K-Theory as an Eilenberg-Mac Lane Spectrum

To Sasha Merkurjev, on the occasion of his 60 th birthday

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Abstract. For an additive Waldhausen category linear over a ring k, the corresponding K-theory spectrum is a module spectrum over the K-theory spectrum of k. Thus if k is a finite field of characteristic p, then after localization at p, we obtain an Eilenberg-Mac Lane spectrum – in other words, a chain complex. We propose an elementary and direct construction of this chain complex that behaves well in families and uses only methods of homological algebra (in particular, the notions of a ring spectrum and a module spectrum are not used).

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Introduction.

Various homology and cohomology theories in algebra or algebraic geometry usually take as input a ring A or an algebraic variety X, and produce as output a certain chain complex; the homology groups of this chain complex are by definition the homology or cohomology groups of A or X. Higher algebraic K-groups are very different in this respect – by definition, the groups $K_\bullet(A)$ are homotopy groups of a certain spectrum $\mathcal{K}(A)$. Were it possible to represent $K_\bullet(A)$ as homology groups of a chain complex, one would be able to study it by means of the well-developed and powerful machinery of homological algebra. However, this is not possible: the spectrum $\mathcal{K}(A)$ is almost never a spectrum of the Eilenberg-Mac Lane type.

If one wishes to turn $\mathcal{K}(A)$ into an Eilenberg-Mac Lane spectrum, one needs to complete it or to localize it in a certain set of primes. The cheapest way to do it is of course to localize in *all* primes – rationally, every spectrum is an Eilenberg-Mac Lane spectrum, and the difference between spectra and complexes disappears. The groups $K_\bullet(A) \otimes \mathbb{Q}$ are then the primitive elements in the homology groups $H_\bullet(BGL_\infty(A), \mathbb{Q})$, and this allows for some computations using homological methods. In particular, $K_\bullet(A) \otimes \mathbb{Q}$ has been computed by Borel when A is a number field, and the relative K-groups $K_\bullet(A, I) \otimes \mathbb{Q}$ of a $\mathbb{Q}$-algebra A with respect to a nilpotent ideal $I \subset A$ have been computed in full generality by Goodwillie [Go].

However, there is at least one other situation when $\mathcal{K}(A)$ becomes an Eilenberg-Mac Lane spectrum after localization. Namely, if A is a finite field k of characteristic p, then by a famous result of Quillen [Q], the localization $\mathcal{K}(A)_{(p)}$ of the spectrum $\mathcal{K}(A)$ at p is the Eilenberg-Mac Lane spectrum $H(\mathbb{Z}_{(p)})$ corresponding to the ring $\mathbb{Z}_{(p)}$. Moreover, if A is an algebra over k, then $\mathcal{K}(A)$ is a module spectum over $\mathcal{K}(k)$ by a result of Gillet [Gi]. Then $\mathcal{K}(A)_{(p)}$ is a module spectrum over $H(\mathbb{Z}_{(p)})$, thus an Eilenberg-Mac Lane spectrum corresponding in the standard way ([Sh, Theorem 1.1]) to a chain complex $K_\bullet(A)_{(p)}$ of $\mathbb{Z}_{(p)}$-modules. More generally, if we have a k-linear exact or Waldhausen category $\mathcal{C}$, the p-localization $\mathcal{K}(\mathcal{C})_{(p)}$ of the K-theory spectrum $\mathcal{K}(\mathcal{C})$ is also an Eilenberg-Mac Lane spectrum corresponding to a chain complex $K_\bullet(\mathcal{C})_{(p)}$. Moreover, if we have a nilpotent ideal $I \subset A$ in a k-algebra A, then the relative K-theory spectrum $\mathcal{K}(A, I)$ is automatically p-local. Thus $\mathcal{K}(A, I) \cong \mathcal{K}(A, I)_{(p)}$ is an Eilenberg-Mac Lane spectrum "as is", without further modifications.

Unfortunately, unlike in the rational case, the construction of the chain complex $K_\bullet(\mathcal{C})_{(p)}$ is very indirect and uncanonical, so it does not help much in practical computations. One clear deficiency is insufficient functoriality of the construction that makes it difficult to study its behaviour in families. Namely, a convenient axiomatization of the notion of a family of categories indexed by a small category $\mathcal{C}$ is the notion of a cofibered category $\mathcal{C}'/\mathcal{C}$ introduced in [Gr]. This is basicaly a functor $\pi : \mathcal{C}' \to \mathcal{C}$ satisfying some conditions; the conditions insure that for every morphism $f : c \to c'$ in $\mathcal{C}'$, one has a natural transition functor $f_! : \pi^{-1}(c) \to \pi^{-1}(c')$ between fibers of the cofibration π. Cofibration also behave nicely with respect to pullbacks – for any cofibered category $\mathcal{C}'/\mathcal{C}$ and any functor $\gamma : \mathcal{C}_1 \to \mathcal{C}$, we have the induced cofibration $\gamma^*\mathcal{C}' \to \mathcal{C}_1$. Within the context of algebraic K-theory, one would like to start with a cofibration $\pi : \mathcal{C}' \to \mathcal{C}$ whose fibers $\pi^{-1}(c)$, $c \in \mathcal{C}$, are k-linear additive categories, or maybe k-linear exact or Waldhausen categories, and one would like to pack the individual complexes $K_\bullet(\pi^{-1}(c))_{(p)}$ into a single object

$$K(\mathcal{C}'/\mathcal{C})_{(p)} \in \mathcal{D}(\mathcal{C}, \mathbb{Z}_{(p)})$$

in the derived category $\mathcal{D}(\mathcal{C}, \mathbb{Z}_{(p)})$ of the category of functors from $\mathcal{C}$ to $\mathbb{Z}_{(p)}$-modules. One would also like this construction to be functorial with respect to pullbacks, so that for any functor $\gamma : \mathcal{C}_1 \to \mathcal{C}$, one has a natural isomorphism

$$\gamma^* K(\mathcal{C}'/\mathcal{C})_{(p)} \cong K(\gamma^*\mathcal{C}'/\mathcal{C}_1)_{(p)}.$$

In order to achieve this by the usual methods, one has to construct the chain complex $K_\bullet(\mathcal{C})_{(p)}$ in such a way that it is exactly functorial in $\mathcal{C}$. This is probably possible but extremely painful.

The goal of this paper, then, is to present an alternative very simple construction of the objects $K(\mathcal{C}'/\mathcal{C})_{(p)} \in \mathcal{D}(\mathcal{C}, \mathbb{Z}_{(p)})$ that only uses direct homological methods, without any need to even introduce the notion of a ring spectrum. The only thing we need to set up the construction is a commutative ring k and a localization R of the ring $\mathbb{Z}$ in a set of primes S such that $K_i(k) \otimes R = 0$ for $i \geq 1$, and $K_0(k) \otimes R \cong R$. Starting from these data, we produce a family of objects $K^R(\mathcal{C}'/\mathcal{C}) \in \mathcal{D}(\mathcal{C}, R)$ with the properties listed above, and such that if $\mathcal{C}$ is the point category pt, then $K^R(\mathcal{C}'/\mathsf{pt})$ is naturally identified with the K-theory spectrum $\mathcal{K}(\mathcal{C}')$ localized in S.

Although the only example we have in mind is k a finite field of characteristic p, $R = \mathbb{Z}_{(p)}$, we formulate things in bigger generality to emphasize the essential ingredients of the construction. We do not need any information on how the isomorphism $K_0(k) \otimes R \cong R$ comes about, nor on why the higher K-groups vanish. As our entry point to algebraic K-theory, we use the formalism of Waldhausen categories, since it is the most general one available. However, were one to wish to use, for example, Quillen's Q-construction, everything would work with minimal modifications.
Essentially, our approach is modeled on the approach to Topological Hochschild Homology pioneered by M. Jibladze and T. Pirashvili [JP]. The construction

itself is quite elementary. The underlying idea is also rather transparent and would work in much larger generality, but at the cost of much more technology to make things precise. Thus we have decided to present both the idea and its implementation but to keep them separate. In Section 1, we present the general idea of the construction, without making any mathematical statements precise enough to be proved. The rest of the paper is completely indepedent of Section 1. A rather long Section 2 contains the list of preliminaries; everything is elementary and well-known, but we need to recall these things to set up the notation and make the paper self-contained. A short explanation of what is needed and why is contained in the end of Section 1. Then Section 3 gives the exact statement of our main result, Theorem 3.4, and Section 4 contains its proof.

Acknowledgements. It is a pleasure and an honor to dedicate the paper to Sasha Merkurjev, as a birthday present. This is my first attempt to prove anything in algebraic K-theory, a subject I have always regarded with a lot of respect and a bit of trepidation, and if the resulting paper amuses him, I will be very happy. I am grateful to the referee for a thoughtful report and many useful suggestions.

1 Heuristics.

Assume given a commutative ring R, and let $M(R)$ be the category of finitely generated free R-modules. It will be useful to interpret $M(R)$ as the category of matrices: objects are finite sets S, morphisms from S to S' are R-valued matrices of size $S \times S'$.

Every R-module M defines a R-linear additive functor $\widetilde{M}$ from $M(R)$ to the category of R-modules by setting

$$\widetilde{M}(M_1) = \operatorname{Hom}_R(M_1^*, M) \tag{1.1}$$

for any $M_1 \in M(R)$, where we denote by $M_1^* = \operatorname{Hom}_R(M_1, R)$ the dual R-modules. This gives an equivalence of categories between the category R-mod of R-modules, and the category of R-linear additive functors from $M(R)$ to R-mod.

Let us now make the following observation. If we forget the R-module structure on M and treat it as a set, we of course lose information. However, if we do it pointwise with the functor $\widetilde{M}$, we can still recover the original R-module M. Namely, denote by $\operatorname{Fun}(M(R), R)$ the category of all functors from $M(R)$ to R-mod, without any additivity or linearity conditions, and consider the functor $R\text{-mod} \to \operatorname{Fun}(M(R), R)$ that sends M to $\widetilde{M}$. Then it has a left-adjoint functor

$$\mathsf{Add}_R : \operatorname{Fun}(M(R), R) \to R\text{-mod},$$

and for any $M \in R$-mod, we have

$$M \cong \mathsf{Add}_R(R[\widetilde{M}]), \tag{1.2}$$

where $R[\widetilde{M}] \in \mathrm{Fun}(M(R), R)$ sends $M_1 \in M(R)$ to the free R-module $R[\widetilde{M}(M_1)]$ generated by $\widetilde{M}(M_1)$. Indeed, by adjunction, Add_R commutes with colimits, so it suffices to check (1.2) for a finitely generated free R-module M; but then $R[\widetilde{M}]$ is a representable functor, and (1.2) follows from the Yoneda Lemma.

The functor Add_R also has a version with coefficients. If we have an R-algebra R', then for any R'-module M, the functor $\widetilde{M}$ defined by (1.1) is naturally a functor from $M(R)$ to R'-mod. Then by adjunction, we can define the functor

$$\mathsf{Add}_{R,R'} : \mathrm{Fun}(M(R), R') \to R'\text{-mod},$$

and we have an isomorphism

$$\mathsf{Add}_{R,R'}(R'[\widetilde{M}]) \cong M \otimes_R R' \tag{1.3}$$

for any flat R-module M.

What we want to do now is to obtain a homotopical version of the construction above. We thus replace sets with topological spaces. An abelian group structure on a set becomes an infinite loop space structure on a topological space; this is conveniently encoded by a special Γ-space of G. Segal [Se]. Abelian groups become connective spectra. Rings should become ring spectra. As far as I know, Segal machine does not extend directly to ring spectra – to describe ring spectra, one has to use more complicated machinery such as "functors with smash products", or an elaboration on them, ring objects in the category of symmetric spectra of [HSS]. However, in practice, if we are given a connective spectrum $\mathcal{X}$ represented by an infinite loop space X, then a ring spectrum structure on $\mathcal{X}$ gives rise to a multiplication map $\mu : X \times X \to X$, and in ideal situation, this is sufficiently associative and distributive to define a matrix category $\mathsf{Mat}(X)$ analogous to $M(R)$. This should be a small category enriched over topological spaces. Its objects are finite sets S, and the space of morphisms from S to S' is the space $X^{S \times S'}$ of X-valued matrices of size $S \times S'$, with compositions induced by the multiplication map $\mu : X \times X \to X$.

Ideal situations seem to be rare (the only example that comes to mind readily is a simplicial ring treated as an Eilenberg-Mac Lane ring spectrum). However, one might relax the conditions slightly. Namely, in practice, infinite loop spaces and special Γ-spaces often appear as geometric realizations of monoidal categories. The simplest example of this is the sphere spectrum $\mathcal{S}$. One starts with the groupoid $\overline{\Gamma}$ of finite sets and isomorphisms between them, one treats it as a monoidal category with respect to the disjoint union operation, and one produces a special Γ-space with underlying topological space $|\overline{\Gamma}|$, the geometric realization of the nerve of the category $\overline{\Gamma}$. Then by Barratt-Quillen Theorem, up to a stable homotopy equivalence, the corresponding spectrum is exactly $\mathcal{S}$. The sphere spectrum is of course a ring spectrum, and the multiplication operation μ also has a categorical origin: it is induced by the cartesian product functor $\overline{\Gamma} \times \overline{\Gamma} \to \overline{\Gamma}$. This functor is not associative or commutative on the nose, but it is associative and commutative up to canonical isomorphisms. The hypothetical matrix category $\mathsf{Mat}(|\overline{\Gamma}|)$ is then easily constructed as the geometric realization $|\mathcal{Q}\Gamma|$ of a strictification of a 2-category $\mathcal{Q}\Gamma$ whose objects are finite sets S, and whose category $\mathcal{Q}\Gamma(S,S')$ of morphisms from S to S' is the groupoid $\overline{\Gamma}^{S \times S'}$. Equivalently, $\mathcal{Q}\Gamma(S,S')$ is the category of diagrams

$$S \xleftarrow{\ l\ } \widetilde{S} \xrightarrow{\ r\ } S' \tag{1.4}$$

of finite sets, and isomorphisms between these diagrams. Compositions are obtained by taking pullbacks.

Any spectrum is canonically a module spectrum over $\mathcal{S}$. So, in line with the additivization yoga described above, we expect to be able to start with a connective spectrum $\mathcal{X}$ corresponding to an infinite loop space X, produce a functor $X_\bullet$ from $|\mathcal{Q}\Gamma|$ to topological spaces sending a finite set S to X^S, and then recover the infinite loop space structure on X from the functor $X_\bullet$.

This is exactly what happens – and in fact, we do not need the whole 2-category $\mathcal{Q}\Gamma$, it suffices to restrict our attention to the subcategory in $\mathcal{Q}\Gamma$ spanned by diagrams (1.4) with injective map l. Since such diagrams have no non-trivial automorphisms, this subcategory is actually a 1-category. It is equivalent to the category Γ_+ of pointed finite sets. Then restricting $X_\bullet$ to Γ_+ produced a functor from Γ_+ to topological spaces, that is, precisely a Γ-space in the sense of Segal. This Γ-space is automatically special, and one recovers the infinite loop space structure on X by applying the Segal machine.

It is also instructive to do the versions with coefficients, with R being the sphere spectrum, and R' being the Eilenberg-Mac Lane ring spectrum $H(A)$ corresponding to a ring A. Then module spectra over $H(A)$ are just complexes of A-modules, forming the derived category $\mathcal{D}(A)$ of the category A-mod, and functors from Γ_+ to $H(A)$-module spectra are complexes in the category $\operatorname{Fun}(\Gamma_+, A)$ of functors from Γ_+ to A-mod, forming the derived category $\mathcal{D}(\Gamma_+, A)$ of the abelian category $\operatorname{Fun}(\Gamma_+, A)$. One has a tautological functor from A-mod to $\operatorname{Fun}(\Gamma_+, A)$ sending an A-module M to $\widetilde{M} \in \operatorname{Fun}(\Gamma_+, A)$ given by $\widetilde{M}(S) = M[\overline{S}]$, where $\overline{S} \subset S$ is the complement to the distinguished element $o \in S$. This has a left-adjoint functor

$$\mathsf{Add} : \operatorname{Fun}(\Gamma_+, A) \to A\text{-mod},$$

with its derived functor $L^\bullet \mathsf{Add} : \mathcal{D}(\Gamma_+, A) \to \mathcal{D}(A)$. The role of the free A-module $A[S]$ generated by a set S is played by the singular chain complex $C_\bullet(X, A)$ of a topological space, and we expect to start with a special Γ-space $X_+ : \Gamma_+ \to \mathrm{Top}$, and obtain an analog of (1.3), namely, an isomorphism

$$L^\bullet \mathsf{Add}(C_\bullet(X_+, A)) \cong H_\bullet(\mathcal{X}, A),$$

where $H_\bullet(\mathcal{X}, A)$ are the homology groups of the spectum $\mathcal{X}$ corresponding to X_+ with coefficients in A (that is, homotopy groups of the product $\mathcal{X} \wedge A$).

Such an isomorphism indeed exists; we recall a precise statement below in Lemma 4.1.
Moreover, we can be more faithful to the original construction and avoid restricting to $\Gamma_+ \subset \mathcal{Q}\Gamma$. This entails a technical difficulty, since one has to explain what is a functor from the 2-category $\mathcal{Q}\Gamma$ to complexes of A-modules, and define the corresponding derived category $\mathcal{D}(\mathcal{Q}\Gamma, A)$. It can be done in several equivalent ways, see e.g. [Ka2, Section 3.1], and by [Ka2, Lemma 3.4(i)], the answer remains the same – we still recover the homology groups $H_\bullet(\mathcal{X}, A)$.

Now, the point of the present paper is the following. The K-theory spectrum $\mathcal{K}(k)$ of a commutative ring k also comes from a monoidal category, namely, the groupoid $\mathsf{Iso}(k)$ of finitely generated projective k-modules and isomorphisms between them. Moreover, the ring structure on $\mathcal{K}(k)$ also has categorical origin – it comes from the tensor product functor $\mathsf{Iso}(k) \times \mathsf{Iso}(k) \to \mathsf{Iso}(k)$. And if we have some k-linear Waldhausen category $\mathcal{C}$, then the infinite loop space corresponding to the K-theory spectrum $\mathcal{K}(\mathcal{C})$ is the realization of the nerve of a category $S\mathcal{C}$ on which $\mathsf{Iso}(k)$ acts. Therefore one can construct a 2-category $\mathsf{Mat}(k)$ of matrices over $\mathsf{Iso}(k)$, and $\mathcal{C}$ defines a 2-functor $\mathsf{Vect}(S\mathcal{C}) : \mathsf{Mat}(k) \to$ Cat to the 2-category Cat of small categories. At this point, we can forget all about ring spectra and module spectra, define an additivization functor

$$\mathsf{Add} : \mathcal{D}(\mathsf{Mat}(k), R) \to \mathcal{D}(R),$$

and use an analog of (1.3) to recover if not $\mathcal{K}(\mathcal{C})$ then at least $\mathcal{K}(\mathcal{C}) \wedge_{\mathcal{K}(k)} H(R)$, where $H(R)$ is the Eilenberg-Mac Lane spectrum corresponding to R. This is good enough: if R is the localization of $\mathbb{Z}$ in a set of primes S such that $\mathcal{K}(k)$ localized in S is $H(R)$, then $\mathcal{K}(\mathcal{C}) \wedge_{\mathcal{K}(k)} H(R)$ is the localization of $\mathcal{K}(\mathcal{C})$ in S.
The implementation of the idea sketched above requires some preliminaries. Here is a list. Subsection 2.1 discusses functor categories, their derived categories and the like; it is there mostly to fix notation. Subsection 2.2 recalls the basics of the Grothendieck construction of [Gr]. Subsection 2.3 contains some related homological facts. Subsection 2.4 recalls some standard facts about simplicial sets and nerves of 2-categories. Subsection 2.5 discusses 2-categories and their nerves. Subsection 2.6 constructs the derived category $\mathcal{D}(\mathcal{C}, R)$ of functors from a small 2-category $\mathcal{C}$ to the category of modules over a ring R; this material is slightly non-standard, and we have even included one statement with a proof. We use an approach based on nerves, since it is cleaner and does not require any strictification of 2-categories. Then we introduce the 2-categories we will need: Subsection 2.7 is concerned with the 2-category $\mathcal{Q}\Gamma$ and its subcategory $\Gamma_+ \subset \mathcal{Q}\Gamma$, while Subsection 2.8 explains the matrix 2-categories $\mathsf{Mat}(k)$ and the 2-functors $\mathsf{Vect}(\mathcal{C})$. Finally, Subsection 2.9 explains how the matrix and vector categories are defined in families (that is, in the relative setting, with respect to a cofibration in the sense of [Gr]).
Having finished with the preliminaries, we turn to our results. Section 3 contains a brief recollection on K-theory, and then the statement of the main result, Theorem 3.4. Since we do not introduce ring spectra, we cannot really

state that we prove a spectral analog of (1.3). Instead, we construct directly a map $\mathcal{K}(\mathcal{C}) \to \mathcal{K}$ to a certain Eilenberg-Mac Lane spectrum $\mathcal{K}$, and we prove that the map becomes an isomorphism after the appropriate localization. The actual proof is contained in Section 4.

2 Preliminaries.

2.1 Homology of small categories. For any two objects $c, c' \in \mathcal{C}$ in a category $\mathcal{C}$, we will denote by $\mathcal{C}(c,c')$ the set of maps from c to c'. For any category $\mathcal{C}$, we will denote by $\mathcal{C}^o$ the opposite category, so that $\mathcal{C}(c,c') = \mathcal{C}^o(c',c)$, $c,c' \in \mathcal{C}$. For any functor $\pi : \mathcal{C}_1 \to \mathcal{C}_2$, we denote by $\pi^o : \mathcal{C}_1^o \to \mathcal{C}_2^o$ the induced functor between the opposite categories.

For any small category $\mathcal{C}$ and ring R, we will denote by $\operatorname{Fun}(\mathcal{C}, R)$ the abelian category of functors from $\mathcal{C}$ to the category R-mod of left R-modules, and we will denote by $\mathcal{D}(\mathcal{C}, R)$ its derived category. The triangulated category $\mathcal{D}(\mathcal{C}, R)$ has a standard t-structure in the sense of [BBD] whose heart is $\operatorname{Fun}(\mathcal{C}, R)$. For any object $c \in \mathcal{C}$, we will denote by $R_c \in \operatorname{Fun}(\mathcal{C}, R)$ the representable functor given by

$$R_c(c') = R[\mathcal{C}(c,c')], \tag{2.1}$$

where for any set S, we denote by $R[S]$ the free R-module spanned by S. Every object $E \in \mathcal{D}(\mathcal{C}, R)$ defines a functor $\mathcal{D}(E) : \mathcal{C} \to \mathcal{D}(R)$ from $\mathcal{C}$ to the derived category $\mathcal{D}(R)$ of the category R-mod, and by adjunction, we have a quasiisomorphism

$$\mathcal{D}(E)(c) \cong \operatorname{RHom}^{\bullet}(R_c, E) \tag{2.2}$$

for any object $c \in \mathcal{C}$ (we will abuse notation by writing $E(c)$ instead of $\mathcal{D}(E)(c)$). Any functor $\gamma : \mathcal{C} \to \mathcal{C}'$ between small categories induces an exact pullback functor $\gamma^* : \operatorname{Fun}(\mathcal{C}', R) \to \operatorname{Fun}(\mathcal{C}, R)$ and its adjoints, the left and right Kan extension functors $\gamma_!, \gamma_* : \operatorname{Fun}(\mathcal{C}, R) \to \operatorname{Fun}(\mathcal{C}', R)$. The derived functors $L^{\bullet}\gamma_!, R^{\bullet}\gamma_* : \mathcal{D}(\mathcal{C}, R) \to \mathcal{D}(C', R)$ are left resp. right-adjoint to the pullback functor $\gamma^* : \mathcal{D}(C', R) \to \mathcal{D}(\mathcal{C}, R)$. The homology resp. cohomology of a small category $\mathcal{C}$ with coefficients in a functor $E \in \operatorname{Fun}(\mathcal{C}, R)$ are given by

$$H_i(\mathcal{C}, E) = L^i\tau_! E, \qquad H^i(\mathcal{C}, E) = R^i\tau_* E, \qquad i \geq 0,$$

where $\tau : \mathcal{C} \to \mathsf{pt}$ is the tautological projection to the point category pt.

Assume that the ring R is commutative. Then for any $E \in \operatorname{Fun}(\mathcal{C}, R)$, $T \in \operatorname{Fun}(\mathcal{C}^o, R)$, the *tensor product* $E \otimes_{\mathcal{C}} T$ is the cokernel of the natural map

$$\bigoplus_{f : c \to c'} E(c) \otimes_R T(c') \xrightarrow{E(f) \otimes \mathsf{id} - \mathsf{id} \otimes T(f)} \bigoplus_{c \in \mathcal{C}} E(c) \otimes_R T(c).$$

Sending E to $E \otimes_{\mathcal{C}} T$ gives a right-exact functor from $\operatorname{Fun}(\mathcal{C}, R)$ to R-mod; we denote its derived functors by $\operatorname{Tor}_i^{\mathcal{C}}(E, T)$, $i \geq 1$, and we denote by $E \overset{\mathsf{L}}{\otimes} T$ the derived tensor product. If $T(c)$ is a free R-module for any $c \in \mathcal{C}$, then $- \otimes_{\mathcal{C}} T$ is left-adjoint to an exact functor $\mathcal{H}om(T, -) : R\text{-mod} \to \operatorname{Fun}(\mathcal{C}, R)$ given by

$$\mathcal{H}om(T, E)(c) = \operatorname{Hom}(T(c), E), \qquad c \in \mathcal{C}, E \in R\text{-mod}. \tag{2.3}$$

Being exact, $\mathcal{H}om(T, -)$ induces a functor from $\mathcal{D}(R)$ to $\mathcal{D}(\mathcal{C}, R)$; this functor is right-adjoint to the derived tensor product functor $- \overset{\mathsf{L}}{\otimes}_{\mathcal{C}} T$. For example, if $T = R$ is the constant functor with value R, then we have

$$H_\bullet(\mathcal{C}, E) \cong \operatorname{Tor}^{\mathcal{C}}_\bullet(E, R)$$

for any $E \in \operatorname{Fun}(\mathcal{C}, R)$.

2.2 Grothendieck construction. A morphism $f : c \to c'$ in a category $\mathcal{C}'$ is called *cartesian* with respect to a functor $\pi : \mathcal{C}' \to \mathcal{C}$ if any morphism $f_1 : c_1 \to c'$ in $\mathcal{C}'$ such that $\pi(f) = \pi(f_1)$ factors uniquely as $f_1 = f \circ g$ for some $g : c_1 \to c$. A functor $\pi : \mathcal{C}' \to \mathcal{C}$ is a *prefibration* if for any morphism $f : c \to c'$ in $\mathcal{C}$ and object $c'_1 \in \mathcal{C}'$ with $\pi(c'_1) = c'$, there exists a cartesian map $f_1 : c_1 \to c'_1$ in $\mathcal{C}'$ with $\pi(f_1) = f$. A prefibration is a *fibration* if the composition of two cartesian maps is cartesian. A functor $F : \mathcal{C}' \to \mathcal{C}''$ between two fibrations $\mathcal{C}', \mathcal{C}''/\mathcal{C}$ is *cartesian* if it commutes with projections to $\mathcal{C}$ and sends cartesian maps to cartesian maps. For any fibration $\mathcal{C}' \to \mathcal{C}$, a subcategory $\mathcal{C}'_0 \subset \mathcal{C}'$ is a *subfibration* if the induced functor $\mathcal{C}'_0 \to \mathcal{C}$ is a fibration, and the embedding functor $\mathcal{C}'_0 \to \mathcal{C}'$ is cartesian over $\mathcal{C}$.

A fibration $\pi : \mathcal{C}' \to \mathcal{C}$ is called *discrete* if its fibers $\pi_c = \pi^{-1}(c)$, $c \in \mathcal{C}$ are discrete categories. For example, for any $c \in \mathcal{C}$, let $\mathcal{C}/c$ be the category of objects $c' \in \mathcal{C}$ equipped with a map $c' \to c$. Then the forgetful functor $\varphi : \mathcal{C}/c \to \mathcal{C}$ sending $c' \to c$ to c' is a discrete fibration, with fibers $\varphi_{c'} = \mathcal{C}(c', c)$, $c' \in \mathcal{C}$.

For any functor $F : \mathcal{C}^o \to \operatorname{Cat}$ to the category Cat of small categories, let $\operatorname{Tot}(F)$ be the category of pairs $\langle c, s \rangle$ of an object $c \in \mathcal{C}$ and an object $s \in F(c)$, with morphisms from $\langle c, s \rangle$ to $\langle c', s' \rangle$ given by a morphism $f : c \to c'$ and a morphism $s \to F(f)(s')$. Then the forgetful functor $\pi : \operatorname{Tot}(F) \to \mathcal{C}$ is a fibration, with fibers $\pi_c \cong F(c)$, $c \in \mathcal{C}$. If F is a functor to sets, so that for any $c \in \mathcal{C}$, $F(c)$ is a discrete category, then the fibration π is discrete.

Conversely, for any fibration $\pi : \mathcal{C}' \to \mathcal{C}$ with of small categories, and any object $c \in \mathcal{C}$, let $\operatorname{Gr}(\pi)(c)$ be the category of cartesian functors $\mathcal{C}/c \to \mathcal{C}'$. Then $\operatorname{Gr}(\pi)(c)$ is contravariantly functorial in c and gives a functor $\operatorname{Gr}(\pi) : \mathcal{C}^o \to \operatorname{Cat}$. The two constructions are inverse, in the sense that we have a natural cartesian equivalence $\operatorname{Tot}(\operatorname{Gr}(\pi)) \cong \mathcal{C}'$ for any fibration $\pi' : \mathcal{C}' \to \mathcal{C}$, and a natural pointwise equivalence $F \to \operatorname{Gr}(\operatorname{Tot}(F))$ for any $F : \mathcal{C}^o \to \operatorname{Cat}$. In particular, we have equivalences

$$\pi_c \cong \operatorname{Gr}(\pi)(c), \qquad c \in \mathcal{C}.$$

These equivalences of categories are not isomorphisms, so that the fibers π_c themselves do not form a functor from $\mathcal{C}^o$ to Cat – they only form a pseudofunctor in the sense of [Gr] (we do have a transition functor $f^* : \pi_{c'} \to \pi_c$

for any morphism $f : c \to c'$ in $\mathcal{C}$, but this is compatible with compositions only up to a canonical isomorphism). Nevertheless, for all practical purposes, a fibered category over $\mathcal{C}$ is a convenient axiomatization of the notion of a family of categories contravariantly indexed by $\mathcal{C}$.

For any fibration $\pi : \mathcal{C}' \to \mathcal{C}$ of small categories, and any functor $\gamma : \mathcal{C}_1 \to \mathcal{C}$ from a small category $\mathcal{C}_1$, we can define a category $\gamma^*\mathcal{C}'$ and a functor $\pi_1 : \gamma^*\mathcal{C}' \to \mathcal{C}_1$ by taking the cartesian square

$$\begin{CD} \gamma^*\mathcal{C}' @>{\gamma'}>> \mathcal{C}' \\ @V{\pi_1}VV @VV{\pi}V \\ \mathcal{C}_1 @>{\gamma}>> \mathcal{C} \end{CD} \tag{2.4}$$

in Cat. Then π_1 is also a fibration, called the *induced fibration*. The corresponding pseudofunctor $\mathrm{Gr}(\pi_1) : \mathcal{C}_1^o \to \mathrm{Cat}$ is the composition of the functor γ and $\mathrm{Gr}(\pi)$.

For covariantly indexed families, one uses the dual notion of a cofibration: a morphism f is cocartesian with respect to a functor π if it is cartesian with respect to π^o, a functor π is a cofibration if π^o is a fibration, a functor $F : \mathcal{C}' \to \mathcal{C}''$ between two cofibrations is cocartesian if F^o is cartesian, and a subcategory $\mathcal{C}'_0 \subset \mathcal{C}'$ is a subcofibration if $(\mathcal{C}'_0)^o \subset (\mathcal{C}')^o$ is a subfibration. The Grothendieck construction associates cofibrations over $\mathcal{C}$ to functors from $\mathcal{C}$ to Cat. We have the same notion of an induced cofibration. Functors to Sets $\subset$ Cat correspond to discrete cofibrations; the simplest example of such is the projection

$$\rho_c : c\backslash\mathcal{C} \to \mathcal{C} \tag{2.5}$$

for some object $c \in \mathcal{C}$, where $c\backslash\mathcal{C} = (\mathcal{C}^o/c)^o$ is the category of objects $c' \in \mathcal{C}$ equipped with a map $c \to c'$.

2.3 Base change. Assume given a cofibration $\pi : \mathcal{C}' \to \mathcal{C}$ of small categories and a functor $\gamma : \mathcal{C}_1 \to \mathcal{C}$, and consider the cartesian square (2.4). Then the isomorphism $\gamma'^* \circ \pi^* \cong \pi_1^* \circ \gamma^*$ induces by adjunction a base change map

$$L^\bullet\pi_{1!} \circ \gamma'^* \to \gamma^* \circ L^\bullet\pi_!.$$

This map is an isomorphism (for a proof see e.g. [Ka1]). In particular, for any object $c \in \mathcal{C}$, any ring R, and any $E \in \mathrm{Fun}(\mathcal{C}', R)$, we have a natural identification

$$L^\bullet\pi_! E(c) \cong H_\bullet(\pi_c, E|_c), \tag{2.6}$$

where $E|_c \in \mathrm{Fun}(\pi_c, R)$ is the restriction to the fiber $\pi_c \subset \mathcal{C}'$. If the cofibration π is discrete, then this shows that $L^i\pi_! E = 0$ for $i \geq 1$, and

$$\pi_! E(c) = \bigoplus_{c' \in \pi_c} E(c').$$

For example, for the discrete cofibration ρ_c of (2.5) and the constant functor $R \in \operatorname{Fun}(c\backslash\mathcal{C}, R)$, we obtain an identification

$$R_c \cong \rho_{c!}R \cong L^{\bullet}\rho_{c!}R, \tag{2.7}$$

where $R_c \in \operatorname{Fun}(\mathcal{C}, R)$ is the representable functor (2.1). For fibrations, we have exactly the same statements with left Kan extensions replaced by right Kan extensions, and sums replaced by products.

Moreover, assume that R is commutative, and assume given an object $T \in \operatorname{Fun}((\mathcal{C}')^o, R)$ that inverts all maps f in $\mathcal{C}'$ cocartesian with respect to π – that is, $T(f)$ is invertible for any such map. Then we can define the relative tensor product functor $- \otimes_\pi T : \operatorname{Fun}(\mathcal{C}', R) \to \operatorname{Fun}(\mathcal{C}, R)$ by setting

$$(E \otimes_\pi T)(c) = E|_c \otimes_{\pi_c} T|_c$$

for any $E \in \operatorname{Fun}(\mathcal{C}', R)$. This has individual derived functors $\operatorname{Tor}^{\pi}_{\bullet}(-, T)$ and the total derived functor $- \overset{\mathsf{L}}{\otimes}_\pi T$. For any $c \in \mathcal{C}$, we have

$$(E \overset{\mathsf{L}}{\otimes}_\pi T)(c) \cong E|_c \overset{\mathsf{L}}{\otimes}_{\pi_c} T|_c. \tag{2.8}$$

If $T(c)$ is a free R-module for any $c \in \mathcal{C}'$, then we also have the relative version

$$\mathcal{H}om_\pi(T, -) : \operatorname{Fun}(\mathcal{C}, R) \to \operatorname{Fun}(\mathcal{C}', R)$$

of the functor (2.3); it is exact and right-adjoint to $- \otimes_\pi T$, resp. $- \overset{\mathsf{L}}{\otimes}_\pi T$. In the case $T = R$, we have $E \overset{\mathsf{L}}{\otimes}_\pi R \cong L^{\bullet}\pi_! E$, and the isomorphism (2.8) is the isomorphism (2.6).

2.4 Simplicial objects. As usual, we denote by Δ the category of finite non-empty totally ordered sets, a.k.a. finite non-empty ordinals, and somewhat unusually, we denote by $[n] \in \Delta$ the set with n elements, $n \geq 1$. A simplicial object in a category $\mathcal{C}$ is a functor from Δ^o to $\mathcal{C}$; these form a category denoted $\Delta^o\mathcal{C}$. For any ring R and $E \in \operatorname{Fun}(\Delta^o, R) = \Delta^o R$-mod, we denote by $C_{\bullet}(E)$ the normalized chain complex of the simplicial R-module E. The homology of the complex $C_{\bullet}(E)$ is canonically identified with the homology $H_{\bullet}(\Delta^o, E)$ of the category Δ^o with coefficients in E. Even stronger, sending E to $C_{\bullet}(E)$ gives the *Dold-Kan equivalence*

$$\mathsf{N} : \operatorname{Fun}(\Delta^o, R) \to C_{\geq 0}(R)$$

between the category $\operatorname{Fun}(\Delta^o, R)$ and the category $C_{\geq 0}(R)$ of complexes of R-modules concentrated in non-negative homological degrees. The inverse equivalence is given by the denormalization functor $\mathsf{D} : C_{\geq 0}(R) \to \operatorname{Fun}(\Delta^o, R)$ right-adjoint to N.

For any simplicial set X, its homology $H_{\bullet}(X, R)$ with coefficients in a ring R is the homology of the chain complex

$$C_{\bullet}(X, R) = C_{\bullet}(R[X]),$$

where $R[X] \in \mathrm{Fun}(\Delta^o, R)$ is given by $R[X]([n]) = R[X([n])]$, $[n] \in \Delta$. By adjunction, for any simplicial set X and any complex $E_\bullet \in C_{\geq 0}(R)$, a map $C_\bullet(X,R) \to E_\bullet$ gives rise to a map of simplicial sets

$$X \longrightarrow R[X] \longrightarrow \mathsf{D}(E_\bullet), \tag{2.9}$$

where we treat simplicial R-modules $R[X]$ and $\mathsf{D}(E_\bullet)$ as simplicial sets. Conversely, every map of simplicial sets $X \to \mathsf{D}(E_\bullet)$ gives rise to a map $C_\bullet(X,R) \to E_\bullet$. In particular, if we take $X = \mathsf{D}(E_\bullet)$, we obtain the *assembly map*

$$C_\bullet(\mathsf{D}(E_\bullet), R) \to E_\bullet. \tag{2.10}$$

The constructions are mutually inverse: every map of complexes of R-modules $C_\bullet(X,R) \to E_\bullet$ decomposes as

$$C_\bullet(X,R) \longrightarrow C_\bullet(\mathsf{D}(E_\bullet), R) \longrightarrow E_\bullet, \tag{2.11}$$

where the first map is induced by the tautological map (2.9), and the second map is the assembly map (2.10).

Applying the Grothendieck construction to a simplicial set X, we obtain a category $\mathrm{Tot}(X)$ with a discrete fibration $\pi : \mathrm{Tot}(X) \to \Delta$. We then have a canonical identification

$$H_\bullet(\mathrm{Tot}(X)^o, R) \cong H_\bullet(\Delta^o, \pi_! R) \cong H_\bullet(\Delta^o, R[X]), \tag{2.12}$$

so that $H_\bullet(X,R)$ is naturally identified with the homology of the small category $\mathrm{Tot}(X)^o$ with coefficients in the constant functor R.

The *nerve* of a small category $\mathcal{C}$ is the simplicial set $N(\mathcal{C}) \in \Delta^o$ Sets such that for any $[n] \in \Delta$, $N(\mathcal{C})([n])$ is the set of functors from the ordinal $[n]$ to $\mathcal{C}$. Explicitly, elements in $N(\mathcal{C})([n])$ are diagrams

$$c_1 \longrightarrow \dots \longrightarrow c_n \tag{2.13}$$

in $\mathcal{C}$. We denote by $\mathcal{N}(\mathcal{C}) = \mathrm{Tot}(N(\mathcal{C}))$ the corresponding fibered category over Δ. Then by definition, objects of $\mathcal{N}(\mathcal{C})$ are diagrams (2.13), and sending such a diagram to c_n gives a functor

$$q : \mathcal{N}(\mathcal{C}) \to \mathcal{C}. \tag{2.14}$$

Say that a map $f : [n] \to [m]$ in Δ is *special* if it identifies $[n]$ with a terminal segment of the ordinal $[m]$. For any fibration $\pi : \mathcal{C}' \to \Delta$, say that a map f in $\mathcal{C}'$ is *special* if it is cartesian with respect to π and $\pi(f)$ is special in Δ, and say that a functor $F : \mathcal{C}' \to \mathcal{E}$ to some category E is *special* if it $F(f)$ is invertible for any special map f in $\mathcal{C}'$. Then the functor q of (2.14) is special, and any special functor factors uniquely through q. In particular, $\mathrm{Fun}(\mathcal{C}, R)$ is naturally identified the full subcategory in $\mathrm{Fun}(\mathcal{N}(\mathcal{C}), R)$ spanned by special functors. Moreover, on the level of derived categories, say that $E \in \mathcal{D}(\mathcal{C}', R)$

is *special* if so is $\mathcal{D}(E) : \mathcal{C}' \to \mathcal{D}(R)$, and denote by $\mathcal{D}_{sp}(\mathcal{C}', R) \subset \mathcal{D}(\mathcal{C}', R)$ the full subcategory spanned by special objects. Then the pullback functor

$$q^* : \mathcal{D}(\mathcal{C}, R) \to \mathcal{D}(\mathcal{N}(\mathcal{C}), R) \tag{2.15}$$

induces an equivalence between $\mathcal{D}(\mathcal{C}, R)$ and $\mathcal{D}_{sp}(\mathcal{N}(\mathcal{C}), R)$. In particular, we have a natural isomorphism

$$H_\bullet(\mathcal{C}, R) \cong H_\bullet(\mathcal{N}(\mathcal{C}), R), \tag{2.16}$$

and by (2.12), the right-hand side is also canonically identified with the homology $H_\bullet(N(\mathcal{C}), R)$ of the simplicial set $N(\mathcal{C})$.
The *geometric realization functor* $X \mapsto |X|$ is a functor from Δ^o Sets to the category Top of topological spaces. For any simplicial set X and any ring R, the homology $H_\bullet(X, R)$ is naturally identified with the homology $H_\bullet(|X|, R)$ of its realization, and the isomorphism (2.16) can also be deduced from the following geometric fact: for any simplicial set X, we have a natural homotopy equivalence

$$|N(\mathrm{Tot}(X))| \cong |X|.$$

Even stronger, the geometric realization functor extends to a functor from Δ^o Top to Top, and for any small category $\mathcal{C}$ equipped with a fibration $\pi : \mathcal{C} \to \Delta$, we have a natural homotopy equivalence

$$|N(\mathcal{C})| \cong ||N(\mathrm{Gr}(\pi))||, \tag{2.17}$$

where $N(\mathrm{Gr}(\pi)) : \Delta^o \to \Delta^o$ Sets is the natural bisimplicial set corresponding to π, and $||-||$ in the right-hand side stands for the geometric realization of its pointwise geometric realization. Geometric realization commutes with products by the well-known Milnor Theorem, so that in particular, (2.17) implies that for any self-product $\mathcal{C} \times_\Delta \cdots \times_\Delta \mathcal{C}$, we have a natural homotopy equivalence

$$|N(\mathcal{C} \times_\Delta \cdots \times_\Delta \mathcal{C})| \cong |N(\mathcal{C})| \times \cdots \times |N(\mathcal{C})|. \tag{2.18}$$

2.5 2-categories. We will also need to work with 2-categories, and for this, the language of nerves is very convenient, since the nerve of a 2-category can be converted into a 1-category by the Grothendieck construction.
Namely, recall that a 2-category[2] $\mathcal{C}$ is given by a class of objects $c \in \mathcal{C}$, a collection of morphism categories $\mathcal{C}(c, c')$, $c, c' \in \mathcal{C}$, a collection of identity objects $\mathsf{id}_c \in \mathcal{C}(c, c)$ for any $c \in \mathcal{C}$, and a collection of composition functors

$$m_{c,c',c''} : \mathcal{C}(c, c') \times \mathcal{C}(c', c'') \to \mathcal{C}(c, c''), \qquad c, c', c'' \in \mathcal{C} \tag{2.19}$$

equiped with associativity and unitality isomorphisms, subject to standard higher contraints (see [Be]). A 1-category is then a 2-category $\mathcal{C}$ with discrete

[2]We use "2-category" to mean "weak 2-category" a.k.a. "bicategory"; we avoid current usage that seems to reserve "2-category" for "strict 2-category", a rather unnatural notion with no clear conceptual meaning.

$\mathcal{C}(c,c')$, $c,c' \in \mathcal{C}$. For any 2-category $\mathcal{C}$ and any $[n] \in \Delta$, one can consider the category

$$N(\mathcal{C})_n = \coprod_{c_1,\dots,c_n \in \mathcal{C}} \mathcal{C}(c_1,c_2) \times \dots \times \mathcal{C}(c_{n-1},c_n).$$

If $\mathcal{C}$ is a small 1-category, then $N(\mathcal{C})_n = N(\mathcal{C})([n])$ is the value of the nerve $N(\mathcal{C}) \in \Delta^o\,\mathrm{Sets}$ at $[n] \in \Delta$, and the structure maps of the functor $N(\mathcal{C}) : \Delta^o \to \mathrm{Sets}$ are induced by the composition and unity maps in $\mathcal{C}$. In the general case, the composition and unity functors turn $N(\mathcal{C})$ into a pseudofunctor from Δ^o to Cat. We let

$$\mathcal{N}(\mathcal{C}) = \mathrm{Tot}(N(\mathcal{C}))$$

be the corresponding fibered category over Δ, and call it the *nerve* of the 2-category $\mathcal{C}$.
The associativity and unitality isomorphisms in $\mathcal{C}$ give rise to the compatibility isomorphisms of the pseudofunctor $N(\mathcal{C})$, so that they are encoded by the fibration $\mathcal{N}(\mathcal{C}) \to \Delta$. One can in fact use this to give an alternative definition of a 2-category, see e.g. [Ka3], but we will not need this. However, it is useful to note what happens to functors. A 2-*functor* $F : \mathcal{C} \to \mathcal{C}'$ between 2-categories $\mathcal{C}$, $\mathcal{C}'$ is given by a map F between their classes of objects, a collection of functors

$$F(c,c') : \mathcal{C}(c,c') \to \mathcal{C}'(F(c),F(c')), \qquad c,c' \in \mathcal{C}, \tag{2.20}$$

and a collection of isomorphisms $F(c,c)(\mathsf{id}_c) \cong \mathsf{id}_{F(c)}$, $c \in \mathcal{C}$, and

$$m_{F(c),F(c'),F(c'')} \circ (F(c,c') \times F(c',c'')) \cong F(c,c'') \circ m_{c,c',c''}, \qquad c,c',c'' \in \mathcal{C},$$

again subject to standard higher constraints. Such a 2-functor gives rise to a functor $\mathcal{N}(F) : \mathcal{N}(\mathcal{C}) \to \mathcal{N}(\mathcal{C}')$ cartesian over Δ, and the correspondence between 2-functors and cartesian functors is one-to-one.
The category Cat is a 2-category in a natural way, and the Grothendieck construction generalizes directly to 2-functors from a 2-category $\mathcal{C}$ to Cat. Namely, say that a cofibration $\pi : \mathcal{C}' \to \mathcal{N}(\mathcal{C})$ is *special* if for any special morphism $f : c \to c'$ in $\mathcal{N}(\mathcal{C})$, the transition functor $f_1 : \pi_c \to \pi_{c'}$ is an equivalence. Then 2-functors $F : \mathcal{C} \to \mathrm{Cat}$ correspond to special cofibrations $\mathrm{Tot}(F) \to \mathcal{N}(\mathcal{C})$, and the correspondence is again one-to-one. If $\mathcal{C}$ is actually a 1-category, then a 2-functor $F : \mathcal{C} \to \mathrm{Cat}$ is exactly the same thing as a pseudofunctor $\overline{F} : \mathcal{C} \to \mathrm{Cat}$ in the sense of the usual Grothendieck construction, and we have $\mathrm{Tot}(F) \cong q^* \,\mathrm{Tot}(\overline{F})$, where q is the functor of (2.14) (one easily checks that every special cofibration over $\mathcal{N}(\mathcal{C})$ arises in this way).
The simplest example of a 2-functor from a 2-category $\mathcal{C}$ to Cat is the functor $\mathcal{C}(c,-)$ represented by an object $c \in \mathcal{C}$. We denote the corresponding special cofibration by

$$\widetilde{\rho}_c : \mathcal{N}(c\backslash\mathcal{C}) \to \mathcal{N}(\mathcal{C}). \tag{2.21}$$

If $\mathcal{C}$ is a 1-category, then $\widetilde{\rho}_c = q^*\rho_c$, where ρ_c is the discrete cofibration (2.5)

2.6 Homology of 2-categories. To define the derived category of functors from a small 2-category $\mathcal{C}$ to complexes of modules over a ring R, we use its nerve $\mathcal{N}(\mathcal{C})$, with its fibration $\pi : \mathcal{N}(\mathcal{C}) \to \Delta$ and the associated notion of a special map and a special object.

Definition 2.1. For any ring R and small 2-category $\mathcal{C}$, the *derived category of functors from $\mathcal{C}$ to R*-mod is given by

$$\mathcal{D}(\mathcal{C}, R) = \mathcal{D}_{sp}(\mathcal{N}(\mathcal{C}), R).$$

Recall that if $\mathcal{C}$ is a 1-category, then $\mathcal{D}_{sp}(\mathcal{N}(\mathcal{C}), R)$ is identified with $\mathcal{D}(\mathcal{C}, R)$ by the functor q^* of (2.15), so that the notation is consistent. Since the truncation functors with respect to the standard t-structure on $\mathcal{D}(\mathcal{N}(\mathcal{C}), R)$ send special objects to special objects, this standard t-structure induces a t-structure on $\mathcal{D}(\mathcal{C}, R) \subset \mathcal{D}(\mathcal{N}(\mathcal{C}), R)$ that we also call standard. We denote its heart by $\mathrm{Fun}(\mathcal{C}, R) \subset \mathcal{D}(\mathcal{C}, R)$; it is equivalent to the category of special functors from $\mathcal{N}(\mathcal{C})$ to R-mod. If $\mathcal{C}$ is a 1-category, every special functor factors uniquely through q of (2.14), so that the notation is still consistent.

Lemma 2.2. *For any 2-category $\mathcal{C}$, the embedding $\mathcal{D}(\mathcal{C}, R) \subset \mathcal{D}(\mathcal{N}(\mathcal{C}), R)$ admits a left and a right-adjoint functors $L^{sp}, R^{sp} : \mathcal{D}(\mathcal{N}(\mathcal{C}), R) \to \mathcal{D}(\mathcal{C}, R)$. For any object $c \in \mathcal{C}$ with the correspoding object $n(c) \in N(\mathcal{C})_1 \subset \mathcal{N}(\mathcal{C})$, we have*

$$L^{sp} R_{n(c)} \cong L^{\bullet} \widetilde{\rho}_{c!} R,$$

where $\widetilde{\rho}_c$ is the special cofibration (2.21)*, and R in the right-hand side is the constant functor.*

Proof. Say that a map f in $\mathcal{D}(\mathcal{N}(\mathcal{C}))$ is *co-special* if $\pi(f) : [n] \to [n']$ sends the initial object of the ordinal $[n]$ to the initial object of the ordinal $[n']$. Then as in the proof of [Ka2, Lemma 4.8], it is elementary to check that special and co-special maps in $\mathcal{N}(C)$ form a complementary pair in the sense of [Ka2, Definition 4.3], and then the adjoint functor L^{sp} is provided by [Ka2, Lemma 4.6]. Moreover, $L^{sp} \circ L^{sp} \cong L^{sp}$, and L^{sp} is an idempotent comonad on $\mathcal{D}(\mathcal{N}(\mathcal{C}), R)$, with algebras over this comonad being exactly the objects of $\mathcal{D}(\mathcal{C}, R)$. Moreover, by construction of [Ka2, Lemma 4.6], $L^{sp} : \mathcal{D}(\mathcal{N}(C), R) \to \mathcal{D}(\mathcal{N}(\mathcal{C}), R)$ has a right-adjoint functor $R^{sp} : \mathcal{D}(\mathcal{N}(\mathcal{C}), R) \to \mathcal{D}(\mathcal{N}(\mathcal{C}), R)$. By adjunction, R^{sp} is an idempotent monad, algebras over this monad are objects in $\mathcal{D}(\mathcal{C}, R)$, and R^{sp} factors through the desired functor $\mathcal{D}(\mathcal{N}(\mathcal{C}, R)) \to \mathcal{D}(\mathcal{C}, R)$ right-adjoint to the embedding $\mathcal{D}(\mathcal{C}, R) \subset \mathcal{D}(\mathcal{N}(\mathcal{C}), R)$. Finally, the last claim immediately follows by the same argument as in the proof of [Ka2, Theorem 4.2]. $\square$

For any 2-functor $F : \mathcal{C} \to \mathcal{C}'$ between small 2-categories, the corresponding functor $\mathcal{N}(F)$ sends special maps to special maps, so that we have a pullback functor

$$F^* = \mathcal{N}(F)^* : \mathcal{D}(\mathcal{C}', R) \to \mathcal{D}(\mathcal{C}, R).$$

By Lemma 2.2, F^* has a left and a right-adjoint functor $F_!$, F_*, given by

$$F_! = L^{sp} \circ L^{\bullet}\mathcal{N}(F)_!, \qquad F_* = R^{sp} \circ R^{\bullet}\mathcal{N}(F)_*.$$

For any object $c \in \mathcal{C}$, we denote

$$R_c = L^{sp} R_{n(c)} \cong L^{\bullet}\widetilde{\rho}_{c!}R \in \mathcal{D}(\mathcal{C}, R). \tag{2.22}$$

If $\mathcal{C}$ is a 1-category, then this is consistent with (2.1) by (2.7). In the general case, by base change, we have a natural identification

$$R_c(c') \cong H_{\bullet}(\mathcal{C}(c, c'), R) \tag{2.23}$$

for any $c' \in \mathcal{C}$, an analog of (2.1). Moreover, by adjunction, we have a natural isomorphism

$$E(c) \cong \operatorname{Hom}(R_c, E) \tag{2.24}$$

for any $E \in \mathcal{D}(\mathcal{C}, R)$, a generalization of (2.2).

2.7 Finite sets. The first example of a 2-category that we will need is the following. Denote by Γ the category of finite sets. Then objects of the 2-category $\mathcal{Q}\Gamma$ are finite sets $S \in \Gamma$, and for any two $S_1, S_2 \in \Gamma$, the category $\mathcal{Q}\Gamma(S_1, S_2)$ is the groupoid of diagrams

$$S_1 \xleftarrow{\;l\;} S \xrightarrow{\;r\;} S_2 \tag{2.25}$$

in Γ and isomorphisms between them. The composition functors (2.19) are obtained by taking fibered products.

We can also define a smaller 2-category $\Gamma_+ \subset \mathcal{Q}\Gamma$ by keeping the same objects and requiring that $\Gamma_+(S_1, S_2)$ consists of diagrams (2.25) with injective map l. Then since such diagrams have no non-trivial automorphisms, Γ_+ is actually a 1-category. It is equivalent to the category of finite pointed sets. The equivalence sends a set S with a disntiguished element $o \in S$ to the complement $\overline{S} = S \setminus \{o\}$, and a map $f : S \to S'$ goes to the diagram

$$\overline{S} \xleftarrow{\;i\;} f^{-1}(\overline{S}') \xrightarrow{\;f\;} \overline{S}',$$

where $i : f^{-1}(\overline{S}') \to \overline{S}$ is the natural embedding. For any $n \geq 0$, we denote by $[n]_+ \in \Gamma_+$ the set with n non-distinguished elements (and one distinguished element o). In particular, $[0]_+ = \{o\}$ is the set with the single element o.

To construct 2-functors from $\mathcal{Q}\Gamma$ to Cat, recall that for any category $\mathcal{C}$, the *wreath product* $\mathcal{C} \wr \Gamma$ is the category of pairs $\langle S, \{c_s\}\rangle$ of a set $S \in \Gamma$ and a collection of objects $c_s \in \mathcal{C}$ indexed by elements $s \in S$. Morphisms from $\langle S, \{c_s\}\rangle$ to $\langle S', \{c'_s\}\rangle$ are given by a morphism $f : S \to S'$ and a collection of morphisms $c_s \to c'_{f(s)}$, $s \in S$. Then the forgetful functor $\rho : \mathcal{C} \wr \Gamma \to \Gamma$ is a fibration whose fiber over $S \in \Gamma$ is the product $\mathcal{C}^S$ of copies of the category

$\mathcal{C}$ numbered by elements $s \in S$, and whose transition functor $f^* : \mathcal{C}^{S_2} \to \mathcal{C}^{S_1}$ associated to a map $f : S_1 \to S_2$ is the natural pullback functor.
Assume that the category $\mathcal{C}$ has finite coproducts (including the coproduct of an empty collection of objects, namely, the initial object $0 \in \mathcal{C}$). Then all the transition functors f^* of the fibration ρ have left-adjoint functors $f_!$, so that ρ is also a cofibration. Moreover, for any diagram (2.25) in Γ, we have a natural functor

$$r_! \circ l^* : \mathcal{C}^{S_1} \to \mathcal{C}^{S_2}. \tag{2.26}$$

This defines a 2-functor $\mathsf{Vect}(\mathcal{C}) : \mathcal{Q}\Gamma \to \mathrm{Cat}$ – for any finite set $S \in \Gamma$, we let $\mathsf{Vect}(\mathcal{C})(S) = \mathcal{C}^S$, and for any $S_1, S_2 \in \Gamma$, the functor $\mathsf{Vect}(\mathcal{C})(S_1, S_2)$ of (2.20) sends a diagram (2.25) to the functor induced by (2.26). Moreover, for any subcategory $w(\mathcal{C}) \subset \mathcal{C}$ with the same objects as $\mathcal{C}$ and containing all isomorphisms, the collection of subcategories $\mathsf{Vect}(w(\mathcal{C}))(S) = w(\mathcal{C})^S \subset \mathcal{C}^S$ defines a subfunctor $\mathsf{Vect}(w(\mathcal{C})) \subset \mathsf{Vect}(\mathcal{C})$.
Restricting the 2-functor $\mathsf{Vect}(\mathcal{C})$ to the subcategory $\Gamma_+ \subset \mathcal{Q}\Gamma$ and applying the Grothendieck construction, we obtain a cofibration over Γ_+ that we denote by $\rho_+ : (\mathcal{C} \wr \Gamma)_+ \to \Gamma_+$. For any subcategory $w(\mathcal{C})$ with the same objects an containng all isomorphisms, we can do the same procedure with the subfunctor $\mathsf{Vect}(w(\mathcal{C})) \subset \mathsf{Vect}(\mathcal{C})$; this gives a subcofibration $(w(\mathcal{C}) \wr \Gamma)_+ \subset (\mathcal{C} \wr \Gamma)_+$, and in particular, ρ_+ restricts to a cofibration

$$\rho_+ : (w(\mathcal{C}) \wr \Gamma)_+ \to \Gamma_+. \tag{2.27}$$

Explicitly, the fiber of the cofibration ρ_+ over a pointed set $S \in \Gamma_+$ is identified with $w(\mathcal{C})^{\overline{S}}$, where $\overline{S} \subset S$ is the complement to the distiguished element. The transition functor corresponding to a pointed map $f : S \to S'$ sends a collection $\{c_s\} \in w(\mathcal{C})^{\overline{S}}$, $s \in \overline{S}$ to the collection $c'_{s'}$, $s' \in \overline{S}'$ given by

$$c'_{s'} = \bigoplus_{s \in f^{-1}(s')} c_s, \tag{2.28}$$

where $\oplus$ stands for the coproduct in the category $\mathcal{C}$.

2.8 Matrices and vectors. Now more generally, assume that we are given a small category $\mathcal{C}_0$ with finite coproducts and initial object, and moreover, $\mathcal{C}_0$ is a unital monoidal category, with a unit object $1 \in \mathcal{C}_0$ and the tensor product functor $- \otimes -$ that preserves finite coproducts in each variable. Then we can define a 2-category $\mathsf{Mat}(\mathcal{C}_0)$ in the following way:

(i) objects of $\mathsf{Mat}(\mathcal{C}_0)$ are finite sets $S \in \Gamma$,

(ii) for any $S_1, S_2 \in \Gamma$, $\mathsf{Mat}(\mathcal{C}_0)(S_1, S_2) \subset \mathcal{C}^{S_1 \times S_2}$ is the groupoid of isomorphisms of the category $\mathcal{C}^{S_1 \times S_2}$,

(iii) for any $S \in \Gamma$, $\mathsf{id}_S \in \mathsf{Mat}(\mathcal{C}_0)(S, S)$ is given by $\mathsf{id}_S = \delta_!(p^*(1))$, where $p : S \to \mathsf{pt}$ is the projection to the point, and $\delta : S \to S \times S$ is the diagonal embedding, and

(iv) for any $S_1, S_2, S_2 \in \Gamma$, the composition functor m_{S_1,S_2,S_3} of (2.19) is given by
$$m_{S_1,S_2,S_3} = p_{2!} \circ \delta_2^*,$$
where $p_2 : S_1 \times S_2 \times S_3 \to S_1 \times S_3$ is the product $p_2 = \mathsf{id} \times p \times \mathsf{id}$, and analogously, $\delta_2 = \mathsf{id} \times \delta \times \mathsf{id}$.

In other words, objects in $\mathsf{Mat}(\mathcal{C}_0)(S_1, S_2)$ are matrices of objects in $\mathcal{C}$ indexed by $S_1 \times S_2$, and the identity object and the composition functors are induces by those of $\mathcal{C}$ by the usual matrix multiplication rules. The associativity and unitality isomorphisms are also induced by those of $\mathcal{C}_0$. It is straightforward to check that this indeed defines a 2-category; to simplify notation, we denote its nerve by
$$\mathcal{M}at(\mathcal{C}_0) = \mathcal{N}(\mathsf{Mat}(\mathcal{C}_0)).$$
Moreover, assume given another small category $\mathcal{C}$ with finite coproducts, and assume that $\mathcal{C}$ is a unital right module category over the unital monoidal category $\mathcal{C}_0$ – that is, we have the action functor
$$- \otimes - : \mathcal{C} \times \mathcal{C}_0 \to \mathcal{C}, \tag{2.29}$$
preserving finite coproducts in each variable and equipped with the relevant unitality and asociativity isomorphism. Then we can define a 2-functor $\mathsf{Vect}(\mathcal{C}, \mathcal{C}_0)$ from $\mathsf{Mat}(\mathcal{C}_0)$ to Cat that sends $S \in \Gamma$ to $\mathcal{C}^S$, and sends an object $M \in \mathsf{Mat}(\mathcal{C}_0)(S_1, S_2)$ to the functor $\mathcal{C}^{S_1} \to \mathcal{C}^{S_2}$ induced by (2.29) via the usual rule of matrix action on vectors. We denote the corresponding special cofibration over $\mathcal{M}at(\mathcal{C}_0)$ by $\mathcal{V}ect(\mathcal{C}, \mathcal{C}_0)$. Moreover, given a subcategory $w(\mathcal{C}) \subset \mathcal{C}$ with the same objects and containing all the isomorphisms, we obtain a subfunctor $\mathsf{Vect}(w(\mathcal{C}), \mathcal{C}_0) \subset \mathsf{Vect}(\mathcal{C}, \mathcal{C}_0)$ given by
$$\mathsf{Vect}(w(\mathcal{C}), \mathcal{C}_0)(S) = w(\mathcal{C})^S \subset \mathcal{C}^S = \mathsf{Vect}(\mathcal{C}, \mathcal{C}_0)(S).$$
We denote the corresponding subcofibration by
$$\mathcal{V}ect(w(\mathcal{C}), \mathcal{C}_0) \subset \mathcal{V}ect(\mathcal{C}, \mathcal{C}_0).$$
If we take $\mathcal{C}_0 = \Gamma$, and let $- \otimes -$ be the cartesian product, then $\mathsf{Mat}(\mathcal{C}_0)$ is exactly the category $\mathcal{Q}\Gamma$ of Subsection 2.7. Moreover, any category $\mathcal{C}$ that has finite coproducts is automatically a module category over Γ with respect to the action functor
$$c \otimes S = \bigoplus_{s \in S} c, \qquad c \in \mathcal{C}, S \in \Gamma,$$
and we have $\mathsf{Vect}(\mathcal{C}, \Gamma) = \mathsf{Vect}(\mathcal{C})$, $\mathsf{Vect}(w(\mathcal{C}), \Gamma) = \mathsf{Vect}(w(\mathcal{C}))$. This example is universal in the following sense: for any associative unital category $\mathcal{C}_0$ with finite coproducts, we have a unique coproduct-preserving tensor functor $\Gamma \to \mathcal{C}_0$, namely $S \mapsto 1 \otimes S$, so that we have a canonical 2-functor
$$e : \mathcal{Q}\Gamma \to \mathsf{Mat}(\mathcal{C}_0). \tag{2.30}$$
For any $\mathcal{C}_0$-module category $\mathcal{C}$ with finite coproducts, we have a natural equivalence $e \circ \mathsf{Vect}(\mathcal{C}, \mathcal{C}_0) \cong \mathsf{Vect}(\mathcal{C})$, and similarly for $w(\mathcal{C})$.

2.9 THE RELATIVE SETTING. Finally, let us observe that the 2-functors $\mathsf{Vect}(\mathcal{C}, \mathcal{C}_0)$, $\mathsf{Vect}(w(\mathcal{C}), \mathcal{C}_0)$ can also be defined in the relative situation. Namely, assume given a cofibration $\pi : \mathcal{C} \to \mathcal{C}'$ whose fibers π_c, $c \in \mathcal{C}'$ have finite coproducts. Moreover, assume that $\mathcal{C}$ is a module category over $\mathcal{C}_0$, and the action functor (2.29) commutes with projections to $\mathcal{C}'$, thus induces $\mathcal{C}_0$-module category structures on the fibers π_c of the cofibration π. Furthermore, assume that the induced action functors on the fibers π_c preserve finite coproducts in each variable. Then we can define a natural 2-functor $\mathsf{Vect}(\mathcal{C}/\mathcal{C}', \mathcal{C}_0) : \mathsf{Mat}(\mathcal{C}_0) \to \mathrm{Cat}$ by setting

$$\mathsf{Vect}(\mathcal{C}/\mathcal{C}', \mathcal{C}_0)(S) = \mathcal{C} \times_{\mathcal{C}'} \cdots \times_{\mathcal{C}'} \mathcal{C} \tag{2.31}$$

where the terms in the product in the right-hand side are numbered by elements of the finite set S. As in the absolute situation, the categories $\mathsf{Mat}(\mathcal{C}_0)(S_1, S_2)$ act by the vector multiplication rule. We denote by

$$\mathcal{V}ect(\mathcal{C}/\mathcal{C}', \mathcal{C}_0) \to \mathcal{M}at(\mathcal{C}_0)$$

the special cofibration corresponding to the 2-functor $\mathsf{Vect}(\mathcal{C}/\mathcal{C}', \mathcal{C}_0)$, and we observe that the cofibration π induces a natural cofibration

$$\mathcal{V}ect(\mathcal{C}/\mathcal{C}', \mathcal{C}_0) \to \mathcal{C} \tag{2.32}$$

whose fiber over $c \in \mathcal{C}$ is naturally identified with $\mathcal{V}ect(\pi_c, \mathcal{C}_0)$. Moreover, if we have a subcategory $w(\mathcal{C}) \subset \mathcal{C}$ with the same objects that contains all the isomorphisms, and $w(\mathcal{C}) \subset \mathcal{C}$ is a subcofibration, then we can let

$$\mathsf{Vect}(w(\mathcal{C})/\mathcal{C}', \mathcal{C}_0)(S) = w(\mathcal{C}) \times_{\mathcal{C}'} \cdots \times_{\mathcal{C}'} w(\mathcal{C}) \subset \mathsf{Vect}(\mathcal{C}/\mathcal{C}', \mathcal{C}_0)(S)$$

for any finite set $S \in \Gamma$, and this gives a subfunctor $\mathsf{Vect}(w(\mathcal{C})/\mathcal{C}', \mathcal{C}_0) \subset \mathsf{Vect}(\mathcal{C}/\mathcal{C}', \mathcal{C}_0)$ and a subcofibration $\mathcal{V}ect(w(\mathcal{C})/\mathcal{C}', \mathcal{C}_0) \subset \mathcal{V}ect(\mathcal{C}/\mathcal{C}', \mathcal{C}_0)$. The cofibration (2.32) then induces a cofibration

$$\mathcal{V}ect(w(\mathcal{C})/\mathcal{C}', \mathcal{C}_0) \to \mathcal{C} \tag{2.33}$$

whose fibers are identified with $\mathcal{V}ect(w(\pi_c), \mathcal{C}_0)$, $c \in \mathcal{C}$. As in the absolute case, in the case $\mathcal{C}_0 = \Gamma$, we simplify notation by setting $\mathcal{V}ect(w(\mathcal{C})/\mathcal{C}') = \mathcal{V}ect(w(\mathcal{C})/\mathcal{C}', \Gamma)$, and we denote by

$$((w(\mathcal{C})/\mathcal{C}') \wr \Gamma)_+ \to \Gamma_+ \tag{2.34}$$

the induced cofibration over $\Gamma_+ \subset \mathcal{Q}\Gamma$.

Analogously, if $\pi : \mathcal{C} \to \mathcal{C}'$ is a fibration, then the same constructions go through, except that $w(\mathcal{C}) \subset \mathcal{C}$ has to be a subfibration, and the functors (2.32), (2.33) are also fibrations, with the same identification of the fibers.

3 Statements.

3.1 Generalities on K-theory. To fix notations and terminology, let us summarize very briefly the definitions of algebraic K-theory groups.
First assume given a ring k, let $k\text{-mod}^{fp} \subset k\text{-mod}$ be the category of finitely generated projective k-modules, and let $\mathsf{Iso}(k) \subset k\text{-mod}^{fp}$ be the groupoid of finitely generated projective k-modules and their isomorphisms. Explicitly, we have

$$\mathsf{Iso}(k) \cong \coprod_{P \in k\text{-mod}^{fp}} [\mathsf{pt}/\mathrm{Aut}(P)],$$

where the sum is over all isomorphism classes of finitely generated projective k-modules, $\mathrm{Aut}(P)$ is the automorphism group of the module P, and for any group G, $[\mathsf{pt}/G]$ stands for the groupoid with one object with automorphism group G. The category $k\text{-mod}^{fp}$ is additive. In particular, it has finite coproducts. Since $\mathsf{Iso}(k) \subset k\text{-mod}^{fp}$ contains all objects and all the isomorphisms, we have the cofibration

$$\rho_+ : (\mathsf{Iso}(k) \wr \Gamma)_+ \to \Gamma_+$$

of (2.27). Its fiber $(\rho_+)_{[1]_+}$ over the set $[1]_+ \in \Gamma_+$ is $\mathsf{Iso}(k)$, and the fiber $(\rho_+)_S$ over a general $S \in \Gamma_+$ is the product $\mathsf{Iso}(k)^{\overline{S}}$. Applying the Grothendieck construction and taking the geometric realization of the nerve, we obtain a functor

$$|N(\mathrm{Gr}(\rho_+))| : \Gamma_+ \to \mathrm{Top}$$

from Γ_+ to the category Top of topological spaces, or in other terminology, a Γ-space. Then (2.28) immediately shows that this Γ-space is special in the sense of the Segal machine [Se], thus gives rise to a spectrum $\mathcal{K}(k)$. The algebraic K-groups $K_\bullet(k) = \pi_\bullet\mathcal{K}(k)$ are by definition the homotopy groups of this spectrum.
For a more general K-theory setup, assume given a small category $\mathcal{C}$ with the subcategories $c(\mathcal{C}), w(\mathcal{C}) \subset \mathcal{C}$ of cofibrations and weak equivalences, and assume that $\langle \mathcal{C}, c(\mathcal{C}), w(\mathcal{C})\rangle$ is a Waldhausen category. In particular, $\mathcal{C}$ has finite coproducts and the initial object $0 \in \mathcal{C}$. Then one lets $E\mathcal{C}$ be the category of pairs $\langle [n], \varphi\rangle$ of an object $[n] \in \Delta$ and a functor $\varphi : [n] \to \mathcal{C}$, with morphisms from $\langle [n], \varphi\rangle$ to $\langle [n'], \varphi'\rangle$ given by a pair $\langle f, \alpha\rangle$ of a map $f : [n] \to [n']$ and a map $\alpha : \varphi' \circ f \to \varphi$. Further, one lets $\widetilde{S\mathcal{C}} \subset E\mathcal{C}$ be the full subcategory spanned by pairs $\langle [n], \varphi\rangle$ such that φ factors through $c(\mathcal{C}) \subset \mathcal{C}$ and sends the initial object $o \in [n]$ to $0 \in \mathcal{C}$. The forgetful functor $s : \widetilde{S\mathcal{C}} \to \Delta$ sending $\langle [n], \varphi\rangle$ to $[n]$ is a fibration; explicitly, its fiber over $[n] \in \Delta$ is the category of diagrams (2.13) such that all the maps are cofibrations, and $c_1 = 0$. Finally, one says that a map f in $\widetilde{S\mathcal{C}}$ is *admissible* if in its canonical factorization $f = g \circ f'$ with $s(f) = s(f')$ and f' cartesian with respect to s, the morphism g pointwise lies in $w(\mathcal{C}) \subset \mathcal{C}$. Then by definition, $S\mathcal{C} \subset \widetilde{S\mathcal{C}}$ is the subcategory with the same objects and admissible maps between them. This is again a fibered category over Δ, with the fibration $S\mathcal{C} \to \Delta$ induced by the forgetful functor s. The

K-groups $K_\bullet(\mathcal{C})$ are given by

$$K_i(\mathcal{C}) = \pi_{i+1}(|N(S\mathcal{C})|), \qquad i \geq 0.$$

Moreover, since $\mathcal{C}$ has finite coproducts, the fibers of the fibration $\widetilde{S\mathcal{C}} \to \Delta$ also have finite coproducts, and since $S\mathcal{C} \subset \widetilde{S\mathcal{C}}$ contains all objects and all isomorphisms, we can form the cofibration

$$\rho_+ : ((S\mathcal{C}/\Delta) \wr \Gamma)_+ \to \Gamma_+ \tag{3.1}$$

of (2.34). Its fibers are the self-products $S\mathcal{C} \times_\Delta \cdots \times_\Delta S\mathcal{C}$. Then by (2.18),

$$|N(\mathrm{Gr}(\rho_+))| : \Gamma_+ \to \mathrm{Top}$$

is a special Γ-space, so that $|N(S\mathcal{C})|$ has a natural infinite loop space structure and defines a connective spectrum. The K-theory spectrum $\mathcal{K}(\mathcal{C})$ is given by $\mathcal{K}(\mathcal{C}) = \Omega|N(S\mathcal{C})|$.

Remark 3.1. Our definition of the category $S\mathcal{C}$ differs from the usual one in that the fibers of the fibration s are opposite to what one gets in the usual definition. This is harmless since passing to the opposite category does not change the homotopy type of the nerve, and this allows for a more succint definition.

The main reason we have reproduced the S-construction instead of using it as a black box is the following observation: the construction works just as well in the relative setting. Namely, let us say that a *family of Waldhausen categories* indexed by a category $\mathcal{C}'$ is a category $\mathcal{C}$ equipped with a cofibration $\pi : \mathcal{C} \to \mathcal{C}'$ with small fibers, and two subcofibrations $c(\mathcal{C}), w(\mathcal{C}) \subset \mathcal{C}$ such that for any $c \in \mathcal{C}'$, the subcategories

$$c(\pi_c) = c(\mathcal{C}) \cap \pi_c \subset \pi_c, \qquad w(\pi_c) = w(\mathcal{C}) \cap \pi_c \subset \pi_c$$

in the fiber π_c of the cofibration π turn it into a Waldhausen category. Then given such a family, one defines the category $E\mathcal{C}$ exactly as in the absolute case, and one lets $\widetilde{S(\mathcal{C}/\mathcal{C}')} \subset E\mathcal{C}$ be the full subcategory spanned by $\widetilde{S\pi_c} \subset E\pi_c \subset E\mathcal{C}$, $c \in \mathcal{C}'$. Further, one observes that the forgetful functor $s : \widetilde{S(\mathcal{C}/\mathcal{C}')} \to \Delta$ is a fibration, and as in the absolute case, one let $S(\mathcal{C}/\mathcal{C}') \subset \widetilde{S(\mathcal{C}/\mathcal{C}')}$ be the subcategory spanned by maps f in whose canonical factorization $f = g \circ f'$ with $s(f) = s(f')$ and f' cartesian with respect to s, the morphism g pointwise lies in $w(\mathcal{C}) \subset \mathcal{C}$. One then checks easily that the cofibration π induces a cofibration

$$S(\mathcal{C}/\mathcal{C}') \to \mathcal{C}'$$

whose fiber over $c \in \mathcal{C}'$ is naturally identified with $S\pi_c$. This cofibration is obviously functorial in $\mathcal{C}'$: for any functor $\gamma : \mathcal{C}'' \to \mathcal{C}'$ with the induced cofibration $\gamma^*\mathcal{C} \to \mathcal{C}''$, we have $S(\gamma^*\mathcal{C}/\mathcal{C}'') \cong \gamma^* S(\mathcal{C}/\mathcal{C}')$.

3.2 The setup and the statement. Now assume given a commutative ring k, so that $k\text{-mod}^{fp}$ is a monoidal category, and a Waldhausen category $\mathcal{C}$ that is additive and k-linear, so that $\mathcal{C}$ is a module category over $k\text{-mod}^{fp}$. Then all the fibers of the fibration $S\mathcal{C} \to \Delta$ are also module categories over $k\text{-mod}^{fp}$. To simplify notation, denote

$$\mathcal{M}at(k) = \mathcal{M}at(k\text{-mod}^{fp}), \quad \mathbb{K}(\mathcal{C}, k) = \mathcal{V}ect(S\mathcal{C}/\Delta, k\text{-mod}^{fp}).$$

More generally, assume given a family $\pi : \mathcal{C} \to \mathcal{C}'$ of Waldhausen categories, and assume that all the fibers of the cofibration π are additive and k-linear, and transition functors are additive k-linear functors. Then $\mathcal{C}$ is a $k\text{-mod}^{fp}$-module category over $\mathcal{C}$, and we can form the cofibration

$$\mathbb{K}(\mathcal{C}/\mathcal{C}', k) = \mathcal{V}ect(S(\mathcal{C}/\mathcal{C}')/\Delta, k\text{-mod}^{fp}) \to \mathcal{C}' \times \mathcal{M}at(k).$$

Denote by

$$\widetilde{\pi} : \mathbb{K}(\mathcal{C}/\mathcal{C}', k) \to \mathcal{C}', \qquad \varphi : \mathbb{K}(\mathcal{C}/\mathcal{C}', k) \to \mathcal{M}at(k) \tag{3.2}$$

its compositions with the projections to $\mathcal{C}'$ resp. $\mathcal{M}at(k)$. Then the fiber of the cofibration $\widetilde{\pi}$ over $c \in \mathcal{C}'$ is naturally idenitified with the category $\mathbb{K}(\pi_c, k)$.

Definition 3.2. Let R be the localization of $\mathbb{Z}$ in a set of primes. A commutative ring k is *R-adapted* if $K_i(k) \otimes R = 0$ for $i \geq 1$, and $K_0(k) \otimes R \cong R$ as a ring.

Example 3.3. Let k be a finite field of characteristic $\mathsf{char}(k) = p$, and let $R = \mathbb{Z}_{(p)}$ be the localization of $\mathbb{Z}$ in the prime ideal $p\mathbb{Z} \subset \mathbb{Z}$. Then k is R-adapted by the famous theorem of Quillen [Q].

Assume given an R-adapted commutative ring k. Any additive map $K_0(k) \to R$ induces a map of spectra

$$\mathcal{K}(k) \to H(R), \tag{3.3}$$

where $H(R)$ is the Eilenberg-Mac Lane spectrum corresponding to R, so that fixing an isomorphism $K_0(k) \otimes R \cong R$ fixes a map (3.3). Do this, and for any $P \in k\text{-mod}^{fp}$, denote by $\mathsf{rk}(P) \in R$ the image of its class $[P] \in K_0(k) \otimes R$ under the isomorphism we have fixed. Let $M(R)$ be the category of free finitely generated R-modules, and let $T \in \operatorname{Fun}(M(R)^o, R)$ be the functor sending a free R-module M to $M^* = \operatorname{Hom}_R(M, R)$. Equivalently, objects in $M(R)$ are finite sets S, and morphisms from S_1 to S_2 are elements in the set $R[S_1 \times S_2]$. In this description, sending $P \in k\text{-mod}^{fp}$ to $\mathsf{rk}(P)$ defines a 2-functor $rk : \mathsf{Mat}(k) \to M(R)$. By abuse of notation, we denote

$$\mathsf{rk} = q \circ \mathcal{N}(rk) : \mathcal{M}at(k) \to \mathcal{N}(M(R)) \to M(R).$$

Since the projection φ of (3.2) obviously inverts all maps cocartesian with respect to the cofibration $\widetilde{\pi}$, the pullback $\varphi^{o*}\,\mathsf{rk}^{o*}\,T \in \operatorname{Fun}(\mathbb{K}(\mathcal{C}/\mathcal{C}', k), R)$ also inverts all such maps. Therefore we are in the situation of Subsection 2.3, and we have a well-defined object

$$K^R_{\bullet}(\mathcal{C}/\mathcal{C}', k) = \mathbb{Z} \overset{\mathsf{L}}{\otimes}_{\widetilde{\pi}} \varphi^{o*} \operatorname{\mathsf{rk}}^{o*} T \in \mathcal{D}(\mathcal{C}', R), \tag{3.4}$$

where $\mathbb{Z}$ on the left-hand side of the product is the constant functor with value $\mathbb{Z}$. If $\mathcal{C}' = \mathsf{pt}$ is the point category, we simplify notation by letting $K^R_{\bullet}(\mathcal{C}, k) = K^R_{\bullet}(\mathcal{C}/\mathsf{pt}, k)$. The object $K^R_{\bullet}(\mathcal{C}/\mathcal{C}', k)$ is clearly functorial in $\mathcal{C}'$: for any functor $\gamma : \mathcal{C}'' \to \mathcal{C}'$, we have a natural isomorphism

$$\gamma^* K^R_{\bullet}(\mathcal{C}/\mathcal{C}', k) \cong K^R_{\bullet}(\gamma^*\mathcal{C}/\mathcal{C}'', k).$$

In particular, the value of $K^R_{\bullet}(\mathcal{C}/\mathcal{C}', k)$ at an object $c \in \mathcal{C}'$ is naturally identified with $K^R_{\bullet}(\pi_c, k)$. Here is, then, the main result of the paper.

Theorem 3.4. *Assume given a k-linear additive small Waldhausen category $\mathcal{C}$, and a ring R that is k-adapted in the sense of Definition 3.2, and let $\mathcal{K}^R(\mathcal{C}, k)$ be the Eilenberg-Mac Lane spectum associated to the complex $K^R_{\bullet}(\mathcal{C}, k)$ of* (3.4). *Then there exists a natural map of spectra*

$$\nu : \mathcal{K}(\mathcal{C}) \to \mathcal{K}^R(\mathcal{C}, k)$$

that induces an isomorphism of homology with coefficients in R.

Here a "spectrum" is understood as an object of the stable homotopy category without choosing any particular model for it. In practice, what we produce in proving Theorem 3.4 is two special Γ-spaces in the sense of the Segal machine representing the source and the target of our map ν, and we produce ν as a map of Γ-spaces. Note that our complex $\mathcal{K}^R_{\bullet}(\mathcal{C}, k)$ is concentrated in non-negative homological degrees. For such a complex, the simplest way to construct the corresponding Eilenberg-Mac Lane spectrum is to apply the Dold-Kan equivalence, and take the realization of the resulting simplicial abelian group — it is then trivially a special Γ-space. This is exactly what we do. As usual, we define "homology with coefficients in R" of a spectrum X by

$$H_{\bullet}(X, R) = \pi_{\bullet}(X \wedge H(R)).$$

If R is the localization of $\mathbb{Z}$ in the set of primes S, then by the standard spectral sequence argument, Theorem 3.4 implies that ν becomes a homotopy equivalence after localizing at the same set of primes S.

4 Proofs.

4.1 Additive functors. Before we prove Theorem 3.4, we need a couple of technical facts on the categories $\mathcal{D}(\mathsf{Mat}(k), R)$, $\mathcal{D}(M(R), R)$. Recall that we have a natural 2-functor $e : \mathcal{Q}\Gamma \to \mathsf{Mat}(k)$ of (2.30). Composing it with the natural embedding $\Gamma_+ \to \mathcal{Q}\Gamma$, we obtain a 2-functor

$$i : \Gamma_+ \to \mathsf{Mat}(k).$$

Composing further with the 2-functor $rk : \mathsf{Mat}(k) \to M(R)$, we obtain a functor

$$\bar{i} : \Gamma_+ \to M(R).$$

Explicitly, $\bar{i}$ sends a finite pointed set S to its reduced span

$$\bar{i}(S) = \overline{R[S]} = R[S]/R \cdot \{o\},$$

where $o \in S$ is the distinguished element. The object $T \in \operatorname{Fun}(M(R)^o, R)$ gives by pullback objects $\mathsf{rk}^{o*} T \in \operatorname{Fun}(\mathcal{M}at(k)^o, R)$, $\bar{i}^{o*} T \in \operatorname{Fun}(\Gamma_+^o, R)$. For any $E \in \mathcal{D}(\Gamma_+, R)$, denote

$$H_\bullet^\Gamma(E) = \operatorname{Tor}_\bullet^{\Gamma_+}(E, \bar{i}^* T). \tag{4.1}$$

Say that an object $E \in \mathcal{D}(\Gamma_+, R)$ is *pointed* if $E([0]_+) = 0$, where $[0]_+ = \{o\} \in \Gamma_+$ is the pointed set consisting of the distinguished element.

Lemma 4.1. (i) *For any two pointed objects $E_1, E_2 \in \mathcal{D}(\Gamma_+, R)$, we have $H_\bullet^\Gamma(E_1 \overset{\mathsf{L}}{\otimes} E_2) = 0$.*

(ii) *Assume given a spectrum X represented by a Γ-space $|X| : \Gamma_+ \to \mathrm{Top}$ special in the sense of Segal, and let $C_\bullet(|X|, R) \in \mathcal{D}(\Gamma_+, R)$ be the object obtained by taking pointwise the singular chain homology complex $C_\bullet(-, R)$. Then there exists a natural identification*

$$H_\bullet^\Gamma(C_\bullet(|X|, R)) \cong H_\bullet(X, R).$$

Proof. Although both claims are due to T. Pirashvili, in this form, (i) is [Ka4, Lemma 2.3], and its corollary (ii) is [Ka4, Theorem 3.2]. □

The category Γ_+ has coproducts – for any $S, S' \in \Gamma_+$, their coproduct $S \vee S' \in \Gamma_+$ is the disjoint union of S and S' with distinguished elements glued together. The embedding $S \to S \vee S'$ admits a canonical retraction $p : S \vee S' \to S$ identical on S and sending the rest to the distiguished element, and similarly, $S' \to S \vee S'$ has a canonical retraction $p' : S \vee S' \to S'$.

Definition 4.2. An object $E \in \mathcal{D}(\Gamma_+, R)$ is *additive* if for any $S, S' \in \Gamma_+$, the natural map

$$E(S \vee S') \to E(S) \oplus E(S') \tag{4.2}$$

induced by the retractions p, p' is an isomorphism. An object E in the category $\mathcal{D}(\mathsf{Mat}(k), R)$ resp. $\mathcal{D}(M(R), R)$ is *additive* if so is $i^* E$ resp. $\bar{i}^* E$.

We denote by $\mathcal{D}_{add}(\Gamma_+, R)$, $\mathcal{D}_{add}(\mathsf{Mat}(k), R)$, $\mathcal{D}_{add}(M(R), R)$ the full subcategories in $\mathcal{D}(\Gamma_+, R)$, $\mathcal{D}(\mathsf{Mat}(k), R)$, $\mathcal{D}(M(R), R)$ spanned by additive objects. In fact, $\mathcal{D}_{add}(\Gamma_+, R)$ is easily seen to be equivalent to $\mathcal{D}(R)$. Indeed, $[0]_+ \in \Gamma_+$ is a retract of $[1]_+ \in \Gamma_+$, so that we have a canonical direct sum decomposition

$$R_1 \cong t \oplus R_0$$

for a certain $t \in \mathrm{Fun}(\Gamma_+, R)$, where to simplify notation, we denote $R_n = R_{[n]_+} \in \mathrm{Fun}(\Gamma_+, R)$, $n \geq 0$. Then for any pointed $E \in \mathcal{D}(\Gamma_+, R)$, the adjunction map induces a map

$$t \otimes M \to E, \tag{4.3}$$

where $M = E([1])_+ \in \mathcal{D}(R)$. Any additive object is automatically pointed, and the map (4.3) is an isomorphism if and only if E is additive. We actually have $t \otimes M \cong \mathcal{H}om(\overline{i}^{o*}T, M) \cong \overline{i}^* \mathcal{H}om(T, M)$, so that the equivalence $\mathcal{D}(R) \cong \mathcal{D}_{add}(\Gamma_+, R)$ is realized by the functor

$$\overline{i}^* \circ \mathcal{H}om(T, -) : \mathcal{D}(R) \xrightarrow{\sim} \mathcal{D}_{add}(\Gamma_+, R) \subset \mathcal{D}(\Gamma_+, R).$$

4.2 Adjunctions. By definition, $\overline{i}^*$ and i^* preserve additivity – namely, $\overline{i}^*$ sends $\mathcal{D}_{add}(M(R), R) \subset \mathcal{D}(M(R), R)$ into $\mathcal{D}_{add}(\Gamma_+, R) \subset \mathcal{D}(\Gamma_+, R)$, and i^* sends $\mathcal{D}_{add}(\mathsf{Mat}(k), R) \subset \mathcal{D}(M(R), R)$ into $\mathcal{D}_{add}(\Gamma_+, R) \subset \mathcal{D}(\Gamma_+, R)$. It turns out that their adjoint functors $R^\bullet \overline{i}_*$, i_* also preserve additivity.

Lemma 4.3. (i) *For any additive $\overline{E} \in \mathcal{D}(\Gamma_+, R)$, the objects $R^\bullet \overline{i}_* \overline{E} \in \mathcal{D}(M(R), R)$ and $i_* \overline{E} \in \mathcal{D}(\mathsf{Mat}(k), R)$ are additive.*

(ii) *For any additive $E \in \mathrm{Fun}(\mathsf{Mat}(k), R) \subset \mathcal{D}(\mathsf{Mat}(k), R)$, the adjunction unit map $E \to i_* i^* E$ is an isomorphism in homological degree 0 with respect to the standard t-structure.*

Proof. For the first claim, let $E = R^\bullet \overline{i}_* \overline{E}$, and note that we may assume that $\overline{E} = \overline{i}^* \mathcal{H}om(T, M)$ for some $M \in D(R)$. Then by adjunction, for any finite set S, we have

$$\begin{aligned} E(\overline{i}(S)) \cong \mathrm{Hom}(R_{\overline{i}(S)}, E) = \mathrm{Hom}(R_{\overline{i}(S)}, R^\bullet \overline{i}_* \overline{E}) \cong \\ \cong \mathrm{Hom}(\overline{i}^* R_{\overline{i}(S)}, \overline{E}) \cong \mathrm{Hom}(H^\Gamma_\bullet(\overline{i}^* R_{\overline{i}(S)}), M), \end{aligned}$$

where R_S is the representable functor (2.1), and $H^\Gamma_\bullet(-)$ is as in (4.1). Thus to to check that (4.2) is an isomorphism, we need to check that the natural map

$$H^\Gamma_\bullet(\overline{i}^* R_{\overline{i}(S)}) \oplus H^\Gamma_\bullet(\overline{i}^* R_{\overline{i}(S')}) \to H^\Gamma_\bullet(R_{\overline{i}(S \vee S')})$$

induced by the projections p, p' is an isomorphism. For any $S, S_1 \in \Gamma_+$, we have

$$\overline{i}^* R_{\overline{i}(S)}(S_1) \cong R[\overline{S} \times \overline{S}_1]. \tag{4.4}$$

In particular, $\overline{i}^* R_{\overline{i}(S)}([0]_+) \cong R$ indepedently of S, and the tautological projection $S \to [0]_+$ induces a functorial map

$$t : \overline{i}^* R_{\overline{i}([0]_+)} \to \overline{i}^* R_{\overline{i}(S)} \cong R$$

in $\mathrm{Fun}(\Gamma_+, R)$ identical after evaluation at $[0]_+ \in \Gamma_+$. Moreover, we have

$$\overline{i}^* R_{\overline{i}(S \vee S')} \cong \overline{i}^* R_{\overline{i}(S)} \otimes \overline{i}^* R_{\overline{i}(S')}, \tag{4.5}$$

and under these identifications, the projections p, p' induce maps $\mathsf{id} \otimes t$ resp. $t \otimes \mathsf{id}$. Then to finish the proof, in suffices to invoke Lemma 4.1 (i).
For the object $i_*\overline{E}$, the argument is the same, but we need to replace the representable functors $R_{\bar{i}(S)}$, $R_{\bar{i}(S')}$, $R_{\bar{i}(S \vee S')}$ by their 2-category versions of (2.22), and (4.4) becomes the isomorphism

$$i^* R_{i(S)} \cong H_{\bullet}(\mathsf{Iso}(k)^{\overline{S} \times \overline{S_1}}, R)$$

provided by (2.23). The corresponding version of (4.5) then follows from the Künneth formula.
For the second claim, note that since we have already proved that $i_*i^*\overline{E}$ is additive, it suffices to prove that the natural map

$$E([1]_+) \to i_*i^*E([1]_+)$$

is an isomorphism in homological degree 0. Again by Lemma 4.1 (ii) and adjunction, this amount to checking that the natural map

$$H_0(\mathcal{K}(k), R) \to R$$

induced by the rank map rk is an isomorphism. This follows from Definition 3.2 and Hurewicz Theorem. $\square$

By definition, the functor rk^* also sends additive objects to additive objects, but here the situation is even better.

Lemma 4.4. *The functor* $\mathsf{rk}_* : \mathcal{D}(\mathsf{Mat}(k), R) \to \mathcal{D}(M(R), R)$ *sends additive objects to additive objects, and* rk^*, rk_* *induce mutually inverse equivalences between* $\mathcal{D}_{add}(\mathsf{Mat}(k), R)$ *and* $\mathcal{D}_{add}(M(R), R)$.

Proof. Assume for a moment that we know that for any additive $E \in \mathcal{D}(\mathsf{Mat}(k), R)$, $\mathsf{rk}_* E$ is additive, and the adjunction counit map $\mathsf{rk}^* \mathsf{rk}_* E \to E$ is an isomorphism. Then for any additive $E \in \mathcal{D}_{add}(M(R), R)$, the cone of the adjunction unit map $E \to \mathsf{rk}_* \mathsf{rk}^* E$ is annihilated by rk^*. Since the functor rk^* is obviously conservative, $E \to \mathsf{rk}_* \mathsf{rk}^* E$ then must be an isomorphism, and this would prove the claim.
It remains to prove that for any $E \in \mathcal{D}_{add}(\mathsf{Mat}(k), R)$, $\mathsf{rk}_* E$ is additive, and the map $\mathsf{rk}^* \mathsf{rk}_* E \to E$ is an isomorphism. Note that we have

$$E \cong \lim_{\stackrel{\leftarrow}{n}}{}^{\bullet} \tau_{\geq -n} E,$$

where $\tau_{\geq -n}E$ is the truncation with respect to the standard t-structure. If E is additive, then all its truncations are additive, and by adjunction, rk_* commutes with derived inverse limits. Moreover, since derived inverse limit commutes with finite sums, it preserves the additivity condition. Thus it suffices to prove the statement under assumption that E is bounded from below with respect

to the standard t-structure. Moreover, it suffices to prove it separately in each homological degree n.
Since rk^* is obviously exact with respect to the standard t-structure, rk_* is right-exact by adjunction, and the statement is trivially true for $E \in \mathcal{D}^{\geq n+1}(\mathsf{Mat}(k), R)$. Therefore by induction, we may assume that the statement is proved for $E \in \mathcal{D}^{\geq m+1}_{add}(\mathsf{Mat}(k), R)$ for some m, and we need to prove it for $E \in \mathcal{D}^{\geq m}_{add}(\mathsf{Mat}(k), R)$. Let $\overline{E} = i^*E$. Since E is additive, $\overline{E}$ is also additive, so that $i_*\overline{E}$ is additive by Lemma 4.3 (i). The functor i_* is also right-exact with respect to the standard t-structures by adjunction, and by Lemma 4.3 (ii), the cone of the adjunction map

$$E \to i_*i^*E = i_*\overline{E}$$

lies in $\mathcal{D}^{\geq m+1}_{add}(\mathsf{Mat}(k), R)$. Therefore it suffices to prove the statement for $i_*\overline{E}$ instead of E. Since $\mathsf{rk}_* \, i_*\overline{E} \cong R^\bullet \overline{i}_*\overline{E}$ is additive by Lemma 4.3 (i), it suffices to prove that the adjunction map

$$\mathsf{rk}^* \, \overline{i}_*\overline{E} \cong \mathsf{rk}^* \, \mathsf{rk}_* \, i_*\overline{E} \to i_*\overline{E}$$

is an isomorphism. Moreover, since both sides are additive, it suffices to prove it after evaluating at $i([1]_+)$. We may assume that $\overline{E} = \mathcal{H}om(\overline{i}^*T, M)$ for some $M \in \mathcal{D}(R)$, so that by adjunction, this is equivalent to proving that the natural map

$$H^\Gamma_\bullet(i^*R_{i([1]_+)}) \to H^\Gamma_\bullet(\overline{i}^*R_{\overline{i}([1]_+)})$$

is an isomorphism. But as in the proof of Lemma 4.3, this map is the map

$$H^\Gamma_\bullet(C_\bullet(\mathsf{Iso}(k)^{\overline{S}}, R)) \to H^\Gamma_\bullet(R[\overline{S}])$$

induced by the functor rk, and by Lemma 4.1 (ii), it is identified with the map of homology

$$H_\bullet(\mathcal{K}(k), R) \to H_\bullet(H(R), R)$$

induced by the map of spectra (3.3). This map is an isomorphism by Definition 3.2. $\square$

4.3 Proof of the theorem. We can now prove Theorem 3.4. We begin by constructing the map. To simplify notation, let $K = K^R_\bullet(\mathcal{C}, k) \in \mathcal{D}(R)$, and let

$$E = L^\bullet\pi_{2!}R \in \mathcal{D}(\mathsf{Mat}(k), \mathbb{Z}) \subset \mathcal{D}(\mathcal{M}at(k), \mathbb{Z}).$$

Then by the projection formula, we have a natural quasiisomorphism

$$K \cong E \overset{\mathsf{L}}{\otimes}_{\mathcal{M}at(k)} \mathsf{rk}^{o*} T,$$

so that by adjunction, we obtain a natural map

$$v : E \to \mathcal{H}om(\mathsf{rk}^{o*} T, K). \qquad (4.6)$$

Restricting with respect to the 2-functor $i : \Gamma_+ \to \mathsf{Mat}(k)$, we obtain a map

$$\overline{v} : \overline{E} \to i^* \mathcal{H}om(\mathsf{rk}^{o*} T, K) \cong \mathcal{H}om(\overline{i}^{o*} T, K), \tag{4.7}$$

where we denote $\overline{E} = i^* E$. Now note that over $i(\mathcal{N}(\Gamma_+)) \subset \mathcal{M}at(k)$, the cofibration $\varphi : \mathbb{K}(\mathcal{C}, k) \to \mathcal{M}at(k)$ restricts to the special cofibration corresponding to the cofibration ρ_+ of (3.1). Therefore by base change, we have $\overline{E} \cong L^{\bullet}\rho_{+!}R$. Then to compute $\overline{E}$, we can apply the Grothendieck construction to the cofibration ρ_+ and use base change; this shows that $\overline{E} \in \mathcal{D}(\Gamma_+, R)$ can be represented by the homology complex

$$E_{\bullet} = C_{\bullet}(N(\mathrm{Gr}(\rho_+)), R).$$

Choose a complex $\overline{K}_{\bullet}$ representing $\mathcal{H}om(\overline{i}^* T, K) \in \mathcal{D}(\Gamma_+, R)$ in such a way that the map $\overline{v}$ of (4.7) is represented by a map of complexes

$$\overline{v}_{\bullet} : E_{\bullet} \to \overline{K}_{\bullet}.$$

Replacing $\overline{K}_{\bullet}$ with its truncation if necessary, we may assume that it is concentrated in non-negative homological degrees. Applying the Dold-Kan equivalence pointwise, we obtain a functor $\mathsf{D}(\overline{K}_{\bullet})$ from Γ_+ to simplicial abelian groups. We can treat it as a functor to simplicial sets, and take pointwise the tautological map (2.9); this results in a map

$$\overline{\nu} : N(\mathrm{Gr}(\rho_+)) \to \mathsf{D}(\overline{K}_{\bullet}) \tag{4.8}$$

of functors from Γ_+ to simplicial sets. Taking pointwise geometric realization, we obtain a map of Γ-spaces, hence of spectra. By definition, the Γ-space $|N(\mathrm{Gr}(\rho_+))|$ corresponds to the spectrum $\mathcal{K}(\mathcal{C})$. Since $\overline{K}_{\bullet}$ represents the additive object $\overline{i}^* \mathcal{H}om(T, K) \in \mathcal{D}(\Gamma_+, R)$, the isomorphisms (4.2) induce weak equivalences of simplicial sets

$$\mathsf{D}(\overline{K}_{\bullet})(S \vee S') \cong \mathsf{D}(\overline{K}_{\bullet})(S) \times \mathsf{D}(\overline{K}_{\bullet})(S'),$$

so that the Γ-space $|\,\mathsf{D}(\overline{K}_{\bullet})|$ is special. It gives the Eilenberg-MacLane spectrum $\mathcal{K}$ corresponding to $K \cong \overline{K}_{\bullet}([1]_+) \in \mathcal{D}(R)$. Thus the map of spectra induced by $\overline{\nu}$ of (4.8) reads as

$$\mathcal{K}(\mathcal{C}) \to \mathcal{K}. \tag{4.9}$$

This is our map.

To prove the theorem, we need to show that the map $\overline{\nu}$ induces an isomorphism on homology with coefficients in R. Let $\overline{S} \in \mathcal{D}(\Gamma_+, R)$ be the object represented by the chain complex $C_{\bullet}(\mathsf{D}(\overline{K}_{\bullet}), R)$. Then by Lemma 4.1 (ii), it suffices to prove that the map

$$H^{\Gamma}_{\bullet}(\overline{E}) \to H^{\Gamma}_{\bullet}(\overline{S}) \tag{4.10}$$

induced by (4.8) is an isomorphism. Moreover, note that we can apply the procedure above to the map v of (4.6) instead of its restriction $\overline{v}$ of (4.7). This results in a map of functors

$$N(\mathrm{Gr}(\varphi)) \to \mathsf{D}(K_\bullet),$$

where $K_\bullet$ is a certain complex representing $\mathcal{H}om(\mathsf{rk}^* T, K) \in \mathcal{D}(\mathsf{Mat}(k), R)$. If we denote by $S \in \mathcal{D}(\mathsf{Mat}(k), R)$ the object represented by $C_\bullet(\mathsf{D}(K_\bullet), R)$ and let

$$\nu : E \to S \tag{4.11}$$

be the map induced by the map v, then we have $S_0 \cong i^*S$, $i^*\nu$ is the map induced by $\overline{v}$ of (4.8), and (4.10) becomes the map

$$H^\Gamma_\bullet(i^*\nu) : H^\Gamma_\bullet(i^*E) \to H^\Gamma_\bullet(i^*S).$$

By adjunction and Lemma 4.3 (i), it then suffices to prove that for any additive $N \in \mathcal{D}(\mathsf{Mat}(k), R)$, the map

$$\mathrm{Hom}(S, N) \to \mathrm{Hom}(E, N)$$

induced by the map $\nu : E \to S$ is an isomorphism. By Lemma 4.4, we may assume that $N \cong \mathsf{rk}^* \widetilde{N}$ for some additive $\widetilde{N} \in \mathcal{D}(M(R), R)$, and by induction on degree, we may further assume that $\widetilde{N}$ lies in a single homological degree. But since R is a localization of $\mathbb{Z}$, any additive functor from $M(R)$ to R-modules is R-linear, thus of the form $\mathcal{H}om(T, M)$ for some R-module M. Thus we may assume $\widetilde{N} = \mathcal{H}om(T, M)$ for some $M \in \mathcal{D}(R)$. Again by adjunction, it then suffices to prove that the map

$$E \overset{\mathsf{L}}{\otimes}_{\mathcal{M}at(k)} \mathsf{rk}^{o*} T \to S \overset{\mathsf{L}}{\otimes}_{\mathcal{M}at(k)} \mathsf{rk}^{o*} T$$

induced by the map ν of (4.11) is an isomorphism. But the adjunction map v of (4.6) has the decomposition (2.11) that reads as

$$E \xrightarrow{\ \nu\ } S \xrightarrow{\ \kappa\ } \mathcal{H}om(\mathsf{rk}^{o*} T, K),$$

where κ is the assembly map (2.10) for the complex $K_\bullet$. Thus to finish the proof, it suffices to check the following.

Lemma 4.5. *For any object $K \in \mathcal{D}(R)$ represented by a complex $K_\bullet$ of flat R-modules concentrated in non-negative homological degrees, denote by $\widetilde{S} \in \mathcal{D}(M(R), R)$ the object represented by the complex $C_\bullet(\mathsf{D}(\mathcal{H}om(T, K_\bullet)), R)$, let $S = \mathsf{rk}^* \widetilde{S}$, and let*

$$\mathsf{rk}^* \kappa : S \to \mathsf{rk}^* \mathcal{H}om(T, K) \cong \mathcal{H}om(\mathsf{rk}^{o*} T, K)$$

be the pullback of the assembly map $\kappa : \widetilde{S} \to \mathcal{H}om(T, K)$. Then the map

$$S \overset{\mathsf{L}}{\otimes}_{\mathcal{M}at(k)} \mathsf{rk}^{o*} T \to K$$

adjoint to $\mathsf{rk}^* \kappa$ is an isomorphism in $\mathcal{D}(R)$.

Proof. For any $M \in R$-mod, we can consider the functor $\mathcal{H}om(T, M)$ as a functor from $M(R)$ to sets, and we have the assembly map

$$R[\mathcal{H}om(T, M)] \to \mathcal{H}om(T, M). \tag{4.12}$$

If M is finitely generated and free, then by definition, we have

$$\begin{aligned} R[\mathcal{H}om(T, M)](M_1) = R[\mathcal{H}om(T, M)(M_1)] &= R[\operatorname{Hom}(M_1^*, M)] \\ &\cong R[\operatorname{Hom}(M^*, M_1)] \end{aligned}$$

for any $M_1 \in M(R)$, so that $R[\mathcal{H}om(T, M)] \cong R_{M^*}$ is a representable functor. Therefore $\operatorname{Tor}_i^{M(R)}(R[\mathcal{H}om(T, M)], T)$ vanishes for $i \geq 1$, and the map

$$R[\mathcal{H}om(T, M)] \overset{\mathsf{L}}{\otimes}_{M(R)} T \cong R[\mathcal{H}om(T, M)] \otimes_{M(R)} T \to M$$

adjoint to the assembly map (4.12) is an isomorphism. Since $- \overset{\mathsf{L}}{\otimes} -$ commutes with filtered direct limits, the same is true for an R-module M that is only flat, not necessarily finitely generated or free.

Moreover, consider the product $\Delta^o \times M(R)$, with the projections $\tau : \Delta^o \times M(R) \to M(R)$, $\tau' : \Delta^o \times M(R) \to \Delta^o$. Then for any simplicial pointwise flat R-module $M \in \operatorname{Fun}(\Delta^o, R)$, the map

$$a : R[\mathcal{H}om(\tau^* T, M)] \overset{\mathsf{L}}{\otimes}_{\tau'} \tau^* T \to M \tag{4.13}$$

adjoint to the assembly map $R[\mathcal{H}om(\tau^* T, M)] \to \mathcal{H}om(\tau^* T, M)$ is also an isomorphism. Apply this to $M = \mathsf{D}(K_\bullet)$, and note that we have

$$K \cong L^\bullet \tau_! M, \qquad \widetilde{S} \cong L^\bullet \tau_! R[\mathcal{H}om(\tau^* T, M)],$$

and the map $\widetilde{S} \overset{\mathsf{L}}{\otimes}_{M(R)} T \to K$ adjoint to the assembly map κ is exactly $L^\bullet \tau_!(a)$, where a is the map (4.13). Therefore it is also an isomorphism.

To finish the proof, it remains to show that the natural map

$$\widetilde{S} \overset{\mathsf{L}}{\otimes}_{M(R)} T \to \mathsf{rk}^* \widetilde{S} \overset{\mathsf{L}}{\otimes}_{\mathsf{Mat}(k)} \mathsf{rk}^{o*} T = S \overset{\mathsf{L}}{\otimes}_{\mathsf{Mat}(k)} \mathsf{rk}^{o*} T$$

is an isomorphism. By adjunction, it suffices to show that the natural map

$$\operatorname{Hom}(\widetilde{S}, E) \to \operatorname{Hom}(\widetilde{S}, \mathsf{rk}_* \mathsf{rk}^* E) \cong \operatorname{Hom}(S, \mathsf{rk}^* E)$$

is an isomorphism for any additive $E \in \mathcal{D}(M(R), R)$, and this immediately follows from Lemma 4.4. $\square$

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