathfrak{D}$ are isomorphic.
2. Moreover, one could modify the axiomatics of $\mathfrak{D}$ and consider instead a category that would only contain the triangulated subcategory of DM^{eff}_{gm} generated by motives of smooth varieties of dimension $\le n$ (for a fixed $n > 0$). Our results and arguments below can be easily carried over to this setting (with minor modifications; it is also useful here to weaken condition 8 in the Proposition). This makes sense since these 'geometric pieces' of DM^{eff}_{gm} are self-dual with respect to Poincare duality (at least, if $\operatorname{char} k = 0$); see §6.4 below. See also Remark 4.5.2(2).
Alternatively, we can weaken the condition for the functor $DM^{eff}_{gm} \to \mathfrak{D}$ to be a full embedding. For example, it could be interesting to consider the version of $\mathfrak{D}$ for which this functor kills $DM^{eff}_{gm}(n)$ (for some fixed $n > 0$).
Lastly note that we do not really need condition 2 in its full generality (below).

Now we derive some consequences from the axiomatics listed.

Corollary 3.1.3. *1. For any $Z \in Obj DM^{eff}_{gm} \subset Obj\mathfrak{D}$, any $X : L \to \mathfrak{D}'$ we have $\mathfrak{D}(p(\varprojlim_{l\in L} X_l), Z) = \varinjlim_{l\in L} \mathfrak{D}(p(X_l), Z)$.*

2. *For any $T \in Obj\mathfrak{D}$, all functors $Y : I \to \mathfrak{D}'$ we have functorial short exact sequences*
$$\{0\} \to \varprojlim{}^1 \mathfrak{D}(T, p(Y_i)[-1]) \to \mathfrak{D}(T, p(\varprojlim Y_i)) \to \varprojlim \mathfrak{D}(T, p(Y_i)) \to \{0\};$$
here $\varprojlim^1$ is the (first) derived functor of $\varprojlim = \varprojlim_I$.

3. *For all functors $X : L \to C^b(SmCor)$, $Y : I \to C^b(SmCor)$, we have functorial short exact sequences*
$$\begin{aligned}\{0\} \to \varprojlim{}^1_{i\in I}(\varinjlim_{l\in L} \mathfrak{D}(M_{gm}(X_l), M_{gm}(Y_i)[-1])) \to \\ \mathfrak{D}(p(\varprojlim_{l\in L} j(X_l)), p(\varprojlim_{i\in I} j(Y_i))) \to \\ \varprojlim_{i\in I}(\varinjlim_{l\in L} \mathfrak{D}(M_{gm}(X_l), M_{gm}(Y_i))) \to \{0\}.\end{aligned} \tag{28}$$

4. *$\mathfrak{D}$ is idempotent complete.*

Proof. 1. If $Z \in M_{gm}(C^b(SmCor))$, then the assertion is exactly Proposition 3.1.1(7).

It remains to note that any $Z \in ObjDM_{gm}^{eff}$ is a retract of some object coming from $C^b(SmCor)$.

2. Since inverse limits and their derived functors do not change when we replace a projective system by any unbounded subsystem, we can assume that L consists of some i_n as in (27).

Now, (27) yields a long exact sequence

$$\cdots \to \prod_{i\in I} \mathfrak{D}(T, p(Y_i)[-1]) \xrightarrow{f} \prod_{i\in I} \mathfrak{D}(T, p(Y_i)[-1]) \to \mathfrak{D}(T, p(\varprojlim_{i\in I} Y_i))$$
$$\to \prod_{i\in I} \mathfrak{D}(T, p(Y_i)) \xrightarrow{g} \prod_{i\in I} \mathfrak{D}(T, p(Y_i)) \to \dots,$$

here f and g are induced by e in (27).

It is easily seen that $\operatorname{Ker} g \cong \varprojlim \mathfrak{D}(T, M_{gm}(Y_m))$.

Lastly, Remark A.3.6 of [21] allows to identify $\operatorname{Coker} f$ with $\varprojlim^1 \mathfrak{D}(T, M_{gm}(Y_m)[-1])$.

3. Immediate from the previous assertions.

4. Since $\mathfrak{D}'$ is closed with respect to all inverse limits, it is closed with respect to all (small) products. Then Proposition 3.1.1(2) yields that $\mathfrak{D}$ is also closed with respect to all products. Now, Remark 1.6.9 of [21] yields the result (in fact, the proof uses only countable products).

□

We will often call the objects of $\mathfrak{D}$ *comotives.*

3.2 Pro-schemes and their comotives

Now we have certain inverse limits for objects (coming from) $C^b(SmCor)$; this allows to define (reasonable) comotives for certain schemes that are not (necessarily) of finite type over k (and for their disjoint unions). We also define certain Tate twists of those.

We will call certain ind-schemes over k *pro-schemes.* An ind-scheme V/k is a pro-scheme if it is a countable disjoint union of schemes, such that each of them is a projective limit of smooth varieties of dimension $\le c_V$ for some fixed $c_V \ge 0$ (in the category of schemes) with connecting morphisms being open dense embeddings. One may say that a pro-scheme is a countable disjoint union of countable intersections of smooth varieties. Note that (the spectrum of) any function field over k is a pro-scheme; any smooth k-variety is a pro-scheme also.

We have the operation of countable disjoint union for pro-schemes of bounded dimension.
Now, we would like to present a (not necessarily connected) pro-scheme V as projective limits of smooth varieties V_i. This is easy if V is connected (cf. Lemma 3.2.9 of [9]). In the general case one should allow (formally) zero morphisms between connected components of V_i (for distinct i). So we consider a new category $SmVar'$ containing the category of all smooth varieties as a (non-full!) subcategory. We take $ObjSmVar' = SmVar$; for any smooth connected varieties $X, Y \in SmVar$ we have $SmVar'(X,Y) = Mor_{Var}(X,Y) \cup \{0\}$; the composition of a zero morphism with any other one is zero; $SmVar'(\sqcup_i X_i, \sqcup_j Y_j) = \sqcup_{i,j} SmVar'(X_i, Y_j)$. $SmVar'$ can be embedded into $SmCor$ (certainly, this embedding is not full).
We will write $V = \varprojlim V_i$ (this is not possible in the category of ind-schemes, but works in Pro $-SmVar'$). Note that the set of connected components of V is the inductive limit of the corresponding sets for V_i.
Now, for any pro-scheme $V = \varprojlim V_i$, any $s \geq 0$, we introduce the following notation: $M_{gm}(V)(s) = p(\varprojlim(j(V_i)(s))) \in Obj\mathfrak{D}$ (see Proposition 3.1.1); we will denote $M_{gm}(V)(0)$ by $M_{gm}(V)$ and call $M_{gm}(V)$ the comotif of V. This notation should be considered as formal i.e. we do not define Tate twists on $\mathfrak{D}$ (till §5.4.3).
Obviously, if $V \in SmVar$, its comotif (and its twists) coincides with its motif (and its twists), so we can use the same notation for them.
If $\underline{A}$ is a category closed with respect to filtered direct limits, $H' : DM_{gm}^{eff} \to \underline{A}$ is a functor, we can (formally) extend it to co-motives in question; we set:

$$H(M_{gm}(V)(s)[n]) = \varinjlim H'(M_{gm}(V_i)(s)[n]). \tag{29}$$

Remark 3.2.1. 1. For a general H' this notation should be considered as formal. Yet in the case $H' = (-,Y) : \mathfrak{D} \to Ab$, $Y \in ObjDM_{gm}^{eff} \subset Obj\mathfrak{D}$, we have $H(M_{gm}(V)(i)[n]) = \mathfrak{D}(M_{gm}(V)(i)[n], X)$; see Corollary 3.1.3(1), i.e. (29) yields the value of a well-defined functor $\mathfrak{D} \to Ab$ at $M_{gm}(V)(s)[n]$. We will only need H' of this sort till §4.3.
More generally, there exists such an H if $\underline{A}$ satisfies AB5 and H' is cohomological; we will call the corresponding H an *extended cohomology theory*, see Remark 4.3.2 below.
2. Let V^j be a countable set of pro-schemes (of bounded dimensions). Then $M_{gm}(\sqcup V^j) = \prod M_{gm}(V^j)$ by Proposition 3.1.1(2).
Besides, for any H' as in (29) we have $H(M_{gm}(\sqcup V^j)(s)[n]) = \bigoplus H(M_{gm}(V^j)(s)[n])$.

Below we will need some conventions for pro-schemes.
For pro-schemes $U = \varprojlim U_i$ and $V = \varprojlim V_j$ we will call an element of $\varprojlim_{i\in I}(\varinjlim_{j\in J} SmCor(U_i, V_j))$ an open embedding if it can be obtained as a double limit of open embeddings $U_i \to V_j$ (as varieties). If $U = V \setminus W$ for some pro-scheme W, we will say that W is a closed sub-pro-scheme of V. Note that in this case any connected component of W is a closed subscheme of some

connected component of V; yet some components of V could contain an infinite set of connected components of W.
For $V = \sqcup V^j$, V^j are connected pro-schemes, we will call the maximum of the transcendence degrees of function fields of V^j the dimension of V (note that this is finite). We will say that a sub-pro-scheme $U = \sqcup U^m$, U^m are connected, is everywhere of codimension r (resp. $\geq r$, for some fixed $r \geq 0$) in $V = \sqcup V^j$ if for every induced embedding $U^m \to V^j$ the difference of their dimensions (defined as above) is r (resp. $\geq r$).
We will call the inverse limit of the sets of points of V_i of a fixed codimension $s \geq 0$ the set of points of V of codimension s (note that any element of this set indeed defines a point of some connected component of V).

3.3 Primitive schemes: reminder

In [29] M. Walker proved that primitive schemes in the case of an infinite k have 'motivic' properties similar to those of smooth semi-local schemes (in the sense of §4.4 of [26]). Since we don't want to discriminate the case of a finite k, we will modify slightly the standard definition of primitive schemes.

Definition 3.3.1. If k is infinite then a (pro-)scheme is called primitive if all of its connected components are affine and their coordinate rings R_j satisfy the following primitivity criterion: for any $n > 0$ every polynomial in $R_j[X_1, \dots, X_n]$ whose coefficients generate R_j as an ideal over itself, represents an R_j-unit.
If k is finite, then we will call a pro-scheme primitive whenever all of its connected components are semi-local (in the sense of §4.4 of [26]).

Remark 3.3.2. Recall that in the case of infinite k all semi-local k-algebras satisfy the primitivity criterion (see Example 2.1 of [29]).

Below we will mostly use the following basic property of primitive schemes.

Proposition 3.3.3. *Let S be a primitive pro-scheme, let S_0 be the collection of all of its generic points; F is a homotopy invariant presheaf with transfers. Then $F(S) \subset F(S_0)$; here we define F on pro-schemes as in (29).*

Proof. We can assume that S is connected (so it is a smooth primitive scheme). Hence in the case of infinite k our assertion follows from Theorem 4.19 of [29]. Now, if k is finite, then S_0 is semi-local (by our convention); so we may apply Corollary 4.18 of [26] instead.

□

3.4 Basic motivic properties of primitive schemes

We will call a primitive pro-scheme just a primitive scheme. We prove certain motivic properties of primitive schemes (in the form in which we will need them below).

Proposition 3.4.1. *For $F \in ObjDM_-^{eff}$ we define $H'(-) = DM_-^{eff}(-, F)$ on DM_{gm}^{eff}; we also define $H(M_{gm}(V)(i)[n])$ as in (29). Let S be a primitive scheme, $m \geq 0$, $i \in \mathbb{Z}$.*
1. Let $F \in DM_-^{eff\, t \leq -1}$ (t is the homotopy t-structure, that we considered in §1.3). Then $H(M_{gm}(S)(m)[m]) = \{0\}$.
2. More generally, for any $F \in ObjDM_-^{eff}$ we have $H([M_{gm}(S)(m)[m]) \cong F^0_{-m}(S)$ where $F^0 = F^{t=0}$, F^0_{-m} is the m-th Tate twist of F^0 (see Definition 1.4.1).

Proof. 1. We consider the homotopy invariant presheaf with transfers $F_{-m} : X \mapsto DM_-^{eff}(M_{gm}(X)(m)[m], F)$. We should prove that $F_{-m}(S) = 0$ (here we extend F_{-m} to pro-schemes in the usual way i.e. as in (29)).
(29) also yields that $F_{-m}(\sqcup S_i) = \bigoplus F_{-m}(S_i)$. Hence by Proposition 3.3.3, it suffices to consider the case of S being (the spectrum of) a function field over k. Since F_{-m} is represented by an object of $DM_-^{eff\, t \leq -1}$ (see Proposition 1.4.2(2)), it suffices to note that any field is a Henselian scheme i.e. a point in the Nisnevich topology.
2. By Proposition 1.4.2, for any $X \in SmVar$ we have $M_{gm}(X)(m)[m] \perp DM_-^{eff\, t \geq 1}$. Hence we can assume $F \in DM_-^{eff\, t \leq 0}$.
Next, using assertion 1, we can easily reduce the situation to the case $F = F^{t=0} \in ObjHI$ (by considering the t-decomposition of $F[-1]$). In this case the statement is immediate from Proposition 1.4.2(1).

$\square$

Lemma 3.4.2. *Let $U \to U'$ be an open dense embedding of smooth varieties.*
1. We have $\mathrm{Cone}(M_{gm}(U) \to M_{gm}(U')) \in DM_-^{eff\, t \leq -1}$.
2. Let S be primitive. Then for any $n, m, i \geq 0$ the map

$$\mathfrak{D}(M_{gm}(S)(m)[m], M_{gm}(U)(n)[n+i]) \to \mathfrak{D}(M_{gm}(S)(m)[m], M_{gm}(U')(n)[n+i])$$

is surjective.

Proof. 1. We denote $\mathrm{Cone}(M_{gm}(U) \to M_{gm}(U')) \in DM_-^{eff\, t \leq -1}$ by C. Obviously, $C \in DM_-^{eff\, t \leq 0}$. Let H denote $C^{t=0}$ ($H \in ObjHI$). By Corollary 4.19 of [26], we have $H(U) \subset H(U')$. Next, from the long exact sequence $\{0\}(= DM_-^{eff}(M_{gm}(U)[1], H)) \to DM_-^{eff}(C, H) \to DM_-^{eff}(U', H) \to DM_-^{eff}(U, H) \to \dots$ we obtain $C \perp H$. Then the long exact sequence $\dots \to DM_-^{eff}(C^{t \leq -1}[2], H) \to DM_-^{eff}(H, H) \to DM_-^{eff}(C, H) \to \dots$ yields $H = 0$.
2. It suffices to check that $M_{gm}(S)(m)[m] \perp C(n)[n+i]$. Since $M_{gm}(U)(n)[n]$ is canonically a retract of $M_{gm}(U \times G_m^n)$, we can assume that $n = 0$.
Now the claim follows immediately from assertion 1 and Proposition 3.4.1(1).

$\square$

3.5 On morphisms between comotives of primitive schemes

We will need the fact that certain 'positive' morphism groups are zero.
Let $n, m, \geq 0$, $i > 0$, $Y = \varprojlim Y_l$ $(l \in L)$, be any pro-scheme, X be a primitive scheme.

Proposition 3.5.1. *1. The natural homomorphism*

$$\mathfrak{D}(M_{gm}(X)(m)[m], M_{gm}(Y)[n](n)) \to \\ \to \varprojlim_l (\varinjlim_{X \subset Z, Z \in SmVar} DM_{gm}^{eff}(Z(m)[m], M_{gm}(Y_l)(n)[n]))$$

is surjective.

2. $M_{gm}(X)(m)[m] \perp M_{gm}(Y)[n+i](n)$.

Proof. Note first that by the definition of the Tate twist (1), it can be lifted to $C^b(SmCor)$.

1. This is immediate from the short exact sequence (28).

2. By Remark 3.2.1(2), we may suppose that Y is connected. Then we apply (28) again. The corresponding $\varprojlim$-term is zero by Proposition 3.4.1(1). Lastly, the surjectivity proved in Lemma 3.4.2(2) yields that the corresponding $\varprojlim^1$-term is zero. Indeed, the groups $\mathfrak{D}(M_{gm}(X)(m)[m], M_{gm}(Y_l)[n+i-1](n))$ obviously satisfy the Mittag-Leffler condition; see §A.3 of [21].

   In fact, one could easily deduce the assertion from the results of the previous subsection and (27) directly (we do not need much of the theory of higher limits in this paper).

□

Remark 3.5.2. In fact, this statement, as well as all other properties of (primitive) pro-schemes that we need, are also true for not necessary countable disjoint unions of (primitive) pro-schemes. This observation could be used to extend the main results of the paper to a somewhat larger category; yet such an extension does not seem to be important.

3.6 The Gysin distinguished triangle for pro-schemes; 'Gersten' Postnikov towers for comotives of pro-schemes

We prove that we can pass to countable homotopy limits in Gysin distinguished triangles.

Proposition 3.6.1. *Let Z, X be pro-schemes, Z a closed subscheme of X (everywhere) of codimension r. Then for any $s \geq 0$ the natural morphism $M_{gm}(X \setminus Z)(s) \to M_{gm}(X)(s)$ extends to a distinguished triangle (in $\mathfrak{D}$): $M_{gm}(X \setminus Z)(s) \to M_{gm}(X)(s) \to M_{gm}(Z)(r+s)[2r]$.*

Proof. First assume $s = 0$.
We can assume $X = \varprojlim X_i$, $Z = \varprojlim Z_i$ for $i \in I$, where $X_i, Z_i \in SmVar$, Z_i is closed everywhere of codimension r in X_i for all $i \in I$.
We take $Y_i = j(X_i \setminus Z_i \to X_i)$, $Y = p(\varprojlim_{i\in I} Y_i)$. By parts 4 and 5 of Proposition 3.1.1 we have a distinguished triangle $M_{gm}(X \setminus Z) \to M_{gm}(X) \to Y$.
It remains to prove that $Y \cong M_{gm}(Z)(r)[2r]$. Proposition 2.4.5 of [9] (a functorial form of the Gysin distinguished triangle for Voevodsky's motives) yields that $p(Y_i) \cong M_{gm}(Z_i)(r)[2r]$; moreover, the connecting morphisms $p(Y_i) \to p(Y_{i+1})$ are obtained from the corresponding morphisms $M_{gm}(Z_i) \to M_{gm}(Z_{i+1})$ by tensoring by $\mathbb{Z}(r)[2r]$. It remains to recall: by Proposition 3.1.1(9), the isomorphism class of a homotopy limit in $\mathfrak{D}$ can be completely described in terms of (objects and morphisms) of $\mathfrak{D}$ (i.e. we don't have to consider the lifts of objects and morphisms to $\mathfrak{D}'$). This yields the result.
Now, since $M_{gm}(X \times G_m) = M_{gm}(X) \bigoplus M_{gm}(X)(1)[1]$ for any $X \in SmVar$ (hence this is also true for pro-schemes), the assertion for the case $s = 0$ yields the general case easily. □

Now we will construct a certain Postnikov tower $Po(X)$ for X being the (twisted) comotif of a pro-scheme Z that will be related to the coniveau spectral sequences (for cohomology) of Z; our method was described in §1.5 above. Note that we consider the general case of an arbitrary pro-scheme Z (since in this paper pro-schemes play an important role); yet this case is not much distinct from the (partial) case of $Z \in SmVar$.

Corollary 3.6.2. *We denote the dimension of Z by d (recall the conventions of §3.2).*
For all $i \geq 0$ we denote by Z^i the set of points of Z of codimension i.
For any $s \geq 0$ there exists a Postnikov tower for $X = M_{gm}(Z)(s)[s]$ such that $l = 0$, $m = d+1$, $X_i \cong \prod_{z\in Z^i} M_{gm}(z)(i+s)[2i+s]$.

Proof. As above, it suffices to prove the statement for $s = 0$. Since any product of distinguished triangles is distinguished, we can assume Z to be connected.
We consider a projective system L whose elements are sequences of closed subschemes $\varnothing = Z_{d+1} \subset Z_d \subset Z_{d-1} \subset \cdots \subset Z_0$. Here $Z_0 \in SmVar$, $Z_l \in Var$ for $l > 0$, Z is open in Z_0 (see §3.2; Z_0 is connected; in the case when $Z \in SmVar$ we only take $Z_0 = Z$); for all $j > 0$ we have: Z_j is everywhere of codimension $\geq j$ in Z_0; all irreducible components of all Z_j are everywhere of codimension $\geq j$ in Z_0; and Z_{j+1} contains the singular locus of Z_j (for $j \leq d$).
The ordering in L is given by open embeddings of varieties $U_j = Z_0 \setminus Z_j$ for $j > 0$. For $l \in L$ we will denote the corresponding sequence by $\varnothing = Z^l_{d+1} \subset Z^l_d \subset Z^l_{d-1} \subset \cdots \subset Z^l_0$. Note that L is countable!
By the previous proposition, for any j we have a distinguished triangle $M_{gm}(\varprojlim(Z^l_0 \setminus Z^l_j)) \to M_{gm}(\varprojlim(Z^l_0 \setminus Z^l_{j+1})) \to M_{gm}(\varprojlim(Z^l_j \setminus Z^l_{j+1})(j)[2j])$.
It remains to compute the last term; we fix some j.
We have $\varprojlim_{l\in L'}(Z^l_j \setminus Z^l_{j+1})) \cong \prod_{z\in Z^i} M_{gm}(z)$. Indeed, for all $l \in L$ the variety $Z^l_j \setminus Z^l_{j+1}$ is the disjoint union of some locally closed smooth subschemes of

Z_0^l of codimension j; for any $z_0 \in Z^j$ for $l \in L$ large enough z_0 is contained in $Z_j^l \setminus Z_{j+1}^l$ as an open sub-pro-scheme, and the inverse limit of connected components of $Z_j^l \setminus Z_{j+1}^l$ containing z_0 is exactly z_0. Now, we can apply the functor $X \mapsto M_{gm}(X)(j)[2j]$ to this isomorphism. We obtain $M_{gm}(\varprojlim(Z_j^l \setminus Z_{j+1}^l)(j)[2j]) \cong \prod_{z \in Z^i} M_{gm}(z)(i)$. This yields the result.

$\square$

Remark 3.6.3. 1. Alternatively, one could construct $Po(X)$ for the (twisted) comotif of a pro-scheme $T = \varprojlim T^l$ as the inverse limit of the Postnikov towers for T^l (constructed as above yet with fixed $Z_0^l = T^l$); certainly, to this end one should pass to the limit in $\mathfrak{D}'$. It is easily seen that one would get the same tower this way.

2. Certainly, if we shift a Postnikov tower for $M_{gm}(Z)(s)[s]$ by $[j]$ for some $j \in \mathbb{Z}$, we obtain a Postnikov tower for $M_{gm}(Z)(s)[s+j]$. We didn't formulate assertion 2 for these shifts only because we wanted X^p to belong to $\mathfrak{D}_s^{w=0}$ (see Proposition 4.1.1 below).

3. Since the calculation of X^i used Proposition 3.1.1(9), our method cannot describe connecting morphisms between them (in $\mathfrak{D}$). Yet one can calculate the 'images' of the connecting morphisms in $\mathfrak{D}^{naive}$; see §1.5 and §6.1.

4 Main motivic results

The results of the previous section combined with those of §2.2 allow us to construct (in §4.1) a certain *Gersten weight structure* w on a certain triangulated $\mathfrak{D}_s$: $DM_{gm}^{eff} \subset \mathfrak{D}_s \subset \mathfrak{D}$. Its main property is that comotives of function fields over k (and their products) belong to $\underline{Hw}$. It follows immediately that the Postnikov tower $Po(X)$ provided by Corollary 3.6.2 is a *weight Postnikov tower* with respect to w. Using this, in §4.2 we prove: if S is a primitive scheme, S_0 is its dense sub-pro-scheme, then $M_{gm}(S)$ is a direct summand of $M_{gm}(S_0)$; $M_{gm}(K)$ (for a function field K/k) contains (as retracts) comotives of primitive schemes whose generic point is K, as well as twisted comotives of residue fields of K (for all geometric valuations).

In §4.3 we (easily) translate these results to cohomology; in particular, the cohomology of (the spectrum of) K contains direct summands corresponding to the cohomology of primitive schemes whose generic point is K, as well as twisted cohomology of residue fields of K. Here one can consider any cohomology theory $H : \mathfrak{D}_s \to \underline{A}$; one can obtain such an H by extending to $\mathfrak{D}_s$ any cohomological $H' : DM_{gm}^{eff} \to \underline{A}$ if $\underline{A}$ satisfies AB5 (by means of Proposition 1.2.1). Note: in this case the cohomology of pro-schemes mentioned is calculated in the 'usual' way.

In §4.4 we consider weight spectral sequences corresponding to (the Gersten weight structure) w. We observe that these spectral sequences generalize naturally the classical coniveau spectral sequences. Besides, for a fixed $H : \mathfrak{D}_s \to \underline{A}$ our (generalized) coniveau spectral sequence converging to $H^*(X)$ (where X

could be a motif or just an object of $\mathfrak{D}_s$) is $\mathfrak{D}_s$-functorial in X (i.e. it is motivically functorial for objects of DM_{gm}^{eff}); this fact is non-trivial even when restricted to motives of smooth varieties.

In §4.5 we prove that there exists a nice duality $\mathfrak{D}^{op} \times DM_-^{eff} \to Ab$ (extending the bi-functor $DM_-^{eff}(-,-) : DM_{gm}^{eff\,op} \times DM_-^{eff} \to Ab$); the Gersten weight structure w (on $\mathfrak{D}_s$) is left orthogonal to the homotopy t-structure t on DM_-^{eff} with respect to it. This allows to apply Theorem 2.6.1: in the case when H comes from $Y \in ObjDM_-^{eff}$ we prove the isomorphism (starting from E_2) of (the coniveau) $T(H,X)$ with the spectral sequence corresponding to the t-truncations of Y. We describe $ObjDM_{gm}^{eff} \cap \mathfrak{D}_s^{w\le i}$ in terms of t (for DM_-^{eff}). We also note that our results allow to describe torsion motivic cohomology in terms of (torsion) étale cohomology (see Remark 4.5.4(4)).

In §4.6 we define the coniveau spectral sequence (starting from E_2) for cohomology of a motif X over a not (necessarily) countable perfect base field l as the limit of the corresponding coniveau spectral sequences over countable perfect subfields of definition for X. This definition is compatible with the classical one (for X being the motif of a smooth variety); so we obtain motivic functoriality of classical coniveau spectral sequences over a general base field.

In §4.7 we prove that the Chow weight structure for DM_{gm}^{eff} (introduced in §6 of [6]) could be extended to $\mathfrak{D}$ (certainly, the corresponding weight structure w_{Chow} differs from w). We will call the corresponding weight spectral sequences *Chow-weight* ones; note that they are isomorphic to classical (i.e. Deligne's) weight spectral sequences when the latter are defined.

In §4.8 we use the results §2.7 to compare coniveau spectral sequences with Chow-weight ones. We always have a comparison morphism; it is an isomorphism if H is a *birational* cohomology theory.

In §4.9 we consider the category of birational comotives $\mathfrak{D}_{bir}$ (a certain 'completion' of birational motives of [15]) i.e. the localization of $\mathfrak{D}$ by $\mathfrak{D}(1)$. It turns our that w and w_{Chow} induce the same weight structure w'_{bir} on $\mathfrak{D}_{bir}$. Conversely, starting from w'_{bir} one can glue 'from slices' the weight structures induced by w and w_{Chow} on $\mathfrak{D}/\mathfrak{D}(n)$ for all $n > 0$. Furthermore, these structures belong to an interesting family of weight structures indexed by a single integral parameter; other terms of this family could be also interesting!

4.1 The Gersten weight structure for $\mathfrak{D}_s \supset DM_{gm}^{eff}$

Now we describe the main weight structure of this paper. Unfortunately, the author does not know whether it is possible to define the Gersten weight structure (see below) on the whole $\mathfrak{D}$. Yet for our purposes it is quite sufficient to define the corresponding weight structure on a certain triangulated subcategory $\mathfrak{D}_s \subset \mathfrak{D}$ containing DM_{gm}^{eff} (and comotives of all pro-schemes).

In order to make the choice of $\mathfrak{D}_s \subset \mathfrak{D}$ compatible with extensions of scalars, we bound certain dimensions of objects of $\underline{Hw}$.

We will denote by H the full subcategory of $\mathfrak{D}$ whose objects are all countable products $\prod_{l\in L} M_{gm}(K_l)(n_l)[n_l]$; here K_l are (the spectra of) function fields over k, $n_l \geq 0$; we assume that the transcendence degrees of K_l/k and n_l are bounded.

Proposition 4.1.1. *1. Let $\mathfrak{D}_s$ be the Karoubi-closure of $\langle H\rangle$ in $\mathfrak{D}$. Then $\underline{C} = \mathfrak{D}_s$ can be endowed with a unique weight structure w such that $\underline{Hw}$ contains H.*
2. $\underline{Hw}$ is the idempotent completion of H.
3. $\mathfrak{D}_s$ contains DM_{gm}^{eff} as well as all $M_{gm}(Z)(l)$ for Z being a pro-scheme, $l \geq 0$.
4. For any primitive S, $i \geq 0$, we have $M_{gm}(S)(i)[i] \in \mathfrak{D}_s^{w=0}$.
5. Let Z be a pro-scheme, $s \geq 0$. Then $M_{gm}(Z)(s)[s] \in \mathfrak{D}_s^{w\leq 0}$; the Postnikov tower for $M_{gm}(Z)(s)[s]$ given by Corollary 3.6.2 is a weight Postnikov tower for it.

Proof. 1. By Proposition 3.5.1(2), H is negative (since any object of H is a finite sum of $M_{gm}(X_i)(m_i)$ for some primitive pro-schemes X_i, $m_i \in \mathbb{Z}$). Besides, $\mathfrak{D}$ is idempotent complete (see Corollary 3.1.3(4)); hence $\mathfrak{D}_s$ and $\mathfrak{D}_s^{w=0}$ also are. Hence we can apply Theorem 2.2.1(18) (for $D = H$).
2. Also immediate from Theorem 2.2.1(18).
3. $M_{gm}(Z)(l) \in Obj\mathfrak{D}_s$ by Corollary 3.6.2; in particular, this is true for $Z \in SmVar$. It remains to note that DM_{gm}^{eff} is the Karoubization of $\langle M_{gm}(U) : U \in SmVar\rangle$ in $\mathfrak{D}$.
4. It suffices to note that $M_{gm}(S)(i)[i]$ belongs both to $\mathfrak{D}_s^{w\leq 0}$ and to $\mathfrak{D}_s^{w\geq 0}$ by Theorem 2.2.1(20). Here we use Proposition 3.5.1(2) again.
5. We have $X^i \in \mathfrak{D}_s^{w=0}$. Hence Theorem 2.2.1(14) yields the result. Note here that we have $Y_0 = 0$ in the notation of Definition 2.1.2(9).

□

We will call w the *Gersten* weight structure, since it is closely connected with Gersten resolutions of cohomology (cf. §4.5 below). By default, below w will denote the Gersten weight structure.

Remark 4.1.2. 1. $\underline{Hw}$ is idempotent complete since $\mathfrak{D}_s$ is.
2. In fact, one could easily prove similar statements for $\underline{C}$ being just $\langle H\rangle$ (instead of its Karoubization in $\mathfrak{D}$). Certainly, for this version of $\underline{C}$ we will only have $\underline{C} \supset M_{gm}(K^b(SmCor))$.
Besides, note that for any function field K'/k, any $r \geq 0$, there exists a function field K/k such that $M_{gm}(K')(r)[r]$ is a retract of $M_{gm}(K)$ (see Corollary 4.2.2 below). Hence it suffices take H being the full subcategory of $\mathfrak{D}$ whose objects are $\prod_{l\in L} M_{gm}(K_l)$ (for bounded transcendence degrees of K_l/k).
3. The proposition implies that $\mathfrak{D}_s$ is exactly the Karoubization in $\mathfrak{D}$ of the triangulated category generated by comotives of all pro-schemes.
4. The author does not know whether one can describe weight decompositions for arbitrary objects of DM_{gm}^{eff} explicitly. Still, one can say something about these weight decompositions and weight complexes using their functoriality properties. In particular, knowing weight complexes for $X, Y \in ObjDM_{gm}^{eff}$ (or just $\in ObjDM^s$) one can describe the weight complex of $X \to Y$ up to a

homotopy equivalence as the corresponding cone (see Lemma 6.1.1 below). Besides, let $X \to Y \to Z$ be a distinguished triangle (in $\mathfrak{D}$). Then for any choice of $(X^{w\le 0}, X^{w\ge 1})$ and $(Z^{w\le 0}, Z^{w\ge 1})$ there exists a choice of $(Y^{w\le 0}, Y^{w\ge 1})$ such that there exist distinguished triangles $X^{w\le 0} \to Y^{w\le 0} \to Z^{w\le 0}$ and $X^{w\ge 1} \to Y^{w\ge 1} \to Z^{w\ge 1}$; see Lemma 1.5.4 of [6]. In particular, we obtain that j maps complexes (over $SmCor$) concentrated in degrees $\le j$ into $\mathfrak{D}_s^{w\le j}$ (we will prove a stronger statement in Remark 4.5.4(4) below). If $X \in ObjDM_{gm}^{eff}$ comes from a complex over $SmCor$ whose connecting morphisms satisfy certain codimension restrictions, these observations could be extended to an explicit description of a weight decomposition for it; cf. §7.4 of [6].

4.2 Direct summand results for comotives

Proposition 4.1.1 easily implies the following interesting result.

Theorem 4.2.1. *1. Let S be a primitive scheme; let S_0 be its dense sub-pro-scheme. Then $M_{gm}(S)$ is a direct summand of $M_{gm}(S_0)$.*
2. Suppose moreover that $S_0 = S \setminus T$ where T is a closed subscheme of S everywhere of codimension $r > 0$. Then we have $M_{gm}(S_0) \cong M_{gm}(S) \bigoplus M_{gm}(T)(r)[2r-1]$.

Proof. We can assume that S and S_0 are connected.
1. By Proposition 4.1.1(5), we have: $M_{gm}(S_0), M_{gm}(S) \in \mathfrak{D}_s^{w\le 0}$; $M_{gm}(\operatorname{Spec}(k(S)))$ could be assumed to be the zeroth term of their weight complexes for a choice of weight complexes compatible with some negative Postnikov weight towers for them; the embedding $S_0 \to S$ is compatible with $id_{M_{gm}(\operatorname{Spec}(k(S)))}$ (since we have a commutative triangle $\operatorname{Spec} k(S) \to S_0 \to S$ of pro-schemes). Hence Theorem 2.2.1(16) yields the result.
2. By Proposition 3.6.1 we have a distinguished triangle $M_{gm}(S_0) \to M_{gm}(S) \to M_{gm}(T)(r)[2r]$. By parts 4 and 5 of Proposition 4.1.1 we have $M_{gm}(S_0) \in \mathfrak{D}_s^{w\le 0}$, $M_{gm}(S) \in \mathfrak{D}_s^{w=0}$, $M_{gm}(T)(r)[2r] \in \mathfrak{D}_s^{w\le -r} \subset \mathfrak{D}_s^{w\le -1}$. Hence Theorem 2.2.1(8) yields the result.

□

Corollary 4.2.2. *1. Let S be a connected primitive scheme, let S_0 be its generic point. Then $M_{gm}(S)$ is a retract of $M_{gm}(S_0)$.*
2. Let K be a function field over k. Let K' be the residue fields for a geometric valuation v of K of rank r. Then $M_{gm}(K')(r)[r]$ is a retract of $M_{gm}(K)$.

Proof. 1. This is just a partial case of part 1 of the the theorem.
2. Obviously, it suffices to prove the statement in the case $r = 1$. Next, K is the function field of some normal projective variety over k. Hence there exists a $U \in SmVar$ such that: $k(U) = K$, v yields a non-empty closed subscheme of U (since the singular locus has codimension ≥ 2 in a normal variety). It easily follows that there exists a pro-scheme S (i.e. an inverse limit of smooth varieties) whose only points are the spectra of K and K_0. So, S is local, hence it is primitive.

By part 2 of the theorem, we have

$$M_{gm}(\operatorname{Spec} K) = M_{gm}(S) \bigoplus M_{gm}(\operatorname{Spec} K_0)(1)[1];$$

this concludes the proof.

□

Remark 4.2.3. 1. Note that we do not construct any explicit splitting morphisms in the decompositions above. Probably, one cannot choose any canonical splittings here (in the general case); so there is no (automatic) compatibility for any pair of related decompositions. Respectively, though comotives of (spectra of) function fields contain tons of direct summands, there seems to be no general way to decompose them into indecomposable summands.
2. Yet Proposition 3.6.1 easily yields that $M_{gm}(\operatorname{Spec} k(t)) \cong \mathbb{Z} \bigoplus \prod_E M_{gm}(E)(1)[1]$; here E runs through all closed points of $\mathbb{A}^1$ (considered as a scheme over k).

4.3 On cohomology of pro-schemes, and its direct summands

The results proved above immediately imply similar assertions for cohomology. We also construct a class of cohomology theories that respect homotopy limits.

Proposition 4.3.1. *Let $H : \mathfrak{D}_s \to \underline{A}$ be cohomological, S be a primitive scheme.*
1. Let S_0 be a dense sub-pro-scheme of S. Then $H(M_{gm}(S))$ is a direct summand of $H(M_{gm}(S_0))$.
2. Suppose moreover that $S_0 = S \setminus T$ where T is a closed subscheme of S of codimension $r > 0$. Then we have $H(M_{gm}(S_0)) \cong H(M_{gm}(S)) \bigoplus H(M_{gm}(T)(r)[2r-1])$.
3. Let S be connected, S_0 be the generic point of S. Then $H(M_{gm}(S))$ is a retract of $H(M_{gm}(S_0))$ in $\underline{A}$.
4. Let K be a function field over k. Let K' be the residue field for a geometric valuation v of K of rank r. Then $H(M_{gm}(K')(r)[r])$ is a retract of $H(M_{gm}(K))$ in $\underline{A}$.
5. Let $H' : DM_{gm}^{eff} \to \underline{A}$ be a cohomological functor, let $\underline{A}$ satisfy AB5. Then Proposition 1.2.1 allows to extend H' to a cohomological functor $H : \mathfrak{D} \to \underline{A}$ that converts inverse limits in $\mathfrak{D}'$ to the corresponding direct limits in $\underline{A}$.

Proof. 1. Immediate from Theorem 4.2.1(1).
2. Immediate from Theorem 4.2.1(2).
3. Immediate from Corollary 4.2.2(1).
4. Immediate from Corollary 4.2.2(2).
5. Immediate from Proposition 1.2.1; note that DM_{gm}^{eff} is skeletally small. Here in order to prove that H converts homotopy limits into direct limits we use part I2 of loc.cit. and Proposition 3.1.1(7).

□

Remark 4.3.2. 1. In the setting of assertion 5 we will call H an *extended* cohomology theory.
Note that for $H' = DM_{gm}^{eff}(-,Y)$, $Y \in Obj DM_{gm}^{eff}$, we have $H = \mathfrak{D}(-,Y)$; see (4).
2. Now recall that for any pro-scheme Z, any $i \geq 0$, $M_{gm}(Z)(i)$ (by definition) could be presented as a countable homotopy limit of geometric motives. Moreover, the same is true for all small countable products of $M_{gm}(Z_l)(i)$. Hence if H is extended, then the cohomology of $\prod M_{gm}(Z_l)(i)$ is the corresponding direct limit; this coincides with the definition given by (29) (cf. Remark 3.2.1). In particular, one can apply the results of Proposition 4.3.1 to the usual étale cohomology of pro-schemes mentioned (with values in Ab or in some category of Galois modules).
3. If H' is also a tensor functor (i.e. it converts tensor product in DM_{gm}^{eff} into tensor products in $D(\underline{A})$), then certainly the cohomology of $M_{gm}(K')(r)[r]$ is the corresponding tensor product of $H^*(M_{gm}(K'))$ with $H^*(\mathbb{Z}(r)[r])$. Note that the latter one is a retract of $H^*(G_m^r)$; we obtain the Tate twist for cohomology this way.

4.4 Coniveau spectral sequences for cohomology of (co)motives

Let $H : \mathfrak{D}_s^{op} \to \underline{A}$ be a cohomological functor, $X \in Obj\mathfrak{D}_s$.

Proposition 4.4.1. *1. Any choice of a weight spectral sequence $T(H,X)$ (see Theorem 2.4.2) corresponding to the Gersten weight structure w is canonical and $\mathfrak{D}_s$-functorial in X starting from E_2.*
2. $T(H,X)$ converges to $H(X)$.
3. Let H be an extended theory (see Remark 4.3.2), $X = M_{gm}(Z)$ for $Z \in SmVar$. Then any choice of $T(H,X)$ starting from E_2 is canonically isomorphic to the classical coniveau spectral sequence (converging to the H-cohomology of Z; see §1 of [8]).

Proof. 1. This is just a partial case of Theorem 2.4.2(I).
2. Immediate since w is bounded; see part I2 of loc.cit.
3. Recall that in the proof of Corollary 3.6.2 a certain Postnikov tower $Po(X)$ for X was obtained from certain 'geometric' Postnikov towers (in $j(C^b(SmCor))$) by passing to the homotopy limit. Now, the coniveau spectral sequence (for the H-cohomology of Z) in §1.2 of [8] was constructed by applying H to the same geometric towers and then passing to the inductive limit (in $\underline{A}$). Furthermore, Remark 4.3.2(2) yields that the latter limit is (naturally) isomorphic to the spectral sequence obtained via H from $Po(X)$. Next, since $Po(X)$ is a weight Postnikov tower for X (see Proposition 4.1.1(5)), we obtain that the latter spectral sequence is one of the possible choices for $T(H,X)$.
Lastly, assertion 1 yields that all other possible $T(H,X)$ (they depend on the choice of a weight Postnikov tower for X) starting from E_2 are also canonically isomorphic to the classical coniveau spectral sequence mentioned.

□

Remark 4.4.2. 1. Hence we proved (in particular) that classical coniveau spectral sequences (for cohomology theories that could be factorized through motives; this includes étale and singular cohomology of smooth varieties) are DM_{gm}^{eff}-functorial (starting from E_2); we also obtain such a functoriality for the coniveau filtration for cohomology! These facts are far from being obvious from the usual definition of the coniveau filtration and spectral sequences, and seem to be new (in the general case). So, we justified the title of the paper.
We also obtain certain coniveau spectral sequences for cohomology of singular varieties (for cohomology theories that could be factorized through DM_{gm}^{eff}; in the case $\operatorname{char} k > 0$ one also needs rational coefficients here).
2. Assertion 3 of the proposition yields a nice reason to call (any choice of) $T(H, X)$ a *coniveau spectral sequence* (for a general $H, \underline{A}$, and $X \in Obj\mathfrak{D}_s$); this will also distinguish (this version of) T from weight spectral sequences corresponding to other weight structures. We will give more justification for this term in Remark 4.5.4 below. So, the corresponding filtration could be called the (generalized) coniveau filtration.

4.5 An extension of results of Bloch and Ogus

Now we want to relate coniveau spectral sequences with the homotopy t-structure (in DM_{-}^{eff}). This would be a vast extension of the seminal results of §6 of [5] (i.e. of the calculation by Bloch and Ogus of the E_2-terms of coniveau spectral sequences) and of §6 of [11].
We should relate t (for DM_{-}^{eff}) and w; it turns out that they are orthogonal with respect to a certain quite natural nice duality.

Proposition 4.5.1. *For any $Y \in ObjDM_{-}^{eff}$ we extend $H' = DM_{-}^{eff}(-, Y)$ from DM_{gm}^{eff} to $\mathfrak{D} \supset \mathfrak{D}_s$ by the method of Proposition 1.2.1; we define $\Phi(X, Y) = H(X)$. Then the following statements are valid.*
1. Φ is a nice duality (see Definition 2.5.1).
2 w is left orthogonal to the homotopy t-structure t (on DM_{-}^{eff}) with respect to Φ.
3. $\Phi(-, Y)$ converts homotopy limits (in $\mathfrak{D}'$) into direct limits in Ab.

Proof. 1. By Proposition 2.5.6(1), the restriction of Φ to $DM_{gm}^{eff\,op} \times DM_{-}^{eff}$ is a nice duality. It remains to apply part 3 of loc.cit.
2. In the notation of Proposition 2.5.3, we take for D the set of all small products $\prod_{l \in L} M_{gm}(K_l)(n_l)[n_l] \in Obj\mathfrak{D}_s$; here $M_{gm}(K_l)$ denote comotives of (spectra of) some function fields over k, $n_l \geq 0$ and the transcendence degrees of K_l/k are bounded (cf. §4.1). Then D, Φ satisfy the assumptions of the proposition by Proposition 3.4.1(2) (see also Remark 4.3.2(2)).
3. Immediate from Proposition 4.3.1(3).

□

Remark 4.5.2. 1. Suppose that we have an inductive family $Y_i \in ObjDM_{-}^{eff}$ connected by a compatible family of morphisms with some $Y \in DM_{-}^{eff}$ such

that: for any $Z \in ObjDM^{eff}_{gm}$ we have $DM^{eff}_-(Z,Y) \cong \varinjlim DM^{eff}_-(Z,Y_i)$ (via these morphisms $Y_i \to Y$). In such a situation it is reasonable to call Y a homotopy colimit of Y_i.
The definition of Φ in the proposition easily implies: for any $X \in Obj\mathfrak{D}$ we have $\Phi(X,Y) = \varinjlim \Phi(X,Y_i)$. So, one may say that all objects of $\mathfrak{D}$ are 'compact with respect to Φ', whereas part 3 of the proposition yields that all objects of DM^{eff}_- are 'cocompact with respect to Φ'. Note that no analogues of these nice properties can hold in the case of an adjacent weight and t-structure (defined on a single triangulated category).
2. Now, we could have replaced DM^{eff}_{gm} by DM_{gm} everywhere in the 'axiomatics' of $\mathfrak{D}$ (in Proposition 3.1.1). Then the corresponding category $\mathfrak{D}_{gm}$ could be used for our purposes (instead of $\mathfrak{D}$), since our arguments work for it also. Note that we can extend Φ to a nice duality $\mathfrak{D}^{op}_{gm} \times DM^{eff}_- \to Ab$; to this end it suffices for $Y \in ObjDM^{eff}_-$ to extend H' to DM_{gm} in the following way: $H'(X(-n)) = DM^{eff}_-(X,Y(n))$ for $X \in ObjDM^{eff}_{gm} \subset ObjDM_{gm}$, $n \geq 0$.
Moreover, the methods of §5.4.3 allow to define an invertible Tate twist functor on $\mathfrak{D}_{gm}$.

Corollary 4.5.3. *1. If H is represented by a $Y \in ObjDM^{eff}_-$ (via our Φ) then for a (co)motif X our coniveau spectral sequence $T(H,X)$ starting from E_2 could be naturally expressed in terms of the cohomology of X with coefficients in t-truncations of Y (as in Theorem 2.6.1).*
In particular, the coniveau filtration for $H^*(X)$ could be described as in part 2 of loc.cit.
2. For $U \in ObjDM^{eff}_{gm}$, $i \in \mathbb{Z}$, we have $U \in \mathfrak{D}^{w\leq i}_s \iff U \in DM^{eff\,t\leq i}_-$.

Proof. 1. Immediate from Proposition 4.5.1.
2. By Theorem 2.2.1(20), we should check whether $Z \perp U$ for any $Z = \prod_{l\in L} M_{gm}(K_l)(n_l)[n_l + r]$, where K_l are function fields over k, $n_l \geq 0$ and the transcendence degrees of K_l/k are bounded, $r > 0$ (see Proposition 4.1.1(2)). Moreover, since U is cocompact in $\mathfrak{D}$, it suffices to consider $Z = M_{gm}(K')(n)[n+r]$ (K'/k is a function field, $n \geq 0$). Lastly, Corollary 4.2.2(2) reduces the situation to the case $Z = M_{gm}(K)$ (K/k is a function field).
Hence (25) implies: $U \in \mathfrak{D}^{w\leq i}_s$ whenever for any $j > i$, any function field K/k, the stalk of $U^{t=j}$ at K is zero. Now, if $U \in DM^{eff\,t\leq i}_-$ then $U^{t=j} = 0$ for all $j > i$; hence all stalks of $U^{t=j}$ are zero. Conversely, if all stalks of $U^{t=j}$ at function fields are zero, then Corollary 4.19 of [26] yields $U^{t=j} = 0$ (see also Corollary 4.20 of loc.cit.); if $U^{t=j} = 0$ for all $j > i$ then $U \in DM^{eff\,t\leq i}_-$.

□

Remark 4.5.4. 1. Our comparison statement is true for Y-cohomology of an arbitrary $X \in ObjDM^{eff}_{gm}$; this extends to motives Theorem 6.4 of [11] (whereas the latter essentially extends the results of §6 of [5]). We obtain one more reason to call T (in this case) the coniveau spectral sequence for (cohomology of) motives.

2. If $Y \in ObjHI$, then $E_2(T)$ yields the Gersten resolution for Y (when X varies); this is why we called w the Gersten weight structure.
3. Now, let Y represent étale cohomology with coefficients in $\mathbb{Z}/l\mathbb{Z}$, l is prime to $\operatorname{char} k$ (Y is actually unbounded from above, yet this is not important). Then the t-truncations of Y represent $\mathbb{Z}/l\mathbb{Z}$-motivic cohomology by the (recently proved) Beilinson-Lichtenbaum conjecture (see [28]; this paper is not published at the moment). Hence Proposition 2.5.4(1) yields some new formulae for $\mathbb{Z}/l\mathbb{Z}$-motivic cohomology of X and for the 'difference' between étale and motivic cohomology. Note also that the virtual t-truncations (mentioned in loc.cit.) are exactly the D_2-terms of the alternative exact couple for $T(H, X)$ and for the version of the exact couple used in the current paper respectively (i.e. we consider exact couples coming from the two possible versions for a weight Postnikov tower for X, as described in Remark 2.1.3). See also §7.5 of [6] for more explicit results of this sort. It could also be interesting to study coniveau spectral sequences for singular cohomology; this could yield a certain theory of 'motives up to algebraic equivalence'; see Remark 7.5.3(3) of loc.cit. for more details.
5. Assertion 2 of the corollary yields that $\mathfrak{D}_s^{w\le 0} \cap ObjDM_{gm}^{eff}$ is large enough to recover w (in a certain sense); in particular, this assertion is similar to the definition of adjacent structures (see Remark 2.5.7). In contrast, $\mathfrak{D}_s^{w\ge 0} \cap ObjDM_{gm}^{eff}$ seems to be too small.

4.6 Base field change for coniveau spectral sequences; functoriality for an uncountable k

It can be easily seen (and well-known) that for any perfect field extension l/k there exist an extension of scalars functor $DM_{gm}^{eff}{}_k \to DM_{gm}^{eff}{}_l$ compatible with the extension of scalars for smooth varieties (and for $K^b(SmCor)$). In 5.4.2 below we will prove that this functor could be expanded to a functor $\mathrm{Ext}_{l/k} : \mathfrak{D}_k \to \mathfrak{D}_l$ that sends $M_{gm,k}(X)$ to $M_{gm,l}(X_l)$ for a pro-scheme X/k; this extension procedure is functorial with respect to embeddings of base fields. Moreover, $\mathrm{Ext}_{l/k}$ maps $\mathfrak{D}_{sk}$ into $\mathfrak{D}_{sl}$. Note the existence of base change for comotives does not follow from the properties of $\mathfrak{D}$ listed in Proposition 3.1.1; yet one can define base change for our model of comotives (described in §5 below) and (probably) for any other possible reasonable version of $\mathfrak{D}$.
Now we prove that base change for comotives yields base change for coniveau spectral sequences; it also allows to prove that these spectral sequences are motivically functorial for not necessary countable base fields.
In order to make the limit in Proposition 4.6.1(2) below well-defined, we assume that for any $X \in ObjDM_{gm}^{eff}$ there is a fixed representative Y, Z, p chosen, where: $Z, Y \in C^b(SmCor)$, $M_{gm}(Y) \cong M_{gm}(Z)$, $p \in C^b(SmCor)(Y, Z)$ yields a direct summand of $M_{gm}(Y)$ in DM_{gm}^{eff} that is isomorphic to X. We also assume that all the components of (X, Y, p) have fixed expressions in terms of algebraic equations over k; so one may speak about fields of definition for X.

Proposition 4.6.1. *Let l be a perfect field, $H : \mathfrak{D}_l \to \underline{A}$ be any cohomological functor (for an abelian $\underline{A}$). For any perfect $k \subset l$ we denote $H \circ \operatorname{Ext}_{l/k} : \mathfrak{D}_k \to \underline{A}$ by H_k.*

1. Let l be countable. Then for any $X \in Obj\mathfrak{D}_k$ the method of Proposition 2.7.3(II) yields some morphism $N_{l/k} : T_{w_k}(H_k, X) \to T_{w_l}(H, \operatorname{Ext}_{l/k}(X))$; this morphism is unique and $\mathfrak{D}_k$-functorial in X starting from E_2.

The correspondence $(l, k) \mapsto N_{l/k}$ is associative with respect to extensions of countable fields (starting from E_2); cf. part I3 of loc.cit.

2. Let l be a not (necessarily) countable perfect field, let $\underline{A}$ satisfy AB5.

For $X \in ObjDM_{gm}^{eff}{}_l$ we define $T_w(H, X) = \varinjlim_k T_{w_k}(H_k, X_k)$. Here we take the limit with respect to all perfect $k \subset l$ such that k is countable, X is defined over k; the connecting morphisms are given by the maps $N_{-/-}$ mentioned in assertion 1; we start our spectral sequences from E_2. Then $T_w(H, X)$ is a well-defined spectral sequence that is $DM_{gm}^{eff}{}_l$-functorial in X.

3. If $X = M_{gm,l}(Z)$, $Z \in SmVar$, H is as an extended theory, and $\underline{A}$ satisfies AB5, the spectral sequence given by the previous assertion is canonically isomorphic to the classical coniveau spectral sequence (for (H, Z); considered starting from E_2).

Proof. 1. By Proposition 2.7.3(II) it suffices to check that $\operatorname{Ext}_{l/k}$ is left weight-exact (with respect to weight structures in question). We take D being the class of all small products $\prod_{l \in L} M_{gm}(K_l)$, where $M_{gm}(K_l)$ denote comotives of (spectra of) function fields over k of bounded transcendence degree. Proposition 4.1.1 and Corollary 4.2.2(2) yield that any $X \in \mathfrak{D}_{s_k}^{w=0}$ is a retract of some element of D. It suffices to check that for any $X = \prod_{l \in L} M_{gm,k}(K_l)$ we have $\operatorname{Ext}_{l/k} X \in \mathfrak{D}_{s_l}^{w_l \le 0}$; here we recall that w_k is bounded and apply Lemma 2.7.5.

Now, X is the comotif of a certain pro-scheme, hence the same is true for $\operatorname{Ext}_{l/k} X$. It remains to apply Proposition 4.1.1(5).

2. By the associativity statement in the previous assertion, the limit is well-defined. Since $\underline{A}$ satisfies AB5, we obtain a spectral sequence indeed. Since we have k-motivic functoriality of coniveau spectral sequences over each k, we obtain l-motivic functoriality in the limit.

3. Again (as in the proof of Proposition 4.4.1(3)) we recall that the classical coniveau spectral sequence for this case is defined by applying H to 'geometric' Postnikov towers (coming from elements of L as in the proof of Corollary 3.6.2) and then passing to the limit (in $\underline{A}$) with respect to L. Our assertion follows easily, since each $l \in L$ is defined over some perfect countable $k \subset l$; the limit of the spectral sequences with respect to the subset of L defined over a fixed k is exactly $T_{w_k}(H_k, X_k)$ since H sends homotopy limits to inductive limits in $\underline{A}$ (being an extended theory).

Here we certainly use the functoriality of T starting from E_2.

□

Remark 4.6.2. 1. For a general $X \in ObjDM^{eff}_{gm}$ we only have a canonical choice of base change maps (for $T(H_{k_l}, X)$) starting from E_2; this is why we start our spectral sequence from the E_2-level.
2. Assertion 2 of the proposition is also valid for any comotif defined over a (perfect) countable subfield of l. Unfortunately, this does not seem to include comotives of function fields over l (of positive transcendence degrees, if l is not countable).

4.7 The Chow weight structure for $\mathfrak{D}$

Till the end of the section, we will either assume that $\operatorname{char} k = 0$, or that we deal with motives, comotives, and cohomology with rational coefficients (we will use the same notation for motives with integral and rational coefficients; cf. §6.3 below).
We prove that $\mathfrak{D}$ supports a weight structure that extends the Chow weight structure of DM^{eff}_{gm} (see §6.5 and Remark 6.6.1 of [6], and also [7]).
In this subsection we do not require k to be countable.

Proposition 4.7.1. *1. There exists a Chow weight structure on DM^{eff}_{gm} that is uniquely characterized by the condition that all $M_{gm}(P)$ for $P \in SmPrVar$ belong to its heart; it could be extended to a weight structure w_{Chow} on $\mathfrak{D}$.*
2. The heart of w_{Chow} is the category H_{Chow} of arbitrary small products of (effective) Chow motives.
3. We have $X \in \mathfrak{D}^{w_{Chow}\geq 0}$ if and only if $\mathfrak{D}(X, Y[i]) = \{0\}$ for any $Y \in ObjChow^{eff}$, $i > 0$.
4. There exists a t-structure t_{Chow} on $\mathfrak{D}$ that is right adjacent to w_{Chow} (see Remark 2.5.7). Its heart is the opposite category to $Chow^{eff*}$ (i.e. it is equivalent to $(\mathrm{AddFun}(Chow^{eff}, Ab))^{op}$).
5. w_{Chow} respects products i.e. $X_i \in \mathfrak{D}^{w_{Chow}\leq 0} \implies \prod X_i \in \mathfrak{D}^{w_{Chow}\leq 0}$ and $X_i \in \mathfrak{D}^{w_{Chow}\geq 0} \implies \prod X_i \in \mathfrak{D}^{w_{Chow}\geq 0}$.
6. For $\prod X_i$ there exists a weight decomposition: $\prod X_i \to \prod X_i^{w\leq 0} \to \prod X_i^{w\geq 1}$.
7. If $H : \mathfrak{D} \to \underline{A}$ is an extended theory, then the functor that sends X to the derived exact couple for $T_{w_{Chow}}(H, X)$ (see Theorem 2.4.2) converts all small products into direct sums.

Proof. 1. It was proved in (Proposition 6.5.3 and Remark 6.6.1 of) [6] that there exists a unique weight structure w'_{Chow} on DM^{eff}_{gm} such that $M_{gm}(P) \in \mathfrak{D}^{w'_{Chow}=0}$ for all $P \in SmPrVar$. Moreover, the heart of this structure is exactly $Chow^{eff} \subset DM^{eff}_{gm}$.
Now, DM^{eff}_{gm} is generated by $Chow^{eff}$. It easily follows that $\{M_{gm}(P),\ P \in SmPrVar\}$ weakly cogenerates $\mathfrak{D}$. Then the dual (see Theorem 2.2.1(1)) of Theorem 4.5.2(I2) of [6] yields that w'_{Chow} could be extended to a weight structure w_{Chow} for $\mathfrak{D}$. Moreover, the dual to part II1 of loc.cit. yields that for this extension we have: $\underline{Hw}_{Chow}$ is the idempotent completion of H_{Chow}.

2. It remains to prove that H_{Chow} is idempotent complete. This is obvious since $Chow^{eff}$ is.
3. This is just the dual of (27) in loc.cit.
4. The dual statement to part I2 of loc.cit. (cf. Remark 1.1.3(1)) yields the existence of t_{Chow}. Applying the dual of Theorem 4.5.2(II1) of [6] we obtain for the heart of t: $\underline{Ht}_{Chow} \cong (Chow_*^{eff})^{op}$.
5. Theorem 2.2.1(2) easily yields that $\mathfrak{D}^{w_{Chow}\le 0}$ is stable with respect to products. The stability of $\mathfrak{D}^{w_{Chow}\ge 0}$ with respect to products follows from assertion 3; here we recall that all objects of $Chow^{eff}$ are cocompact in $\mathfrak{D}$.
6. Immediate from the previous assertion; note that any small product of distinguished triangles is distinguished (see Remark 1.2.2 of [21]).
7. Since H is extended, it converts products in $\mathfrak{D}$ into direct sums in $\underline{A}$. Hence for any $X_i \in Obj\mathfrak{D}$ there exist a choice of exact couples for the corresponding weight spectral sequences for X_i and $\prod X_i$ that respects products i.e such that $D_1^{pq}T_{w_{Chow}}(H, \prod X_i) \cong \bigoplus_i D_1^{pq}T_{w_{Chow}}(H, X_i)$ and $E_1^{pq}T_{w_{Chow}}(H, \prod X_i) \cong \bigoplus_i E_1^{pq}T_{w_{Chow}}(H, X_i)$ (for all $p.q \in \mathbb{Z}$; this isomorphism is also compatible with the connecting morphisms of couples). Since $\underline{A}$ satisfies AB5, we obtain the isomorphism desired for D_2 and E_2-terms (note that those are uniquely determined by H and X).

□

Remark 4.7.2. 1. In Remark 2.4.3 of [6] it was shown that weight spectral sequences corresponding to the Chow weight structure are isomorphic to the classical (i.e. Deligne's) weight spectral sequences when the latter are defined (i.e. for singular or étale cohomology of varieties). Yet in order to specify the choice of a weight structure here we will call these spectral sequences *Chow-weight* ones.
2. All the assertions of the Proposition could be extended to arbitrary triangulated categories with negative families of cocompact weak cogenerators (sometimes one should also demand all products to exist; in assertion 7 we only need H to convert all products into direct sums).
3. Since (effective) Chow motives are cocompact in $\mathfrak{D}$, $\underline{Hw}_{Chow}$ is the category of 'formal products' of $Chow^{eff}$ i.e. $\mathfrak{D}(\prod_{l\in L} X_l, \prod_{i\in I} Y_i) = \prod_{i\in I}(\oplus_{l\in L} Chow^{eff}(X_l, Y_i))$ for $X_l, Y_l \in ObjChow^{eff} \subset Obj\mathfrak{D}$ (cf. Remark 4.5.3(2) of [6]).
4. Recall (see §7.1 of ibid.) that DM_-^{eff} supports (adjacent) Chow weight and t-structures (we will denote them by w'_{Chow} and t'_{Chow}, respectively). One could also check that these structures are right orthogonal to the corresponding Chow structures for $\mathfrak{D}$. Hence, applying Proposition 2.5.4(1) repeatedly one could relate the compositions of truncations (on $\mathfrak{D}_s \subset \mathfrak{D}$) via w and via t_{Chow} (resp. via w and via w_{Chow}) with truncations via t and via w'_{Chow} (resp. via t and via t'_{Chow}) on DM_-^{eff}; cf. §8.3 of [6]. One could also apply w_{Chow}-truncations and then w-truncations (i.e. compose truncations in the opposite order) when starting from an object of DM_{gm}^{eff}. Recall also that truncations via

t_{Chow} (and their compositions with t-truncations) are related with unramified cohomology; see Remark 7.6.2 of ibid.

4.8 Comparing Chow-weight and coniveau spectral sequences

Now we prove that Chow-weight and coniveau spectral sequences are naturally isomorphic for birational cohomology theories.

Proposition 4.8.1. *1. w_{Chow} for $\mathfrak{D}$ dominates w (for $\mathfrak{D}_s$) in the sense of §2.7.*
2. Let $H : DM_{gm}^{eff} \to \underline{A}$ be an extended cohomology theory in the sense of Remark 4.3.2; suppose that H is birational i.e. that $H(M_{gm}(P)(1)[i]) = 0$ for all $P \in SmPrVar$, $i \in \mathbb{Z}$. Then for any $X \in Obj\mathfrak{D}_s$ the Chow-weight spectral sequence $T_{w_{Chow}}(H, X)$ (corresponding to w_{Chow}) is naturally isomorphic starting from E_2 to (our) coniveau spectral sequence $T_w(H, X)$ via the comparison morphism M given by Proposition 2.7.3(I1).

Proof. 1. Let D be the class of all countable products $\prod_{l\in L} M_{gm}(K_l)$, where $M_{gm}(K_l)$ denote comotives of (spectra of) function fields over k of bounded transcendence degree. Proposition 4.1.1 and Corollary 4.2.2(2) yield that any $X \in \mathfrak{D}_s^{w=0}$ is a retract of some element of D. It suffices to check that any $X = \prod_{l\in L} M_{gm}(K_l)$ belongs to $\mathfrak{D}^{w_{Chow}\geq 0}$; here we recall that w is bounded and apply Lemma 2.7.5.
By Proposition 4.7.1(5), we can assume that L consists of a single element. In this case we have $\mathfrak{D}(M_{gm}(K_l), M_{gm}(P)[i]) = 0$ (this is a trivial case of Proposition 3.5.1); hence loc.cit. yields the result.
2. We take the same D and X as above.
Let $\operatorname{char} k = 0$. We choose $P_l \in SmPrVar$ such that K_l are their function fields. Since all $M_{gm}(P_l)$ are cocompact in $\mathfrak{D}$, we have a natural morphism $X \to \prod M_{gm}(P_l)$. By Proposition 2.7.3(I2), it suffices to check that $\operatorname{Cone}(X \to \prod M_{gm}(P_l)) \in \mathfrak{D}^{w_{Chow}\geq 0}$, $H(X) \cong H(\prod M_{gm}(P_l))$, and $E_2^{**}T_{w_{Chow}}(H, \operatorname{Cone}(X \to \prod M_{gm}(P_l))) = 0$.
By Proposition 4.7.1(7) we obtain: it suffices again to verify these statements in the case when L consists of a single element. Now, we have $\operatorname{Spec}(K_l) = \varprojlim M_{gm}(U)$ for $U \in SmVar$, $k(U) = K_l$. Therefore (27) yields: it suffices to verify assertions required for $Z = M_{gm}(U \to P)$ instead, where $U \in SmVar$, U is open in $P \in SmPrVar$.
The Gysin distinguished triangle for Voevodsky's motives (see Proposition 2.4.5 of [9]) easily yields by induction that $Z \in ObjDM_{gm}^{eff}(1)$.
Since $Chow^{eff}$ is $-\otimes \mathbb{Z}(1)[2]$-stable, we obtain that there exists a w_{Chow}-Postnikov tower for Z such that all of its terms are divisible by $\mathbb{Z}(1)$; this yields the vanishing of $E_2^{**}T_{w_{Chow}}(H, Z)$. Lastly, the fact that $Z \in DM_{gm}^{eff\, w'_{Chow}\geq 0}$ was (essentially) proved by easy induction (using the Gysin triangle) in the proof of Theorem 6.2.1 of [7].
In the case $\operatorname{char} k > 0$, de Jong's alterations allow to replace $M_{gm}(P_l)$ in the reasoning above by some Chow motives (with rational coefficients); see

Appendix B of [14]; we will not write down the details here.

□

Remark 4.8.2. Assertion 2 is not very actual for cohomology of smooth varieties since any $Z \in SmPrVar$ is birationally isomorphic to $P \in SmPrVar$ (at least for $\operatorname{char} k = 0$). Yet the statement becomes more interesting when applied for $X = M^c_{gm}(Z)$.

4.9 Birational motives; constructing the Gersten weight structure by gluing; other possible weight structures

An alternative way to prove Proposition 4.8.1(2) is to consider (following [15]) the category of *birational comotives*. It satisfies the following properties:
(i) All birational cohomology theories factorize through it.
(ii) Chow and Gersten weight structures induce the same weight structure on it (see Definition 2.7.1(4)).
(iii) More generally, for any $n \geq 0$ Chow and Gersten weight structures induce weight structures on the localizations $\mathfrak{D}(n)/\mathfrak{D}(n+1) \cong \mathfrak{D}_{bir}$ (we call these localizations *slices*) that differ only by a shift.
Moreover, one could 'almost recover' original Chow and Gersten weight structures starting from this single weight structure.
Now we describe the constructions and facts mentioned in more detail. We will be rather sketchy here, since we will not use the results of this subsection elsewhere in the paper. Possibly, the details will be written down in another paper.
As we will show in §5.4.3 below, the Tate twist functor could be extended (as an exact functor) from DM^{eff}_{gm} to $\mathfrak{D}$; this functor is compatible with (small) products.

Proposition 4.9.1. *I The functor $-\otimes \mathbb{Z}(1)[1]$ is weight-exact with respect to w on $\mathfrak{D}_s$; $-\otimes \mathbb{Z}(1)[2]$ is weight-exact with respect to w_{Chow} on $\mathfrak{D}$ (we will say that w is $-\otimes \mathbb{Z}(1)[1]$-stable, and w_{Chow} is $-\otimes \mathbb{Z}(1)[2]$-stable).*
II Let $\mathfrak{D}_{bir}$ denote the localization of $\mathfrak{D}$ by $\mathfrak{D}(1)$; B is the localization functor. We denote $B(\mathfrak{D}_s)$ by $\mathfrak{D}_{s,bir}$.
1. w_{Chow} induces a weight structure w'_{bir} on $\mathfrak{D}_{bir}$. Besides, w induces a weight structure w_{bir} on $\mathfrak{D}_{s,bir}$.
2. We have $\mathfrak{D}_{s,bir}^{w_{bir}\leq 0} \subset \mathfrak{D}_{bir}^{w'_{bir}\leq 0}$, $\mathfrak{D}_{s,bir}^{w_{bir}\geq 0} \subset \mathfrak{D}_{bir}^{w'_{bir}\geq 0}$ (i.e. the embedding $(\mathfrak{D}_{s,bir}, w_{bir}) \to (\mathfrak{D}_{bir}, w'_{bir})$ is weight-exact).
3. For any pro-scheme U we have $B(M_{gm}(U)) \in \mathfrak{D}_{s,bir}^{w_{bir}=0}$.

Proof. I This is easy, since the functors mentioned obviously map the corresponding hearts (of weight structures) into themselves.
II 1. By assertion I, w_{Chow} induces a weight structure on $\mathfrak{D}(1)$ (i.e. $\mathfrak{D}(1)$ is a triangulated category, $Obj\mathfrak{D}(1) \cap \mathfrak{D}^{w_{Chow}\leq 0}$ and $Obj\mathfrak{D}(1) \cap \mathfrak{D}^{w_{Chow}\geq 0}$ yield a weight structure on it). Hence by Proposition 8.1.1(1) of [6] we obtain existence

(and uniqueness) of w'_{bir}. The same argument also implies the existence of some w_{bir} on $\mathfrak{D}_{s,bir}$.
2. Now we compare w_{bir} with w'_{bir}. Since w is bounded, w_{bir} also is (see loc.cit.). Hence it suffices to check that $\underline{Hw}_{bir} \subset \underline{Hw'}_{bir}$ (see Theorem 2.2.1(19)).
Moreover, it suffices to check that for $X = \prod_{l\in L} M_{gm}(K_l)$ we have $B(X) \in \mathfrak{D}_{bir}^{w'_{bir}=0}$ (since $\mathfrak{D}_{bir}^{w'_{bir}=0}$ is Karoubi-closed in $\mathfrak{D}_{bir}$, here we also apply Proposition 4.7.1(2)). As in the proof of Proposition 4.8.1(2), we will consider the case $\operatorname{char} k = 0$; the case $\operatorname{char} k = p$ is treated similarly. Then we choose $P_l \in SmPrVar$ such that K_l are their function fields; we have a natural morphism $X \to \prod M_{gm}(P_l)$. It remains to check that $\operatorname{Cone}(X \to \prod M_{gm}(P_l)) \in \mathfrak{D}_s(1)$.
Now, since $\mathfrak{D}_s(1)$ and the class of distinguished triangles are closed with respect to small products, it suffices to consider the case when L consists of a single element. In this case the statement is immediate from Corollary 3.6.2.
3. Immediate from Corollary 3.6.2.

□

Remark 4.9.2. 1. Assertion II easily implies Proposition 4.8.1(2).
Indeed, any extended birational H (as in loc.cit.) could be factorized as $G \circ B$ for a cohomological $G : \mathfrak{D}_{bir} \to \underline{A}$. Since B is weight-exact with respect to w_{Chow} (and its restriction to $\mathfrak{D}_s$ is weight-exact with respect to w), (the trivial case of) Proposition 2.7.3(I2) implies that for any $X \in Obj\mathfrak{D}$ (any choice) of $T_{w'_{bir}}(G, B(X))$ is naturally isomorphic starting from E_2 to any choice of $T_{w_{Chow}}(H, X)$; for any $X \in Obj\mathfrak{D}_s$ (any choice) of $T_{w_{bir}}(G, B(X))$ is naturally isomorphic starting from E_2 to any choice of $T_w(H, X)$.
It is also easily seen that the isomorphism $T_{w_{Chow}}(H, X) \to T_w(H, X)$ is compatible with the comparison morphism M (see loc.cit.).
2. The proof of existence of w_{bir} and of assertion 3 works with integral coefficients even if $\operatorname{char} k > 0$. Hence we obtain that that the category image $B(M_{gm}(U))$, $U \in SmVar$, is negative. We can apply this statement in $\underline{C}$ being the idempotent completion of $B(DM_{gm}^{eff})$ i.e. in the category of birational comotives. Hence Theorem 2.2.1(18) yields: there exists a weight structure for $\underline{C}$ whose heart is the category of birational Chow motives (defined as in §5 of [15]). Note also that one can pass to the inductive limit with respect to base change in this statement (cf. §4.6); hence one does not need to require k to be countable.

Now we explain that w and w_{Chow} could be 'almost recovered' from $(\mathfrak{D}_{bir}, w'_{bir})$. Exactly the same reasoning as above shows that for any $n > 0$ the localization of $\mathfrak{D}$ by $\mathfrak{D}(n)$ could be endowed with a weight structure w'_n compatible with w_{Chow}, whereas the localization of $\mathfrak{D}_s$ by $\mathfrak{D}_s(n)$ could be endowed with a weight structure w_n compatible with w.
Next, we have a short exact sequence of triangulated categories $\mathfrak{D}/\mathfrak{D}(n) \xrightarrow{i_*} \mathfrak{D}/\mathfrak{D}(n+1) \xrightarrow{j^*} \mathfrak{D}_{bir}$. Here the notation for functors comes from the 'classical' gluing data setting (cf. §8.2 of [6]); i_* could be given by $-\otimes \mathbb{Z}(1)[s]$ for any

$s \in \mathbb{Z}$, j^* is just the localization. Now, if we choose $s = 2$ then i_* is weight-exact with respect to w'_n and w'_{n+1}; if we choose $s = 1$ then the restriction of i_* to $\mathfrak{D}_s/\mathfrak{D}_s(n)$ is weight-exact with respect to w_n and w_{n+1}.

Next, an argument similar to the one used in §8.2 of [6] shows: for any short exact sequence $\underline{D} \xrightarrow{i_*} \underline{C} \xrightarrow{j^*} \underline{E}$ of triangulated categories, if $\underline{D}$ and $\underline{E}$ are endowed with weight structures, then there exist at most one weight structure on $\underline{C}$ such that both i_* and j^* are weight-exact (see also Lemma 4.6 of [3] for the proof of a similar statement for t-structures). Hence one can recover w_n and w'_n from (copies of) w'_{bir}; the main difference between them is that the first one is $-\otimes \mathbb{Z}(1)[1]$-stable, whereas the second one is $-\otimes \mathbb{Z}(1)[2]$-stable. It is quite amazing that weight structures corresponding to spectral sequences of quite distinct geometric origin differ just by [1] here! If one calls the filtration of $\mathfrak{D}$ by $\mathfrak{D}(n)$ the *slice filtration* (this term was already used by A. Huber, B. Kahn, M. Levine, V. Voevodsky, and other authors for other 'motivic' categories), then one may say that w_n and w'_n could be recovered from slices; the difference between them is 'how we shift the slices'.

Moreover, Theorem 8.2.3 of [6] shows: if both adjoints to i_* and j^* exist, then one can use this gluing data in order to glue (any pair) of weight structures for $\underline{D}$ and $\underline{E}$ into a weight structure for $\underline{C}$. So, suppose that we have a weight structure $w_{n,s}$ for $\mathfrak{D}/\mathfrak{D}(n)$ that is $-\otimes(1)[s]$-stable and compatible with w'_{bir} on all slices (in the sense described above; so $w'_n = w_{n,2}$, w_n is the restriction of $w_{n,1}$ to $\mathfrak{D}_s/\mathfrak{D}_s(n)$, and all $w_{1,s}$ coincide with w'_{bir}). General homological algebra (see Proposition 3.3 of [18]) yields that all the adjoints required do exist in our case. Hence one can construct $w_{n+1,s}$ for $\mathfrak{D}/\mathfrak{D}(n+1)$ that satisfies similar properties. So, $w_{n,s}$ exist for all $n > 0$ and all $s \in \mathbb{Z}$. Hence Gersten and Chow weight structures (for $\mathfrak{D}_s/\mathfrak{D}_s(n) \subset \mathfrak{D}/\mathfrak{D}(n)$) are members of a rather natural family of weight structures indexed by a single integral parameter. It could be interesting to study other members of it (for example, the one that is $-\otimes\mathbb{Z}(1)$-stable), though possibly w'_n is the only member of this family whose heart is cocompactly generated.

This approach could allow to construct w in the case of a not necessarily countable k. Note here that the system of $\mathfrak{D}_s/\mathfrak{D}_s(n)$ yields a fine approximation of $\mathfrak{D}_s$. Indeed, if $X \in SmPrVar$, $n \geq \dim X$, then Poincare duality yields: for any $Y \in Obj DM^{eff}_{gm}$ we have $DM^{eff}_{gm}(Y(n), M_{gm}(X)) \cong DM^{eff}_{gm}(Y \otimes X(n - \dim X)[-2\dim X], \mathbb{Z})$; this is zero if $n > \dim X$ since $\mathbb{Z}$ is a birational motif. Hence (by Yoneda's lemma) for any $n > 0$ the full subcategory of DM^{eff}_{gm} generated by motives of varieties of dimension less than n fully embeds into $DM^{eff}_{gm}/DM^{eff}_{gm}(n) \subset \mathfrak{D}/\mathfrak{D}(n)$.

It follows that the restrictions of $w_{n,s}$ to a certain series of (sufficiently small) subcategories of $\mathfrak{D}/\mathfrak{D}(n)$ are induced by a single $-\otimes(1)[s]$-stable weight structure w_s for the corresponding subcategory of $\mathfrak{D}$. Here for the corresponding subcategory of $\mathfrak{D}/\mathfrak{D}(n)$ (or $\mathfrak{D}$) one can take the union of the subcategories of $\mathfrak{D}/\mathfrak{D}(n)$ (resp. $\mathfrak{D}$) generated (in an appropriate sense) by comotives of (smooth) varieties of dimension $\leq r$ (with r running through all natural num-

bers). Note that this subcategory of $\mathfrak{D}$ contains DM_{gm}^{eff}.
We also relate briefly our results with the (conjectural) picture for t-structures described in [3]. There another (geometric) filtration for motives was considered; this filtration (roughly) differs from the filtration considered above by (a certain version of) Poincare duality. Now, conjecturally the gr_n of the category of birational motives with rational coefficients (cf. §4.2 of ibid.) should be (the homotopy category of complexes over) an abelian semisimple category. Hence it supports a t-structure which is simultaneously a weight structure. This structure should be the building block of all relevant weight and t-structures for comotives. Certainly, this picture is quite conjectural at the present moment.

Remark 4.9.3. The author also hopes to carry over (some of) the results of the current paper to relative motives (i.e. motives over a base scheme that is not a field), relative comotives, and their cohomology. One of the possible methods for this is the usage of gluing of weight structures (see §8.2 of [6], especially Remark 8.2.4(3) of loc.cit.). Possibly for this situation the 'version of $\mathfrak{D}$' that uses motives with compact support (see §6.4 below) could be more appropriate.

5 The construction of $\mathfrak{D}$ and $\mathfrak{D}'$; base change and Tate twists

Now we construct our categories $\mathfrak{D}'$ and $\mathfrak{D}$ using the differential graded categories formalism.
In §5.1 we recall the definitions of differential graded categories, modules over them, shifts and cones (of morphisms).
In §5.2 we recall main properties of the derived category of (modules over) a differential graded category.
In §5.3 we define $\mathfrak{D}'$ and $\mathfrak{D}$ as the categories opposite to the corresponding categories of modules; then we prove that they satisfy the properties required.
In §5.4 we use the differential graded modules formalism to define base change for motives (extension and restriction of scalars). This yields: our results on direct summands of comotives (and cohomology) of function fields (proved above) could be carried over to pro-schemes obtained from them via base change.
We also define tensoring of comotives by motives, as well as a certain 'co-internal Hom' (i.e. the corresponding left adjoint functor to $X \otimes -$ for $X \in ObjDM_{gm}^{eff}$). These results do not require k to be countable.

5.1 DG-categories and modules over them

We recall some basic definitions; cf. [16] and [12].
An additive category A is called graded if for any $P, Q \in ObjA$ there is a canonical decomposition $A(P,Q) \cong \oplus_i A^i(P,Q)$ defined; this decomposition satisfies $A^i(*,*) \circ A^j(*,*) \subset A^{i+j}(*,*)$. A differential graded category (cf. [12]) is a graded category endowed with an additive operator $\delta : A^i(P,Q) \to A^{i+1}(P,Q)$ for all $i \in \mathbb{Z}, P, Q \in ObjA$. δ should satisfy the equalities $\delta^2 = 0$ (so $A(P,Q)$ is a complex of abelian groups); $\delta(f \circ g) = \delta f \circ g + (-1)^i f \circ \delta g$ for any $P, Q, R \in ObjA$, $f \in A^i(P,Q)$, $g \in A(Q,R)$. In particular, $\delta(id_P) = 0$.

We denote δ restricted to morphisms of degree i by δ^i.

Now we give a simple example of a differential graded category.

For an additive category B we consider the category $B(B)$ whose objects are the same as for $C(B)$ whereas for $P=(P^i)$, $Q=(Q^i)$ we define $B(B)^i(P,Q)=\prod_{j\in\mathbb{Z}} B(P^j,Q^{i+j})$. Obviously $B(B)$ is a graded category. We will also consider a full subcategory $B^b(B)\subset B(B)$ whose objects are bounded complexes.

We set $\delta f=d_Q\circ f-(-1)^i f\circ d_P$, where $f\in B^i(P,Q)$, d_P and d_Q are the differentials in P and Q. Note that the kernel of $\delta^0(P,Q)$ coincides with $C(A)(P,Q)$ (the morphisms of complexes); the image of δ^{-1} are the morphisms homotopic to 0.

Note also that the opposite category to a differential graded category becomes differential graded also (with the same gradings and differentials) if we define $f^{op}\circ g^{op}=(-1)^{pq}(g\circ f)^{op}$ for g,f being composable homogeneous morphisms of degrees p and q, respectively.

For any differential graded A we define an additive category $H(A)$ (some authors denote it by $H^0(A)$); its objects are the same as for A; its morphisms are defined as

$$H(A)(P,Q)=\operatorname{Ker}\delta^0_A(P,Q)/\operatorname{Im}\delta^{-1}_A(P,Q).$$

In the case when $H(A)$ is triangulated (as a full subcategory of the category $\mathcal{K}(A)$ described below) we will say that A is a (differential graded) *enhancement* for $H(A)$.

We will also need $Z(A)$: $Obj Z(A)=Obj A$; $Z(A)(P,Q)=\operatorname{Ker}\delta^0_A(P,Q)$. We have an obvious functor $Z(A)\to H(A)$. Note that $Z(B(B))=C(B)$; $H(B(B))=K(B)$.

Now we define (left differential graded) modules over a small differential graded category A (cf. §3.1 of [16] or §14 of [12]): the objects DG-Mod(A) are those additive functors of the underlying additive categories $A\to B(Ab)$ that preserve gradings and differentials for morphisms. We define DG-Mod$(A)^i(F,G)$ as the set of transformations of additive functors of degree i; for $h\in$ DG-Mod$(A)^i(F,G)$ we define $\delta^i(h)=d_G\circ f-(-1)^i f\circ d_F$. We have a natural Yoneda embedding $Y: A^{op}\to$ DG-Mod(A) (one should apply Yoneda's lemma for the underlying additive categories); it is easily seen to be a full embedding of differential graded categories.

Now we define shifts and cones in DG-Mod(A) componentwisely. For $F\in Obj\,$DG-Mod(A) we set $F[1](X)=F(X)[1]$. For $h\in\operatorname{Ker}\delta^0_{\text{DG-Mod}(A)}(F,G)$ we define the object Cone(h): Cone$(h)(X)=$ Cone$(F(X)\to G(X))$ for all $X\in Obj A$.

Note that for $A=B(B)$ both of these definitions are compatible with the corresponding notions for complexes (with respect to the Yoneda embedding).

We have a natural triangle of morphisms in $\delta^0_{\text{DG-Mod}(A)}$:

$$P\xrightarrow{f}P'\to\operatorname{Cone}(f)\to P[1]. \tag{30}$$

5.2 The derived category of a differential graded category

We define $\mathcal{K}(A) = H(\text{DG-Mod}(A))$. It was shown in §2.2 of [16] that $\mathcal{K}(A)$ is a triangulated category with respect to shifts and cones of morphisms that were defined above (i.e. a triangle is distinguished if it is isomorphic to those of the form (30)).
We will say that $f \in \operatorname{Ker} \delta^0_{\text{DG-Mod}(A)}(F, G)$ is a *quasi-isomorphism* if for any $X \in ObjA$ it yields an isomorphism $F(X) \to F(Y)$. We define $\mathcal{D}(A)$ as the localization of $\mathcal{K}(A)$ with respect to quasi-isomorphisms; so it is a triangulated category. Note that quasi-isomorphisms yield a localizing class of morphisms in $K(A)$. Moreover, the functor $X \to H^0(F(X)) : \mathcal{K}(A) \to Ab$ is corepresented by $\text{DG-Mod}(A)(X, -) \in Obj\mathcal{K}(A)$; hence for any $X \in ObjA$, $F \in Obj\mathcal{K}(A)$ we have

$$\mathcal{D}(A)(Y(X), F) \cong \mathcal{K}(A)(Y(X), F). \tag{31}$$

Hence we have an embedding $H(A)^{op} \to \mathcal{D}(A)$.
We define $\mathcal{C}(A)$ as $Z(\text{DG-Mod}(A))$. It is easily seen that $\mathcal{C}(A)$ is closed with respect to (small filtered) direct limits, and $\varinjlim F_l$ is given by $X \to \varinjlim F_l(X)$.
Now we recall (briefly) that differential graded modules admit certain 'resolutions' (i.e. any object is quasi-isomorphic to a *semi-free* one in the terms of §14 of [12]).

Proposition 5.2.1. *There exists a full triangulated $K' \subset \mathcal{K}(A)$ such that the projection $K(A) \to D(A)$ induces an equivalence $K' \approx \mathcal{D}(A)$. K' is closed with respect to all (small) coproducts.*

Proof. See §14.8 of [12] □

Remark 5.2.2. In fact, there exists a (Quillen) model structure for $\mathcal{C}(A)$ such that $\mathcal{D}(A)$ its homotopy category; see Theorem 3.2 of [16]. Moreover (for the first model structures mentioned in loc.cit) all objects of $\mathcal{C}(A)$ are fibrant, all objects coming from A are cofibrant. For this model structure two morphisms are homotopic whenever they become equal in $\mathcal{K}(A)$. So, one could take K' whose objects are the cofibrant objects of $\mathcal{C}(A)$.
Using these facts, one could verify most of Proposition 3.1.1 (for $\mathfrak{D}'$ and $\mathfrak{D}$ described below).

5.3 The construction of $\mathfrak{D}'$ and $\mathfrak{D}$; the proof of Proposition 3.1.1

It was proved in §2.3 of [4] (cf. also [19] or §8.3.1 of [7]) that DM^{eff}_{gm} could be described as $H(A)$, where A is a certain (small) differential graded category. Moreover, the functor $K^b(SmCor) \to DM^{eff}_{gm}$ could be presented as $H(f)$, where $f : B^b(SmCor) \to A$ is a differential graded functor. We will not describe the details for (any of) these constructions since we will not need them.
We define $\mathfrak{D}' = \mathcal{C}(A)^{op}$, $\mathfrak{D} = \mathcal{D}(A)^{op}$, p is the natural projection. We verify that these categories satisfy Proposition 3.1.1. Assertion 10 follows from the fact

that any localization of a triangulated category that possesses an enhancement is enhanceable also (see §§3.4–3.5 of [12]).
The embedding $H(A)^{op} \to \mathcal{D}(A)$ yields $DM_{gm}^{eff} \subset \mathfrak{D}'$. Since all objects coming from A are cocompact in $\mathcal{K}(A)^{op}$, Proposition 5.2.1 yields that the same is true in $\mathfrak{D}$. We obtain assertion 1.
$\mathfrak{D}'$ is closed with respect to inverse limits since $\mathcal{C}(A)$ is closed with respect to direct ones. Since the projection $\mathcal{C}(A) \to \mathcal{K}(A)$ respects coproducts (as well as all other (filtered) colimits), Proposition 5.2.1 yields that p respects products also. We obtain assertion 2.
The descriptions of $\mathcal{C}(A)$ and $\mathcal{D}(A)$ yields all the properties of shifts and cones required. This yields assertions 3, 4, and 6. Since $\mathcal{D}(A)$ is a localization of $\mathcal{K}(A)$, we also obtain assertion 5.
Next, since $\mathcal{D}(A)$ is a localization of $\mathcal{K}(A)$ with respect to quasi-isomorphisms, we obtain assertion 8.
Recall that filtered direct limits of exact sequences of abelian groups are exact. Hence for any $X \in Obj A \subset Obj \mathfrak{D}'$, $Y : L \to$ DG-Mod(A) we have

$$\mathcal{K}(A)(\text{DG-Mod}(A)(X,-), \varinjlim_l Y_l) = H^0((\varinjlim Y_l)(A))$$
$$= H^0(\varinjlim(Y_l(A))) = \varinjlim H^0(Y_l(A)) = \varinjlim_l \mathcal{K}(A)(\text{DG-Mod}(A)(X,-), Y_l).$$

Applying (31) we obtain assertion 7.
It remains to verify assertion 9 of loc.cit. Since the inverse limit with respect to a projective system is isomorphic to the inverse limit with respect to any its unbounded subsystem, and the same is true for $\varprojlim_1$ in the countable case, we can assume that I is the category of natural numbers, i.e. we have a sequence of F_i connected by morphisms.

In this case we have functorial morphisms $\varprojlim F_i \xrightarrow{f} \prod F_l \xrightarrow{g} \prod F_i$ as in (27). Hence it suffices to check that these morphisms yield a distinguished triangle in $\mathfrak{D}$. Note that $g \circ f = 0$; hence g could be factorized through a morphism $h : \operatorname{Cone} f \to \prod F_i$ in $\mathfrak{D}'$. Since for any $X \in Obj A$ the morphism $h^* : \prod_{\mathfrak{D}'} F_i(X) \to \operatorname{Cone} F(X)$ is a quasi-isomorphism, h becomes an isomorphism in $\mathfrak{D}$. This finishes the proof.

Remark 5.3.1. 1. Note that the only part of our argument when we needed k to be countable in the proof of assertion 9 of loc.cit.
2. The constructions of A (i.e. of the 'enhancement' for DM_{gm}^{eff} mentioned above) that were described in [4], [19], and in [7], are easily seen to be functorial with respect to base field change (see below). Still, the constructions mentioned are distinct and far from being the only ones possible; the author does not know whether all possible $\mathfrak{D}$ are isomorphic. Still, this makes no difference for cohomology (of pro-schemes); see Remark 4.3.2.
Moreover, note that assertion 10 of Proposition 3.1.1 was not very important for us (without if we would only have to consider a certain *weakly exact* weight complex functor in §6.1 below; see §3 of [6]). The author doubts that this condition follows from the other parts of Proposition 3.1.1.

5.4 Base change and Tate twists for comotives

Our differential graded formalism yields certain functoriality of comotives with respect to embeddings of base fields. We construct both extension and restriction of scalars (the latter one for the case of a finite extension of fields only). The construction of base change functors uses induction for differential graded modules. This method also allows to define certain tensor products and $Co-\underline{Hom}$ for comotives. In particular, we obtain a Tate twist functor which is compatible with (29) (and a left adjoint to it).
We note that the results of this subsection (probably) could not be deduced from the 'axioms' of $\mathfrak{D}$ listed in Proposition 3.1.1; yet they are quite natural.

5.4.1 Induction and restriction for differential graded modules: reminder

We recall certain results of §14 of [12] on functoriality of differential graded modules. These extend the corresponding (more or less standard) statements for modules over differential graded algebras (cf. §14.2 of ibid.).
If $f : A \to B$ is a functor of differential graded categories, we have an obvious *restriction* functor $f^* : \mathcal{C}(B) \to \mathcal{C}(A)$. It is easily seen that f^* also induces functors $\mathcal{K}(B) \to \mathcal{K}(A)$ and $\mathcal{D}(B) \to \mathcal{D}(A)$. Certainly, the latter functor respects homotopy colimits (i.e. the direct limits from $\mathcal{C}(B)$).
Now, it is not difficult to construct an *induction* functor $f_* : \text{DG-Mod}(A) \to \text{DG-Mod}(B)$ which is left adjoint to f^*; see §14.9 of ibid. By Example 14.10 of ibid, for any $X \in ObjA$ this functor sends $X^* = A(X,-)$ to $f(X)^*$.
f_* also induces functors $\mathcal{C}(A) \to \mathcal{C}(B)$ and $\mathcal{K}(A) \to \mathcal{K}(B)$. Restricting the latter one to the category of semi-free modules K' (see Proposition 5.2.1) one obtains a functor $Lf_* : \mathcal{D}(A) \to \mathcal{D}(B)$ which is also left adjoint to the corresponding f^*; see §14.12 of [12]. Since all functors of the type X^* are semi-free by definition, we have $Lf_*(X^*) = A(X,-) = Lf(X)^*$. It can also be shown that Lf_* respects direct limits of objects of A^{op} (considered as A-modules via the Yoneda embedding). In the case of countable limits this follows easily from the definition of semi-free modules and the expression of the homotopy colimit in $\mathcal{D}(A)$ as $\varinjlim X_i = \text{Cone}(\bigoplus X_i \to \bigoplus X_i)$ (this is just the dual to (27)). For uncountable limits, one could prove the fact using a 'resolution' of the direct limit similar to those described in §A3 of [21].

5.4.2 Extension and restriction of scalars for comotives

Now let l/k be an extension of perfect fields.
Recall that $\mathfrak{D}'$ and $\mathfrak{D}$ were described (in §5.3) in terms of modules over a certain differential graded category A. It was shown in [19] that the corresponding version of A is a tensor (differential graded) category; we also have an extension of scalars functor $A_k \to A_l$. It is most probable that both of these properties hold for the version of A described in [4] (note that they obviously hold for

$B^b(SmCor)$). Moreover, if l/k is finite, then we have the functor of restriction of scalars in inverse direction.
So, the induction for the corresponding differential graded modules yields an exact functor of extension of scalars $\mathrm{Ext}_{l/k}: \mathfrak{D}_k \to \mathfrak{D}_l$. The reasoning above shows that $\mathrm{Ext}_{l/k}$ is compatible with the 'usual' extension of scalars for smooth varieties (and complexes of smooth correspondences). Moreover, since $\mathrm{Ext}_{l/k}$ respects homotopy limits, this compatibility extends to comotives of pro-schemes and their products. It can also be easily shown that $\mathrm{Ext}_{l/k}$ respects Tate twists.
We immediately obtain the following result.

Proposition 5.4.1. *Let k be countable (and perfect), let $l \supset k$ be a perfect field.*
1. Let S be a connected primitive scheme over k, let S_0 be its generic point. Then $M_{gm}(S_l)$ is a retract of $M_{gm}(S_{0l})$ in $\mathfrak{D}_l$.
2. Let K be a function field over k. Let K' be the residue field for a geometric valuation v of K of rank r. Then $M_{gm}(K'_l(r)[r])$ is a retract of $M_{gm}(K_l)$ in $\mathfrak{D}_l$.

As in 4.3, this result immediately implies similar statements for any cohomology of pro-schemes mentioned (constructed from a cohomological $H: DM^{eff}_{gm}{}_l \to A$ via Proposition 1.2.1).
Next, if l/k is finite, induction for differential graded modules applied to the restriction of scalars for A's also yields a restriction of scalars functor $\mathrm{Res}_{l/k}: \mathfrak{D}_l \to \mathfrak{D}_k$. Similarly to $\mathrm{Ext}_{l/k}$, this functor is compatible with restriction of scalars for smooth varieties, pro-schemes, and complexes of smooth correspondences; it also respects Tate twists.
It follows: l/k is finite, then $\mathrm{Ext}_{l/k}$ maps $\mathfrak{D}_{sk}$ to $\mathfrak{D}_{sl}$; $\mathrm{Res}_{l/k}$ maps $\mathfrak{D}_{sl}$ to $\mathfrak{D}_{sk}$. Besides, if we also assume l to be countable, then both of these functors respect weight structures (i.e. they map ${\mathfrak{D}_s}_k^{w\le 0}$ to ${\mathfrak{D}_s}_l^{w\le 0}$, ${\mathfrak{D}_s}_k^{w\ge 0}$ to ${\mathfrak{D}_s}_l^{w\ge 0}$, and vice versa).

Remark 5.4.2. It seems that one can also define restriction of scalars via restriction of differential graded modules (applied to the extension of scalars for A's). To this end one needs to check the corresponding adjunction for DM^{eff}_{gm}; the corresponding (and related) statement for the motivic homotopy categories was proved by J. Ayoub. This would allow to define $\mathrm{Res}_{l/k}$ also in the case when l/k is infinite; this seems to be rather interesting if l is a function field over k. Note that $\mathrm{Res}_{l/k}$ (in this case) would (probably) also map ${\mathfrak{D}_s}_l^{w\le 0}$ to ${\mathfrak{D}_s}_k^{w\le 0}$ and ${\mathfrak{D}_s}_l^{w\ge 0}$ to ${\mathfrak{D}_s}_k^{w\ge 0}$ (if l is countable).

5.4.3 Tensor products and 'co-internal Hom' for comotives; Tate twists

Now, for $X \in ObjA$ we apply restriction and induction of differential graded modules for the functor $X \otimes - : A \to A$. Induction yields a certain functor

$X \otimes - : \mathfrak{D} \to \mathfrak{D}$, whereas restriction yields its left adjoint which we will denote by $Co - \underline{Hom}(X, -) : \mathfrak{D} \to \mathfrak{D}$. Both of them respect homotopy limits. Also, $X \otimes -$ is compatible with tensoring by X on DM_{gm}^{eff}. Besides, the isomorphisms classes of these functors only depend on the quasi-isomorphism class of X in DG-Mod(A). Indeed, it is easily seen that both $X \otimes Y$ and $Co - \underline{Hom}(X, Y)$ are exact with respect to X if we fix Y; since they are obviously zero for $X = 0$, it remains to note that quasi-isomorphic objects could be connected by a chain of quasi-isomorphisms.

Now suppose that X is a Tate motif i.e. $X = \mathbb{Z}(m)[n]$, $m > 0$, $n \in \mathbb{Z}$. Then we obtain that the formal Tate twists defined by (29) are the true Tate twists i.e. they are given by tensoring by X on $\mathfrak{D}$. Then recall the Cancellation Theorem for motives: (see Theorem 4.3.1 of [25], and [27])): $X \otimes -$ is a full embedding of DM_{gm}^{eff} into itself. Then one can deduce that $X \otimes -$ is fully faithful on $\mathfrak{D}$ also (since all objects of $\mathfrak{D}$ come from semi-free modules over A). Moreover, $Co - \underline{Hom}(X, -) \circ (X \otimes -)$ is easily seen to be isomorphic to the identity on $\mathfrak{D}$ (for such an X).

6 Supplements

We describe some more properties of comotives, as well as certain possible variations of our methods and results. We will be somewhat sketchy sometimes.

In §6.1 we define an additive category $\mathfrak{D}^{gen}$ of generic motives (a variation of those studied in [9]). We also prove that the exact conservative *weight complex* functor (that exists by the general theory of weight structures) could be modified to an exact conservative $WC : \mathfrak{D}_s \to K^b(\mathfrak{D}^{gen})$. Besides, we prove assertions on retracts of the pro-motif of a function field K/k, that are similar to (and follow from) those for its comotif.

In §6.2 we prove that HI has a nice description in terms of $\underline{Hw}$. This is a sort of Brown representability: a cofunctor $\underline{Hw} \to Ab$ is representable by a (homotopy invariant) sheaf with transfers whenever it converts all small products into direct sums. This result is similar to the corresponding results of §4 of [6] (on the connection between the hearts of adjacent structures).

In §6.3 we note that our methods could be used for motives (and comotives) with coefficients in an arbitrary commutative unital ring R; the most important cases are rational (co)motives and 'torsion' (co)motives.

In §6.4 we note that there exist natural motives of pro-schemes with compact support in DM_{-}^{eff}. It seems that one could construct alternative $\mathfrak{D}$ and $\mathfrak{D}'$ using this observation (yet this probably would not affect our main results significantly).

We conclude the section by studying which of our arguments could be extended to the case of an uncountable k.

6.1 The weight complex functor; relation with generic motives

We recall that the general formalism of weight structures yields a conservative exact weight complex functor $t : \mathfrak{D}_s \to K^b(\underline{Hw})$; it is compatible with Definition 2.1.2(9). Next we prove that one can compose it with a certain 'projection' functor without losing the conservativity.

Lemma 6.1.1. *There exists an exact conservative functor $t : \mathfrak{D}_s \to K^b(\underline{Hw})$ that sends $X \in Obj\mathfrak{D}_s$ to a choice of its weight complex (coming from any choice of a weight Postnikov tower for it).*

Proof. Immediate from Remark 6.2.2(2) and Theorem 3.3.1(V) of [6] (note that $\mathfrak{D}_s$ has a differential graded enhancement by Proposition 3.1.1(10)).

□

Now, since all objects of $\underline{Hw}$ are retracts of those that come via p from inverse limits of objects of $j(C^b(SmCor))$, we have a natural additive functor $\underline{Hw} \to \mathfrak{D}^{naive}$ (see §1.5). Its categorical image will be denoted by $\mathfrak{D}^{gen}$; this is a slight modification of Deglise's category of generic motives. We will denote the 'projection' $\underline{Hw} \to \mathfrak{D}^{gen}$ and $K^b(\underline{Hw}) \to K^b(\mathfrak{D}^{gen})$ by pr.

Theorem 6.1.2. *1. The functor $WC = pr \circ t : \mathfrak{D}_s \to K^b(\mathfrak{D}^{gen})$ is exact and conservative.*
2. Let S be a connected primitive scheme, let S_0 be its generic point. Then $pr(M_{gm}(S))$ is a retract of $pr(M_{gm}(S_0))$ in $\mathfrak{D}^{gen}$.
3. Let K be a function field over k. Let K' be the residue field for some geometric valuation v of K of rank r. Then $pr(M_{gm}(K')(r)[r])$ is a retract of $pr(M_{gm}(K))$ in $\mathfrak{D}^{gen}$.

Proof. 1. The exactness of WC is obvious (from Lemma 6.1.1). Now we check that WC is conservative.
By Proposition 3.1.1(8), it suffices to check: if $WC(X)$ is acyclic for some $X \in Obj\mathfrak{D}_s$, then $\mathfrak{D}(X, Y) = 0$ for all $Y \in ObjDM^{eff}_{gm}$. We denote the terms of $t(X)$ by X^i.
We consider the coniveau spectral sequence $T(H, X)$ for the functor $H = \mathfrak{D}(-, Y)$ (see Remark 4.4.2). Since $WC(X)$ is acyclic, we obtain that the complexes $\mathfrak{D}(X^{-i}, Y[j])$ are acyclic for all $j \in \mathbb{Z}$. Indeed, note that the restriction of a functor $\mathfrak{D}(X^{-i}, -)$ to DM^{eff}_{gm} could be expressed in terms of $pr(X^{-i})$; see Remark 3.2.1. Hence $E_2(T)$ vanishes. Since T converges (see Proposition 4.4.1(2)) we obtain the claim.
2. Immediate from Corollary 4.2.2(1).
3. Immediate from Corollary 4.2.2(2).

□

Remark 6.1.3. For $X = M_{gm}(Z)$, $Z \in SmVar$, it easily seen that $WC(X)$ could be described as a 'naive' limit of complexes of motives; cf. §1.5.
Now, the terms of $t(X)$ are just the factors of (some possible) weight Postnikov tower for X; so one can calculate them (at least, up to an isomorphism) for

$X = M_{gm}(Z)$. Unfortunately, it seems difficult to describe the boundary for $t(X)$ completely since $\underline{Hw}$ is finer than $\mathfrak{D}^{gen}$.

6.2 The relation of the heart of w with HI ('Brown representability')

In Theorem 4.4.2(4) of [6], for a pair of adjacent structures (w,t) for $\underline{C}$ (see Remark 2.5.7) it was proved that $\underline{Ht}$ is a full subcategory of $\underline{Hw}_*(= \mathrm{AddFun}(\underline{Hw}^{op}, Ab))$. This result cannot be extended to arbitrary orthogonal structures since our definition of a duality did not include any non-degenerateness conditions (in particular, Φ could be 0). Yet for our main example of orthogonal structures the statement is true; moreover, HI has a natural description in terms of $\underline{Hw}$. This statement is very similar to a certain Brown representability-type result (for adjacent structures) proved in Theorem 4.5.2(II.2) of ibid.
Note that $\underline{Hw}$ is closed with respect to arbitrary small products; see Proposition 4.1.1(2).

Proposition 6.2.1. *HI is naturally isomorphic to a full abelian subcategory $\underline{Hw}'_*$ of $\underline{Hw}_*$ that consists of functors that convert all products in $\underline{Hw}$ into direct sums (of the corresponding abelian groups).*

Proof. First, note that for any $G \in ObjDM_-^{eff}$ the functor $\mathfrak{D} \to Ab$ that sends $X \in Obj\mathfrak{D}$ to $\Phi(X,G)$ (Φ is the duality constructed in Proposition 4.5.1) is cohomological. Moreover, it converts homotopy limits into injective limits (of the corresponding abelian groups); hence its restriction to $\underline{Hw}$ belongs to $\underline{Hw}'_*$. We obtain an additive functor $DM_{gm}^{eff} \to \underline{Hw}'_*$. In fact, it factorizes through HI (by (25)). For $G \in ObjHI$ we denote the functor $\underline{Hw} \to Ab$ obtained by G'.
Next, for any (additive) $F : \underline{Hw}^{op} \to Ab$ we define $F' : \mathfrak{D}_s \to Ab$ by:

$$F'(X) = (\mathrm{Ker}(F(X^0) \to F(X^{-1}))/\,\mathrm{Im}(F(X^1) \to F(X^0)); \qquad (32)$$

here X^i is a weight complex for X. It easily seen from Lemma 6.1.1 that F' is a well-defined cohomological functor. Moreover, Theorem 2.2.1(19) yields that F' vanishes on $\mathfrak{D}_s^{w\le -1}$ and on $\mathfrak{D}_s^{w\ge 1}$ (since it vanishes on $\mathfrak{D}_s^{w=i}$ for all $i \neq 0$). Hence F' defines an additive functor $F'' = F' \circ M_{gm} : SmCor^{op} \to Ab$ i.e. a presheaf with transfers. Since $M_{gm}(Z) \cong M_{gm}(Z \times \mathbb{A}^1)$ for any $Z \in SmVar$, F'' is homotopy invariant. We should check that F'' is actually a (Nisnevich) sheaf. By Proposition 5.5 of [26], it suffices to check that F'' is a Zariski sheaf. Now, the the Mayer-Vietoris triangle for motives (§2 of [25]) yields: to any Zariski covering $U \coprod V \to U \cup V$ there corresponds a long exact sequence

$$\dots \to F'(M_{gm}(U \cap V)[1]) \to F''(U \cup V) \to F''(U) \bigoplus F''(V) \to F''(U \cap V) \to \dots$$

Since $M_{gm}(U \cap V) \in \mathfrak{D}_s^{w\le 0}$ by part 5 of Proposition 4.1.1, we have $F'(M_{gm}(U \cap V)[1]) = \{0\}$; hence F'' is a sheaf indeed.

So, $F \mapsto F''$ yields an additive functor $\underline{Hw}_* \to HI$.
Now we check that the functor $G \mapsto G'$ (described above) and the restrictions of $F \mapsto F''$ to $\underline{Hw}'_* \subset \underline{Hw}_*$ yield mutually inverse equivalences of the categories in question.
(24) immediately yields that the functor $HI \to HI$ that sends $G \in ObjHI$ to $(G')''$ is isomorphic to id_{HI}.
Now for $F \in Obj\underline{Hw}'_*$ we should check: for any $P \in \mathfrak{D}_s^{w=0}$ we have a natural isomorphism $(F'')'(P) \cong F(P)$. Since $\underline{Hw}$ is the idempotent completion of H, it suffices to consider P being of the form $\prod_{l \in L} M_{gm}(K_l)(n_l)[n_l]$ (here K_l are function fields over k, $n_l \geq 0$; n_l and the transcendence degrees of K_l/k are bounded); see part 2 of Proposition 4.1.1. Moreover, since F converts products into direct sums, it suffices to consider $P = M_{gm}(K')(n)[n]$ (K'/k is a function field, $n \geq 0$). Lastly, part 2 of Corollary 4.2.2 reduces the situation to the case $P = M_{gm}(K)$ (K/k is a function field). Now, by the definition of the functor $G \mapsto G'$, we have $(F'')'(M_{gm}(K)) = \varinjlim_{l \in L} F''(M_{gm}(U_l))$, where $K = \varprojlim_{l \in L} U_l$, $U_l \in SmVar$. We have $F''(U_l) = \operatorname{Ker} F(M_{gm}(K)) \to F(\prod_{z \in U_l^1} M_{gm}(z)(1)[1])$; here U_l^1 is the set of points of U_l of codimension 1. Since $F(\prod_{z \in U_l^1} M_{gm}(z)(1)[1]) = \oplus_{z \in U_l^1} F(M_{gm}(z)(1)[1])$; we have $\varinjlim_{l \in L} F(\prod_{z \in U_l^1} M_{gm}(z)(1)[1]) = \{0\}$; this yields the result.

□

6.3 Motives and comotives with rational and torsion coefficients

Above we considered (co)motives with integral coefficients. Yet, as was shown in [20], one could do the theory of motives with coefficients in an arbitrary commutative associative ring with a unit R. One should start with the naturally defined category of R-correspondences: $Obj(SmCor_R) = SmVar$; for X, Y in $SmVar$ we set $SmCor_R(X, Y) = \bigoplus_U R$ for all integral closed $U \subset X \times Y$ that are finite over X and dominant over a connected component of X. Then one obtains a theory of motives that would satisfy all properties that are required in order to deduce the main results of this paper. So, we can define R-comotives and extend our results to them.
A well-known case of motives with coefficients are the motives with rational coefficients (note that $\mathbb{Q}$ is a flat $\mathbb{Z}$-algebra). Yet, one could also take $R = \mathbb{Z}/n\mathbb{Z}$ for any n prime to $\operatorname{char} k$.
So, the results of this paper are also valid for rational (co)motives and 'torsion' (co)motives.
Still, note that there could be idempotents for R-motives that do not come from integral ones. In particular, for the naturally defined rational motivic categories we have $DM_{gm}^{eff}\mathbb{Q} \neq DM_{gm}^{eff} \otimes \mathbb{Q}$; also $Chow^{eff}\mathbb{Q} \neq Chow^{eff} \otimes \mathbb{Q}$ (here $Chow^{eff}\mathbb{Q} \subset DM_{gm}^{eff}\mathbb{Q}$ denote the corresponding R-hulls). Certainly, this does not matter at all in the current paper.

6.4 Another possibility for $\mathfrak{D}$; motives with compact support of pro-schemes

In the case $\operatorname{char} k = 0$, Voevodsky developed a nice theory of motives with compact support that is compatible with Poincare duality; see Theorem 4.3.7 of [25]. Moreover, the explicit constructions of [25] yield that the functor of motif with compact support $M^c_{gm} : SmVar^{op} \to DM^{eff}_{gm}$ is compatible with a certain $j^c : SmVar^{op}_{fl} \to C^-(Shv(SmCor))$ (which sends X to the Suslin complex of $L^c(X)$, see §4.2 loc.cit.); this observation was kindly communicated to the author by Bruno Kahn). This allows to define $j^c(V)$ for a pro-scheme V as the corresponding direct limit (in $C(Shv(SmCor))$).

Starting from this observation, one could try to develop an analogue of our theory using the functor M^c_{gm}. One could consider $\mathfrak{D} = DM^{eff\,op}_-$; then it would contain $DM^{eff\,op}_{gm}$ as the full category of cocompact objects. It seems that our arguments could be carried over to this context. One can construct some $\mathfrak{D}'$ for this $\mathfrak{D}$ using certain differential graded categories.

Though motives with compact support are Poincare dual to ordinary motives of smooth varieties (up to a certain Tate twist), we do not have a covariant embedding $DM^{eff}_{gm} \to \mathfrak{D}$ (for this 'alternative' $\mathfrak{D}$), since (the whole) DM^{eff}_{gm} is not self-dual. Still, DM^{eff}_{gm} has a nice embedding into (Voevodsky's) self-dual category DM_{gm}; it contains an exhausting system of self-dual subcategories. Hence this alternative $\mathfrak{D}$ would yield a theory that is compatible with (though not 'isomorphic' to) the theory developed above.

Since the alternative version of $\mathfrak{D}$ is closely related with $DM^{eff\,op}_-$, it seems reasonable to call its objects comotives (as we did for the objects of 'our' $\mathfrak{D}$).

These observations show that one can dualize all the direct summands results of §4 to obtain their natural analogues for motives of pro-schemes with compact support. Indeed, to prove them we may apply the duals of our arguments in §4 without any problem; see part 2 of Remark 3.1.2. Note that we obtain certain direct summand statements for objects of DM^{eff}_- this way. This is an advantage of our 'axiomatic' approach in §3.1.

One could also take $\mathfrak{D}^{op} = \cup_{n\in\mathbb{Z}} DM^{eff}_{gm}(-n)$ (more precisely, this is the direct limit of copies of DM^{eff}_{gm} with connecting morphisms being $-\otimes\mathbb{Z}(1)$). Then we have a covariant embedding $DM^{eff}_{gm} \to DM_{gm} \to \mathfrak{D}$.

Note that both of these alternative versions of $\mathfrak{D}$ are not closed with respect to all (countable) products, and so not closed with respect to all (filtered countable) homotopy limits; yet they contain all products and homotopy limits that are required for our main arguments.

6.5 What happens if k is uncountable

We describe which of the arguments above could be applied in the case of an uncountable k (and for which of them the author has no idea how to achieve this). The author warns that he didn't check the details thoroughly here.

As we have already noted above, it is no problem to define $\mathfrak{D}$, $\mathfrak{D}'$, or even $\mathfrak{D}_s$ for any k. The main problem here that (if k is uncountable) the comotives of generic points of varieties (and of other pro-schemes) can usually be presented only as uncountable homotopy limits of motives of varieties. The general formalism of inverse limits (applied to the categories of modules over a differential graded category) allows to extend to this case all parts of Proposition 3.1.1 expect part 9. This actually means that instead of the short exact sequence (28) one obtains a spectral sequence whose E_1-terms are certain $\varprojlim^j$; here $\varprojlim^j$ is the j's derived functor of $\varprojlim_I$; cf. Appendix A of [21]. This does not seem to be catastrophic; yet the author has absolutely no idea how to control higher projective limits in the proof of Proposition 3.5.1; note that part 2 of loc.cit. is especially important for the construction of the Gersten weight structure.
Besides, the author does not know how to pass to an uncountable homotopy limit in the Gysin distinguished triangle. It seems that to this end one either needs to lift the functoriality of the (usual) motivic Gysin triangle to $\mathfrak{D}'$, or to find a way to describe the isomorphism class of an uncountable homotopy limit in $\mathfrak{D}$ in terms of $\mathfrak{D}$-only (i.e. without fixing any lifts to $\mathfrak{D}'$; this seems to be impossible in general). So, one could define the 'Gersten' weight tower for a comotif of a pro-scheme as as the homotopy limit of 'geometric towers' (as in the proof of Corollary 3.6.2); yet it seems to be rather difficult to calculate factors of such a tower. It seems that the problems mentioned do not become simpler for the alternative versions of $\mathfrak{D}$ described in §6.4. So, currently the author does not know how to prove the direct summand results of §4.2 if k is uncountable (they even could be wrong). The problem here that the splittings of §4.2 are not canonical (see Remark 4.2.3), so one cannot apply a limit argument (as in §4.6) here.
It seems that constructing the Gersten weight structure is easier for $\mathfrak{D}_s/\mathfrak{D}_s(n)$ (for some $n > 0$); see §4.9.
Lastly, one can avoid the problems with homotopy limits completely by restricting attention to the subcategory of Artin-Tate motives in DM_{gm}^{eff} (i.e. the triangulated category generated by Tate twists of motives of finite extensions of k, as considered in [30]). Note that coniveau spectral sequences for cohomology of such motives (could be chosen to be) very 'economic'.

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