Zero Cycles on Singular Varieties and Their Desingularisations

To Sasha Merkurjev on the occasion of his 60th birthday

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Abstract. We use pro cdh-descent of K-theory to study the relationship between the zero cycles on a singular variety X and those on its desingularisation X'. We prove many cases of a conjecture of S. Bloch and V. Srinivas, and relate the Chow groups of X to the Kerz–Saito Chow group with modulus of X' relative to its exceptional fibre.

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0 Introduction

Let $X' \to X$ be a desingularisation of a d-dimensional, integral variety over a field k, with exceptional fibre $E \hookrightarrow X$. Letting rE denote the r^{th} infinitesimal thickening of E, we denote by $F^dK_0(X', rE)$ the subgroup of the relative K-group $K_0(X', rE)$ generated by the cycle classes of closed points of $X' \setminus E$, for each $r \geq 1$. This inverse system

$$F^dK_0(X', E) \longleftarrow F^dK_0(X', 2E) \longleftarrow F^dK_0(X', 3E) \longleftarrow \cdots$$

was first studied by S. Bloch and V. Srinivas [16], in the case of normal surfaces, as a means of relating zero cycles on the singular variety X to zero cycles on the smooth variety X'. They conjectured [pg. 6, op. cit.] in 1985 that this inverse system would eventually stabilise, i.e., $F^dK_0(X', rE) \stackrel{\simeq}{\to} F^dK_0(X', (r-1)E)$ for $r \gg 1$, with stable value equal to $F^dK_0(X)$, the subgroup of $K_0(X)$ generated by cycle classes of smooth, closed points of X.

The Bloch–Srinivas conjecture was proved for normal surfaces by A. Krishna and Srinivas [9, Thm. 1.1], and later extended to higher dimensional, Cohen–Macaulay varieties with isolated singularities in characteristic zero by Krishna [6, Thm. 1.1] [7, Thm. 1.2]. The conjecture has not been previously verified in any case of non-isolated singularities, nor for any higher dimensional varieties in finite characteristic.

The primary goal of this paper is to prove the following cases of the Bloch–Srinivas conjecture for varieties which are regular in codimension one:

Theorem 0.1. *Let $\pi : X' \to X$ be a desingularisation of a d-dimensional, quasi-projective, integral variety X over an infinite, perfect field k which is assumed to have strong resolution of singularities. Let $E \hookrightarrow X$ be a closed embedding covering the exceptional fibre, and assume that* $\operatorname{codim}(X, \pi(E)) \geq 2$. *Then the associated Bloch–Srinivas conjecture is*

- *(i) true up to $(d-1)!$-torsion;*
- *(ii) true if X is projective, $k = k^{alg}$, and* $\operatorname{char} k = 0$*;*
- *(iii) true if X is projective, $k = k^{alg}$, and $d \leq$* $\operatorname{char} k \neq 0$*;*
- *(iv) true if X is affine and $k = k^{alg}$;*
- *(v) true "up to a finite group" if $k = k^{alg}$ and X_{sing} is contained in an affine open of X;*
- *(vi) true if $\pi(E)$ is finite;*
- *(vii) true if the cycle class map $CH_0(X) \to F^d K_0(X)$ is an isomorphism.*

The group $CH_0(X)$ appearing in part (vii) of Theorem 0.1 is the Levine–Weibel Chow group of zero cycles of the singular variety X [10, 12]; it will be reviewed in Section 1.1.

Part (iv) of the Theorem, combined with arguments of Krishna [7] and R. Murthy [15], has concrete applications to Chow groups of cones and to the structure of modules and ideals of graded algebras; see Theorem 1.17 and Corollaries 1.18 and 1.19.

This paper is intended partly to justify the author's pro cdh-descent theorem for K-theory [13]; indeed, the results of Theorem 0.1 are obtained in Section 1.2 as corollaries of the following general result, which itself is an immediate consequence of pro cdh-descent:

Theorem 0.2. *Let $\pi : X' \to X$ be a desingularisation of a d-dimensional, quasi-projective, integral variety over an infinite, perfect field k which is assumed to have strong resolution of singularities. Let $E \hookrightarrow X$ be a closed embedding covering the exceptional fibre. Then:*

- *(i) There exists a unique homomorphism $BS_r : F^d K_0(X', rE) \to F^d K_0(X)$ for $r \gg 1$ which is compatible with cycle classes of closed points.*

(ii) The associated Bloch–Srinivas conjecture is true if and only if the canonical map $F^dK_0(X, rY) \to F^dK_0(X)$ is an isomorphism for $r \gg 1$, where $Y := \pi(E)_{red}$.

Section 2 concerns Chow groups of zero cycles with modulus. If X is a smooth, projective variety over a field k and D is an effective divisor on X, then the Chow group with modulus $CH_0(X; D)$ is defined to be the free abelian group on the closed points of $X \setminus D$, modulo rational equivalence coming from closed curves C which are not contained in $|D|$ and rational functions $f \in k(C)^\times$ which are $\equiv 1 \bmod D$. This Chow group is central in M. Kerz and S. Saito's [5] higher dimensional class field theory.

It is natural to formulate an analogue of the Bloch–Srinivas conjecture for the Chow groups with modulus given by successive thickenings of the exceptional fibre of a desingularisation. We will explain this further in Section 2, where we prove it in the following cases:

Theorem 0.3. *Let $\pi : X' \to X$ be a desingularisation of a d-dimensional, quasi-projective, integral variety over an algebraically closed field k which is assumed to have strong resolution of singularities. Let D be an effective Cartier divisor on X covering the exceptional fibre, and assume that* $\operatorname{codim}(X, \pi(D)) \geq 2$.

Then the inverse system

$$CH_0(X'; D) \longleftarrow CH_0(X'; 2D) \longleftarrow CH_0(X'; 3D) \longleftarrow \cdots$$

eventually stabilises with stable value equal to $CH_0(X)$, assuming that either

(i) X is projective and $\operatorname{char} k = 0$; or

(ii) X is projective and $d \leq \operatorname{char} k \neq 0$; or

(iii) X is affine.

Whenever the assertions of Theorem 0.3 can be proved for a singular, projective variety X over a *finite* field (e.g., for surfaces, as we shall see in Remark 2.8), it has applications to the class field theory of X; in particular, it shows that there is a reciprocity isomorphism of finite groups $CH_0(X)^0 \xrightarrow{\simeq} \pi_1^{\mathrm{ab}}(X_{\mathrm{reg}})^0$. See Remark 2.7 for further details.

We prove Theorem 0.3 by reducing it to the analogous assertion in K-theory, which is precisely the Bloch–Srinivas conjecture, and then applying Theorem 0.1. This reduction is through the construction of a new cycle class homomorphism

$$CH_0(X; D) \longrightarrow F^dK_0(X, D),$$

which is valid for any effective Cartier divisor D on a smooth variety X. This also allows us to prove the following result, which appears related to a special case of a conjecture of Kerz and Saito [5, Qu. V]:

Theorem 0.4. *With notation and assumptions as in Theorem 0.3, the cycle class homomorphism*

$$CH_0(X';rD) \longrightarrow F^dK_0(X';rD)$$

is an isomorphism for $r \gg 1$.

Notation, conventions, etc.

A field k will be called *good* if and only if it is infinite, perfect, and has strong resolution of singularities, e.g., $\operatorname{char} k = 0$ suffices. A *k-variety* means simply a finite type k-scheme; further assumptions will be specified when required, and the reference to k with occasionally be omitted. Our conventions about "desingularisations" can be found at the start of Section 1.2.
A *curve* over k is a one-dimensional, integral k-variety. Given a closed point $x \in C_0$, there is an associated order function $\operatorname{ord}_x : k(X)^\times \to \mathbb{Z}$ characterised by the property that $\operatorname{ord}_x(t) = \operatorname{length}_{\mathcal{O}_{C,x}}(\mathcal{O}_{C,x}/t\mathcal{O}_{C,x})$ for any non-zero $t \in \mathcal{O}_{C,x}$; when C is smooth ord_x is the usual valuation associated to x.
An *effective divisor* D on X is by definition a closed subscheme whose defining sheaf of ideals $\mathcal{O}_X(-D)$ is an invertible $\mathcal{O}_X$-module, or, equivalently, is locally defined by a single non-zero-divisor; its associated support is denoted by $|D|$, but we write $X \setminus D$ in place of $X \setminus |D|$ for simplicity.
Given a closed embedding $Y = \operatorname{Spec} \mathcal{O}_X/\mathcal{I} \hookrightarrow X$, its r^{th} infinitesimal thickening is denoted by $rY = \operatorname{Spec} \mathcal{O}_X/\mathcal{I}^r$.
A *pro abelian group* $\{A_r\}_r$ is an inverse system of abelian groups, with morphisms given by the rule

$$\operatorname{Hom}_{\operatorname{Pro} Ab}(\{A_r\}_r, \{B_s\}_s) := \varprojlim_s \varinjlim_r \operatorname{Hom}_{Ab}(A_r, B_s).$$

The category of pro abelian groups is abelian; we refer to [1, App.] for more details.

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1 Zero cycles of desingularisations

In this section we prove cases of the Bloch–Srinivas conjecture relating zero cycles on a singular variety to those on its desingularisation.
There will be an important distinction between closed subsets $S \subseteq X$ and closed subschemes $Y \hookrightarrow X$; in an attempt to keep this clear we will use the

differentiating notation $\subseteq$ and $\hookrightarrow$ just indicated. Any closed subscheme $Y \hookrightarrow X$ has an associated support $|Y| \subseteq X$, though we will continue to write $X \setminus Y$ rather than $X \setminus |Y|$ for the associated open complement, and any closed subset $S \subseteq X$ has an associated reduced closed subscheme $S_{\mathrm{red}} \hookrightarrow X$. The singular locus of X is denoted by $X_{\mathrm{sing}} \subseteq X$.

1.1 Review of the Levine–Weibel Chow group

We begin by reviewing the Levine–Weibel Chow group of zero cycles [10, 12], restricting to the situation that the singularities of X are in codimension ≥ 2, since this is sufficient for our applications. Unless specified otherwise, k is an arbitrary field.

Definition 1.1. Let X be an integral k-variety which is regular in codimension one, and $S \subseteq X$ any closed subset containing X_{sing}. Then the associated *Levine–Weibel Chow group of zero cycles* is

$$CH_0(X;S) := \frac{\text{free abelian group on closed points of } X \setminus S}{\langle (f)_C \,:\, C \hookrightarrow X \text{ a curve not meeting } S, \text{ and } f \in k(C)^\times \rangle}$$

where $(f)_C := \sum_{x \in C_0} \mathrm{ord}_x(f)\, x$ as usual. In particular, $CH_0(X) := CH_0(X; X_{\mathrm{sing}})$.

Remark 1.2. Several remarks should be made:

(i) The group $CH_0(X;S)$ we have just defined can actually only reasonably be called the Levine–Weibel Chow group of zero cycles if we assume that $\mathrm{codim}(X,S) \geq 2$. But it is convenient to introduce the notation in slightly greater generality since it will be useful in Section 2.

(ii) An inclusion of closed subsets $S \subseteq S'$ of X, both containing X_{sing}, induces a canonical surjection $CH_0(X;S') \twoheadrightarrow CH_0(X;S)$. This surjection is an isomorphism if X is quasi-projective and S, S' have codimension ≥ 2, by a moving lemma [12, pg. 113].

(iii) Suppose that X is a smooth k-variety and that $S \subseteq X$ is a closed subset. Then there is a canonical surjection $CH_0(X;S) \twoheadrightarrow CH_0(X;\emptyset) = CH_0(X)$, which will be an isomorphism if S has codimension ≥ 2 and X is quasi-projective, by the aforementioned moving lemma.

(iv) Suppose that $X' \to X$ is a proper morphism which restricts to an isomorphism $X' \setminus S' \xrightarrow{\simeq} X \setminus S$ for some closed subsets $S \subseteq X$, $S' \subseteq X'$ containing the singular loci. Then the induced map $CH_0(X;S) \to CH_0(X';S')$ is an isomorphism. Indeed, both sides are generated by the closed points of $X' \setminus S' = X \setminus S$, and closed curves on X not meeting S correspond to closed curves on X' not meeting S'.

To review the relationship between $CH_0(X)$ and K-theory, we must first explain the cycle class map. Let X be a k-variety, and $i : Y \hookrightarrow X$ a fixed closed subscheme. If $j : C \hookrightarrow X$ is a closed subscheme with image disjoint from both $|Y|$ and X_{sing}, then j is of finite Tor dimension since it factors as $C \hookrightarrow X_{\mathrm{reg}} \to X$, and it is moreover proper; thus the pushforward map $j_* : K(C) \to K(X)$ on the K-theory spectra is well-defined. Moreover, the projection formula [19, Prop. 3.18] associated to the pullback diagram

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & C \arrow[d, "j"] \\ Y \arrow[r, "i"'] & X \end{tikzcd}$$

shows that the composition $K(C) \xrightarrow{j_*} K(X) \xrightarrow{i^*} K(Y)$ is null-homotopic, and thus there is an induced pushforward $j_* : K(C) \to K(X, Y)$. The *cycle class* of C in $K_0(X, Y)$ is defined to be

$$[C] := j_*([\mathcal{O}_C]) \in K_0(X, Y).$$

Although this appears to depend a priori on a chosen null-homotopy, it was shown by K. Coombes [4] that the "obvious choices of homotopies" yield a class which is functorial with respect to both X and Y, and so we will follow Coombes' choices. A codimension filtration on $K_0(X, Y)$ is now defined by

$$F^p K_0(X, Y) := \langle [C] : C \hookrightarrow X \text{ an integral closed subscheme of } X \text{ of codim} \geq p \\ \text{disjoint from } |Y| \text{ and } X_{\mathrm{sing}} \rangle$$

In particular, $F^d K_0(X, Y)$ is the subgroup of $K_0(X, Y)$ generated by the cycle classes of smooth, closed points of $X \setminus Y$. The following is standard:

Lemma 1.3. *Let notation be as immediately above. If $j : C \hookrightarrow X$ is a closed embedding of a curve into X not meeting $|Y|$ or X_{sing}, and $f \in k(C)^\times$, then $\sum_{x \in C_0} \mathrm{ord}_x(f)[x] = 0$ in $K_0(X, Y)$.*

Proof. One has $\sum_{x \in C_0} \mathrm{ord}_x(f)[x] = j_*([\mathcal{O}_C] - [f\mathcal{O}_C]) = j_*(0) = 0$. ∎

Now suppose that X is a d-dimensional, integral k-variety which is regular in codimension one, let $Y \hookrightarrow X$ be a closed subscheme, and let $S \subseteq X$ be a closed subset containing both $|Y|$ and X_{sing}. It follows from Lemma 1.3 that the cycle class homomorphism

$$CH_0(X; S) \longrightarrow F^d K_0(X, Y), \quad x \longmapsto [x]$$

is well-defined. In particular, taking $S = X_{\mathrm{sing}}$ and $Y = \emptyset$ yields the cycle class homomorphism

$$[\ ] : CH_0(X) \longrightarrow F^d K_0(X),$$

which is evidently surjective. Moreover, as part of a general Riemann–Roch theory, M. Levine [11, 10] constructed a Chern class $ch_0 : F^dK_0(X) \to CH_0(X)$ such that the compositions $[\ ] \circ ch_0$ and $ch_0 \circ [\ ]$ are both multiplication by $(-1)^{d-1}(d-1)!$. In particular, $[\ ] : CH_0(X) \to F^dK_0(X)$ is an isomorphism if $d = 2$.

We complete our review of the Levine–Weibel Chow group of zero cycles by presenting the higher dimensional cases in which the cycle class homomorphism can be shown to be an isomorphism:

Theorem 1.4 (Barbieri Viale, Levine, Srinivas). *Let X be a d-dimensional, integral, quasi-projective variety over an algebraically closed field k which is regular in codimension one. Then the cycle class homomorphism $CH_0(X) \to F^dK_0(X)$ is*

(i) *an isomorphism if X is projective and* $\operatorname{char} k = 0$*;*

(ii) *an isomorphism if X is projective and $d \leq \operatorname{char} k \neq 0$;*

(iii) *an isomorphism if X is affine and* $\operatorname{char} k$ *is arbitrary;*

(iv) *a surjection with finite kernel if X_{sing} is contained in an affine open subscheme of X and* $\operatorname{char} k = 0$*;*

(v) *a surjection with finite kernel if X_{sing} is contained in an affine open subscheme of X and $d \leq \operatorname{char} k \neq 0$;*

Proof. Thanks to the existence of Levine's Chern class ch_0, it is enough to check that $CH_0(X)$ has no $(d-1)!$-torsion in cases (i)–(ii), that it has only a finite amount of $(d-1)!$-torsion in cases (iv)–(v), and that it has no torsion in case (iii).

Then (i) and (ii) are [10, Thm. 3.2], while (iv) and (v) are [2, Thm. A]. Finally, (iii) in characteristic zero (and when $d \leq \operatorname{char} k \neq 0$) is [10, Corol. 2.7], and so it remains only to deal with the following case: assuming that X is an integral, affine variety which is regular in codimension one, over an algebraically closed field of finite characteristic, we must show that $CH_0(X)$ is torsion-free. This is true for the normalisation $\widetilde{X}$ by [18], and so it remains only to check that $CH_0(X) \overset{\simeq}{\to} CH_0(\widetilde{X})$. But since X is assumed to be regular in codimension one, there are closed subsets $S \subseteq X$, $S' \subseteq \widetilde{X}$ (given by the conductor ideal, for example) of codimension ≥ 2, containing the singular loci, and such that the morphism $\widetilde{X} \to X$ restricts to an isomorphism $\widetilde{X} \setminus S' \overset{\simeq}{\to} X \setminus S$. Then, in the commutative diagram

$$\begin{tikzcd} CH_0(\widetilde{X}; S') \arrow[r] & CH_0(\widetilde{X}) \\ CH_0(X; S) \arrow[r] \arrow[u] & CH_0(X) \arrow[u] \end{tikzcd}$$

the horizontal arrows are isomorphisms by Remark 1.2(ii), while the left vertical arrow is an isomorphism by Remark 1.2(iv). Hence the right vertical arrow is an isomorphism, as required. ∎

1.2 The Bloch–Srinivas conjecture

Before we can carefully state the Bloch–Srinivas conjecture we must first fix some terminology concerning desingularisations. Given an integral variety X, a *desingularisation* is any proper, birational morphism $\pi : X' \to X$ where X' is smooth; in particular, we allow the desingularisation to change the smooth locus of X, though it is not clear if this is ever important in practice. There exists a smallest closed subset $S \subseteq X$ with the property that $X' \setminus \pi^{-1}(S) \xrightarrow{\simeq} X \setminus S$, and $\pi^{-1}(S)$ is known as the *exceptional set* of the resolution; setting $E := \pi^{-1}(S)_{\mathrm{red}}$ yields the *exceptional fibre* $E \hookrightarrow X'$. Corollaries 1.10–1.15 will require that $\pi(|E|)$ has codimension ≥ 2 in X, which in particular implies that X is regular in codimension one.

If $X' \to X$ is a desingularisation of an integral variety X, with exceptional fibre $E \hookrightarrow X'$, then Bloch and Srinivas [16, pg. 6] made the following conjecture in 1985:

Conjecture 1.5 (Bloch–Srinivas). The inverse system

$$F^dK_0(X',E) \longleftarrow F^dK_0(X',2E) \longleftarrow F^dK_0(X',3E) \longleftarrow \cdots$$

stabilises, with stable value $F^dK_0(X)$.

Remark 1.6. To be precise, Bloch and Srinivas stated their conjecture in the case of a normal surface X over an algebraically closed field, assuming that the desingularisation did not alter the smooth locus of X. If Conjecture 1.5 is false because it has been formulated in excessive generality, it is the author's fault. In fact, we will consider Conjecture 1.5 in greater generality still, by replacing the exceptional fibre E by any reduced closed subscheme $E \hookrightarrow X'$ whose support contains the exceptional set (henceforth "covers the exceptional set").

We interpret part of the Bloch–Srinivas conjecture as an implicit statement that there exists a cycle class homomorphism

$$BS_r : F^dK_0(X',rE) \longrightarrow F^dK_0(X)$$

for $r \gg 1$ which is compatible with cycle classes of closed points $x \in X' \setminus E$, i.e., $BS_r([x]) = [x]$. Such a map BS_r is unique if it exists.

Our main technical theorem, which is an immediate consequence of the author's pro cdh-descent theorem for K-theory [13], proves the existence of the maps BS_r in full generality, and reduces the Bloch–Srinivas conjecture to the study of the K-theory of X:

Theorem 1.7. *Let X be a d-dimensional, integral variety over a good field k; let $\pi : X' \to X$ be a desingularisation, $E \hookrightarrow X'$ any reduced closed subscheme covering the exceptional set, and set $Y := \pi(|E|)_{red}$. Then:*

(i) For $r \gg 1$, the canonical map $F^d K_0(X, rY) \to F^d K_0(X)$ factors through the surjection $F^d K_0(X, rY) \to F^d K_0(X', rE)$, i.e., there exists a commutative diagram

$$\begin{tikzcd} F^d K_0(X', rE) \arrow[r] \arrow[dr, dashed, "\exists\, BS_r"] & F^d K_0(X') \\ F^d K_0(X, rY) \arrow[u, two heads] \arrow[r] & F^d K_0(X) \arrow[u] \end{tikzcd}$$

(ii) The following are equivalent:

(a) The associated Bloch–Srinivas conjecture is true, i.e., BS_r is an isomorphism for $r \gg 1$.

(b) The canonical map $F^d K_0(X, rY) \to F^d K_0(X)$ is an isomorphism for $r \gg 1$.

(c) The canonical map $F^d K_0(X, rY) \to F^d K_0(X)$ is an isomorphism for all $r \geq 1$.

Proof. There is an abstract blow-up square

$$\begin{tikzcd} Y' \arrow[r] \arrow[d] & X' \arrow[d, "\pi"] \\ Y \arrow[r] & X \end{tikzcd}$$

where $Y' := X' \times_X Y$; note that Y' is a nilpotent thickening of E. By pro cdh-descent for K-theory [13, Thm. 0.1] (it is here that the field k is required to be good), the canonical homomorphism of pro abelian groups

$$\{K_0(X, rY)\}_r \longrightarrow \{K_0(X', rY')\}_r \cong \{K_0(X', rE)\}_r$$

is an isomorphism. Restricting to the codimension filtration we deduce that the homomorphism

$$\{F^d K_0(X, rY)\}_r \longrightarrow \{F^d K_0(X', rE)\}_r \tag{†}$$

is injective; but each map $F^d K_0(X, rY) \to F^d K_0(X', rE)$ is evidently surjective, since both sides are generated by the closed points of $X \setminus Y = X' \setminus E$. Thus (†) is an isomorphism.

By definition of an isomorphism of pro abelian groups, this implies that for any $s \geq 1$ there exists $r \geq s$ and a homomorphism $F^d K_0(X', rE') \to F^d K_0(X, sY)$ making the diagram commute:

$$\begin{tikzcd} F^d K_0(X', rE) \arrow[dr, dashed, "\exists"] & \\ F^d K_0(X, rY) \arrow[u, two heads] \arrow[r, two heads] & F^d K_0(X, sY) \end{tikzcd}$$

Note that the vertical and horizontal arrows are surjective, since the groups are generated by the closed points of $X \setminus Y = X' \setminus E$. This diagram shows that the canonical map $F^dK_0(X, rY) \to F^dK_0(X)$ factors through the surjection $F^dK_0(X, rY) \to F^dK_0(X', rE)$, proving (i).
This gives a commutative diagram

$$\begin{tikzcd}
F^dK_0(X', rE) \arrow[dr] \arrow[drr, bend left, "BS_r"] & & \\
F^dK_0(X, rY) \arrow[u, two heads] \arrow[r, two heads] & F^dK_0(X, sY) \arrow[r] & F^dK_0(X)
\end{tikzcd}$$

from which a simple diagram chase yields the following implications (valid for any $s \geq 1$ and $r \gg s$):

$F^dK_0(X, rY) \to F^dK_0(X)$ is an isomorphism $\implies$ BS_r is an isomorphism.
BS_r is an isomorphism $\implies$ $F^dK_0(X, sY) \to F^dK_0(X)$ is an isomorphism.

The equivalence of (a)–(c) follow, completing the proof. □

REMARK 1.8. Suppose that the desingularisation $X' \to X$ does not change the smooth locus of X and that E is equal to the exceptional fibre (this is probably the most important case of the conjecture). Then Theorem 1.7 states that the associated Bloch–Srinivas conjecture is true if and only if $F^dK_0(X, rY) \overset{\simeq}{\twoheadrightarrow} F^dK_0(X)$ for $r \gg 1$, where $Y = (X_{\mathrm{sing}})_{\mathrm{red}}$.
In particular, under these additional hypotheses on X' and E we see that the Bloch–Srinivas conjecture depends only on X, and not on the chosen desingularisation. Even in the case of arbitrary desingularisations and general E covering the exceptional set, Theorem 1.7 shows that the associated Bloch–Srinivas conjecture depends only on X and $\pi(|E|)$.

REMARK 1.9. The proof of Theorem 1.7 also shows the following: the inverse system $F^dK_0(X', rE)$, $r \geq 1$, stabilises if and only if the inverse system $F^dK_0(X, rY)$, $r \geq 1$, stabilises, in which case the canonical map $F^dK_0(X, rY) \to F^dK_0(X', rE)$ is an isomorphism for $r \gg 1$.

The following corollary recovers all previously known cases of the Bloch–Srinivas conjecture (normal surfaces [9, Thm. 1.1]; Cohen–Macaulay varieties with isolated singularities in characteristic zero [6, Thm. 1.1] [7, Thm. 1.2]; note that in these cases one can use the reduction ideal trick of Weibel [20] to avoid assuming that k has resolution of singularities, c.f., Remark 2.8):

COROLLARY 1.10. *Let X be a d-dimensional, integral variety over a good field k; let $\pi : X' \to X$ be a desingularisation, and $E \hookrightarrow X'$ any reduced closed subscheme covering the exceptional set. Assume $\pi(|E|)$ is finite and $d \geq 2$. Then the associated Bloch–Srinivas conjecture is true.*

Proof. Set $Y := \pi(|E|)_{\mathrm{red}}$. According to Theorem 1.7, it is necessary and sufficient to show that the canonical map $F^dK_0(X, rY) \to F^dK_0(X)$ is an isomorphism for all $r \geq 1$. But this follows from [6, Lem. 3.1] since rY is zero dimensional. □

The next corollary proves the Bloch–Srinivas conjecture under the assumption that the cycle class homomorphism $CH_0(X) \to F^dK_0(X)$ is an isomorphism:

Corollary 1.11. *Let X be a d-dimensional, integral, quasi-projective variety over a good field k; let $\pi : X' \to X$ be a desingularisation, and $E \hookrightarrow X'$ any reduced closed subscheme covering the exceptional set. Assume* $\operatorname{codim}(X, \pi(|E|)) \geq 2$ *and that the cycle class map $CH_0(X) \to F^dK_0(X)$ is an isomorphism.*
Then the associated Bloch–Srinivas conjecture is true.

Proof. Set $Y = \pi(|E|)_{\mathrm{red}}$. According to Theorem 1.7, it is necessary and sufficient to show that the canonical map $F^dK_0(X, rY) \to F^dK_0(X)$ is an isomorphism for all $r \geq 1$. To prove this we consider the commutative diagram

$$\begin{tikzcd} F^dK_0(X, rY) \arrow[r] & F^dK_0(X) \\ CH_0(X; |Y|) \arrow[r] \arrow[u] & CH_0(X) \arrow[u] \end{tikzcd}$$

The right vertical arrow is an isomorphism by assumption, the bottom horizontal arrow is an isomorphism by Remark 1.2(ii), and the left vertical arrow is a surjection since the domain and codomain are generated by the closed points of $X \setminus Y$. It follows that the top horizontal arrow (and left vertical arrow – we will need this in the proof of Theorem 2.5) is an isomorphism, as desired. □

In particular, we have proved the Bloch–Srinivas conjecture for projective varieties over an algebraically closed field of characteristic zero which are regular in codimension one:

Corollary 1.12. *Let X be a d-dimensional, integral variety over an algebraically closed field k which has strong resolution of singularities; let $\pi : X' \to X$ be a desingularisation, and $E \hookrightarrow X'$ any reduced closed subscheme covering the exceptional set. Assume* $\operatorname{codim}(X, \pi(|E|)) \geq 2$ *and that one of the following is true:*

(i) X *is projective and* $\operatorname{char} k = 0$*; or*
(ii) X *is projective and* $d \leq \operatorname{char} k \neq 0$*; or*
(iii) X *is affine and* $\operatorname{char} k$ *is arbitrary.*

Then the associated Bloch–Srinivas conjecture is true.

Proof. This follows from Corollary 1.11 and the results of Levine and Srinivas recalled in Theorem 1.4. □

Remark 1.13. It seems plausible that some descent or base change technique should eliminate the requirement in Corollary 1.12 that k be algebraically closed.

We can also solve the Bloch–Srinivas conjecture up to $(d-1)!$-torsion whenever X is regular in codimension one:

Corollary 1.14. *Let X be a d-dimensional, integral, quasi-projective variety over a good field k; let $\pi : X' \to X$ be a desingularisation, and $E \hookrightarrow X'$ any reduced closed subscheme covering the exceptional set. Assume* $\operatorname{codim}(X, \pi(|E|)) \geq 2$.
Then the associated Bloch–Srinivas conjecture is true up to $(d-1)!$-torsion, i.e., the maps

$$BS_r : F^d K_0(X', rE) \otimes \mathbb{Z}[\tfrac{1}{(d-1)!}] \longrightarrow F^d K_0(X) \otimes \mathbb{Z}[\tfrac{1}{(d-1)!}]$$

are isomorphisms for $r \gg 1$.

Proof. Set $Y = \pi(|E|)_{\mathrm{red}}$. By a trivial modification of Theorem 1.7, it is necessary and sufficient to show that the canonical map $F^d K_0(X, rY) \to F^d K_0(X)$ is an isomorphism for all $r \geq 1$ after inverting $(d-1)!$. This follows exactly as in Corollary 1.11, since the cycle class map $CH_0(X) \to F^d K_0(X)$ is an isomorphism after inverting $(d-1)!$, thanks to the existence of Levine Chern class $ch_0 : F^d K_0(X) \to CH_0(X)$. □

The next result solves the Bloch–Srinivas conjecture up to a finite group when the singular locus X_{sing} has codimension ≥ 2 and is contained in an affine open of X. Note that the "obvious" cases in which this happens, namely when X_{sing} is finite or X itself is affine, are already largely covered by Corollaries 1.10 and 1.12(iii) respectively:

Corollary 1.15. *Let X be a d-dimensional, integral, quasi-projective variety over an algebraically closed field k which has strong resolution of singularities; let $\pi : X' \to X$ be a desingularisation, and $E \hookrightarrow X'$ any reduced closed subscheme covering the exceptional set. Assume* $\operatorname{codim}(X, \pi(|E|)) \geq 2$*, that X_{sing} is contained in an affine open of X, and moreover that $d \leq \operatorname{char} k$ if $\operatorname{char} k \neq 0$. Then the maps*

$$BS_r : F^d K_0(X', rE) \longrightarrow F^d K_0(X)$$

are surjective with finite kernel for $r \gg 1$, and the inverse system $F^d K_0(X', rE)$, $r \geq 1$, stabilises.

Proof. Set $Y = \pi(|E|)_{\mathrm{red}}$. We concatenate commutative diagrams we have already considered in Theorem 1.7 and Corollary 1.11:

$$\begin{tikzcd}
F^dK_0(X',rE) \arrow[dr, bend left, "BS_r"] & \\
F^dK_0(X,rY) \arrow[u, two heads] \arrow[r] & F^dK_0(X) \\
CH_0(X;|Y|) \arrow[u, two heads] \arrow[r, "\cong"] & CH_0(X) \arrow[u]
\end{tikzcd}$$

The left vertical arrows are surjective since the groups are generated by the closed points of $X \setminus Y = X' \setminus E$; the bottom horizontal arrow is an isomorphism by Remark 1.2(ii); the right vertical arrow is surjective with finite kernel Λ by the result of Barbieri Viale recalled in Theorem 1.4.

A simple diagram chase shows that BS_r is surjective and that its kernel Λ_r is naturally a quotient of Λ. Since Λ is finite, this tower of quotients Λ_r must eventually stabilise, completing the proof. $\square$

Remark 1.16. We finish our discussion of the Bloch–Srinivas conjecture with a remark about SK_1. Let $\pi : X' \to X$, E, Y, k be as in the statement of Theorem 1.7, and assume X is quasi-projective and $\operatorname{codim}(X, Y) \geq 2$.

The maps $F^dK_0(X, rY) \to F^dK_0(X)$ are surjective for all $r \geq 1$ (by Remark 1.2(ii) and existence of the cycle class maps); hence we may add

(b′) The canonical map $F^dK_0(X, rY) \to F^dK_0(X)$ is injective for $r \gg 1$.

to the list of equivalent conditions in Theorem 1.7(ii).

Next, it follows from [6, Lem. 3.1] that (b′) (hence the associated Bloch–Srinivas conjecture) would follow from showing that $\partial(SK_1(rY)) = 0$, where $\partial : K_1(rY) \to K_0(X, rY)$ is the boundary map and $SK_1(rY) := \operatorname{Ker}(K_1(rY) \twoheadrightarrow H^0(rY, \mathcal{O}_{rY}^\times))$; equivalently, it is enough to show that $SK_1(X) \to SK_1(rY)$ is surjective. Using the arguments of Theorem 1.7 it would even be enough to show, for each $r \gg 1$, that

$$\operatorname{Im}(SK_1(sY) \to SK_1(rY)) \subseteq \operatorname{Im}(SK_1(X) \to SK_1(rY))$$

for some $s \geq r$. It is not clear whether one should expect this to be true.

We finish the section with some consequence of the Bloch–Srinivas conjecture. The following result about Chow groups of cones was conjectured by Srinivas [17, §3] in 1987; it was proved by Krishna [7, Thm. 1.5] under the assumption that the cone X was normal and Cohen–Macaulay, and we will combine his argument with Theorem 1.7 to establish the result in general; due to the failure of Kodaira vanishing in finite characteristic we must restrict to characteristic zero:

Theorem 1.17. *Let $Y \hookrightarrow \mathbb{P}^N_k$ be a d-dimensional, smooth, projective variety over an algebraically closed field k of characteristic zero; assume $d > 0$ and $H^d(Y, \mathcal{O}_Y(1)) = 0$, and let X be the affine cone over Y. Then $CH_0(X) = 0$.*

Proof. We may resolve X, which has a unique singular point, to obtain X' which is a line bundle over Y, whose zero section is the exceptional fibre of the resolution $X' \to X$. By Corollary 1.10 or 1.12(iii), we know that $CH_0(X) \cong F^{d+1}K_0(X', rY)$ for $r \gg 1$; moreover, $CH_0(X')$ surjects onto $F^dK_0(X')$, and $CH_0(X') = 0$ since X' is a line bundle, so $F^dK_0(X') = 0$. So it is enough to show that the canonical map $F^{d+1}K_0(X', rY) \to F^{d+1}K_0(X')$ is an isomorphism. According to Krishna's proof of [7, Cor. 8.5], this would follows from knowing that:

(i) $H^d(X', \mathcal{K}_{d,X'}) \otimes k^\times \longrightarrow H^d(Y, \mathcal{K}_{d,Y}) \otimes k^\times$ is surjective; and

(ii) $H^d\left(rY, \frac{\Omega^d_{(rY,Y)}}{d\Omega^{d-1}_{(rY,Y)}}\right) = 0$ for $r \gg 1$.

Condition (i) is satisfied since the zero section $Y \hookrightarrow X'$ is split by the line bundle structure map $X' \to Y$. Condition (ii) is deduced from the Akizuki–Nakano vanishing theorem, as explained in Lem. 9.1 and the proof of Thm. 1.5 in [7]. □

Corollary 1.18. *Let Y, k be as in the previous theorem, and let A be its homogeneous coordinate ring. Then every projective module over A of rank at least d has a free direct summand of rank one.*

Proof. This follows from Theorem 1.17 using a result of R. Murthy [15, Cor. 3.9]. □

Corollary 1.19. *Let k be an algebraically closed field of characteristic zero, and $f \in k[\underline{t}] := k[t_0, \dots, t_d]$ a homogenous polynomial of degree at most $d+1$ which defines a smooth hypersurface in $\mathbb{P}^d_k$. Then every smooth closed point of* $\operatorname{Spec} k[\underline{t}]/\langle f \rangle$ *is a complete intersection.*
In other words, if $\mathfrak{m}$ is any maximal ideal of $k[\underline{t}]$ containing f other than the origin, then $\mathfrak{m} = \langle f, f_1, \dots, f_d \rangle$ for some $f_1, \dots, f_d \in k[\underline{t}]$.

Proof. This also follows from Theorem 1.17 thanks to Murthy [15, Thm. 4.4]. □

2 Chow groups with modulus

If X is a smooth variety over a field k, and D is an effective divisor on X, then the Chow group $CH_0(X; |D|)$ from Definition 1.1 may be a rather coarse invariant, as there may not be enough curves on X avoiding the codimension-one subset $|D|$. Of greater interest is $CH_0(X; D)$, the Chow group of zero cycles on X with modulus D, which we will define precisely in Definition 2.1; note

the notational difference, indicating that $CH_0(X;D)$ depends not only on the support of D, but on its schematic, and possibly non-reduced, structure.
According to the higher dimensional class field theory of M. Kerz and S. Saito, when k is finite and X is proper over k, the group $CH_0(X;D)$ classifies the abelian étale covers of $X \setminus D$ whose ramification is bounded by D; we refer the reader to [5] for details since we will not require any of their results.
We now turn to definitions, and refer again to [op. cit.] for a more detailed exposition. Let C be a smooth curve over a field k, and D an effective divisor on C; writing $D = \sum_{x\in|D|} m_x x$ as a Weil divisor, we let

$$k(C)^\times_D := \{f \in k(C)^\times : \mathrm{ord}_x(f-1) \geq m_x \text{ for all } x \in |D|\}$$

denote the rational functions on C which are $\equiv 1 \bmod D$. More generally, if X is a smooth variety over k and D is an effective divisor on X, then for any curve $C \hookrightarrow X$ which is not contained in $|D|$ we write

$$k(C)^\times_D := k(\widetilde{C})^\times_{\phi^* D},$$

where $\phi : \widetilde{C} \to C \hookrightarrow X$ is the resulting map from the normalisation $\widetilde{C}$ to X; evidently $k(C)^\times_D = k(C)^\times$ if C does not meet $|D|$.
The Chow group with modulus is defined as follows:

Definition 2.1. Let X be a smooth variety over k, and D an effective divisor on X. Then the associated *Chow group of zero cycles of X with modulus D* is

$$CH_0(X;D) := \frac{\text{free abelian group on closed points of } X \setminus D}{\langle (f)_C \ : \ C \hookrightarrow X \text{ a curve not contained in } |D|, \text{ and } f \in k(C)^\times_D \rangle}$$

where $(f)_C = \sum_{x\in C_0} \mathrm{ord}_x(f)\, x$.

If we were to define

$$k(C)^\times_{|D|} := \begin{cases} k(C)^\times & \text{if } C \text{ does not meet } |D|, \\ 1 & \text{if } C \text{ meets } |D|, \end{cases}$$

and repeat Definition 2.1 with $|D|$ in place of D, then the resulting group $CH_0(X;|D|)$ would coincide with that defined in Definition 1.1. Since $k(C)^\times_{|D|} \subseteq k(C)^\times_D$, we thus obtain a canonical surjection

$$CH_0(X;|D|) \twoheadrightarrow CH_0(X;D).$$

One sense in which $CH_0(X;D)$ is a more refined invariant than $CH_0(X;|D|)$ is that the cycle class homomorphism $CH_0(X;|D|) \to K_0(X,D)$ of Section 1.1 factors through $CH_0(X;D)$. There does not appear to be a proof of this important result in the literature, so we give one here, beginning with a much stronger result in the case of curves:

Lemma 2.2. *Let C be a smooth curve over a field k, and D an effective divisor on C. Then the canonical map*

$$\text{free abelian group on closed points of } C \setminus D \longrightarrow K_0(C,D), \quad x \longmapsto [x]$$

induces an injective cycle class homomorphism

$$CH_0(C;D) \longrightarrow K_0(C,D),$$

which is an isomorphism if $D \neq 0$ (and has cokernel $= \mathbb{Z}$ if $D = 0$).

Proof. The Zariski descent spectral sequence for the K-theory of C relative to D degenerates to short exact sequences, since $\dim C = 1$, yielding in particular

$$0 \longrightarrow H^1(C, \mathcal{K}_{1,(C,D)}) \longrightarrow K_0(C,D) \longrightarrow H^0(C, \mathcal{K}_{0,(C,D)}) \longrightarrow 0.$$

Here $\mathcal{K}_{i,(C,D)}$ is by definition the Zariski sheafification on C of the presheaf $U \mapsto K_i(U, U \times_C D)$.
To describe these terms further we make some standard comments about the long exact sequence of sheaves

$$\mathcal{K}_{2,C} \to \mathcal{K}_{2,D} \to \mathcal{K}_{1,(C,D)} \to \mathcal{K}_{1,C} \to \mathcal{K}_{1,D} \to \mathcal{K}_{0,(C,D)} \to \mathcal{K}_{0,C} \to \mathcal{K}_{0,D}.$$

Firstly, $\mathcal{K}_{1,C} \cong \mathcal{O}_C^\times$ and $\mathcal{K}_{1,D} \cong \mathcal{O}_D^\times$, so the map $\mathcal{K}_{1,C} \to \mathcal{K}_{1,D}$ is surjective; moreover, the sheaves $\mathcal{K}_{2,C}$ and $\mathcal{K}_{2,D}$ are generated by symbols, and so the map $\mathcal{K}_{2,C} \to \mathcal{K}_{2,D}$ is also surjective. It follows that $\mathcal{K}_{1,(C,D)} \cong \operatorname{Ker}(\mathcal{O}_C^\times \to \mathcal{O}_D^\times) =: \mathcal{O}_{(C,D)}^\times$ and that $H^0(C, \mathcal{K}_{0,(C,D)}) = \operatorname{Ker}(H^0(C, \mathcal{K}_{0,C}) \to H^0(D, \mathcal{K}_{0,D}))$. Secondly, $\mathcal{K}_{0,C} \cong \mathbb{Z}$ via the rank map, and so $H^0(C, \mathcal{K}_{0,C}) \cong \mathbb{Z}$; similarly, $H^0(D, \mathcal{K}_{0,D}) \cong \bigoplus_{x \in |D|} \mathbb{Z}$ via the rank map. If $D \neq 0$, we deduce that the map $H^0(C, \mathcal{K}_{0,C}) \to H^0(D, \mathcal{K}_{0,D})$ is injective and so $H^0(C, \mathcal{K}_{0,(C,D)}) = 0$; while if $D = 0$ then evidently $H^0(X, \mathcal{K}_{0,(C,D)}) = H^0(X, \mathcal{K}_{0,C}) \cong \mathbb{Z}$.
In conclusion, it remains only to construct the cycle class isomorphism

$$CH_0(C;D) \xrightarrow{\simeq} H^1(C, \mathcal{O}_{(C,D)}^\times).$$

We will do this via a standard Gersten resolution.
Given an open subscheme $U \subseteq C$ containing $|D|$, let $j_U : U \to C$ denote the open inclusion. Then the canonical map $\mathcal{O}_{(C,D)}^\times \to j_{U*} j_U^* \mathcal{O}_{C,D}^\times$ fits into an exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_{(C,D)}^\times \longrightarrow j_{U*} j_U^* \mathcal{O}_{(C,D)}^\times \xrightarrow{(\mathrm{ord}_x)_x} \bigoplus_{x \in C \setminus U} i_{x*}\mathbb{Z} \longrightarrow 0,$$

where $i_{x*}\mathbb{Z}$ is a skyscraper sheaf at the closed point x. This remains exact after taking the filtered colimit over all open U containing $|D|$, yielding

$$0 \longrightarrow \mathcal{O}_{(C,D)}^\times \longrightarrow k(C)_D^\times \xrightarrow{(\mathrm{ord}_x)_x} \bigoplus_{x \in C_0 \setminus D} i_{x*}\mathbb{Z} \longrightarrow 0,$$

where $k(C)_D^\times$ denotes a constant sheaf by abuse of notation. This latter sequence is a flasque resolution of $\mathcal{O}^\times_{(C,D)}$, and using it to compute cohomology yields a natural isomorphism

$$\operatorname{coker}\big(k(C)_D^\times \xrightarrow{(\operatorname{ord}_x)_x} \bigoplus_{x\in C_0\setminus D}\mathbb{Z}\big) \xrightarrow{\simeq} H^1(C,\mathcal{O}^\times_{(C,D)}).$$

But the left side of this isomorphism is precisely $CH_0(C;D)$, thereby completing the proof. ☐

Proposition 2.3. *Let X be a smooth variety over a field k, and D an effective divisor on X. Then the canonical map*

$$\text{free abelian group on closed points of } X\setminus D \longrightarrow K_0(X,D), \quad x\longmapsto [x]$$

descends to a cycle class homomorphism

$$CH_0(X;D)\longrightarrow K_0(X,D).$$

Proof. We must show that if $C\hookrightarrow X$ is a curve not contained in $|D|$ and $f\in k(C)_D^\times$, then $\sum_{x\in C_0}\operatorname{ord}_x(f)[x]=0$ in $K_0(X,D)$. We will deduce this from Lemma 2.2 once we have verified a suitable pushforward formalism.
Let $\phi:\widetilde{C}\to C\hookrightarrow X$ be the resulting map from the normalisation $\widetilde{C}$ to X, and consider the following pullback square:

$$\begin{tikzcd} \phi^*D \arrow[r, "j'"] \arrow[d, "\phi'"'] & \widetilde{C} \arrow[d, "\phi"] \\ D \arrow[r, "j"'] & X \end{tikzcd}$$

We claim that ϕ and j are Tor-independent; that is, if y is a closed point of $\widetilde{C}$ such that $x:=\phi(y)$ lies in $|D|$, we must show that $\operatorname{Tor}^i_{\mathcal{O}_{X,x}}(\mathcal{O}_{D,x},\mathcal{O}_{\widetilde{C},y})=0$ for all $i>0$. But since D is an effective Cartier divisor, there exists a non-zero-divisor $t\in\mathcal{O}_{X,x}$ such that $\mathcal{O}_{D,x}=\mathcal{O}_{X,x}/t\mathcal{O}_{X,x}$; thus the only possible non-zero higher Tor is Tor^1, which equals the $\phi^*(t)$-torsion of $\mathcal{O}_{\widetilde{C},y}$; this could only be non-zero if $\phi^*(t)=0$ in $\mathcal{O}_{\widetilde{C},y}$, but this would contradict the condition that C does not lie in $|D|$. This proves the desired Tor-independence.
Moreover, ϕ is a finite morphism and X is assumed to be smooth, whence ϕ is proper and of finite Tor-dimension. Therefore the projection formula [19, Prop. 3.18] (or [4, Thm. 4.4]) states that the diagram

$$\begin{tikzcd} K(\widetilde{C}) \arrow[r, "j'^*"] \arrow[d, "\phi_*"'] & K(\phi^*D) \arrow[d, "\phi'_*"] \\ K(X) \arrow[r, "j^*"'] & K(D) \end{tikzcd}$$

is well-defined and commutes up to homotopy; so there is an induced pushforward map
$$\phi_* : K(\widetilde{C}, \phi^* D) \longrightarrow K(X, D),$$
which by functoriality of pushforwards (as in Section 1.1 we must appeal to [4, §4–5] to know that the obvious choices of homotopies yield a functorial construction) satisfies $\phi_*[x] = [\phi(x)]$ for any $x \in \widetilde{C}_0$. Therefore
$$\begin{aligned}\sum_{x \in C_0} \mathrm{ord}_x(f)[x] &= \sum_{x \in \widetilde{C}_0} \mathrm{ord}_{\phi(x)}(f)[\phi(x)] \\ &= \phi_*\Big(\sum_{x \in \widetilde{C}_0} \mathrm{ord}_x(f)[x]\Big) \\ &= \phi_*(0) \\ &= 0,\end{aligned}$$
where $\sum_{x \in \widetilde{C}_0} \mathrm{ord}_x(f)[x] \in K_0(\widetilde{C}, \phi^* D)$ vanishes by Lemma 2.2. □

Remark 2.4. F. Binda [3] has independently proved Proposition 2.3, as well as constructing cycle class homomorphisms $CH_0(X; D; n) \to K_n(X, D)$ for the higher Chow groups with modulus.

Let X be a d-dimensional, smooth variety over k. Given effective divisors $D' \geq D$ with the same support, the inclusions $k(C)^\times_{D'} \subseteq k(C)^\times_D$ induce a canonical surjection $CH_0(X; D') \twoheadrightarrow CH_0(X; D)$. This applies in particular when $D' = rD$ is a thickening of D. Combining this observation with Proposition 2.3 we obtain a commutative diagram of inverse systems of Chow groups and relative K-groups (recall the definition of $F^d K_0$ from Section 1.1) in which all maps are surjective (since every group is generated by the closed points of $X \setminus D$):

$$\begin{tikzcd}
F^d K_0(X, D) & F^d K_0(X, 2D) \arrow[l, two heads] & F^d K_0(X, 3D) \arrow[l, two heads] & F^d K_0(X, 4D) \arrow[l, two heads] & \cdots \arrow[l, two heads] \\
CH_0(X; D) \arrow[u, two heads] & CH_0(X; 2D) \arrow[l, two heads] \arrow[u, two heads] & CH_0(X; 3D) \arrow[l, two heads] \arrow[u, two heads] & CH_0(X; 4D) \arrow[l, two heads] \arrow[u, two heads] & \cdots \arrow[l, two heads] \\
 & & CH_0(X; |D|) \arrow[ull, two heads] \arrow[ul, two heads] \arrow[u, two heads] \arrow[ur, two heads] \arrow[urr, two heads, dashed] & &
\end{tikzcd}$$

There are two natural questions to consider concerning this diagram. Firstly, a question seemingly related to a conjecture of Kerz and Saito [5, Qu. V] is whether the cycle class homomorphism
$$\{CH_0(X; rD)\}_r \longrightarrow \{F^d K_0(X; rD)\}_r$$
is an isomorphism of pro abelian groups, perhaps at least ignoring $(d-1)!$-torsion.

Secondly, changing notation, now suppose that $X' \to X$ is a desingularisation of an integral variety X, whose exceptional fibre is an effective Cartier divisor D. Then, as a Chow-theoretic analogue of the Bloch–Srinivas conjecture, we ask whether the inverse system

$$CH_0(X';D) \longleftarrow CH_0(X';2D) \longleftarrow CH_0(X';3D) \longleftarrow \cdots$$

eventually stabilises, with stable value most likely equal to the Levine–Weibel Chow group $CH_0(X)$ of X.

The following theorem simultaneously answers cases of these two questions, working under almost identical hypotheses to Corollary 1.11:

Theorem 2.5. *Let X be a d-dimensional, integral, quasi-projective variety over a good field k; let $\pi : X' \to X$ be a desingularisation, and D any effective Cartier divisor on X whose support contains the exceptional set. Assume* $\operatorname{codim}(X, \pi(|D|)) \geq 2$ *and that the cycle class map $CH_0(X) \to F^d K_0(X)$ is an isomorphism.*
Then $CH_0(X) \cong CH_0(X';|D|)$, and the canonical maps

$$CH_0(X';|D|) \longrightarrow CH_0(X';rD) \longrightarrow F^d K_0(X';rD)$$

are isomorphisms for $r \gg 1$.

Proof. Let $Y \hookrightarrow X$ be the reduced closed subscheme with support $\pi(|D|)$; this has codimension ≥ 2 and covers X_{sing}. Consider the following commutative diagram, which exists for any $r \gg 1$:

$$\begin{tikzcd}
CH_0(X';|D|) \arrow[r] & CH_0(X';rD) \arrow[r] & F^d K_0(X';rD) \arrow[d, "BS_r"] \\
CH_0(X;|Y|) \arrow[u] \arrow[r] & CH_0(X) \arrow[r] & F^d K_0(X)
\end{tikzcd}$$

The bottom right horizontal arrow is an isomorphism by assumption; the bottom left horizontal arrow is an isomorphism by Remark 1.2(ii); the left vertical arrow is an isomorphism by Remark 1.2(iv); the right vertical arrow is an isomorphism by Corollary 1.11. Since the two top horizontal arrows are surjective, it follows that they are isomorphisms. $\square$

Corollary 2.6. *Let X be a d-dimensional, integral variety over an algebraically closed field k which has strong resolution of singularities; let $\pi : X' \to X$ be a desingularisation, and D any effective Cartier divisor on X whose support contains the exceptional set. Assume* $\operatorname{codim}(X, \pi(|D|)) \geq 2$ *and that one of the following is true:*

(i) *X is projective and* $\operatorname{char} k = 0$*; or*

(ii) *X is projective and* $d \leq \operatorname{char} k \neq 0$*; or*

(iii) *X is affine.*

Then $CH_0(X) \cong CH_0(X'; |D|)$, and the canonical maps

$$CH_0(X'; |D|) \longrightarrow CH_0(X'; rD) \longrightarrow F^d K_0(X'; rD)$$

are isomorphisms for $r \gg 1$.

Proof. This follows from Theorem 2.5 and the results of Levine and Srinivas recalled in Theorem 1.4. □

Remark 2.7 (Class field theory of singular varieties). In this remark we explain how the CH_0 isomorphism of Theorem 2.5 over a finite field $\mathbb{F}_q$ can be interpreted as part of an unramified class field theory for singular, projective varieties.

Let X be a projective variety over $\mathbb{F}_q$ which is regular in codimension one; suppose that a desingularisation $\pi : X' \to X$ exists, that D is an effective Cartier divisor on X whose support contains the exceptional set, and that $\operatorname{codim}(X, \pi(|D|)) \geq 2$. Write $U := X' \setminus D = X \setminus \pi(|D|)$.

The Kerz–Saito class group [5] of U is $C(U) := \varprojlim_r CH_0(X'; rD)$, and their class field theory provides a reciprocity isomorphism $C(U)^0 \xrightarrow{\simeq} \pi_1^{\mathrm{ab}}(U)^0$, where the superscripts 0 denote degree-0 subgroups. Assuming that the conclusions of Theorem 2.5 are true in this setting, we deduce that $C(U) = CH_0(X'; rD) \cong CH_0(X)$ for $r \gg 1$. Kerz–Saito prove moreover that each group $CH_0(X'; rD)^0$ is finite.

In particular, this would prove finiteness of $CH_0(X)^0$, which is known in the smooth case thanks to the unramified class field theory of S. Bloch, K. Kato and Saito, et al. It would also yield a reciprocity isomorphism

$$CH_0(X)^0 \xrightarrow{\simeq} \pi_1^{\mathrm{ab}}(U)^0, \quad [x] \mapsto \mathrm{Frob}_x$$

However, since the canonical map $\pi_1^{\mathrm{ab}}(U) \to \pi_1^{\mathrm{ab}}(X)$ is surjective but generally not an isomorphism, we would obtain in general only a surjective reciprocity map

$$CH_0(X)^0 \longrightarrow \pi_1^{\mathrm{ab}}(X)^0,$$

indicating that the Levine–Weibel Chow group $CH_0(X)$ is not the correct class group for unramified class field theory of a singular variety.

Remark 2.8 (The case of surfaces). If X is an integral, projective surface over $\mathbb{F}_q$ which is regular in codimension one, then we have actually proved the observations of Remark 2.7 unconditionally: $CH_0(X)$ is isomorphic to the Kerz–Saito class group $C(X_{\mathrm{reg}})$, its degree-0 subgroup is finite, and there is a reciprocity isomorphism

$$CH_0(X)^0 \xrightarrow{\simeq} \pi_1^{\mathrm{ab}}(X_{\mathrm{reg}})^0$$

of finite groups. This was brought to the author's attention by [8], in which Krisha reproduced the argument while being unaware of the present paper.

To prove this we must only check that Theorem 2.5 is true for surfaces over finite fields. In fact, we will let X be a 2-dimensional, integral, quasi-projective variety over an *arbitrary* field k which is regular in codimension one. Then X admits a resolution of singularities $\pi : X' \to X$ with exceptional set equal to exactly $\pi^{-1}(X_{\mathrm{sing}})$; let $E := \pi^{-1}(X_{\mathrm{sing}})_{\mathrm{red}}$ and $Y := (X_{\mathrm{sing}})_{\mathrm{red}}$.
Then Theorem 1.7 is true for the data $X' \to X$, Y, E. Indeed, it is only necessary to establish the isomorphism (†) occurring in the proof, which may be broken into the two isomorphisms

$$\{F^d K_0(X, rY)\}_r \xrightarrow{\cong} \{F^d K_0(\widetilde{X}, \widetilde{X} \times_X rY)\}_r \xrightarrow{\cong} \{F^d K_0(X', rE)\}_r,$$

where $\widetilde{X} \to X$ denotes the normalisation of X. The second of these isomorphisms is due to Krishna and Srinivas [9, Thm. 1.1]; the first isomorphism follows from the isomorphism $\{K_0(X, rY)\}_r \xrightarrow{\cong} \{K_0(\widetilde{X}, \widetilde{X} \times_X rY)\}_r$, which is a case of the author's pro-excision theorem [14, Corol. 0.4 & E.g. 2.5], and the obvious surjectivity just as in the proof of Theorem 1.7.
Now assume further (perhaps after blowing-up X' at finitely many points) that there is an effective divisor D on X' with support $\pi^{-1}(X_{\mathrm{sing}})$. Since the cycle class map $CH_0(X) \to F^d K_0(X)$ is automatically an isomorphism (as we remarked immediately before Theorem 1.4), it follows that the assertions of Theorem 2.5 are also true, as required: $CH_0(X) \cong CH_0(X'; |D|)$, and the canonical maps $CH_0(X'; |D|) \to CH_0(X'; rD) \to F^d K_0(X'; rD)$ are isomorphisms for $r \gg 1$.

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