Euler, Mei-Ko Kwan, Königsberg, and a Chinese Postman

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2010 Mathematics Subject Classification: 00-02, 01A05, 05C38, 90-03
Keywords and Phrases: Eulerian graphs, Chinese Postman Problem

Looking at the world's history, nothing very important happened in 1736. There was one exception, at least for mathematicians. Leonhard Euler wrote an article [3] with the title "Solutio Problematis ad Geometriam Situs Pertinentis", a paper of 13 pages with 21 short paragraphs, published in St. Petersburg, Russia. The paper looks like treating a certain puzzle, and it did not receive much attention for a long period of time. Moreover, in his own research Euler never returned to this particular topic. In retrospect, his article on the bridges of Königsberg laid the foundations of graph theory, a new branch of mathematics, that is today permeating almost every other science, is employed even in daily life, has become a powerful modeling language and a tool that is of particular importance in discrete mathematics and optimization. Euler could have become the father of combinatorial optimization, but he missed this opportunity. A young Chinese mathematician was the first to consider an optimization version of Euler's bridges problem which was later called the Chinese Postman Problem in his honor.

Readers interested in graph theory papers of historic relevance should consult [1] which contains a collection of 37 important articles, translated into English; [3] is the first one in this collection.

Leonhard Euler: When did he solve the Königsberg bridges problem?

We refrain from saying here more than a few words about the life of Leonhard Euler. Almost infinitely many books and papers describe aspects of his work. The article [5] in this book sketches some of the important steps of his career. Clifford Truesdell's (1919-2000) estimate that Euler produced about one third of all the mathematical literature of the 18^{th} century indicates his distinguished role. But Euler's interests went far beyond mathematics. He made significant contributions to engineering, cartography, music theory, philosophy, and theology.

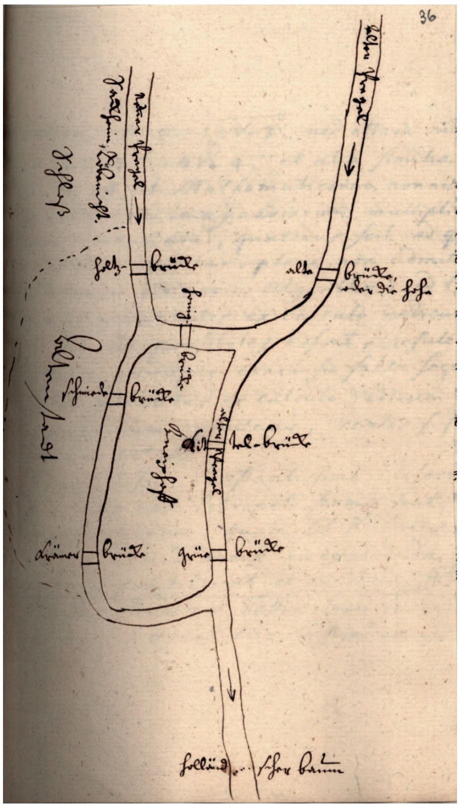

Figure 1: Ehler's drawing of Königsberg, 1736

There is almost no book in graph theory today that does not show a copy of the map of Regiomonti in Borussia (Königsberg in Prussia, today, Kaliningrad in Russia) that one can find in Euler's article and that explains how Euler abstracted the concept of a graph from this map. Fig. 1 shows the *real original drawing* that we obtained from W. Velminski who made a copy for his book [9] in the St. Petersburg archive from the original Ehler letter mentioned below.

It is not known for sure when and from whom Euler learned about the Königsberg bridges for the first time. (Euler, as far as one knows, never visited Königsberg.) What is known is that he corresponded with Karl Leonhard Gottlieb Ehler about this problem (variations of the name in the literature: Carl instead of Karl and Ehlers instead of Ehler), where Ehler acted as an intermediary between Euler and the mathematician Heinrich Kühn from Danzig. Ehler was a mathematics enthusiast; and he was the mayor of Danzig from 1740 to 1753. A list of 20 letters exchanged in the period 1735 to 1742 between these two can be found at `http://eulerarchive.maa.org/correspondence/correspondents/Ehler.html`. The article [8] investigates three letters that deal with the Königsberg bridges and also shows a copy of Fig. 1. This drawing is from the first of these letters, dated March 9, 1736. One may infer from this letter that Euler and Ehler had already discussed the bridges problem, but if Euler had known the details of the problem, it would not have been necessary for Ehler to produce this drawing. And so it is not unreasonable to assume that Euler learned the problem through this letter. This reasoning, though, contradicts the statement in the minutes of the St. Petersburg Academy that Euler presented the Königsberg bridges problem to the Academy on August 26, 1735. Velminski claims in [9] that this date may be a misprint.

Confusion occurs also with respect to the publication date of Euler's paper. It is contained in the 1736 Academy volume, but the publication was delayed so that the volume only appeared in 1741. What is known, due to still existing letters, see [8], is that Euler outlined his solution of the problem in letters to Giovanni J. Marinoni (March 13, 1736) and to Ehler (April 3, 1736). And so we prefer to regard 1736 as the birth year of graph theory in which the following problem was addressed:

The Königsberg Bridges Problem (*briefly* KBP):
Is it possible for a pedestrian to walk across all seven bridges in Königsberg without crossing any bridge twice?

Euler could have worked hard to solve this particular problem instance by checking cases, but he, and this distinguishes true mathematicians from puzzle solvers, tries to solve this problem type, once and for all, for all possible instances and not just for Königsberg. He, thus, formulated what we call the

Eulerian Path *(or Walk)* Problem (briefly EPP):
Is it possible to traverse a graph passing through every edge exactly once?

Euler's results

Here is a sketch of what Euler did in his paper.

Euler mentions the "almost unknown" *geometriam situs*, a term introduced by Leibniz and today usually translated into topology or graph theory, and says that "this branch is concerned only with the determination of position and its properties; it does not involve distances, nor calculations made with them." He claims that the bridges problem belongs to this area.

He states the EPP verbally, introduces the symbols a, b, c... for the bridges (the edges of the graph) and the symbols A, B, C,... for the areas of Königsberg linked by the bridges (the nodes of the graph). (The terms graph, node, vertex, and edge did not exist yet.) He also denotes an edge by a pair of nodes, such as a=AB, introduces the notation ABD for a path that links the nodes A and D via the sequence of edges AC and CD, and defines path length. He even discusses notational difficulties with parallel edges. Graph theory notation and notational trouble have not much changed since 1736!

Euler also states that solving the problem for Königsberg by enumeration is possible but too laborious and hopeless for EPP in general.

Euler then argues that a solution of KBP must have a representation by a sequence AB... of 8 letters/nodes from the 4 letters A,B,C,D (with side constraints) and counts node degrees along a path. Degree counting for KBP results in: node A must appear 3 times in the sequence (path), nodes B, C, D must appear twice each, but the sequence must have length 8. This is a contradiction, and KBP is solved. There is no such path!

Now follows a verbal statement of what we today call

Euler's Theorem:
A graph has an Eulerian path if and only if it has 0 or 2 nodes of odd degree.

Euler does not mention connectivity, it appears that he assumes that a graph has to be connected.

Afterwards Euler discusses various cases and a more general example. And then he states and proves what one can truly call the

First Theorem of Graph Theory:
In any graph, the sum of node degrees is equal to twice the number of edges.

And he continues with the

Second Theorem of Graph Theory:
In any graph, the number of nodes of odd degree is even.

Euler remarks that KBP could be solved if all bridges were doubled, and then states his theorem formally, copied from [3]:

> Si fuerint plures duabus regiones, ad quas ducentium pontium numerus eſt impar, tum certo affirmari poteſt, talem tranſitum non dari. Si autem ad duas tantum regiones ducentium pontium numerus eſt impar, tunc tranſitus fieri poterit, ſi modo curſus in altera harum regionum incipiatur. Si denique nulla omnino fuerit regio, ad quam pontes numero impares conducant, tum tranſitus deſiderato modo inſtitui poterit, in quacunque regione ambulandi initium ponatur. Hac igitur data regula problemati propoſito pleniſſime ſatisfit.

Euler, though, has shown so far only that if a graph has more than two nodes of odd degree then there is no Eulerian path. He then argues:

> *When it has been determined that such a journey can be made, one still has to find how it should be arranged. For this I use the following rule: let those pairs of bridges which lead from one*

> *area to another mentally be removed (*deletion of pairs of parallel edges*), thereby considerably reducing the number of bridges; it is then an easy task to construct the required route across the remaining bridges, and the bridges which have been removed will not significantly alter the route found, as will become clear after a little thought. I do not therefore think it worthwhile to give any further details concerning the finding of the routes.*

We do not doubt that Euler knew how to construct an Eulerian path, but the text above is not what one could call a proof. Those who have taught Euler's theorem in class know the problem. It is really difficult to provide a short sequence of convincing arguments. Hand waving in front of the blackboard usually does the trick! The theory of algorithms did not exist in his time, and Euler did not have the concept of recursion, for instance, to describe his thoughts. In a formal sense, thus, Euler did not prove his characterization of Eulerian graphs. It took 140 further years to get it done.

Carl Hierholzer

The final step of the proof has an interesting story of its own. The first full proof of Euler's theorem was given by C. Hierholzer (1840–1871). He outlined his proof in 1871 to friends but passed away before he had written it up. Christian Wiener re-composed the proof from memory with the help of Jacob Lüroth. The resulting paper [4] was published in 1873 and contains what is now sometimes called the Hierholzer algorithm for the construction of an Eulerian path or cycle.

Euler and optimization

If one glances through Euler's publications, it looks like one topic seems to have permeated his work: the idea of minima and maxima. Just read the introduction to this book. One could have guessed that, after having characterized the existence of an Eulerian path or cycle in a graph, he would have raised (and tried to answer) one of the questions: How many edges does one have to add to a graph or how many edges does one have to double so that an Eulerian path or cycle exist? More generally, if one considers walking distances in addition, Euler could have asked: What is the shortest walk covering every edge at least once? He came close to this issue, since he mentioned that one can solve KBP by doubling all edges. If he had done this next step, we could call Euler rightfully the "father of combinatorial optimization". Euler missed this opportunity. It took 224 years until an optimization version of the Eulerian graph problem was considered, and this was in China.

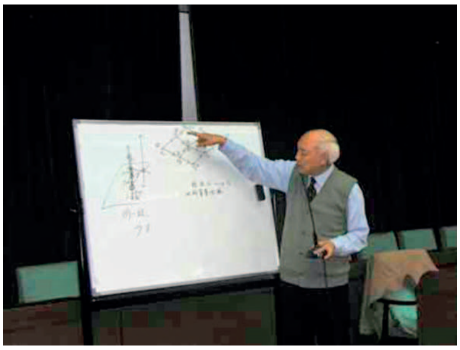

Figure 2: Mei-Ko Kwan

Mei-Ko Kwan and the Chinese Postman Problem

Before going to 1960 we take a step back in history. The great Chinese philosopher Confucius (551 BC – 479 BC) was born in the city of Qufu in Shandong Province. As the homeland of Confucius, Shandong has played a major role in Chinese history. During the Great Leap Forward movement (1958-1960), Chinese scientists were encouraged to solve real-world problems to help Chairman Mao's ambitious campaign to rapidly transform the country from an agrarian economy into a modern communist society. At that time, many mathematicians in China were engaged in real-world applications, and in particular, carried out operations research (OR) activities, focusing on problems such as transportation and production planning. Shandong, one of the few provinces where early Chinese OR application activities took place, is in fact the birthplace of the Chinese Postman Problem.

In 1960, the 26 years old Mei-Ko Kwan (modern PinYin spelling: Mei-Gu Guan), a young lecturer at Shandong Normal University, published his paper [6], in which he stated the following problem.

Chinese Postman Problem*:*
A postman has to deliver letters to a given neighborhood. He needs to walk through all the streets in the neighborhood and back to the post-office. How can he design his route so that he walks the shortest distance?

Due to this paper and other contributions to optimization, Mei-Ko Kwan became one of the leading experts on mathematical programming in China. He was, for instance, the president of Shandong Normal University from 1984 to 1990, and from 1990 to 1995, director of the OR department of Fudan University, the best university in Shanghai. In 1995, Mei-Ko Kwan moved to Australia and has worked at the Royal Melbourne Institute of Technology.

By calling a node of a graph odd or even if the number of edges incident to the node is odd or even, Kwan converted the Chinese postman problem into the following optimization problem on a graph:

PROBLEM
Given a connected graph where 2n of the nodes are odd and all other nodes are even. Suppose we need to add some edges to the graph with the following property: the number of edges added to any odd node is odd and that added to any even node is even. We need to minimize the total length of the added edges.

The main theoretical result Kwan proved in [6] is the following theorem:

THEOREM:
For a set of added edges it is necessary and sufficient to be an optimal solution for the above problem if the following two conditions hold:
(1) Between any two nodes, no more than one edge is added.
(2) In any cycle of the extended graph, the total length of the added edges is not greater than half of the total length of the cycle.

His proof is constructive; this way Kwan [6] also proposed a method for finding a solution to the Chinese Postman Problem. Fig. 3 shows two drawings copied from his original paper [6]. In the left diagram, the dotted lines are the added edges, while the right diagram shows an optimal solution:

图 4 图 5

Figure 3

Kwan's original paper was published in Chinese. Two years later the paper [6] was translated into English [7], which attracted the attention of Jack Edmonds. Edmonds was the one who introduced this interesting problem to the optimization community outside China, and he was also the first person to name it Chinese Postman Problem. Moreover, J. Edmonds and E. L. Johnson proved in a beautiful paper [2] that the Chinese Postman Problem can be reduced to matching, and thus, that it is solvable in polynomial time. This result was out of reach for mathematicians of the 18^{th} century; even for Kwan this was not an issue since modern complexity theory did not yet exist in 1960.

But if Euler had known linear programming and complexity theory, who knows?

REFERENCES

[1] N. L. Biggs, E. K. Lloyd and R. J. Wilson, *Graph theory 1736–1936*, Reprint with corrections, Clarendon Press, 1998.

[2] J. Edmonds and E. L. Johnson, Matching, Euler tours and the Chinese Postman. *Mathematical Programming* 5 (1973) 88–124.

[3] L. Euler, Solutio Problematis ad Geometriam Situs Pertinentis, *Commentarii Academiae Scientiarum Imperialis Petropolitanae* 8 (1736/1741) 128–140.

[4] C. Hierholzer, Über die Möglichkeit, einen Linienzug ohne Wiederholung und ohne Unterbrechung zu umfahren, *Mathematische Annalen* VI (1873) 30–32.

[5] E. Knobloch, Euler and infinite speed, this volume.

[6] Mei-Ko Kwan, Programming method using odd or even pints, *Acta Mathematica Sinica* 10 (1960) 263–266 (in Chinese).

[7] Mei-Ko Kwan, Graphic programming using odd or even points, *Chinese Mathematics* 1 (1962) 273–277.

[8] H. Sachs, M. Stiebitz and R. J. Wilson, An Historical Note: Euler's Königsberg Letters, *Journal of Graph Theory* 12 (1988) 133–139.

[9] W. Velminski, *Leonhard Euler: Die Geburt der Graphentheorie*, Kadmos, Berlin, 2008.

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