William Karush and the KKT Theorem

Richard W. Cottle

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1 Prologue

This chapter is mainly about William Karush and his role in the Karush-Kuhn-Tucker theorem of nonlinear programming. It tells the story of fundamental optimization results that he obtained in his master's thesis: results that he neither published nor advertised and that were later independently rediscovered and published by Harold W. Kuhn and Albert W. Tucker. The principal result – which concerns necessary conditions of optimality in the problem of minimizing a function of several variables constrained by inequalities – first became known as the Kuhn–Tucker theorem. Years later, when awareness of Karush's pioneering work spread, his name was adjoined to the name of the theorem where it remains to this day. Still, the recognition of Karush's discovery of this key result left two questions unanswered: why was the thesis not published? and why did he remain silent on the priority issue? After learning of the thesis work, Harold Kuhn wrote to Karush stating his intention to set the record straight on the matter of priority, and he did so soon thereafter. In his letter to Karush, Kuhn posed these two questions, and Karush answered them in his reply. These two letters are quoted below.

Although there had long been optimization problems calling for the maximization or minimization of functions of several variables subject to constraints, it took the advent of linear programming to inspire the name "nonlinear programming." This term was first used as the title of a paper [30] by Harold W. Kuhn and Albert W. Tucker. Appearing in 1951, the paper contained many results, but interest focused on the one declaring conditions that must be satisfied by a solution of the

> Maximum Problem. *To find an x^0 that maximizes $g(x)$ constrained by $Fx \geq 0$, $x \geq 0$.*

In this formulation of the problem, Fx denotes a mapping from R^n to R^m with component functions f_i, $i = 1, \ldots, m$. The function g and the f_i were all assumed to be differentiable.

A further assumption was immediately imposed. Kuhn and Tucker called it the *constraint qualification*. The precise statement of the Kuhn-Tucker constraint qualification is somewhat complicated, but it's purpose is easy enough to understand. It is used in assuring the existence of the nonnegative Lagrange multipliers, $u_1, \ldots, u_m$, which appear in the theorem statement. A simpler constraint qualification is the condition that the gradients of the active constraints at x^0 be linearly independent. Citing a paper of Fritz John [16] at this point, Kuhn and Tucker then went ahead and constructed the *Lagrangian function*

$$\phi(x, u) = g(x) + u'Fx$$

in which u denotes a vector of nonnegative Lagrange multipliers. With these assumptions in place, and the symbols ϕ_x^0 and ϕ_u^0 denoting the partial gradients of ϕ at (x^0, u^0) with respect to x and u, their result was

Theorem 1. *In order that x^0 be a solution of the maximum problem, it is necessary that x^0 and some u^0 satisfy conditions*

$$\phi_x^0 \leq 0, \qquad \phi_x^{0\prime} x^0 = 0, \qquad x^0 \geq 0 \tag{1}$$

$$\phi_u^0 \geq 0, \qquad \phi_u^{0\prime} u^0 = 0, \qquad u^0 \geq 0 \tag{2}$$

for $\phi(x, u) = g(x) + u'Fx$.

The equations and inequalities stated in (1) and (2) became known as the *Kuhn–Tucker conditions* for the stated maximum problem while the result itself became known as the *Kuhn–Tucker theorem*.

Unbeknownst to Kuhn and Tucker, their theorem and several others in their paper had been establlshed in 1939 by William Karush in his master's degree thesis [18]. At that time, Karush was a graduate student at the University of Chicago mathematics department which was noted for its preoccupation with a topic called *the calculus of variations*.

The fundamental problem in the calculus of variations is to find a function $\varphi(x)$ belonging to an admissible set of functions that minimizes the integral

$$I = \int_X^{\bar{X}} F\big(x, \varphi(x), \varphi'(x)\big)\, dx \tag{3}$$

where X, Y, $\bar{X}$, $\bar{Y}$ with $X < \bar{X}$ are given real numbers, such that $\varphi(X) = Y$, $\varphi(\bar{X}) = \bar{Y}$, and $F(x, y, z)$ is a given function of three independent variables. With each admissible function $\varphi(x)$ there is an associated real number I. Accordingly, when φ is regarded as an independent variable, I is a functional: a numerical-valued function of φ. (See Pars [34].)

Much of the research in the calculus of variations concentrated on necessary and sufficient conditions for relative minima in (specializations of) these problems. Karush's master's thesis dealt with a truly finite-dimensional version of this class of problems. He called the work "Minima of Functions of Several Variables with Inequalities as Side Conditions." In stating the problems he proposed to analyze, Karush first made reference to those of the familiar Lagrangian type where a point $x = (x_1, x_2, \ldots, x_n)$ satisying a system of equations

$$g_\alpha(x) = 0 \qquad (\alpha = 1, 2, \ldots, m)$$

is to be found so as to minimize a given function $f(x_1, x_2, \ldots, x_n)$. Saying that the necessary and sufficient conditions for a relative minimum in this equality-constrained minimization problem had already been satisfactorily treated, Karush then announced

> This paper proposes to take up the corresponding problem in the class of points x satisfying the inequalities
>
> $$g_\alpha(x) \geq 0 \qquad (\alpha = 1, 2, \ldots, m)$$
>
> where m may be less than, equal to, or greater than n.

Karush's minimization problem is clearly one of nonlinear programming in the sense of Kuhn and Tucker. It takes only a little bit of elementary manipulation and notation changing to cast the Kuhn–Tucker maximization problem in the form of a minimization problem studied by Karush. One slight (and insignificant) difference between the two papers is that Karush seems to assume his functions are of class C^1 (or C^2 for second-order results).

The precursor of (Kuhn and Tucker's) Theorem 1 appears in Karush's thesis as Theorem 3.2. Both the Kuhn–Tucker paper and the Karush paper point out the importance of the gradients of the active constraints (those satisfied as equations) at a relative maximum or minimum, respectively. Both papers make use of the notion of admissible arcs, both papers make use of linear inequality theory (even Farkas's lemma), and both papers address the need for a constraint qualification. Where the papers differ is that the Kuhn–Tucker paper was published and Karush's was not submitted for publication. Instead, it remained almost totally unknown for close to 30 years. This article tells more of the story about William Karush, his master's thesis, and its place in optimization.

2 Introduction

For roughly four decades, the result originally known as the Kuhn–Tucker (KT) Theorem has been called the Karush-Kuhn–Tucker (KKT) Theorem in recognition of the fact that in 1939 William Karush had produced the same result in his Master of Science degree thesis [18] at the mathematics department of the University of Chicago.[1] The Kuhn–Tucker paper [30] containing the eponymous theorem was published in 1951 having been presented the preceding year

[1] Actually, both the thesis and the KT paper contain separate theorems on first-order and second-order necessary conditions and sufficient conditions for local optimality.

at the Symposium on Mathematical Statistics and Probability held at the University of California, Berkeley.

Nearly every textbook covering nonlinear programming relates this fact but gives no more information than what is stated above. There are, however, publications that give a much more specific account of this history. For instance, Harold Kuhn (coauthor of the Kuhn–Tucker paper [30]) has written at least three others [27], [28], and [29] in which he "sets the record straight" about the earlier work by Karush in his master's thesis. In these three articles[2] Kuhn relates that he first became aware of Karush's earlier work from Akira Takayama's 1974 monograph *Mathematical Economics* [36]. Kuhn has much more to say than just this. He gives a brief overview of publications prior to 1974 that cite the work of Karush. These include Pennisi [35], El-Hodiri [10], [11], and Fiacco and McCormick [13]. Both Takayama [36, pages 61 and 101], [37, pages 65 and 105], and Kuhn [27, pp. 10–11] present the key points regarding literature that very well could have influenced Karush.

Moreover, it is worth reiterating a point already plain by Kuhn: namely, that Karush's MS thesis also contains what we know today as Fritz John's Theorem, a result that appeared in a 1948 paper [16] later cited by Kuhn and Tucker [30] but not actually declared there because it was inserted when the paper was in galley proof. John makes no mention of Karush's work even though his research might be viewed as close to the mathematical school of thought from which Karush emerged. Kuhn [27, p. 15] tells the interesting story of John's experience in the process of attempting to publish his paper. The three cited papers by Kuhn are very informative, yet somewhat limited in scope. There is more to say on how Takayama became aware of Karush's Master of Science thesis – and about the thesis itself.

I am grateful to Professor Kuhn for introducing me to the writings of Professor Tinne Hoff Kjeldsen, a professor of mathematics and historian of mathematical science at the University of Roskilde in Roskilde, Denmark. I wrote to her at once. She soon replied and kindly sent me a batch of her papers [23], [24], [25], and [26] on this subject. For most people, the most easily found of these papers is certain to be the rewarding journal article [24].

Professor Kjeldsen provided something else of enormous interest. In February 1975, as Harold Kuhn was preparing for his first historic effort to set the priority record straight, he wrote to William Karush announcing this intention. Copies of their correspondence were given to Kjeldsen when she visited Kuhn at Princeton to gather information for her doctoral dissertation. In 2012, when I came along requesting copies of this correspondence, they were no longer in Kuhn's possession, having been discarded in the process of vacating his mathematics department office at Princeton. Fortunately, Professor Kjeldsen had copies of this valuable correspondence and graciously shared them with me. On March 7, 2012 I returned them (electronically) to Professor Kuhn. Among

[2] Except for their typesetting method and their Introductions, the first two of these articles are very much alike; the third is more autobiographical in nature. Here, for reasons of brevity and historical precedence, the earliest one [27] will be used for most citations.

other things, this correspondence addresses two questions that virtually all observers would ask: why didn't Karush publish his MS thesis and why didn't he make its existence known after the appearance of the Kuhn–Tucker paper, some 11 or 12 years later? Kuhn covers the main facts on this story in [27]. Karush's answers to these and other questions from Kuhn are revealed below.[3]

What else does this chapter have to offer? In light of the widely known and available literature on nonlinear programming and the herein repeatedly cited historical papers by Kuhn and Kjeldsen, it seems unnecessary to spell out all the Karush-Kuhn–Tucker theorems with an analysis of whose paper had what, especially because Kuhn has so usefully reproduced the similar content of Karush's thesis in [27]. And because the published Kuhn–Tucker paper can be found in many university libraries as well as online at `https://projecteuclid.org`, I have chosen to concentrate on a few other aspects of Karush's MS thesis. To obtain a proper appreciation of this work, one must consider it as a product of the milieu in which it was created, namely the research of the University of Chicago mathematicians devoted to the calculus of variations. Some of this has been done in [36], [27], and [24]. In truth, the exposition given here is much briefer than it could be.

Quite a lot has been written about the careers of Harold W. Kuhn and Albert W. Tucker (see, for example, [24, p. 342], [2, Chapters 29 and 6], and a multitude of web sites including [38]), what then remains to be given is a bio-sketch of William Karush. Even this can be found on the web, but primarily in thumbnail form. The bio-sketch of Karush in this paper includes his image (which cannot ordinarily be seen elsewhere). As a bonus, the paper also exhibits an image of Fritz John (one can be found on the web). While both the biographical information and the concluding reference list provided here are necessarily condensed, they may prove to be the main contributions of this article and provide an incentive to explore this subject in greater depth.

3 On Karush's Master's Thesis

Dated December, 1939, the body of William Karush's master's thesis is a 25-page document centered between two pages of front matter (the title page and table of contents) and two pages of back matter (the list of references and a half-page vita). In the vita Karush provides information on his date and place of birth, his prior education, and the (sur)names of ten faculty members under whom he studied at the University of Chicago. He acknowledges them all for "the helpful part they played in his mathematical development" and then singles out Professor Lawrence M. Graves, thanking him "for his guidance as a teacher and in the writing of this dissertation." The work is composed of six sections, of which the first is an introduction to the class of problems under investigation, and the second presents preliminary results on systems of linear inequalities (about eight pages in all). The remaining four sections take up

[3] Kjeldsen [24, pp. 337–338] quotes a portion of this correspondence as well.

necessary conditions and sufficient conditions involving only first derivatives and then the same issues involving second derivatives.

Karush's results are given in the Appendix of Kuhn's paper [27]. Not given, however, is Karush's list of references. The following is a replica thereof.

LIST OF REFERENCES

1. Bliss, G. A., Normality and Abnormality in the Calculus of Variations, Transactions of the American Mathematical Society, vol. 43 (1938), pp. 365-376.

2. Dines, L. L., Systems of Linear Inequalities, Annals of Mathematics, vol. 23 (1922), p. 212.

3. Dines and McCoy, On Linear Inequalities, Transactions of the Royal Society of Canada, vol. 27 (1933), pp. 37-70.

4. Farkas, J. I., Theorie der einfachen Ungleichungen, Crelle, vol. 124 (1902), p. 1.

Stylistic inconsistency aside, three aspects of this list are peculiar. The first is that it contains only one publication from the calculus of variations. To a slight extent, this topic will be discussed in another section of this article. The second is that W.B. Carver, *not* L.L. Dines, is the author of the paper listed as Reference 2. The third (very minor) oddity is the insertion of a middle initial on the name of Farkas. His forename is given as "Julius" on the original German language paper, though in his native Hungary it would have been "Gyorgy." And speaking of names, "Crelle" is a common nickname used for "Journal für die reine und angewandte Mathematik" which in 1826 was founded and edited by August Leopold Crelle in Berlin.

As stated above, the questions of why the thesis was not published and why its author remained silent on the subject after the publication of the Kuhn–Tucker paper were discussed in very cordial correspondence between Harold Kuhn and William Karush. I now take the liberty of quoting from some (almost the entirety) of it. On February 4, 1975 Kuhn wrote:

> In March I am talking at an AMS Symposium on "Nonlinear Programming - A Historical View." Last summer I learned through reading Takayama's Mathematical Economics of your 1939 Master's Thesis and have obtained a copy. First, let me say that you have clear priority on the results known as the Kuhn–Tucker conditions (including the constraint qualification). I intend to set the record as straight as I can in my talk. You could help me if you would be kind enough to give me whatever details you remember regarding the writing of your thesis. Of special interest to me would be answers to the following questions: Who was your advisor (or other faculty influences)? Who set the problem? Why was the thesis never published? (One possibility would be to attempt (at least partial) publication as an appendix to my survey.)

> Dick Cottle, who organized the session, has been told of my plans to rewrite history and says "you must be a saint" not to complain about the absence of recognition. Al Tucker remembers you from RAND, wonders why you never called this to his attention and sends his best regards,

In his friendly reply, dated February 10, 1975, Karush said:

> Thank you for your most gracious letter. I appreciate your thoughtfulness in wanting to draw attention to my early work. If you ask why I did not bring up the matter of priority before, perhaps the answer lies in what is now happening – I am not only going to get credit for my work, but I am going to crowned a "saint"!
>
> I wrote my master's thesis at the University of Chicago under Lawrence M. Graves, who also proposed the problem. Those were the final years of the school of classical calculus of variations at the University and I suppose that the problem was given to me as a finite-dimensional version of research going on in the calculus of variations with inequalities as side conditions. Gilbert A. Bliss was chairman of the department, and Magnus R. Hestenes was a young member of the faculty; both of these men influenced me, and in fact I wrote my doctoral thesis later under Hestenes on isoperimetric problems and index theorems in the calculus of variations (this work <u>was</u> published after the war). The thought of publication never occurred to me at the time I wrote the master's thesis. I was a struggling graduate student trying to meet the requirements for going on to my Ph.D. and Graves never brought up the question of publication. I imagine nobody at that time anticipated the future interest in the problem,
>
> That does not answer the question of why I did not point to my work in later years when nonlinear programming took hold and flourished. The thought of doing this did occur to me from time to time, but I felt rather diffident about that early work and I don't think I have a strong necessity to be "recognized". In any case, the master's thesis lay buried until a few years ago when Hestenes urged me to look at it again to see if it shouldn't receive its proper place in history – he expressed an interest in setting the record straight in some publication of his own. So I did look at the thesis again, and I looked again at your work with Tucker. I concluded that you two had exploited and developed the subject so much further than I, that there was no justification for announcing to the world, "Look what I did, first." I expressed my feelings to Magnus Hestenes in the past year and that closed the matter as far as I was concerned. I will tell Magnus of your AMS Symposium talk and I am sure he will be glad of it.

This refreshing exchange of letters would seem to represent the last word on the subject. In the period from 1939 to 1942: Karush was, as he testified, busy working on a doctoral thesis and WWII broke out. It has been asserted that publication was curtailed during the war due to a shortage of paper. In any case, [18] was just a master's thesis, part of the degree requirements, and was a finite-dimensional version of results already in print. As Kjeldsen's contextualized historical analysis [24] of the matter emphasizes, it was a little ahead of its time, particularly of the post-WWII period.

There remains the question: How did Takayama learn of Karush's work? Takayama's book [36], and subsequently Kuhn's papers [27], [28], and [29] suggest how this happened. Takayama heard about it from Mohamed A. El-Hodiri [12] who (in 1963) had found a reference to [18] in a paper by Louis L. Pennisi [35]. El-Hodiri related this information to Leo Hurwicz among others and incorporated the Karush/John/Kuhn–Tucker results into his own writings [10], [11]. Strangely *missing* from the literature of the 1960s is a reference to Karush's MS thesis (and the KT paper) in the book [14] by Magnus Hestenes. Nine years later, Hestenes's book [15] gave Karush his due.

4 The Chicago School

William Karush began his undergraduate education in Chicago at Central Y.M.C.A. College.[4] He spent two years there after which he transferred to the University of Chicago, receiving the Bachelor of Science degree there in June, 1938. His graduate studies began there in October that same year. The mathematics department at the University of Chicago was known as a bastion of the study of the calculus of variations. The history of the department and this powerful tradition have been chronicled in numerous articles, many available online. For our purposes, the works of Kuhn [27] and Kjeldsen [24] are more than adequate starting points, relating directly as they do to our subject. Kjeldsen's article in particular goes into greater detail about the history and reputation of the department. She reports how it was thought (even by some Chicago mathematicians) to be exceptionally narrow with its concentration on the calculus of variations.

Nevertheless, the Chicago mathematics department maintained a grand heritage. It is instructive (one might say fruitful) to trace a portion of the mathematical tree that leads to William Karush's master's thesis. As stated above, the problem was set Lawrence M. Graves, and the work was carried out under his supervision. Graves's Ph.D. thesis advisor was Gilbert A. Bliss who was Chairman of the mathematics department at the time. Bliss was a powerful figure in the study of calculus of variations. He supervised the Ph.D. theses of many other mathematicians who are well known in mathematical programming circles today. They include, Lloyd Dines, Magnus Hestenes, Alston Householder, Edward McShane, and Frederick Valentine (who was advised

[4] In 1945, this institution became Roosevelt University.

by Graves in addition to Bliss). Bliss's Ph.D. thesis was supervised by Oskar Bolza whose Ph.D. was obtained in Göttingen under the supervision of C. Felix Klein. Three more such steps lead us from Klein to Julius Plücker and Rudolf Lipschitz (jointly) to Christian Ludwig Gerling to Carl Friedrich Gauß. This impressive lineage can be reconstructed using the Mathematics Genealogy Project [33].

Returning now to the master's thesis of Karush, it is important to note that the results have been described by Takayama [36, pages 61] as a finite-dimensional versions of counterparts from Valentine's doctoral dissertation [40] completed in 1937. Indeed, even Karush (in his previously quoted letter to Kuhn) said, "I suppose that the problem was given to me as a finite-dimensional version of research going on in the calculus of variations with inequalities as side conditions." Pennisi was, it seems, the first to cite Karush's thesis, albeit briefly. In [35, section 3] which is called "The problem with a finite number of variables", Pennisi asserts

> For the normal case, which is the only one we consider, our results are more general than those of Karush.

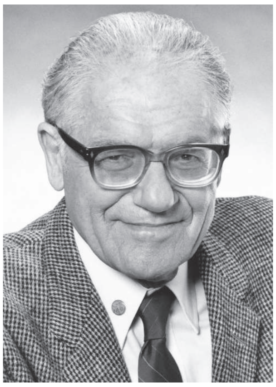
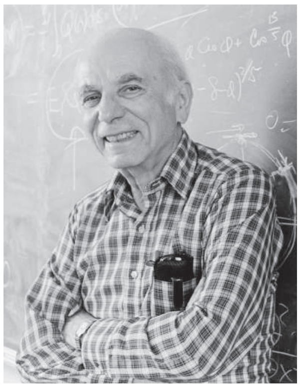
Pennisi refers to Valentine's Ph.D. thesis [40], but does not speak of [18] as a finite-dimensional version of it, Nonetheless, it is interesting to note that Valentine, Karush, and Pennisi were all supervised by Graves at the University of Chicago.

The title of Valentine's doctoral dissertation "The Problem of Lagrange with Differential Inequalities as Added Side Conditions" uses some rather common terminology of the time. Many research papers focused on "The Problem of Lagrange." Another commonly treated subject was "The Problem of Bolza." The phrase "added side conditions" is how these contemporary mathematicians spoke of what we call "constraints." This kind of terminology is found in the title of Fritz John's paper as well.

More broadly the introduction of inequalities as "side conditions" had been going on for some time at the University of Chicago and elsewhere, and not just by Fritz John. In the calculus of variations literature, one finds inequalities as side conditions in Bolza's 1913 paper [7]. Moreover, as noted by Kuhn [27], the type of modified Lagrangian function that we associate with Fritz John had been used by Bliss [5] many years earlier. In fact, Bliss himself used it well before 1938, for example, in the notes for his lectures [3] given in the Summer Quarter of 1925. Before that, Courant and Hilbert [9, p. 143] used this type of Lagrangian function and commented that if the multiplier associated with the minimand (objective function) is nonzero, then the conventional Lagrangian function can be recovered.

5 A Biographical Sketch of William Karush

William Karush was born in Chicago, Illinois on March 1, 1917. His parents Sam and Tillie (formerly Shmuel and Tybel) were fairly recent immigrants,

William Karush, circa 1987 Fritz John at NYU, circa 1987

Harold Kuhn and Albert Tucker, 1980
at von Neumann Prize presentation

(Printed with permission of Larry Karush; NYU; Harold Kuhn and Alan Tucker.)

having come to the United States from Bialystok which was then under Russian control. (It is now in Poland.) As a child, William was known as "Willie;" his older brother Fred was called "Freddie" [39]. They eventually had two younger siblings, Jack and Esther. Of the four, only Esther is still living. Willie outgrew this diminutive name and became known as "Will." He attended public schools in Chicago, graduating from Murray F. Tuley High School in June, 1934. From that point on, his Bachelor of Science, Master of Science, and Doctor of Philosophy were all earned at the University of Chicago in 1938, 1939, and 1942, respectively [18].

Based on an entry in the 17th Edition of *American Men & Women of Science* [1, p. 215], the table below gives a summary of the positions held by William Karush. The table does not make explicit the fact that during World War II, Karush worked on the Manhattan Project which culminated in the atomic

Table 1: Employment Chronology of William Karush [1]

Year	Position	Employer
1942–43	Mathematician	Geographical Laboratory, Carnegie Inst. of Washington
1943–45	Physicist	Metallurgical Laboratory, University of Chicago
1945–56	Instructor to Associate Professor	Mathematics Deptartment, University of Chicago
1956–57	Member, Senior Staff	Ramo-Wooldridge Corporation
1958–62	Sr. Operations Research Scientist	System Development Corporation
1962–67	Principal Scientist	System Development Corporation
1967–87	Professor of Mathematics	California State University, Northridge
1987–97	Emeritus Professor of Mathematics	California State University, Northridge
Concurrent Positions		
1949–52	Mathematician	Inst. Numerical Anal., Nat. Bur. Standards, UCLA
1953	Member, Technical Staff	Research & Development Labs., Hughes Aircraft
1954–55	Member, Technical Staff	Ramo-Wooldridge Corporation
1955–56	Ford Faculty Fellow	University of California, Los Angeles

bombs that the United States used on Hiroshima and Nagasaki, Japan. As it happens, though, William Karush was one of 155 scientists of the Manhattan Project of Oak Ridge, Tennessee who in 1945 put their names to the so-called Szilárd Petition which was drafted by physicist Léo Szilárd "and asked President Harry S. Truman to consider an observed demonstration of the power of the atomic bomb first, before using it against people" [41]. The petition never reached Truman. In later years, Will Karush became an outspoken peace advocate [32]. The portrait of him presented here shows Karush wearing a "Beyond War" pin on his shirt collar.

In general, William Karush listed his research interests as belonging to operations research, calculus of variations, and applied mathematics. His published works in operations research include papers in mathematical programming, queueing, and dynamic programming. He is also known for having edited two different dictionaries of mathematics [20], [22].

As is evident from the table above, Karush had a varied career: part of it in industry, and a somewhat larger part in academia. At the University of Chicago (1945–56) he rose from instructor to associate professor. He took a leave of absence in southern California and never returned to the University of Chicago. Eleven years later, he joined the faculty of California State University (at the time called "San Fernando Valley College") as a full professor where his duties involved the teaching of undergraduate-level mathematics. He taught there until 1987 at which time he retired and became an emeritus professor.

Will Karush and his wife, Rebecca, were close friends of Richard E. Bellman of dynamic programming fame. For a number of years, Rebecca was a technical typist for Bellman. Will and Rebecca had two children, Larry and Barbara, both of whom live in California. Larry is a musician (see [17]). Barbara is a retired school teacher. In January 1991, Will and Rebecca took a short vacation in Palm Springs, California. One evening after dinner, Rebecca was struck by a car and fatally injured. Will Karush lived until February 22, 1997, one week before his 80th birthday. He died of complications from surgery.

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Many people deserve credit for helping me to produce this view of William Karush and the Karush-Kuhn-Tucker Theorem. Accordingly, it is a great pleasure to acknowledge that I received various sorts of information and assistance from John R. Birge, Harold W. Kuhn, Tinne Hoff Kjeldsen, Mohamed A. El-Hodiri, Larry Karush, Esther Diamond, Stephen D. Brady, Philip Wolfe, Saul I. Gass, Kenneth J. Arrow, Ilan Adler, Werner Horn, William Watkins, George Biriuk, Malcolm Soule, Joel L. Zeitlin, Efrem Ostrow, Ingram Olkin, Edwin Knihnicki, Margaret H. Wright, April E. Bacon, Joseph B. Keller, and library and departmental staff from coast to coast. As usual, the flaws are mine alone.

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Richard W. Cottle
Department of Management
Science and Engineering
Stanford University
Stanford, California 94305-4121
USA
`rwc@stanford.edu`